# What are Your Powers? - Truth Set Algebras

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#### Abstract

The paper studies the interplay between modalities representing four different types of multistep strategies in the imperfect information setting. It introduces a new "truth set algebra" technique for proving undefinability, which is significantly different from the existing techniques based on bisimulation. The newly proposed technique is used to prove the undefinability of each of the four modalities through a combination of the three others.

#### Introduction

In this paper, we study multistep strategies in single-agent transition systems with imperfect information. An example



Figure 1: A transition system.

of such a system is depicted in Figure 1. This system has seven states: a, b, c, d, u, v, and w. In each state, the agent has two possible commands: "go" and "stop". In the figure, the result of the command "go" in each state is shown using an arrow. For example, the command "go" in state b results in a transition of the system into state c. The command "stop" always terminates the execution of the strategy. We assume that atomic proposition p is true only in state v.

Multistep strategies may either *maintain* or *achieve* a condition. Strategies to achieve a condition, in turn, can be divided into those that stop once the condition is achieved and those that "achieve in passing" – they might continue the execution after the condition is achieved. An example of the latter for the transition system depicted in Figure 1 is the strategy "execute command 'go' 100 times", which can be used to "achieve in passing" condition p from each of the states in our system except for state a. In the rest of this paper, we only consider strategies to achieve a condition, and we require that the strategy stop once the desired condition is achieved.

There are several classes of strategies that we consider in this paper. The first is the class of *linear* strategies. Those are strategies that can be represented by a sequence of commands terminating with "stop". For example, linear strategy "go, go, go, go, stop" could be used in state b to achieve condition p. Indeed, execution of this sequence of commands starting from state b eventually transitions the system to state v, where atomic proposition p is true. At the same time, no linear strategy to achieve condition p exists in state a.

Let us now assume that the agent cannot distinguish the states of the same colour. For readers with mono-colour printers, we represent different-coloured states by different shapes. Because the agent cannot distinguish states a and b, in state b the agent has a strategy to achieve p, but does not know this.

Note that the agent also cannot distinguish states c and u. The agent has strategies to achieve p in both of these states, but these strategies are different. In state c, the strategy is "go, go, ..., go, stop" for any integer  $k \ge 0$ . At the same  $\frac{3k+3}{2}$ 

time, in state u, the strategy is "
$$\underbrace{go, go, \dots, go}_{3k+1}$$
, stop" for any

integer  $k \ge 0$ . Because no integer can be congruent modulo 3 to number 3 and number 1 at the same time there is no uniform linear strategy that succeeds in states c and u.

Thus, in state *c* the agent *has a strategy*, she *knows that she has a strategy*, but she *does not know what the strategy is*.

Finally, note that the agent can distinguish state d from every state except state w. In either of the states d and w, she can use the strategy "go, go, stop" to achieve condition p. Thus, in state d the agent knows how she can achieve p. We write this as:  $d \Vdash Lp$ . In general, we write  $s \Vdash L\varphi$  if in state s the agent knows a linear strategy to achieve  $\varphi$ .

A sound and complete logical system for a modality that captures knowledge of a linear strategy to achieve a condition  $\varphi$  has been first proposed by Wang (2018). Although the syntax and semantics of his modality are somewhat different from those of our modality L, the definition of linear strategies that he considers is essentially the same as ours.

Observe that the agent needs to know the colour of the initial state in order to decide which linear strategy to use, but she does not need to know the colour of the states the system is passing through in order to execute a linear strategy. Linear strategies can be executed by a programmable device not equipped with any sensors. For this reason, we will say that these strategies are *sensorless*. At the same time, the agent executing a linear strategy must have a memory to remember which command should be executed next.

Another important class of strategies that we consider is *reactive strategies*. In essence, they are the opposite of the linear strategies because they do not rely on the memory, but are *sensor-informed* in the sense that they select an action based on the current observations. We illustrate this class of strategies using the transition system depicted in Figure 2. Because a reactive strategy cannot expect the agent to have



Figure 2: A transition system.

memory, such a strategy must choose the same action each time the system is in a state of any given colour. In our example, a memoryless strategy must use the same action in red (diamond) states u and w. Note that to achieve condition p from state a, the system must pass through indistinguishable states u and w and in these two states, the agent must execute the commands "go" and "stop", respectively. Therefore, there is no reactive strategy to achieve condition p from state a.

