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Second-order gravitational self-force in a highly regular gauge

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Abstract

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Gravitational-wave emission from extreme-mass-ratio inspirals (EMRIs) is expected to be a key source for the Laser Interferometer Space Antenna (LISA), a future space-based gravitational-wave detector. In this thesis, we detail an approach to model these systems through a perturbative method known as gravitational self-force theory. Accurate EMRI science requires us to go to second order in perturbation theory, which introduces a number of obstacles. One major problem that we focus on ameliorating in this thesis is the strong divergence encountered on the worldline of the small object. This divergence creates a severe computational cost in numerical simulations and hinders the rapid calculations that are required for waveform generation for LISA. However, building on previous work by Pound [Phys. Rev. D 95, 104056 (2017)], we develop a class of "highly regular" gauges with a weaker singularity structure. We calculate all orders of the metric perturbations required for numerical implementation and generate fully covariant and generic coordinate-expansion expressions for the metric perturbations in this class of gauges. Not only will the weaker divergences enable quicker numerical calculations, they also allow us to rigorously derive a pointlike second-order stress-energy tensor for the small object. We demonstrate that the form of this second-order stress-energy tensor is valid in any smoothly related gauge and, using a specific distributional definition, also valid in a widely used gauge in self-force calculations, the Lorenz gauge. This stress-energy tensor can then be used as part of the source when solving for the full, physical fields at second order and we outline how this can be done through the introduction of a counter term that cancels the most singular part of the second-order source in the Lorenz gauge. Finally, we present the calculation of the gauge vector required to transform from the Lorenz gauge to the highly regular gauge and provide it in mode-decomposed form for the case of quasicircular orbits in Schwarzschild spacetime. While this work is motivated by EMRIs, much of the work in this thesis is valid for a small object in any vacuum background spacetime with an external lengthscale much larger than the size of the small object.

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Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:
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Signed:.. Date:..................

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To Irene, Islwyn, Muriel, Ronald, and Christine

Definitions and Abbreviations

$\mathsf{g}_{\mu\nu}$	Metric tensor of the full spacetime
$g_{\mu\nu}$	Metric tensor of the background spacetime
$\tilde{g}_{\mu\nu}$	Effective metric constructed from $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$
δ_{ab}	Three-dimensional Euclidean metric
$h_{\mu\nu}$	Perturbations to the background spacetime
$h^{n}_{\mu\nu}$	nth order perturbation of the background spacetime
	In Ch. 3.5 onwards: perturbation in the highly regular gauge
$h^{n'}_{\mu\nu}$	Perturbation in the lightcone rest gauge
$h^{n*}_{\mu\nu}$	Perturbation in the Lorenz gauge
$\tilde{A}^{\mu_1\cdots}_{\nu_1\cdots}$	Tensor defined with respect to $\tilde{g}_{\mu\nu}$
${}^{n}A^{\mu_1\cdots}_{\nu_1\cdots}$	Prescript counting powers of ϵ after expanding acceleration, $a^\mu = \sum_{n\geq 1}\epsilon^n f^\mu_n$
$\mathring{A}^{\mu_1\cdots}_{\nu_1\cdots}$	Recollection of terms after acceleration expansion, i.e. $\mathring{h}^{n}_{\mu\nu} = \sum_{i=0}^{n} {}^{i}h^{n-i}_{\mu\nu}$
∇ or ;	Covariant derivative with respect to $g_{\mu\nu}$
L	Multi-indices, i.e. $L \coloneqq i_1 \dots i_l$
(L) or Sym_L	Symmetrisation over L
$[L]$	Antisymmetrisation over L
$\langle L\rangle$ or STF_L	Symmetric trace-free (STF) combination over L with respect to δ_{ab}
$\hat{T}^L$	STF tensor, i.e. $T^{\langle L\rangle}$
	In Ch. 2.2.1 and Ch. 6:
$A^{\mu\cdots}_{\nu\cdots}$	Tensor evaluated at x^μ, a point off the worldline
$A^{\mu'\cdots}_{\nu'\cdots}$	Tensor evaluated at $x^{\mu'}$, a point on the worldline
$A^{\bar{\mu}\cdots}_{\bar{\nu}\cdots}$	Tensor evaluated at $x^{\bar{\mu}}$ which is connected to x^μ by a geodesic intersecting the worldline orthogonally

Chapter 1

Introduction

Gravitational waves are ripples in the fundamental fabric of spacetime and were first predicted by Einstein in 1916 [1, 2] (translated in [3, 4]). These propagate at the speed of light, a fact which has only recently been experimentally confirmed [5] with the advent of the field of *gravitational-wave astronomy*. Gravitational waves are emitted from sources that feature accelerations of objects with non-symmetric properties. Astrophysical systems exhibiting this feature include supernovae collapse, rotating neutron stars or, primarily of interest to us, neutron star/black hole binaries [6, 7].

One particular class of binary sources is that of *extreme-mass-ratio inspirals* (EMRIs) [8]. In an EMRI, a black hole or neutron star of hundreds or thousands of solar masses slowly inspirals into a supermassive black hole over the course of a year, continuously emitting gravitational waves. The primary motivation for the work in this thesis is the modelling of these EMRI systems.

However, much of the work is more general than just being applicable to EMRIs. Many of the results in this thesis are valid for an object in a vacuum spacetime whose size is much smaller than some external lengthscale, e.g. the orbital separation between two bodies. In the case of an EMRI, this lengthscale is the large mass of the central black hole. The EMRI problem connects back to fundamental questions in general relativity, such as how extended, self-gravitating bodies move through spacetime and whether we can idealise them as point particles. In this thesis, we aim to tackle foundational problems in order to reduce the challenge of modelling these systems.

1.1 Extreme-mass-ratio inspirals

In an EMRI, an object of mass $m \sim 1$–$10^2 M_\odot$ slowly spirals into an object of mass $M \sim 10^5$–$10^7 M_\odot$. The smaller object is a compact object, such as a black hole or neutron star, whereas the larger object is a supermassive black hole, believed to exist in the centre of most galaxies [9–11]. In EMRI modelling, the geometry of the central black hole is taken to be described by the Kerr metric [12]. Some examples of EMRI formation processes are described in Refs. [6, Sec. 4.2, 13, Ch. 3.2], but the main formation channel is believed to be direct capture of a nearby compact object by the supermassive black hole. In this process, if a compact object travels sufficiently close to the supermassive black hole, then its orbit can become bound to it.

To the lowest approximation, the trajectory of the bound small object is described by the geodesic of a test particle in the Kerr spacetime. These geodesics involve complicated and intricate motion and are described by three constants of motion (up to initial conditions): the orbital energy, E, the azimuthal angular momentum, L_z, and the Carter constant, Q [14]. The motion is *ergodic* (space-filling) in an approximately torus shaped region surrounding the large black hole where the axis of revolution of the torus coincides with the spin axis of the black hole and is triperiodic over all three spatial dimensions. It should be noted that this description of the motion is no longer true when the ratio between two of the frequencies is a rational number. One instead encounters *resonant* motion where the trajectory of the small object becomes periodic and traces out the two-dimensional surface of a self-intersecting cylinder [15]. While geodesics in the Kerr spacetime are complex, they can be described in analytical form [16].

However, this approximation is not sufficient for tracking the motion of the small object [17]. The extended nature and motion of the smaller body interacts with the gravitational field of the supermassive black hole. This alters the trajectory of the smaller body and moves it away from geodesic motion by exerting a *self-force*, so-called as it is caused by the presence of the small object. It is this force that drives the inspiral by acting on the smaller object and dissipating energy and angular momentum from the system. These quantities are then radiated away as gravitational waves. The result of this process is that the smaller object slowly inspirals into the larger one. As the mass ratio, $\epsilon := m/M$, is very small, the inspiral occurs over a long timescale, with the smaller object expected to complete $\epsilon^{-1} \sim 10^5$ intricate orbits before plunging into the central black hole [18, 19].

Due to the large number of orbits occurring near to the supermassive black hole, the gravitational waves emitted are expected to provide an excellent picture of the geometry of the black hole in the strong-gravity regime [8, 18]. The result is that we may test predictions from general relativity, such as the 'no hair' theorem. This theorem states that a stationary, uncharged and isolated black hole can be completely described using only two parameters: its mass and its spin [20, Ch. 12.3, 21, Ch. 33.2]. These two parameters then dictate all the higher multipole moments of the black hole. A deviation from this would indicate that the spacetime geometry of these black holes is not described by the Kerr metric [6]. More possible tests are given in Refs. [6, Sec. 6, 19, Ch. III, 22, 23], and references therein, and include (but are not limited to) whether the larger object is in fact a black hole or is some other exotic object, such as a boson star, and tests of other gravity theories beyond general relativity.

1.2 Detection of gravitational waves

In 2015, the Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors provided the first direct measurement of gravitational waves from the merger of two black holes [24]. Since the original discovery, there have been an additional 89 candidates detected by the LIGO and Virgo detectors [25–28]. All but one of the detections have been of systems with the individual objects having masses in the range of order $1\,M_\odot$ to $10^1\,M_\odot$.[1] Although the majority of detections have involved binary systems containing objects of comparable sizes, a handful of detections [28–31] have had unequal mass ratios, even approaching $\sim 1:27$ in the case of GW191219_163120 [28].

While the frequency band covered by LIGO and Virgo is 10^1 Hz to 10^3 Hz [7], gravitational waves from EMRI systems are emitted at a frequency of order 10^{-3} Hz [7], outside of this band. Earth based detectors are limited at low frequencies by noise coming from seismic events and, as such, cannot detect gravitational wave signals emitted from EMRIs. The space-based Laser Interferometer Space Antenna (LISA) [32, 33], scheduled to launch in the mid-2030's [34], has been designed to cover the frequency band needed to detect EMRIs. The expected number of detections is very uncertain and has a wide range, but it is expected at least a few will be detected throughout the mission lifetime, with a potential upper bound in the thousands [8].

EMRIs will be a key source for LISA, but they are not the only family of sources that are hoped to be detected.[2] The two main others are massive black hole binaries and galactic binaries [6, 13]. Massive black hole binaries feature the collision of two black holes of order $10^4\,M_\odot$ to $10^7\,M_\odot$ with mass ratios ranging from equal up to $\sim 1:100$. Galactic binary sources are events in our own galaxy featuring two stellar-mass compact objects. LISA data will also aid ground-based detectors by providing information on upcoming equal-mass binary black hole collisions, such as the estimated time of merger and approximate location in the sky, long before they enter the LIGO/Virgo frequency band [37]. Figure 1.1 illustrates the respective sensitivity bands for LISA, LIGO, and Virgo compared with emission frequencies of different sources. Data from LISA will also be used to explore cosmological questions such as testing the expansion and acceleration of the universe, the existence of cosmic strings, and the nature of the early universe, among other questions [22, 23, 39].

[1] GW190426_190642 featured a collision of two black holes of individual masses $106\,M_\odot$ and $76\,M_\odot$.

[2] EMRI modelling takes the small object to be compact, but non-compact objects, such as brown dwarfs, may be detectable by LISA if they orbit Sagittarius A* in the Galactic Centre [35, 36].

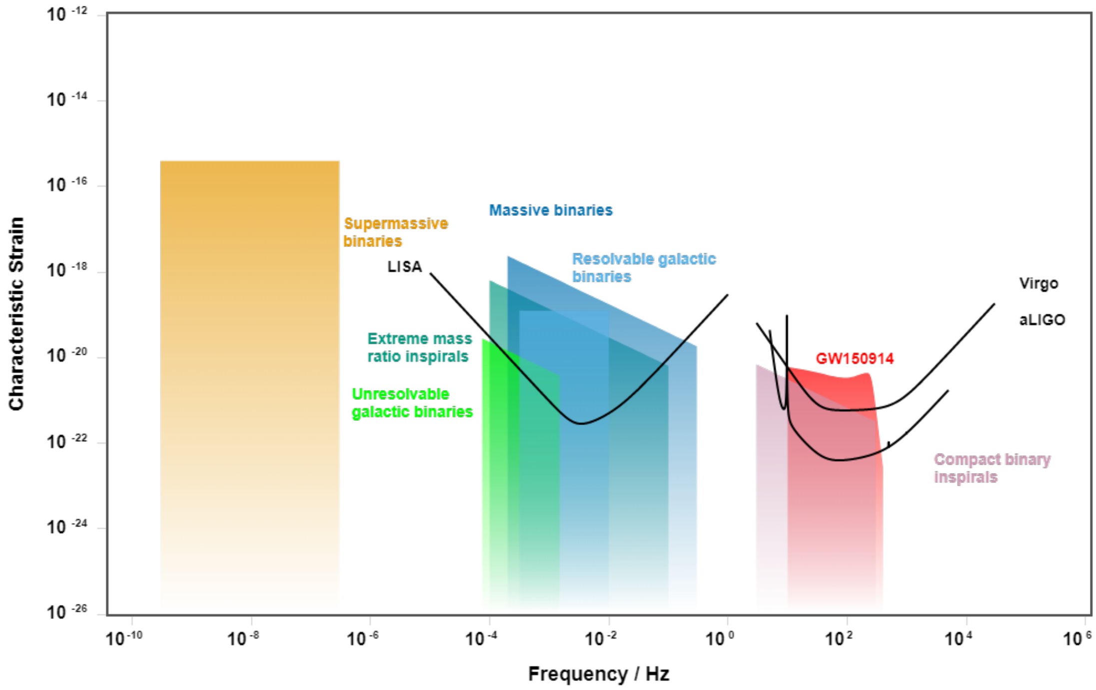

FIGURE 1.1: Figure showing the various detection bands for LISA, Virgo and Advanced LIGO (the current version of the detector). The characteristic strain is related to the time light takes to travel across the detector. The black lines are the sensitivity curves, a lower value means the detector is more sensitive at that frequency. Generated using Ref. [38], based on Ref. [7].

Methods to detect EMRI signals in the raw LISA data are currently in development, as new challenges appear when compared to LIGO/Virgo sources [13, 40]. The primary method of detecting gravitational waves in the LIGO/Virgo data is through the use of matched filtering [41, 42]. Roughly speaking, in this method, one precomputes a template bank of possible gravitational wave signals with different source properties and matches them against the data output from the gravitational wave detector. This allows one to extract the signal even for signals highly dominated by the noise associated with the detector. However, the exact same methods used for LIGO/Virgo are unlikely to be used for LISA data owing to the increased complexity of the EMRI waveforms, necessitating of order $\sim 10^{40}$ gravitational waveform templates in the template bank [43]. For comparison, the second LIGO/Virgo observing run used a template bank with $\sim 10^5$ templates [44, 45].

To aid in developing data analysis techniques for LISA, so-called "kludge" models, which rapidly generate waveforms, have been used [46–48]. While these can not be used for actual parameter extraction from real LISA data, they are useful for developing methods to find EMRI signals in the LISA data [49, 50]. An additional complication is that, due to the long length of an EMRI inspiral, it is possible that the gravitational wave signals from multiple EMRIs will be captured at that same time. Not only that, signals from the other sources previously mentioned will be detected at the same time, necessitating one to disentangle the individual contributions when analysing the final data collected [51, 52]. One must also take into account potential degeneracies that may occur in the signals in order to extract the correct system parameters [53].

1.3 Waveform modelling methods

In order to precisely extract the parameters from EMRIs in the data collected by LISA, we require accurate models for the gravitational waves that are produced in EMRI systems. The long inspiral time and the complexity of the orbits in EMRIs makes modelling them extremely challenging, however.

The traditional technique for modelling compact binary mergers, such as the comparable-mass binaries observed by LIGO/Virgo, is numerical relativity [54–56]. In a numerical relativity setup, one seeks to exactly solve the full nonlinear Einstein field equations (EFEs) using numerical methods. The majority of publicly available simulations have had mass ratios ranging from equal up to 1 : 15 [57–59], although there have been some simulations of systems with a 1 : 100 mass ratio [60, 61]. This greater mass-ratio is still far below that needed for EMRIs. Unfortunately, numerical relativity is unsuitable for use in EMRI systems as it is prohibitively computationally expensive over the length scales involved [15]: that being the large number of orbits and the much smaller size of the small object. The first condition requires one to track the small object over a very long time period and the second condition requires one to keep track of behaviour on both the scale of the large black hole and that of the smaller object. By a simple scaling argument, each of these conditions increases the runtime of the numerical simulation by a factor of ϵ^{-1}, leading to a final overall factor of ϵ^{-2}. For a typical EMRI, this would increase the time needed to complete a numerical simulation by a factor of 10^{10}.

Another technique that has been used to model compact binary systems is post-Newtonian theory [62]. This is a weak-field and slow-motion approximation that expands the EFEs in terms of the small parameter $v/c \ll 1$, where v is the velocity of the body and c is the speed of light. However, as a large part of the inspiral in an EMRI occurs near to the supermassive black hole and in the strong-gravity regime with relativistic velocities, it is not possible to use post-Newtonian theory to model the motion of the small object [15].

As a result, we use an alternative approach, that of *gravitational self-force theory* [15, 63–65]. As mentioned in Ch. 1.1, the self-force refers to the process by which changes in an external field caused by an object's dynamics propagate back and affect the motion of the very same object. This can be modelled with a perturbative method, which we review in Ch. 2.1, that expands the metric describing the geometry of the full spacetime, $\mathsf{g}_{\mu\nu}$, around a known, background metric, $g_{\mu\nu}$, with perturbations, $h^n_{\mu\nu}$, caused by the presence of the small object. The disparate sizes of the small and large object lead to a natural perturbative parameter, the mass ratio between the two objects, $\epsilon := m/M \ll 1$. This perturbative expansion is written as

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \mathcal{O}\left(\epsilon^3\right). \tag{1.1}$$

In the case of an EMRI, the background metric describes the geometry of the large black hole if it were isolated in space and is taken to be either the Schwarzschild or Kerr metric.

To the leading order in the mass ratio, as mentioned in Ch. 1.1, the small object's worldline, γ, is a geodesic of the background spacetime, $g_{\mu\nu}$. The metric perturbations then alter the motion at higher orders and exert a self-force on the body, moving it away from a background geodesic. This can be written as

$$\frac{D^2 z^\alpha}{d\tau^2} = \epsilon f^\alpha_1 + \epsilon^2 f^\alpha_2 + \mathcal{O}\left(\epsilon^3\right), \tag{1.2}$$

which reduces to the geodesic equation when $\epsilon \to 0$. In Eq. (1.2), z^α are coordinates on the worldline, γ, τ is the proper time in the background metric, $g_{\mu\nu}$, $D/\,d\tau := u^\mu \nabla_\mu$ is the covariant derivative along the worldline and is compatible with $g_{\mu\nu}$, $u^\alpha := dz^\alpha/d\tau$ is the four-velocity and f^α_n is the nth-order self-force. The self-force (or at least part of it) causes the orbit to evolve at a rate of $\dot{E}/E \sim \epsilon$, resulting in an inspiral over the radiation reaction time, $t_{rr} \sim E/\dot{E} \sim 1/\epsilon$ [64].

As alluded to in the previous paragraph, the self-force does not only contain dissipative (time-antisymmetric) effects, it includes conservative (time-symmetric) ones as well. One can think of the conservative parts of the force as determining the corrections to the instantaneous orbital frequencies of the orbit with the dissipative parts controlling their slow evolution [17, 65, 66]. Examples of conservative effects are the periastron advance of the small object's orbit [67, 68] and the *Detweiler redshift* [69, 70], related to the surface gravity of the small object if it is a black hole [15, 71].

A challenge is that we are required to go to at least *second order* in our perturbations in order to model the waveforms accurately. This is a result of the requirement that for us to extract information from the data gathered by LISA, the phase of the waveform must be accurate to within a fraction of 1 radian. A rough argument is provided by Ref. [72], which states that if we have a worldline $z(t)$ with calculated acceleration a that has error δa, then our worldline will have an azimuthal phase error of $\delta z \sim t^2 \delta a$. Avoiding any errors of $\mathcal{O}(1)$ in our worldline (and therefore our gravitational-wave phase) requires that $\delta z \ll 1$. EMRI systems evolve over a timescale of $1/\epsilon$, implying that we require $\delta z \sim \delta a/\epsilon^2 \ll 1$. This then implies that we require $\delta a \ll \epsilon^2$. Thus, to accurately model an EMRI, we need to go to second order in self-force theory so that $\delta a \sim \epsilon^3$.

A more precise argument was made by Hinderer and Flanagan [17]. The orbital parameters, $J_B = \{E, L_z, Q\}$, slowly evolve over the radiation reaction time, $t_{rr} \sim 1/\epsilon$. This motives the introduction of a 'slow time', $\tilde{t} = \epsilon t$, so that $J_B = J_B(\tilde{t})$. The orbital frequencies, $\Omega_A = \{\Omega_r, \Omega_\theta, \Omega_\phi\}$ in the case of Kerr, are functions of the orbital parameters, $J_B(\tilde{t})$, and have perturbative expansions,

$$\Omega_A(J_B, \epsilon) = \Omega_A^{(0)}(J_B) + \epsilon\Omega_A^{(1)}(J_B) + \mathcal{O}\Big(\epsilon^2\Big), \tag{1.3}$$

where $\Omega_A^{(n\geq1)}$ are the nth order corrections to $\Omega_A^{(0)}$ due to the conservative part of the self-force. The orbital frequencies evolve with respect to the time, t, as

$$\frac{d\Omega_A}{dt} = \epsilon F_A^{(1)}(J_B) + \epsilon^2 F_A^{(2)}(J_B) + \mathcal{O}\Big(\epsilon^3\Big). \tag{1.4}$$

where $F_A^{(n)}$ is constructed from the nth-order dissipative force. These can then be related to the orbital phases by

$$\varphi_A = \int \Omega_A \, dt\,, \tag{1.5}$$

so that

$$\varphi_A = \frac{1}{\epsilon}\Big(\varphi_A^{(0)}(\tilde{t}) + \epsilon\varphi_A^{(1)}(\tilde{t}) + \mathcal{O}(\epsilon^2)\Big), \tag{1.6}$$

where $\varphi_A^{(0)}$ is constructed from $\Omega_A^{(0)}$ and $F_A^{(1)}$ and $\varphi_A^{(1)}$ is constructed from $\Omega_A^{(1)}$ and $F_A^{(2)}$. One can see this through noting that an integration over t introduces a factor of $1/\epsilon$ through $dt = dt/d\tilde{t}\, d\tilde{t} = \epsilon^{-1}\, d\tilde{t}$. Therefore, to calculate the orbital phases with an error much less than order-ϵ^0 requires the entirety of the first-order self-force and the dissipative part of the second-order self-force. It should be emphasised that the conservative piece of the first-order self-force and the dissipative piece of the second-order self-force are on equal footing: even if one has the entirety of the first-order self-force (both dissipative and conservative parts), if one does not have the dissipative piece of the second-order self-force then one cannot correctly track the motion of the small object.

Returning to our perturbative expansion for $\mathsf{g}_{\mu\nu}$ from Eq. (1.1), when we substitute this into the Einstein field equations,

$$G_{\mu\nu}[\mathsf{g}] = 8\pi T_{\mu\nu}, \tag{1.7}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor (with $T_{\mu\nu} = 0$ in the vacuum EFE), we get

$$G_{\mu\nu}[g + \epsilon h^1 + \epsilon^2 h^2 + \ldots] = 8\pi T_{\mu\nu}. \tag{1.8}$$

We will review how to perform expansions of Eq. (1.8) in Ch. 2.1 but through second order in the mass ratio, the Einstein field equations have the form

$$\delta G_{\mu\nu}[h^1] = 8\pi T^1_{\mu\nu}, \tag{1.9}$$
$$\delta G_{\mu\nu}[h^2] = 8\pi T^2_{\mu\nu} - \delta^2 G_{\mu\nu}[h^1, h^1], \tag{1.10}$$

where we take the background metric to satisfy the vacuum Einstein field equations,

$$G_{\mu\nu}[g] = 0, \tag{1.11}$$

and

$$\delta G_{\mu\nu}[h] = h^{\alpha}{}_{(\mu;\nu)\alpha} + g_{\mu\nu} h^{\alpha}{}_{[\alpha;\beta]}{}^{\beta} - \frac{1}{2}(h_{\mu\nu;\alpha}{}^{\alpha} + h^{\alpha}{}_{\alpha;\mu\nu}), \tag{1.12}$$

is the linearised Einstein tensor, constructed from terms linear in $h_{\mu\nu}$ when expanding the field equations, and

$$\begin{aligned}
\delta^2 G_{\mu\nu}[h,h] = {} & \frac{1}{2} h_{\mu\nu;\alpha} h^{\alpha\beta}{}_{;\beta} - \frac{1}{4} h_{\beta}{}^{\beta;\alpha} h_{\mu\nu;\alpha} + h_{\mu\nu} h_{\alpha}{}^{[\alpha;\beta]}{}_{\beta} - h_{\mu}{}^{\alpha;\beta} h_{\nu[\alpha;\beta]} \\
& + \frac{1}{2} h_{\alpha\beta;(\mu} h^{\alpha\beta}{}_{;\nu)} - h_{\alpha\beta}{}^{;\beta} h^{\alpha}{}_{(\mu;\nu)} + \frac{1}{4} h_{\alpha\beta;\mu} h^{\alpha\beta}{}_{;\nu} \\
& + h^{\alpha\beta}(h_{\nu[\mu;\alpha]\beta} - h_{\alpha[\mu;|\nu|\beta]}) + g_{\mu\nu}\Big(h_{\alpha}{}^{[\beta;\alpha]} h_{\beta\rho}{}^{;\rho} + \frac{1}{8} h^{\rho}{}_{\rho;\beta} h_{\alpha}{}^{\alpha;\beta} \\
& + \frac{1}{4} h_{\alpha\rho;\beta} h^{\alpha\beta;\rho} - \frac{3}{8} h_{\alpha\beta;\rho} h^{\alpha\beta;\rho} - h^{\alpha\beta}[h^{\rho}{}_{[\rho;\alpha]\beta} + h_{\alpha[\beta;\rho]}{}^{\rho}]\Big)
\end{aligned} \tag{1.13}$$

is the second-order Einstein tensor which is constructed from terms quadratic in $h^1_{\mu\nu}$ at order ϵ^2 in Eq. (1.8). Inspecting Eq. (1.10), we see that we can think of the first-order perturbations as ‘sourcing’ those at second order. It is (roughly) these equations and the equation of motion for the small object, from Eq. (1.2), that we must solve.

1.4 History of self-force

In this chapter, we review the historical development of the self-force problem in gravitational physics before moving on to give the current status of gravitational self-force research. A number of extensive reviews of the self-force literature already exist, which we signpost here: Ref. [15] provides a non-technical overview of the entire development of gravitational self-force methods, Refs. [63, 64] provide a technical review of foundational concepts used in gravitational self-force, Ref. [73] provides an overview of the computational methods that can be used, and Ref. [65] describes the analytic perturbative methods used.

