A refracted Lévy process with delayed dividend pullbacks

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Abstract

The threshold dividend strategy, under which dividends are paid only when the insurer's surplus exceeds a pre-determined threshold, has received considerable attention in risk theory. However, in practice, it seems rather unlikely that an insurer will immediately pull back the dividend payments as soon as its surplus level drops below the dividend threshold. Hence, in this paper, we propose a refracted Lévy risk model with delayed dividend pullbacks triggered by a certain Poissonian observation scheme. Leveraging the extensive literature on fluctuation identities for spectrally negative Lévy processes, we obtain explicit expressions for two-sided exit identities of the proposed insurance risk process. Also, penalties are incorporated into the analysis of dividend payouts as a mechanism to penalize for the volatility of the dividend policy and account for an investor's typical preference for more stable cash flows. An explicit expression for the expected (discounted) dividend payouts net of penalties is derived. The criterion for the optimal threshold level that maximizes the expected dividend payouts is also discussed. Finally, several numerical examples are considered to assess the impact of dividend delays on ruin-related quantities. We numerically show that dividend strategies with more steady dividend payouts can be preferred (over the well-known threshold dividend strategy) when penalty fee become too onerous.

Keywords: Spectrally negative Lévy process; refraction; threshold dividend strategy; delayed dividend pullbacks

1 Introduction

In recent years, the refracted Lévy process has drawn considerable interest in the field of insurance mathematics (see, e.g., Kyprianou and Loeffen[16], Renaud [27], Czarna and Kaszubowski [9] and references therein). We recall that on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, the refracted Lévy process is a risk process $\widetilde{U} = \{\widetilde{U}_t\}_{t\geq 0}$ defined as

$$\widetilde{U}_t = X_t - \delta \int_0^t \mathbf{1}_{\{\widetilde{U}_s > b\}} \mathrm{d}s, \quad t \geq 0, \tag{1}$$

where $X = \{X_t\}_{t\geq 0}$ is a spectrally negative Lévy process, $\delta \geq 0$ is the refraction rate and $b \in \mathbb{R}$ is the refraction level. For the insurance risk process (1), various fluctuation identities related to classic and some exotic ruin times were obtained; see, e.g., Kyprianou and Loeffen [16] for the classic ruin time,

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Kyprianou et al. [17], Renaud [27] and Landriault et al. [18] for results on occupation times and Renaud [27] and Lkabous et al. [22] for Parisian ruin times. The reader is also invited to consult Pérez and Yamazaki [26] for the study of a joint refracted and reflected Lévy process, and Czarna et al. [11] on the topic of multi-refracted Lévy processes.

It is well known that the refracted Lévy process naturally arises in the context of the so-called threshold dividend strategy (see, e.g., Lin and Pavlova [21], Yang et al. [29], and Albrecher and Hartinger [3]). Indeed, under such a strategy, dividends are paid when the insurer's surplus exceeds a pre-determined critical level, and dividends stop as soon as the insurer's surplus drops below the critical level. In the formulation (1), it is understood that a dividend rate of δ is paid whenever the insurer's surplus process $\widetilde{U}$ exceeds level b, while no dividend is paid when the insurer's surplus lies below level b. However, in practice, it seems rather unlikely that an insurer will adopt a dividend strategy whose dividend pullbacks are as reactive as the one implied by the refracted Lévy process. Among other reasons, dividend pullbacks are generally regarded by the market as a signal of an entity's financial distress. More generally, investors are usually risk averse and tend to prefer stocks with steadier (i.e., less volatile) dividend payouts. As a possible remedy, the *ratcheting* strategy is proposed in the literature (see, e.g., Albrecher et al. [1, 2]), under which the dividend payment stream does not decrease over time. One problem with this strategy is that the insurer cannot make dividend cuts even in financial distress.

Hence, in this paper, we propose a refracted Lévy process with delayed dividend pullbacks. Heuristically speaking, this process will be such that dividend payouts will not immediately stop when the underlying risk process drops below the pre-determined dividend threshold $b > 0$. More specifically, dividend payouts will continue even if the insurer's surplus process drops below level b as long as such excursions (below b) are considered "short" (relative to a given pre-specified *grace period*). If the surplus does not revisit the threshold level b before the end of this *grace period*, dividend payouts stop and may only resume when the insurer's surplus creeps again above the threshold level b.

In light of recent contributions on Poissonian observations in the field of insurance mathematics (see, e.g., Landriault et al. [19] and Albrecher et al. [4]), we formally define the surplus process $U = \{U_t\}_{t\geq 0}$ of interest as

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{\left\{\{U_s \geq b\} \cup \left\{U_s < b, s-g_s < \mathrm{e}_\lambda^{g_s}, s \geq \kappa_b^+\right\}\right\}} \mathrm{d}s, \quad t \geq 0, \tag{2}$$

where

$$\kappa_a^{+(-)} = \inf\{t \geq 0 \colon U_t > (<) a\}, \quad a \in \mathbb{R},$$

and $g_t := \sup\{0 \leq s \leq t : U_s \geq b\}$, with the convention $\sup \emptyset = 0$. Also, let G be the set of left-end points of excursions below b, and for each $g \in G$ we consider an iid copy of a generic independent (of U) exponential random variable e_λ with mean $1/\lambda > 0$. Note that $U_s < b$ implies that $g_s \in G$. Hence, $\mathrm{e}_\lambda^{g_t}$ is the length of the *grace period* associated to the excursion of U below b that started at time g_t. Figure 1 displays a sample path of U to illustrate the dynamics of the surplus process with delayed dividend pullbacks.

For notational convenience, we define the binary dividend-paying process $Q^b = \{Q_t^b\}_{t\geq 0}$ as

$$Q_t^b = \begin{cases} 1, & \text{if } \{U_t \geq b\} \text{ or } \left\{U_t < b, t - g_t < \mathrm{e}_\lambda^{g_t}, \text{and } t \geq \kappa_b^+\right\}, \\ 0, & \text{otherwise,} \end{cases}$$

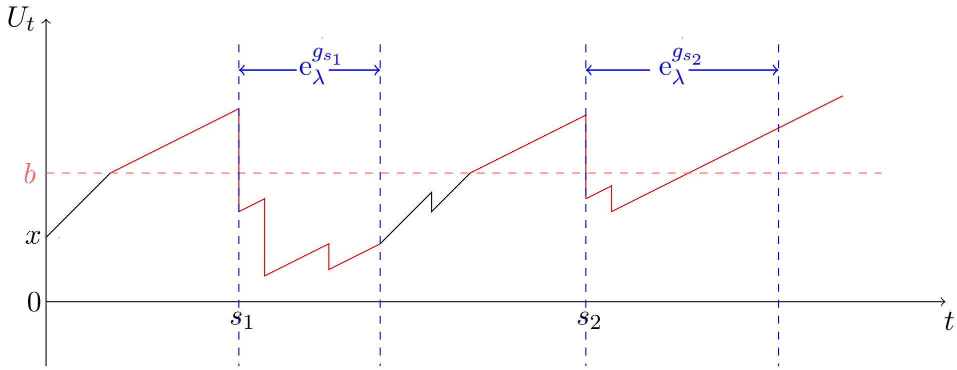


Figure 1: A sample path of the risk model U

which allows to rewrite (2) as

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{\{Q_s^b = 1\}} \mathrm{d}s, \quad t \geq 0.$$

It can be shown that the two-dimensional process $\{(U_t, Q_t^b)\}_{t\geq 0}$ is a strong Markov process, which will be heavily relied upon in the analysis of the refracted Lévy process (with delays) U. In what follows, we shall denote by

$$\mathbb{P}_x(\cdot) := \begin{cases} \mathbb{P}\left(\cdot | U_0 = x, Q_0^b = 0\right), & \text{if } x < b; \\ \mathbb{P}\left(\cdot | U_0 = x, Q_0^b = 1\right), & \text{if } x \geq b. \end{cases}$$

Moreover, $\mathbb{E}_x$ will be the expectation operator associated to $\mathbb{P}_x$. For simplicity, we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. We note that the Lévy risk model under a ratcheting dividend strategy considered in Albrecher et al. [2] is a special case of the risk model (2) where the delay rate $\lambda = 0$. Also, note that there are obvious parallels to be drawn between the refracted Lévy process (with delays) U and the work of Li et al. [20] on the non-refracted Lévy risk model with hybrid observation schemes.

In the literature of dividend payment strategies, some researchers also consider incorporating delays between the decision to pay dividends and its implementation, see, e.g., Dassios and Wu [13] and Czarna and Palmowski [10]. In contrast to pulling back dividends, the willingness and ability to pay dividends often send the market a positive message about the insurer's current performance and future prospects. To demonstrate its financial strength, the insurer may want to pay dividends in a prompt manner. As a generalization of the risk model U, it may be of practical interest to consider a refracted Lévy process with delays in both implementing dividend payments and pulling back dividends, where the delay rate of paying dividends is greater than the delay rate of dividend pullbacks λ. In this paper, to isolate the effects of delayed dividend pullbacks, we focus on the analysis of the relatively tractable risk model U.