At the same time, there is a reactive strategy to achieve condition p from state b. The strategy consists of executing the command "go" in each non-red state and the command "stop" in each red state. Because the agent cannot distinguish purple (circle) states a and b, in state b the agent has a reactive strategy, but she does not know that she does.

Next, note that the agent has a reactive strategy to achieve condition p from state a. The strategy again consists of executing the command "go" in each non-red (non-diamond) state and the command "stop" in each red state. The agent also has a reactive strategy to achieve condition p from state t. The strategy consists of executing the command "stop" in each state. Because the agent can distinguish state c from all states except for state t, in states c and t the agent knows that she has a reactive strategy to achieve condition p. At the same time, because there is no uniform reactive strategy that works in both of these states, in either of these states, she does not know what the strategy is. Finally, note that there is a uniform reactive strategy to achieve condition p from the yellow (triangle) states d and v. Namely, "go" in each non-red (non-diamond) state and the command "stop" in each red state. Thus, in state d (and state v) the agent knows a reactive strategy to achieve p. We write this as  $d \Vdash \mathsf{R}p$ . In general, we write  $s \Vdash \mathsf{R}\varphi$  if in state s the agent knows a reactive strategy to achieve  $\varphi$ . A modal logic for modality R has been proposed by Fervari et al. (2017).

So far, we have discussed sensorless strategies with perfect recall and sensor-informed memoryless strategies. We call such strategies linear and reactive, respectively. The

	Sensorless	Sensor-informed
Memoryless	Knowledge (K)	Reactive (R)
Perfect Recall	Linear (L)	Unrestricted (U)

Figure 3: Four different strategic modalities.

knowledge of a linear strategy is denoted by modality L and the knowledge of a reactive strategy is denoted by modality R. In this paper, we also consider *sensorless memoryless* strategies and *sensor-informed perfect recall* strategies, see Figure 3. One of our contributions is a general framework that formally defines all four of these classes of strategies in a uniform way.

An example of a *sensorless memoryless* strategy is one that immediately stops. In our formal framework the *knowledge* of a sensorless memoryless strategy to achieve  $\varphi$  is equivalent to the *knowledge* of  $\varphi$  being true in the current state. We prove this result later as Lemma 2.

Finally, let us consider the class of *sensor-informed per-fect recall* strategies. This is the widest class of strategies that we consider: among others, it includes the linear, reactive, and sensorless memoryless strategies. For this reason we refer to sensor-informed perfect recall strategies as just *unrestricted* strategies. We denote the knowledge of an unrestricted strategy to achieve condition  $\varphi$  by U $\varphi$ , see Figure 3.



Figure 4: A transition system.

As an example, consider the transition system depicted in Figure 4. Clearly, there is no unrestricted strategy to achieve condition p from state a. There is an unrestricted strategy to achieve condition p from state b ("stop at the second red state that you encounter"). Thus, because states a and b have the same colours, in state b the agent has an unrestricted strategy to achieve p, but she does not know this. In indistinguishable states c and v the agent has an unrestricted strategy to achieve condition p (strategy "go, stop" in state c and strategy "stop" in state c), but she does not have a uniform unrestricted strategy to do it. Thus, in state c (and state v) she knows that she has an unrestricted strategy but she does not know what the strategy is. Finally, in indistinguishable states d and u, the agent has the same unrestricted strategy to achieve p ("stop at the second red state that you encounter"). Thus, in state d she knows an unrestricted strategy to achieve p. We denote this by  $d \Vdash Up$ .

## **Transition Systems**

In our introductory examples, the agent had only two commands: "go" and "stop". In general, the transition systems that we consider are endowed with an arbitrary (possibly empty) set of "actions" A and a distinct command  $stop \notin A$ . **Definition 1** A sextuple  $(W, \sim, A, s, \delta, \pi)$  is called a transition system if

- 1. W is a (possibly empty) set of states,
- 2. ~ is an indistinguishability equivalence relation on W,
- 3. A is a set of "actions",
- 4.  $stop \notin A$  is a "stop command",
- 5.  $\delta: W \times A \rightarrow W$  is a "transition function",

6.  $\pi(p) \subseteq W$  for each propositional variable p.

**Definition 2** A history is a sequence  $w_0, a_0, w_1, \ldots, a_{n-1}, w_n$  such that  $\delta(w_{i-1}, a_{i-1}) = w_i$  for each  $i \leq n$ .