We should emphasise though that the work presented here, and in this thesis more generally, is much broader than just seeking to solve the EMRI problem. We are interested in how an object moves in any vacuum background spacetime that features a large external lengthscale. In the case of EMRIs, this is the curvature caused by the presence of the large black hole at the centre of the system.

1.4.1 Formal development of the self-force problem in physics

The problem of how a moving body interacts with the field that it is in has long been studied in physics; see Ch. 2 of Ref. [15] and Sec. 18 of Ref. [63] for an overview. In the case of electromagnetism, the Abraham–Lorentz equation was derived to show how a non-relativistic particle's motion is changed by the self-force. When an electromagnetic particle accelerates, it emits radiation. This emission causes a back reaction which alters the motion of the object. This was extended to the relativistic flat-space case by Dirac [74] and later, to the curved-space case by DeWitt and Brehme [75]. One can interpret the self-force in gravity in a similar way. As the small object inspirals during an EMRI, the systems emits gravitational waves which can be interpreted as causing a recoil in the small object and altering its motion. One must also consider that when gravitational waves are emitted, they can scatter off the spacetime curvature and come back to interact with the small object at a later time.

Just as a small, charged particle moves as a test particle when within an external electromagnetic field, at leading order in gravitational self-force theory, the small object follows a geodesic of the external background. However, as our small object has some spatial extent and its own gravitational field, we can not use a test-particle approximation at all orders. Instead, we treat the small object as causing perturbations at $\mathcal{O}(\epsilon)$ and higher in the background spacetime. To find the form of these perturbations, we use the technique of *matched asymptotic expansions*. This technique will be further outlined in Ch. 2.1.4, but we summarise it here. When sufficiently close to the small object, the expansion from Eq. (1.1) breaks down as the effects of the small object's gravity dominate. One then introduces a second asymptotic expansion which zooms in on the small object. These two expansions are then matched at some appropriate lengthscale where the gravitational effects from each body are comparable. This matching procedure determines the form of the metric perturbations, $h_{\mu\nu}$, in a region near to – but outside – the small object.

This method was used by D'Eath [76] to find that the form of the perturbations is, at first order, entirely equivalent to the solution of the linearised Einstein equation (1.9) when sourced by a point particle,

$$T^1_{\mu\nu} = m \int_\gamma u_\mu u_\nu \frac{\delta^4(x-z)}{\sqrt{-g}}\, d\tau\,. \tag{1.14}$$

Here, γ is the worldline that represents the mean motion of the small object in the background spacetime. It was also employed by Mino, Sasaki, and Tanaka [77] to find the first-order equation of motion when including the effect of the gravitational self-force. The equation of motion is named the *MiSaTaQuWa equation* after the original three discoverers and Quinn and Wald [78], who derived the same equation using an alternative approach. It is given by

$$\frac{D^2 z^\mu}{d\tau^2} = -\frac{\epsilon}{2} g^{\mu\nu}\Big(2h^{\rm tail}_{\nu\rho\sigma} - h^{\rm tail}_{\rho\sigma\nu}\Big)u^\rho u^\sigma + \mathcal{O}\Big(\epsilon^2\Big), \tag{1.15}$$

where

$$h^{\rm tail}_{\mu\nu\rho} = m\int_{-\infty}^{\tau^-} \nabla_\rho \bar{G}^+_{\mu\nu\mu'\nu'} u^{\mu'} u^{\nu'}\, d\tau\,, \tag{1.16}$$

where ∇_ρ is the covariant derivative compatible with the background metric. In Eq. (1.16), $G^+_{\mu\nu\mu'\nu'}$ is the retarded Green's function associated with a wave equation derived from the linearised Einstein equation in the *Lorenz gauge* and the bar denotes the trace-reversal [64]. The Lorenz gauge condition involves taking the trace-reverse of the perturbations,

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma} \tag{1.17}$$

and imposing that it has vanishing divergence,

$$\nabla^\mu \bar{h}_{\mu\nu} = 0. \tag{1.18}$$

This results in the linearised Einstein tensor (1.12) taking the form

$$E_{\mu\nu}[\bar{h}] = -\frac{1}{2}\Box\bar{h}_{\mu\nu} - R_\mu{}^\rho{}_\nu{}^\sigma \bar{h}_{\rho\sigma}, \tag{1.19}$$

where $\Box := g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian or box operator. The tail piece, from Eq. (1.16), can be interpreted as being determined by the gravitational waves previously emitted by the small object. The waves have deflected off the curvature of the spacetime and have come back to collide with the small object at a later time. The remaining part of $h^1_{\mu\nu}$ is the direct piece, $h^{\rm direct}_{\mu\nu}$, that is determined by the waves travelling along the past lightcone of the small object.

A few years after the publication of the MiSaTaQuWa equation, Detweiler and Whiting [79, 80] showed that it could be expressed in the form

$$\frac{D^2 z^\mu}{d\tau^2} = -\frac{\epsilon}{2} g^{\mu\nu}\Big(2h^{\rm R1}_{\nu\rho;\sigma} - h^{\rm R1}_{\rho\sigma;\nu}\Big)u^\rho u^\sigma + \mathcal{O}\Big(\epsilon^2\Big). \tag{1.20}$$

Their approach was to split the first-order perturbation into two pieces: a regular field, $h^{\rm R1}_{\mu\nu}$, and a singular field, $h^{\rm S1}_{\mu\nu}$. The Detweiler–Whiting regular field can be thought of as an effectively external field to the small object which is smooth and valid for all r while satisfying the linearised vacuum Einstein equations

$$\delta G_{\mu\nu}[h^{\rm R1}] = 0. \tag{1.21}$$

Similar to the tail piece, the regular field can be interpreted as containing the previously emitted gravitational waves but with the extra condition of Eq. (1.21). As the regular field satisfies a field equation, it is possible determine how it propagates through spacetime.

The small object then travels as a geodesic in an effective metric, $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{\mathrm{R1}}_{\mu\nu}$, so that [80]

$$\frac{\tilde{D}^2 z^\mu}{d\tilde{\tau}^2} = \mathcal{O}\Big(\epsilon^2\Big), \tag{1.22}$$

where the tilde refers to all objects being defined with respect to $\tilde{g}_{\mu\nu}$. Eq. (1.22) is exactly equivalent to Eq. (1.20), and this correspondence is known as the *generalised equivalence principle* [81]. This will be expanded upon further in Ch. 3.2. The Detweiler–Whiting singular field can be interpreted as being caused by the object's 'self-field' and contains information about the small object's multipole moments. It satisfies the linearised Einstein equations with a point particle source (1.14),

$$\delta G_{\mu\nu}[h^{\mathrm{S1}}] = 8\pi T^1_{\mu\nu}, \tag{1.23}$$

and diverges on the worldline of the small object. This schematically has the form,

$$h^{\mathrm{S1}}_{\mu\nu} \sim \frac{m}{r}, \tag{1.24}$$

at leading order; here, r is the proper spatial distance to the worldline of the small object. The singular field has the interpretation of a Coulomb-like field [15, 64]; that is, as in the case of a point mass in Newtonian gravity, it is generated by the small object but has no effect on its motion.

However, it is generally no longer valid to discuss objects in terms of point particles beyond linear perturbative order in general relativity due to the non-linear nature of the Einstein field equations [82]. This manifests itself at second order in the non-integrability of the $\delta^2 G_{\mu\nu}$ term in Eq. (1.10). The first-order perturbation, $h^1_{\mu\nu}$, behaves like $\sim 1/r$ near the small object's worldline, where r is the proper distance from the worldline. The second-order Einstein tensor has the form $\delta^2 G[h^1] \sim (\partial h^1)^2 + h^1 \partial^2 h^1 \sim 1/r^4$. This is not well defined as a distribution for two reasons: it is non-integrable on any domain containing $r = 0$, and neither is it uniquely representable as a linear operator acting on another distribution.[3] However, a resolution to this problem at second order will be illustrated in Ch. 5 with a rigorous derivation of a second-order stress-energy tensor for a pointlike object.

It is important to note that Eq. (1.20) and thus the generalised equivalence principle apply only to spherically symmetric objects with zero spin. However, since then, it has been shown [85–87] that the equation of motion for a *spinning* body is given by

$$\frac{\tilde{D}^2 z^\mu}{d\tilde{\tau}^2} = -\frac{\epsilon}{2m} \tilde{R}^\mu{}_{\alpha\beta\gamma} \tilde{u}^\alpha \tilde{s}^{\beta\gamma} + \mathcal{O}\Big(\epsilon^2\Big), \tag{1.25}$$

where, again, the tilde refers to quantities related to the effective metric and $s^{\beta\gamma}$ is the spin tensor. The new term is the Mathisson–Papapetrou spin force [88–90]. Eq. (1.25) is

[3] See, e.g. Refs. [83, 84] for introductory references on distribution theory.

the equation of motion derived by Dixon [91] for extended, spinning test bodies,

$$\frac{D^2 z^\mu}{d\tau^2} = -\frac{1}{2m} R^\mu{}_{\alpha\beta\gamma} u^\alpha s^{\beta\gamma}, \tag{1.26}$$

but with the body now travelling in the effective metric, $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{\mathrm{R1}}_{\mu\nu}$. This shows that the small object follows a geodesic – up to corrections due to spin, etc. – in an effective spacetime caused by its perturbations.

As discussed before, we require self-force results up to second order to perform precise parameter extraction from LISA data. In light of this, work was done [92–94] to extend the MiSaTaQuWa equation (1.20) up to second order. This was completed by Pound [95] who found the following expression for the second-order equation of motion,

$$\frac{D^2 z^\mu}{d\tau^2} = -\frac{1}{2}(g^{\mu\alpha} + u^\mu u^\alpha)\left(g_\alpha{}^\delta - h^{\mathrm{R}\delta}_\alpha\right)\left(2h^{\mathrm{R}}_{\delta\beta;\gamma} - h^{\mathrm{R}}_{\beta\gamma;\delta}\right)u^\beta u^\gamma + \mathcal{O}\left(\epsilon^3\right) \tag{1.27}$$

and showed that the generalised equivalence principle still holds to $\mathcal{O}(\epsilon^3)$, that is

$$\frac{\tilde{D}^2 z^\mu}{d\tilde{\tau}^2} = \mathcal{O}\left(\epsilon^3\right). \tag{1.28}$$

Here, $h^{\mathrm{R}}_{\mu\nu} = \epsilon h^{\mathrm{R1}}_{\mu\nu} + \epsilon^2 h^{\mathrm{R2}}_{\mu\nu}$, and has the same properties as the $h^{\mathrm{R1}}_{\mu\nu}$ described under Eq. (1.20): it is smooth on the worldline, γ, and the effective metric, $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$, is a solution to the vacuum EFEs,

$$G_{\mu\nu}[g + h^{\mathrm{R}}] = \mathcal{O}\left(\epsilon^3\right), \tag{1.29}$$

at all points in space. The higher order self-field, $h^{\mathrm{S}}_{\mu\nu}$, has the same general property as the previously defined $h^{\mathrm{S1}}_{\mu\nu}$ and provides information about the small object's higher-order multipole moments [72]. It should be noted that the split into regular and singular fields is not unique [96] but we choose the split in this thesis to correspond to the one made in Refs. [72, 81, 95] which satisfies the properties listed above. That is, that the regular field satisfies the vacuum field equations from Eq. (1.29), the regular field is smooth on the worldline, and that the equation of motion is a geodesic in the effective metric, $\tilde{g}_{\mu\nu}$. In addition to the non-uniqueness of the split, neither $h^{\mathrm{R}}_{\mu\nu}$ nor $h^{\mathrm{S}}_{\mu\nu}$ represent the true physical field; only their sum $h_{\mu\nu} = h^{\mathrm{R}}_{\mu\nu} + h^{\mathrm{S}}_{\mu\nu}$ does.

The method of matched asymptotic expansions and the split into regular and singular fields has the effect of 'skeletonising' the body [64]. The true, physical body is replaced by a singularity with multipole moments matching that of the small object. The singularity then moves in an effective geometry dictated by the background metric and the regular field.

While the original result for the equation of motion from Eq. (1.27) was derived in the Lorenz gauge, it was also shown that this equation of motion is true in any gauge

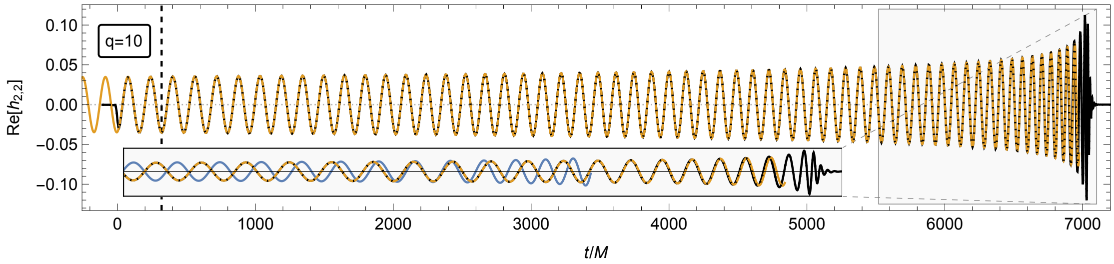

FIGURE 1.2: Waveform generated in Ref. [107] using second-order gravitational self-force methods for a nonspinning binary system with mass ratio 1 : 10. The orange line is the waveform generated by the second-order self-force calculation while the black line is the waveform for the equivalent system generated using numerical relativity. The blue line in the inset is the waveform generated using only first-order self-force methods. Reproduced from Ref. [107].

smoothly related to this gauge and also in a certain class of highly regular gauges, which are of central importance to this thesis. The generalised equivalence principle will be discussed further in Ch. 3.2.

1.4.2 Current status of the implementations of self-force theory in black hole spacetimes

There has been a concerted international effort to implement the self-force formalism before LISA's launch. Currently, it is possible to calculate a full inspiral, driven by the first-order self-force, with a spinning small object on a generic orbit in a Schwarzschild background [97–100]. One can calculate the first-order self-force on any generic bound orbit in Kerr [101] with inspirals performed for equatorial [102] and generic [103] orbits. Some of the code written in service of performing self-force calculations has been made available in the BLACK HOLE PERTURBATION TOOLKIT [104].

At second order, it is only in the last few years that the first complete numerical calculation has been performed, that being the computation of the binding energy in a quasicircular orbit around a Schwarzschild black hole [105]. Since then, the gravitational wave energy flux for the same type of system has been calculated [106] and, in December of 2021, the first full waveforms using second-order gravitational self-force were published [107]. The waveform for the mass ratio of 1 : 10 is provided in Fig. 1.2. This shows excellent agreement between second-order self-force calculations (orange line) and numerical relativity ones (black line) for the same system. The waveforms match deep into the inspiral, only deviating just before the transition to plunge. The blue line in Fig. 1.2 gives the waveform if only first-order dissipative effects are included, demonstrating the need for full, second-order calculations to model these systems.

However, much work remains to bring self-force calculations to the required state before the expected launch of LISA in the mid-2030's, where the ultimate goal is generic orbits in the Kerr spacetime. Recent work has been undertaken on incorporating effects such as

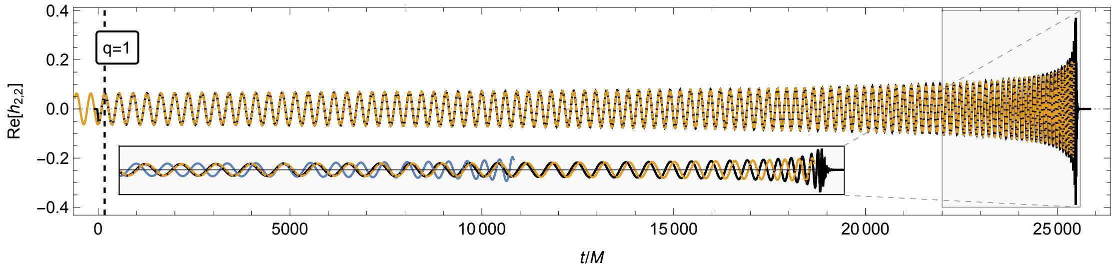

FIGURE 1.3: As in Fig. 1.2, this is the waveform generated in Ref. [107] using second-order gravitational self-force methods for a nonspinning binary system but this time with mass ratio 1 : 1. The orange line is the waveform generated by the second-order self-force calculation while the black line is the waveform for the equivalent system generated using numerical relativity. The blue line in the inset is the waveform generated using only first-order self-force methods. Reproduced from Ref. [107].

the spin of the small object [108–112] into self-force models as this has an $\mathcal{O}(1)$ impact on the gravitational-wave phase. New methods for calculating the metric perturbations of Kerr have been developed [113, 114] and beyond that, methods for rapidly generating waveform templates over the entire parameter space to allow for data analysis to be performed have been presented [115–117].

Second-order self-force has other applications outside of accurate waveform generation for LISA. Information from gravitational self-force models has been used to refine effective one-body (EOB) models [67, 118, 119], and second-order self-force calculations could fully fix post-Newtonian (PN) and post-Minkowskian (PM) two-body dynamics to fifth PN order and sixth PM order [120]. This is one [121] and two [122, 123] orders higher than the current state of the art, respectively.

There is also an increasing body of evidence that the self-force formalism may be applicable to binary systems outside of the usual EMRI regime of $\epsilon \ll 1$ [106, 107, 124–128]. In fact, this is potentially true even up to comparable mass ratios, $\epsilon \approx 1$, in certain areas of the parameter space [106, 107, 129]. Fig 1.3 shows the waveform calculated in Ref. [107] for a 1 : 1 mass ratio inspiral demonstrating remarkable agreement, even late into the inspiral. The applicability of self-force methods for more comparable mass ratios is particularly relevant with the announcement of the LIGO-Virgo Collaboration detecting a binary with mass ratio $\sim 1:27$ [28], possibly indicating that gravitational self-force models could be used for current ground-based detectors.

Self-force methods have traditionally been used for bound orbits where the small object spirals into the large central object. However, recent work has used them in unbound, scattering orbits [130, 131], which showed good agreement when compared to existing post-Minkowskian calculations of the same systems.

1.5 Thesis goals and outline

The MiSaTaQuWa equation (1.15), the Detweiler–Whiting reformulation (1.20), and the second-order equation of motion (1.27) were all originally presented in the Lorenz gauge. While the Lorenz gauge has advantages, it still suffers from the non-integrability of the second-order Einstein tensor as discussed on page 11. As $\delta^2 G_{\mu\nu}$ is unbounded and ill-defined on the worldline of the object, this prohibits us from straightforwardly writing down a field equation for $h^2_{\mu\nu}$ that includes a well-defined stress-energy source, $T^2_{\mu\nu}$.[4]

The behaviour of $\delta^2 G_{\mu\nu}$ also introduces problems when attempting to construct the source and to solve the second-order field equations numerically away from the worldline, such as in a puncture scheme. In a puncture scheme, one truncates the singular field at some order in distance, so that $h^{\mathcal{P}}_{\mu\nu} \approx h^{\mathrm{S}}_{\mu\nu}$. One then enforces that the puncture field vanishes at some suitable distance from the worldline. We define a residual field constructed from the physical field and the puncture field, defined as $h^{\mathcal{R}}_{\mu\nu} := h_{\mu\nu} - h^{\mathcal{P}}_{\mu\nu}$. Substituting this into the perturbed vacuum field equations (Eqs. (1.9)–(1.10) with $T^n_{\mu\nu} = 0$), one moves the puncture fields to the right-hand side of the equation and solves for the residual field. Combining the residual and puncture fields allows one to recover the physical field. We discuss this method more in Ch. 3.3.

The problem is that of *infinite mode coupling* [132], which we summarise here. To take advantage of the symmetries of the spacetime, one decomposes this into a suitable basis of harmonics. For example, in Schwarzschild, one could choose Barack–Lousto–Sago tensor spherical harmonics [133, 134], so that the metric perturbations can be decomposed as

$$h^n_{\mu\nu} = \sum_{i\ell m} h^n_{i\ell m}(t,r) Y^{i\ell m}_{\mu\nu}(\theta,\phi). \tag{1.30}$$

With the modes written as such, to calculate a single mode of $\delta^2 G_{\mu\nu}[h^1, h^1]$ requires one to calculate the infinite sum of products of first-order modes [132, 135],

$$\delta^2 G_{i\ell m} = \sum_{i_1\ell_1 m_1 i_2\ell_2 m_2} \mathcal{D}^{i_1\ell_1 m_1 i_2\ell_2 m_2}_{i\ell m}[h^1_{i_1\ell_1 m_1}, h^1_{i_2\ell_2 m_2}], \tag{1.31}$$

where $\mathcal{D}^{i_1\ell_1 m_1 i_2\ell_2 m_2}_{i\ell m}[h_{i_1\ell_1 m_1}, h_{i_2\ell_2 m_2}]$ is a differential operator. As discussed earlier, the second-order Einstein tensor diverges as $\sim 1/\Delta r^4$ at the worldline of the object, where Δr is the distance to the worldline. After decomposing into modes and integrating over two of the dimensions, one finds that Eq. (1.31) acts as

$$\delta^2 G_{i\ell m}[h^1, h^1] \sim \frac{1}{\Delta r^2}. \tag{1.32}$$

[4] A solution to this problem is presented in Ch. 5.2 but requires a number of technical subtleties and knowledge of the local form of the field.

However, the modes of the first-order field are finite on the worldline [136, 137], meaning that we are attempting to reconstruct a divergent function through summing up finite modes. Thus to get convergence requires one to calculate an arbitrarily *large* number of modes of the first-order fields to calculate even one second-order mode.

A way to circumvent this problem was provided by Miller et al. [132]. Instead of summing over modes, as in Eq. (1.31), one expands the first-order field into regular and singular pieces. When doing this, the second-order Einstein tensor in the source of the second-order field equations takes the form

$$\delta^2 G_{\mu\nu}[h^1, h^1] = \delta^2 G_{\mu\nu}[h^{\rm R1}, h^{\rm R1}] + 2\delta^2 G_{\mu\nu}[h^{\rm R1}, h^{\rm S1}] + \delta^2 G_{\mu\nu}[h^{\rm S1}, h^{\rm S1}], \quad \Delta r > 0. \tag{1.33}$$

One then replaces the regular and singular fields in Eq. (1.33) with the residual and puncture fields. The $\delta^2 G_{i\ell m}[h^{\mathcal{R}1}, h^{\mathcal{R}1}]$ and $\delta^2 G_{i\ell m}[h^{\mathcal{R}1}, h^{\mathcal{P}1}]$ terms are sufficiently well-behaved that one may compute the modes directly from the modes of the first-order residual and puncture fields. As described in Ref. [132], the problem is entirely caused by the slow converge of the modes of $\delta^2 G_{i\ell m}[h^{\mathcal{P}1}, h^{\mathcal{P}1}]$ as this is the term that causes the second-order Einstein tensor to diverge as $\sim 1/\Delta r^4$. Instead of summing up the products of the modes of $h^{\mathcal{P}1}_{\mu\nu}$, Miller et al. [132] directly calculate $\delta^2 G_{\mu\nu}[h^{\mathcal{P}1}, h^{\mathcal{P}1}]$ in four dimensions using the four dimensional expression for $h^{\mathcal{P}1}_{\mu\nu}$ and then decompose this quantity into modes. Unfortunately, while this makes the calculation of the modes of the source possible, it is incredibly computationally expensive and takes up almost all of the code runtime when implemented (such as in Ref. [105]). This is due to having to calculate the modes by numerically integrating the complete four-dimensional expression on a grid of r and Δr values. This will not be efficiently extendible when approaching problems involving more complicated dynamics, such as generic orbits in Kerr.

One motivation for the introduction of the class of highly regular gauges by Pound [81] was to try to avoid this infinite mode coupling problem. In these gauges, the second-order singular field $h^{\rm S2}_{\mu\nu}$ behaves like $\sim m^2/r$, one order less divergent than its behaviour $\sim m^2/r^2$ in a generic gauge. One can divide the second-order singular field into two pieces: a 'singular times regular' piece, $h^{\rm SR}_{\mu\nu} \sim m h^{\rm R1}_{\mu\nu}/\Delta r$, and a 'singular times singular' piece, $h^{\rm SS}_{\mu\nu} \sim m^2 \Delta r^0$. By simple order counting of m and $h^{\rm R1}_{\mu\nu}$, we see that, in the second-order Einstein field equations, $h^{\rm SS}_{\mu\nu}$ is sourced by $\delta^2 G_{\mu\nu}[h^{\rm S1}, h^{\rm S1}]$, as they both feature terms $\sim m^2$, and that $h^{\rm SR}_{\mu\nu}$ is sourced by $\delta^2 G_{\mu\nu}[h^{\rm R1}, h^{\rm S1}]$ as both expressions have terms of the form $\sim m h^{\rm R1}_{\mu\nu}$. Although the $h^{\rm SR}_{\mu\nu}$ term appears more divergent, as expanded on later in this thesis, its source, $\delta^2 G_{\mu\nu}[h^{\rm R1}, h^{\rm S1}]$, can be well-defined as a distribution and it is the 'singular times singular' term that causes the most issues. Acting on the 'singular times singular' piece with the Einstein tensor, we see that $\delta G_{\mu\nu}[h^{\rm SS}] \sim m^2/\Delta r^2$. Therefore, we know that the most singular piece of the second-order Einstein tensor can only act as badly $\delta^2 G_{\mu\nu}[h^{\rm S1}, h^{\rm S1}] \sim 1/r^2$ instead of $\sim 1/r^4$. This means that when decomposing into modes, the individual modes of the second-order Einstein tensor can behave, at worst, as $\delta^2 G_{i\ell m} \sim \log|\Delta r|$. While this is still divergent, it is much weaker than that found in the Lorenz gauge.

The ultimate goal of this thesis is to further develop the highly regular gauge and take advantage of its weaker singularity structure. We start in Ch. 2 by providing a summary of mathematical techniques that are essential to the rest of the thesis. These include performing local covariant and coordinate expansions near the worldline of an object and a description of Fermi–Walker coordinates, coordinates that are tethered to an accelerated worldline. As a demonstration of these tools, we show how to perform coordinate expansions of the Detweiler–Whiting singular field for a scalar charge, given in covariant form in Ref. [138], through five total orders in distance, one more than the current state of the art.

In Ch. 3, we detail concepts important to gravitational self-force research. We describe the self-consistent formalism that we use throughout this thesis. This is based on expanding the metric perturbations in powers of ϵ but, crucially, not expanding the dependence on the accelerated worldline. Next, we follow Ref. [139] and provide a derivation of the generalised equivalence principle and then give an overview of the puncture scheme that was previously mentioned. In the final parts of the chapter, we follow the method used by Pound [81] to calculate the form of the metric perturbations in a lightcone rest gauge which preserves the local structure of the lightcone while keeping the small object at rest in spacetime. Finally, we describe the method used by Pound [81] to derive the leading-order piece of the second-order singular field in the highly regular gauge.