A quantity that has drawn much interest in the context of the refracted Lévy process is the total discounted dividend payouts until ruin, whose expectation was derived by Kyprianou and Loeffen [16] for the refracted Lévy process $\widetilde{U}$. Research on the optimality of dividend strategies is also very relevant in this context. We refer the reader to the work by e.g., Loeffen and Renaud [24], Czarna and Palmowski [10] and Renaud [28] for more details. It is generally assumed that maximizing the expected discounted dividend payouts until ruin is the optimality criterion. However, it is well known that investors have their own set of preferences, and as a result, their objectives may not necessarily be in line with maximizing the expected discounted dividend payouts until ruin. For instance, it is usually the case that investors are risk averse and reward steadier dividend payouts or alternatively, penalize for changes in dividend payouts. This could be the case as investors may have to rebalance their portfolios when a change in dividend payouts occurs (for instance, to maintain a given percentage of their portfolio in dividend-paying assets). Hence, in this paper, we propose to incorporate penalties into the analysis of dividend payouts for the risk model U by assuming that a lump-sum penalty fee of ζ is applied whenever the dividend payout rate changes. The expected (discounted) dividend payouts net of penalties is defined as

$$D^b(x) := \delta \mathbb{E}_x \left[\int_0^{\kappa_0^-} \mathrm{e}^{-qt} \mathbf{1}_{\{Q_t^b=1\}} \mathrm{d}t \right] - \zeta \mathbb{E}_x \left[\sum_{0 \le t < \kappa_0^-} \mathrm{e}^{-qt} \mathbf{1}_{\{\Delta Q_t^b \neq 0\}} \right], \tag{3}$$

where $\Delta Q_t^b := Q_t^b - Q_{t-}^b$ and $q \geq 0$ can be interpreted as a force of interest. Note that $D^b(x)$ reduces to the expected discounted dividend payouts until ruin when $\zeta = 0$. We remark that the second term on the right-hand side of (3) is used to penalize for the volatility of the dividend policy. An explicit expression for $D^b(x)$ is given in Theorem 7. This is followed by a numerical study involving the quantity $D^b(x)$ in Section 3. More precisely, we identify a set of dividend strategies with identical ruin probability (for which the insurer is presumably indifferent), and pick the one maximizing $D^b(x)$ for an investor whose objective is consistent with this criterion. We observe that as the penalty fee ζ increases, a dividend strategy with less reactive dividend pullbacks (i.e., the dividend pullback rate λ decreases) is preferred. We note that a similar exercise can be performed for other choices of $D^b(x)$ such as the one related to the traditional mean-variance criterion in the field of mathematical finance and insurance risk management, see, e.g., Basak and Chabakauri [6], Björk et al. [8] and Dai et al. [12]. In theory, one could also incorporate transaction costs incurred on each dividend payment into the model (see, e.g., Avanzi et al. [5] and Loeffen [23]). For simplicity, we limit our analysis to $D^b(x)$ as defined in Eq. (3).

The rest of the paper is organized as follows. In the rest of Section 1, we recall some background on spectrally negative Lévy processes. Some useful fluctuation identities are postponed to Section B. Section 2 contains the main results of the paper, while Section 3 presents a few numerical examples. All technical proofs are postponed to the Appendix A.

1.1 Preliminaries

On the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let $X = \{X_t\}_{t\geq 0}$ be a spectrally negative Lévy process (SNLP), that is a process with stationary and independent increments and no positive jumps (we exclude the case where X has monotone paths). The SNLP X can be fully characterized via its Laplace exponent $\psi : [0, \infty) \to \mathbb{R}$ defined as

$$\psi(s) := \ln \mathbb{E} \left[\mathrm{e}^{sX_1} \right], \quad s \geq 0,$$

which admits the Lévy-Khintchine representation

$$\psi(s) = \gamma s + \frac{1}{2}\sigma^2 s^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-sz} - 1 + sz \mathbf{1}_{(0,1)}(z) \right) \Pi(\mathrm{d}z),$$

where $\gamma \in \mathbb{R}$ and $\sigma \geq 0$, and Π is a σ-finite measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} (1 \wedge z^2) \Pi(\mathrm{d}z) < \infty.$$

It is known that ψ is strictly convex with $\psi(0) = 0$ and $\psi(\infty) = \infty$ and is infinitely differentiable on $(0, \infty)$.

The measure Π is called the Lévy measure of X, while (γ, σ, Π) is referred to as the Lévy triplet of X. Note that for convenience we define the Lévy measure in such a way that it is a measure on the positive half line instead of the negative half line. Further, note that $\mathbb{E}[X_1] = \psi'(0+)$.

1.2 Scale functions

For an arbitrary SNLP, there exists a function $\Phi\colon [0,\infty) \to [0,\infty)$ defined by $\Phi(q) = \sup\{s \geq 0 \mid \psi(s) = q\}$ (its right-inverse) such that

$$\psi(\Phi(q)) = q, \quad q \geq 0.$$

We have that $\Phi(q) = 0$ if and only if $q = 0$ and $\psi'(0+) \geq 0$.

We now recall the definition of the q-scale function $W^{(q)}$. For $q \geq 0$, the q-scale function of the process X is defined as the continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-sy} W^{(q)}(y)\mathrm{d}y = \frac{1}{\psi(s) - q}, \quad \text{for } s > \Phi(q).$$

This function is unique, positive and strictly increasing for $x \geq 0$. We extend $W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. We write $W = W^{(0)}$ when $q = 0$. The initial value of $W^{(q)}$ is known to be

$$W^{(q)}(0) = \begin{cases} 1/c, & \text{when } \sigma = 0 \text{ and } \int_{(0,1)} z\Pi(\mathrm{d}z) < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

where $c := \gamma + \int_{(0,1)} z\Pi(\mathrm{d}z) > 0$ is the drift of X and we used the following definition: $W^{(q)}(0) = \lim_{x\downarrow 0} W^{(q)}(x)$. For all $q \geq 0$, $W^{(q)} \in C^1(0,\infty)$ if X is a process of unbounded variation, or if the Lévy measure Π is atomless when X is of bounded variation.

Also, fluctuation identities frequently rely on the scale function $Z^{(q)}$ defined as

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y, \quad x \in \mathbb{R},$$

for which a generalized form is given by

$$Z^{(q)}(x,\theta) = \mathrm{e}^{\theta x}\left(1 - (\psi(\theta) - q)\int_0^x \mathrm{e}^{-\theta y} W^{(q)}(y)\mathrm{d}y\right), \quad x \in \mathbb{R}, \tag{4}$$

for $\theta \geq 0$. It is immediate that $Z^{(q)}(x,0) = Z^{(q)}(x)$ and $Z^{(q)}(x,\theta) = \mathrm{e}^{\theta x}$ for $x \leq 0$.

For convenience, we also introduce a second SNLP $Y = \{Y_t\}_{t\geq 0}$ (independent of X) with Laplace exponent $\psi(s) - \delta s$, whose right inverse is denoted by $\varphi(\cdot)$. Let $\mathbb{W}^{(q)}$ (and $\mathbb{Z}^{(q)}$) be the counterpart of $W^{(q)}$ (and $Z^{(q)}$) for the SNLP Y. It is well known that

$$\lim_{x\to\infty} \frac{\mathbb{W}^{(q)}(x+y)}{\mathbb{W}^{(q)}(x)} = \mathrm{e}^{\varphi(q)y} \text{ and } \lim_{x\to\infty} \frac{\mathbb{Z}^{(q)}(x)}{\mathbb{W}^{(q)}(x)} = \frac{q}{\varphi(q)}. \tag{5}$$

It is well known that a SNLP takes the form of a strictly positive drift minus a pure jump subordinator if it has paths of bounded variation. To avoid the case where Y has monotone path, we assume that $0 \leq \delta < c$ if X has paths of bounded variation throughout the paper.

We also recall the *second generation* scale functions introduced in Loeffen et al. [25], that is, for $p, p+q \geq 0$ and $a, x \in \mathbb{R}$,

$$\begin{aligned}\overline{\mathbb{W}}_a^{(p,q)}(x) &:= \mathbb{W}^{(p+q)}(x) - q\int_0^a \mathbb{W}^{(p+q)}(x-y)\mathbb{W}^{(p)}(y)\mathrm{d}y \\ &= \mathbb{W}^{(p)}(x) + q\int_a^x \mathbb{W}^{(p+q)}(x-y)\mathbb{W}^{(p)}(y)\mathrm{d}y, \qquad (6)\\ \overline{\mathbb{Z}}_a^{(p,q)}(x) &:= \mathbb{Z}^{(p+q)}(x) - q\int_0^a \mathbb{W}^{(p+q)}(x-y)\mathbb{Z}^{(p)}(y)\mathrm{d}y \\ &= \mathbb{Z}^{(p)}(x) + q\int_a^x \mathbb{W}^{(p+q)}(x-y)\mathbb{Z}^{(p)}(y)\mathrm{d}y. \qquad (7)\end{aligned}$$

Note that the expressions on the right-hand side of Eq. (6) and that of Eq. (7) can be shown to be equivalent, respectively, by using the following identities taken from Loeffen et al. [25]: for $p, q \geq 0$ and $x \in \mathbb{R}$,

$$(p-q)\int_0^x \mathbb{W}^{(q)}(x-y)\mathbb{W}^{(p)}(y)\mathrm{d}y = \mathbb{W}^{(p)}(x) - \mathbb{W}^{(q)}(x),$$

and

$$(p-q)\int_0^x \mathbb{W}^{(q)}(x-y)\mathbb{Z}^{(p)}(y)\mathrm{d}y = \mathbb{Z}^{(p)}(x) - \mathbb{Z}^{(q)}(x).$$

Similarly, we denote the counterparts of $\overline{\mathbb{W}}_a^{(p,q)}(x)$ and $\overline{\mathbb{Z}}_a^{(p,q)}(x)$ for the SNLP X by $\overline{W}_a^{(p,q)}(x)$ and $\overline{Z}_a^{(p,q)}(x)$, respectively.

For any $a \in \mathbb{R}$, we define the following stopping times

$$\tau_a^{+(-)} = \inf\{t \geq 0\colon X_t > (<)a\} \quad \text{and} \quad \nu_a^{+(-)} = \inf\{t \geq 0\colon Y_t > (<)a\},$$

with the convention that $\inf \emptyset = \infty$. The two-sided exit identities for X, Y and the refracted SNLP $\widetilde{U}$ are well known; see Theorem 8.1 of Kyprianou [15] and Theorem 4 of Kyprianou and Loeffen [16]. For completeness, we recall these results in Appendix B.

An extensive body of literature has recently emerged on "delayed" first passage times in which some grace period is given to the process before the first passage time is triggered/recognized. A common example is the so-called Parisian ruin time (below a critical level b) defined as

$$\nu_b^\lambda = \inf\{t \geq 0 : Y_t < b, t - \tilde{g}_t > \mathrm{e}_\lambda^{\tilde{g}_t}\},$$

where $\tilde{g}_t = \sup\{s \leq t\colon Y_s \geq b\}$. In this context, each excursion below level b of Y is accompanied by an iid independent (of Y) exponentially distributed random variable $\mathrm{e}_\lambda^{\tilde{g}_t}$ with mean $1/\lambda > 0$. Identities related to the Parisian ruin time are also provided in Appendix B.

