BRIDGING THE FIRST AND LAST PASSAGE TIMES FOR LÉVY MODELS

DAVID LANDRIAULT, BIN LI, MOHAMED AMINE LKABOUS, AND ZIJIA WANG

Abstract. Research in classical ruin theory has largely focused on the first passage time analysis of a surplus process below level 0. Recently, inspired by numerous applications in finance, physics, and optimization, there has been an accrued interest in the analysis of the last passage time (below level 0). In this paper, we aim to bridge the first and the last passage times and unify their analyses. For this purpose, we consider negative excursions of an underlying process in two manners, cumulative and noncumulative, and introduce two random times, denoted by s_r and l_r, where r can be interpreted as a measure of a decision maker's tolerance to negative excursions. Our analysis focuses on spectrally negative Lévy processes, for which we derive the Laplace transform and some distributional quantities of these random times in terms of standard scale functions. An application to credit risk management is considered at the end.

1. Introduction

In applied probability, an extensive literature exists on the analysis of first passage times for a variety of stochastic processes. The study of first passage times has also found numerous applications in many fields. In quantitative risk management, decision makers are concerned with institution's solvency when an adverse event occurs, and thus most attention has been paid to study the distribution of the first time an underlying process X drops below level 0, i.e.,

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\}.$$

On the other hand, the last passage time (below level 0), defined as

$$g = \sup\{t \geq 0 : X_t \leq 0\}, \tag{1}$$

with the convention $\sup\emptyset = 0$, is also of great importance in both theoretical and applied probability. Early theoretical works on last passage times include Getoor and Sharpe [12], Maisonneuve [29], Syski [36], Monrad [30], Janson [17], Salminen [33], Doney [8], and Lachal [19], among others. More recently, for Lévy processes, Sato and Watanabe [34] considered the study of the moments of last exit times. Chiu and Yin [7] derive the joint Laplace transform of first and last passage times for spectrally negative Lévy processes. Baurdoux [2] further consider the last passage time over an independent exponential time horizon defined as

$$g^q = \sup\{0 \leq t \leq \mathrm{e}_q : X_t \leq 0\},$$

where e_q is an exponential random variable with rate $q > 0$, independent of X. Li et al. [24] generalize the results in [2] by studying the joint Laplace transform of the last exit time, the value of the process at the last exit time, and the occupation time until the last exit time.

Date: January 30, 2023.

Key words and phrases. First passage time, last passage time, occupation time, Parisian time, spectrally negative Lévy processes, scale functions.

Cai and Li [6] derive the Laplace transform of occupation times of intervals until last passage times for spectrally negative Lévy processes.

It is worth noting that last passage times have found many applications in finance, physics, and optimization. For instance, last passage times are applied in modelling default risks (see the seminal paper by Elliott et al. [11]) and insider trading (e.g., Imkeller [16]). Madan et al. [28] discover the link between European option pricing and distributions of some last passage times. Egami and Kevkhishvili [10] study last passage times of diffusions with applications to credit risk management. Another interesting topic in optimization is to find optimal stopping times to approximate last passage times. For example, Baurdoux and Pedraza [3, 4] obtained a stopping time that is close in L^1 and L^p sense to the last passage time g for a spectrally negative Lévy process. Other applications of last passage times can be found in physics (e.g., [13] and [1]) and reliability theory (e.g., [31]).

In risk management, the first passage time (τ_0^-) is typically used to model the event of insolvency and the last passage time (g) is to characterize the last time that a company is in an insolvent position. Note that negative surplus may be inevitable for a start-up company or for a given subsidiary of a very large company because of high initial expenses, and it can take a few years for a startup to be eventually profitable. Moreover, the risk tolerance level to negative surplus of different businesses can be very different too. It is certainly debatable whether a company should rely on risk measures involving only the first or the last passage time to formulate a comprehensive approach on decisions related to business expansion, investment, capital injections, dividend payments, etc.

In this paper, we propose a possible remedy by introducing two types of random times to bridge the first and last passage times and unify their analyses. We consider the length of negative excursions of an underlying process in two manners, cumulative and noncumulative, and use them to model a company's financial distress.

More specifically, the first random time is called *occupation-type first-last passage time* defined as

$$s_r = \sup\left\{t \geq \tau_0^- : \mathcal{O}_t > r \text{ and } X_t \leq 0\right\}, \quad r > 0, \tag{2}$$

where $\mathcal{O}_t = \int_0^t \mathbf{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s$ is the *occupation time*[1] which represents the cumulative length of negative excursions of the surplus process X below level 0 up to time t. If the set in (2) is empty, we follow the convention that the supremum is reached at the smallest point, i.e., $\sup\emptyset = \tau_0^-$. For a given sample path ω, it is seen from (2) that

$$s_r(\omega) = \tau_0^-(\omega),$$

if the total amount of time $X(\omega)$ stays below level 0 does not exceed r (i.e., $\mathcal{O}_g \leq r$), and

$$s_r(\omega) = g(\omega),$$

if the total amount of time $X(\omega)$ stays below level 0 exceeds r (i.e., $\mathcal{O}_g > r$). As such, s_r behaves like a "binary distribution" taking values in τ_0^- or g. Heuristically, we have $s_r \to \tau_0^-$ when $r \to \infty$, and $s_r \to g$ if $r \to 0$. This result will be formally proved in Propositions 17 and 18. The parameter $r > 0$ can be interpreted as a decision maker's *risk tolerance level to negative surplus*. A smaller r implies a lower risk tolerance level to negative surplus and thus more weight is put on the last passage time g. Recall that the last passage time g represents the ultimate time a company turns to be profitable. A company with low tolerance level of

[1] There is an extensive literature on occupation times in applied probability and more specifically, in insurance mathematics. See, e.g., [27] and [20].

negative surplus will wait until very close to that time to consider more aggressive strategic decisions such as hiring new employees, expanding the business, and paying dividends to shareholders. On the other hand, it is common for start-up companies to experience negative surplus at the onset and their associated tolerance level should likely to be high.

The second random time is the so-called *Parisian-type first-last passage time* defined as

$$l_r = \sup\left\{t \geq \tau_0^- :\ U_t > r\right\}, \quad r > 0, \tag{3}$$

where $U_t := t - g_t$ with $g_t := \sup\{0 \leq s \leq t : X_s \geq 0\}$. Note that U_t corresponds to the length of the current excursion of the process X below 0 at time t, a quantity which is known to play an important role in the definition of Parisian ruin times in [25] and [23]. Intuitively, l_r corresponds to the *ending time of negative excursions longer than* r. After l_r, the surplus process X may still experience periods of negative surplus but none of these negative excursions will individually last longer than r time units. It is seen from (3) that $\tau_0^- \leq l_r < g$, a.s. and l_r may have a mass point at τ_0^- if all the negative excursions are shorter than r. The distinguishing feature of s_r is that, for a given sample path ω, $l_r(\omega)$ may be such that $\tau_0^-(\omega) \leq l_r(\omega) < g(\omega)$. Hence, l_r provides a smoother bridge (than s_r) between the first and last passage times. Similarly, the parameter r in (3) can be interpreted as a decision maker's risk tolerance to negative surplus. Furthermore, it will be shown that $l_r \to \tau_0^-$ if $r \to \infty$ and $l_r \to g$ if $r \to 0$ (see Propositions 17 and 18 for more details).

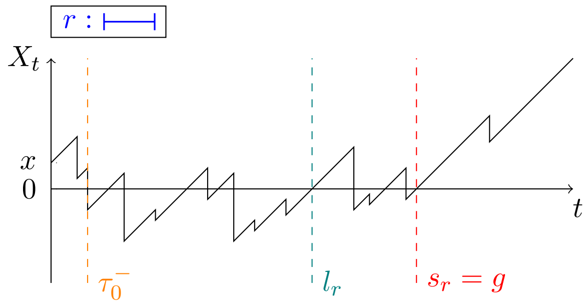
From the definitions of l_r and s_r, one can see that

$$\tau_0^- \leq l_r \leq s_r \leq g, \quad \text{a.s.} \tag{4}$$

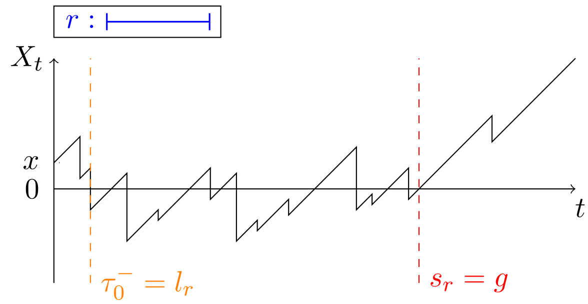
for all $r \geq 0$. Figure 1 plots three possible cases with different choices of r for a sample path of X. Panel (A) represents the case that r is shorter than the longest individual negative excursion, which implies $\tau_0^- < l_r < s_r = g$. Panel (B) represents the case that r is longer than the longest individual negative excursion but shorter than the aggregate total negative excursion ($\mathcal{O}_g$), which implies $\tau_0^- = l_r < s_r = g$. Panel (C) represents the case r is longer than the aggregate total negative excursion, which implies $\tau_0^- = l_r = s_r < g$.

The main contributions of the paper are summarized as follows. First, we derive the analytical distribution for the two random times, namely s_r and l_r, through their Laplace transforms. Secondly, via the analysis of s_r and l_r, some new results on the joint distributions of $(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q})$ and $(U_{\mathrm{e}_q}, X_{\mathrm{e}_q})$ for spectrally negative Lévy processes are derived, which are of interest on their own in fluctuation theory. Recall that $\mathcal{O}_t = \int_0^t \mathbf{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s$ and $U_t = t - g_t$, where $g_t = \sup\{0 \leq s \leq t : X_s \geq 0\}$. For the joint distribution of $(\mathcal{O}_t, X_t)$, explicit expressions for a Brownian risk process with drift and a Cramér-Lundberg process with exponential jumps will also be derived. Third, we show the convergence of s_r and l_r to the first passage time τ_0^- and the last passage time g when $r \to \infty$ and $r \to 0$, respectively. As such, our analysis of two random times enables us to bridge the first and the last passage times in terms of the level r. By varying different levels of r, a decision maker can form a more comprehensive view on the periods with financial distress (i.e., negative surplus). Fourth, we consider an application of our results on the two random times to a credit risk management model introduced in Egami and Kevkhishvili [10].

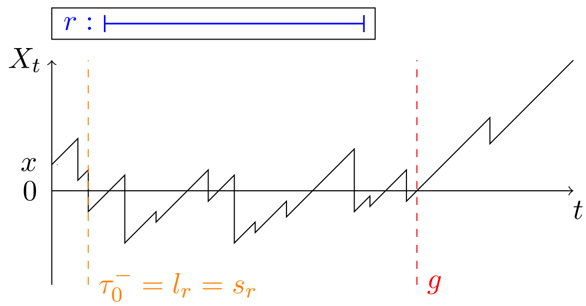
The rest of the paper is organized as follows. Section 2 presents the necessary background material on spectrally negative Lévy processes and scale functions. The main results of this paper as they pertain to s_r and l_r are derived in Sections 3 and 4, respectively. Section 5 proves the convergence of the random times s_r and l_r to the first and last passage times when



(A) $\tau_0^-(\omega) < l_r(\omega) < s_r(\omega) = g(\omega)$



(B) $\tau_0^-(\omega) = l_r(\omega) < s_r(\omega) = g(\omega)$



(C) $\tau_0^-(\omega) = l_r(\omega) = s_r(\omega) < g(\omega)$

FIGURE 1. A sample path of X with different choices of r.

$r \to 0$ or ∞. In Section 6, we consider an application to credit risk management using the two random times.

2. Preliminaries on spectrally negative Lévy processes

First, we present the necessary background material on spectrally negative Lévy processes. A Lévy insurance risk process X is a process with stationary and independent increments and no positive jumps. To avoid trivialities, we exclude the case where X has monotone paths. As the Lévy process X has no positive jumps, its Laplace transform exists: for all $\lambda, t \geq 0$,

$$\mathbb{E}\left[\mathrm{e}^{\lambda X_t}\right] = \mathrm{e}^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = \gamma\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{0}\left(\mathrm{e}^{\lambda z} - 1 - \lambda z \mathbf{1}_{\{z>-1\}}\right)\Pi(\mathrm{d}z),$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$, and where Π is a σ-finite measure on $(-\infty, 0)$ called the Lévy measure of X which is assumed to satisfy

$$\int_{-\infty}^{0}(1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

Throughout, we will use the standard Markovian notation: the law of X when starting from $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

We now present the definition of the scale functions W_q and Z_q of X. First, recall that there exists a function $\Phi\colon [0,\infty) \to [0,\infty)$ defined by $\Phi_q = \sup\{\lambda \geq 0 \mid \psi(\lambda) = q\}$ (the right-inverse of ψ) such that

$$\psi(\Phi_q) = q, \quad q \geq 0.$$

When $\mathbb{E}[X_1] > 0$, we have

$$\lim_{q\to 0} \frac{q}{\Phi_q} = \psi'(0+) = \mathbb{E}[X_1]. \tag{5}$$

We have that $\Phi(q) = 0$ if and only if $q = 0$ and $\psi'(0+) \geq 0$.

For $q \geq 0$, the q-scale function of the process X is defined as the continuous function on $[0,\infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda y} W_q(y)\mathrm{d}y = \frac{1}{\psi_q(\lambda)}, \quad \text{for } \lambda > \Phi_q, \tag{6}$$

where $\psi_q(\lambda) = \psi(\lambda) - q$. This function is unique, positive and strictly increasing for $x \geq 0$ and is further continuous for $q \geq 0$. We extend W_q to the whole real line by setting $W_q(x) = 0$ for $x < 0$. We write $W = W_0$ when $q = 0$.