In Ch. 4 of this thesis, we continue this calculation and derive the form of $h^{S2}_{\mu\nu}$ through order r. This is the order required for numerical implementation, such as in a puncture scheme. As discussed in Ref. [81], the weaker divergence in the highly regular gauge means that the second-order Einstein tensor is well defined as a distribution. Therefore, one may write a field equation for $h^2_{\mu\nu}$ that is well defined at all points in spacetime. This results in us being able to rigorously derive a pointlike second-order stress-energy tensor for the small object, which we detail in Ch. 5. In fact, when combined with the first-order stress-energy tensor from Eq. (1.14), we show that this is nothing more than the stress-energy tensor of a point-particle in an effective spacetime, $\tilde{g}_{\mu\nu}$, given by

$$\tilde{T}^{\mu\nu} = \epsilon m \int_\gamma \tilde{u}^\mu \tilde{u}^\nu \frac{\delta^4(x-z)}{\sqrt{-\tilde{g}}}\, d\tilde{\tau}\,. \tag{1.34}$$

This form of the stress-energy tensor was originally conjectured by Detweiler [93] to be true in all gauges and, as such, we name $\tilde{T}^{\mu\nu}$ the *Detweiler stress-energy.* Our analysis shows that it is valid in the class of highly regular gauges, and we later show that the functional form of this stress-energy is retained in any smoothly related gauge. We then demonstrate that, using a specific distributional definition for $\delta^2 G^{\mu\nu}$, we can derive the Detweiler stress-energy tensor in the Lorenz gauge, and we show that, again, it has the same functional form as in the highly regular gauge. These two chapters (excluding

Ch. 5.3 onwards) along with various other parts of this thesis dealing with the highly regular gauge are also available in the paper Ref. [140]. In Ch. 5.3 we demonstrate how our distributional definitions can be used to formulate field equations to directly solve for the second-order field in the Lorenz gauge. This is done in the case of both the Einstein field equations in Ch. 5.3.1 and the Teukolsky equation in Ch. 5.3.2, which is a reformulation of the EFEs in terms of scalar quantities.

Following this, in Ch. 6, we convert our specific coordinate expressions for the metric perturbations in the highly regular gauge into fully covariant form using the techniques described in Ch. 2.2 and Ref. [96]. These are then expanded into a generic coordinate form and provided in the Additional Material as a MATHEMATICA notebook [141]. With a fully generic coordinate form, one may write the highly regular gauge metric perturbations in any coordinate basis adapted to the problem being studied.

In Ch. 7, we derive the gauge vector required to transform Lorenz gauge data calculated in Ref. [105] for quasicircular orbits in a Schwarzschild spacetime into the highly regular gauge. This could then be used as input for either the second-order Einstein field equations or second-order Teukolsky equation from Ch. 5.3. We begin by recapping the highly regular gauge conditions before deriving the covariant form of the gauge vector for the transformation between the highly regular gauge and the Lorenz gauge and then expanding it into generic coordinate form. Part of the gauge vector was found at leading order by Spiers [142] but we provide all terms through the order required to ensure our gauge conditions are satisfied on the worldline. We project this into the Newman–Penrose formalism that allows one to rewrite tensorial quantities in terms of spin-weighted scalars. The essential ingredients of the Newman–Penrose formalism are provided in App. C.

To perform the mode decomposition into spin-weighted spherical harmonics (presented in App. D), we use the method of Refs. [137, 143] where we introduce a rotated coordinate system in which the small object is always located at the north pole. This has the advantage of drastically reducing the number of m modes that need to be calculated to perform the mode decomposition as all but the lowest few m modes vanish when evaluated here. Which m modes are non-vanishing depends on a number of factors, including the spin-weight of the object being considered, but overall we only need to calculate a handful of m modes instead of potentially hundreds. We then convert the modes in our rotated coordinates into modes in the standard Schwarzschild coordinates through the use of the Wigner-D matrix. The modes in the rotated coordinate system are provided in the Additional Material [141]. In the last part of the chapter, we calculate how one can construct the metric perturbations needed to transform the Lorenz gauge data into the highly regular gauge using expressions from Ref. [135].

Finally, we sum up the work in this thesis and present future avenues for research in Ch. 8.

1.6 Conventions

The definitions in this section are given in table format in the Definitions and Abbreviations section.

In this thesis, we use geometric units so that $G = c = 1$. Greek indices run from 0 to 3 and are raised and lowered with the background metric, $g_{\mu\nu}$, which has signature $(-,+,+,+)$. Latin indices run from 1 to 3 and are raised and lowered with the Euclidean metric, δ_{ab}. Uppercase Latin indices denote multi indices, that is $L := i_1 \dots i_l$. Parentheses and square brackets around indices denote symmetrisation/antisymmetrisation, respectively. Angled brackets, such as $\langle L\rangle$, denote the symmetric trace-free (STF) combination of the enclosed indices with respect to δ_{ab}. Additionally, we define $\hat{T}^L := T^{\langle L\rangle}$, for a generic tensor T. In some cases, we additionally use the notation Sym_L or STF_L to denote symmetrisation and the STF combination over the indices L, respectively.

We use a comma/semicolon or ∂/∇ to denote partial/covariant differentiation respectively. The covariant derivative is compatible with $g_{\mu\nu}$ unless otherwise stated. We denote the four-velocity of the small object as

$$u^\mu := \frac{dz^\mu}{d\tau}, \tag{1.35}$$

where τ is proper time in $g_{\mu\nu}$, and the directional derivative as

$$\frac{D}{d\tau} := u^\mu \nabla_\mu. \tag{1.36}$$

Terms written in a serif font are exact quantities, e.g. $\mathsf{g}_{\mu\nu}$ is the full, exact metric describing the physical spacetime. A prime symbol on the perturbation, $h^{n'}_{\mu\nu}$, denotes quantities in the lightcone rest gauge, and a star, $h^{n*}_{\mu\nu}$, denotes quantities in the Lorenz gauge. No prime, $h^n_{\mu\nu}$, indicates terms in the highly regular gauge. A prescript, ${}^nA^{\mu_1\cdots}_{\nu_1\dots}$, on a tensor counts the power of ϵ coming from substituting the expansion of the acceleration $a^\mu = \sum_{n\geq 1}\epsilon^n f^\mu_n$ into $A^{\mu_1\cdots}_{\nu_1\dots}$. An overset ring, $\mathring{A}^{\mu_1\cdots}_{\nu_1\dots}$, indicates terms that have been re-expanded for small acceleration and then re-collected at each order in ϵ, i.e. $\mathring{h}^n_{\mu\nu} = \sum_{i=0}^n {}^ih^{n-i}_{\mu\nu}$ (where, for this purpose, $h^0_{\mu\nu} := g_{\mu\nu}$). Tildes placed over a tensor, $\tilde{A}^{\mu_1\cdots}_{\nu_1\dots}$, denote quantities defined with respect to the effective metric, $\tilde{g}_{\mu\nu}$.

Additionally, we say that $f(x) = \mathcal{O}(g(x))$ if

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = k, \tag{1.37}$$

where k is some (potentially zero) constant and that $f(x) = o(g(x))$ if

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 0. \tag{1.38}$$

This thesis makes use of Mathematica [144] and the computer algebra package xAct [145–151] throughout. We make use of the SpinWeightedSpheroidalHarmonics package from the Black Hole Perturbation Toolkit [104] as well.

Chapter 2

Mathematical preliminaries

This chapter will give an overview of some of the mathematical techniques used in the analysis of the gravitational self-force. Firstly, we start with a recap of perturbation theory and how we can, in theory, extend a perturbed quantity to any order we require before moving on to discuss gauge freedom in perturbation theory. We provide an overview of the method of matched asymptotic expansions and how it has been used to determine the form of the metric perturbations. We then outline local expansion methods that can be used in general relativity. These will be used extensively throughout the rest of the thesis when analysing fields near to the small object and are of crucial importance to this work. We detail how one can perform both covariant and generic coordinate expansions of various tensorial quantities. Finally, we detail a useful coordinate system when working near the worldline of an object, that of Fermi–Walker coordinates.

2.1 Perturbation theory in general relativity

A perturbed metric is one in which we have a small deviation from an exact solution to the Einstein field equations. This perturbed metric can be written as

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{2.1}$$

where $g_{\mu\nu}$ is an exact solution and

$$h_{\mu\nu} = \sum_{n\geq 1}^{\infty} \epsilon^n h^n_{\mu\nu}[\gamma] \tag{2.2}$$

is a series for the perturbation, $h_{\mu\nu}$, with ϵ giving the 'size' of the perturbation [20, Ch. 7.5, 139]. We can then recover a specific perturbation order by differentiating with respect to our parameter,

$$h^n_{\mu\nu} = \frac{1}{n!}\left.\frac{d^n \mathsf{g}_{\mu\nu}}{d\epsilon^n}\right|_{\epsilon=0}. \tag{2.3}$$

This is similar for an arbitrary tensor, $A^{\mu\nu\cdots}_{\rho\sigma\ldots}$, constructed from $\mathsf{g}_{\mu\nu}$. We can expand in powers of $h_{\mu\nu}$, i.e.

$$A^{\mu\nu\cdots}_{\rho\sigma\ldots}[g+h] = A^{\mu\nu\cdots}_{\rho\sigma\ldots}[g] + \delta A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h] + \delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h,h] + \delta^3 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h,h,h] + \mathcal{O}\Big(h^4\Big), \quad (2.4)$$

where $\delta^n A^{\mu\nu\cdots}_{\rho\sigma\ldots}$ is nth-order in $h_{\mu\nu}$ and linear in each of its arguments. We can then define the operators in a similar way to Eq. (2.3), as

$$\delta^n A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h,\ldots,h] = \frac{1}{n!}\left.\frac{d^n A^{\mu\nu\cdots}_{\rho\sigma\ldots}[g+\lambda h]}{d\lambda^n}\right|_{\lambda=0}, \quad (2.5)$$

where we introduce λ as a formal order counting parameter to count powers of $h_{\mu\nu}$.

Often, we are interested in operators constructed from products of different perturbations, i.e. $\delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^1,h^2]$. To handle this, we adopt the definition [139, App. C]

$$\delta^n A^{\mu\nu\cdots}_{\rho\sigma\ldots}[f_1,\ldots,f_n] := \frac{1}{n!}\left.\frac{d^n}{d\lambda_1\ldots d\lambda_n} A^{\mu\nu\cdots}_{\rho\sigma\ldots}[g_{\mu\nu}+\lambda_1 f_1+\cdots+\lambda_n f_n]\right|_{\lambda_i=0}, \quad (2.6)$$

which ensures that $\delta^n A$ is symmetric in all arguments. To see where these terms featuring different perturbations come from, we substitute Eq. (2.2) into Eq. (2.4) to obtain a power series in ϵ,

$$\begin{aligned} A^{\mu\nu\cdots}_{\rho\sigma\ldots}[g+h] = {} & A^{\mu\nu\cdots}_{\rho\sigma\ldots}[g] + \epsilon\delta A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^1] + \epsilon^2\left(\delta A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^2] + \delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^1,h^1]\right) \\ & + \epsilon^3\left(\delta A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^3] + 2\delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^1,h^2] + \delta^3 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^1,h^1,h^1]\right) + \mathcal{O}\Big(\epsilon^4\Big). \end{aligned} \quad (2.7)$$

The final line shows coupling between different orders of the metric perturbation but we can also split our metric perturbations at each order up into separate fields. That is, if we take $\delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^1,h^1]$ and we split $h^1_{\mu\nu} = h^{A1}_{\mu\nu} + h^{B1}_{\mu\nu}$, then, using our definition in Eq. (2.6), we arrive at

$$\delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^1,h^1] = \delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^{A1},h^{A1}] + 2\delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^{A1},h^{B1}] + \delta^2 A^{\mu\nu\cdots}_{\rho\sigma\ldots}[h^{B1},h^{B1}]. \quad (2.8)$$

This is precisely what we have done in Eq. (1.33) when expanding the metric perturbation into singular and regular fields.

2.1.1 Second-order Ricci tensor in a vacuum background

As an example, we shall compute the form of the Ricci tensor, $\mathsf{R}_{\mu\nu}$, of $\mathsf{g}_{\mu\nu}$ to second order when $g_{\mu\nu}$ is a solution to the vacuum Einstein equations, that is,

$$R_{\mu\nu} = 0. \quad (2.9)$$

Here, tensors written in sans-serif font are constructed from the full metric, $\mathsf{g}_{\mu\nu}$, whereas those in a serif font are constructed from the background metric, $g_{\mu\nu}$. As we are always interested in a vacuum background in this thesis, this is entirely equivalent to deriving the second-order Einstein field equations that we discussed in the previous chapter, as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.10}$$

where R is the Ricci scalar, so that $G_{\mu\nu} = 0 \iff R_{\mu\nu} = 0$.

To calculate $\mathsf{R}_{\mu\nu}$, we must find an expression for it in terms of other 'known' quantities. Firstly, we define $\widehat{\nabla}_\mu$ and ∇_μ to be the covariant derivatives compatible with $\mathsf{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively. The difference between these two covariant derivatives is given by [20, Ch. 3.1]

$$\widehat{\nabla}_\mu A^{\nu_1\dots\nu_p}_{\rho_1\dots\rho_q} = \nabla_\mu A^{\nu_1\dots\nu_p}_{\rho_1\dots\rho_q} + \sum_i C^{\nu_i}{}_{\mu\sigma} A^{\nu_1\dots\sigma\dots\nu_p}_{\rho_1\dots\rho_q} - \sum_j C^{\sigma}{}_{\mu\rho_j} A^{\nu_1\dots\nu_p}_{\rho_1\dots\sigma\dots\rho_q}, \tag{2.11}$$

where

$$\begin{aligned} C^{\rho}{}_{\mu\nu} &= \frac{1}{2}\mathsf{g}^{\rho\sigma}\Big(\nabla_\mu \mathsf{g}_{\nu\sigma} + \nabla_\nu \mathsf{g}_{\mu\sigma} - \nabla_\sigma \mathsf{g}_{\mu\nu}\Big) \\ &= \frac{1}{2}\Big(g^{\rho\sigma} + \hat{h}^{\rho\sigma}\Big)(\nabla_\mu h_{\nu\sigma} + \nabla_\nu h_{\mu\sigma} - \nabla_\sigma h_{\mu\nu}) \end{aligned} \tag{2.12}$$

is the difference between the Christoffel symbols in $\mathsf{g}_{\mu\nu}$ and $g_{\mu\nu}$. The second line follows from using Eq. (2.1) to write the full metric with indices down in terms of the background metric and the metric perturbation, introducing the definition,

$$\hat{h}^{\mu\nu} := \mathsf{g}^{\mu\nu} - g^{\mu\nu}, \tag{2.13}$$

for the difference between the exact and background metrics with indices up, and noting that $\nabla_\mu g_{\nu\rho} = 0$. We should note at this time that this is still an exact expression, we have performed no truncation of $h_{\mu\nu}$. Additionally, $\hat{h}^{\mu\nu} \neq g^{\mu\rho} g^{\nu\sigma} h_{\rho\sigma}$: it instead satisfies $\mathsf{g}_{\mu\nu}\mathsf{g}^{\nu\rho} = \delta^\rho_\nu$.

Secondly, we recall that one way to define the Riemann curvature tensor for a spacetime with no torsion is how the commutator of the covariant derivative acts on a one-form, that is [20, Ch. 3.2]

$$\widehat{\nabla}_\mu \widehat{\nabla}_\nu \omega_\rho - \widehat{\nabla}_\nu \widehat{\nabla}_\mu \omega_\rho = \mathsf{R}_{\mu\nu\rho}{}^{\sigma} \omega_\sigma. \tag{2.14}$$

Now, we expand the left-hand side of Eq. (2.14) in terms of ∇_μ and the connection coefficients from Eq. (2.12), so that

$$\begin{aligned} \widehat{\nabla}_\mu \widehat{\nabla}_\nu \omega_\rho = {} & \nabla_\mu \nabla_\nu \omega_\rho - \Big(\nabla_\mu C^{\sigma}{}_{\nu\rho}\Big)\omega_\sigma - C^{\sigma}{}_{\nu\rho}\nabla_\mu \omega_\sigma - C^{\sigma}{}_{\mu\rho}\nabla_\nu\omega_\sigma \\ & - C^{\sigma}{}_{\mu\nu}\nabla_\sigma\omega_\rho + C^{\sigma}{}_{\mu\nu} C^{\gamma}{}_{\rho\sigma}\omega_\gamma + C^{\sigma}{}_{\mu\rho} C^{\gamma}{}_{\nu\sigma}\omega_\gamma. \end{aligned} \tag{2.15}$$

We can now substitute this into Eq. (2.14) to show that

$$\mathsf{R}_{\mu\nu\rho}{}^{\sigma} = R_{\mu\nu\rho}{}^{\sigma} - 2\nabla_{[\mu} C^{\sigma}{}_{\nu]\rho} + 2C^{\gamma}{}_{\rho[\mu} C^{\sigma}{}_{\nu]\gamma}, \tag{2.16}$$

where $R_{\mu\nu\rho}{}^{\sigma}$ is the Riemann tensor for $g_{\mu\nu}$ and we have noted that ω_μ is arbitrary and can be omitted. To find the Ricci tensor, we contract over ν and σ, so

$$\mathsf{R}_{\mu\sigma\rho}{}^{\sigma} = \mathsf{R}_{\mu\rho} = -2\nabla_{[\mu} C^{\sigma}{}_{\sigma]\rho} + 2C^{\gamma}{}_{\rho[\mu} C^{\sigma}{}_{\sigma]\gamma}, \tag{2.17}$$

where the Ricci curvature term of the background disappears by Eq. (2.9).

We stress that, up until now, all expressions that we have derived are exact, featuring no expansions of the metric perturbations. To calculate the explicit expression for Eq. (2.17), we substitute in Eq. (2.12) and keep all terms up to and including $\mathcal{O}(h^2)$. We also require the leading-order expansion of $\hat{h}^{\mu\nu}$ from Eq. (2.13) in terms of $h_{\mu\nu}$. This is quite simply given by

$$\hat{h}^{\mu\nu} = -h^{\mu\nu} + \mathcal{O}\Big(h^2\Big), \tag{2.18}$$

which can be verified by substitution into $\mathsf{g}_{\mu\nu}\mathsf{g}^{\nu\rho} = \delta^{\rho}_{\mu} + \mathcal{O}(h^2)$. This gives rise to the final expression

$$\begin{aligned}\mathsf{R}_{\mu\nu} = {} & h_{\rho(\mu;\nu)}{}^{\rho} - \frac{1}{2}\Big(h_{;\nu\mu} + h_{\mu\nu;\rho}{}^{\rho}\Big) - h^{\rho\sigma} h_{\sigma(\mu;\nu)\rho} + h^{\sigma}{}_{\nu;}{}^{\rho} h_{\mu[\sigma;\rho]} \\ & + \frac{1}{2} h^{\rho\sigma}{}_{;\rho}\Big(h_{\mu\nu;\sigma} - 2h_{\sigma(\mu;\nu)}\Big) + \frac{1}{2} h^{\rho\sigma}\Big(h_{\rho\sigma;\nu\mu} + h_{\mu\nu;\sigma\rho}\Big) \\ & + \frac{1}{4}\Big(h^{\rho\sigma}{}_{;\mu} h_{\rho\sigma;\nu} - h^{;\sigma}\Big(h_{\mu\nu;\sigma} + 2h_{\sigma(\mu;\nu)}\Big)\Big) + \mathcal{O}\Big(h^3\Big),\end{aligned} \tag{2.19}$$

where we have switched to semi-colon notation for compactness and $h := h^{\mu}{}_{\mu} := g^{\mu\nu} h_{\mu\nu}$. We can then 'pick off' the pieces we require: e.g. following the notation from Eq. (2.6), we have

$$\delta R_{\mu\nu}[h] = h_{\rho(\mu;\nu)}{}^{\rho} - \frac{1}{2}\Big(h_{;\nu\mu} + h_{\mu\nu;\rho}{}^{\rho}\Big) \tag{2.20}$$

$$\begin{aligned}\delta^2 R_{\mu\nu}[h,h] = {} & h^{\sigma}{}_{\nu;}{}^{\rho} h_{\mu[\sigma;\rho]} - h^{\rho\sigma} h_{\sigma(\mu;\nu)\rho} + \frac{1}{2} h^{\rho\sigma}{}_{;\rho}\Big(h_{\mu\nu;\sigma} - 2h_{\sigma(\mu;\nu)}\Big) \\ & + \frac{1}{2} h^{\rho\sigma}\Big(h_{\rho\sigma;\nu\mu} + h_{\mu\nu;\sigma\rho}\Big) + \frac{1}{4}\Big(h^{\rho\sigma}{}_{;\mu} h_{\rho\sigma;\nu} - h^{;\sigma}\Big(h_{\mu\nu;\sigma} - 2h_{\sigma(\mu;\nu)}\Big)\Big).\end{aligned} \tag{2.21}$$

It is important to note that, in general, it is *not* true that $g^{\mu\nu}\delta^n A_{\nu\rho\ldots}{}^{\cdots} = \delta^n A^{\mu}{}_{\rho\ldots}{}^{\cdots}$ as we can miss terms by not taking into account the expansion of the full metric. Care must be taken when raising and lowering indices in perturbation theory, and we must define exactly what we mean when we perform index manipulations.

2.1.2 Linear and quadratic Einstein tensors

One may also perform a similar operation on the Einstein tensor,

$$\mathsf{G}^{\mu\nu}[g+h] = \delta G^{\mu\nu}[h] + \delta^2 G^{\mu\nu}[h,h] + \mathcal{O}\Big(|h|^3\Big), \tag{2.22}$$

to show that

$$\delta G^{\mu\nu}[h] = h_\alpha{}^{(\mu;\nu)\alpha} + g^{\mu\nu} h^\alpha{}_{[\alpha;\beta]}{}^\beta - \frac{1}{2}(h^{\mu\nu;\alpha}{}_\alpha + h_\alpha{}^{\alpha;\mu\nu}), \tag{2.23}$$

and

$$\begin{aligned}\delta^2 G^{\mu\nu}[h,h] = {} & \frac{1}{2} h^{\mu\nu}{}_{;\alpha} h^{\alpha\beta}{}_{;\beta} - \frac{1}{4} h_\beta{}^{\beta;\alpha} h^{\mu\nu}{}_{;\alpha} + h^{\mu\nu} h_\alpha{}^{[\alpha;\beta]}{}_\beta + h^{\mu\alpha;\beta} h^\nu{}_{[\alpha;\beta]} + \frac{1}{2} h^\beta{}_{\beta;\alpha} h^{\alpha(\mu;\nu)} \\ & - h_{\alpha\beta}{}^{;\beta} h^{\alpha(\mu;\nu)} + \frac{1}{4} h_{\alpha\beta}{}^{;\mu} h^{\alpha\beta;\nu} + h_{\alpha\beta}(h^{\nu[\mu;\alpha]\beta} - h^{\alpha[\mu;|\nu|\beta]}) \\ & + g^{\mu\nu}\Big(h_\alpha{}^{[\beta;\alpha]} h_{\beta\rho}{}^{;\rho} + \frac{1}{8} h^\rho{}_{\rho;\beta} h_\alpha{}^{\alpha;\beta} + \frac{1}{4} h_{\alpha\rho;\beta} h^{\alpha\beta;\rho} - \frac{3}{8} h_{\alpha\beta;\rho} h^{\alpha\beta;\rho} \\ & - h^{\alpha\beta}[h^\rho{}_{[\rho;\alpha]\beta} + h_{\alpha[\beta;\rho]}{}^\rho]\Big) - 2h^{(\mu}{}_\rho \delta G^{\nu)\rho}[h]. \end{aligned} \tag{2.24}$$

We can use Eq. (2.6) to ensure that $\delta^2 G^{\mu\nu}[h,h]$ is symmetric and bilinear by defining

$$\delta^2 G^{\mu\nu}[h^\flat, h^\sharp] \coloneqq \frac{1}{2}\frac{d^2}{d\lambda_1\, d\lambda_2} G^{\mu\nu}[g + \lambda_1 h^\flat + \lambda_2 h^\sharp]\bigg|_{\lambda_i = 0}, \tag{2.25}$$

which reduces to Eq. (2.24) when $h^\flat_{\mu\nu} = h^\sharp_{\mu\nu} = h_{\mu\nu}$. We also define a linear operator

$$Q^{\mu\nu}_\flat[h^\sharp] \coloneqq \delta^2 G^{\mu\nu}[h^\flat, h^\sharp], \tag{2.26}$$

which is the term bilinear in $h^\flat_{\mu\nu}$ and $h^\sharp_{\mu\nu}$ if we expand $G^{\mu\nu}[g + h^\flat + h^\sharp]$ in powers of $h^\sharp_{\mu\nu}$ and its derivatives. That is,

$$G^{\mu\nu}[g + h^\flat + h^\sharp] = G^{\mu\nu}[g + h^\flat] + Q^{\mu\nu}_\flat[h^\sharp] + \mathcal{O}\Big(|h^\sharp|^2, |h^\flat|^2|h^\sharp|\Big). \tag{2.27}$$