An example of a history for the transition system depicted in Figure 4 is b, go, u, go, c. For any history  $h = w_0, a_0, w_1, \ldots, a_{n-1}, w_n$ , by end(h) we mean state  $w_n$ .

**Definition 3** A strategy is a function that maps histories into  $A \cup \{stop\}$ .

As usual, we capture knowledge as an equivalence relation E on the states. In this paper, we consider two possible equivalence relations E: the indistinguishability relation  $\sim$ from Definition 1 and the total relation  $W \times W$ . We use these relations to define sensor-informed and sensorless strategies, respectively.

**Definition 4** For any equivalence relation E on the set W, the class  $St_E$  of memoryless E-informed strategies is the set of all strategies s such that for any histories h and h', if

$$h = w_0, a_0, w_1, \dots, a_{n-1}, w_n, h' = w'_0, a'_0, w'_1, \dots, a'_{m-1}, w'_m,$$

and  $w_n E w'_m$ , then s(h) = s(h').

**Definition 5** For any equivalence relation E on the set W, the class  $St_E^*$  of perfect recall E-informed strategies is the set of all strategies s such that for any histories h and h', if

$$\begin{aligned} h &= w_0, a_0, w_1, \dots, a_{n-1}, w_n, \\ h' &= w'_0, a_0, w'_1, \dots, a_{n-1}, w'_n, \end{aligned}$$

 $w_1 E w'_1, \ldots, w_{n-1} E w'_{n-1}$ , and  $w_n E w'_n$ , then s(h) = s(h').

**Definition 6** For any state  $w_0$  and any strategy s, the set  $Play(w_0, s)$  includes all histories  $w_0, a_0, w_1, \ldots, a_{n-1}, w_n$  such that

1. 
$$s(w_0, a_0, w_1, \dots, a_{i-1}, w_i) = a_i$$
 for each  $i \le n-1$ ,  
2.  $s(w_0, a_0, w_1, \dots, a_{n-1}, w_n) = stop$ .

**Definition 7** Let  $A \xrightarrow{s} B$  if for each  $a \in A$  there is a history  $h \in Play(a, s)$  such that  $end(h) \in B$ .

**Lemma 1** For each state  $a \in A$ , each strategy s, and each history  $h \in Play(a, s)$ , if  $A \xrightarrow{s} B$ , then  $end(h) \in B$ .

Informally,  $A \xrightarrow{s} B$  means that s is a strategy that navigates from each state in set A to a state in set B.

# Syntax and Semantics

The language  $\Phi$  that we consider in this paper is defined by the following grammar, where p is a propositional variable,

$$\varphi := p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathsf{K}\varphi \mid \mathsf{L}\varphi \mid \mathsf{M}\varphi \mid \mathsf{S}\varphi$$

In the definition below, by [w] we denote the equivalence class of a state  $w \in W$  with respect to relation  $\sim$ . Note that  $\{w\} \xrightarrow{s} B$  means that strategy s achieves one of the states in set B from state w. At the same time,  $[w] \xrightarrow{s} B$  means that strategy s achieves one of the states in set B from each state indistinguishable from w. In other words,  $[w] \xrightarrow{s} B$  means that in state w the agent knows that strategy s achieves one of the states in set B from the current state.

**Definition 8** For any transition system  $(W, \sim, A, s, \delta, \pi)$ and any formula  $\varphi$ , the truth set  $\llbracket \varphi \rrbracket \subseteq W$  is defined recursively as follows:

- $$\begin{split} &I. \quad \llbracket p \rrbracket = \pi(p), \\ &2. \quad \llbracket \neg \varphi \rrbracket = W \setminus \llbracket \varphi \rrbracket, \\ &3. \quad \llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket, \\ &4. \quad \llbracket \mathsf{K} \varphi \rrbracket = \{ w \mid \exists s \in St_{W \times W}(\llbracket w \rrbracket \xrightarrow{s} \llbracket \varphi \rrbracket) \}, \\ &5. \quad \llbracket \mathsf{L} \varphi \rrbracket = \{ w \mid \exists s \in St_{W \times W}^*(\llbracket w \rrbracket \xrightarrow{s} \llbracket \varphi \rrbracket) \}, \end{split}$$
- 6.  $\llbracket \mathsf{M}\varphi \rrbracket = \{ w \mid \exists s \in St_{\sim}(\llbracket w \rrbracket \xrightarrow{s} \llbracket \varphi \rrbracket) \},\$
- 7.  $\llbracket \mathsf{S}\varphi \rrbracket = \{ w \mid \exists s \in St^*_{\sim}(\llbracket w \rrbracket \xrightarrow{s} \llbracket \varphi \rrbracket) \}.$

Perhaps the most unexpected result in this paper is the observation that modality K is exactly the standard in epistemic logic knowledge modality associated with indistinguishibility relation  $\sim$ . We show this in the lemma below.