We also define another scale function $Z_q(x,\theta)$ by

$$Z_q(x,\theta) = \mathrm{e}^{\theta x}\left(1 - \psi_q(\theta)\int_0^x \mathrm{e}^{-\theta y} W_q(y)\mathrm{d}y\right), \quad x \geq 0, \tag{7}$$

and $Z_q(x,\theta) = \mathrm{e}^{\theta x}$ for $x < 0$. We denote the derivative of $Z_q(x,\theta)$ with respect to x by

$$Z_q'(x,\theta) = \theta Z_q(x,\theta) - \psi_q(\theta) W_q(x). \tag{8}$$

A *second generation* scale function was introduced by Loeffen et al. [27], that is, for $p, p+q \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}\mathcal{W}_a^{(p,q)}(x) &= W_p(x) + q\int_a^x W_{p+q}(x-y) W_p(y)\,\mathrm{d}y \\ &= W_{p+q}(x) - q\int_0^a W_{p+q}(x-y) W_p(y)\,\mathrm{d}y,\end{aligned} \tag{9}$$

Note that the two expressions on the right-hand side of (9) can be shown to be equivalent using the following identity from [27]: for $p, q \geq 0$ and $x \in \mathbb{R}$,

$$(p-q)\int_0^x W_q(x-y)W_p(y)\mathrm{d}y = W_p(x) - W_q(x). \tag{10}$$

The derivative of $\mathcal{W}_a^{(p,q)}$ with respect to x is given by

$$\mathcal{W}_a^{(p,q)\prime}(x) = W_{p+q}'(x) - q\int_0^a W_{p+q}'(x-y) W_p(y)\,\mathrm{d}y. \tag{11}$$

We also recall the following function introduced by Loeffen et al. [26] defined as

$$\Lambda^{(q)}(x,z) = \int_0^\infty W_q(x+u)\,\frac{u}{z}\mathbb{P}(X_z \in \mathrm{d}u),$$

and we write $\Lambda = \Lambda^{(0)}$ when $q = 0$. We also denote the partial derivative of $\Lambda^{(q)}$ with respect to x by

$$\Lambda^{(q)\prime}(x,z) = \frac{\partial \Lambda^{(q)}}{\partial x}(x,z) = \int_0^\infty W_q'(x+u)\,\frac{u}{z}\mathbb{P}(X_z \in \mathrm{d}u). \tag{12}$$

We also recall the definition of the first-passage time of X above a level $b \in \mathbb{R}$ defined as

$$\tau_b^+ = \inf\{t > 0 \colon X_t > b\},$$

with the convention $\inf \emptyset = \infty$. It is well known that

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \infty\}} \right] = \mathrm{e}^{-\Phi_q b}, \quad b > 0. \tag{13}$$

Finally, we recall the following identities for the exit times τ_0^- and g. From Theorem 8.1 of [18],

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} \mathbf{1}_{\{\tau_0^- < \infty\}} \right] = Z_q(x) - \frac{q}{\Phi_q} W_q(x), \quad q \geq 0, x \in \mathbb{R}, \tag{14}$$

where $Z_q(x) = Z_q(x,0)$. From Lemma 2.2 of [27].

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} W\left(X_{\tau_0^-} + z \right) \mathbf{1}_{\{\tau_0^- < \infty\}} \right] = \mathcal{W}_x^{(q,-q)}(x+z) - W_q(x) Z(z, \Phi_q). \tag{15}$$

Also, by Theorem 3.1 of [7], it is known that

$$\mathbb{E}_x\left[\mathrm{e}^{-qg} \mathbf{1}_{\{g>0\}} \right] = \mathbb{E}\left[X_1\right] \left(\Phi_q' \mathrm{e}^{\Phi_q x} - W_q(x) \right), \quad q \geq 0, x \in \mathbb{R}, \tag{16}$$

if $\mathbb{E}\left[X_1\right] > 0$. We refer the reader to [18] for more details on spectrally negative Lévy processes and fluctuation identities.

3. Occupation-type first-last passage time

3.1. **Distribution of s_r.** We begin our analysis with the occupation-type first-last passage time s_r defined in (2). We recall that s_r is a binary distribution taking values in τ_0^- and g. More specifically,

$$s_r = \begin{cases} \tau_0^-, & \text{if } \mathcal{O}_\infty \leq r, \\ g, & \text{if } \mathcal{O}_\infty > r, \end{cases}$$

where, from Corollary 5 of [20],

$$\mathbb{P}_x\left(\mathcal{O}_\infty \leq r\right) = \mathbb{E}[X_1] \left(W(x) + \int_0^r \Lambda'(x,s)\, \mathrm{d}s \right). \tag{17}$$

We note that an expression for the Laplace transform and distribution function of s_r are respectively given in Theorem 2 and Corollary 4. We first provide a preliminary result related to the joint distribution of $\left(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q}\right)$, which will be used in the proof of Theorem 2.

Lemma 1. *For $q > 0$, $x \in \mathbb{R}$ and $y, z \geq 0$,*

$$\begin{aligned} \mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} \in \mathrm{d}z, X_{\mathrm{e}_q} \in \mathrm{d}y\right) =& q\left(\mathrm{e}^{-\Phi_q y} W_q(x) - W_q(x-y)\right) \delta_0(\mathrm{d}z)\, \mathrm{d}y \\ &+ q\mathrm{e}^{-\Phi_q y} \mathrm{e}^{-qz} \left(\Lambda^{(q)\prime}(x,z) - \Phi_q \Lambda^{(q)}(x,z) \right) \mathrm{d}z \mathrm{d}y, \end{aligned} \tag{18}$$

where $\delta_0(.)$ is the Dirac mass at 0.

Proof. From [14], we have the following potential measure discounted by its joint occupation time over the half line $(-\infty, 0)$, that is : for $\lambda \geq 0$, $q > 0$ and $x, y \in \mathbb{R}$,

$$\mathbb{E}_x\left[\mathrm{e}^{-\lambda \mathcal{O}_{\mathrm{e}_q}}, X_{\mathrm{e}_q} \in \mathrm{d}y \right] = q\left(\frac{\Phi_{q+\lambda} - \Phi_q}{\lambda} Z_q(x, \Phi_{\lambda+q}) Z_{\lambda+q}(-y, \Phi_q) - \mathcal{W}_x^{(q,\lambda)}(x-y) \right) \mathrm{d}y. \tag{19}$$

For $y \geq 0$, $\mathcal{W}_x^{(q,\lambda)}(x-y) = W_q(x-y)$ and $Z_{\lambda+q}(-y,\Phi_q) = \mathrm{e}^{-\Phi_q y}$, and Eq. (19) reduces to

$$\mathbb{E}_x\left[\mathrm{e}^{-\lambda \mathcal{O}_{\mathrm{e}_q}}, X_{\mathrm{e}_q} \in \mathrm{d}y\right] = q\mathrm{e}^{-\Phi_q y}\left(\frac{\Phi_{q+\lambda} Z_q(x,\Phi_{\lambda+q})}{\lambda} - \Phi_q \frac{Z_q(x,\Phi_{\lambda+q})}{\lambda}\right) - qW_q(x-y). \quad (20)$$

Finally, substituting the following identities (see, e.g., [20]):

$$\frac{Z_q(x,\Phi_{\lambda+q})}{\lambda} = \int_0^\infty \mathrm{e}^{-\lambda z}\left(\mathrm{e}^{-qz}\Lambda^{(q)}(x,z)\right)\mathrm{d}z, \quad (21)$$

and

$$\frac{\Phi_{\lambda+q} Z_q(x,\Phi_{\lambda+q}) - \lambda W_q(x)}{\lambda} = \int_0^\infty \mathrm{e}^{-\lambda z}\left(\mathrm{e}^{-qz}\Lambda^{(q)\prime}(x,z)\right)\mathrm{d}z, \quad (22)$$

into (20), one easily obtains Eq. (18) by Laplace transform inversion. ■

From Lemma 1, it is clear that

$$\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} = 0, X_{\mathrm{e}_q} \in \mathrm{d}y\right) = q\left(\mathrm{e}^{-\Phi_q y} W_q(x) - W_q(x-y)\right)\mathrm{d}y, \quad (23)$$

for $x, y \geq 0$. Also, it is worth noting that (23) corresponds to $\mathbb{P}_x\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^- > \mathrm{e}_q\right)$, the q-potential measure of X killed on exiting $[0,\infty)$.

We now derive an expression for the Laplace transform of s_r.

Theorem 2. *For $q, r > 0$, $x \in \mathbb{R}$ and $\mathbb{E}[X_1] > 0$,*

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-qs_r}\right] =& \mathbb{E}[X_1]\int_r^\infty \mathrm{e}^{-qz}\left(\Lambda^{(q)\prime}(x,z) - \Phi_q\Lambda^{(q)}(x,z)\right)\mathrm{d}z \\ &+ \mathbb{E}[X_1]\int_0^r\int_0^\infty\left(\mathcal{W}_x^{(q,-q)\prime}(x+z) - W_q(x)Z'(z,\Phi_q)\right)\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)\,\mathrm{d}s. \quad (24)\end{aligned}$$

Proof. Using the fact that $\left\{s_r = \tau_0^-\right\} = \{\mathcal{O}_\infty \leq r\}$ and $\left\{\tau_0^- < s_r < \mathrm{e}_q\right\} = \{\mathcal{O}_{\mathrm{e}_q} > r, X_s > 0 \text{ for all } s \geq \mathrm{e}_q\}$, it follows that

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-qs_r}\right] =& \mathbb{E}_x\left[\mathrm{e}^{-qs_r}\mathbf{1}_{\{s_r > \tau_0^-\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{s_r = \tau_0^-\}}\right] \\ =& \mathbb{E}_x\left[\mathbb{P}_{X_{\mathrm{e}_q}}\left(\tau_0^- = \infty\right)\mathbf{1}_{\{\mathcal{O}_{\mathrm{e}_q} > r\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\mathcal{O}_\infty \leq r\}}\right] \\ =& \mathbb{E}[X_1]\int_0^\infty\int_r^\infty W(y)\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} \in \mathrm{d}z, X_{\mathrm{e}_q} \in \mathrm{d}y\right) + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\mathcal{O}_\infty \leq r\}}\right], \quad (25)\end{aligned}$$

where the second equality follows from the Markov property (applied at time e_q) and the third equality follows from the fact that $\mathbb{P}_x\left(\tau_0^- = \infty\right) = \mathbb{E}[X_1]W(x)$.

From (17) and Tonelli's theorem, we get

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\mathcal{O}_\infty \leq r\}}\right] =& \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{P}_{X_{\tau_0^-}}(\mathcal{O}_\infty \leq r)\right] \\ =& \mathbb{E}[X_1]\int_0^r \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\Lambda'\left(X_{\tau_0^-}, s\right)\right]\mathrm{d}s \\ =& \mathbb{E}[X_1]\int_0^r\int_0^\infty \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}W'\left(X_{\tau_0^-} + z\right)\mathbf{1}_{\{\tau_0^- < \infty\}}\right]\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)\,\mathrm{d}s \\ =& \mathbb{E}[X_1]\int_0^r\int_0^\infty\left(\mathcal{W}_x^{(q,-q)\prime}(x+z) - W_q(x)Z'(z,\Phi_q)\right)\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)\,\mathrm{d}s,\end{aligned} \quad (26)$$

where (12) is applied in the third equality and the derivative of (15) is applied in the last equality. Substituting (18) and (26) into (25) completes the proof of Theorem 2. ■

In the following remark, we prove that s_r converges in distribution to τ_0^- (as $r \to \infty$) and g (as $r \to 0$) by showing that the Laplace transform of s_r reduces to (14) and (16) in their respective limiting cases.

Remark 3. First, as $r \to \infty$,

$$\begin{aligned}\lim_{r\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-q s_r}\right] =& \mathbb{E}\left[X_1\right]\int_0^\infty \int_0^\infty \left(\mathcal{W}_x^{(q,-q)\prime}(x+z) - W_q(x)Z'(z,\Phi_q)\right)\frac{z}{s}\mathbb{P}\left(X_s \in \mathrm{d}z\right)\mathrm{d}s\\ =&\mathbb{E}[X_1]\lim_{\theta\to 0}\int_0^\infty \mathrm{e}^{-\theta s}\int_0^\infty \left(\mathcal{W}_x^{(q,-q)\prime}(x+z) - W_q(x)Z'(z,\Phi_q)\right)\frac{z}{s}\mathbb{P}\left(X_s \in \mathrm{d}z\right)\mathrm{d}s.\end{aligned} \tag{27}$$

Applying Kendall's identity, and Eqs. (8) and (39) of [21], we obtain

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\theta s}\int_0^\infty \left(\mathcal{W}_x^{(q,-q)\prime}(x+z) - W_q(x)Z'(z,\Phi_q)\right)\frac{z}{s}\mathbb{P}\left(X_s \in \mathrm{d}z\right)\mathrm{d}s\\ =&\int_0^\infty \mathrm{e}^{-\Phi_\theta z}\mathcal{W}_x^{(-q,q)\prime}(x+z)\,\mathrm{d}z - W_q(x)\int_0^\infty \mathrm{e}^{-\Phi_\theta z}Z'(z,\Phi_q)\mathrm{d}z\\ =&\Phi_\theta\int_0^\infty \mathrm{e}^{-\Phi_\theta z}\mathcal{W}_x^{(q,-q)}(x+z)\,\mathrm{d}z - W_q(x) - W_q(x)\left(\frac{\Phi_q(\theta-q)}{\theta(\Phi_\theta-\Phi_q)} - \frac{q}{\theta}\right)\\ =&\Phi_\theta\frac{Z_q(x,\Phi_\theta)}{\theta} - W_q(x)\frac{\Phi_\theta(\theta-q)}{\theta(\Phi_\theta-\Phi_q)}.\end{aligned} \tag{28}$$

Substituting (28) into (27) and applying (5), one deduces that the Laplace transform of s_r reduces to (14) as $r \to \infty$.