In Ch. 5, we make extensive use of the adjoints of these quantities. The precise definition of the adjoint is given later in the thesis in Ch. 5.2.2. The linearised Einstein tensor is self-adjoint [152], $\delta G^{\dagger\mu\nu}[h] = \delta G^{\mu\nu}[h]$. The adjoint of $Q^{\mu\nu}_\flat$ is

$$\begin{aligned} Q^{\dagger\mu\nu}_\flat[\phi] = \frac{1}{2}\bigg[& \phi^{\alpha\beta}\Big(h^{\mu\nu}_\flat{}_{;\alpha\beta} - 2h^{(\mu}_\flat{}_{\alpha;}{}^{\nu)}{}_\beta + g^{\mu\nu}\Big\{h^\flat_{\alpha\rho;\beta}{}^\rho - \tfrac{1}{2} h^\flat_{\alpha\beta;\rho}{}^\rho\Big\} + h^\flat_{\alpha\beta;}{}^{(\mu\nu)}\Big) \\ & - h^{\alpha\beta}_\flat\Big(2\phi_\alpha{}^{(\mu;\nu)}{}_\beta - \phi^{\mu\nu}{}_{;\alpha\beta} - \phi_{\alpha\beta;}{}^{(\mu\nu)} + g^{\mu\nu}\Big\{\phi^\rho{}_{\rho;\alpha\beta} + \phi_{\alpha\beta;\rho}{}^\rho - 2\phi_\alpha{}^\rho{}_{;\rho\beta}\Big\}\Big) \\ & + \phi^{\alpha(\mu} h^{|\beta|}_\flat{}_{\beta;}{}^{\nu)}{}_\alpha + 2h^{\alpha(\mu}_\flat \phi^{|\beta|}{}_{\beta;}{}^{\nu)}{}_\alpha + \phi^\beta{}_{\beta;\alpha}\Big(h^{\alpha(\mu;\nu)}_\flat - \tfrac{1}{2} h^{\mu\nu;\alpha}_\flat\Big) \\ & - \frac{1}{2} g^{\mu\nu}\Big(\phi^\rho{}_{\rho;\beta} h^\flat_\alpha{}^{\alpha;\beta} + 2h^\flat_\alpha{}^{\beta;\alpha}\Big\{\phi^\rho{}_{\rho;\beta} - 2\phi_{\beta\rho;}{}^\rho\Big\} + \phi^\alpha{}_\alpha h^\flat_\beta{}^{\beta;\rho}{}_\rho - 2\phi^{\alpha\rho;\beta} h^\flat_{\alpha\beta;\rho} \\ & + 3\phi^{\alpha\beta;\rho} h^\flat_{\alpha\beta;\rho}\Big) + h^\flat_\alpha{}^{\beta;\alpha}\Big(\phi^{\mu\nu}{}_{;\beta} - 2\phi_\beta{}^{(\mu;\nu)}\Big) + \phi^{\mu\nu}\Big(h^\flat_{\alpha\beta;}{}^{\alpha\beta} - \tfrac{1}{2} h^\flat_\alpha{}^{\alpha;\beta}{}_\beta\Big) \end{aligned}$$

$$
\begin{aligned}
&- 2\phi^{\alpha(\mu}h_\flat^{\nu)\beta}{}_{;\alpha\beta} - 2h_\flat^{\mu\nu}\phi_\alpha{}^{[\alpha;\beta]}{}_\beta + h_\flat^{\mu\nu;\alpha}\phi_{\alpha\beta;}{}^\beta - 2h_\flat^{\alpha(\mu}\phi^{\nu)\beta}{}_{;\alpha\beta} + 2\phi^{\alpha(\mu}h_\flat^{\nu)}{}_{\alpha;\beta}{}^\beta \\
&- \phi^\alpha{}_\alpha\Big(h_\flat^{\beta(\mu;\nu)}{}_\beta - \tfrac{1}{2}h_\flat^{\mu\nu;\beta}{}_\beta\Big) - 2\phi^{\alpha(\mu}h^\flat_\alpha{}^{|\beta|;\nu)}{}_\beta + \frac{1}{2}\phi^{\mu\nu}{}_{;\beta}h^\flat_\alpha{}^{\alpha;\beta} + \phi_\beta{}^{\beta;(\mu}h^\flat_\alpha{}^{|\alpha|;\nu)} \\
&+ \phi^{\alpha\beta;(\mu}h^\flat_{\alpha\beta;}{}^{\nu)} - \phi^{\beta(\mu}{}_{;\beta}h^\flat_\alpha{}^{|\alpha|;\nu)} - 2\phi_{\alpha\beta;}{}^\beta h_\flat^{\alpha(\mu;\nu)} \\
&+ 4\operatorname*{Sym}_{\mu\nu}\Big(h_\flat^{\mu\alpha}\phi_\alpha{}^{[\nu;\beta]}{}_\beta + \phi^\mu{}_{[\alpha;\beta]}h_\flat^{\nu\alpha;\beta}\Big)\Big]. \qquad (2.28)
\end{aligned}
$$

As mentioned at the end of Ch. 2.1.1, it is generally not true that one can raise and lower indices on perturbed tensors with the background metric. However, one can move indices by perturbing the metric contracted with our perturbed tensor. For example,

$$\mathsf{G}_\mu{}^\nu[g+h] = \delta(g_{\mu\rho}G^{\rho\nu})[h] + \delta^2(g_{\mu\rho}G^{\rho\nu})[h,h] + \mathcal{O}\Big(|h|^3\Big), \qquad (2.29)$$

where

$$\delta(g_{\mu\rho}G^{\rho\nu})[h] = g_{\mu\rho}\delta G^{\rho\nu}[h], \qquad (2.30)$$

$$\delta^2(g_{\mu\rho}G^{\rho\nu})[h,h] = g_{\mu\rho}\delta^2 G^{\rho\nu}[h,h] + h_{\mu\rho}\delta G^{\rho\nu}[h], \qquad (2.31)$$

and

$$\mathsf{G}_{\mu\nu}[g+h] = \delta(g_{\mu\rho}g_{\nu\sigma}G^{\rho\sigma})[h] + \delta^2(g_{\mu\rho}g_{\nu\sigma}G^{\rho\sigma})[h,h] + \mathcal{O}\Big(|h|^3\Big), \qquad (2.32)$$

where

$$\delta(g_{\mu\rho}g_{\nu\sigma}G^{\rho\sigma})[h] = g_{\mu\rho}g_{\nu\sigma}\delta G^{\rho\sigma}[h], \qquad (2.33)$$

$$\delta^2(g_{\mu\rho}g_{\nu\sigma}G_{\rho\sigma})[h,h] = g_{\mu\rho}g_{\nu\sigma}\delta^2 G^{\rho\sigma}[h,h] + 2h_{\rho(\mu}g_{\nu)\sigma}\delta G^{\rho\sigma}[h]. \qquad (2.34)$$

These expressions can be used for any symmetric rank-2 tensor which vanishes on the background.

2.1.3 Gauge freedom in perturbation theory

In this section, we outline the concept of gauge freedom in perturbation theory in general relativity. Perturbations are inherently tied to gauge choices in GR [153] and this fact can be exploited when performing calculations using perturbed quantities. Different gauges may have properties that make them more amenable to certain calculations than others. For example, the highly regular gauge mentioned in the introduction (and which will be expanded on in upcoming chapters) has the benefit of reducing the singular nature of the second-order source to the EFEs, allowing one to rigorously derive the second-order stress-energy tensor. Other gauges have been chosen for other useful properties: the Lorenz gauge is often chosen as it reduces the perturbed Einstein equations into a sequence of hyperbolic wave equations [15] whereas the radiation gauge [154] allows one to reconstruct the first-order metric perturbations from certain scalar fields.

It is first important to note that gauge freedom in perturbation theory is distinct from gauge freedom in the full theory of general relativity. In full GR, the gauge freedom is given by the group of diffeomorphisms [20, Ch. 10.2 & App. C]. If we have two manifolds, $\mathcal{R}$ and $\mathcal{S}$, then a smooth map, $\phi : \mathcal{R} \to \mathcal{S}$, is diffeomorphic if it is bijective with a smooth inverse. The existence of a diffeomorphism between two manifolds implies that they have the same structure and, thinking in physical terms, mean that they represent the same spacetime geometry. This is equivalent to saying that if the components of two metrics can be related via a coordinate transformation, then they physically represent the same solution to the Einstein field equations.

In contrast, gauge freedom in perturbation theory is the freedom to choose the map between points in the background spacetime and points in the perturbed spacetime [153]. There are two ways to think of gauge transformations in perturbation theory in general relativity, either the *active view* or the *passive view* [139]. The passive view treats gauge transformations as infinitesimal changes in coordinates. This means that coordinates at a point q, $x^\mu(q)$, transform as [139]

$$x^\mu(q) \to x'^\mu(q) = x^\mu(q) - \epsilon\xi_1^\mu - \epsilon^2\left(\xi_2^\mu - \frac{1}{2}\xi_1^\nu\partial_\nu\xi_1^\mu\right) + \mathcal{O}\left(\epsilon^3\right), \tag{2.35}$$

where ξ_n are independent vector fields [153].

However, we will make most use of the active view. As stated in Ch. 2.1, we are looking at small perturbations or fluctuations away from a known, background spacetime. Strictly speaking, however, we are considering a family of metrics parametrised by ϵ [139]. We can think of this as 'adding' an extra dimension to our manifold, so that along with our usual spacetime dimensions, we have a parameter that tells us 'how close' we are to our background spacetime. More concretely, we are looking at a five-dimensional manifold, $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, where the value on $\mathbb{R}$ is given by ϵ, which picks out a certain 'copy' of $\mathcal{M}$, labelled as $\mathcal{M}_\epsilon$ [153].

We refer to a *gauge choice* as the choice of identification map between our background spacetime and the physical spacetime, which we denote $\phi_\epsilon^X : \mathcal{M}_0 \to \mathcal{M}_\epsilon$, where ϕ_ϵ^X is generated by the vector field X and is a diffeomorphism forming a one-parameter group [139, 153]. If $A^{\mu\cdots}_{\nu\ldots}(q)$ is an arbitrary tensor on $\mathcal{M}_\epsilon$ at point q, then we can construct a tensor on $\mathcal{M}_0$ using the pullback by our identification map, $\phi_\epsilon^{X*}A^{\mu\cdots}_{\nu\ldots}(p)$, where $p = \phi_\epsilon^{X*}(q)$ is a point on $\mathcal{M}_0$. In fact, we can approximate $\phi_\epsilon^{X*}A^{\mu\cdots}_{\nu\ldots}$ at p by using a Taylor expansion around $A^{\mu\cdots}_{\nu\ldots}$ [139, 153],

$$\begin{aligned}\phi_\epsilon^{X*}A^{\mu\cdots}_{\nu\ldots}(p) &= \sum_{n\geq 0}\frac{\epsilon^n}{n!}\frac{d^n}{d\epsilon^n}\left(\phi_\epsilon^{X*}A^{\mu\cdots}_{\nu\ldots}\right)\Big|_{\epsilon=0}(p)\\ &= \sum_{n\geq 0}\frac{\epsilon^n}{n!}\mathcal{L}_X^n\left(A^{\mu\cdots}_{\nu\ldots}\right)(p),\end{aligned} \tag{2.36}$$

where the first line follows from assuming $A^{\mu\cdots}_{\nu\cdots}$ is analytic and the second line from using the normal definition of the Lie derivative and proof by induction. We can then define the nth-order perturbation to be [139]

$$A^{(X)\mu\ldots}_{(n)\ \nu\ldots}(p) := \frac{1}{n!}\mathcal{L}^n_X(A^{\mu\cdots}_{\nu\cdots})(p), \tag{2.37}$$

where all $A^{(X)\mu\ldots}_{(n)\ \nu\ldots}$ are defined on $\mathcal{M}_0$, so that

$$\phi^{X*}_\epsilon A^{\mu\cdots}_{\nu\cdots}(p) = \sum_{n\geq 0}\epsilon^n A^{(X)\mu\ldots}_{(n)\ \nu\ldots}(p), \tag{2.38}$$

where $A^{(X)\mu\ldots}_{(0)\ \nu\ldots} := A^{\mu\cdots}_{\nu\cdots}|_{\epsilon=0}$.

A natural question to ask is what happens if we wish to work with a different identification map, say ϕ^Y_ϵ, defined the same way as before but generated by a vector field Y? This will take our point in our background spacetime, p, to a different point on $\mathcal{M}_\epsilon$, $q' = \phi^Y_\epsilon(p)$. Moving from one map to another is known as a *gauge transformation*. We define the gauge transformation of an nth-order perturbation as [139]

$$\Delta A_{(n)\nu\ldots}^{\ \ \ \mu\ldots}(p) = \frac{1}{n!}\Big(A^{(Y)\mu\ldots}_{(n)\ \nu\ldots} - A^{(X)\mu\ldots}_{(n)\ \nu\ldots}\Big)(p), \tag{2.39}$$

Eq. (4.6) from Ref. [153] gives an explicit generating formula for gauge transformations as the difference of two pullbacks, that being[1]

$$\begin{aligned}\phi^{Y*}_\epsilon A^{\mu\cdots}_{\nu\cdots}(p) - \phi^{X*}_\epsilon A^{\mu\cdots}_{\nu\cdots}(p) &= \epsilon\mathcal{L}_{\xi_1}(\phi^{X*}_\epsilon A^{\mu\cdots}_{\nu\cdots})(p)\\ &+ \frac{\epsilon^2}{2}\Big(\mathcal{L}^2_{\xi_1} + 2\mathcal{L}_{\xi_2}\Big)\Big(\phi^{X*}_\epsilon A^{\mu\cdots}_{\nu\cdots}\Big)(p) + \mathcal{O}\Big(\epsilon^3\Big),\end{aligned} \tag{2.40}$$

up to second order, where [139]

$$\xi_1 = Y - X \tag{2.41}$$

$$\xi_2 = \frac{1}{2}[X, Y]. \tag{2.42}$$

We then substitute (2.38) into (2.40), equate order-by-order in ϵ and, by using (2.39), we see that

$$\Delta A_{(1)\nu\ldots}^{\ \ \ \mu\ldots}(p) = \mathcal{L}_{\xi_1}A^{\mu\cdots}_{\nu\cdots}(p), \tag{2.43}$$

$$\Delta A_{(2)\nu\ldots}^{\ \ \ \mu\ldots}(p) = \frac{1}{2}\Big(\mathcal{L}^2_{\xi_1} + 2\mathcal{L}_{\xi_2}\Big)A^{\mu\cdots}_{\nu\cdots}(p) + \mathcal{L}_{\xi_1}A^{(X)\mu\ldots}_{(1)\ \nu\ldots}. \tag{2.44}$$

[1]The factor of 2 before $\mathcal{L}_{\xi_2}$ in our equation comes from differing conventions for ξ_2.

2.1.4 Matched asymptotic expansions

The use of matched asymptotic expansions [155] is key in the analysis of gravitational self-force and has been used to derive a number of famous results such as the MiSaTaQuWa equation (1.15). A regular asymptotic expansion of a function $f(x, \epsilon)$ is defined as [155]

$$f(x;\epsilon) = \sum_{n=0}^{N} \phi_n(\epsilon) f_n(x) + \mathcal{O}(\phi_N) \quad \text{as} \quad \epsilon \to 0, \tag{2.45}$$

where $\phi_n(\epsilon)$ is an element of an asymptotic sequence satisfying

$$\phi_{n+1}(\epsilon) = o(\phi_n(\epsilon)) \quad \text{as} \quad \epsilon \to 0. \tag{2.46}$$

However, there may be cases (such as in the self-consistent formalism in Ch. 3.1) where we wish to keep some ϵ dependence inside of our expansion coefficients. Thus we write [155, 156]

$$f(x,\epsilon) = \sum_{n=0}^{N} \phi_n(\epsilon) f_n(x,\epsilon) + \mathcal{O}(\phi_N) \quad \text{as} \quad \epsilon \to 0, \tag{2.47}$$

where $f(x,\epsilon) = \mathcal{O}(1)$ and $f(x,\epsilon) \neq o(1)$. This is known as a *general asymptotic expansion*. In our case, we are identifying $f_n(x,\epsilon)$ with $h^n_{\mu\nu}(x;z)$ and our asymptotic sequence is the set $\{\epsilon^n\}$ with $n = 0, 1, 2, \ldots$.

As an example, we illustrate how the method can be used to find the form of the metric perturbations. Taking our small object to be in a spacetime described by the metric $\mathsf{g}_{\mu\nu}(x,\epsilon)$, we can expand the metric as

$$\mathsf{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + \epsilon h^1_{\mu\nu}(x,\epsilon) + \epsilon^2 h^2_{\mu\nu}(x,\epsilon) + \mathcal{O}\Big(\epsilon^3\Big), \tag{2.48}$$

where $g_{\mu\nu}$ is the external background vacuum metric that describes the spacetime as if the small object were not present. The perturbations are then caused by the presence and motion of the small object.

When expanding the metric in Eq. (2.48), we have assumed that the effect of the perturbations is small everywhere. However, if there is a region in which this is not true, then our expansion from Eq. (2.47) is no longer valid. This situation occurs in our EMRI system. To see this, consider the region of spacetime near to the small object. If we get near enough then the gravitational effects that we feel will be dominated by those caused by the small object. Intuitively this makes sense: if we consider the Earth orbiting the Sun then the gravity on Earth and nearby in space is primarily dictated by the Earth's gravitational field.

More mathematically [64], if the distance, r, to the object is $\sim \epsilon$, then any terms $\sim m/r$ reduce in order and become the same 'size' as the background metric. This causes the expansion in Eq. (2.48) to break down. To account for this, we introduce a second

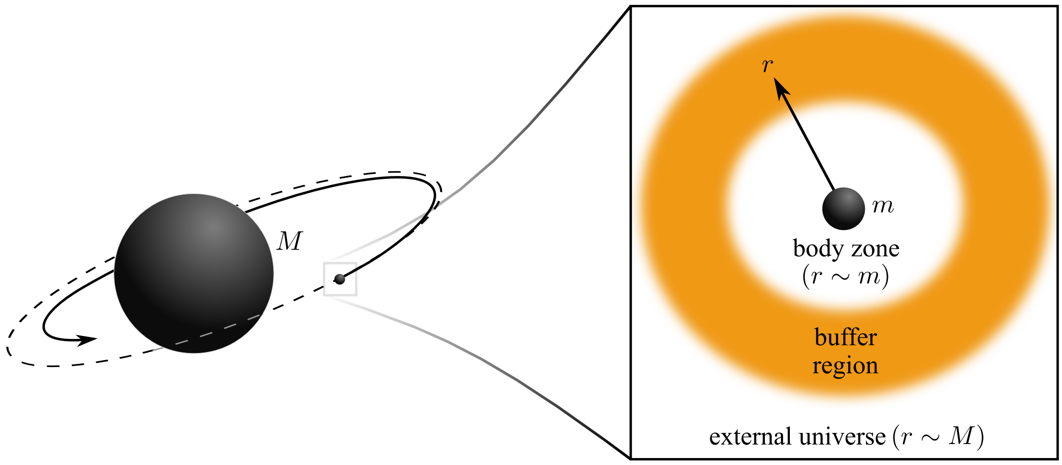

FIGURE 2.1: Diagram illustrating the buffer region in the case of a small-mass-ratio binary. We split up spacetime into three regions: one where the small object's gravity is dominant, the *body zone*; one in which the supermassive black hole's gravity dominates, the *external universe*; and the *buffer region* where the gravitational effect of both objects is significant. Reproduced from Ref. [15].

asymptotic expansion that uses a scaled distance,

$$\tilde{r} := \frac{r}{\epsilon}. \tag{2.49}$$

Now when we take the limit as $\epsilon \to 0$ at fixed $\tilde{r}$, we keep the scale of the small object fixed and send the external universe to infinity. This is in contrast to our original $\epsilon \to 0$ limit, which fixes the external universe and sends the size of the small object to zero.

In our new expansion near the small object, we rewrite our full spacetime metric so that [81]

$$\mathsf{g}_{\mu\nu}(r,\epsilon) = g^{\text{obj}}_{\mu\nu}(\tilde{r}) + \epsilon H^1_{\mu\nu}(\tilde{r}) + \epsilon^2 H^2_{\mu\nu}(\tilde{r}) + \mathcal{O}\left(\epsilon^3\right), \tag{2.50}$$

where $g^{\text{obj}}_{\mu\nu}$ is the metric the small object would have if it were isolated in spacetime. Both Eq. (2.48) and Eq. (2.50) are expansions of the same spacetime and we refer to them as the outer and inner expansions, respectively. This refers to where we are expanding around: either the large object or the small object. However, as they are expansions of the same spacetime, they must agree at some suitable length scale. We call this region the *buffer region* which is given by the range $\epsilon \ll r \ll 1$; see Fig. 2.1.

To perform the matching process, we re-expand both metrics in terms of r and ϵ and match the coefficients that appear at each order in r and ϵ. By requiring that the outer and inner expansion are well behaved (i.e. that there are no negative powers of ϵ in either expansion), we constrain the powers of r and $\tilde{r}$ that appear in our expansions. Following the argument in Ref. [81], if $h^n_{\mu\nu} = \sum r^p h^{n,p}_{\mu\nu}$, where the sum is over all p, then terms with $p < -n$ would have to match terms in the inner expansion with inverse powers of ϵ. These types of terms have been explicitly ruled out by imposing that our expansions be well behaved in the appropriate limits. This argument also applies to the inner expansions but with r^p replaced by $1/\tilde{r}^p$. Therefore, we find that the expansions in r and $\tilde{r}$ must be given by [81]

$$g_{\mu\nu} = \sum_{p\geq 0} r^p g^p_{\mu\nu}, \tag{2.51}$$

$$g^{\text{obj}}_{\mu\nu} = \sum_{p\geq 0} \frac{1}{\tilde{r}^p} g^{\text{obj},p}_{\mu\nu}, \tag{2.52}$$

$$h^n_{\mu\nu} = \sum_{p\geq -n} r^p h^{n,p}_{\mu\nu}, \tag{2.53}$$

$$H^n_{\mu\nu} = \sum_{p\geq -n} \frac{1}{\tilde{r}^p} H^{n,p}_{\mu\nu}, \tag{2.54}$$

where $\ln(r)$ terms may appear but have been absorbed in to the coefficients for visual clarity.

Which terms must match is best illustrated in a tableau form [157] as

$$\begin{array}{ccccccc} & & g^{\text{obj}} & \epsilon H^1 & \epsilon^2 H^2 & & \\ & & \wr & \wr & \wr & & \\ g & \sim & g^0 = g^{\text{obj},0} & r\left(g^1 = H^{1,-1}\right) & r^2\left(g^2 = H^{2,-2}\right) & \mathcal{O}(r^3), \\ \epsilon h^1 & \sim & \frac{\epsilon}{r}\left(h^{1,-1} = g^{\text{obj},1}\right) & \epsilon\left(h^{1,0} = H^{1,0}\right) & \epsilon r\left(h^{1,1} = H^{2,-1}\right) & \mathcal{O}(\epsilon r^2), \\ \epsilon^2 h^2 & \sim & \frac{\epsilon^2}{r^2}\left(h^{2,-2} = g^{\text{obj},2}\right) & \frac{\epsilon^2}{r}\left(h^{2,-1} = H^{1,1}\right) & \epsilon^2\left(h^{2,0} = H^{2,0}\right) & \mathcal{O}(\epsilon^2 r), \\ & & \mathcal{O}\left(\frac{\epsilon^3}{r^3}\right), & \mathcal{O}\left(\frac{\epsilon^3}{r^2}\right), & \mathcal{O}\left(\frac{\epsilon^3}{r}\right), & & \end{array} \tag{2.55}$$

where all terms are now re-expressed in terms of r and ϵ instead of $\tilde{r}$ and ϵ as in Eqs. (2.52) and (2.54). Thus we can see at what orders which perturbations must match with one another in order for the expansions to be consistent. As an example, we see from Eq. (2.52) that $g^{\text{obj}}_{\mu\nu}$ is asymptotically flat and thus can be written completely in terms of multipole moments [158, 159]. Broadly, this means that in the buffer region, it has the form [81]

$$g^{\text{obj}}_{\mu\nu} \sim 1 + \frac{\epsilon}{r} m + \frac{\epsilon^2}{r^2}\left(m^2 + M_i n^i + \epsilon_{ijk} S^j n^k\right) + \mathcal{O}\left(\frac{\epsilon^3}{r^3}\right), \tag{2.56}$$

where m/S^i is the Arnowitt–Deser–Misner (ADM) mass/angular momentum of $g^{\text{obj}}_{\mu\nu}$ and M_i is its mass dipole moment. We see from Eq. (2.55) that this immediately constrains the forms of $h^{1,-1}_{\mu\nu}$ and $h^{2,-2}_{\mu\nu}$ to be

$$h^{1,-1}_{\mu\nu} \sim \frac{m}{r}, \tag{2.57}$$

$$h^{2,-2}_{\mu\nu} \sim \frac{m^2 + M_i n^i + \epsilon_{ijk} S^j n^k}{r^2}, \tag{2.58}$$

meaning that the leading-order behaviour of the perturbations at each order in ϵ in the outer expansion is fully determined by the multipole moments of the small object. There are two potential next steps to find the form of the perturbations: one may solve the vacuum Einstein field equations order-by-order in ϵ and r for the perturbations in the outer expansion, $h^n_{\mu\nu}$ [72, 87]; or one could solve the field equations in the inner expansion for $H^n_{\mu\nu}$ order-by-order and then express this in the buffer region in terms of the outer expansion [81]. This second option will be discussed further in Ch. 3.4 when discussing the metric perturbations in a lightcone rest gauge.

2.2 Local expansion methods

In this section, we outline the methods of performing covariant and coordinate expansions of tensorial quantities to analyse their behaviour off the worldline in terms of quantities defined on the worldline. We also detail a useful coordinate system known as Fermi–Walker coordinates that we will make heavy use of throughout the thesis. The descriptions of covariant expansions using bitensors and of Fermi–Walker coordinates follows that of Ref. [63].

2.2.1 Covariant and coordinate expansions

As stated previously, we are interested in analysing the behaviour of tensor fields near to the worldline of the small object in our EMRI system. To do so, we require a method to express tensors evaluated at points in the field away from the worldline, x^μ, in terms of quantities evaluated at points on the worldline, $x^{\mu'}$. We assume that x^μ and $x^{\mu'}$ are sufficiently close and are linked by a unique geodesic, β. While the motivation is inspecting the behaviour of fields near the small object in EMRIs, these methods are entirely generic to fields near a worldline.

We first outline how one may perform a covariant expansion of tensor fields near a worldline in Ch. 2.2.1.1 following the explanation in Ref. [63]. In Ch. 2.2.1.2, we show how one may then write these expansions in a generic coordinate form and give the example of the expansion of the Detweiler–Whiting singular scalar field in the Schwarzschild spacetime, given in Ref. [138]. The Detweiler–Whiting singular scalar field, Φ^S, is the scalar field analogue of the Detweiler–Whiting singular gravitational field, $h^{\mathrm{S1}}_{\mu\nu}$, discussed in Ch. 1.4.1 [80]. The coordinate expansion methods have been used previously in, e.g. Refs. [138, 160], however, we present the coordinate expansion one order in distance higher than the current state of the art [138].

2.2.1.1 Covariant expansions using bitensors

In this section, we outline how one may construct local covariant expansions of tensor fields. Our explanation of the method follows that of Refs. [63, Part I, 161, Ch. 2, 162]. To do this, we introduce the concept of a bitensor, a tensor which is a function of two spacetime points. One important bitensor that we will make extensive use of is Synge's world function [63, Ch. 3, 161, Ch. 2],

$$\sigma(x,x') = \frac{\varepsilon}{2}\left(\int_\beta ds\right)^2, \tag{2.59}$$

where β is the unique geodesic connecting x^μ and $x^{\mu'}$, s is an affine parameter and $\varepsilon = \mp 1$ for time/spacelike geodesics. This gives half the geodesic distance squared between the points x^μ and $x^{\mu'}$.