Lemma 2 
$$\llbracket \mathsf{K}\varphi \rrbracket = \{w \in W \mid [w] \subseteq \llbracket \varphi \rrbracket\}$$

PROOF. ( $\subseteq$ ). Consider any state  $w \in [\![K\varphi]\!]$ . Towards the proof of statement  $[w] \subseteq [\![\varphi]\!]$ , consider any world  $w' \in W$  such that  $w \sim w'$ . It suffices to show that  $w' \in [\![\varphi]\!]$ .

By item 4 of Definition 8, the assumption  $w \in \llbracket \mathsf{K} \varphi \rrbracket$  implies that there exists a memoryless strategy

$$s \in St_{W \times W} \tag{1}$$

such that

$$[w] \stackrel{s}{\to} \llbracket \varphi \rrbracket. \tag{2}$$

Recall that  $w \sim w'$ . Hence, by Definition 7, there is a history

$$h = w', a_0, w_1, \dots, a_{n-1}, w_n \in Play(w', s)$$
 (3)

such that  $end(h) \in \llbracket \varphi \rrbracket$ .

Note that  $(w', end(h)) \in W \times W$ . Thus, by Definition 4, statement (1) implies that s(h') = s(h), where h' is a single-state history consisting of state w'. Note that s(h) = stop by item 2 of Definition 6 and statement (3). Hence, s(h') = stop. Then,  $h' \in Play(w', s)$  by Definition 6 and because h' is a single-state history consisting of state w'. Thus, by Lemma 1, statement (2) implies that  $end(h') \in [\![\varphi]\!]$ . Therefore,  $w' \in [\![\varphi]\!]$  again because h' is a single-state history consisting of state w'.

 $(\supseteq)$  Consider any world w such that

$$[w] \subseteq \llbracket \varphi \rrbracket \tag{4}$$

It suffices to show that  $w \in [\![K\varphi]\!]$ . Let s be a strategy that maps each history into command *stop*. Note that

$$s \in St_{W \times W} \tag{5}$$

by Definition 4.

# Claim 1 $[w] \xrightarrow{s} \llbracket \varphi \rrbracket$ .

*Proof of Claim.* Consider any state  $w' \in [w]$ . Let h be the history consisting of a single state w'. By Definition 7, it suffices to show that  $h \in Play(w', s)$  and  $w' \in [\![\varphi]\!]$ . The former is true by Definition 6 because strategy s maps each history into command *stop*. The latter is true by the assumption  $w' \in [w]$  and statement (4).

By item 4 of Definition 8, Claim 1 and statement (5) imply that  $w \in [\![K\varphi]\!]$ .

# **Undefinability Results**

To understand any set of notions, it is important to see how they relate to each other. One way to do this is to give a complete axiomatisation of the properties describing the interplay between the notions. The other way is to study if and how these notions can be defined through each other. In this section, we show that none of the modalities K, L, M, and S can be defined through a combination of the others. In the next section, we discuss some of the properties that connect these modalities.

## Undefinability of L via K, M and S

In this subsection, we show that modality L cannot be defined through modalities K, M and S. The standard way to prove undefinability is to use bisimulation (Fervari, Velázquez-Quesada, and Wang 2021). In this paper, we introduce a new method which is significantly simpler. We call it the "truth set algebra" technique. We use it to prove all four undefinability results in this paper.

For our first undefinability result, consider the transition system depicted in Figure 5. Without loss of generality, we assume that the set of propositional variables consists of a single variable p. By  $\Phi^{-L}$  we denote the set of all formulae in language  $\Phi$  that do not use modality L. Our technique consists of proving that  $[Lp] \notin \{ [\varphi] \mid \varphi \in \Phi^{-L} \}$ . This implies that formula Lp is not semantically equivalent to any expression in the language  $\Phi^{-L}$ .



Figure 5: A transition system.