Now, we move on to the limiting case where $r \to 0$. With the help of (21) and (22), it follows that

$$\begin{aligned}\lim_{r\to 0}\mathbb{E}_x\left[\mathrm{e}^{-q s_r}\right] =&\mathbb{E}\left[X_1\right]\int_0^\infty \mathrm{e}^{-qz}\left(\Lambda^{(q)\prime}(x,z) - \Phi_q\Lambda^{(q)}(x,z)\right)\mathrm{d}z\\ =&\mathbb{E}\left[X_1\right]\cdot\lim_{\lambda\to 0}\left(\frac{\left(\Phi_{\lambda+q}-\Phi_q\right)Z_q\left(x,\Phi_{\lambda+q}\right) - \lambda W_q(x)}{\lambda}\right)\\ =&\mathbb{E}\left[X_1\right]\left(\Phi_q'\mathrm{e}^{\Phi_q x} - W_q(x)\right),\end{aligned} \tag{29}$$

where $Z_q(x,\Phi_q) = \mathrm{e}^{\Phi_q x}$ and $\lim_{\lambda\to 0}\frac{\Phi_{\lambda+q}-\Phi_q}{\lambda} = \Phi_q'$ are applied in the last equation. Eq. (29) corresponds to the Laplace transform of g given in (16).

In fact, it can be shown that s_r converges to τ_0^- and g (as $r \to \infty$ and $r \to 0$, respectively) $\mathbb{P}_x$ almost surely. We refer the reader to Section 5 for the proof of this result.

The next result on the distribution of s_r is an immediate consequence of Eq. (25).

Corollary 4. *For $t, r > 0$, $x \in \mathbb{R}$ and $\mathbb{E}\left[X_1\right] > 0$,*

$$\mathbb{P}_x\left(s_r \le t\right) = \mathbb{E}\left[X_1\right]\int_0^\infty\int_r^\infty W(y)\mathbb{P}_x\left(\mathcal{O}_t \in \mathrm{d}s, X_t \in \mathrm{d}y\right) + \mathbb{P}_x\left(\mathcal{O}_\infty \le r, \tau_0^- \le t\right), \tag{30}$$

where

$$\mathbb{P}_x\left(\mathcal{O}_\infty \le r, \tau_0^- \le t\right) = \mathbb{E}\left[X_1\right]\int_0^r\int_0^\infty \mathbb{E}_x\left[W'\left(X_{\tau_0^-}+z\right)\mathbf{1}_{\{\tau_0^-\le t\}}\right]\frac{z}{s}\mathbb{P}\left(X_s \in \mathrm{d}z\right)\mathrm{d}s.$$

For completeness, we also consider the random time s^θ defined as

$$s^\theta = \sup\left\{t \geq \tau_0^- : \mathcal{O}_t > \mathrm{e}_\theta \text{ and } X_t \leq 0\right\}, \tag{31}$$

where the parameter r in s_r is replaced by an independent (of X) exponential rv e_θ. The following theorem gives an explicit expression (in terms of scale functions) of the Laplace transform of s^θ.

Theorem 5. *For $q, \theta > 0$, $x \in \mathbb{R}$ and $\mathbb{E}[X_1] > 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-qs^\theta}\right] = \mathbb{E}[X_1]\left(\Phi_q' \mathrm{e}^{\Phi_q x} - \frac{\Phi_{q+\theta} - \Phi_q}{\theta} Z_q(x, \Phi_{q+\theta}) + \frac{\Phi_\theta}{\theta} Z_q(x, \Phi_\theta) - \frac{\Phi_\theta}{\theta}\frac{\theta - q}{\Phi_\theta - \Phi_q} W_q(x)\right). \tag{32}$$

Proof. Similar to the proof of Theorem 2, from the strong Markov property of X, it follows that

$$\begin{aligned}
\mathbb{E}_x\left[\mathrm{e}^{-qs^\theta}\right] =& \mathbb{E}_x\left[\mathbb{P}_{X_{\mathrm{e}_q}}\left(\tau_0^- = \infty\right)\mathbf{1}_{\{\mathcal{O}_{\mathrm{e}_q} > \mathrm{e}_\theta\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\mathcal{O}_\infty \leq \mathrm{e}_\theta\}}\right] \\
=& \mathbb{E}[X_1]\mathbb{E}_x\left[W(X_{\mathrm{e}_q})\mathbf{1}_{\{\mathcal{O}_{\mathrm{e}_q} > \mathrm{e}_\theta\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-\theta\mathcal{O}_\infty}\right]\right].
\end{aligned} \tag{33}$$

Using the potential measure of X (see Corollary 8.9 of [18]) and (20), it is straightforward to show that

$$\begin{aligned}
\mathbb{E}_x\left[W(X_{\mathrm{e}_q})\mathbf{1}_{\{\mathcal{O}_{\mathrm{e}_q} > \mathrm{e}_\theta\}}\right] =& \mathbb{E}_x\left[W(X_{\mathrm{e}_q})\right] - \mathbb{E}_x\left[\mathrm{e}^{-\theta\mathcal{O}_{\mathrm{e}_q}}W(X_{\mathrm{e}_q})\right] \\
=& \Phi_q' \mathrm{e}^{\Phi_q x} - \frac{\Phi_{q+\theta} - \Phi_q}{\theta} Z_q(x, \Phi_{q+\theta}).
\end{aligned} \tag{34}$$

Moreover, from Theorem 1 of [23], we have

$$\begin{aligned}
\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-\theta\mathcal{O}_\infty}\right]\right] =& \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-\theta\tau_0^+}\mathbb{E}\left[\mathrm{e}^{-\theta\mathcal{O}_\infty}\right]\right]\right] \\
=& \psi'(0+)\frac{\Phi_\theta}{\theta}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^- + \Phi_\theta X_{\tau_0^-}}\right] \\
=& \psi'(0+)\frac{\Phi_\theta}{\theta}\left(Z_q(x, \Phi_\theta) - \frac{\theta - q}{\Phi_\theta - \Phi_q} W_q(x)\right).
\end{aligned} \tag{35}$$

Substituting (34) and (35) into (33) completes the proof of Theorem 5. ■

Once again, it can be shown that s^θ converges in distribution to τ_0^- (and g) as $\theta \to 0(\infty)$.

Remark 6. By noting that $Z_q(x, \Phi_q) = \mathrm{e}^{\Phi_q x}$ and using (5), one observes from (32) that the Laplace transform of s^θ converges to the Laplace transform of τ_0^- as $\theta \to 0$. On the other hand, applying the initial value theorem,

$$\lim_{\theta\to\infty} \frac{\Phi_{q+\theta} - \Phi_q}{\theta} Z_q(x, \Phi_{q+\theta}) = W_q(x)$$

and

$$\lim_{\theta\to\infty} \frac{\Phi_\theta}{\theta} Z_q(x, \Phi_\theta) = W_q(x).$$

Then, one concludes that (32) reduces to (16) as $\theta \to \infty$.

As shown in Corollary 4, evaluating $\mathbb{P}_x(s_r \le t)$ boils down to deriving explicit expressions for $\mathbb{P}_x(\mathcal{O}_t \in \mathrm{d}s, X_t \in \mathrm{d}y)$ and $\mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}y, \tau_0^- \le t\right)$. In what follows, we provide their characterizations for two special cases of SNLPS, namely a Brownian motion with drift or a Cramér-Lundberg process with exponential claims.

3.2. Examples.

3.2.1. *Brownian risk model.* Let $X_t = \mu t + B_t$, where $\mu > 0$ and $\{B_t\}_{t\ge 0}$ is a standard Brownian motion. Using Formula 2.0.2 of [5], we obtain

$$\mathbb{P}_x\left(X_{\tau_0^-} \in \mathrm{d}y, \tau_0^- \le t\right) = x\int_0^t \frac{1}{\sqrt{2\pi}z^{3/2}} \exp\left(-\frac{(x+\mu z)^2}{2z}\right) \mathrm{d}z\delta_0(\mathrm{d}y)$$

for $x \ge 0$. We recall that, for this risk process, the scale function $W_q (q \ge 0)$ is given by

$$W_q(x) = \frac{1}{\Phi_q + \mu}\left(\mathrm{e}^{\Phi_q x} - \mathrm{e}^{-(\Phi_q+2\mu)x}\right), \quad x \ge 0, \tag{36}$$

where

$$\Phi_q = \left(\sqrt{\mu^2 + 2q\sigma^2} - \mu\right)\sigma^{-2}.$$

Using Lemma 1, an expression for the joint distribution of $(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q})$ is provided in the next corollary. This corresponds to Formulas 1.5.6 on page 258 of [5].

Corollary 7. *For $q > 0$ and $y, z \ge 0$,*

$$\begin{aligned}
&\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} \in \mathrm{d}z, X_{\mathrm{e}_q} \in \mathrm{d}y\right) \\
=& q\mathrm{e}^{\mu(y-z)} A_{(\Phi_q+\mu)^2/2}(x, y, z)\, \mathrm{d}z\mathrm{d}y \\
&+ \left\{\frac{q}{\Phi_q+\mu}\left(\mathrm{e}^{-(\Phi_q+2\mu)(x-y)}\mathbf{1}_{\{x>y\}} - \mathrm{e}^{-(\Phi_q+2\mu)x - \Phi_q y} + \mathrm{e}^{\Phi_q(x-y)}\mathbf{1}_{\{x\le y\}}\right)\right\}\delta_0(\mathrm{d}z),
\end{aligned} \tag{37}$$

where

$$A_\lambda(x, y, z) = \begin{cases} \frac{\sqrt{2}\mathrm{e}^{-y\sqrt{2\lambda} - \lambda z - x^2/(2z)}}{\sqrt{\pi z}} - \sqrt{2\lambda}\mathrm{e}^{-(y+x)\sqrt{2\lambda}} Erfc\left(\frac{\sqrt{2z}\lambda - x}{\sqrt{2\lambda}}\right), & \text{for } x \le 0, \\ \mathrm{e}^{-(y+x)\sqrt{2\lambda}}\left(\frac{\sqrt{2}\mathrm{e}^{-\lambda z}}{\sqrt{\pi z}} - \sqrt{2\lambda} Erfc\left(\sqrt{\lambda z}\right)\right), & \text{for } x > 0. \end{cases}$$

Proof. First, we note that the first term on the right-hand side of Eq. (18) can be evaluated using (36). Now, we want to evaluate $\Lambda^{(q)\prime}(x, s) - \Phi_q \Lambda^{(q)}(x, s)$ for $x \in \mathbb{R}$. Given that X_s has a normal distribution with mean μs and variance s, we obtain

$$\Lambda^{(q)}(x, s) = \int_{(-x)\vee 0}^{\infty} W_q(x+z) \frac{z}{s\sqrt{2\pi s}} \mathrm{e}^{\frac{-(z-\mu s)^2}{2s}} \mathrm{d}z, \quad x \in \mathbb{R}.$$

For $x > 0$,

$$\begin{aligned}
\Lambda^{(q)}(x, s) \quad = \quad & \frac{W_q(x)}{2}\left(\frac{2\mathrm{e}^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s}} + \mathrm{e}^{qs}\left(\Phi_q - 2(\Phi_q+\mu)\mathcal{N}\left(-\frac{s(\Phi_q+\mu)}{\sqrt{s}}\right)\right)\right) \\
& + \frac{\mathrm{e}^{qs}}{2(\Phi_q+\mu)}\left((\Phi_q+2\mu)\mathrm{e}^{\Phi_q x} + \Phi_q \mathrm{e}^{-(\Phi_q+2\mu)x}\right),
\end{aligned}$$

and its derivative is given by

$$\Lambda^{(q)\prime}(x,s) = \frac{W_q'(x)}{2}\left(\frac{2e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s}} + e^{qs}\left(\Phi_q - 2(\Phi_q+\mu)\mathcal{N}\left(-\frac{s(\Phi_q+\mu)}{\sqrt{s}}\right)\right)\right)$$
$$+\frac{(\Phi_q+2\mu)\Phi_q e^{qs}}{2(\Phi_q+\mu)}\left(e^{\Phi_q x} - e^{-(\Phi_q+2\mu)x}\right),$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution. For $x \le 0$, we obtain

$$\Lambda^{(q)}(x,s) = e^{qs}\left(e^{\Phi_q x}\mathcal{N}\left(\frac{x+s(\Phi_q+\mu)}{\sqrt{s}}\right) + e^{-(\Phi_q+2\mu)x}\mathcal{N}\left(\frac{x-s(\Phi_q+\mu)}{\sqrt{s}}\right)\right),$$

and $\Lambda^{(q)\prime}(x,s)$ is given by

$$\Lambda^{(q)\prime}(x,s) = e^{qs}\left(\Phi_q e^{\Phi_q x}\mathcal{N}\left(\frac{x+s(\Phi_q+\mu)}{\sqrt{s}}\right) - (\Phi_q+2\mu)e^{-(\Phi_q+2\mu)x}\mathcal{N}\left(\frac{x-s(\Phi_q+\mu)}{\sqrt{s}}\right)\right)$$
$$+e^{qs}\left(\frac{e^{\Phi_q x}e^{-(x+s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}} + \frac{e^{-(\Phi_q+2\mu)x}e^{-(x-s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}}\right).$$

Then, for $x \le 0$,

$$\Lambda^{(q)\prime}(x,s) - \Phi_q\Lambda^{(q)}(x,s) = e^{qs}\left(\frac{e^{\Phi_q x}e^{-(x+s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}} + \frac{e^{-(\Phi_q+2\mu)x}e^{-(x-s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}}\right)$$
$$-2(\Phi_q+\mu)e^{qs}e^{-(\Phi_q+2\mu)x}\mathcal{N}\left(\frac{x-s(\Phi_q+\mu)}{\sqrt{s}}\right), \tag{38}$$

and for $x \ge 0$,

$$\Lambda^{(q)\prime}(x,s) - \Phi_q\Lambda^{(q)}(x,s)$$
$$= e^{-(\Phi_q+2\mu)x}\left(\Phi_q e^{qs} + \frac{2e^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s}} + e^{qs}\left(\Phi_q - 2(\Phi_q+\mu)\mathcal{N}\left(-\frac{s(\Phi_q+\mu)}{\sqrt{s}}\right)\right)\right). \tag{39}$$

Rearranging and rewriting (38) and (39) using the complementary error function $Erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt = 2\mathcal{N}(-x\sqrt{2})$, we recover the result. ■

Now, if we take the inverse wrt to q of the joint distribution (37), we obtain the following result which coincides with Formulas 1.5.8 on page 258 of [5].