We denote derivatives of Synge's world function as $\sigma_{\mu'} := \nabla_{\mu'}\sigma(x,x') = \partial_{\mu'}\sigma(x,x')$. Note also that we may take derivatives of Synge's world function at the unprimed coordinates as well, giving $\sigma_\mu := \nabla_\mu\sigma(x,x') = \partial_\mu\sigma(x,x')$. This can be generalised to higher and higher derivatives, e.g. $\sigma_{\mu'\nu'} := \nabla_{\nu'}\nabla_{\mu'}\sigma$ or $\sigma_{\mu'\nu} := \nabla_\nu\nabla_{\mu'}\sigma$. The indices of σ tell us its tensorial structure at both x^μ and $x^{\mu'}$, that is, $\sigma_{\mu'\nu'}$ is a rank-2 tensor at $x^{\mu'}$ but a scalar at x^μ. Likewise, $\sigma_{\mu'\nu}$ is a covector at both x^μ and $x^{\mu'}$. This property demonstrates that we can always commute primed and unprimed indices as the existence of one does not affect the tensorial rank at the other point.

Derivatives of Synge's world function also satisfy the useful identity

$$g_{\alpha\beta}\sigma^\alpha\sigma^\beta = g_{\alpha'\beta'}\sigma^{\alpha'}\sigma^{\beta'} = 2\sigma. \tag{2.60}$$

By taking derivatives of Eq. (2.60) and then the limit as x^μ goes to $x^{\mu'}$, one may derive local covariant expansions of $\sigma_{\alpha'\ldots\alpha\ldots}$ in terms of quantities defined on the worldline. To see an example, we start by introducing the standard notation for the coincidence limit [161],

$$[A^{\alpha\ldots\alpha'\ldots}_{\beta\ldots\beta'\ldots}] := \lim_{x^\mu\to x^{\mu'}} A^{\alpha\ldots\alpha'\ldots}_{\beta\ldots\beta'\ldots}(x,x'). \tag{2.61}$$

It immediately follows from Eqs. (2.59)–(2.60) that

$$[\sigma] = [\sigma_\alpha] = [\sigma_{\alpha'}] = 0. \tag{2.62}$$

as, if the length of β goes to 0, then the integral in (2.59) vanishes. Taking primed derivatives of Eq. (2.60), we see

$$\sigma_{\mu'} = \sigma^{\nu'}\sigma_{\nu'\mu'}, \tag{2.63}$$

which implies that

$$[\sigma_{\mu'\nu'}] = g_{\mu'\nu'}. \tag{2.64}$$

This can be repeated to find higher and higher derivatives of $\sigma(x,x')$ [162],

$$[\sigma_{\mu'\nu'\rho'}] = 0, \tag{2.65}$$

$$[\sigma_{\mu'\nu'\alpha'\beta'}] = \frac{2}{3}R_{\mu'(\alpha'\beta')\nu'}. \tag{2.66}$$

Another object we will use is that of the parallel propagator, $g^{\mu'}{}_{\mu}(x,x')$ [63, Ch. 5, 161, Ch. 2, 162]. The parallel propagator parallel transports a tensor from $x^{\mu'}$ to x^{μ} along β. For instance, the vector $A^{\mu}(x)$ can be transported from/to $A^{\mu'}(x')$ via

$$A^{\mu}(x) = g^{\mu}{}_{\mu'}(x,x')A^{\mu'}(x'), \tag{2.67}$$

$$A^{\mu'}(x') = g^{\mu'}{}_{\mu}(x',x)A^{\mu}(x), \tag{2.68}$$

respectively. These expressions hold for covectors as well and tensors with any number of indices with the inclusion of an appropriate number of parallel propagators, e.g.

$$A^{\alpha\beta}{}_{\mu}{}^{\nu}(x) = g^{\alpha}{}_{\alpha'}g^{\beta}{}_{\beta'}g^{\mu'}{}_{\mu}g^{\nu}{}_{\nu'}A^{\alpha'\beta'}{}_{\mu'}{}^{\nu'}(x'). \tag{2.69}$$

It also has the properties that when contracted with itself, it returns the Kronecker delta,

$$g^{\mu}{}_{\mu'}g^{\mu'}{}_{\nu} = \delta^{\mu}_{\nu}, \tag{2.70}$$

and is symmetric in indices and arguments,

$$g_{\mu}{}^{\mu'}(x,x') = g^{\mu'}{}_{\mu}(x',x). \tag{2.71}$$

This allows us to write the parallel propagator as $g^{\mu'}_{\mu}$ as the actual index order is not relevant. When contracted with Synge's world function, it gives

$$\sigma_{\mu} = -g^{\mu'}{}_{\mu}\sigma_{\mu'}, \tag{2.72}$$

$$\sigma_{\mu'} = -g^{\mu}{}_{\mu'}\sigma_{\mu}, \tag{2.73}$$

and its derivative contracted with Synge's world function vanishes for all combinations of primed and unprimed indices,

$$g^{\mu'}{}_{\mu;\nu}\sigma^{\nu} = 0. \tag{2.74}$$

As we did for Synge's world function with Eq. (2.60), we can calculate different covariant expansions by repeatedly differentiating Eq. (2.74) and taking the coincidence limit. For example [162],

$$[g^{\mu}{}_{\nu'}] = \delta^{\mu'}_{\nu'}, \tag{2.75}$$

$$[g^{\mu}{}_{\nu';\alpha'}] = 0, \tag{2.76}$$

$$[g^{\mu}{}_{\nu';\alpha\beta}] = -\frac{1}{2}R^{\mu'}{}_{\nu'\alpha'\beta'}. \tag{2.77}$$

Combining the previous definitions, we can then express an arbitrary tensor $A^{\mu}{}_{\nu}$, evaluated at x, in terms of quantities evaluated at x' as

$$A^{\mu}{}_{\nu}(x) = g^{\mu}{}_{\mu'} g_{\nu}{}^{\nu'} \Big(A^{(0)\mu'}{}_{\nu'}(x') + \lambda A^{(1)\mu'}{}_{\nu'\alpha'}(x') \sigma^{\alpha'} + \frac{\lambda^2}{2} A^{(2)\mu'}{}_{\nu';\alpha'\beta'}(x') \sigma^{\alpha'} \sigma^{\beta'} \Big) + \mathcal{O}\Big(\lambda^3\Big), \quad (2.78)$$

where λ is a formal order counting parameter to be set to unity at the end of the calculation. The unknown coefficients, $A^{(N)\mu'}{}_{\nu'\alpha'_1 \ldots \alpha'_n}$, can be found in the same manner as before by repeated differentiation and taking of the coincidence limit. As an example, we seek the covariant expansion of $\sigma_{\mu'\nu'}$. We first expand, as in Eq. (2.78) but without the need for parallel propagators, as

$$\sigma_{\mu'\nu'} = \sigma^{(0)}_{\mu'\nu'} + \lambda \sigma^{(1)}_{\mu'\nu'\alpha'} \sigma^{\alpha'} + \frac{\lambda^2}{2} \sigma^{(2)}_{\mu'\nu'\alpha'\beta'} \sigma^{\alpha'} \sigma^{\beta'} + \mathcal{O}\Big(\lambda^3\Big). \quad (2.79)$$

We know from Eq. (2.64), that $A^{(0)}_{\mu'\nu'} = g_{\mu'\nu'}$. Taking a primed derivative and the coincidence limit gives that

$$\sigma^{(1)}_{\mu'\nu'\alpha'} = [\sigma_{\mu'\nu'\alpha'}] = 0, \quad (2.80)$$

$$\sigma^{(2)}_{\mu'\nu'\alpha'\beta'} = [\sigma_{\mu'\nu'\alpha'\beta'}] = \frac{2}{3} R_{\mu'(\alpha'\beta')\nu'}, \quad (2.81)$$

meaning that

$$\sigma_{\mu'\nu'} = g_{\mu'\nu'} + \frac{\lambda^2}{3} R_{\mu'\alpha'\beta'\nu'} \sigma^{\alpha'} \sigma^{\beta'} + \mathcal{O}\Big(\lambda^3\Big). \quad (2.82)$$

This can be repeated for any required covariant quantity. Ref. [163] provides a semi-recursive method for calculating expansions of Synge's world function and the parallel propagator, along with many other covariant quantities.

2.2.1.2 Coordinate expansions of covariant quantities

In order to implement the covariant expansions in a specific calculation, one must first write them in a chosen coordinate system. This necessitates re-expanding all the covariant quantities in terms of coordinate differences,

$$\Delta x^{\alpha'} := x^{\alpha} - x^{\alpha'}, \quad (2.83)$$

where $\Delta x^{\alpha'} \sim \lambda$. A derivative of $\Delta x^{\alpha'}$ at $x^{\mu'}$ then gives

$$\Delta x^{\alpha'}{}_{,\beta'} = -\delta^{\alpha'}_{\beta'}. \quad (2.84)$$

This leaves us with coefficients evaluated at $x^{\mu'}$, as in Eq. (2.78), contracted into certain combinations of $\Delta x^{\alpha'}$.

As stated in the introduction to this section, as an example of how to re-expand a covariant quantity into a generic expansion, we will re-expand the Detweiler–Whiting singular scalar field. This was presented in Ref. [138] through order λ^4 in both covariant form and in a specific coordinate system in the Schwarzschild spacetime but, to our knowledge, has never been presented explicitly as a generic coordinate expansion past order λ^2. Instead of satisfying the linearised Einstein field equations with point particle source, as in the gravitational case in Eq. (1.23), the Detweiler–Whiting singular scalar field satisfies the inhomogeneous scalar wave equation

$$\Box\Phi^S = -4\pi q \int \delta^4(x,x')\, d\tau\,, \tag{2.85}$$

where q is the scalar charge of the particle and $\delta^4(x,x')$ is the covariant Dirac delta function. As in the gravitational case, one can construct a regular field from this quantity,

$$\Phi^R := \Phi - \Phi^S, \tag{2.86}$$

which satisfies the homogeneous wave equation,

$$\Box\Phi^R = 0, \tag{2.87}$$

and can be used to calculate the scalar self-force,

$$F^\mu = q\nabla^\mu\Phi^R. \tag{2.88}$$

Detweiler and Whiting [80] demonstrated that the scalar singular field can be written in terms of certain bitensors, $U(x,x')$ and $V(x,x')$, as

$$\Phi^S = \frac{q}{2}\left[\frac{U(x,x'(\tau))}{u^{\mu'}\sigma_{\mu'}}\right]_{\tau_-}^{\tau_+} + \frac{q}{2}\int_{\tau_-}^{\tau_+} V(x,x'(\tau))\, d\tau\,, \tag{2.89}$$

where $\tau_\pm$ is the proper time that the worldline of the small object intersects the future/past lightcones of the point x^μ. The specific forms of $U(x,x')$ and $V(x,x')$ need not be considered as Heffernan et al. [138] derived a local covariant expansion of Eq. (2.89), using the techniques illustrated in the previous chapter (along with those of Ref. [164]). This is provided in Eqs (4.1)–(4.3) of Ref. [138] and is given by

$$\begin{aligned}\Phi = q\bigg(&\frac{1}{\lambda\boldsymbol{\rho}} + \lambda\frac{\boldsymbol{r}^2-\boldsymbol{\rho}^2}{6\boldsymbol{\rho}^3}R_{u\sigma u\sigma} + \frac{\lambda^2}{24\boldsymbol{\rho}^3}\Big[(\boldsymbol{r}^2-3\boldsymbol{\rho}^2)\boldsymbol{r}\dot{R}_{u\sigma u\sigma} - (\boldsymbol{r}^2-\boldsymbol{\rho}^2)R_{u\sigma u\sigma;\sigma}\Big]\\ &+ \frac{\lambda^3}{360\boldsymbol{\rho}^5}\Phi^{(3)}\bigg) + \mathcal{O}\big(\lambda^4\big),\end{aligned} \tag{2.90}$$

where

$$
\begin{aligned}
\boldsymbol{\Phi}^{(3)} = {}& 15(\boldsymbol{r}^2-\boldsymbol{\rho}^2)^2 R_{u\sigma u\sigma}R_{u\sigma u\sigma} + \boldsymbol{\rho}^2\Big[(\boldsymbol{r}^2-\boldsymbol{\rho}^2)(3R_{u\sigma u\sigma;\sigma\sigma}+4R_{u\sigma\sigma\alpha'}R_{u\sigma\sigma}{}^{\alpha'}) \\
&+(\boldsymbol{r}^4-6\boldsymbol{r}^2\boldsymbol{\rho}^2-3\boldsymbol{\rho}^4)(4R_{u\sigma u\alpha'}R_{u\sigma u}{}^{\alpha'}+3\ddot{R}_{u\sigma u\sigma})+\boldsymbol{r}(\boldsymbol{r}^2-3\boldsymbol{\rho}^2) \\
&\times(16R_{u\sigma u}{}^{\alpha'}R_{u\sigma\sigma\alpha'}-3R_{u\sigma u\sigma;u\sigma})\Big]+\boldsymbol{\rho}^4\Big(2R_u{}^{\alpha'}{}_u{}^{\beta'}\Big[(\boldsymbol{r}^2+\boldsymbol{\rho}^2)R_{\sigma\alpha'\sigma\beta'} \\
&+2\boldsymbol{r}(\boldsymbol{r}^2+3\boldsymbol{\rho}^2)R_{u\alpha'\sigma\beta'}\Big]+2R_u{}^{\alpha'}{}_\sigma{}^{\beta'}\Big[2\boldsymbol{r}R_{\sigma\alpha'\sigma\beta'}+(\boldsymbol{r}^2+\boldsymbol{\rho}^2)R_{u\beta'\sigma\alpha'}\Big] \\
&+(\boldsymbol{r}^4+6\boldsymbol{r}^2\boldsymbol{\rho}^2+\boldsymbol{\rho}^4)R_{u\alpha' u\beta'}R_u{}^{\alpha'}{}_u{}^{\beta'}+2(\boldsymbol{r}^2+\boldsymbol{\rho}^2)R_{u\alpha'\sigma\beta'}R_u{}^{\alpha'}{}_\sigma{}^{\beta'} \\
&+R_{\sigma\alpha'\sigma\beta'}R_\sigma{}^{\alpha'}{}_\sigma{}^{\beta'}\Big).
\end{aligned} \tag{2.91}
$$

Ref. [138] also provides the order-λ^4 term in the covariant expansion but we do not consider that for our coordinate expansion.

In this example, we push the generic coordinate expansion of the covariant scalar singular field expression one order higher to order λ^3. To do so requires us to perform a coordinate expansion of a derivative of Synge's world function through $\mathcal{O}(\lambda^5)$, one order higher than the current state of the art [138]. Additionally, while it is not required for an expansion of the scalar field, we also calculate a coordinate expansion of the parallel propagator as it is required for work later in the thesis.

To find a coordinate expansion of Synge's world function, we exploit the fact that it satisfies the identity from Eq. (2.60). We make the following ansatz as an expansion for Synge's world function,

$$
\begin{aligned}
\sigma(x,x') = {}& \sum_{n=2}^{\infty}\lambda^n A^{(n-1)}_{\alpha'_1\cdots\alpha'_n}(x')\Delta x^{\alpha'_1}\cdots\Delta x^{\alpha'_n} \\
={}& \lambda^2 A^{(1)}_{\alpha'\beta'}(x')\Delta x^{\alpha'}\Delta x^{\beta'}+\lambda^3 A^{(2)}_{\alpha'\beta'\gamma'}(x')\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'} \\
&+\lambda^4 A^{(3)}_{\alpha'\beta'\gamma'\mu'}(x')\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\mu'} \\
&+\lambda^5 A^{(4)}_{\alpha'\beta'\gamma'\mu'\nu'}(x')\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\mu'}\Delta x^{\nu'} \\
&+\lambda^6 A^{(5)}_{\alpha'\beta'\gamma'\mu'\nu'\rho'}(x')\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\mu'}\Delta x^{\nu'}\Delta x^{\rho'}+\mathcal{O}\Big(\lambda^7\Big);
\end{aligned} \tag{2.92}
$$

see Refs. [138, 160] for similar expansions but with different conventions for $\Delta x^{\alpha'}$. The primed derivative is then given by

$$
\begin{aligned}
\sigma_{\mu'}(x,x') = {}& -2\lambda A^{(1)}_{\mu'\alpha'}\Delta x^{\alpha'}+\lambda^2\Big(A^{(1)}_{\alpha'\beta',\mu'}-3A^{(2)}_{\mu'\alpha'\beta'}\Big)\Delta x^{\alpha'}\Delta x^{\beta'} \\
&+\lambda^3\Big(A^{(2)}_{\alpha'\beta'\gamma',\mu'}-4A^{(3)}_{\mu'\alpha'\beta'\gamma'}\Big)\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'} \\
&+\lambda^4\Big(A^{(3)}_{\alpha'\beta'\gamma'\delta',\mu'}-5A^{(4)}_{\mu'\alpha'\beta'\gamma'\delta'}\Big)\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\delta'} \\
&+\lambda^5\Big(A^{(4)}_{\alpha'\beta'\gamma'\delta'\iota',\mu'}-6A^{(5)}_{\mu'\alpha'\beta'\gamma'\delta'\iota'}\Big)\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\delta'}\Delta x^{\iota'}+\mathcal{O}\Big(\lambda^6\Big),
\end{aligned} \tag{2.93}
$$

We then substitute Eqs. (2.92)–(2.93) into the identity for Synge's world function from Eq. (2.60) and solve order-by-order. The expressions for $A^{(n)}_{\alpha'^1\cdots\alpha'^n}$ are

$$A^{(1)}_{\alpha'\beta'} = \frac{1}{2} g_{\alpha'\beta'}, \tag{2.94a}$$

$$A^{(2)}_{\alpha'\beta'\gamma'} = \frac{1}{2} g_{\delta'(\alpha'}\Gamma^{\delta'}_{\beta'\gamma')}, \tag{2.94b}$$

$$\begin{aligned} A^{(3)}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{72}\Big(& R_{\alpha'[\gamma'\delta']\beta'} + 3g_{\alpha'\iota'}\Gamma^{\iota'}_{\gamma'\delta',\beta'} + 9g_{\iota'(\beta'}\Gamma^{\iota'}_{\gamma'\delta'),\alpha'} + 9g_{\iota'\mu'}\Gamma^{\iota'}_{\alpha'(\beta'}\Gamma^{\mu'}_{\gamma'\delta')} \\ & + 6g_{\mu'(\alpha'}\Gamma^{\mu'}_{\beta')\iota'}\Gamma^{\iota'}_{\gamma'\delta'} + 6g_{\mu'(\gamma'}\Gamma^{\iota'}_{\delta')\beta'}\Gamma^{\mu'}_{\alpha'\iota'}\Big), \end{aligned} \tag{2.94c}$$

$$\begin{aligned} A^{(4)}_{\alpha'\beta'\gamma'\delta'\iota'} = \frac{1}{120}\Big(& 5g_{(\alpha'|\rho'|}\Gamma^{\rho'}_{\delta'\iota',\gamma'\beta')} + 5\Gamma^{\rho'}_{(\alpha'\beta'}g_{\gamma'|\kappa'|}\Gamma^{\kappa'}_{\iota'|\rho'|,\delta')} + 10\Gamma^{\rho'}_{(\alpha'\beta'}g_{|\rho'\kappa'|}\Gamma^{\kappa'}_{\delta'\iota',\gamma')} \\ & + 10\Gamma^{\rho'}_{(\alpha'|\kappa'|}g_{\beta'|\rho'|}\partial_{\gamma'}\Gamma^{\kappa'}_{\delta'\iota')} + 3\Gamma^{\rho'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\mu'}_{\iota')\rho'}g_{\kappa'\mu'} + 7\Gamma^{\rho'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\mu'}_{\iota')\kappa'}g_{\rho'\mu'} \\ & + 5\Gamma^{\rho'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\mu'|}\Gamma^{\mu'}_{\delta'|\rho'|}g_{\iota')\kappa'}\Big) \end{aligned} \tag{2.94d}$$

$$\begin{aligned} A^{(5)}_{\alpha'\beta'\gamma'\delta'\iota'\mu'} = \frac{1}{7200}\Big(& 60g_{(\alpha'|\zeta'|}\Gamma^{\zeta'}_{\iota'\mu',\delta'\gamma'\beta')} - 90\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}R_{\iota'|\zeta'|\mu')\kappa'} \\ & + 120\Gamma^{\zeta'}_{(\alpha'\beta'}g_{\gamma'|\kappa'|}\Gamma^{\kappa'}_{\mu'|\zeta'|,\iota'\delta')} - 60\Gamma^{\zeta'}_{(\alpha'\beta'}g_{\gamma'|\kappa'}\Gamma^{\kappa'}_{\iota'\mu',\delta')\zeta'} \\ & + 150\Gamma^{\zeta'}_{(\alpha'\beta'}g_{|\zeta'\kappa'|}\Gamma^{\kappa'}_{\iota'\mu',\delta'\gamma')} + 180\Gamma^{\zeta'}_{(\alpha'|\kappa'|}g_{\beta'|\zeta'|}\Gamma^{\kappa'}_{\iota'\mu',\delta'\gamma')} \\ & + 180g_{(\alpha'|\zeta'|}\Gamma^{\kappa'}_{\gamma'\delta',\beta'}\Gamma^{\zeta'}_{\mu'|\kappa'|,\iota')} + 100g_{\zeta'\kappa'}\Gamma^{\zeta'}_{\beta'\gamma',\alpha'}\Gamma^{\kappa'}_{\iota'\mu',\delta')} \\ & - 60\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}g_{\iota'|\nu'|}\Gamma^{\nu'}_{|\zeta'\kappa'|,\mu')} - 3\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}g_{\iota'|\nu'|}\Gamma^{\nu'}_{\mu')\kappa',\zeta'} \\ & + 3\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}g_{\iota'|\nu'|}\Gamma^{\nu'}_{\mu')\zeta',\kappa'} + 105\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}g_{|\zeta'\nu'|}\Gamma^{\nu'}{}_{\mu'|\kappa'|,\iota')} \\ & + 45\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}{}_{\gamma'\delta'}g_{|\kappa'\nu'|}\Gamma^{\nu'}_{\mu'|\zeta'|,\iota')} + 120\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}{}_{\gamma'|\zeta'|}g_{\delta'|\nu'|}\Gamma^{\nu'}_{\mu'|\kappa'|,\iota')} \\ & + 180\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\nu'|}g_{\delta'|\kappa'|}\Gamma^{\nu'}_{\mu'|\zeta'|,\iota')} + 200\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\zeta'}g_{\kappa'\nu'|}\Gamma^{\nu'}{}_{\iota'\mu',\delta')} \\ & + 300\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\nu'}g_{\zeta'\kappa'|}\Gamma^{\nu'}_{\iota'\mu',\delta')} - 60\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}{}_{|\zeta'\nu'|}g_{\gamma'|\kappa'|}\Gamma^{\nu'}{}_{\iota'\mu',\delta')} \\ & + 180\Gamma^{\zeta'}_{(\alpha'|\kappa'|}\Gamma^{\kappa'}{}_{\beta'|\nu'|}g_{\gamma'|\zeta'|}\Gamma^{\nu'}_{\iota'\mu',\delta')} + 30\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\nu'}_{\iota'|\zeta'|}\Gamma^{\rho'}_{\mu')\kappa'}g_{\nu'\rho'} \\ & + 70\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\nu'}_{\iota'|\kappa'|}\Gamma^{\rho'}_{\mu')\zeta'}g_{\nu'\rho'} + 70\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\nu'}_{\iota'|\rho'|}\Gamma^{\rho'}_{\mu')\zeta'}g_{\kappa'\nu'} \\ & + 80\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\nu'}_{\iota'|\rho'|}\Gamma^{\rho'}_{\mu')\kappa'}g_{\zeta'\nu'} + 3\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\nu'}_{\iota'|\zeta'}\Gamma^{\rho'}_{\kappa'\nu'|}g_{\mu')\rho'} \\ & - 3\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'\delta'}\Gamma^{\nu'}_{\iota'|\kappa'}\Gamma^{\rho'}_{\zeta'\nu'|}g_{\mu')\rho'} - 48\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\zeta'|}\Gamma^{\nu'}_{\delta'|\rho'|}\Gamma^{\rho'}_{\iota'|\kappa'|}g_{\mu')\nu'} \\ & + 24\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\nu'|}\Gamma^{\nu'}_{\delta'|\zeta'|}\Gamma^{\rho'}_{\iota'|\kappa'|}g_{\mu')\rho'} + 84\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\nu'|}\Gamma^{\nu'}_{\delta'|\rho'|}\Gamma^{\rho'}_{\iota'|\zeta'|}g_{\mu')\kappa'} \\ & - 60\Gamma^{\zeta'}_{(\alpha'\beta'}\Gamma^{\kappa'}_{\gamma'|\nu'|}g_{\delta'|\kappa'|}g^{\nu'\rho'}R_{\iota'|\zeta'|\mu')\rho'}\Big). \end{aligned} \tag{2.94e}$$