2 Main results

Of particular interest for the refracted Lévy process (with delays) U are the two-sided exit quantities

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right],$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right],$$

which will be derived in Theorem 2, and can be viewed as the counterparts of Theorem 4 in Kyprianou and Loeffen [16] for the refracted Lévy process $\widetilde{U}$. We later consider the expected (discounted) dividend payouts net of penalties defined in (3).

To better formulate the results, we define the following auxiliary functions: for $q, \lambda, x, b \geq 0$,

$$\xi_b^{(q,\lambda)}(x) = \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) - \left(W^{(q)}(x) + \lambda\int_0^b W^{(q)}(y)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y\right), \tag{8}$$

and

$$\alpha_b^{(q,\lambda)}(x) = \overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \left(Z^{(q)}(x) + \lambda\int_0^b Z^{(q)}(y)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y\right). \tag{9}$$

The next proposition provides alternative expressions for the above auxiliary functions.

Proposition 1. *For $q, \lambda, x, b \geq 0$,*

$$\xi_b^{(q,\lambda)}(x) = \delta\int_{[0,b)}\left(\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) - \mathbb{W}^{(q)}(x-y)\right)W^{(q)}(\mathrm{d}y) + \mathbb{W}^{(q)}(x) - W^{(q)}(x), \tag{10}$$

and

$$\alpha_b^{(q,\lambda)}(x) = \delta q\int_0^x W^{(q)}(y)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y. \tag{11}$$

We note that $W^{(q)}(\mathrm{d}y)$ is the measure (defined on $[0,\infty)$) associated with $W^{(q)}(a,b] := W^{(q)}(b) - W^{(q)}(a)$ for $-\infty < a \leq b < \infty$ (we refer readers to Chapter 8 of Kyprianou [15] for more details).

In Theorem 2, we provide the two-sided exit results for the refracted Lévy process (with delays) U.

Theorem 2. *For $q, \lambda \geq 0$ and $0 \leq x, b \leq a$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right] = \frac{\mathcal{U}_b^{(q,\lambda)}(x)}{\mathcal{U}_b^{(q,\lambda)}(a)}, \tag{12}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right] = \mathcal{V}_b^{(q,\lambda)}(x) - \frac{\mathcal{U}_b^{(q,\lambda)}(x)}{\mathcal{U}_b^{(q,\lambda)}(a)}\mathcal{V}_b^{(q,\lambda)}(a), \tag{13}$$

where

$$\mathcal{U}_b^{(q,\lambda)}(x) = W^{(q)}(x) + \mathbf{1}_{\{x\geq b\}}\left(\xi_b^{(q,\lambda)}(x) - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{W}^{(q+\lambda)}(b)}\xi_b^{(q,\lambda)}(b)\right), \tag{14}$$

and

$$\mathcal{V}_b^{(q,\lambda)}(x) = Z^{(q)}(x) + \mathbf{1}_{\{x\geq b\}} \left(\alpha_b^{(q,\lambda)}(x) - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{W}^{(q+\lambda)}(b)} \alpha_b^{(q,\lambda)}(b) \right). \tag{15}$$

Remark 3. For $\delta = 0$, we have $\mathbb{W}^{(q)}(x) = W^{(q)}(x)$ and $\overline{\mathbb{W}}_x^{(q,\lambda)}(y) = \overline{W}_x^{(q,\lambda)}(y)$, which in turn implies from Eqs. (10) and (11) that $\xi_b^{(q,\lambda)}(x) = \alpha_b^{(q,\lambda)}(x) = 0$. Hence, $\mathcal{U}_b^{(q,\lambda)}(x) = W^{(q)}(x)$ and $\mathcal{V}_b^{(q,\lambda)}(x) = Z^{(q)}(x)$ for all $x \geq 0$, and as expected, (12) and (13) reduce to the classical two-sided exit results (65) and (66), respectively.

Remark 4. We now compare the scale function $w_b^{(q)}(x)$ for the refracted Lévy process $\{\widetilde{U}_t\}_{t\geq 0}$ to its counterpart $\mathcal{U}_b^{(q,\lambda)}(x)$ for the process U by showing that $\mathcal{U}_b^{(q,\lambda)}(x) \geq w_b^{(q)}(x)$ for all $x \geq 0$. From Eq. (14) (together with Eqs. (10) and (74)) and the representation (71) for $w_b^{(q)}(x)$, it is not difficult to show that

$$\mathcal{U}_b^{(q,\lambda)}(x) - w_b^{(q)}(x) = \delta \int_0^b W^{(q)\prime}(y) \left\{ \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{W}^{(q+\lambda)}(b)} \mathbb{W}^{(q+\lambda)}(b-y) \right\} \mathrm{d}y, \tag{16}$$

for $x \geq b$, and $\mathcal{U}_b^{(q,\lambda)}(x) = w_b^{(q)}(x) = W^{(q)}(x)$ for $x \in [0, b)$. Using (69), one observes that

$$\frac{\mathbb{W}^{(q+\lambda)}(b-y)}{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)} = \mathbb{E}\left[\mathrm{e}^{-q\nu_{x-b}^+} \mathbf{1}_{\left\{\nu_{x-b}^+ < \nu_0^\lambda \wedge \nu_{-(b-y)}^-\right\}} \right],$$

for all $y \in [0, b)$, which implies that $\mathcal{U}_b^{(q,\lambda)}(x) \geq w_b^{(q)}(x)$ for all $x \geq 0$. It immediately follows that

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+} \mathbf{1}_{\left\{\kappa_a^+ < \kappa_0^-\right\}} \right] \leq \mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_a^+} \mathbf{1}_{\left\{\tilde{\kappa}_a^+ < \tilde{\kappa}_0^-\right\}} \right],$$

for $x \leq b$.

Corollary 5. *For $q, \lambda > 0$ and $x, b \geq 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-} \mathbf{1}_{\left\{\kappa_0^- < \infty\right\}} \right] = \mathcal{V}_b^{(q,\lambda)}(x) - \mathcal{L}_b^{(q,\lambda)} \mathcal{U}_b^{(q,\lambda)}(x), \tag{17}$$

where

$$\mathcal{L}_b^{(q,\lambda)} = \frac{\delta q \mathrm{e}^{\varphi(q)b} \int_b^\infty \mathrm{e}^{-\varphi(q)y} W^{(q)}(y)\mathrm{d}y + \frac{\delta q \mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}{\lambda \mathbb{W}^{(q+\lambda)}(b)} W^{(q)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q)) \left(1 - \frac{\xi_b^{(q,\lambda)}(b)}{\mathbb{W}^{(q+\lambda)}(b)}\right) - \lambda \int_0^b W^{(q)}(y) \mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))\mathrm{d}y} - \frac{\delta q}{\lambda}. \tag{18}$$

In addition, under the condition $0 \leq \delta < \mathbb{E}(X_1)$, letting $q \to 0$ one has the ruin probability

$$\mathbb{P}_x\left(\kappa_0^- < \infty\right) = 1 - \mathcal{L}_b^{(0,\lambda)} \mathcal{U}_b^{(0,\lambda)}(x), \tag{19}$$

where

$$\mathcal{L}_b^{(0,\lambda)} = \frac{\psi'(0+) - \delta}{1 + \delta\left(\mathbb{W}^{(\lambda)}(b) - W(b)\right) - \left(\delta + \frac{\mathbb{Z}^{(\lambda)}(b)}{\mathbb{W}^{(\lambda)}(b)}\right) \xi_b^{0,\lambda}(b)}. \tag{20}$$

Remark 6. It can be seen from (14) that $\mathcal{U}_b^{(0,0)}(x) = \frac{W(b)\mathbb{W}(x)}{\mathbb{W}(b)}$ for $x \geq b$ and $\mathcal{U}_b^{(0,0)}(x) = W(x)$ for $x < b$. As a result of Corollary 5, for $\lambda = 0$ (which corresponds to the ratcheting strategy),

$$\mathbb{P}_x\left(\kappa_0^- < \infty\right) = \begin{cases} 1 - \left(\psi'(0+) - \delta\right)\mathbb{W}(x), & x \geq b, \\ 1 - \frac{\psi'(0+)-\delta}{W(b)} W(x)\mathbb{W}(b), & x < b. \end{cases}$$

Note that the ruin probability can be written as $1 - \mathbb{P}_x(\tau_b^+ < \tau_0^-)\mathbb{P}_b(\nu_0^- = \infty)$ for $x < b$ by using the classical exit identities (65) and (67).

Now, we turn our attention to the expected discounted dividends net of penalties paid until ruin.

Theorem 7. *For $q > 0$ and $\lambda, b, x \geq 0$,*

$$D^b(x) = \frac{\mathcal{B}_{b,\zeta}^{(q,\lambda)}(b) - \zeta}{W^{(q)}(b) - \mathcal{A}_b^{(q,\lambda)}(b)} \mathcal{A}_b^{(q,\lambda)}(x) + \mathbf{1}_{\{x \geq b\}} \mathcal{B}_{b,\zeta}^{(q,\lambda)}(x), \tag{21}$$

where

$$\mathcal{A}_b^{(q,\lambda)}(x) = \begin{cases} W^{(q)}(x), & x < b, \\ \lambda \int_0^b W^{(q)}(y) \left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) \right) \mathrm{d}y, & x \geq b, \end{cases} \tag{22}$$

and

$$\begin{aligned} \mathcal{B}_{b,\zeta}^{(q,\lambda)}(x) = &\frac{\delta}{q} + (\zeta\lambda - \delta) \left\{ \frac{\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x)}{q+\lambda} - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\varphi(q)} + \frac{\lambda \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\mathbb{Z}^{(q+\lambda)}(b)}{\varphi(q)\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))(q+\lambda)} \right\} \\ &+ \frac{\lambda\left(\zeta + \frac{\delta}{q}\right)}{q+\lambda} \left(\frac{q\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\varphi(q)\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} - \mathbb{Z}^{(q)}(x-b) \right). \end{aligned} \tag{23}$$

Define by

$$V(b,x) := \delta \mathbb{E}_x \left[\int_0^{\kappa_0^-} \mathrm{e}^{-qt} \mathbf{1}_{\{Q_t^b = 1\}} \mathrm{d}t \right],$$

the expectation of discounted dividends under the delayed dividend pullbacks strategy. By letting $\zeta = 0$ in Eq. (21), one obtains

$$V(b,x) = \frac{\mathcal{B}_b^{(q,\lambda)}(b)}{W^{(q)}(b) - \mathcal{A}_b^{(q,\lambda)}(b)} \mathcal{A}_b^{(q,\lambda)}(x) + \mathbf{1}_{\{x \geq b\}} \mathcal{B}_b^{(q,\lambda)}(x), \tag{24}$$

for $b, x \geq 0$, where $\mathcal{B}_b^{(q,\lambda)}(x) := \mathcal{B}_{b,0}^{(q,\lambda)}(x)$.