Theorem 8. *For a fixed time $t > 0$ and $y \ge 0$,*

$$\mathbb{P}_x(\mathcal{O}_t \in dz, X_t \in dy) = \mathbb{P}_x\left(X_t \in dy, \tau_0^- > t\right)\delta_0(dz) + e^{\mu(y-x)-\mu^2 t/2}B(t,x,y,z)\,dz dy,$$

where,

$$\mathbb{P}_x\left(X_t \in dy, \tau_0^- > t\right) = \frac{1}{\sqrt{2\pi t}}\left(e^{-(y-x-\mu t)^2} - e^{\mu(y-x)-\mu^2 t/2-(x+y)^2/(2t)}\right)dy,$$

and

$$B(t,x,y,z) = \left(\frac{y\sqrt{t-z}}{\pi t^2\sqrt{z}} - \frac{x\sqrt{z}}{\pi t^2\sqrt{t-z}}\right)e^{-\frac{y^2}{2(t-z)}-\frac{x^2}{2z}}$$

$$+\left(\frac{1}{\sqrt{2\pi}t^{3/2}}-\frac{(x+y)^2}{\sqrt{2\pi}t^{5/2}}\right)\mathrm{e}^{-\frac{(x+y)^2}{2t}}Erfc\left(\frac{y\sqrt{z}}{\sqrt{2t\,(t-z)}}-\frac{x\sqrt{t-z}}{\sqrt{2tz}}\right),$$

for $x\leq 0$, *and*

$$B\,(t,x,y,z)=\frac{(x+y)\sqrt{t-z}}{\pi t^2\sqrt{z}}\mathrm{e}^{-\frac{(x+y)^2}{2(t-z)}}$$
$$+\left(\frac{1}{\sqrt{2\pi}t^{3/2}}-\frac{(x+y)^2}{\sqrt{2\pi}t^{5/2}}\right)\mathrm{e}^{-\frac{(x+y)^2}{2t}}Erfc\left(\frac{(x+y)\sqrt{z}}{\sqrt{2t\,(t-z)}}\right),$$

for $x\geq 0$.

3.2.2. *Cramér-Lundberg risk model with exponential claims.* Let $X=\{X_t,t\geq 0\}$ be a Cramér-Lundberg risk model with exponential claims, i.e.

$$X_t=X_0+ct-\sum_{i=1}^{N_t}C_i,$$

where $N=\{N_t,t\geq 0\}$ is a Poisson process with rate $\eta>0$ and $\{C_i\}_{i\in\mathbb{N}^+}$ is a sequence of iid exponential rv's with mean $1/\alpha$, independent of N. In this case, the law of X is given by

$$\mathbb{P}\left(X_t\in \mathrm{d}z\right)=\mathrm{e}^{-\eta t}\left(\delta_0(\mathrm{d}z-ct)+\mathrm{e}^{-\alpha(ct-z)}\sqrt{\frac{\eta\alpha t}{ct-z}}I_1(2\sqrt{\eta\alpha t(ct-z)})\mathrm{d}z\right),\quad z\leq ct,\quad (40)$$

where I_ν represents the modified Bessel function of the first kind of order ν and it can be computed using (see, e.g., Supplement 4 in [32] for more details)

$$I_v(z)=\left(\frac{1}{2}z\right)^v\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(v+k+1)}.$$

Also, from Eq. (20) of [22], it is already known that

$$\mathbb{P}_x\left(-X_{\tau_0^-}\in \mathrm{d}y,\tau_0^-\leq t\right)$$
$$=\int_0^t\left\{\int_0^\infty \eta\alpha\mathrm{e}^{-\alpha(z+y)}\left(\mathbb{P}_x\left(X_w\in\mathrm{d}z\right)-\int_0^w\frac{z}{w-s}\mathbb{P}\left(X_{w-s}\in\mathrm{d}z\right)f(x+cs,s)\mathrm{d}s\right)\right\}\mathrm{d}w\mathrm{d}y,$$

for $x\geq 0$, where

$$f(x+cs,s)=\mathrm{e}^{-\alpha x-(\eta+\alpha c)s}\sqrt{\frac{\eta\alpha s}{x+cs}}I_1(2\sqrt{\eta\alpha s(x+cs)}).$$

For the process X, its scale function $W_q(q\geq 0)$ is given by

$$W_q(x)=\frac{1}{\sqrt{\Delta_q}}\left((\alpha+\Phi_q)\mathrm{e}^{\Phi_q x}-(\alpha+\theta_q)\mathrm{e}^{\theta_q x}\right),\quad x\geq 0,\quad (41)$$

where

$$\Phi_q=\frac{1}{2c}\left(q+\eta-c\alpha+\sqrt{\Delta_q}\right),\quad (42)$$
$$\theta_q=\frac{1}{2c}\left(q+\eta-c\alpha-\sqrt{\Delta_q}\right),\quad (43)$$
$$\Delta_q=(q+\eta-c\alpha)^2+4c\alpha q=(q+\eta+c\alpha)^2-4c\alpha\eta.\quad (44)$$

From Lemma 1, we now derive an expression for the joint distribution of $\left(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q}\right)$.

Corollary 9. *For $q>0$, $x\geq -cs$ and $y\geq 0$,*

$$
\begin{aligned}
&\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q}\in\mathrm{d}s, X_{\mathrm{e}_q}\in\mathrm{d}y\right)\\
=&\mathbb{P}_x\left(X_{\mathrm{e}_q}\in\mathrm{d}y,\tau_0^->\mathrm{e}_q\right)\delta_0(\mathrm{d}s)+q\mathrm{e}^{\theta_q x-\Phi_q y+(c\theta_q-\eta-q)s}(\alpha+\theta_q)\mathrm{d}s\mathrm{d}y\\
&+q\sum_{m=0}^{\infty}\frac{(\alpha\eta s)^{m+1}\mathrm{e}^{\theta_q x-\Phi_q y+(c\theta_q-\eta-q)s}}{(\alpha+\theta_q)^m m!(m+1)!}\left(f_{m+1}(s)-\frac{f_{m+2}(s)}{(\alpha+\theta_q)cs}\right)\mathrm{d}s\mathrm{d}y, \qquad (45)
\end{aligned}
$$

where $f_m(s):=\gamma(m,(\alpha+\theta_q)(cs+0\wedge x))$ and $\gamma(m,x)=\int_0^x \mathrm{e}^{-z}z^{m-1}\mathrm{d}z$ is the incomplete gamma function. In addition,

$$
\begin{aligned}
&\mathbb{P}_x\left(X_{\mathrm{e}_q}\in\mathrm{d}y,\tau_0^->\mathrm{e}_q\right)\\
=&\frac{q}{\sqrt{\Delta_q}}\left\{(\alpha+\theta_q)\mathrm{e}^{\theta_q x}\left(\mathbf{1}_{\{y\leq x\}}\mathrm{e}^{-\theta_q y}-\mathrm{e}^{-\Phi_q y}\right)+\mathbf{1}_{\{y>x\}}(\alpha+\Phi_q)\mathrm{e}^{\Phi_q(x-y)}\right\}\mathrm{d}y.
\end{aligned}
$$

Proof. Using Eq. (41), it is straightforward to show that

$$
\begin{aligned}
&\left(\mathrm{e}^{-\Phi_q y}W_q(x)-W_q(x-y)\right)\delta_0(\mathrm{d}s)\,\mathrm{d}y\\
=&\frac{1}{\sqrt{\Delta_q}}\left\{(\alpha+\theta_q)\mathrm{e}^{\theta_q x}\left(\mathbf{1}_{\{y\leq x\}}\mathrm{e}^{-\theta_q y}-\mathrm{e}^{-\Phi_q y}\right)+\mathbf{1}_{\{y>x\}}(\alpha+\Phi_q)\mathrm{e}^{\Phi_q(x-y)}\right\}\delta_0(\mathrm{d}s)\mathrm{d}y. \qquad (46)
\end{aligned}
$$

As for the second term of (18), using the relationship that $\Phi_q-\theta_q=\sqrt{\Delta_q}/c$, we obtain

$$
\begin{aligned}
&\Lambda^{(q)'}(x,s)-\Phi_q\Lambda^{(q)}(x,s)\\
=&\int_{0\vee -x}^{\infty}\left(W_q'(x+z)-\Phi_q W_q(x+z)\right)\frac{z}{s}\mathbb{P}\left(X_s\in\mathrm{d}z\right)\\
=&\frac{(\alpha+\theta_q)\mathrm{e}^{\theta_q x}}{\sqrt{\Delta_q}}\int_{0\vee -x}^{\infty}\mathrm{e}^{\theta_q z}\left(\Phi_q-\theta_q\right)\frac{z}{s}\mathbb{P}\left(X_s\in\mathrm{d}z\right)\\
=&\frac{(\alpha+\theta_q)\mathrm{e}^{\theta_q x}}{cs}\int_{0\vee -x}^{\infty}\mathrm{e}^{\theta_q z}z\mathbb{P}\left(X_s\in\mathrm{d}z\right). \qquad (47)
\end{aligned}
$$

Applying Eq. (40), the integral on the right-hand side of (47) becomes

$$
\begin{aligned}
&\int_{0\vee -x}^{\infty}\mathrm{e}^{\theta_q z}z\mathbb{P}\left(X_s\in\mathrm{d}z\right)\\
=&\mathrm{e}^{-\eta s}\int_{0\vee -x}^{cs}\mathrm{e}^{\theta_q z}z\left(\delta_0(\mathrm{d}z-cs)+\mathrm{e}^{-\alpha(cs-z)}\sqrt{\frac{\eta\alpha s}{cs-z}}I_1(2\sqrt{\eta\alpha s(cs-z)})\mathrm{d}z\right)\\
=&\mathrm{e}^{(c\theta_q-\eta)s}\left\{cs+\sum_{m=0}^{\infty}\frac{(\alpha\eta s)^{m+1}}{(\alpha+\theta_q)^{m+1}m!(m+1)!}\left(cs\cdot f_{m+1}(s)-\frac{f_{m+2}(s)}{\alpha+\theta_q}\right)\right\}. \qquad (48)
\end{aligned}
$$

Substituting (46), (47) and (48) into (18) completes the proof of Corollary 9. ■

For a fixed $t>0$, we can invert Eq. (18) wrt q to obtain the joint distribution of $(\mathcal{O}_t, X_t)$ as follows.

Theorem 10. *For $y,s\geq 0, t>0$ and $x\geq -cs$,*

$$\mathbb{P}_x\left(\mathcal{O}_t\in\mathrm{d}s, X_t\in\mathrm{d}y\right)$$

$$=\left\{\mathbb{P}_x\left(X_t\in \mathrm{d}y\right)-y\mathrm{e}^{-\alpha x}\int_0^t\frac{\mathrm{e}^{-(\alpha c+\eta)u}}{t-u}\mathbb{P}\left(X_{t-u}\in \mathrm{d}y\right)\sqrt{\frac{\alpha\eta u}{x+cu}}I_1(2\sqrt{\alpha\eta u(x+cu)})\mathrm{d}u\right\}\delta_0(\mathrm{d}s)$$
$$+\mathbf{1}_{\{t\geq s+\frac{y}{c}\}}\alpha\eta\cdot\exp\left(-\eta(t-s)-c\alpha v-\alpha x\right)$$
$$\times\int_{0\vee -x}^{\infty}\frac{z\mathrm{e}^{-\alpha z}}{cs}\left\{I_0\left(w\right)-\frac{v\cdot I_2\left(w\right)}{\frac{x+z}{c}+t-s}\right\}\mathbb{P}\left(X_s\in \mathrm{d}z\right)\mathrm{d}s\mathrm{d}y, \tag{49}$$

where $w=a\sqrt{v^2+\frac{v(x+y+z)}{c}}$, $v=t-s-\frac{y}{c}$ *and* $a=2\sqrt{c\alpha\eta}$.

Proof. From Lemma 1, one observes that

$$\begin{aligned}&\mathbb{P}_x\left(\mathcal{O}_t\in \mathrm{d}s, X_t\in \mathrm{d}y\right)\\ =&\mathbb{P}_x\left(X_t\in \mathrm{d}y, t<\tau_0^-\right)\delta_0(\mathrm{d}s)+\mathcal{L}_q^{-1}\left\{\mathrm{e}^{-\Phi_q y}\mathrm{e}^{-qs}\left(\Lambda^{(q)\prime}\left(x,s\right)-\Phi_q\Lambda^{(q)}\left(x,s\right)\right)\right\}(t)\mathrm{d}s\mathrm{d}y,\end{aligned} \tag{50}$$

where $\mathcal{L}_q^{-1}$ denotes the inverse Laplace transform wrt q.

In the Cramér-Lundberg risk process, it is known from Landriault et al. [22] that

$$\begin{aligned}&\mathbb{P}_x\left(X_t\in \mathrm{d}y, t<\tau_0^-\right)\\ =&\mathbb{P}_x\left(X_t\in \mathrm{d}y\right)-y\mathrm{e}^{-\alpha x}\int_0^t\frac{\mathrm{e}^{-(\alpha c+\eta)u}}{t-u}\mathbb{P}\left(X_{t-u}\in \mathrm{d}y\right)\sqrt{\frac{\alpha\eta u}{x+cu}}I_1(2\sqrt{\alpha\eta u(x+cu)})\mathrm{d}u,\end{aligned} \tag{51}$$

for $t>0, y\geq 0$. Hence, we are left with the evaluation of the second term on the right-hand side of Eq. (50).