These are similar to the expansions appearing in Refs. [160, Eq. (2.10), 138, Eq. (3.10)] but here, we have a slightly different definition for $\Delta x^{\alpha'}$ and we take the derivatives at $x^{\mu'}$ instead of x^{μ}. Taking the primed derivative of the appropriate quantities and then substituting these and Eq. (2.94) into Eq. (2.93) gives us the final expression for the coordinate expansion of Synge's world function,

$$\sigma_{\alpha'} = \sum_{n=1}^{\infty} \lambda^n \sigma^{(n)}_{\alpha'}, \tag{2.95}$$

where the first five orders are given by

$$\sigma^{(1)}_{\alpha'} = -\, g_{\alpha'\beta'}\Delta x^{\beta'}, \tag{2.96a}$$

$$\sigma^{(2)}_{\alpha'} = -\,\frac{1}{2} g_{\alpha'\delta'}\Gamma^{\delta'}_{\beta'\gamma'}\Delta x^{\beta'}\Delta x^{\gamma'}, \tag{2.96b}$$

$$\sigma^{(3)}_{\alpha'} = -\,\frac{1}{6}\Big(g_{\alpha'\iota'}\Gamma^{\iota'}_{\beta'\gamma',\delta'} + g_{\alpha'\mu'}\Gamma^{\iota'}_{\beta'\gamma'}\Gamma^{\mu'}_{\delta'\iota'}\Big)\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\delta'}, \tag{2.96c}$$

$$\begin{aligned}\sigma^{(4)}_{\alpha'} = &-\,\frac{1}{24}\Big[\Gamma^{\nu'}_{\beta'\gamma'}\Big(g_{\alpha'\mu'}\Gamma^{\kappa'}_{\delta'\nu'}\Gamma^{\mu'}_{\iota'\kappa'} - R_{\alpha'\delta'\iota'\nu'} + g_{\alpha'\kappa'}\Gamma^{\kappa'}_{\delta'\nu',\iota'}\Big)\\ &+ g_{\alpha'\nu'}\Big(2\Gamma^{\nu'}_{\beta'\kappa'}\Gamma^{\kappa'}_{\gamma'\delta',\iota'} + \Gamma^{\nu'}_{\beta'\gamma',\delta'\iota'}\Big)\Big]\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\delta'}\Delta x^{\iota'},\end{aligned} \tag{2.96d}$$

$$\begin{aligned}\sigma^{(5)}_{\alpha'} = &\,\frac{1}{360}\Big[9\Gamma^{\kappa'}_{\alpha'\mu'}\Gamma^{\nu'}_{\beta'\gamma'}g_{\delta'\kappa'}g^{\mu'\rho'}R_{\iota'\nu'\varsigma'\rho'} + 7R_{\alpha'\iota'\varsigma'\kappa'}\Gamma^{\kappa'}_{\beta'\gamma',\delta'} - 3g_{\alpha'\kappa'}\Big(3\Gamma^{\kappa'}_{\beta'\mu',\gamma'}\Gamma^{\mu'}_{\delta'\iota',\varsigma'}\\ &+ 3\Gamma^{\kappa'}_{\beta'\mu'}\Big[\Gamma^{\mu'}_{\gamma'\nu'}\Gamma^{\nu'}_{\delta'\iota',\varsigma'} + \Gamma^{\mu'}_{\gamma'\delta',\iota'\varsigma'}\Big] + \Gamma^{\kappa'}_{\beta'\gamma',\delta'\iota'\varsigma'}\Big) + 3\Gamma^{\kappa'}_{\alpha'\beta'}\Big(2\Gamma^{\mu'}_{\gamma'\nu'}g_{\delta'\mu'}\\ &\cdot\Big[g^{\nu'\rho'}R_{\iota'\kappa'\varsigma'\rho'} - \Gamma^{\nu'}_{\iota'\kappa',\varsigma'}\Big] - 2g_{\delta'\nu'}\Gamma^{\mu'}_{\gamma'\kappa'}\Gamma^{\nu'}_{\iota'\mu',\varsigma'} + \Gamma^{\mu'}_{\gamma'\delta'}\Big[5R_{\iota'\kappa'\varsigma'\mu'} + 2g_{\iota'\nu'}\Gamma^{\nu'}_{\kappa'\mu',\varsigma'}\Big]\\ &+ 2g_{\gamma'\mu'}\Big[\Gamma^{\mu'}_{\kappa'\nu'}\Gamma^{\nu'}_{\delta'\iota',\varsigma'} + 2\Gamma^{\mu'}_{\delta'[\iota',\kappa']\varsigma'}\Big]\Big) + \Gamma^{\kappa'}_{\beta'\gamma'}\Big(-\Gamma^{\mu'}_{\delta'\kappa'}\Big[3g_{\alpha'\rho'}\Gamma^{\nu'}_{\iota'\mu'}\Gamma^{\rho'}_{\varsigma'\nu'} - 10R_{\alpha'\iota'\varsigma'\mu'}\\ &+ 9g_{\iota'\nu'}\Gamma^{\nu'}_{\varsigma'\mu',\alpha'} + 6g_{\alpha'\nu'}\Gamma^{\nu'}_{\iota'\mu',\varsigma'}\Big] + 3\Gamma^{\mu'}_{\delta'\nu'}\Big[-3g_{\iota'\mu'}\Gamma^{\nu'}_{\varsigma'\kappa',\alpha'} + g_{\alpha'\mu'}\Big\{g^{\nu'\rho'}R_{\iota'\kappa'\varsigma'\rho'}\\ &- 3\Gamma^{\nu'}_{\iota'\kappa',\varsigma'}\Big\}\Big] + 3\Gamma^{\mu'}_{\delta'\iota'}\Big[4R_{\alpha'\kappa'\varsigma'\mu'} + 3g_{\varsigma'\nu'}\Gamma^{\nu'}_{\kappa'\mu',\alpha'} + g_{\alpha'\nu'}\Gamma^{\nu'}_{\kappa'\mu',\varsigma'}\Big] + 9R_{\alpha'\delta'\iota'\kappa';\varsigma'}\\ &+ 9g_{\delta'\mu'}\Big[\Gamma^{\mu'}_{\kappa'\nu'}\Gamma^{\nu'}_{\iota'\varsigma',\alpha'} + 2\Gamma^{\mu'}_{\iota'[\varsigma',\kappa']\alpha'}\Big] + 3g_{\alpha'\mu'}\Big[\Gamma^{\mu'}_{\kappa'\nu'}\Gamma^{\nu'}_{\delta'\iota',\varsigma'} - 2\Gamma^{\mu'}_{\delta'\kappa',\iota'\varsigma'} + \Gamma^{\mu'}_{\delta'\iota',\varsigma'\kappa'}\Big]\Big)\Big]\\ &\cdot\Delta x^{\beta'}\Delta x^{\gamma'}\Delta x^{\delta'}\Delta x^{\iota'}\Delta x^{\varsigma'}.\end{aligned} \tag{2.96e}$$

The expression for $\sigma^{(5)}_{\alpha'}$ appears here for the first time and is one order past the current state of the art. As a check on these expressions, we can substitute Eq. (2.96) into the identity for Synge's world function from Eq. (2.60). This has been correctly passed for the first four orders, but we have been unable to check the highest order due to the complexity of the expression that appears[2].

Before proceeding, we define

$$\boldsymbol{r} := u_{\mu'}\sigma^{\mu'}, \tag{2.97}$$

$$\boldsymbol{\rho} := \sqrt{P_{\mu'\nu'}\sigma^{\mu'}\sigma^{\nu'}}, \tag{2.98}$$

for notational simplicity[3]. Here, $P^{\mu\nu} := g^{\mu\nu} + u^{\mu}u^{\nu}$ is the projection operator. This means that

$$\sigma^{\mu'}\sigma_{\mu'} = 2\sigma(x, x') = \boldsymbol{\rho}^2 - \boldsymbol{r}^2. \tag{2.99}$$

Here, $\boldsymbol{r}$ gives a notion of the difference in proper time while $\boldsymbol{\rho}$ denotes a difference in proper distance. These two quantities appear in the scalar puncture field necessitating

[2]The resulting expansion of the scalar field in Eq. (2.105) uses Eq. (2.96e). The expression in Eq. (2.105) matches the expression derived using an alternative approach that has found the singular scalar field up to $\mathcal{O}(\lambda^{14})$ for circular orbits in the Schwarzschild spacetime [165]. The specific details of this method are not considered here but it leads us to believe that Eq. (2.96e) is correct.

[3]We use $\boldsymbol{r}$ in agreement with Refs. [96, 138, 164] but we use $\boldsymbol{\rho}$ to match Refs. [137, 143] instead of $\boldsymbol{s}$ as in Ref. [96].

us to find their coordinate expansions. The expression for $\boldsymbol{r}$ is trivial as it just requires us to contract the four-velocity into Eq. (2.95), so that, at leading order,

$$\boldsymbol{r} = -\lambda \boldsymbol{r}_0 + \mathcal{O}\Big(\lambda^2\Big). \tag{2.100}$$

where, in analogy with Eq. (2.97), we define the four-velocity contracted with the coordinate difference as

$$\boldsymbol{r}_0 \coloneqq u_{\mu'} \Delta x^{\mu'}, \tag{2.101}$$

We write the expansion of $\boldsymbol{\rho}$ as a power series,

$$\boldsymbol{\rho} = \sum_{n=1}^{\infty} \lambda^n \boldsymbol{\rho}^{(n)}, \tag{2.102}$$

and define

$$\boldsymbol{\rho}_0 \coloneqq \sqrt{P_{\mu'\nu'} \Delta x^{\mu'} \Delta x^{\nu'}}. \tag{2.103}$$

We then proceed to substitute our coordinate expansion for $\sigma_{\alpha'}$ from Eq. (2.95) into the definition for $\boldsymbol{\rho}$ from Eq. (2.102) and collect terms at each order in λ. The first five orders of the expansion are given by

$$\boldsymbol{\rho}^{(1)} = \boldsymbol{\rho}_0, \tag{2.104a}$$

$$\boldsymbol{\rho}^{(2)} = \frac{1}{2\boldsymbol{\rho}_0}\Big(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Big), \tag{2.104b}$$

$$\begin{aligned}\boldsymbol{\rho}^{(3)} = &-\frac{1}{8\boldsymbol{\rho}_0^3}\Big(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Big)^2 + \frac{1}{24\boldsymbol{\rho}_0}\Big(3\Gamma^{u}_{\Delta\Delta}{}^2 + 4\Gamma^{\Delta}_{\Delta\Delta,\Delta} + 4\Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0 + 4\boldsymbol{r}_0\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{u}_{\alpha'\Delta}\\ &+ 4\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\Delta} + 3g_{\alpha'\beta'}\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\beta'}_{\Delta\Delta}\Big),\end{aligned} \tag{2.104c}$$

$$\begin{aligned}\boldsymbol{\rho}^{(4)} = &\frac{1}{16\boldsymbol{\rho}_0^5}\Big(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Big)^3 - \frac{1}{48\boldsymbol{\rho}_0^3}\Big(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Big)\Big(3\Gamma^{u}_{\Delta\Delta}{}^2 + \Gamma^{\alpha'}_{\Delta\Delta}\Big[4\Gamma^{\Delta}_{\alpha'\Delta} + 3g_{\alpha'\beta'}\Gamma^{\beta'}_{\Delta\Delta}\\ &+ 4\Gamma^{u}_{\alpha'\Delta}\boldsymbol{r}_0\Big] + 4\Big[\Gamma^{\Delta}_{\Delta\Delta,\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0\Big]\Big) + \frac{1}{24\boldsymbol{\rho}_0}\Big(2\Gamma^{u}_{\Delta\Delta}\Gamma^{u}_{\Delta\Delta,\Delta} + 2\Gamma^{\alpha'}_{\Delta\Delta,\Delta}\Gamma^{\Delta}_{\alpha'\Delta} + \Gamma^{\Delta}_{\Delta\Delta,\Delta\Delta}\\ &+ 2\Gamma^{\alpha'}_{\Delta\Delta,\Delta}\Gamma^{u}_{\alpha'\Delta}\boldsymbol{r}_0 + \Gamma^{u}_{\Delta\Delta,\Delta\Delta}\boldsymbol{r}_0 + \Gamma^{\alpha'}_{\Delta\Delta}\Big[2\Gamma^{u}_{\alpha'\Delta}\Gamma^{u}_{\Delta\Delta} + \Gamma^{\Delta}_{\Delta\alpha',\Delta} + 2g_{\alpha'\beta'}\Gamma^{\beta'}_{\Delta\Delta,\Delta} + \Gamma^{u}_{\Delta\alpha',\Delta}\boldsymbol{r}_0\\ &+ R_{\alpha'\Delta u\Delta}\boldsymbol{r}_0\Big] + \Gamma^{\alpha'}_{\Delta\beta'}\Gamma^{\beta'}_{\Delta\Delta}\Big[\Gamma^{\Delta}_{\alpha'\Delta} + 2g_{\alpha'\gamma'}\Gamma^{\gamma'}_{\Delta\Delta} + \Gamma^{u}_{\alpha'\Delta}\boldsymbol{r}_0\Big]\Big),\end{aligned} \tag{2.104d}$$

$$\begin{aligned}\boldsymbol{\rho}^{(5)} = &-\frac{5}{128\boldsymbol{\rho}_0^7}\Big(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Big)^4 + \frac{1}{64\boldsymbol{\rho}_0^5}\Big(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Big)^2\Big(3\Gamma^{u}_{\Delta\Delta}{}^2 + 4\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\Delta} + 4\Gamma^{\Delta}_{\Delta\Delta,\Delta}\\ &+ 3g_{\alpha'\beta'}\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\beta'}_{\Delta\Delta} + 4\Big[\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{u}_{\alpha'\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta}\Big]\boldsymbol{r}_0\Big)\\ &+ \frac{1}{1152\boldsymbol{\rho}_0^3}\Bigg(-9\Gamma^{u}_{\Delta\Delta}{}^4 - 48\Gamma^{\alpha'}_{\Delta\Delta,\Delta}(\Gamma^{\Delta}_{\alpha'\Delta} + \Gamma^{u}_{\alpha'\Delta}\boldsymbol{r}_0)(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0) - 24\Gamma^{u}_{\Delta\Delta}{}^2\\ &\times(\Gamma^{\Delta}_{\Delta\Delta,\Delta} + 3\Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0) - 8\Big(2(\Gamma^{\Delta}_{\Delta\Delta,\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0)^2 + 3\Gamma^{\Delta}_{\Delta\Delta}(\Gamma^{\Delta}_{\Delta\Delta,\Delta\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta\Delta}\boldsymbol{r}_0)\Big)\\ &- 24\Gamma^{u}_{\Delta\Delta}\Big(2\Gamma^{u}_{\Delta\Delta,\Delta}\Gamma^{\Delta}_{\Delta\Delta} + \boldsymbol{r}_0(\Gamma^{\Delta}_{\Delta\Delta,\Delta\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta\Delta}\boldsymbol{r}_0)\Big) - 24\Gamma^{\alpha'}_{\Delta\beta'}\Gamma^{\beta'}_{\Delta\Delta}(\Gamma^{\Delta}_{\alpha'\Delta} + 2\Gamma^{\gamma'}_{\Delta\Delta}g_{\alpha'\gamma'}\end{aligned}$$

$$
\begin{aligned}
&+ \Gamma^{u}_{\alpha'\Delta}\boldsymbol{r}_0)(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0) - \Gamma^{\alpha'}_{\Delta\Delta}\Big[8\Gamma^{\Delta}_{\alpha'\Delta}\Big(3{\Gamma^{u}_{\Delta\Delta}}^2 + \Gamma^{\beta'}_{\Delta\Delta}(2\Gamma^{\Delta}_{\beta'\Delta} + 3\Gamma^{\gamma'}_{\Delta\Delta}g_{\beta'\gamma'}) \\
&+ 4(\Gamma^{\Delta}_{\Delta\Delta,\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0)\Big) + 8\Gamma^{u}_{\alpha'\Delta}\Big(6\Gamma^{u}_{\Delta\Delta}\Gamma^{\Delta}_{\Delta\Delta} + 9{\Gamma^{u}_{\Delta\Delta}}^2\boldsymbol{r}_0 + \Gamma^{\beta'}_{\Delta\Delta}\boldsymbol{r}_0(4\Gamma^{\Delta}_{\beta'\Delta} \\
&+ 3\Gamma^{\gamma'}_{\Delta\Delta}g_{\beta'\gamma'} + 2\Gamma^{u}_{\beta'\Delta}\boldsymbol{r}_0) + 4\boldsymbol{r}_0(\Gamma^{\Delta}_{\Delta\Delta,\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0)\Big) + 3\bigg(8\Gamma^{\Delta}_{\Delta\alpha',\Delta}(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0) \\
&+ 8\Big(2\Gamma^{\beta'}_{\Delta\Delta,\Delta}g_{\alpha'\beta'} + (R_{\alpha'\Delta u\Delta} + \Gamma^{u}_{\Delta\alpha',\Delta})\boldsymbol{r}_0\Big)(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0) + \Gamma^{\beta'}_{\Delta\Delta}g_{\alpha'\beta'}(6{\Gamma^{u}_{\Delta\Delta}}^2 \\
&+ 8\Gamma^{\Delta}_{\Delta\Delta,\Delta} + 3\Gamma^{\gamma'}_{\Delta\Delta}\Gamma^{\zeta'}_{\Delta\Delta}g_{\gamma'\zeta'} + 8\Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0)\bigg)\Big]\bigg) \\
&+ \frac{1}{720\boldsymbol{\rho}_0}\bigg(10{\Gamma^{u}_{\Delta\Delta,\Delta}}^2 + 15\Gamma^{u}_{\Delta\Delta}\Gamma^{u}_{\Delta\Delta,\Delta\Delta} + 18\Gamma^{\alpha'}_{\Delta\Delta,\Delta\Delta}\Gamma^{\Delta}_{\alpha'\Delta} + 6\Gamma^{\Delta}_{\Delta\Delta,\Delta\Delta\Delta} + 2\Gamma^{\alpha'}_{\Delta\Delta,\Delta} \\
&\cdot (15\Gamma^{u}_{\alpha'\Delta}\Gamma^{u}_{\Delta\Delta} + 9\Gamma^{\Delta}_{\Delta\alpha',\Delta} + 5\Gamma^{\beta'}_{\Delta\Delta,\Delta}g_{\alpha'\beta'}) + 3\Gamma^{\alpha'}_{\Delta\beta',\Delta}\Gamma^{\beta'}_{\Delta\Delta}(16\Gamma^{\Delta}_{\alpha'\Delta} + 5\Gamma^{\gamma'}_{\Delta\Delta}g_{\alpha'\gamma'}) \\
&+ \Gamma^{\alpha'}_{\Delta\beta'}\Big[6\Gamma^{\beta'}_{\Delta\Delta,\Delta}(3\Gamma^{\Delta}_{\alpha'\Delta} + 5\Gamma^{\gamma'}_{\Delta\Delta}g_{\alpha'\gamma'}) + \Gamma^{\beta'}_{\Delta\Delta}\Big(15\Gamma^{u}_{\alpha'\Delta}\Gamma^{u}_{\Delta\Delta} + 42\Gamma^{\Delta}_{\Delta\alpha',\Delta} \\
&+ 10(\Gamma^{\gamma'}_{\Delta\zeta'}\Gamma^{\zeta'}_{\Delta\Delta} + 2\Gamma^{\gamma'}_{\Delta\Delta,\Delta})g_{\alpha'\gamma'}\Big) + 3\Gamma^{\beta'}_{\Delta\gamma'}\Gamma^{\gamma'}_{\Delta\Delta}(2\Gamma^{\Delta}_{\alpha'\Delta} + 5\Gamma^{\zeta'}_{\Delta\Delta}g_{\alpha'\zeta'})\Big] \\
&+ 2\Big[R_{\alpha'\Delta u\Delta}(10\Gamma^{\alpha'}_{\Delta\beta'}\Gamma^{\beta'}_{\Delta\Delta} + 7\Gamma^{\alpha'}_{\Delta\Delta,\Delta}) + 3\bigg(3\Gamma^{\alpha'}_{\Delta\Delta,\Delta\Delta}\Gamma^{u}_{\alpha'\Delta} + 3\Gamma^{\alpha'}_{\Delta\Delta,\Delta}\Gamma^{u}_{\Delta\alpha',\Delta} + \Gamma^{u}_{\Delta\Delta,\Delta\Delta\Delta} \\
&- 3\Gamma^{\alpha'}_{u\Delta,\Delta}\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\beta'} - 2\Gamma^{\alpha'}_{u\Delta}\Gamma^{\beta'}_{\Delta\Delta,\Delta}\Gamma^{\Delta}_{\alpha'\beta'} - 5\Gamma^{\alpha'}_{u\Delta}\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\beta',\Delta} + 3\Gamma^{\alpha'}_{\Delta\beta',\Delta}\Gamma^{\beta'}_{\Delta\Delta}(\Gamma^{u}_{\alpha'\Delta} + \Gamma^{\Delta}_{\alpha'u}) \\
&+ 2\Gamma^{\alpha'}_{u\Delta}\Gamma^{\beta'}_{\Delta\alpha',\Delta}\Gamma^{\Delta}_{\beta'\Delta} + \Gamma^{\alpha'}_{\Delta\beta'}\Big((\Gamma^{\beta'}_{\Delta\gamma'}\Gamma^{\gamma'}_{\Delta\Delta} + 3\Gamma^{\beta'}_{\Delta\Delta,\Delta})\Gamma^{u}_{\alpha'\Delta} + \Gamma^{\beta'}_{\Delta\Delta}(2\Gamma^{u}_{\Delta\alpha',\Delta} + 3\Gamma^{\Delta}_{u\alpha',\Delta})\Big) \\
&+ 2\Gamma^{\alpha'}_{u\Delta}\Gamma^{\beta'}_{\Delta\alpha'}\Gamma^{\Delta}_{\Delta\beta',\Delta} + 2\Gamma^{\alpha'}_{u\Delta}\Gamma^{\Delta}_{\Delta\alpha',\Delta\Delta} - 2\Gamma^{\alpha'}_{u\Delta}\Gamma^{\Delta}_{\Delta\Delta,\alpha'\Delta} - 4R_{\alpha'u\beta'\Delta}\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{\Delta}_{\gamma'\Delta}g^{\alpha'\gamma'} \\
&- 2R_{\alpha'\Delta\gamma'\Delta}\Gamma^{\alpha'}_{u\Delta}\Gamma^{\Delta}_{\beta'\Delta}g^{\beta'\gamma'}\bigg)\Big]\boldsymbol{r}_0 - 3R_{\alpha'\Delta\beta'\Delta}\Gamma^{\beta'}_{\Delta\Delta}(13\Gamma^{\alpha'}_{\Delta\Delta} + 4\Gamma^{\alpha'}_{u\Delta}\boldsymbol{r}_0) \\
&+ \Gamma^{\alpha'}_{\Delta\Delta}\Big[10\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{u}_{\alpha'\Delta}\Gamma^{u}_{\beta'\Delta} + 20\Gamma^{u}_{\alpha'\Delta}\Gamma^{u}_{\Delta\Delta,\Delta} - 36\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\beta',\Delta} - 6\Big(\Gamma^{\beta'}_{\Delta\Delta,\Delta}\Gamma^{u}_{\alpha'\beta'} \\
&+ \Gamma^{\beta'}_{\Delta\Delta}\Gamma^{u}_{\alpha'\beta',\Delta} - 2\Gamma^{u}_{\Delta\alpha',\Delta\Delta} + \Gamma^{u}_{\Delta\Delta,\alpha'\Delta} + 3\Gamma^{\Delta}_{u\Delta,\alpha'\Delta} - 3\Gamma^{\Delta}_{\Delta\alpha',u\Delta} + (R_{\alpha'\Delta\gamma'\Delta}\Gamma^{u}_{\beta'\Delta} \\
&- 4R_{\alpha'\gamma'\Delta u}\Gamma^{\Delta}_{\beta'\Delta})g^{\beta'\gamma'} + R_{\alpha'u\beta'\Delta}(7\Gamma^{\beta'}_{\Delta\Delta} - \Gamma^{\Delta}_{\gamma'\Delta}g^{\beta'\gamma'})\Big)\boldsymbol{r}_0 + 3\Big(5(R_{\alpha'\Delta u\Delta} + \Gamma^{u}_{\Delta\alpha',\Delta}) \\
&\times \Gamma^{u}_{\Delta\Delta} - 12\Gamma^{\beta'}_{\Delta\Delta,\Delta}\Gamma^{\Delta}_{\alpha'\beta'} + 14\Gamma^{\Delta}_{\Delta\alpha',\Delta\Delta} - 12\Gamma^{\Delta}_{\Delta\Delta,\alpha'\Delta} + 5\Gamma^{\beta'}_{\Delta\Delta,\Delta\Delta}g_{\alpha'\beta'} \\
&- 12R_{\alpha'\Delta\gamma'\Delta}\Gamma^{\Delta}_{\beta'\Delta}g^{\beta'\gamma'} + 6R_{\alpha'\Delta u\Delta;\Delta}\boldsymbol{r}_0\Big)\Big]\bigg),
\end{aligned}
\tag{2.104e}
$$

where we have introduced notation for contractions of $u^{\mu'}$ and $\Delta x^{\mu'}$ so that, $\Gamma^{\Delta}_{u\Delta,\Delta} := \Gamma^{\alpha'}_{\beta'\mu',\nu'}\Delta x_{\alpha'}u^{\beta'}\Delta x^{\mu'}\Delta x^{\nu'}$ or $\dot{R}_{u\Delta u\Delta} := R_{\alpha'\beta'\mu'\nu';\gamma'}\Delta x^{\beta'}\Delta x^{\nu'}u^{\alpha'}u^{\mu'}u^{\gamma'}$, for example. With these quantities in hand, we can proceed to calculate the coordinate expansion of the scalar puncture field from Eq. (2.90).