Remark 8. A small adaptation of Eq. (24) (by letting $\delta = c_2 > 0$ and $\lambda = 0$) leads to the following expression for the expected discounted dividends paid until ruin under the ratcheting strategy,

$$\mathbb{E}_x \left[\int_0^{\kappa_0^-} \mathrm{e}^{-qt} \left(c_1 + c_2 \mathbf{1}_{\{t \geq \kappa_b^+\}} \right) \mathrm{d}t \right]$$

$$
\begin{aligned}
=&\frac{c_1}{q}\left(1-\mathcal{V}_b^{(q,0)}(x)+\mathcal{L}_b^{(q,0)}\mathcal{U}_b^{(q,0)}(x)\right)+c_2\left(\frac{\mathcal{B}_b^{(q,0)}(b)\mathcal{A}_b^{(q,0)}(x)}{W^{(q)}(b)-\mathcal{A}_b^{(q,0)}(b)}+\mathbf{1}_{\{x\geq b\}}\mathcal{B}_b^{(q,0)}(x)\right)\\
=&\begin{cases}\frac{c_1+c_2}{q}\left(1-\mathbb{Z}^{(q)}(x)+\frac{q}{\varphi(q)}\mathbb{W}^{(q)}(x)\right), & 0\leq b\leq x,\\ \frac{c_1}{q}\left\{1-Z^{(q)}(x)+\frac{W^{(q)}(x)}{W^{(q)}(b)}\left(\frac{q\mathbb{W}^{(q)}(b)}{\varphi(q)}-\mathbb{Z}^{(q)}(b)+Z^{(q)}(b)\right)\right\} & \\ +\frac{c_2W^{(q)}(x)}{qW^{(q)}(b)}\left\{1-\mathbb{Z}^{(q)}(b)+\frac{q\mathbb{W}^{(q)}(b)}{\varphi(q)}\right\}, & 0\leq x<b,\end{cases}
\end{aligned}
$$

for $q,x,b,c_1\geq 0$, which recovers Theorem 2.1 of Albrecher et al. [2].

To identify the threshold level b under which $V(b,x)$ is maximized, a natural criterion is to find a solution to the following equation:

$$\frac{\partial V(b,x)}{\partial b}=0,\quad b\geq 0. \tag{25}$$

As discussed in Albrecher et al. [2], the necessity and sufficiency of criterion (25) depends on the analytical properties of $V(b,x)$. In particular, (25) is a necessary condition if the optimal threshold b is strictly positive. The following two propositions are generalizations of Proposition 2.2 and Theorem 2.3 of Albrecher et al. [2].

Proposition 9. *For fixed $\lambda\geq 0$ and $\delta,q>0$, a threshold level $b^*>0$ that maximizes $V(b,x)$ does not depend on x and satisfies the equation*

$$\frac{\mathrm{d}}{\mathrm{d}b}\left\{\frac{\mathcal{B}_b^{(q,\lambda)}(b)}{W^{(q)}(b)-\mathcal{A}_b^{(q,\lambda)}(b)}\right\}\bigg|_{b=b^*}=0. \tag{26}$$

By letting $\lambda=0$, Proposition 9 recovers Proposition 2.2 (with $c_1=0$) of Albrecher et al. [2].

One observes from Eq. (24) that $V(b,x)$ is continuous at $x=b$. However, the continuity of the derivative of $V(b,x)$ with respect to x at $x=b$ may not be preserved. The following proposition provides another way to characterize the optimal threshold.

Proposition 10. *Assume that $W^{(q)}(x)$ is continuously differentiable on $(0,\infty)$ and its derivative is denoted by $W^{(q)\prime}(x)$. For fixed $\delta,\lambda\geq 0$ and $q>0$, a threshold $b^*>0$ that maximizes $V(b,x)$ is exactly the one which makes $V(b,x)$ continuously differentiable at $x=b^*$.*

In the next section, a numerical study of the impact of delayed dividend pullbacks on ruin probabilities and the expected (discounted) dividends net of penalties is performed. Among other things, it is shown that the strategy to delay the dividend pullbacks is preferred under certain model settings.

3 Numerical examples

3.1 Brownian risk processes

In this section, we consider the refracted Lévy process (with delays) U where $X=\{X_t\}_{t\geq 0}$ is a drifted Brownian motion with Laplace exponent $\psi(s)=cs+\frac{1}{2}s^2(s\geq 0)$, and the scale functions are

$$W^{(q)}(x)=\frac{1}{\Phi(q)+c}\left(\mathrm{e}^{\Phi(q)x}-\mathrm{e}^{-(\Phi(q)+2c)x}\right),\quad x\geq 0,$$

and

$$Z^{(q)}(x) = \frac{q}{\Phi(q)+c}\left(\frac{\mathrm{e}^{\Phi(q)x}}{\Phi(q)} + \frac{\mathrm{e}^{-(\Phi(q)+2c)x}}{\Phi(q)+2c}\right), \quad x \geq 0,$$

with $\Phi(q) = \sqrt{c^2+2q} - c$ for $q \geq 0$. The scale functions $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ are the same as the ones for $W^{(q)}$ and $Z^{(q)}$, respectively, with c replaced by $c-\delta$.

Example 1. We conduct a numerical study of the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ (as given in Eq. (19)) under the following parameter setting: $b = c = 1$ and $\delta = 0.5$. In Figure 2, we plot the ruin probability for the refracted Lévy process (with delays) U with different delay rates λ ($\lambda = 0.1, 10, 50, \infty$). One expects that $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ reduces to the ruin probability for the process $\widetilde{U}$ (which is provided in Theorem 5 of Kyprianou and Loeffen [16]) when $\lambda \to \infty$ as this case corresponds to the continuous refraction at b. As a basis of comparison, we also provide the values of the classical ruin probabilities for both processes X and Y.

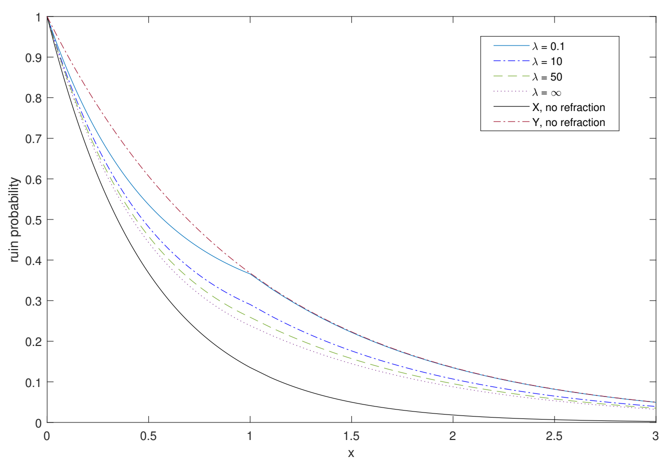
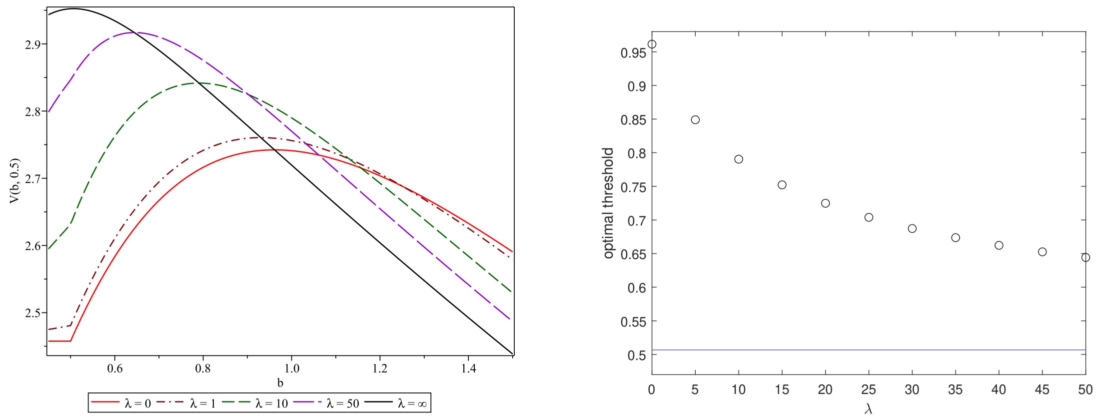


Figure 2: Ruin probabilities with different delay rates

From Figure 2, the following observations are worthy of mention:

- as expected, for a given initial surplus x, the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ decreases as the rate of dividend pullbacks λ increases;
- the classical ruin probabilities for processes X and Y serve as lower and upper bounds, respectively, for the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$;
- the rate of decrease in ruin probability is not smooth at the threshold level b (given that a dividend payout is triggered precisely at $x = b$).