First, using Eqs. (42) – (44), we have

$$\begin{aligned}&(\alpha+\theta_q)\mathrm{e}^{\theta_q(x+z)-\Phi_q y-qs}\\ =&\frac{2\alpha\eta\cdot\exp\left((\eta+c\alpha)s-\alpha(x-y+z)\right)}{a^2}\left(p-\sqrt{p^2-a^2}\right)\exp\left((b-s-\frac{y}{c})p-b\left(\sqrt{p^2-a^2}\right)\right),\end{aligned}$$

where $p:=q+\eta+c\alpha$ and $b:=\frac{x+y+z}{2c}$. From Eq. (D.17) in Drekic [9], we know that the inverse Laplace transform of $\left(p-\sqrt{p^2-a^2}\right)\cdot\exp\left(bp-b\left(\sqrt{p^2-a^2}\right)\right)$ wrt p is

$$\begin{aligned}&\mathcal{L}_p^{-1}\left\{\left(p-\sqrt{p^2-a^2}\right)\exp\left(bp-b\left(\sqrt{p^2-a^2}\right)\right)\right\}(t)\\ =&\frac{a^2}{2}\left\{I_0\left(a\sqrt{t^2+2bt}\right)-\frac{t}{2b+t}I_2\left(a\sqrt{t^2+2bt}\right)\right\}.\end{aligned} \tag{52}$$

Then, it is immediate that

$$\begin{aligned}&\mathcal{L}_q^{-1}\left\{(\alpha+\theta_q)\mathrm{e}^{\theta_q(x+z)-\Phi_q y-qs}\right\}(t)\\ =&\mathbf{1}_{\{t\geq s+\frac{y}{c}\}}\alpha\eta\cdot\exp\left((\eta+c\alpha)(s-t)-\alpha(x-y+z)\right)\\ &\times\left\{I_0\left(a\sqrt{\left(t-s-\frac{y}{c}\right)^2+2b\left(t-s-\frac{y}{c}\right)}\right)-\frac{t-s-\frac{y}{c}}{2b+t-s-\frac{y}{c}}I_2\left(a\sqrt{\left(t-s-\frac{y}{c}\right)^2+2b\left(t-s-\frac{y}{c}\right)}\right)\right\}.\end{aligned} \tag{53}$$

Therefore, letting $v=t-s-\frac{y}{c}$ and using (47) and (53), we obtain

$$\mathcal{L}_q^{-1}\left\{\mathrm{e}^{-\Phi_q y}\mathrm{e}^{-qs}\left(\Lambda^{(q)\prime}\left(x,s\right)-\Phi_q\Lambda^{(q)}\left(x,s\right)\right)\right\}(t)$$

$$
\begin{aligned}
=&\mathcal{L}_q^{-1}\left\{\int_{0\vee -x}^{\infty}(\alpha+\theta_q)\mathrm{e}^{\theta_q(x+z)-\Phi_q y-qs}\frac{z}{cs}\mathbb{P}\left(X_s\in \mathrm{d}z\right)\right\}(t)\\
=&\mathbf{1}_{\{t\geq s+\frac{y}{c}\}}\alpha\eta\cdot\exp\left(-\eta(t-s)-c\alpha v-\alpha x\right)\\
&\times\int_{0\vee -x}^{\infty}\frac{z\mathrm{e}^{-\alpha z}}{cs}\left\{I_0\left(w\right)-\frac{v\cdot I_2\left(w\right)}{\frac{x+z}{c}+t-s}\right\}\mathbb{P}\left(X_s\in\mathrm{d}z\right).
\end{aligned}
\tag{54}
$$

Substituting (51) and (54) into Eq. (50) completes the proof. ■

4. Parisian-type first-last passage times

In this section, we focus on the Parisian type first-last passage time l_r defined in (3) and it corresponds to the right endpoint of the last negative excursion which lasts a longer than a fixed implementation delay r. In a special case, if the maximum duration of negative excursions is less than r, l_r will be equal to the classical ruin time τ_0^-; see Figure 1 for illustrations. Also, we note that the random time l_r has ties with the Parisian ruin time with delay $r>0$ defined as

$$
\kappa_r=\inf\left\{t>0\colon U_t>r\right\},
$$

where $U_t=t-g_t$ with $g_t=\sup\{0\leq s\leq t: X_s\geq 0\}$. We recall that, for a spectrally negative Lévy insurance risk process X, Loeffen et al. [25] obtained an elegant expression for the probability of Parisian ruin, that is for $\mathbb{E}\left[X_1\right]>0$ and $x\in\mathbb{R}$,

$$
\mathbb{P}_x\left(\kappa_r<\infty\right)=1-r\mathbb{E}[X_1]\frac{\Lambda(x,r)}{\int_0^{\infty}z\mathbb{P}(X_r\in\mathrm{d}z)}. \tag{55}
$$

In the rest of this section, we denote by

$$
\mathbb{P}_{u,x}(\cdot):=\mathbb{P}(\cdot|U_0=u,X_0=x),
$$

for all $(u,x)\in\mathcal{S}:=\{(0,\infty)\times(-\infty,0)\cup\{0\}\times[0,\infty)\}$ and by $\mathbb{E}_{u,x}$ its corresponding expectation operator.

Theorem 11. *For $r,q>0$, $(u,x)\in\mathcal{S}$ and $\mathbb{E}[X_1]>0$,*

$$
\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right] \;=\; r\mathbb{E}\left[X_1\right]\frac{\mathbb{E}_{u,x}\left[\Lambda\left(X_{\mathrm{e}_q},r-U_{\mathrm{e}_q}\right)\mathbf{1}_{\left\{U_{\mathrm{e}_q}<r,\tau_0^-<\mathrm{e}_q\right\}}\right]}{\int_0^{\infty}z\mathbb{P}\left(X_r\in\mathrm{d}z\right)}. \tag{56}
$$

Proof. We first note that

$$
\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right]=\mathbb{P}_{u,x}\left(l_r<\mathrm{e}_q\right),
$$

where e_q is an independent exponential rv with rate $q>0$. From the definition of l_r, we point out the following equivalence between the following two events:

$$
\left\{l_r<\mathrm{e}_q\right\}=\left\{\tau_0^-<\mathrm{e}_q,U_t\leq r\text{ for all }t\geq\mathrm{e}_q\right\}.
$$

Hence,

$$
\begin{aligned}
\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right]&=\int_0^r\int_{-\infty}^{\infty}\mathbb{P}_{u,x}\left(\tau_0^-<\mathrm{e}_q,U_t\leq r\text{ for all }t\geq\mathrm{e}_q,X_{\mathrm{e}_q}\in\mathrm{d}y,U_{\mathrm{e}_q}\in\mathrm{d}s\right)\\
&=\int_0^r\int_{-\infty}^{\infty}\mathbb{P}_{s,y}\left(\kappa_r=\infty\right)\mathbb{P}_{u,x}\left(\tau_0^-<\mathrm{e}_q,X_{\mathrm{e}_q}\in\mathrm{d}y,U_{\mathrm{e}_q}\in\mathrm{d}s\right),
\end{aligned}
\tag{57}
$$

where the strong Markov property of X at e_q is applied in (57). Given that that $U_{\mathrm{e}_q}=0$ when $X_{\mathrm{e}_q}\geq 0$, (57) can be written as

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right]=\int_0^\infty \mathbb{P}_y\left(\kappa_r=\infty\right)\mathbb{P}_x\left(\tau_0^-<\mathrm{e}_q, X_{\mathrm{e}_q}\in \mathrm{d}y\right)$$
$$+\int_0^r\int_{-\infty}^0 \mathbb{P}_{s,y}\left(\kappa_r=\infty\right)\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, U_{\mathrm{e}_q}\in \mathrm{d}s\right). \tag{58}$$

Using (55) (note that $\Lambda(0,r)=1$) and the fact that $\mathbb{P}_y\left(\tau_0^+\leq r\right)=\Lambda(y,r)$ for $y<0$ (see [25] for more details), we obtain

$$\mathbb{P}_{s,y}\left(\kappa_r=\infty\right)=\begin{cases}\mathbb{P}_y\left(\kappa_r=\infty\right)=r\mathbb{E}[X_1]\frac{\Lambda(y,r)}{\int_0^\infty z\mathbb{P}(X_r\in \mathrm{d}z)}, & \text{if } y\geq 0,\\ \mathbb{P}_y\left(\tau_0^+\leq r-s\right)\mathbb{P}\left(\kappa_r=\infty\right)=r\mathbb{E}[X_1]\frac{\Lambda(y,r-s)}{\int_0^\infty z\mathbb{P}(X_r\in \mathrm{d}z)}, & \text{if } y<0.\end{cases} \tag{59}$$

Finally, substituting (59) into (58) yields the desired result. ■

From (56), we observe that the joint distribution of $(X_{\mathrm{e}_q}, U_{\mathrm{e}_q})$ plays a pivotal role in the characterization of the Laplace transform of l_r. Given that $U_{\mathrm{e}_q}=0$ when $X_{\mathrm{e}_q}\geq 0$, we have

$$\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, U_{\mathrm{e}_q}=0\right)=\mathbb{P}_x\left(X_{\mathrm{e}_q}\in \mathrm{d}y\right), \tag{60}$$

for any $(u,x)\in\mathcal{S}$ and $y\geq 0$. Hence, we are left with the identification of the joint distribution of $(X_{\mathrm{e}_q}, U_{\mathrm{e}_q})$ only for $U_{\mathrm{e}_q}>0$ and $X_{\mathrm{e}_q}<0$.

Lemma 12. *For $q,s>0$, $y<0$, and $(u,x)\in\mathcal{S}$,*

$$\begin{aligned}&\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, U_{\mathrm{e}_q}>s\right)\\ =&\mathrm{e}^{-qs}\int_{-\infty}^0 \mathbb{P}_x\left(X_{\mathrm{e}_q}\in \mathrm{d}z, \tau_0^+<\mathrm{e}_q\right)\mathbb{P}_z\left(X_s\in \mathrm{d}y, s<\tau_0^+\right)\\ &+\mathbf{1}_{\{s\geq u\}}\int_{s-u}^\infty q\mathrm{e}^{-qt}\mathbb{P}_x\left(X_t\in \mathrm{d}y, t<\tau_0^+\right)\mathrm{d}t+\mathbf{1}_{\{s<u\}}\mathbb{P}_x\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \mathrm{e}_q<\tau_0^+\right),\end{aligned} \tag{61}$$

where

$$\mathbb{P}_{-z}\left(X_s\in \mathrm{d}y, s<\tau_0^+\right)\mathrm{d}z=\mathbb{P}_{-z}\left(X_s\in \mathrm{d}y\right)\mathrm{d}z-\int_0^s \frac{z}{t}\mathbb{P}\left(X_t\in \mathrm{d}z\right)\mathbb{P}\left(X_{s-t}\in \mathrm{d}y\right)\mathrm{d}t, \quad z\geq 0, \tag{62}$$

and

$$\mathbb{P}_x\left(\tau_0^+<\mathrm{e}_q, -X_{\mathrm{e}_q}\in \mathrm{d}z\right)=\begin{cases} q\left(\Phi_q'\mathrm{e}^{\Phi_q(x+z)}-W_q(x+z)\right)\mathrm{d}z, & x>0,\\ q\mathrm{e}^{\Phi_q x}\left(\Phi_q'\mathrm{e}^{\Phi_q z}-W_q(z)\right)\mathrm{d}z, & x<0.\end{cases} \tag{63}$$

Proof. To derive the joint distribution of $(X_{\mathrm{e}_q}, U_{\mathrm{e}_q})$, we consider separately the cases where $\left\{\tau_0^+>\mathrm{e}_q\right\}$ (where τ_0^+ is assumed to be 0 a.s. when $x\geq 0$) and $\left\{\tau_0^+<\mathrm{e}_q\right\}$.

First, for the first case $\left\{\tau_0^+>\mathrm{e}_q\right\}$, it follows that

$$\begin{aligned}&\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+>\mathrm{e}_q, U_{\mathrm{e}_q}>s\right)\\ =&\mathbb{P}_x\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+>\mathrm{e}_q, \mathrm{e}_q+u>s\right)\\ =&\mathbf{1}_{\{s\geq u\}}\int_{s-u}^\infty q\mathrm{e}^{-qt}\mathbb{P}_x\left(X_t\in \mathrm{d}y, t<\tau_0^+\right)\mathrm{d}t+\mathbf{1}_{\{s<u\}}\mathbb{P}_x\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+>\mathrm{e}_q\right),\end{aligned} \tag{64}$$

for $s>0$ and $y<0$. We note that $\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+>\mathrm{e}_q, U_{\mathrm{e}_q}>s\right)$ is understood to be 0 when $x\geq 0$.

Now, for the second case $\left\{\tau_0^+<\mathrm{e}_q\right\}$, from the definition of U_t,

$$\begin{aligned}\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+\leq \mathrm{e}_q, U_{\mathrm{e}_q}>s\right) &= \mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+\leq \mathrm{e}_q - s, U_{\mathrm{e}_q}>s\right)\\ &= \mathrm{e}^{-qs}\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+\leq \mathrm{e}_q - s, U_{\mathrm{e}_q}>s\,|\mathrm{e}_q>s\right). \end{aligned} \tag{65}$$

Given that $\mathrm{e}_q-s\,|\mathrm{e}_q>s$ is also exponential with rate q, Eq. (65) can be rewritten as

$$\begin{aligned}\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+\leq \mathrm{e}_q, U_{\mathrm{e}_q}>s\right) &= \mathrm{e}^{-qs}\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q+s}\in \mathrm{d}y, \tau_0^+\leq \mathrm{e}_q, U_{\mathrm{e}_q+s}>s\right)\\ &= \mathrm{e}^{-qs}\int_{-\infty}^0 \mathbb{P}_{u,x}\left(X_{\mathrm{e}_q+s}\in \mathrm{d}y, \tau_0^+\leq \mathrm{e}_q, X_{\mathrm{e}_q}\in \mathrm{d}z, U_{\mathrm{e}_q+s}>s\right)\end{aligned}$$

Using the strong Markov property of X at e_q, it follows that

$$\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in \mathrm{d}y, \tau_0^+\leq \mathrm{e}_q, U_{\mathrm{e}_q}>s\right) = \mathrm{e}^{-qs}\int_{-\infty}^0 \mathbb{P}_x\left(\tau_0^+\leq \mathrm{e}_q, X_{\mathrm{e}_q}\in \mathrm{d}z\right)\mathbb{P}_z\left(X_s\in \mathrm{d}y, s<\tau_0^+\right). \tag{66}$$

Combining (66) and (64) leads to (61).