To show  $\llbracket Lp \rrbracket \notin \{\llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-L}\}$ , it suffices to establish that  $\llbracket Lp \rrbracket = \{c\}$  and that  $\{\llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-L}\} \subseteq A$ , where

$$A = \{ \emptyset, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, e\}, \\ \{a, b, c, d, e\}, \{d\}, \{d, e\}, \{e\} \}.$$

Lemma 3  $[\![Lp]\!] = \{c\}.$ 

PROOF. First, note that linear strategies that achieve p from state a make 4 + 2k "go"s followed by "stop", where  $k \ge 0$ . For state b, the number of "go"s is 3 + 2k. Because sets  $\{4 + 2k \mid k \ge 0\}$  and  $\{3 + 2k \mid k \ge 0\}$  are disjoint, there is no uniform strategy that achieves p from indistinguishable states a and b. Thus,  $a, b \notin [\![Lp]\!]$ . A similar argument can be made about states d and e using the sets  $\{1 + 2k \mid k \ge 0\}$  and  $\{2k \mid k \ge 0\}$ , respectively.

State c is distinguishable from all other states. A linear strategy to achieve p from that state is "go, go, stop".

**Lemma 4** 
$$\{ \llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-\mathsf{L}} \} \subseteq A$$

PROOF. We prove the statement by structural induction on formula  $\varphi$ . Note that  $\llbracket p \rrbracket = \{e\} \in A$ . It is also easy to see that set A is closed with respect to the complement and the intersection. Hence, by Definition 8, it suffices to show that if  $\llbracket \varphi \rrbracket \in A$ , then  $\llbracket K \varphi \rrbracket, \llbracket M \varphi \rrbracket, \llbracket S \varphi \rrbracket \in A$ . We do this by explicitly computing the latter truth sets for each possible value of the truth set  $\llbracket \varphi \rrbracket$  from set A. The results of this computation are given in Figure 6. For example, value *abc* in row S and column e of that table means that if  $\llbracket \varphi \rrbracket = \{e\}$ , then  $\llbracket S \varphi \rrbracket = \{a, b, c\}$ . Below, we justify three of the entries from the table and leave the rest as an exercise for the meticulous readers.

	Ø	abc	abcd	abce	abcde	d	de	e
K M S	Ø Ø Ø	$egin{array}{c} abc\ abc\ abc \end{array}$	$egin{array}{c} abc\ abc\ abc \end{array}$	$egin{array}{c} abc\ abc\ abc\ abc\ \end{array}$	abcde abcde abcde	$\substack{arnothing abc}{abc}$	de abcde abcde	$ \substack{ \varnothing \\ abc } $

Figure 6:	Towards th	he proof of	Lemma 4
0		· r · · ·	

First, we show that if  $\llbracket \varphi \rrbracket = \{d\}$ , then  $\llbracket M \varphi \rrbracket = \{a, b, c\}$ . Indeed, memoryless strategy "*stop* if the current state is red (diamond), otherwise *go*" is a uniform strategy that achieves set  $\{d\}$  from indistinguishable states *a* and *b*. The same strategy also works from state *c*. Hence, *a*, *b*, *c*  $\in \llbracket M \varphi \rrbracket$ . Next, note that there is no uniform memoryless strategy to reach set  $\{d\}$  that works from both of the indistinguishable states *e* and *d*. Indeed, such a strategy would need to "stop" in state *d* and to "go" in state *e*. This is impossible because memoryless strategy must return the same action in indistinguishable states *d* and *e*. Thus, *d*,  $e \notin \llbracket M \varphi \rrbracket$ .

Second, we show that if  $\llbracket \varphi \rrbracket = \{e\}$ , then  $\llbracket M \varphi \rrbracket = \emptyset$ . Indeed, there is no memoryless strategy to reach set  $\{e\}$  from any of the states a, b, c, d because any such strategy must return action "stop" in the destination state e. Then, being memoryless, it has to return the same action "stop" in each red state. Hence, it must return "stop" in state d. But any strategy that returns "stop" in state d will never reach set  $\{e\}$  from any of the states a, b, c, d, see Figure 5. Hence,  $a, b, c, d \notin \llbracket M \varphi \rrbracket$ . Finally, note that although there is a memoryless strategy ("stop") to reach set  $\{e\}$  from state e, this strategy is not uniform because it does not reach set  $\{e\}$  from indistinguishable (from e) state d. Thus,  $e \notin \llbracket M \varphi \rrbracket$ .