In the original paper, Eq. (2.90) was written in terms of the Weyl tensor, $C_{\alpha\beta\mu\nu}$, but here we rewrite in terms of the Riemann tensor. We are free to do so as we work on a vacuum background where the Weyl and Riemann tensors coincide. We now substitute our expansions for $\sigma_{\mu'}$ from Eq. (2.95), $\boldsymbol{r}$ from Eq. (2.100) and $\boldsymbol{\rho}$ from Eq. (2.102) into

Eq. (2.90) to find the coordinate expansion. This is given by

$$\begin{aligned}
\Phi/q = {} & \frac{1}{\lambda\boldsymbol{\rho}_0} - \frac{1}{2\boldsymbol{\rho}_0^3}\Big(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Big) - \frac{\lambda}{24\boldsymbol{\rho}_0^5}\Big(3{\Gamma^{u}_{\Delta\Delta}}^2\boldsymbol{\rho}_0^2 - 9{\Gamma^{\Delta}_{\Delta\Delta}}^2 - 18\Gamma^{u}_{\Delta\Delta}\Gamma^{\Delta}_{\Delta\Delta}\boldsymbol{r}_0 \\
& - 9{\Gamma^{u}_{\Delta\Delta}}^2{\boldsymbol{r}_0}^2 + 4\Gamma^{\Delta}_{\Delta\Delta,\Delta}\boldsymbol{\rho}_0^2 + 4\Gamma^{u}_{\Delta\Delta,\Delta}\boldsymbol{r}_0\boldsymbol{\rho}_0^2 - 4R_{u\Delta u\Delta}{\boldsymbol{r}_0}^2\boldsymbol{\rho}_0^2 + 4\boldsymbol{r}_0\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{u}_{\alpha'\Delta}\boldsymbol{\rho}_0^2 \\
& + 4\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\Delta}\boldsymbol{\rho}_0^2 + 3\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\beta'}_{\Delta\Delta}g_{\alpha'\beta'}\boldsymbol{\rho}_0^2 + 4R_{u\Delta u\Delta}\boldsymbol{\rho}_0^4\Big) \\
& - \frac{\lambda^2}{48{\boldsymbol{\rho}_0}^7}\Big[15{\Gamma^{\Delta}_{\Delta\Delta}}^3 + 45\Gamma^{u}_{\Delta\Delta}{\Gamma^{\Delta}_{\Delta\Delta}}^2\boldsymbol{r}_0 + 45{\Gamma^{u}_{\Delta\Delta}}^2\Gamma^{\Delta}_{\Delta\Delta}{\boldsymbol{r}_0}^2 + 15{\Gamma^{u}_{\Delta\Delta}}^3{\boldsymbol{r}_0}^3 - 9{\Gamma^{u}_{\Delta\Delta}}^2\Gamma^{\Delta}_{\Delta\Delta}{\boldsymbol{\rho}_0}^2 \\
& - 12\Gamma^{\Delta}_{\Delta\Delta}\Gamma^{\Delta}_{\Delta\Delta,\Delta}{\boldsymbol{\rho}_0}^2 - 9{\Gamma^{u}_{\Delta\Delta}}^3\boldsymbol{r}_0{\boldsymbol{\rho}_0}^2 - 12\Gamma^{u}_{\Delta\Delta,\Delta}\Gamma^{\Delta}_{\Delta\Delta}\boldsymbol{r}_0{\boldsymbol{\rho}_0}^2 - 12\Gamma^{u}_{\Delta\Delta}\Gamma^{\Delta}_{\Delta\Delta,\Delta}\boldsymbol{r}_0{\boldsymbol{\rho}_0}^2 \\
& - 12\Gamma^{u}_{\Delta\Delta}\Gamma^{u}_{\Delta\Delta,\Delta}{\boldsymbol{r}_0}^2{\boldsymbol{\rho}_0}^2 + 12\Gamma^{\Delta}_{\Delta\Delta}R_{u\Delta u\Delta}{\boldsymbol{r}_0}^2{\boldsymbol{\rho}_0}^2 + 12\Gamma^{u}_{\Delta\Delta}R_{u\Delta u\Delta}{\boldsymbol{r}_0}^3{\boldsymbol{\rho}_0}^2 \\
& - 12(\Gamma^{\Delta}_{\Delta\Delta} + \Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0)\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\Delta}{\boldsymbol{\rho}_0}^2 - 9\Gamma^{\Delta}_{\Delta\Delta}\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\beta'}_{\Delta\Delta}g_{\alpha'\beta'}{\boldsymbol{\rho}_0}^2 - 9\Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\beta'}_{\Delta\Delta}g_{\alpha'\beta'}{\boldsymbol{\rho}_0}^2 \\
& + 4\Gamma^{u}_{\Delta\Delta}\Gamma^{u}_{\Delta\Delta,\Delta}{\boldsymbol{\rho}_0}^4 - 4\Gamma^{\Delta}_{\Delta\Delta}R_{u\Delta u\Delta}{\boldsymbol{\rho}_0}^4 - 12\Gamma^{u}_{\Delta\Delta}R_{u\Delta u\Delta}\boldsymbol{r}_0{\boldsymbol{\rho}_0}^4 - 8R_{\alpha' u\Delta u}\Gamma^{\alpha'}_{\Delta\Delta}{\boldsymbol{r}_0}^2{\boldsymbol{\rho}_0}^4 \\
& - 2R_{u\Delta u\Delta;\Delta}{\boldsymbol{r}_0}^2{\boldsymbol{\rho}_0}^4 + 2\dot{R}_{u\Delta u\Delta}{\boldsymbol{r}_0}^3{\boldsymbol{\rho}_0}^4 + 2\boldsymbol{r}_0R_{\alpha'\Delta u\Delta}\Gamma^{\alpha'}_{\Delta\Delta}{\boldsymbol{\rho}_0}^4 + 2\boldsymbol{r}_0\Gamma^{\alpha'}_{\Delta\beta'}\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{u}_{\alpha'\Delta}{\boldsymbol{\rho}_0}^4 \\
& + 4\boldsymbol{r}_0\Gamma^{\alpha'}_{\Delta\Delta,\Delta}\Gamma^{u}_{\alpha'\Delta}{\boldsymbol{\rho}_0}^4 + 2\Gamma^{\alpha'}_{\Delta\beta'}\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{\Delta}_{\alpha'\Delta}{\boldsymbol{\rho}_0}^4 + 4\Gamma^{\alpha'}_{\Delta\Delta,\Delta}\Gamma^{\Delta}_{\alpha'\Delta}{\boldsymbol{\rho}_0}^4 + 4\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{\beta'}_{\Delta\Delta,\Delta}g_{\alpha'\beta'}{\boldsymbol{\rho}_0}^4 \\
& + 2\Gamma^{\Delta}_{\Delta\Delta,\Delta\Delta}{\boldsymbol{\rho}_0}^4 + 4\Gamma^{\alpha'}_{\Delta\beta'}\Gamma^{\beta'}_{\Delta\Delta}\Gamma^{\gamma'}_{\Delta\Delta}g_{\alpha'\gamma'}{\boldsymbol{\rho}_0}^4 + 2\Gamma^{\Delta}_{\Delta\alpha',\Delta}\Gamma^{\alpha'}_{\Delta\Delta}{\boldsymbol{\rho}_0}^4 + 2\boldsymbol{r}_0\Gamma^{u}_{\Delta\Delta,\Delta\Delta}{\boldsymbol{\rho}_0}^4 \\
& + 2\boldsymbol{r}_0\Gamma^{u}_{\Delta\beta',\Delta}\Gamma^{\beta'}_{\Delta\Delta}{\boldsymbol{\rho}_0}^4 + 8R_{\alpha' u\Delta u}\Gamma^{\alpha'}_{\Delta\Delta}{\boldsymbol{\rho}_0}^6 + 2R_{u\Delta u\Delta;\Delta}{\boldsymbol{\rho}_0}^6 - 6\dot{R}_{u\Delta u\Delta}\boldsymbol{r}_0{\boldsymbol{\rho}_0}^6 \\
& + 4\Gamma^{\alpha'}_{\Delta\Delta}\Gamma^{u}_{\alpha'\Delta}{\boldsymbol{\rho}_0}^2\Big(\Gamma^{u}_{\Delta\Delta}({\boldsymbol{\rho}_0}^2 - 3{\boldsymbol{r}_0}^2) - 3\Gamma^{\Delta}_{\Delta\Delta}\boldsymbol{r}_0\Big)\Big] + \mathcal{O}\Big(\lambda^3\Big). \qquad (2.105)
\end{aligned}$$

Due to the length of the expression, we have not been able to include the order-λ^3 term, which is the order that is one beyond the current state of the art [138]. Instead, we include it, and the other orders, in a MATHEMATICA notebook that can be found in the Additional Material [141].

For the gravitational case, as opposed to the scalar one, we also require an expansion for the parallel propagator. To calculate the coordinate expansion of ${g^{\nu'}}_{\mu}$, we proceed in a similar way to that of $\sigma_{\alpha'}$. To begin, we use the ansatz

$$\begin{aligned}
{g^{\nu'}}_{\mu} = \delta^{\nu'}_{\mu'} + \lambda {G^{(1)\nu'}}_{\mu'\alpha'}\Delta x^{\alpha'} + \lambda^2 {G^{(2)\nu'}}_{\mu'\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'} \\
+ \lambda^3 {G^{(3)\nu'}}_{\mu'\alpha'\beta'\gamma'}\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\gamma'} + \mathcal{O}\Big(\lambda^4\Big) \qquad (2.106)
\end{aligned}$$

and substitute this into the identity for the derivative of the parallel propagator contracted into a derivative of Synge's world function from Eq. (2.74). We proceed to solve this order-by-order to find

$$\begin{aligned}
{G^{(1)\alpha'}}_{\beta'\gamma'} &= \Gamma^{\alpha'}_{\beta'\gamma'}, \qquad (2.107a) \\
{G^{(2)\alpha'}}_{\beta'\gamma'\delta'} &= \frac{1}{2}\Big(\Gamma^{\alpha'}_{\beta'\iota'}\Gamma^{\iota'}_{\gamma'\delta'} + g^{\alpha'\iota'}R_{\beta'(\gamma'\delta')\iota'} + \Gamma^{\alpha'}_{\gamma'\delta',\beta'}\Big), \qquad (2.107b) \\
{G^{(3)\alpha'}}_{\beta'\gamma'\delta'\iota'} &= \frac{1}{6}\underset{\gamma'\delta'\iota'}{\mathrm{Sym}}\Big(\Gamma^{\varsigma'}_{\gamma'\delta'}\Big[3g^{\alpha'\kappa'}R_{\beta'(\iota'\varsigma')\kappa'} + \Gamma^{\alpha'}_{\iota'\varsigma',\beta'}\Big] + \Gamma^{\alpha'}_{\beta'\varsigma'}\Big[\Gamma^{\varsigma'}_{\gamma'\kappa'}\Gamma^{\kappa'}_{\delta'\iota'} + \Gamma^{\varsigma'}_{\gamma'\delta',\iota'}\Big] \\
&\quad - g^{\alpha'\kappa'}\Gamma^{\varsigma'}_{\beta'\gamma'}R_{\delta'\varsigma'\iota'\kappa'} + g^{\alpha'\varsigma'}R_{\beta'\gamma'\delta'\varsigma';\iota'} + \Gamma^{\alpha'}_{\gamma'\varsigma'}\Gamma^{\varsigma'}_{\delta'\iota',\beta'} + \Gamma^{\alpha'}_{\gamma'\delta',\beta'\iota'}\Big). \qquad (2.107c)
\end{aligned}$$

We have checked our expressions by substituting them into Eq. (2.74) and have verified that they satisfy the identity to the appropriate order in λ.

2.2.2 Fermi–Walker coordinates

When analysing the properties of fields near the worldline of the small object, it is advantageous to introduce coordinates that are adapted to the problem. In this chapter, we introduce Fermi–Walker coordinates which are particularly suited to this task. In general relativity, we can always find a coordinate system around a chosen point, p, where the full spacetime metric reduces to the Minkowski metric through the use of Riemann normal coordinates [166, Ch. 1.6]. In other words,

$$g_{\mu\nu}(p) = \eta_{\mu\nu} \quad \text{and} \quad \Gamma^{\mu}_{\nu\rho}(p) = 0. \tag{2.108}$$

This is (a version of) the equivalence principle. However, it is possible to extend this notion from a single point, p, to an entire timelike curve, γ, with the introduction of Fermi–Walker coordinates. Our description of Fermi–Walker coordinates summarises that of Refs. [63, Ch. 9, 166, Ch. 1.11].

To begin, we introduce an orthonormal tetrad (u^μ, e^μ_a) on γ which is defined at the point $z(\tau)$ so that it satisfies

$$\frac{De^\mu_a}{d\tau} = a_\nu e^\nu_a u^\mu, \tag{2.109}$$

$$g_{\mu\nu} u^\mu u^\nu = -1, \tag{2.110}$$

$$g_{\mu\nu} e^\mu_a u^\nu = 0, \tag{2.111}$$

$$g_{\mu\nu} e^\mu_a e^\nu_b = \delta_{ab}, \tag{2.112}$$

where $u^\mu = dz^\mu/d\tau$ is the curve's four-velocity, $a^\mu = D^2 z^\mu/\, d\tau^2$ is the acceleration of γ and $\delta_{ab} = \mathrm{diag}(1,1,1)$ is the three dimensional flat space metric. If γ is a geodesic then a^μ vanishes. Eq. (2.109) ensures that the tetrad basis is Fermi–Walker transported along γ, thus keeping it orthogonal to the worldline as it travels along it. This condition reduces to that of parallel transport when the worldline is a geodesic. Eqs. (2.110)–(2.112) then ensure that it is orthonormal at all points on γ. The dual tetrad, (e^0_μ, e^a_μ), can be defined as satisfying

$$e^0_\mu = -u_\mu, \tag{2.113}$$

$$e^a_\mu = \delta^{ab} g_{\mu\nu} e^\nu_b. \tag{2.114}$$

Eqs. (2.110)–(2.114) then imply that we can write the metric and inverse metric as

$$g_{\mu\nu} = -e^0_\mu e^0_\nu + \delta_{ab} e^a_\mu e^b_\nu, \tag{2.115}$$

$$g^{\mu\nu} = -u^\mu u^\nu + \delta^{ab} e^\mu_a e^\nu_b, \tag{2.116}$$

respectively.

With the orthonormal tetrad constructed, we may now create a local coordinate system so that we may derive the form of the metric near γ. The full technical details are not considered here (see Ref. [63, Chs 9.3–9.5] for more details) but we outline the geometric picture of the coordinate construction. At a point $\bar{x} := z(t)$ on γ, where t gives the proper time, we generate a surface orthogonal to the worldline by emitting spacelike geodesics from $z(t)$ that are orthogonal to γ. We can then label a point on this surface with coordinates x^i so that we have coordinates, (t, x^i), that describe points near to the worldline. The tetrad can be written in terms of Synge's world function as

$$x^0 = t, \tag{2.117}$$

$$x^a = -e^a_{\bar{\alpha}}(\bar{x})\sigma^{\bar{\alpha}}(x, \bar{x}), \tag{2.118}$$

$$\sigma_{\bar{\alpha}}(x, \bar{x}) u^{\bar{\alpha}}(\bar{x}) = 0. \tag{2.119}$$

Alternatively, we can write $x^i = rn^i$, with $r := \sqrt{\delta_{ab} x^a x^b} = \sqrt{2\sigma(x, \bar{x})}$ being the proper distance (along a unique spacelike geodesic orthogonal to γ) from γ to the point being considered and n^i being a unit vector giving the direction that the point lies in respective to γ. We note as well that, as with $\sigma_{\alpha'}$, $r \sim \lambda$ and so counts powers of distance from the worldline. A geometric representation of the Fermi–Walker coordinate construction is given in Fig. 2.2.

Using these coordinates, we can write the metric near γ in the form [81]

$$g_{tt} = -1 - 2a_i x^i - (R_{titj} + a_i a_j)x^i x^j - \frac{1}{3}(4R_{titj}a_k + R_{titj;k})x^i x^j x^k + \mathcal{O}\left(x^4\right), \tag{2.120a}$$

$$g_{ta} = -\frac{2}{3}R_{tiaj}x^i x^j - \frac{1}{12}(4R_{tiaj}a_k + 3R_{tiaj;k})x^i x^j x^k + \mathcal{O}\left(x^4\right), \tag{2.120b}$$

$$g_{ab} = \delta_{ab} - \frac{1}{3}R_{aibj}x^i x^j - \frac{1}{6}R_{aibj;k}x^i x^j x^k + \mathcal{O}\left(x^4\right), \tag{2.120c}$$

where all Riemann terms are evaluated on γ at time t. When evaluating Eq. (2.120) on γ, we immediately see that the metric in Fermi–Walker coordinates reduces to the Minkowski metric. However, the Christoffel symbols at lowest order are not all zero as we would expect when comparing with Eq. (2.108). Instead, $\Gamma^t_{ta} = a_a$ and $\Gamma^a_{tt} = a^a$; both reduce to 0 if γ is a geodesic as $a^a = 0$ [63, Ch. 9.5].

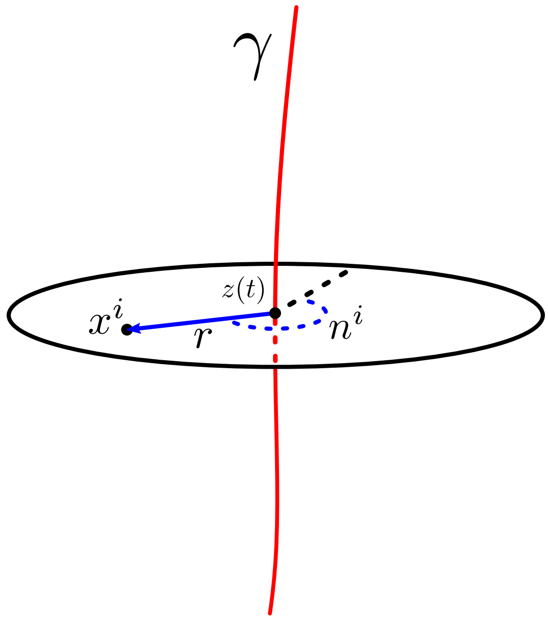

FIGURE 2.2: Visualisation of construction of Fermi–Walker coordinates. At the point $z(t)$, we generate an orthogonal surface and label points on that surface with the coordinate x^i. The quantity r gives the proper distance to x^i and n^i picks out the unique orthogonal geodesic that connects x^i and γ. Based on Fig. 6 from Ref. [63].

As we are looking at a vacuum solution with $R_{\mu\nu} = 0$, we may use the identities from App. D3 of Ref. [167] to write

$$R_{tatb} = \mathcal{E}_{ab}, \tag{2.121a}$$
$$R_{abct} = \epsilon_{ab}{}^{i}\mathcal{B}_{ic}, \tag{2.121b}$$
$$R_{abcd} = -\epsilon_{abi}\epsilon_{cdj}\mathcal{E}^{ij} \tag{2.121c}$$

and the derivatives as

$$R_{tatb;c} = \mathcal{E}_{abc} + \frac{2}{3}\epsilon_{ci(a}\dot{\mathcal{B}}_{b)}{}^{i}, \tag{2.122a}$$
$$R_{abct;d} = \epsilon_{ab}{}^{i}\left(\frac{4}{3}\mathcal{B}_{icd} - \frac{2}{3}\epsilon_{dj(i}\dot{\mathcal{E}}^{j}{}_{c)}\right), \tag{2.122b}$$
$$R_{abcd;e} = -\epsilon_{abi}\epsilon_{cdj}\left(\mathcal{E}^{ij}{}_{e} + \frac{2}{3}\epsilon_{ek}{}^{(i}\dot{\mathcal{B}}^{j)k}\right). \tag{2.122c}$$

The quantities $\mathcal{E}$ and $\mathcal{B}$ are the tidal moments felt by an extended body moving on the world line, γ, where two/three indices refer to the quadrupole/octopole moments respectively. They are symmetric and trace-free, with respect to δ_{ab}, over all indices and only depend on proper time.

For a number of calculations later in this thesis, we are required to integrate expressions in Fermi–Walker coordinates. This requires us to know what the form of the Fermi–Walker surface element is. We require this through order-λ^5 due to the singular nature of the terms we will be integrating.

The surface element on the spacelike hypersurface, S, of constant r is given by [166, Ch. 3]

$$dS_\alpha = N_\alpha \sqrt{|h|}\, dt\, d\theta\, d\phi\,, \tag{2.123}$$

where N_α is the unit normal to S and h is the determinant of the induced metric on S. By construction of the Fermi–Walker coordinates, $N_\alpha = -n_\alpha$. To see this, we write Eq. (2.118) in terms of the parallel propagator, so that

$$x^\alpha = -g^\alpha{}_{\bar\alpha}\sigma^{\bar\alpha}, \tag{2.124}$$

meaning

$$n^\alpha = -\frac{g^\alpha{}_{\bar\alpha}\sigma^{\bar\alpha}}{\sqrt{2\sigma}}, \tag{2.125}$$

and,

$$n^\alpha n_\alpha = \frac{g^\alpha{}_{\bar\alpha}\sigma^{\bar\alpha}}{\sqrt{2\sigma}}\frac{g_\alpha{}^{\bar\beta}\sigma_{\bar\beta}}{\sqrt{2\sigma}} = \frac{\sigma^{\bar\alpha}\sigma_{\bar\alpha}}{2\sigma} = 1. \tag{2.126}$$

Therefore n^α is a unit normal in the full metric, $g_{\mu\nu}$, as well as in the flat space metric, δ_{ij}, with the minus sign in N_α coming from the orientation of the surface.

The square root of the determinant of the induced metric is then given by

$$\sqrt{|h|} = \sqrt{|\eta|}\left(1 + \frac{r^2}{2}\eta^{\mathbf{ij}}j^{(2)}_{\mathbf{ij}} + \frac{r^3}{2}\eta^{\mathbf{ij}}j^{(3)}_{\mathbf{ij}} + \mathcal{O}\Big(r^4\Big)\right), \tag{2.127}$$

where bold Latin indices are (t,θ,ϕ) coordinates on the hypersurface and $j^{(n)}_{\mu\nu}$ is the order-r^n term appearing in the Fermi–Walker metric from Eq. (2.120), so that

$$g_{\mu\nu} = \eta_{\mu\nu} + r^2 j^{(2)}_{\mu\nu} + r^3 j^{(3)}_{\mu\nu} + \mathcal{O}\Big(r^4\Big). \tag{2.128}$$

We now return to Eq. (2.123) and substitute in N_α and $\sqrt{|h|}$, to see that

$$\begin{aligned} N_\alpha\sqrt{|h|} = -n_\alpha r^2 \sin\theta\bigg(1 &+ \frac{r^2}{2}\Big(-j^{(2)}_{tt} + \frac{1}{r^2}\Omega^{AB}j^{(2)}_{AB}\Big) \\ &+ \frac{r^3}{2}\Big(-j^{(3)}_{tt} + \frac{1}{r^2}\Omega^{AB}j^{(3)}_{AB}\Big)\bigg) + \mathcal{O}\Big(r^6\Big), \end{aligned} \tag{2.129}$$

where Latin indices now refer to spatial Fermi–Walker coordinates, (r,θ,ϕ). Substituting Eq. (2.129) and the values for $j^{(n)}_{\mu\nu}$ from Eq. (2.120) into Eq. (2.123), we find the final expression for dS_α to be

$$dS_\alpha = -n_\alpha r^2\left(1 + \frac{r^2}{3}\mathcal{E}_{ab}\hat{n}^{ab} + \frac{r^3}{12}\mathcal{E}_{abc}\hat{n}^{abc}\right) dt\, d\Omega + \mathcal{O}\Big(r^6\Big). \tag{2.130}$$

The Fermi–Walker volume element can be calculated in a similar way and is given by

$$\begin{aligned} dV &= \sqrt{-g}\, d^4x \\ &= \sqrt{-\eta}\left(1 + \frac{r^2}{2}\eta^{\mu\nu} j^{(2)}_{\mu\nu} + \frac{r^3}{2}\eta^{\mu\nu} j^{(3)}_{\mu\nu} + \mathcal{O}\left(r^4\right)\right) \\ &= r^2\left(1 + \frac{r^2}{3}\mathcal{E}_{ab}\hat{n}^{ab} + \frac{r^3}{12}\mathcal{E}_{abc}\hat{n}^{abc}\right) dt\, dr\, d\Omega + \mathcal{O}\left(r^6\right). \end{aligned} \tag{2.131}$$

Chapter 3

Gravitational self-force

In this section we will detail the important concepts used in the study of gravitational self-force. We start, in Ch. 3.1, by describing the self-consistent formalism for expanding the metric perturbations before following Ref. [139] in Ch. 3.2 to show how the generalised equivalence principle can be derived. Ch. 3.3 then explain how the gravitational self-force formalism can be implemented numerically in the puncture scheme that was discussed in the introduction. Finally, in Chs 3.4–3.5, we follow the method in Ref. [81] to determine the form of the metric perturbations in a lightcone rest gauge before detailing the transformation that Pound [81] used to derive the leading-order form of $h^{\rm SR}_{\mu\nu}$ in the highly regular gauge.

3.1 Self-consistent formalism

There are a number of different approaches to self-force calculations, but we will be using the self-consistent formalism in this thesis, as described in Refs. [64, 81, 87, 139]. In the self-consistent approach we expand our perturbation, $h_{\mu\nu}$, in a form similar to Eq. (2.2), but we allow each coefficient $h^n_{\mu\nu}$ to depend on the small object's exact, ϵ-dependent, accelerated worldline $z^\mu(\epsilon)$ [139]:[1]

$$h_{\mu\nu}(x,\epsilon) = \sum_{n>0} \epsilon^n h^n_{\mu\nu}(x;z), \tag{3.1}$$

so that

$$\mathrm{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + \epsilon h^1_{\mu\nu}(x;z) + \epsilon^2 h^2_{\mu\nu}(x;z) + \mathcal{O}\Big(\epsilon^3\Big). \tag{3.2}$$

By doing so, we avoid performing a Taylor series expansion of the small object's worldline

$$z^\mu(\epsilon) = z^\mu_0 + \epsilon z^\mu_1 + \epsilon^2 z^\mu_2 + \mathcal{O}\Big(\epsilon^3\Big). \tag{3.3}$$

[1]Here, we use ϵ as a formal expansion parameter which is set to 1 at the end of the calculation.

If we were to perform a series expansion of the worldline, then all our results would only be valid in the case where the accelerated worldline is very close to the zeroth-order worldline. This assumption will break down at some point, at a time much shorter than the total inspiral time [139]. The result is that each coefficient, $h^n_{\mu\nu}$, encodes the entire ϵ dependence on the worldline, $z^\mu(\epsilon)$, instead of depending on some combination of the Taylor coefficients, z^μ_n, from Eq. (3.3). Eq. (3.1) is an asymptotic series of the type described in Ch. 2.1.4.

As we have not expanded the worldline, the equation of motion is then given by [139]

$$\frac{D^2 z^\mu}{d\tau^2} = f^\mu(\tau, \epsilon) = \sum_{n\geq 0} \epsilon^n f^\mu_n[h^1, \ldots, h^n], \tag{3.4}$$

where each of the f^μ_n inherit dependence on ϵ through their dependence on $h^n_{\mu\nu}(x; z)$.

This approach is advantageous when compared to other treatments as it is valid on asymptotically large spacetime domains, such as the $\sim 1/\epsilon$ timescale that EMRI inspirals occur over [139]. However, as discussed in Ref. [66], there are shortcomings to the self-consistent approach: it would be difficult to numerically implement, and a straightforward implementation would not accurately track the evolution of the spin and mass of the large black hole. In the specific case of binary inspirals, a more practical alternative is provided by a multiscale expansion [17, 65, 66], in which 'fast' and 'slow' variables are used to capture processes that happen on different time scales.

3.2 Generalised equivalence principle

The motion of a test mass in general relativity is governed by the *geodesic equation*. This can be expressed in two equivalent forms, either as

$$\frac{d^2 z^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dz^\nu}{d\tau}\frac{dz^\rho}{d\tau} = 0, \tag{3.5}$$

or as

$$\frac{D^2 z^\mu}{d\tau^2} = 0, \tag{3.6}$$

with $\Gamma^\mu_{\nu\rho}$ being the Christoffel symbols associated with $g_{\mu\nu}$, $z^\mu(\tau)$ being the position of the test mass and τ being the proper time. This is no longer the case when our object has a finite gravitational mass [139]. We now have a perturbation in our spacetime which acts as a force on our object and alters its motion, the *gravitational self-force*. This leads to it no longer travelling on a geodesic in $g_{\mu\nu}$ *or* in $\mathsf{g}_{\mu\nu}$. Instead, it follows the equation of motion given in Eq. (1.27). Given certain conditions (near sphericity and slow spin), we can show that this is equivalent to the object travelling along a geodesic in an effective metric,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\rm R}_{\mu\nu}, \tag{3.7}$$

governed by [95, 139],

$$\frac{\tilde{D}^2 z^\mu}{d\tilde{\tau}^2} = \mathcal{O}\left(\epsilon^3\right), \tag{3.8}$$

where the tilde refers to all objects being defined with respect to $\tilde{g}_{\mu\nu}$. The effective metric has the properties that it is a smooth metric satisfying the Einstein vacuum equations and it is causal on the worldline [81].