It remains to prove Eq. (62) and (63). For Eq. (62), we have that

$$\mathbb{P}_{-z}\left(X_s\in \mathrm{d}y, s<\tau_0^+\right) = \mathbb{P}_{-z}\left(X_s\in \mathrm{d}y\right) - \mathbb{P}_{-z}\left(X_s\in \mathrm{d}y, \tau_0^+<s\right), \tag{67}$$

where by conditioning on τ_0^+ and then using Kendall's identity, the second term on the right-hand side of (67) becomes

$$\begin{aligned}\mathbb{P}_{-z}\left(X_s\in \mathrm{d}y, \tau_0^+<s\right) &= \int_0^s \mathbb{P}\left(X_{s-t}\in \mathrm{d}y\right)\mathbb{P}_{-z}\left(\tau_0^+\in \mathrm{d}t\right)\\ &= \int_0^s \frac{z}{t}\frac{\mathbb{P}\left(X_t\in \mathrm{d}z\right)}{\mathrm{d}z}\mathbb{P}\left(X_{s-t}\in \mathrm{d}y\right)\mathrm{d}t.\end{aligned}$$

Moreover, Eq. (63) is an immediate consequence of Theorem 8.7 and Corollary 8.9 of [18]. ∎

Remark 13. Considering the Brownian risk model in subsection 3.2.1, one can identify the potential measures in Lemma 12. Indeed, from [5] (see equation 1.1.6 on p. 251) we have that for any $x<0$, $y\leq 0$ and $s\geq 0$,

$$\begin{aligned}\mathbb{P}_{-z}(X_s\in \mathrm{d}y, s<\tau_0^+) &= \mathbb{P}_{-z}(X_s\in \mathrm{d}y) - \mathbb{P}_{-z}(X_s\in \mathrm{d}y, \overline{X}_s\geq 0)\\ &= \frac{1}{\sigma\sqrt{s}}\phi\left(\frac{y-\mu s+z}{\sigma\sqrt{s}}\right)\mathrm{d}y - \frac{1}{\sigma\sqrt{s}}\mathrm{e}^{2(\mu/\sigma^2)z}\phi\left(\frac{y-\mu s-z}{\sigma\sqrt{s}}\right)\mathrm{d}y.\end{aligned}$$

where ϕ is the density distribution function of the standard Normal distribution, and

$$\mathbb{P}_x\left(\tau_0^+<\mathrm{e}_q, -X_{\mathrm{e}_q}\in \mathrm{d}z\right) = \begin{cases} q\left(\frac{\mathrm{e}^{\Phi_q(x+z)}}{\sqrt{\mu^2+2\sigma^2 q}} - \frac{1}{\Phi_q+\mu}\left(\mathrm{e}^{\Phi_q(x+z)} - \mathrm{e}^{-(\Phi_q+2\mu)(x+z)}\right)\right)\mathrm{d}z, & x>0,\\ q\mathrm{e}^{\Phi_q x}\left(\frac{\mathrm{e}^{\Phi_q z}}{\sqrt{\mu^2+2\sigma^2 q}} - \frac{1}{\Phi_q+\mu}\left(\mathrm{e}^{\Phi_q z} - \mathrm{e}^{-(\Phi_q+2\mu)z}\right)\right)\mathrm{d}z, & x<0.\end{cases}$$

Next, an expression for the joint Laplace transform of U_{e_q} and X_{e_q} is stated. This result immediately follows from Lemma 12 and Eq. (60). We therefore omit the proof here.

Corollary 14. *For $\nu \geq 0, q > 0 \vee \psi(\nu)$, $p > \psi(\nu) - q$, and $(u, x) \in \mathcal{S}$,*

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\right] = \mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\mathbf{1}_{\{X_{\mathrm{e}_q}\geq 0\}}\right] + \mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\mathbf{1}_{\{X_{\mathrm{e}_q}<0\}}\right], \tag{68}$$

where

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\mathbf{1}_{\{X_{\mathrm{e}_q}\geq 0\}}\right] =& \mathbb{E}_{x}\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbf{1}_{\{X_{\mathrm{e}_q}\geq 0\}}\right] \\ =& \frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q - \nu} + \frac{q\mathrm{e}^{\nu x} - qZ_q(x,\nu)}{q-\psi(\nu)}\end{aligned} \tag{69}$$

and

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\mathbf{1}_{\{X_{\mathrm{e}_q}<0\}}\right] &= \frac{pq}{p+q-\psi(\nu)}\frac{\Phi_q'\mathrm{e}^{\Phi_q x}(\Phi_{p+q}-\nu)}{(\Phi_q-\nu)(\Phi_{p+q}-\Phi_q)} \\ &- \frac{q\mathrm{e}^{-pu}\left(Z_q(x,\Phi_{p+q}) - Z_q(x,\nu)\right)}{p+q-\psi(\nu)} - \frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}.\end{aligned} \tag{70}$$

Note that it is easy to verify that Eq. (70) holds more generally for any $q > 0$.

Remark 15. We recall the quantity $g^q = \sup\{0 \leq t \leq \mathrm{e}_q : X_t \leq 0\}$ studied in [2]. By letting $\nu = 0$ and noting that

$$\mathbb{E}_x\left[\mathrm{e}^{-pU_{\mathrm{e}_q}}\right] = \mathbb{E}_x\left[\mathrm{e}^{-p(\mathrm{e}_q - g^q)}\right] = \frac{q}{p+q}\mathbb{E}_x\left[\mathrm{e}^{pg^{p+q}}\right],$$

Eq. (68) reduces to Theorem 2 in [2].

In the rest of this section, we consider a variation of the random time l_r where the constant parameter r is replaced by independent copies of a generic exponential rv e_θ with rate $\theta > 0$. Specifically, we define l^θ as

$$l^\theta = \sup\left\{t \geq \tau_0^- : U_t > \mathrm{e}_\theta^{g_t}\right\}, \tag{71}$$

where $\mathrm{e}_\theta^{g_t}$ denotes an independent copy of e_θ generated for the negative excursion that began at time g_t.

To study the distribution of l^θ, we first recall the Parisian ruin with exponential delays defined as

$$\kappa^\theta = \inf\left\{t > 0 : U_t > \mathrm{e}_\theta^{g_t}\right\}. \tag{72}$$

An expression for the probability of Parisian ruin with exponential delays was first given in [23], that is, for $\mathbb{E}[X_1] > 0$ and $x \in \mathbb{R}$,

$$\mathbb{P}_x\left(\kappa^\theta < \infty\right) = 1 - \mathbb{E}[X_1]\frac{\Phi_\theta}{\theta}Z(x, \Phi_\theta). \tag{73}$$

Applying Corollary 14, we have the following Laplace transform for l^θ.

Theorem 16. *Assume $\mathbb{E}[X_1] > 0$, for $q, \theta > 0$ and $(u, x) \in \mathcal{S}$,*

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql^\theta}\right] = \frac{\mathbb{E}[X_1]\Phi_\theta}{\theta}\left(\frac{(\theta-q)W_q(x)}{\Phi_q - \Phi_\theta} + \frac{\theta\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_{q+\theta}-\Phi_q} - \mathrm{e}^{-\theta u}\left(Z_q(x,\Phi_{q+\theta}) - Z_q(x,\Phi_\theta)\right)\right). \tag{74}$$

Proof. First, one can observe from the definition of l^θ that the following two events are equivalent:

$$\left\{l^\theta < \mathrm{e}_q\right\} = \left\{\tau_0^- < \mathrm{e}_q, U_t \le \mathrm{e}_\theta^{g_t} \text{ for all } t \ge \mathrm{e}_q\right\}.$$

Using a similar series of arguments as in the derivation of Eq. (58), we obtain

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql^\theta}\right] &= \int_0^\infty \mathbb{P}_y\left(\kappa^\theta = \infty\right)\mathbb{P}_x\left(\tau_0^- < \mathrm{e}_q, X_{\mathrm{e}_q} \in \mathrm{d}y\right)\\ &+ \int_0^\infty \int_{-\infty}^0 \mathbb{P}_{s,y}\left(\kappa^\theta = \infty\right)\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, U_{\mathrm{e}_q} \in \mathrm{d}s, s < \mathrm{e}_\theta\right). \end{aligned} \tag{75}$$

From Eqs. (73) and (13), one deduces that

$$\mathbb{P}_{s,y}\left(\kappa^\theta = \infty\right) = \begin{cases}\mathbb{P}_y\left(\kappa^\theta = \infty\right) = \frac{\mathbb{E}[X_1]\Phi_\theta}{\theta} Z\left(y, \Phi_\theta\right), & \text{if } y \ge 0,\\ \mathbb{P}_y\left(\tau_0^+ \le \mathrm{e}_\theta - s\right)\mathbb{P}\left(\kappa^\theta = \infty\right) = \frac{\mathbb{E}[X_1]\Phi_\theta}{\theta}\mathrm{e}^{-\theta s}\mathrm{e}^{\Phi_\theta y}, & \text{if } y < 0,\end{cases} \tag{76}$$

which allows to rewrite (75) as

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql^\theta}\right] = \frac{\mathbb{E}\left[X_1\right]\Phi_\theta}{\theta}\left(\mathbb{E}_x\left[Z\left(X_{\mathrm{e}_q}, \Phi_\theta\right)\mathbf{1}_{\left\{X_{\mathrm{e}_q}>0, \tau_0^- < \mathrm{e}_q\right\}}\right] + \mathbb{E}_{u,x}\left[\mathrm{e}^{-\theta U_{\mathrm{e}_q} + \Phi_\theta X_{\mathrm{e}_q}}\mathbf{1}_{\left\{X_{\mathrm{e}_q} \le 0\right\}}\right]\right). \tag{77}$$

Using results on potential measures for the SNLP X (see Chapter 8.4 of [18] for more details), one can show that

$$\mathbb{E}_x\left[Z\left(X_{\mathrm{e}_q}, \Phi_\theta\right)\mathbf{1}_{\left\{X_{\mathrm{e}_q}>0, \tau_0^- < \mathrm{e}_q\right\}}\right] = \frac{(q-\theta)\left(\Phi_q' \mathrm{e}^{\Phi_q x} - W_q(x)\right)}{\Phi_q - \Phi_\theta}. \tag{78}$$

Also, from (70), it follows that

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-\theta U_{\mathrm{e}_q} + \Phi_\theta X_{\mathrm{e}_q}}\mathbf{1}_{\left\{X_{\mathrm{e}_q} \le 0\right\}}\right] = \frac{\Phi_q' \mathrm{e}^{\Phi_q x}}{\Phi_q - \Phi_\theta}\left(\frac{\theta(\Phi_{q+\theta} - \Phi_\theta)}{\Phi_{q+\theta} - \Phi_q} - q\right) - \mathrm{e}^{-\theta u}\left(Z_q(x, \Phi_{q+\theta}) - Z_q(x, \Phi_\theta)\right). \tag{79}$$

Substituting (78) and (79) into (77) completes the proof of Theorem 16. ■

Using the same arguments as in Remark 6, it is straightforward to show that l^θ converges in distribution to τ_0^- (and g) as $\theta \to 0(\infty)$. We omit the details here.

5. Additional results on convergence of the first-last passage times

In this section, we study some limiting cases of s_r, s^θ, l_r and l^θ and show their consistency with known results in the literature. We note that the convergence (in distribution) results for s^θ and l^θ have already been discussed in Sections 3 and 4 through their Laplace transforms. In this section, we show that the convergence (in probability) results for s^θ and l^θ can be proved in a more direct way.

We begin by examining the limiting case of s_r, s^θ, l_r and l^θ when $r \to 0$ or $\theta \to \infty$.