Example 2. We now consider the case where $c = 4, \delta = 3, q = 0.9$ and $x = 0.5$. Figure 3(a) plots the expected discounted dividends (as a function of b) for different values of λ. One observes from Figure 3(a) that $V(b, x)$ is strictly concave for $\lambda > 0$, which implies that $b^* > 0$ characterized by Proposition 9 or Proposition 10 is the optimal threshold level. Figure 3(b) plots the optimal threshold level for each fixed delay rate λ, while the blue horizontal line represents the optimal threshold level for the refracted case (which can be calculated by using Eq. (4.2) of Gerber and Shiu [14]). We observe that the optimal threshold level decreases as λ increases.



(a) $V(b, 0.5)$ with different λ

(b) optimal thresholds with different λ

Figure 3: $V(b, 0.5)$ and optimal threshold levels with different delay rates

3.2 Crámer-Lundberg risk processes

We now consider the refracted Lévy process (with delays) U where $X = \{X_t\}_{t\geq 0}$ is a Cramér-Lundberg risk process with exponentially distributed claims, namely

$$X_t = X_0 + ct - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\eta > 0$ and $\{C_i\}_{i\in\mathbb{N}^+}$ is an iid sequence of exponential rv's with mean $1/\alpha$, independent of N. In what follows, we assume that $c > \delta + \eta/\alpha$ so that the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ is not trivially 1. In this case, the Laplace exponent of X is given by $\psi(s) = cs - \eta s/(s+\alpha)$ for $s \geq 0$. Moreover, for $q > 0$ and $x \geq 0$, the scale functions $W^{(q)}$ and $Z^{(q)}$ are

$$W^{(q)}(x) = \frac{(\alpha + \Phi(q))\mathrm{e}^{\Phi(q)x} - (\alpha + \theta_q)\,\mathrm{e}^{\theta_q x}}{\sqrt{(q+\eta-c\alpha)^2 + 4c\alpha q}},$$

and

$$Z^{(q)}(x) = \frac{q\left(\frac{\alpha+\Phi(q)}{\Phi(q)}\mathrm{e}^{\Phi(q)x} - \frac{\alpha+\theta_q}{\theta_q}\mathrm{e}^{\theta_q x}\right)}{\sqrt{(q+\eta-c\alpha)^2 + 4c\alpha q}},$$

where $\Phi(q) = \left(q + \eta - c\alpha + \sqrt{(q+\eta-c\alpha)^2 + 4c\alpha q}\right)/(2c)$ and $\theta_q = (q+\eta-c\alpha)/c - \Phi(q)$. Here again, the scale functions $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ are identically defined as $W^{(q)}$ and $Z^{(q)}$ respectively, but with c replaced by $c - \delta$.

Example 3. To investigate the impact of the dividend delay rate λ on the expected (discounted) dividend payouts net of penalties, namely $D^b(x)$ defined in (3), we consider the refracted Lévy process (with delays)

U under the following parameter setting: $x = 2, b = 5, c = 4, \eta = 2, \alpha = 1$ and $q = 0.1$. Then, we identify different pairs (δ, λ) of dividend strategies whose corresponding ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ (as given in Eq. (19)) are identical (where $\lambda < \infty$). We notice that for a given ruin probability, a dividend strategy with more reactive dividend pullbacks (i.e., a higher dividend pullback rate λ) must be accompanied by a higher dividend rate δ. In other words, all else being equal, a decrease in the dividend pullback rate λ comes at the expense of a decrease in the dividend rate δ. We remark that dividend strategies with smaller λ (or δ) values are less volatile.

In Tables 1 and 2, we provide the values of $D^b(x)$ for the process U with different choices of ζ under the constraint that the ruin probability is 0.22 and 0.32, respectively.

δ	λ	$D^b(x)$			
		$\zeta = 0$	$\zeta = 1$	$\zeta = 1.2$	$\zeta = 1.5$
0.6890	0	4.9026	4.1422	3.9901	**3.7620**
0.80	1.0234	5.5926	4.2947	4.0351	3.6458
0.85	1.8362	5.8996	4.3550	**4.0461**	3.5827
0.90	3.1658	6.2039	4.3973	4.0360	3.4940
0.95	5.7322	6.5050	**4.4097**	3.9906	3.3621
1.00	12.7057	6.8023	4.3777	3.8928	3.1654
1.0595	∞	**7.1493**	4.2369	3.6544	2.7807

Table 1: Impact of λ on $D^b(x)$ when the ruin probability is 0.22.

δ	λ	$D^b(x)$			
		$\zeta = 0$	$\zeta = 0.5$	$\zeta = 0.7$	$\zeta = 1$
1.2952	0	8.6144	8.2336	8.0812	**7.8527**
1.40	0.5837	9.1430	8.4549	8.1796	7.7667
1.45	1.0556	9.3927	8.5375	**8.1954**	7.6823
1.50	1.8172	9.6405	8.5953	8.1771	7.5500
1.55	3.2724	9.8860	**8.6143**	8.1056	7.3425
1.60	7.1910	10.1276	8.5694	7.9461	7.0112
1.6596	∞	**10.4053**	8.3471	7.5237	6.2888

Table 2: Impact of λ on $D^b(x)$ when the ruin probability is 0.32.

From Tables 1 and 2, we observe that:

- In the absence of penalty fee (i.e., $\zeta = 0$), $D^b(x)$ under the threshold dividend strategy ($\lambda \to \infty$) is the largest in both Tables 1 and 2.
- As the penalty fee ζ increases, dividend strategies with smoother dividend payouts (smaller λ/δ) have greater dividend payouts net of penalties. For instance, in Table 2, the dividend strategy with $(\delta = 1.55, \lambda = 3.2724)$ leads the largest $D^b(x)$ when $\zeta = 0.5$, and the strategies with $(\delta = 1.45, \lambda = 1.0556)$ and $(\delta = 1.2952, \lambda = 0)$ are better (in terms of maximizing $D^b(x)$) than other considered strategies when $\zeta = 0.7$ and $\zeta = 1$, respectively. The same observation holds for Table 1.

In conclusion, we note that there is a trade-off between paying dividends at a higher rate (i.e., higher δ) and being able to pay dividends more steadily (i.e., lower λ) under the consideration of penalties. As the penalty fee ζ increases, the above results confirm our intuition that dividend strategies with more steady dividend payouts would be preferred by investors.

A Proofs of the main results

In this appendix, we provide proofs for the main results.

A.1 Proof of Proposition 1

Proof. From (74), it is easy to show that (10) holds for $x < b$. For $x \geq b$, using (6), one deduces that

$$\begin{aligned}
\xi_b^{(q,\lambda)}(x) &= \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) - W^{(q)}(x) \\
&\quad -\lambda \int_0^b W^{(q)}(y) \left[\mathbb{W}^{(q)}(x-y) + \lambda \int_{x-b}^{x-y} \mathbb{W}^{(q)}(z)\mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}z \right] \mathrm{d}y \\
&= \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) - W^{(q)}(x) - \lambda \int_{x-b}^{x} W^{(q)}(x-y)\mathbb{W}^{(q)}(y)\mathrm{d}y \\
&\quad - \lambda^2 \int_{x-b}^{x} \mathbb{W}^{(q)}(z) \left\{ \int_0^{x-z} W^{(q)}(y)\mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}y \right\} \mathrm{d}z. \qquad (27)
\end{aligned}$$

Applying (74) again, the last term of (27) becomes

$$\begin{aligned}
&\lambda^2 \int_{x-b}^{x} \mathbb{W}^{(q)}(z) \left\{ \int_0^{x-z} W^{(q)}(y)\mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}y \right\} \mathrm{d}z \\
=&\lambda \int_{x-b}^{x} \mathbb{W}^{(q)}(z) \left[-W^{(q)}(x-z) + \mathbb{W}^{(q+\lambda)}(x-z) - \delta \int_{[0,x-z)} \mathbb{W}^{(q+\lambda)}(x-z-y)W^{(q)}(\mathrm{d}y) \right] \mathrm{d}z. \qquad (28)
\end{aligned}$$

Substituting (28) into (27) followed by simple algebraic manipulations completes the proof of (10).

We are left with the proof of Eq. (11). Using (75), one can show that (11) clearly holds for $x < b$. For $x \geq b$, by (6) and (75), it follows that

$$\begin{aligned}
\alpha_b^{(q,\lambda)}(x) =&\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - Z^{(q)}(x) - \lambda \int_{x-b}^{x} Z^{(q)}(x-z)\mathbb{W}^{(q)}(z)\mathrm{d}z \\
&-\lambda^2 \int_{x-b}^{x} \int_0^{x-z} Z^{(q)}(y)\mathbb{W}^{(q)}(z)\mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}y\mathrm{d}z \\
&=\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - Z^{(q)}(x) - \lambda \int_{x-b}^{x} Z^{(q)}(x-z)\mathbb{W}^{(q)}(z)\mathrm{d}z \\
&-\lambda \int_{x-b}^{x} \mathbb{W}^{(q)}(z) \left[-\delta q \int_0^{x-z} \mathbb{W}^{(q+\lambda)}(x-y-z)W^{(q)}(y)\mathrm{d}y + \mathbb{Z}^{(q+\lambda)}(x-z) - Z^{(q)}(x-z) \right] \mathrm{d}z \\
&=\mathbb{Z}^{(q)}(x) - Z^{(q)}(x) + \delta q\lambda \int_0^b \int_{x-b}^{x-y} \mathbb{W}^{(q)}(z)\mathbb{W}^{(q+\lambda)}(x-y-z)W^{(q)}(y)\mathrm{d}z\mathrm{d}y \\
&=\mathbb{Z}^{(q)}(x) - Z^{(q)}(x) + \delta q \int_0^b \left\{ \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) - \mathbb{W}^{(q)}(x-y) \right\} W^{(q)}(y)\mathrm{d}y. \qquad (29)
\end{aligned}$$

Combining (29) and (75) completes the proof of (11). ∎

A.2 Proof of Theorem 2

A.2.1 Proof of (12)

Proof. (i) For $0 \leq x < b$, by the strong Markov property of (U, Q^b) and the skip-free upward dynamic of U, it follows that

$$\begin{aligned}
\mathbb{E}_x[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}] =&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_b^+}\mathbf{1}_{\{\kappa_b^+<\kappa_0^-\}}\right]\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right]\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\\
=&\frac{W^{(q)}(x)}{W^{(q)}(b)}\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right],
\end{aligned} \tag{30}$$

where the second equality follows from the fact that $\{X_t, t < \tau_b^+\}$ and $\{U_t, t < \kappa_b^+\}$ have the same distribution with respect to $\mathbb{P}_x$ when $x < b$.