Third, we show that if  $\llbracket \varphi \rrbracket = \{e\}$ , then  $\llbracket S\varphi \rrbracket = \{a, b, c\}$ . This illustrates the difference between modalities M and S, see previous previous paragraph. Strategy "go until the second red (diamond) state"<sup>1</sup>, can be used to reach set  $\{e\}$  from from any of the states a, b, c. Hence,  $a, b, c \in \llbracket M \varphi \rrbracket$ . Because states d and e are indistinguishable, there is no *uniform* strategy to reach set  $\{e\}$  from these two states, see Figure 5. Thus,  $d, e \notin \llbracket M \varphi \rrbracket$ .

### Undefinability of M via K, L, and S

In this subsection, we prove the undefinability of modality M through modalities K, L, and S. To achieve this, we employ the same newly proposed "truth set algebra" technique as described in the previous subsection. This time, we use the transition system depicted in Figure 7. By language  $\Phi^{-M}$  we



Figure 7: A transition system.

denote the set of all formulae in language  $\Phi$  that do not use modality M. Similarly to the previous subsection, we show  $\llbracket Mp \rrbracket \notin \{\llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-M}\}$ , by establishing that  $\llbracket Mp \rrbracket = \{a\}$  and that  $\{\llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-M}\} \subseteq A$ , where

$$A = \{ \emptyset, \{a, d\}, \{a, b, d\}, \{a, b, c, d\}, \\ \{a, c, d\}, \{b\}, \{b, c\}, \{c\} \}.$$

The proofs of the next two lemmas are similar to the proofs of Lemma 3 and Lemma 4 in the previous subsections. The proof of Lemma 6 uses the table given in Figure 8.

Lemma 5  $\llbracket Mp \rrbracket = \{a\}.$ Lemma 6  $\{\llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-M}\} \subseteq A.$ 

	Ø	ad	abd	abcd	acd	b	bc	с
K	Ø	ad	ad	abcd	ad	Ø	bc	Ø
L	Ø	ad	ad	abcd	ad	ad	abcd	ad
S	Ø	ad	ad	abcd	ad	ad	abcd	ad

Figure 8: Towards the proof of Lemma 6.

#### Undefinability of S via K, L, and M



Figure 9: A transition system.

The undefinability of modality S via modalities K, L, and M is provable using again the same technique. Using the transition system depicted in Figure 9 and set

$$A = \{ \varnothing, \{a, b\}, \{a, b, c, e\}, \{a, b, c, d, e\}, \\ \{a, b, d\}, \{c, d, e\}, \{c, e\}, \{d\} \}.$$

<sup>1</sup>Note that this strategy is neither memoryless nor sensorless.

Lemma 7  $\llbracket Sp \rrbracket = \{a\}.$ Lemma 8  $\{\llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-S}\} \subseteq A.$ 

The proof of Lemma 8 uses the table given in Figure 10.

	Ø	ab	abce	abcde	abd	cde	ce	d
Κ	Ø	Ø	ce	abcde	abd	ce	ce	Ø
L	Ø	Ø	abcde	abcde	abcde	abcde	ce	Ø
Μ	Ø	Ø	abcde	abcde	abcde	abcde	abcde	Ø

Figure 10: Towards the proof of Lemma 8.

# Undefinability of K via L, M, and S

The undefinability of modality K via modalities L, M, and S is provable using again the same technique. Using the transition system depicted in Figure 11 and set

$$A = \{ \emptyset, \{a\}, \{a, b\}, \{a, b, c, d\}, \{b\}, \{b, c, d\}, \{d\} \}.$$



Figure 11: A transition system.

Lemma 9  $\llbracket \mathsf{K}p \rrbracket = \{c\}.$ Lemma 10  $\{\llbracket \varphi \rrbracket \mid \varphi \in \Phi^{-\mathsf{K}}\} \subset A.$ 

The proof of Lemma 10 uses the table given in Figure 12.

	Ø	a	ab	abcd	acd	b	bcd	cd
L M S	ØØØ	ØØØ	b b b	$abcd \\ abcd \\ abcd$	$abcd \\ abcd \\ abcd$	ь ь ь	$abcd \\ abcd \\ abcd$	$abcd \\ abcd \\ abcd$

Figure 12: Towards the proof of Lemma 10.

**Note** Literature review and the conclusion will be added in the final version of this paper.

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