Proposition 17. *For any spectrally negative Lévy process X with the condition that $\mathbb{E}\left[X_1\right] > 0$ and $x \in \mathbb{R}$,*

(i) *s_r converges $\mathbb{P}_x$ almost surely to g as $r \to 0$ when $\tau_0^- < \infty$.*
(ii) *s^θ converges in probability to g as $\theta \to \infty$ when $\tau_0^- < \infty$.*
(iii) *l_r converges $\mathbb{P}_x$ almost surely to g as $r \to 0$ when $\tau_0^- < \infty$.*
(iv) *l^θ converges in probability to g as $\theta \to \infty$ when $\tau_0^- < \infty$.*

Proof.
(i) First, we show that s_r converges in probability to g. For any $\epsilon > 0$ and all $x \in \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{P}_x\left(|s_r - g| > \epsilon, \tau_0^- < \infty\right) \\ =&\mathbb{P}_x\left(|s_r - g| > \epsilon, s_r = \tau_0^-, \tau_0^- < \infty\right) + \mathbb{P}_x\left(|s_r - g| > \epsilon, s_r > \tau_0^-, \tau_0^- < \infty\right), \end{aligned}$$

where the second term is zero as s_r coincides with g if $s_r > \tau_0^-$. For all $x \in \mathbb{R}$, we have

$$\begin{aligned} \lim_{r\to 0} \mathbb{P}_x\left(|s_r - g| > \epsilon, s_r = \tau_0^- < \infty\right) &= \lim_{r\to 0} \mathbb{P}_x\left(g - \tau_0^- > \epsilon, s_r = \tau_0^- < \infty\right) \\ &\leq \lim_{r\to 0} \mathbb{P}_x\left(\sigma_r = \infty, \tau_0^- < \infty\right) \\ &= \mathbb{P}_x\left(\tau_0^- = \infty, \tau_0^- < \infty\right) = 0. \end{aligned} \tag{80}$$

where in the last equality we used the fact that $\sigma_r := \inf\{t > 0 \colon \mathcal{O}_t > r\}$ converges $\mathbb{P}_x$ almost surely (a.s.) to the classical ruin time τ_0^- when $r \to 0$ (see Proposition 3.3 in [15]). Therefore, for all $x \in \mathbb{R}$, we deduce that s_r converges to g in probability as $r \to 0$ when $\tau_0^- < \infty$. Also, note that s_r is a non-decreasing function as $r \to 0$, we conclude that s_r converges to g when $\tau_0^- < \infty$ $\mathbb{P}_x$ a.s. as $r \to 0$.
(ii) Similar to (i), for any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned} &\lim_{\theta\to\infty} \mathbb{P}_x\left(|s^\theta - g| > \epsilon, \tau_0^- < \infty\right) \\ =&\lim_{\theta\to\infty} \mathbb{P}_x\left(g - \tau_0^- > \epsilon, s^\theta = \tau_0^-, \tau_0^- < \infty\right) \\ \leq&\lim_{\theta\to\infty} \mathbb{P}_x\left(\sigma_{\mathrm{e}_\theta} = \infty, \tau_0^- < \infty\right) = 0. \end{aligned}$$

(iii) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned} &\lim_{r\to 0} \mathbb{P}_x\left(|l_r - g| > \epsilon, \tau_0^- < \infty\right) \\ =&\lim_{r\to 0} \mathbb{P}_x\left(g - l_r > \epsilon, \kappa_r < \infty, \tau_0^- < \infty\right) + \lim_{r\to 0} \mathbb{P}_x\left(g - l_r > \epsilon, \kappa_r = \infty, \tau_0^- < \infty\right) \\ \leq&\lim_{r\to 0} \mathbb{P}_x\left(\mathbb{P}_{X_{l_r}}\left(\kappa_r = \infty, \tau_0^- < \infty\right)\right) + \lim_{r\to 0} \mathbb{P}_x\left(\kappa_r = \infty, \tau_0^- < \infty\right) \\ =&\lim_{r\to 0} \mathbb{P}_x\left(\mathbb{P}\left(\kappa_r = \infty, \tau_0^- < \infty\right)\right) + \lim_{r\to 0} \mathbb{P}_x\left(\kappa_r = \infty, \tau_0^- < \infty\right), \end{aligned}$$

where the last step is by conditioning on $X_{l_r} = 0$. Since κ_r has the same law as τ_0^- as $r \to 0$ (see Corollary 2.4 of [35]) and l_r is a non-decreasing function as $r \to 0$, we conclude that the above limits all approach zero and thus l_r converges to g a.s. when $\tau_0^- < \infty$ as $r \to 0$.
(iv) Similar to (iii), for any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned} &\lim_{\theta\to\infty} \mathbb{P}_x\left(|l^\theta - g| > \epsilon, \tau_0^- < \infty\right) \\ =&\lim_{\theta\to\infty} \mathbb{P}_x\left(g - l^\theta > \epsilon, \kappa^\theta < \infty, \tau_0^- < \infty\right) + \lim_{\theta\to\infty} \mathbb{P}_x\left(g - l^\theta > \epsilon, \kappa^\theta = \infty, \tau_0^- < \infty\right) \\ \leq&\lim_{r\to 0} \mathbb{P}_x\left(\mathbb{P}_{X_{l^\theta}}\left(\kappa^\theta = \infty, \tau_0^- < \infty\right)\right) + \lim_{\theta\to\infty} \mathbb{P}_x\left(\kappa^\theta = \infty, \tau_0^- < \infty\right) \\ =&\lim_{\theta\to\infty} \mathbb{P}_x\left(\mathbb{P}\left(\kappa^\theta = \infty, \tau_0^- < \infty\right)\right) + \lim_{\theta\to\infty} \mathbb{P}_x\left(\kappa^\theta = \infty, \tau_0^- < \infty\right) = 0. \end{aligned}$$

■

The following proposition is the counterpart to Proposition 17 when $r \to \infty$ or $\theta \to 0$.

Proposition 18. *For any spectrally negative Lévy process X with the condition that $\mathbb{E}[X_1] > 0$ and $x \in \mathbb{R}$,*

(i) *s_r converges $\mathbb{P}_x$ almost surely to τ_0^- as $r \to \infty$.*
(ii) *s^θ converge $\mathbb{P}_x$ in probability to τ_0^- as $\theta \to 0$.*
(iii) *l_r converge $\mathbb{P}_x$ almost surely to τ_0^- as $r \to \infty$.*
(iv) *l^θ converge $\mathbb{P}_x$ in probability to τ_0^- as $\theta \to 0$.*

Proof. As shown in the proof of Proposition 17, it is sufficient to prove the convergence in probability in cases (i) and (iii).
(i) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{r\to\infty} \mathbb{P}_x\left(|s_r - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) &\le \lim_{r\to\infty} \mathbb{P}_x\left(s_r > \tau_0^-, \tau_0^- < \infty\right)\\ &= \lim_{r\to\infty} \mathbb{P}_x\left(\sigma_r < \infty, \tau_0^- < \infty\right)\\ &= \lim_{r\to\infty} \mathbb{P}_x\left(\sigma_r < \infty\right)\\ &=1 - \lim_{r\to\infty} \mathbb{P}_x\left(\int_0^\infty \mathbf{1}_{(-\infty,0)}(X_s)\mathrm{d}s \le r\right) = 0,\end{aligned}$$

the last equation holds because X drifts to $+\infty$ when $\mathbb{E}[X_1] > 0$.
(ii) Similar to (i), for any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{\theta\to 0} \mathbb{P}_x\left(|s^\theta - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) &\le \lim_{\theta\to 0} \mathbb{P}_x\left(\sigma_{\mathrm{e}_\theta} < \infty\right)\\ &=1 - \lim_{\theta\to 0} \mathbb{P}_x\left(\int_0^\infty \mathbf{1}_{(-\infty,0)}(X_s)\mathrm{d}s \le \mathrm{e}_\theta\right) = 0.\end{aligned}$$

(iii) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{r\to\infty} \mathbb{P}_x\left(|l_r - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) &= \lim_{r\to\infty} \mathbb{P}_x\left(l_r - \tau_0^- > \epsilon, \kappa_r < \infty\right)\\ &\le \lim_{r\to\infty} \mathbb{P}_x\left(\kappa_r < \infty\right) \le \lim_{r\to\infty} \mathbb{P}_x\left(\sigma_r < \infty\right) = 0.\end{aligned}$$

(iv) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

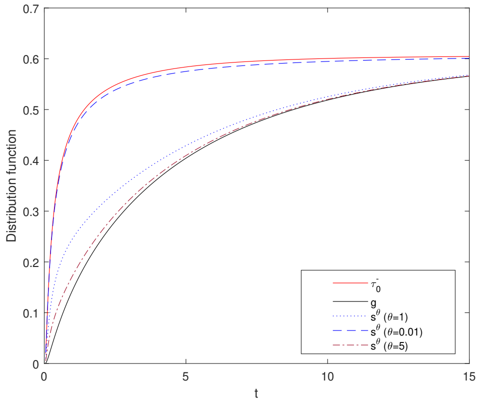
$$\begin{aligned}\lim_{\theta\to 0} \mathbb{P}_x\left(|l^\theta - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) &\le \lim_{\theta\to 0} \mathbb{P}_x\left(l^\theta > \tau_0^-, \tau_0^- < \infty\right)\\ &= \lim_{\theta\to 0} \mathbb{P}_x\left(\kappa^\theta < \infty, \tau_0^- < \infty\right)\\ &= \lim_{\theta\to 0} \mathbb{P}_x\left(\kappa^\theta < \infty\right)\\ &=1 - \lim_{\theta\to 0} \mathbb{P}_x\left(\int_0^\infty \mathbf{1}_{(-\infty,0)}(X_s)\mathrm{d}s \le \mathrm{e}_\theta\right) = 0.\end{aligned}$$

■

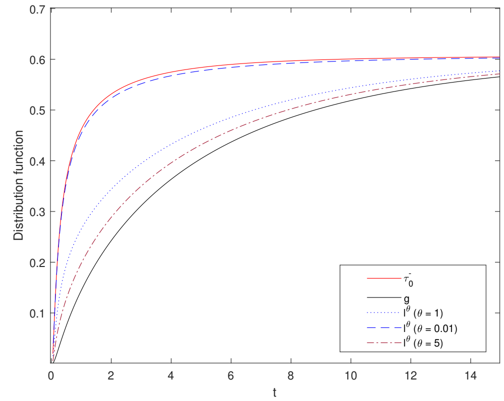
We complement the above analysis with a numerical study of distribution functions of the random times of interest. More specifically, we consider the Brownian risk model $\{X_t\}_{t\ge 0}$ with

$$X_t = x + \mu t + B_t, \quad t \ge 0,$$

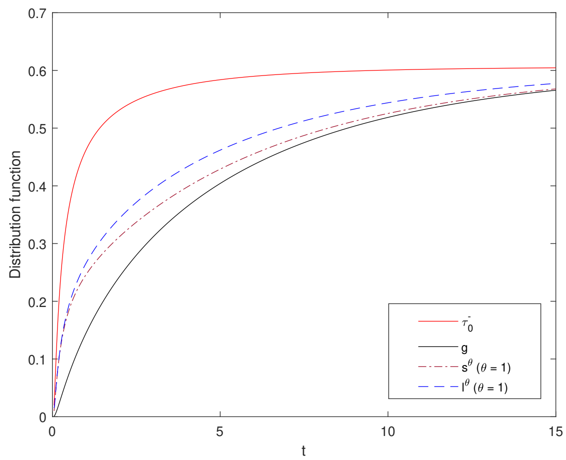
where we choose the parameters to be $x = \mu = 0.5$. Using the known result of $\mathbb{P}_x(\tau_0^- \le t)$ for Brownian risk model and numerically inverting the Laplace transforms of g, s_r, and l_r,



(A) Distribution of s^θ with different θ



(B) Distribution of l^θ with different θ



(C) Comparison of s^θ and l^θ

FIGURE 2. Distribution functions of the random times

respectively, we plot their distribution functions on the same graph for comparison. More specifically, in Figures 2(A) and 2(B), we consider the distribution functions of s^θ and l^θ with different values of θ. We note that the distribution functions of s^θ and l^θ converge to that of the first passage time τ_0^- as $\theta \to 0$, and they converge to the distribution function of g as $\theta \to \infty$, which are consistent with the results discussed in Propositions 17 and 18. This is expected because the randomized risk tolerance level increases on average as θ decreases, and it is more likely that the individual (or the total) negative excursion length is less than the level for a given sample path. To compare the proposed two random times, s^θ and l^θ, we plot the distribution functions of s^θ and l^θ for $\theta = 1$ in Figure 2(C). It can be seen from the graph that

$$\mathbb{P}_x(g \leq t) \leq \mathbb{P}_x(s^\theta \leq t) \leq \mathbb{P}_x(l^\theta \leq t) \leq \mathbb{P}_x(\tau_0^- \leq t) \quad \text{for all } t \geq 0.$$

6. An Application to credit risk management

In this section, we consider an application of our main results to credit risk management. More specifically, we consider the following model introduced in Egami and Kevkhishvili [10].

Assume that a firm's total assets process $\{A_t\}_{t\geq 0}$ follows a geometric Brownian motion with parameters $\nu \in \mathbb{R}$ and $\sigma > 0$, and its debt process $\{D_t\}_{t\geq 0}$ grows at a constant risk-free rate r_f, i.e.,

$$A_t = A_0 \mathrm{e}^{\nu t + \sigma B_t^A} \quad \text{and} \quad D_t = D_0 \mathrm{e}^{r_f t}, \quad t \geq 0,$$

where the initial values are A_0 and D_0 (such that $A_0 > D_0$) respectively, and $\{B_t^A\}_{t\geq 0}$ is a standard Brownian motion. The leverage process $\{R_t\}_{t\geq 0}$ is defined as $R_t := \frac{A_t}{D_t}$ with $R_0 > 1$. The insolvency time of the firm is set as the first time the leverage process drops below level 1. As a precautionary measure, a threshold level $R^* > 1$ is set for the firm to take necessary actions to avoid possible subsequent insolvency. That is to say the firm should monitor the event the leverage process drops below R^*, i.e.,

$$\inf \{t \geq 0 : R_t < R^*\}.$$

The study of $\{R_t\}_{t\geq 0}$ can be reduced to the study of the following process:

$$X_t := \mu t + B_t^A, \quad t \geq 0, \tag{81}$$

where $\mu := \frac{\nu - r_f}{\sigma}$. Then the event $R_t < R^*$ is equivalent to that X drops below level $\alpha := \frac{1}{\sigma} \ln \frac{R^* D_0}{A_0}$, i.e.,

$$\inf \{t \geq 0 : X_t < \alpha\}.$$

In [10], the authors derive the probability density of the last hitting time

$$\lambda_\alpha := \sup\{t \geq 0 : X_t = \alpha\},$$

for a general diffusion process and further consider an interesting example to illustrate how the last hitting time λ_α can provide useful information for risk management.

In what follows, we use the same empirical data for American Apparel Inc., which filed for bankruptcy protection in October 2015, to conduct a numerical study. For completeness, we recall Table 1 taken from [10]. Note that we set $R^* = \frac{1}{0.8} = 1.25$ representing 80% debt/asset ratio.

It is seen from Table 1 that the drift μ declines drastically and turns negative in September 2013. This period with "negative returns" lasts until June 2014. This is in line with the fact that American Apparel Inc. had problems with a new distribution center in 2013. After a short recovery (June to December 2014), the financial situation of American Apparel Inc. continues to sink and eventually files bankruptcy in October 2015.