(ii) Let

$$\kappa_b^\lambda = \inf\{t > 0 : t - g_t > \mathrm{e}_\lambda^{g_t}\},\ b \in \mathbb{R}.$$

For $b \leq x < a$, we first note that $\{Y_t, t < \nu_b^\lambda\}$ and $\{U_t, t < \kappa_b^\lambda\}$ have the same distribution with respect to $\mathbb{P}_x$. Once again, by the strong Markov property of (U, Q^b) and the skip-free upward dynamic of U, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\wedge\kappa_b^\lambda\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_b^\lambda<\kappa_a^+<\kappa_0^-\}}\right]\\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbf{1}_{\{\nu_a^+<\nu_0^-\wedge\nu_b^\lambda\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbb{E}_{Y_{\nu_b^\lambda}}\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\mathbf{1}_{\{\nu_b^\lambda<\nu_0^-\wedge\nu_a^+\}}\right].
\end{aligned} \tag{31}$$

Substituting (30) into (31) together with the help of Eqs. (68) and (69), it follows that

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\\
=&\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}+\frac{\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]}{W^{(q)}(b)}\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}W^{(q)}(Y_{\nu_b^\lambda})\mathbf{1}_{\{\nu_b^\lambda<\nu_0^-\wedge\nu_a^+\}}\right]\\
=&\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}+\frac{\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]}{W^{(q)}(b)}\left(\xi_b^{(q,\lambda)}(x)+W^{(q)}(x)-\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\left(\xi_b^{(q,\lambda)}(a)+W^{(q)}(a)\right)\right).
\end{aligned} \tag{32}$$

Given the expression of $\mathcal{U}_b^{(q,\lambda)}(x)$ as shown in Eq. (14), Eq. (32) can be rewritten as

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]$$

$$=\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}+\frac{\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]}{W^{(q)}(b)\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\left(\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)\mathcal{U}_b^{(q,\lambda)}(x)-\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\mathcal{U}_b^{(q,\lambda)}(a)\right), \tag{33}$$

for $b \leq x < a$. In particular, by letting $x = b$ in Eq. (33), we obtain

$$\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]=\frac{W^{(q)}(b)}{\mathcal{U}_b^{(q,\lambda)}(a)}. \tag{34}$$

Substituting (34) into (30) and (33) completes the proof of (12). ■

A.2.2 Proof of (13)

Proof. We proceed similarly as for the proof of (12). Given the similarity, some details will be omitted here.

(i) For $0 \leq x < b \leq a$, one can deduce that

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]\\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_b^+<\kappa_a^+\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_b^+<\kappa_0^-<\kappa_a^+\}}\right]\\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right]\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]\\
=&Z^{(q)}(x)-\frac{Z^{(q)}(b)-\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]}{W^{(q)}(b)}W^{(q)}(x).
\end{aligned} \tag{35}$$

(ii) For $0 \leq b \leq x < a$, by conditioning on κ_b^λ and making use of the fact that $\{Y_t, t<\nu_b^\lambda\}$ and $\{U_t, t<\kappa_b^\lambda\}$ have the same distribution with respect to $\mathbb{P}_x$, it follows that

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]\\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_b^\lambda<\kappa_0^-<\kappa_a^+\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\wedge\kappa_b^\lambda\}}\right]\\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbb{E}_{Y_{\kappa_b^\lambda}}\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]\mathbf{1}_{\{\nu_b^\lambda<\nu_0^-\wedge\nu_a^+\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbf{1}_{\{\nu_0^-<\nu_a^+\wedge\nu_b^\lambda\}}\right].
\end{aligned} \tag{36}$$

Substituting (35), (68) and (70) into (36) yields

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]\\
=&\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\left(\lambda\int_0^b Z^{(q)}(y)\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a-y)\mathrm{d}y\right)-\lambda\int_0^b Z^{(q)}(y)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y\\
&-\frac{Z^{(q)}(b)-\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]}{W^{(q)}(b)}\left(\mathcal{U}_b^{(q,\lambda)}(x)-\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\mathcal{U}_b^{(q,\lambda)}(a)\right)\\
&+\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x)-\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(a).
\end{aligned} \tag{37}$$

In particular, for $x=b$, substituting (15) and (11) into (37) yields

$$\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\left\{\kappa_0^-<\kappa_a^+\right\}}\right]=Z^{(q)}(b)-\frac{W^{(q)}(b)}{\mathcal{U}_b^{(q,\lambda)}(a)}\mathcal{V}_b^{(q,\lambda)}(a)\,. \tag{38}$$

Substituting (38) into (35) and (37) and together with (11) completes the proof (13). ■

A.3 Proof of Corollary 5

Proof. To prove Eq. (17), one shall identify

$$\lim_{a\to\infty}\frac{\mathcal{V}_b^{(q,\lambda)}(a)}{\mathcal{U}_b^{(q,\lambda)}(a)}, \tag{39}$$

which we will do by looking at the asymptotic behaviour of $\mathcal{U}_b^{(q,\lambda)}(a)$ and $\mathcal{V}_b^{(q,\lambda)}(a)$ in comparison to $\mathbb{W}^{(q)}(a)$.

By the dominated convergence theorem, we first point out that

$$\lim_{a\to\infty}\frac{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}=1+\lambda\int_0^b\mathrm{e}^{-\varphi(q)y}\mathbb{W}^{(q+\lambda)}(y)\mathrm{d}y=\mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right). \tag{40}$$

It follows directly from (40) that

$$\lim_{a\to\infty}\frac{\alpha_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}=\delta q\mathrm{e}^{-\varphi(q)b}\int_0^\infty W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y,$$

and

$$\lim_{a\to\infty}\frac{\xi_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}=\mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right)-\lambda\mathrm{e}^{-\varphi(q)b}\int_0^b W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y.$$

Therefore,

$$\lim_{a\to\infty}\frac{\mathcal{V}_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}=\delta q\mathrm{e}^{-\varphi(q)b}\int_0^\infty W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y-\frac{\mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right)}{\mathbb{W}^{(q+\lambda)}(b)}\alpha_b^{(q,\lambda)}(b), \tag{41}$$

and

$$\begin{aligned}&\lim_{a\to\infty}\frac{\mathcal{U}_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}\\ =&\mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right)\left(1-\frac{\xi_b^{(q,\lambda)}(b)}{\mathbb{W}^{(q+\lambda)}(b)}\right)-\lambda\mathrm{e}^{-\varphi(q)b}\int_0^b W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y.\end{aligned} \tag{42}$$

By simple algebraic manipulations, one can show that the limit (39) is given by Eq. (18).

Moreover, taking the limit as $q \to 0$ in (18) and noting that $\varphi(0) = 0$ and $\lim_{q\to 0} \frac{q}{\varphi(q)} = \psi'(0+) - \delta > 0$ under the security loading condition, it follows that

$$\lim_{q\to 0}\lim_{a\to\infty} \frac{\mathcal{V}_b^{(q,\lambda)}(a)}{\mathcal{U}_b^{(q,\lambda)}(a)} = \lim_{q\to 0} \frac{\delta q \mathrm{e}^{\varphi(q)b} \int_b^\infty \mathrm{e}^{-\varphi(q)y} W^{(q)}(y)\mathrm{d}y}{\mathbb{Z}^{(\lambda)}(b)\left(1 - \frac{\xi_b^{(0,\lambda)}(b)}{\mathbb{W}^{(\lambda)}(b)}\right) - \lambda \int_0^b W(y)\mathbb{Z}^{(\lambda)}(b-y)\mathrm{d}y}$$
$$= \frac{\psi'(0+) - \delta}{\mathbb{Z}^{(\lambda)}(b)\left(1 - \frac{\xi_b^{(0,\lambda)}(b)}{\mathbb{W}^{(\lambda)}(b)}\right) - \lambda \int_0^b W(y)\mathbb{Z}^{(\lambda)}(b-y)\mathrm{d}y}. \tag{43}$$

Note that

$$\mathbb{Z}^{(\lambda)}(b) = 1 + \lambda \int_0^b W(y)\left(\delta \mathbb{W}^{(\lambda)}(b-y) + \mathbb{Z}^{(\lambda)}(b-y)\right)\mathrm{d}y,$$

which can be proved by showing that the Laplace transforms on both sides are equal. Thus, we have

$$\lambda \int_0^b W(y)\mathbb{Z}^{(\lambda)}(b-y)\mathrm{d}y = \mathbb{Z}^{(\lambda)}(b) - 1 + \delta\left(W(b) - \mathbb{W}^{(\lambda)}(b)\right) + \delta \xi_b^{(0,\lambda)}(b). \tag{44}$$

Substituting (44) into (43) and noting that $\mathcal{V}_b^{(0,\lambda)}(x) = 1$ completes the proof of (19). ■

A.4 Proof of Theorem 7

Proof. (1) For $0 \le x < b$, since no dividends is payable until the process reaches level b, it follows that

$$D^b(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\kappa_b^+}\mathbf{1}_{\{\kappa_b^+ < \kappa_0^-\}}\right]\left(D^b(b) - \zeta\right)$$
$$= \frac{W^{(q)}(x)}{W^{(q)}(b)}\left(D^b(b) - \zeta\right). \tag{45}$$

(2) For $x \ge b$, dividends are continuously paid until $(\kappa_0^- \wedge \kappa_b^\lambda)$. If the stopping time κ_b^λ occurs first, the dynamics of U change from the one of process Y to process X and a penalty fee ζ is incurred. By the strong Markov property of (U, Q^b) and Eq. (45), it follows that

$$D^b(x) = \delta \mathbb{E}_x\left[\int_0^{\kappa_0^- \wedge \kappa_b^\lambda} \mathrm{e}^{-qt}\mathrm{d}t\right] - \zeta \mathbb{E}_x\left[\mathrm{e}^{-q\kappa_b^\lambda}\mathbf{1}_{\{\kappa_b^\lambda < \kappa_0^-\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-q\kappa_b^\lambda} D^b(U_{\kappa_b^\lambda})\mathbf{1}_{\{\kappa_b^\lambda < \kappa_0^-\}}\right]$$
$$= \frac{\delta}{q}\left(1 - \mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbf{1}_{\{\nu_0^- < \nu_b^\lambda\}}\right]\right) - \left(\zeta + \frac{\delta}{q}\right)\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbf{1}_{\{\nu_b^\lambda < \nu_0^-\}}\right]$$
$$+ \mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda} W^{(q)}\left(Y_{\nu_b^\lambda}\right)\mathbf{1}_{\{\nu_b^\lambda < \nu_0^-\}}\right]\frac{D^b(b) - \zeta}{W^{(q)}(b)}. \tag{46}$$

Taking limits as $a \to \infty$ in (70) and using Eq. (42) in Loeffen et al. [25], it follows that

$$\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbf{1}_{\{\nu_0^- < \nu_b^\lambda\}}\right] = \overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\frac{\frac{q}{\varphi(q)} + \lambda \int_0^b \mathrm{e}^{-\varphi(q)y}\mathbb{Z}^{(q+\lambda)}(y)\mathrm{d}y}{1 + \lambda \int_0^b \mathrm{e}^{-\varphi(q)y}\mathbb{W}^{(q+\lambda)}(y)\mathrm{d}y}$$

$$= \overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\left(\frac{q+\lambda}{\varphi(q)} - \frac{\lambda \mathbb{Z}^{(q+\lambda)}(b)}{\varphi(q)\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right). \tag{47}$$

By Corollary 1.1 in Baurdoux et al. [7], we obtain

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbf{1}_{\{\nu_b^\lambda<\nu_0^-\}}\right]\\ =&\lambda\int_0^b\left\{\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\mathbb{Z}^{(q+\lambda)}(y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-b+y)\right\}\mathrm{d}y.\end{aligned} \tag{48}$$

Using (4) and (6), one can show that

$$\int_0^b \mathbb{Z}^{(q+\lambda)}(y,\varphi(q))\mathrm{d}y = \frac{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q)) - \frac{\lambda}{q+\lambda}\mathbb{Z}^{(q+\lambda)}(b) - \frac{q}{q+\lambda}}{\varphi(q)}, \tag{49}$$

and

$$\begin{aligned}\int_0^b \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-b+y)\mathrm{d}y =&\frac{\mathbb{Z}^{(q+\lambda)}(x) - \mathbb{Z}^{(q)}(x-b) - \lambda\int_0^{x-b}\mathbb{W}^{(q)}(z)\mathbb{Z}^{(q+\lambda)}(x-z)\mathrm{d}z}{q+\lambda}\\ =&\frac{\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \mathbb{Z}^{(q)}(x-b)}{q+\lambda}.\end{aligned} \tag{50}$$

Substituting (49) and (50) into (48) yields

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbf{1}_{\{\nu_b^\lambda<\nu_0^-\}}\right]\\ =&\lambda\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\varphi(q)}\left(1 - \frac{\lambda\mathbb{Z}^{(q+\lambda)}(b)+q}{(q+\lambda)\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right) - \frac{\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \mathbb{Z}^{(q)}(x-b)}{q+\lambda}\right).\end{aligned} \tag{51}$$

Also, applying (68) and then taking limits as $a\to\infty$ leads to

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}W^{(q)}(Y_{\nu_b^\lambda})\mathbf{1}_{\{\nu_b^\lambda<\nu_0^-\}}\right]\\ =&\frac{\lambda\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\int_0^b W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))\mathrm{d}y}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} + \xi_b^{(q,\lambda)}(x) - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) + W^{(q)}(x).\end{aligned} \tag{52}$$

Substituting (47), (51) and (52) into (45) and (46) completes the proof of (21). ■

A.5 Proof of Proposition 9

Proof. For the case $0\le x<b$, it is straightforward to see from Eq. (22) that Eq. (26) must hold to satisfy the criterion (25). For $x\ge b>0$, by simple algebraic computations, one obtains the following identities:

$$\frac{\partial}{\partial b}\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(y) = \lambda\mathbb{W}^{(q+\lambda)}(y-x+b)\mathbb{W}^{(q)}(x-b), \quad y\ge x-b, \tag{53}$$

$$\frac{\partial}{\partial b}\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(y) = \lambda\mathbb{W}^{(q)}(y-b)\mathbb{Z}^{(q+\lambda)}(b), \quad y\ge b, \tag{54}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))=\varphi(q)\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))+\lambda\mathbb{W}^{(q+\lambda)}(b). \tag{55}$$

For $x\geq b>0$, using Eqs. (22)∼(23) and (53)∼(55), it follows that

$$\begin{aligned}
&\frac{\partial}{\partial b}\mathcal{A}_b^{(q,\lambda)}(x)\\
=&\frac{\partial}{\partial b}\left\{\lambda\int_0^b W^{(q)}(y)\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}-\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\right)\mathrm{d}y\right\}\\
=&\lambda W^{(q)}(b)\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}-\mathbb{W}^{(q)}(x-b)\right)+\lambda\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\int_0^b W^{(q)}(y)\frac{\partial}{\partial b}\left\{\frac{\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right\}\mathrm{d}y\\
&+\lambda^2\mathbb{W}^{(q)}(x-b)\int_0^b W^{(q)}(y)\left(\frac{\mathbb{W}^{(q+\lambda)}(b)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}-\mathbb{W}^{(q+\lambda)}(b-y)\right)\mathrm{d}y\\
=&\lambda W^{(q)}(b)\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}-\mathbb{W}^{(q)}(x-b)\right)-\frac{\lambda\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\mathcal{A}_b^{(q,\lambda)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}+\lambda\mathbb{W}^{(q)}(x-b)\mathcal{A}_b^{(q,\lambda)}(b)\\
=&\lambda\left(W^{(q)}(b)-\mathcal{A}_b^{(q,\lambda)}(b)\right)\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}-\mathbb{W}^{(q)}(x-b)\right),
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial b}\mathcal{B}_b^{(q,\lambda)}(x)\\
=&\delta\lambda\mathbb{W}^{(q)}(x-b)\left(\frac{1-\mathbb{Z}^{(q+\lambda)}(b)}{q+\lambda}+\frac{\mathbb{W}^{(q+\lambda)}(b)}{\varphi(q)}\right)+\frac{\delta\lambda}{(q+\lambda)\varphi(q)}\frac{\partial}{\partial b}\left\{\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\left(1-\mathbb{Z}^{(q+\lambda)}(b)\right)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right\}\\
=&\delta\lambda\mathbb{W}^{(q)}(x-b)\left(\frac{1-\mathbb{Z}^{(q+\lambda)}(b)}{q+\lambda}+\frac{\mathbb{W}^{(q+\lambda)}(b)}{\varphi(q)}\right)-\frac{\delta\lambda}{(q+\lambda)\varphi(q)}\left(\frac{\left(1-\mathbb{Z}^{(q+\lambda)}(b)\right)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\lambda\mathbb{W}^{(q+\lambda)}(b)}{\left(\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))\right)^2}\right.\\
&\left.+\frac{\delta\lambda}{(q+\lambda)\varphi(q)}\left(\frac{\left(\lambda\mathbb{W}^{(q+\lambda)}(b)\mathbb{W}^{(q)}(x-b)-\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\varphi(q)\right)\left(1-\mathbb{Z}^{(q+\lambda)}(b)\right)-(q+\lambda)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\mathbb{W}^{(q+\lambda)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right)\right.\\
=&\lambda\mathcal{B}_b^{(q,\lambda)}(b)\left(\mathbb{W}^{(q)}(x-b)-\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right).
\end{aligned} \tag{57}$$

By Eqs. (56) and (57), it follows that

$$\frac{\mathcal{B}_b^{(q,\lambda)}(b)}{W^{(q)}(b)-\mathcal{A}_b^{(q,\lambda)}(b)}\frac{\partial}{\partial b}\mathcal{A}_b^{(q,\lambda)}(x)+\frac{\partial}{\partial b}\mathcal{B}_b^{(q,\lambda)}(x)=0, \tag{58}$$

for $x\geq b>0$. Combining Eq. (58) with the fact that $\mathcal{A}_b^{(q,\lambda)}(x)>0$ for $x\geq b>0$, one concludes Eqs. (25) and (26) are equivalent for $x\geq b>0$. This completes the proof. ■

A.6 Proof of Proposition 10

Proof. For $x \geq b > 0$, taking partial derivative with respect to x and evaluating at $x = b$, one obtains

$$\frac{\partial}{\partial x}\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\big|_{x=b} = \mathbb{W}^{(q+\lambda)\prime}(b-y) - \lambda\mathbb{W}^{(q+\lambda)}(b-y)\mathbb{W}^{(q)}(0), \quad y \leq b, \tag{59}$$

For $x \geq b > 0$, applying Eqs. (22), (55) and (59), one derives that

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathcal{A}_b^{(q,\lambda)}(x)\big|_{x=b} \\
=&\lambda\int_0^b W^{(q)}(y)\Bigg(\frac{\left(\mathbb{W}^{(q+\lambda)\prime}(b) - \lambda\mathbb{W}^{(q+\lambda)}(b)\mathbb{W}^{(q)}(0)\right)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} \\
&\qquad - \left(\mathbb{W}^{(q+\lambda)\prime}(b-y) - \lambda\mathbb{W}^{(q+\lambda)}(b-y)\mathbb{W}^{(q)}(0)\right)\Bigg)\mathrm{d}y \\
=&\lambda\int_0^b W^{(q)}(y)\left(\frac{\mathbb{W}^{(q+\lambda)\prime}(b)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} - \mathbb{W}^{(q+\lambda)\prime}(b-y)\right)\mathrm{d}y - \lambda\mathbb{W}^{(q)}(0)\mathcal{A}_b^{(q,\lambda)}(b),
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}b}\mathcal{A}_b^{(q,\lambda)}(b) =&\lambda\int_0^b W^{(q)}(y)\left(\frac{\mathbb{W}^{(q+\lambda)\prime}(b)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} - \mathbb{W}^{(q+\lambda)\prime}(b-y)\right)\mathrm{d}y \\
&+\frac{\lambda\mathbb{W}^{(q+\lambda)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\left(W^{(q)}(b) - \mathcal{A}_b^{(q,\lambda)}(b)\right) - \lambda W^{(q)}(b)\mathbb{W}^{(q+\lambda)}(0).
\end{aligned} \tag{61}$$