Figure 3 plots three conditional probabilities of $\mathbb{P}_{-\alpha}\left(g > 1 | \tau_0^- < \infty\right)$, $\mathbb{P}_{-\alpha}\left(l^\theta > 1 | \tau_0^- < \infty\right)$, and $\mathbb{P}_{-\alpha}\left(s^\theta > 1 | \tau_0^- < \infty\right)$. Intuitively, they measure how likely the (scaled) leverage process X takes longer than a year to be "ultimately" recovered given ruin occurs. As such, they can be interpreted as *default probabilities* in general sense. The larger these conditional probabilities, the worse the firm's financial situation is. First, we see large fluctuations and an overall increasing trend for all three measures. This shows that the firm's financial situation is rather unstable and tends to worsen. Second, the conditional probabilities become one in two periods (Sep 2013 - Mar 2014, Mar 2015 - June 2015), which is in line with that the return parameter μ are negative at that time. Third, the conditional probabilities of s^θ and l^θ are lower than that of g, which are consistent with their definitions (see Eq. 4).

	ν	r_f	σ	μ	A_0	D_0	R_0	α
Jun-12	0.5249	0.0021	0.3940	1.3268	214149555	125950000	1.7003	-0.7808
Sep-12	1.0347	0.0017	0.3127	3.3030	290116712	126700000	2.2898	-1.9358
Dec-12	0.1144	0.0016	0.3720	0.3033	223568448	117050000	1.9100	-1.1397
Mar-13	0.6710	0.0014	0.5376	1.2456	362153886	129900000	2.7879	-1.4921
Jun-13	1.3344	0.0015	0.5069	2.6296	349193110	144100000	2.4233	-1.3059
Sep-13	-0.6840	0.0010	0.3325	-2.0604	284917506	141900000	2.0079	-1.4254
Dec-13	-0.5080	0.0013	0.2974	-1.7128	292977497	157550000	1.8596	-1.3356
Mar-14	-0.8271	0.0013	0.7350	-1.1270	203302448	139750000	1.4548	-0.2064
Jun-14	0.1555	0.0011	1.0470	0.1475	265400126	140600000	1.8876	-0.3937
Sep-14	0.7723	0.0013	0.6772	1.1384	272059468	140350000	1.9384	-0.6479
Dec-14	0.3089	0.0025	0.6440	0.4757	321884369	149700000	2.1502	-0.8423
Mar-15	-0.0881	0.0026	0.5476	-0.1657	269423743	154700000	1.7416	-0.6057
Jun-15	-0.8783	0.0028	0.3197	-2.7562	243535792	157850000	1.5428	-0.6583

TABLE 1. American Apparel Inc. parameters. A_0, D_0, R_0 are the values at the end of each indicated month. r_f is the 1-year treasury yield curve rate observed at the end of each. α is calculated by setting $R^* = 1.25$.

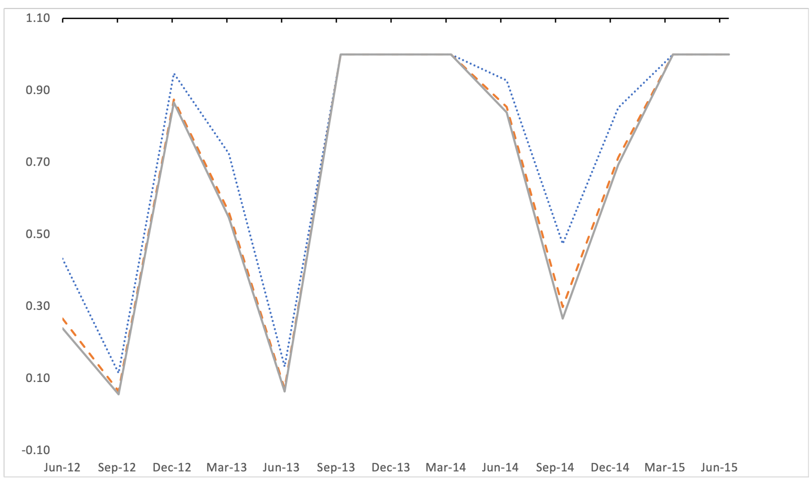


FIGURE 3. The graph displays $\mathbb{P}_{-\alpha}\left(g > 1 | \tau_0^- < \infty\right)$ (blue dotted line), $\mathbb{P}_{-\alpha}\left(l^\theta > 1 | \tau_0^- < \infty\right)$(grey solid line) and $\mathbb{P}_{-\alpha}\left(s^\theta > 1 | \tau_0^- < \infty\right)$ (orange dashed line) for process X defined in Eq. (81) with $\theta = 1$.

We further consider the conditional probabilities with varying threshold levels of R^* and they are presented in Table 2. We choose the end of December 2014 as the starting point, at which the initial level of the leverage process is $R_0 = 2.1502$ and the drift is $\mu = 0.4757$. From the results presented in Table 2, we observe that the conditional probabilities are relatively large overall. Although the current leverage ratio is high (at the level of 2.1502), there are more than 64% probability that the leverage process will take longer than a year to "ultimately" recover to the level of 2.2 (or higher levels). One may conclude from the increase in μ from 2013 to 2014 (see Table 1) that the firm had improved its credit quality, while the measures

R^*	α	$\mathbb{P}_{-\alpha}\left(g > 1 \vert \tau_0^- < \infty\right)$	$\mathbb{P}_{-\alpha}\left(s^1 > 1 \vert \tau_0^- < \infty\right)$	$\mathbb{P}_{-\alpha}\left(l^1 > 1 \vert \tau_0^- < \infty\right)$
2.8	0.41003	0.7513	0.6126	0.5357
2.7	0.35356	0.7362	0.5939	0.5185
2.6	0.29496	0.7202	0.5740	0.5008
2.5	0.2341	0.7030	0.5528	0.4825
2.4	0.1707	0.6849	0.5303	0.4636
2.3	0.1046	0.6655	0.5063	0.4442
2.2	0.0356	0.6450	0.4806	0.4243
2.1	-0.0367	0.6453	0.4771	0.4257
2	-0.1124	0.6678	0.4978	0.4499
1.9	-0.1921	0.6910	0.5203	0.4759
1.8	-0.2760	0.7149	0.5446	0.5039
1.7	-0.3648	0.7393	0.5709	0.5339
1.6	-0.4589	0.7641	0.5991	0.5659
1.5	-0.5592	0.7892	0.6294	0.6000

Table 2. American Apparel Inc. random times probabilities (up to 4 decimal points) for varying R^* by using the end of December 2014 as the starting point. Note that $R_0 = 2.1502$ in December 2014.

in Table 2 indicate that the firm's financial situation may not be as good as it seems, which provides another perspective to comprehensively assess risks.

7. Acknowledgements

The authors would like to thank two anonymous referees for helpful comments and suggestions. This paper is supported by the Society of Actuaries (SOA) Centers of Actuarial Excellence Grants Program. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the SOA. Support from grants from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by David Landriault and Bin Li (grant numbers 341316 and 04338, respectively). Support from the Canada Research Chair Program is gratefully acknowledged by David Landriault. Bin Li acknowledges the support from the National Natural Science Foundation of China (no. 12271171). Mohamed Amine Lkabous acknowledges the support from a start-up grant from the University of Southampton. Zijia Wang acknowledges the support from a start-up grant from The Chinese University of Hong Kong.

References

[1] J.-D. Bao and Y. Jia, *Determination of fission rate by mean last passage time*, Physical Review C **69** (2004), no. 2, 027602.

[2] E. J. Baurdoux, *Last exit before an exponential time for spectrally negative Lévy processes*, Journal of Applied Probability **46** (2009), no. 2, 542–558.

[3] E. J. Baurdoux and J. M. Pedraza, *Predicting the last zero of a spectrally negative Lévy process*, arXiv:1805.12190 [math.PR] (2018).

[4] E. J. Baurdoux and J.M. Pedraza, *Lp optimal prediction of the last zero of a spectrally negative Lévy process*, arXiv:2003.06869[math.PR] (2020).

[5] A. N. Borodin and P. Salminen, *Handbook of Brownian motion-facts and formulae*, Springer Science & Business Media, 2002.

[6] C. Cai and B. Li, *Occupation times of intervals until last passage times for spectrally negative Lévy processes*, Journal of Theoretical Probability **31** (2018), no. 4, 2194–2215.

[7] S. N. Chiu and C. Yin, *Passage times for a spectrally negative Lévy process with applications to risk theory*, Bernoulli **11** (200506), no. 3, 511–522.

[8] R. A. Doney, *Last exit times for random walks*, Stochastic processes and their applications **31** (1989), no. 2, 321–331.

[9] S. Drekic, *"On the joint distributions of the time to ruin, the surplus prior to ruin, and the deficit at ruin in the classical risk model," David Landriault and Gord Willmot, volume 13, no. 2, 2009*, North American Actuarial Journal **13** (2009), no. 3, 404–406.

[10] M. Egami and R. Kevkhishvili, *Time reversal and last passage time of diffusions with applications to credit risk management*, Finance and Stochastics **24** (2020), no. 3, 795–825.

[11] R. J. Elliott, M. Jeanblanc, and M. Yor, *On models of default risk*, Mathematical Finance **10** (2000), no. 2, 179–195.

[12] R. K. Getoor and M. J. Sharpe, *Last exit times and additive functionals*, The Annals of Probability **1** (1973), no. 4, 550–569.

[13] J. A. Given, C.-O. Hwang, and M. Mascagni, *First-and last-passage Monte Carlo algorithms for the charge density distribution on a conducting surface*, Physical Review E **66** (2002), no. 5, 056704.

[14] H. Guérin and J.-F. Renaud, *Joint distribution of a spectrally negative Lévy process and its occupation time, with step option pricing in view*, Advances in Applied Probability **48** (2016), no. 1, 274–297.

[15] H. Guérin and J.-F. Renaud, *On the distribution of cumulative Parisian ruin*, Insurance: Mathematics and Economics **73** (2017), 116–123.

[16] P. Imkeller, *Random times at which insiders can have free lunches*, Stochastics: An International Journal of Probability and Stochastic Processes **74** (2002), no. 1-2, 465–487.

[17] S. Janson, *Moments for first-passage and last-exit times, the minimum, and related quantities for random walks with positive drift*, Advances in applied probability **18** (1986), no. 4, 865–879.

[18] A. E. Kyprianou, *Fluctuations of Lévy processes with applications - Introductory lectures*, Second Edition, Universitext, Springer, Heidelberg, 2014.

[19] A. Lachal, *Last passage for the integral of Brownian motion*, Stochastic processes and their applications **49** (1994), no. 1, 57–64.

[20] D. Landriault, B. Li, and M. A. Lkabous, *On occupation times in the red of Lévy risk models*, Insurance: Mathematics and Economics **92** (2020), 17 –26.

[21] ———, *On the analysis of deep drawdowns for the Lévy insurance risk model*, Insurance: Mathematics and Economics **100** (2021), 147–155.

[22] D. Landriault, B. Li, T. Shi, and D. Xu, *On the distribution of classic and some exotic ruin times*, Insurance: Mathematics and Economics **89** (2019), 38–45.

[23] D. Landriault, J.-F. Renaud, and X. Zhou, *Occupation times of spectrally negative Lévy processes with applications*, Stochastic Processes and their Applications **121** (2011), no. 11, 2629–2641.

[24] Y. Li, C. Yin, and X. Zhou, *On the last exit times for spectrally negative Lévy processes*, Journal of Applied Probability **54** (2017), no. 2, 474–489.

[25] R. L. Loeffen, I. Czarna, and Z. Palmowski, *Parisian ruin probability for spectrally negative Lévy processes*, Bernoulli **19** (2013), no. 2, 599–609.

[26] R. L. Loeffen, Z. Palmowski, and B. A. Surya, *Discounted penalty function at parisian ruin for Lévy insurance risk process*, Insurance: Mathematics and Economics **83** (2018), 190–197.

[27] R. L. Loeffen, J.-F. Renaud, and X. Zhou, *Occupation times of intervals until first passage times for spectrally negative Lévy processes*, Stochastic Processes and their Applications **124** (2014), no. 3, 1408–1435.

[28] D. Madan, B. Roynette, and M. Yor, *Unifying Black–Scholes type formulae which involve Brownian last passage times up to a finite horizon*, Asia-Pacific Financial Markets **15** (2008), no. 2, 97–115.

[29] B. Maisonneuve, *Exit systems*, The Annals of Probability (1975), 399–411.

[30] D. Monrad, *Stable processes: sample function growth at a last exit time*, The Annals of Probability (1977), 455–462.

[31] C. Paroissin and L. Rabehasaina, *First and last passage times of spectrally positive Lévy processes with application to reliability*, Methodology and Computing in Applied Probability **17** (2015), no. 2, 351–372.

[32] A. D. Polyanin and A. V. Manzhirov, *Handbook of integral equations*, Chapman and Hall/CRC, 2008.

[33] P. Salminen, *On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary*, Advances in applied probability **20** (1988), no. 2, 411–426.
[34] K.-I. Sato and T. Watanabe, *Moments of last exit times for Lévy processes*, Annales de l'ihp probabilités et statistiques, 2004, pp. 207–225.
[35] B. A. Surya, *Parisian ruin and resolvent density of terminating spectrally negative Lévy process before first-passage above a level* (2016).
[36] R Syski, *On last-exit time distribution*, Advances in Applied Probability **8** (1976), no. 2, 246–247.

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada
Email address: david.landriault@uwaterloo.ca

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada
Email address: bin.li@uwaterloo.ca

School of Mathematical Sciences, University of Southamton, Highfield, Southampton, SO17 1BJ, UK
Email address: m.a.lkabous@soton.ac.uk

Department of Finance, The Chinese University of Hong Kong, Hong Kong
Email address: zijiawang@cuhk.edu.hk