It follows from Eqs. (60) and (61) that

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathcal{A}_b^{(q,\lambda)}(x)\big|_{x=b} - \frac{\mathrm{d}}{\mathrm{d}b}\mathcal{A}_b^{(q,\lambda)}(b) \\
=&\lambda W^{(q)}(b)\mathbb{W}^{(q+\lambda)}(0) - \lambda\mathbb{W}^{(q)}(0)\mathcal{A}_b^{(q,\lambda)}(b) - \frac{\lambda\mathbb{W}^{(q+\lambda)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\left(W^{(q)}(b) - \mathcal{A}_b^{(q,\lambda)}(b)\right) \\
=&\lambda\left(\mathbb{W}^{(q)}(0) - \frac{\mathbb{W}^{(q+\lambda)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right)\left(W^{(q)}(b) - \mathcal{A}_b^{(q,\lambda)}(b)\right).
\end{aligned} \tag{62}$$

Similarly, it is straightforward (and the details are omitted here) to show that

$$\frac{\partial}{\partial x}\mathcal{B}_b^{(q,\lambda)}(x)\big|_{x=b} - \frac{\mathrm{d}}{\mathrm{d}b}\mathcal{B}_b^{(q,\lambda)}(b) = \lambda\mathcal{B}_b^{(q,\lambda)}(b)\left(\frac{\mathbb{W}^{(q+\lambda)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} - \mathbb{W}^{(q)}(0)\right). \tag{63}$$

If $b^* > 0$ is a threshold level such that $V(b, x)$ is continuously differentiable at $x = b^*$, we have

$$\frac{\mathcal{B}_{b^*}^{(q,\lambda)}(b^*)}{W^{(q)}(b^*) - \mathcal{A}_{b^*}^{(q,\lambda)}(b^*)}W^{(q)\prime}(b^*) = \frac{\mathcal{B}_{b^*}^{(q,\lambda)}(b^*)}{W^{(q)}(b^*) - \mathcal{A}_{b^*}^{(q,\lambda)}(b^*)}\frac{\partial}{\partial x}\mathcal{A}_{b^*}^{(q,\lambda)}(x)\big|_{x=b^*} + \frac{\partial}{\partial x}\mathcal{B}_{b^*}^{(q,\lambda)}(x)\big|_{x=b^*}. \tag{64}$$

Substituting Eqs. (62) and (63) into Eq. (64), it is immediate that

$$\frac{\mathrm{d}}{\mathrm{d}b}\mathcal{B}_b^{(q,\lambda)}(b)\big|_{b=b^*} = \frac{\mathcal{B}_{b^*}^{(q,\lambda)}(b^*)}{W^{(q)}(b^*) - \mathcal{A}_{b^*}^{(q,\lambda)}(b^*)}\left(W^{(q)\prime}(b^*) - \frac{\mathrm{d}}{\mathrm{d}b}\mathcal{A}_b^{(q,\lambda)}(b)\big|_{b=b^*}\right),$$

which is equivalent to Eq. (26). ■

B Fluctuation identities

B.1 Lévy risk processes

For $q \geq 0$ and $x \leq a$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{65}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(a)}Z^{(q)}(a). \tag{66}$$

In particular, the *classical* probability of ruin is given by

$$\mathbb{P}_x\left(\tau_0^- < \infty\right) = 1 - \mathbb{E}\left[X_1\right]W(x), \tag{67}$$

if $\mathbb{E}\left[X_1\right] \geq 0$.

For the SNLP Y, the same results also hold by substituting the scale functions of X by those of Y. For instance,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbf{1}_{\{\nu_a^+<\nu_0^-\}}\right] = \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}.$$

We recall the following useful identities taken from Baurdoux et al. [7] and Landriault et al. [19]. For $q \geq 0, 0 < b < a, x \in [0,a), y \in [0,b)$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbf{1}_{\left\{Y_{\nu_b^\lambda}\in\mathrm{d}y,\nu_b^\lambda<\nu_a^+\wedge\nu_0^-\right\}}\right] = \lambda\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a-y) - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\right)\mathrm{d}y, \tag{68}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbf{1}_{\{\nu_a^+<\nu_b^\lambda\wedge\nu_0^-\}}\right] = \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}. \tag{69}$$

Moreover, using Theorem 1 in Loeffen et al. [25], one can show that for $0 \leq b, x \leq a$

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-q\nu_0^-}\mathbf{1}_{\{\nu_0^-<\nu_a^+\wedge\nu_b^\lambda\}}] \\
=&\mathbb{E}_x[\mathrm{e}^{-q\nu_0^-}\mathrm{e}^{-\lambda\int_0^{\nu_0^-}\mathbf{1}_{\{0<Y_t<b\}}\mathrm{d}t}\mathbf{1}_{\{\nu_0^-<\nu_a^+\}}] \\
=&\mathbb{Z}^{(q+\lambda)}(x) - \lambda\int_b^x \mathbb{W}^{(q)}(x-z)\mathbb{Z}^{(q+\lambda)}(z)\mathrm{d}z \\
-&\frac{\mathbb{Z}^{(q+\lambda)}(a) - \lambda\int_b^a \mathbb{W}^{(q)}(a-z)\mathbb{Z}^{(q+\lambda)}(z)\mathrm{d}z}{\mathbb{W}^{(q+\lambda)}(a) - \lambda\int_b^a \mathbb{W}^{(q)}(a-z)\mathbb{W}^{(q+\lambda)}(z)\mathrm{d}z}\left(\mathbb{W}^{(q+\lambda)}(x) - \lambda\int_b^x \mathbb{W}^{(q)}(x-z)\mathbb{W}^{(q+\lambda)}(z)\mathrm{d}z\right) \\
=&\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(a).
\end{aligned} \tag{70}$$

Note that by (6) we have $\overline{\mathbb{W}}_b^{(q+\lambda,-\lambda)}(x) = \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)$.

B.2 Refracted Lévy processes

For the refracted Lévy process $\{\widetilde{U}_t\}_{t\geq 0}$ defined in (1), we define its first passage times by

$$\tilde{\kappa}_a^{+(-)} = \inf\{t \geq 0 \colon \widetilde{U}_t > (<) a\}, a \in \mathbb{R}.$$

For $q \geq 0$ and $0 \leq x, b \leq a$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_a^+}\mathbf{1}_{\{\tilde{\kappa}_a^+<\tilde{\kappa}_0^-\}}\right] = \frac{w_b^{(q)}(x)}{w_b^{(q)}(a)},$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_0^-}\mathbf{1}_{\{\tilde{\kappa}_0^-<\tilde{\kappa}_a^+\}}\right] = z_b^{(q)}(x) - \frac{w_b^{(q)}(x)}{w_b^{(q)}(a)} z_b^{(q)}(a),$$

where

$$w_b^{(q)}(x) = W^{(q)}(x) + \delta \mathbf{1}_{\{x\geq b\}} \int_b^x \mathbb{W}^{(q)}(x-y) W^{(q)\prime}(y)\mathrm{d}y, \tag{71}$$

and

$$z_b^{(q)}(x) = Z^{(q)}(x) + \delta q \mathbf{1}_{\{x\geq b\}} \int_b^x \mathbb{W}^{(q)}(x-y) W^{(q)}(y)\mathrm{d}y, \tag{72}$$

can be regarded as the scale functions of the refracted process $\widetilde{U}$. For a refracted Lévy process, the probability of *classical* ruin is

$$\mathbb{P}_x\left(\tilde{\kappa}_0^- < \infty\right) = 1 - \left(\frac{\mathbb{E}\left[X_1\right] - \delta}{1 - \delta W(b)}\right) w_b^{(0)}(x),$$

if $0 < \delta < \mathbb{E}\left[X_1\right]$. A thorough derivation and discussion can be found in Kyprianou and Loeffen [16].

Also, for $p, q, x \geq 0$, we have the following useful identity taken from Renaud [27]

$$\begin{aligned}&\delta \int_0^x \mathbb{W}^{(p)}(x-y) W^{(q)}(y)\mathrm{d}y + (p-q) \int_0^x \int_0^y \mathbb{W}^{(p)}(y-z) W^{(q)}(z)\mathrm{d}z\mathrm{d}y \\ =& \int_0^x \mathbb{W}^{(p)}(y)\mathrm{d}y - \int_0^x W^{(q)}(y)\mathrm{d}y.\end{aligned} \tag{73}$$

Differentiating (73) with respect to x yields

$$\begin{aligned}&(q-p)\int_0^x \mathbb{W}^{(p)}(x-y) W^{(q)}(y)\mathrm{d}y \\ =& W^{(q)}(x) - \mathbb{W}^{(p)}(x) + \delta \int_{[0,x)} \mathbb{W}^{(p)}(x-y) W^{(q)}(\mathrm{d}y).\end{aligned} \tag{74}$$

Moreover, rearranging (73), one can show that

$$\int_0^x \mathbb{W}^{(p)}(x-y)\left(\delta W^{(q)}(y) - (q-p)\frac{Z^{(q)}(y)-1}{q}\right)\mathrm{d}y = \frac{\mathbb{Z}^{(p)}(x)-1}{p} - \frac{Z^{(q)}(x)-1}{q},$$

and thus

$$(q-p)\int_0^x \mathbb{W}^{(p)}(x-y) Z^{(q)}(y)\mathrm{d}y = \delta q \int_0^x \mathbb{W}^{(p)}(x-y) W^{(q)}(y)\mathrm{d}y - \mathbb{Z}^{(p)}(x) + Z^{(q)}(x). \tag{75}$$

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