To see that the equation of motion (1.27) is equivalent to the geodesic equation (3.8) we follow the derivation from Ref. [139]. First we start with the geodesic equation in our effective spacetime with a potentially non-affine parameter, s. This is given by

$$\frac{d^2 z^\mu}{ds^2} + \tilde{\Gamma}^\mu{}_{\nu\rho}\frac{dz^\nu}{ds}\frac{dz^\rho}{ds} = \frac{dz^\mu}{ds}\frac{d}{ds}\ln\sqrt{-\tilde{g}_{\nu\rho}\frac{dz^\nu}{ds}\frac{dz^\rho}{ds}}. \tag{3.9}$$

Now substitute in (3.7) and let $s = \tau$, where τ is the proper time in $g_{\mu\nu}$. We can write the left-hand side of (3.9) as

$$\begin{aligned}\frac{d^2 z^\mu}{ds^2} + \tilde{\Gamma}^\mu{}_{\nu\rho}\frac{dz^\nu}{ds}\frac{dz^\rho}{ds} &= u^\nu \frac{\partial u^\mu}{\partial z^\nu} + \tilde{\Gamma}^\mu{}_{\nu\rho}u^\nu u^\rho \\ &= u^\nu\left(\frac{\partial u^\mu}{\partial z^\nu} + \Gamma^\mu_{\nu\rho}u^\rho\right) + \left(\tilde{\Gamma}^\mu{}_{\nu\rho} - \Gamma^\mu_{\nu\rho}\right)u^\nu u^\rho \\ &= u^\nu \nabla_\nu u^\mu + C^\mu{}_{\nu\rho}u^\nu u^\rho \\ &= \frac{D^2 z^\mu}{d\tau^2} + C^\mu{}_{\nu\rho}u^\nu u^\rho,\end{aligned} \tag{3.10}$$

where $C^\mu{}_{\nu\rho}$ is given by (2.12). The right-hand side of (3.9) is now

$$\begin{aligned}u^\mu \frac{\frac{d}{d\tau}\sqrt{1 - h^{\mathrm{R}}_{\nu\rho}u^\nu u^\rho}}{\sqrt{1 - h^{\mathrm{R}}_{\alpha\beta}u^\alpha u^\beta}} &= -\frac{u^\mu u^\sigma}{2(1 - h^{\mathrm{R}}_{\alpha\beta}u^\alpha u^\beta)}\nabla_\sigma\left(h^{\mathrm{R}}_{\nu\rho}u^\nu u^\rho\right) \\ &= -\frac{u^\mu u^\nu}{2}\left(u^\sigma u^\rho h^{\mathrm{R}}_{\nu\rho;\sigma} + h^{\mathrm{R}}_{\alpha\beta}u^\alpha u^\beta u^\sigma u^\rho h^{\mathrm{R}}_{\nu\rho;\sigma} + 2h^{\mathrm{R}}_{\nu\rho}\frac{D^2 z^\rho}{d\tau^2}\right) \\ &\quad + \mathcal{O}\left(h^3\right).\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) and simplifying, we find that (3.8) is equivalent to

$$\frac{D^2 z^\mu}{d\tau^2} = -\frac{1}{2}(g^{\mu\alpha} + u^\mu u^\alpha)\left(g_\alpha{}^\delta - h^{\mathrm{R}\delta}_\alpha\right)\left(2h^{\mathrm{R}}_{\delta\beta;\gamma} - h^{\mathrm{R}}_{\beta\gamma;\delta}\right)u^\beta u^\gamma + \mathcal{O}\left(\epsilon^3\right). \tag{3.12}$$

This is exactly the second-order self force equation of motion (1.27) that was derived by Pound [95] using the method of matched asymptotic expansions. Thus, the small, gravitating object moves on a geodesic of the effective metric (through $\mathcal{O}(\epsilon^2)$).

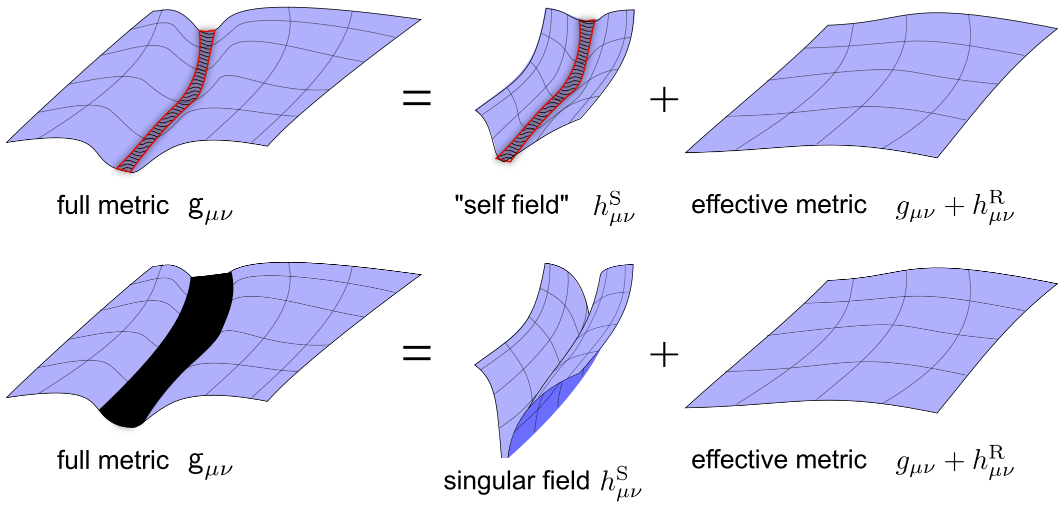

FIGURE 3.1: The idea behind the puncture scheme. The top row illustrates how we split our spacetime in to a self-field, that encodes the multipole structure of the small object, and an effective metric. The puncture scheme is described in the second row where we ignore the actual form of the self field and replace it with a singular field that mimics the curvature outside of the object while leaving the effective metric untouched. Reproduced from Ref. [64].

3.3 Puncture scheme

At second order, starting in Ref. [168], the primary method of solving the field equations is through the use of a *puncture scheme* [73, 169, 170]. This was discussed in the introduction but we expand upon it here. The method involves removing our small object from the spacetime and replacing it with a local singularity. See Fig. 3.1 for a visual representation of the idea. It is this singularity that we now think of as causing the curvature near the original position of the small object and we refer to it as a puncture, denoted as $h^{\mathcal{P}}_{\mu\nu}$. The puncture is obtained by truncating the local expansion of the singular field, $h^{\mathrm{S}}_{\mu\nu}$, so that $h^{\mathcal{P}}_{\mu\nu} \approx h^{\mathrm{S}}_{\mu\nu}$, and imposing that it becomes identically zero at some distance from γ. We define the residual field as

$$h^{\mathcal{R}}_{\mu\nu} := h_{\mu\nu} - h^{\mathcal{P}}_{\mu\nu} \tag{3.13}$$

so that $h^{\mathcal{R}}_{\mu\nu} \approx h^{\mathrm{R}}_{\mu\nu}$ near γ. We are no longer directly solving for the physical field, $h_{\mu\nu}$, instead, we solve for $h^{\mathcal{R}}_{\mu\nu}$ which becomes identical to $h_{\mu\nu}$ in the region where the puncture field vanishes.

We wish to be able to replace $h^{\mathrm{R}}_{\mu\nu}$ with $h^{\mathcal{R}}_{\mu\nu}$ in the equation of motion from Eq. (1.27). This is possible if $h^{\mathcal{R}}_{\mu\nu}$ and its first derivatives are identical to $h^{\mathrm{R}}_{\mu\nu}$. To ensure this, we impose the conditions

$$\lim_{x\to z}\left(h^{\mathcal{P}}_{\mu\nu} - h^{\mathrm{S}}_{\mu\nu}\right) = 0, \tag{3.14}$$

$$\lim_{x\to z}\left(h^{\mathcal{P}}_{\mu\nu,\rho} - h^{\mathrm{S}}_{\mu\nu,\rho}\right) = 0, \tag{3.15}$$

where x^μ is a point near the worldline, $z^\mu := x^{\mu'}$. However, while this condition is sufficient, in practice higher order punctures are used when performing self-force calculations. That is, we ensure that the difference between higher derivatives of $h^{\mathcal{P}}_{\mu\nu}$ and $h^{\mathrm{S}}_{\mu\nu}$ goes to 0 when evaluated on the worldline. This has the effect of improving the regularity of the residual field meaning that any mode decomposition will converge at an accelerated rate [171]. While a higher-order puncture is preferable, it can be very challenging to derive the form of the punctures when moving up through the orders.

Substituting Eq. (3.13) into the vacuum Einstein equations, Eqs (1.9) and (1.10) with $T^n_{\mu\nu} = 0$, we see the fields must satisfy the relations

$$\delta G_{\mu\nu}[h^{\mathcal{R}1}] = -\delta G_{\mu\nu}[h^{\mathcal{P}1}], \qquad r > 0, \tag{3.16}$$

$$\delta G_{\mu\nu}[h^{\mathcal{R}2}] = -\delta^2 G_{\mu\nu}[h^1] - \delta G_{\mu\nu}[h^{\mathcal{P}2}], \qquad r > 0. \tag{3.17}$$

For the total $h^{\mathcal{R}}_{\mu\nu} + h^{\mathcal{P}}_{\mu\nu}$ to match the real physical field, $h^{\mathcal{R}}_{\mu\nu}$ must be a C^k function if the puncture field was truncated at order r^k. This ensures that Eqs. (3.16) and (3.17) are locally integrable on the worldline of the small object. Thus we can write

$$\delta G_{\mu\nu}[h^{\mathcal{R}1}] = -(\delta G_{\mu\nu}[h^{\mathcal{P}1}])^\star, \tag{3.18}$$

$$\delta G_{\mu\nu}[h^{\mathcal{R}2}] = -(\delta^2 G_{\mu\nu}[h^1] + \delta G_{\mu\nu}[h^{\mathcal{P}2}])^\star, \tag{3.19}$$

where the star means that we have promoted the quantities to locally integrable functions by defining them to be zero at $r = 0$ or defining their value to be the limit as $r \to 0$, if it exists.

Thus, the second-order field is determined by solving Eqs. (1.27), (3.18) and (3.19) as a system of coupled equations [64]. Explicitly, we are solving for the position of the worldline, along with $h^{\mathcal{R}}_{\mu\nu}$ inside the worldtube and $h_{\mu\nu}$ outside. This requires a complicated numerical implementation, but the specific details are not considered here.

At first-order, there is a more widely used approach, that of mode-sum regularisation [73, 136, 172, 173]. This method involves decomposing the singular and full fields into spherical harmonic modes, solving the decomposed field equations for the modes of the physical field, and then subtracting the singular field mode-by-mode in order to find the spherical harmonic modes of the regular field. As an equation,

$$h^{\mathrm{R}}_{\mu\nu}(z) = \sum_{i\ell m} [h_{i\ell m}(z) - h^{\mathrm{S}}_{i\ell m}(z)] Y^{i\ell m}_{\mu\nu}(z), \tag{3.20}$$

where $Y^{i\ell m}_{\mu\nu}$ are the Barack–Lousto–Sago tensor harmonics mentioned in the introduction. However, the specific basis used is not relevant. The physical field is determined by solving the linearised Einstein equation from Eq. (1.9) with a point-mass source (1.14). However, there are difficulties in using the mode-sum regularisation method at second order as, generically, the second-order Einstein equation (1.10) diverges too strongly on the worldline. Even if it were possible to compute the physical field, it would not be possible to use mode-sum regularisation to obtain the regular field as the individual modes of $h^{\rm S2}_{\mu\nu}$ diverge on the worldline [15].

3.4 Metric perturbations in a lightcone gauge

In this section, following the method from Ref. [81], we describe the derivation of the metric perturbations in the lightcone gauge using matched asymptotic expansions. The results in this section are all originally from Ref. [81]. To begin, we calculate the form of the inner expansion's metric in a *rest gauge*. The gauge is named as such as the small object is at rest on our worldline. This is possible as we can always find an effective metric in which our small object's worldline is a geodesic [64]. Then, in the buffer region, we express our rest gauge expansion in terms of the outer expansion. Finally, we then transform this to a "practical gauge" that is more suitable for numerical implementation.

To motivate the transformation to this gauge, recall from Eq. (2.55) that the most singular part of the metric perturbations in the outer expansion at each order in ϵ are determined by the multipole moments of the small object. For a non-spinning, spherically symmetric small object, this is described by the Schwarzschild metric and can be written in ingoing Eddington–Finkelstein coordinates as

$$ds^2_{\rm obj} = -\left(1 - \frac{2m}{r}\right) dv^2 + 2\, dv\, dr + r^2\, d\Omega^2\,, \tag{3.21}$$

which we see, immediately, is linear in $1/r$, eliminating any higher order terms. This demonstrates that we can eliminate the term in $h^{\rm S2}_{\mu\nu}$ with the form $\sim m^2/r^2$.

We immediately specialise to a non-spinning, approximately spherically symmetric small object. The inner expansion is given by the metric of a tidally perturbed, non-spinning black hole as presented in Ref. [174] (with conversion to Cartesian Eddington–Finkelstein coordinates from Ref. [81]). This is true whether the small object is a material body or a black hole. To see this, we note that the multipole moments of the small body entirely encode the physical composition of the object. Thus, by our matching tableau in Eq. (2.55), any quadrupole (and higher) corrections will not appear until order ϵ^3 (or higher). The metric is written in the lightcone gauge, with condition

$$H^n_{\mu\mathsf{a}} n^{\mathsf{a}} = 0, \tag{3.22}$$

where the serif font refers to Cartesian advanced Eddington–Finkelstein coordinates, $(\mathsf{v}, \mathsf{x}^{\mathsf{a}})$, defined on the manifold of $g^{\rm obj}_{\mu\nu}$ and $n^{\mathsf{a}} = \mathsf{x}^{\mathsf{a}}/\mathsf{r}$ with $\delta_{\mathsf{ab}} n^{\mathsf{a}} n^{\mathsf{b}} = 1$ and $\mathsf{r} := \sqrt{\delta_{\mathsf{ab}} \mathsf{x}^{\mathsf{a}} \mathsf{x}^{\mathsf{b}}}$. The gauge condition from Eq. (3.22) ensures that lightcones in the background spacetime remain lightcones in the perturbed spacetime. This means that v is constant on lightcones, r is both an affine parameter on the light rays generating these cones and the distance along the null rays from the small object, and n^{a} gives the direction of each individual light ray [63, Ch. 10, 81]. Ref. [174] also refines the gauge to enforce that $H^1_{\mu\nu} = 0$, completely eliminating the middle column of Eq. (2.55). The Eddington–Finkelstein coordinates ensure that no mass dipole term appears at leading order and $H^1_{\mu\nu} = 0$ ensures that no term of that form appears at first subleading order. This enforces that the coordinates are mass centered.

The form of the metric we get for the inner expansion is in terms of $\tilde{\mathsf{r}} := \mathsf{r}/\epsilon$ and is given explicitly by Eqs. (61)–(64) in Ref. [81]. This is then re-expanded for small ϵ at fixed r and then transformed from Eddington–Frankenstein coordinates to Fermi–Walker coordinates using the transformation given by Eq. (65) from Ref. [81]. Our resulting metric is valid in the buffer region and is given by

$$\mathsf{g}_{\mu\nu} = \mathring{g}_{\mu\nu} + \epsilon \mathring{h}^{1'}_{\mu\nu} + \epsilon^2 \mathring{h}^{2'}_{\mu\nu} + \mathcal{O}\Big(\epsilon^3\Big), \tag{3.23}$$

in the outer expansion. The primes indicate that the perturbations are in the lightcone rest gauge. The overset rings indicate that the expansion is organised slightly differently than Eqs. (2.1)–(2.2).

The leading term in Eq. (3.23) is

$$\mathring{g}_{tt} = -1 - r^2 \mathcal{E}_{ab}\hat{n}^{ab} - \frac{r^3}{3}\mathcal{E}_{abc}\hat{n}^{abc} + \mathcal{O}\Big(r^4\Big), \tag{3.24a}$$

$$\mathring{g}_{ta} = -\frac{2}{3}r^2 \mathcal{B}^{bc}\epsilon_{acd}\hat{n}_b{}^d + \frac{r^3}{60}\Big(3\dot{\mathcal{E}}_{ab}\hat{n}^b - 5\dot{\mathcal{E}}_{bc}\hat{n}_a{}^{bc} - 20\mathcal{B}^{bcd}\epsilon_{ab}{}^i\hat{n}_{cdi}\Big) + \mathcal{O}\Big(r^4\Big), \tag{3.24b}$$

$$\begin{aligned}\mathring{g}_{ab} = \delta_{ab} - \frac{r^2}{9}\Big(\mathcal{E}_{ab} - 6\mathcal{E}_{(a}{}^c\hat{n}_{b)c} + 3\mathcal{E}^{cd}\delta_{ab}\hat{n}_{cd}\Big) + \frac{r^3}{90}\Big(30\mathcal{E}_{(a}{}^{cd}\hat{n}_{b)c} - 3\mathcal{E}_{abc}\hat{n}^c \\ - 8\dot{\mathcal{B}}_{(a}{}^d\epsilon_{b)cd}\hat{n}^c + 10\dot{\mathcal{B}}^{cd}\epsilon_{c(a}{}^i\hat{n}_{b)di} - 15\delta_{ab}\mathcal{E}_{cdi}\hat{n}^{cdi}\Big) + \mathcal{O}\Big(r^4\Big),\end{aligned} \tag{3.24c}$$

which is Eq. (2.120) but with the acceleration terms set to zero. We also see from this that $r = 0$ is a geodesic in this new background as there are no acceleration terms in Eq. (3.24), so the Christoffel symbols vanish when evaluated on the worldline. The inner expansion implicitly expanded that acceleration,

$$a^\mu = \sum_{n>0} \epsilon^n f^\mu_n, \tag{3.25}$$

so that any acceleration terms have implicitly been moved to the first- or second-order perturbations. Thus we expand our perturbations in terms of any acceleration terms that may be in them.

The full background metric then becomes

$$g_{\mu\nu} = {}^0g_{\mu\nu} + \epsilon\, {}^1g_{\mu\nu} + \epsilon^2\, {}^2g_{\mu\nu} + \mathcal{O}\Big(\epsilon^3\Big), \tag{3.26}$$

where

$$^{0}g_{\mu\nu} = \mathring{g}_{\mu\nu}, \tag{3.27}$$
$$^{1}g_{\mu\nu} = -2f^{1}_{i}x^{i}\delta^{t}_{\mu}\delta^{t}_{\nu} + \mathcal{O}\Big(r^{3}\Big), \tag{3.28}$$
$$^{2}g_{\mu\nu} = -2f^{2}_{i}x^{i}\delta^{t}_{\mu}\delta^{t}_{\nu} + \mathcal{O}\Big(r^{2}\Big). \tag{3.29}$$

Here we have introduced the notation

$$A = {}^{0}A + \epsilon\, {}^{1}A + \epsilon^{2}\, {}^{2}A + \mathcal{O}\Big(\epsilon^{3}\Big), \tag{3.30}$$

where the prescript is the order of the acceleration term, to denote their re-expansion for small acceleration. The metric perturbations are then written as

$$\mathring{h}^{1'}_{\mu\nu} = {}^{0}h^{1'}_{\mu\nu} + {}^{1}g_{\mu\nu}, \tag{3.31}$$
$$\mathring{h}^{2'}_{\mu\nu} = {}^{0}h^{2'}_{\mu\nu} + {}^{1}h^{1'}_{\mu\nu} + {}^{2}g_{\mu\nu}, \tag{3.32}$$

so that

$$\mathring{h}^{n}_{\mu\nu} = \sum_{i=0}^{n} {}^{i}h^{n-i}_{\mu\nu}, \tag{3.33}$$

where we define ${}^{n}h^{0}_{\mu\nu} \coloneqq {}^{n}g_{\mu\nu}$. This expansion was originally introduced in Ref. [81], where a dagger was used in place of an overset ring.

The first-order term in Eq. (3.23) reads

$$\mathring{h}^{1'}_{\mu\nu} = \mathring{h}^{\mathrm{R}1'}_{\mu\nu} + \mathring{h}^{\mathrm{S}1'}_{\mu\nu}. \tag{3.34}$$

The regular field,

$$\mathring{h}^{\mathrm{R}1'}_{\mu\nu} = {}^{0}h^{\mathrm{R}1'}_{\mu\nu} + {}^{1}g_{\mu\nu}, \tag{3.35}$$

is given by

$${}^{0}h^{\mathrm{R}1'}_{tt} = -r^{2}\delta\mathcal{E}_{ab}\hat{n}^{ab} + \mathcal{O}\Big(r^{3}\Big), \tag{3.36a}$$
$${}^{0}h^{\mathrm{R}1'}_{ta} = -\frac{2}{3}r^{2}\delta\mathcal{B}^{bc}\epsilon_{acd}\hat{n}_{b}{}^{d} + \mathcal{O}\Big(r^{3}\Big), \tag{3.36b}$$
$${}^{0}h^{\mathrm{R}1'}_{ab} = -\frac{1}{9}r^{2}\Big(\delta\mathcal{E}_{ab} - 6\delta\mathcal{E}_{(a}{}^{c}\hat{n}_{b)c} + 3\delta_{ab}\delta\mathcal{E}_{cd}\hat{n}^{cd}\Big) + \mathcal{O}\Big(r^{3}\Big), \tag{3.36c}$$

and Eq. (3.28) where $\delta\mathcal{E}_{ab}$ and $\delta\mathcal{B}_{ab}$ are corrections to the respective tidal moments; these have the identical forms to the tidal terms from the background metric in Eq. (3.24) which implies that this is a smooth vacuum perturbation at $r = 0$. The singular field

$$\mathring{h}^{\mathrm{S}1'}_{\mu\nu} = {}^{0}h^{\mathrm{S}1'}_{\mu\nu}, \tag{3.37}$$

is given by

$$\mathring{h}^{\mathrm{S1}'}_{tt} = \frac{2m}{r} + \frac{11}{3}mr\mathcal{E}_{ab}\hat{n}^{ab} + \frac{1}{12}mr^2\bigg(8\dot{\mathcal{E}}_{ab}\hat{n}^{ab}\bigg[5 - 3\log\bigg(\frac{2m}{r}\bigg)\bigg] + 19\dot{\mathcal{E}}_{abc}\hat{n}^{abc}\bigg) + \mathcal{O}\Big(r^3\Big), \tag{3.38a}$$

$$\begin{aligned}\mathring{h}^{\mathrm{S1}'}_{ta} = {} & \frac{2m}{r}\hat{n}_a + \frac{2}{15}mr\Big(11\mathcal{E}_{ab}\hat{n}^b + 10\mathcal{B}^{bc}\epsilon_{acd}\hat{n}_b{}^d + 15\mathcal{E}_{bc}\hat{n}_a{}^{bc}\Big) + \frac{1}{1260}mr^2\Big(126\dot{\mathcal{E}}_{ab}\hat{n}^b \\ & \times\bigg[25 - 16\log\bigg(\frac{2m}{r}\bigg)\bigg] + 140\dot{\mathcal{B}}^{bc}\epsilon_{acd}\hat{n}_b{}^d\bigg[13 - 12\log\bigg(\frac{2m}{r}\bigg)\bigg] + 1095\mathcal{E}_{abc}\hat{n}^{bc} \\ & + 70\dot{\mathcal{E}}^{bc}\hat{n}_{abc}\bigg[25 - 12\log\bigg(\frac{2m}{r}\bigg)\bigg] + 1400\mathcal{B}^{bcd}\epsilon_{ab}{}^i\hat{n}_{cdi} + 840\mathcal{E}_{bcd}\hat{n}_a{}^{bcd}\Big) + \mathcal{O}\Big(r^3\Big),\end{aligned} \tag{3.38b}$$

$$\begin{aligned}\mathring{h}^{\mathrm{S1}'}_{ab} = {} & \frac{2m}{3r}\Big(\delta_{ab} + 3\hat{n}_{ab}\Big) + \frac{1}{315}mr\Big(154\mathcal{E}_{ab} - 168\mathcal{B}^d_{(a}\epsilon_{b)cd}\hat{n}^c + 480\mathcal{E}^c{}_{(a}\hat{n}_{b)c} + 15\mathcal{E}_{cd}\delta_{ab}\hat{n}^{cd} \\ & + 840\mathcal{B}^{cd}\epsilon_c{}^i{}_{(a}\hat{n}_{b)di} + 105\mathcal{E}_{cd}\hat{n}_{ab}{}^{cd}\Big) + \frac{1}{3780}mr^2\bigg(252\dot{\mathcal{E}}_{ab}\bigg[29 - 20\log\bigg(\frac{2m}{r}\bigg)\bigg] \\ & + 2322\mathcal{E}_{abc}\hat{n}^c - 504\dot{\mathcal{B}}^d{}_{(a}\epsilon_{b)cd}\hat{n}^c\bigg[11 - 12\log\bigg(\frac{2m}{r}\bigg)\bigg] + 1980\dot{\mathcal{E}}^c{}_{(a}\hat{n}_{b)c} + 60\dot{\mathcal{E}}_{cd}\delta_{ab}\hat{n}^{cd} \\ & \times\bigg[59 - 42\log\bigg(\frac{2m}{r}\bigg)\bigg] - 4800\mathcal{B}_{(a|c|}{}^i\epsilon_{b)di}\hat{n}^{cd} - 420\mathcal{E}_{(a}{}^{cd}\hat{n}_{b)cd} + 1680\dot{\mathcal{B}}^{cd}\epsilon_c{}^i{}_{(a}\hat{n}_{b)di} \\ & \times\bigg[4 - 3\log\bigg(\frac{2m}{r}\bigg)\bigg] + 1295\dot{\mathcal{E}}_{cdi}\delta_{ab}\hat{n}^{cdi} + 1260\dot{\mathcal{E}}_{cd}\hat{n}_{ab}{}^{cd} + 5040\mathcal{B}^{cdi}\epsilon_c{}^j{}_{(a}\hat{n}_{b)dij} \\ & + 315\mathcal{E}_{cdi}\hat{n}_{ab}{}^{cdi}\bigg) + \mathcal{O}\Big(r^3\Big).\end{aligned} \tag{3.38c}$$

The regular field features terms of the form $\sim \delta\mathcal{E}_{ab}$ or $\delta\mathcal{B}_{ab}$, while the singular field features terms that have explicit factors of m. This ensures that the regular field satisfies the vacuum Einstein equations, as laid out in Ch. 1.4.

The second-order term in Eq. (3.23) reads

$$\mathring{h}^{2'}_{\mu\nu} = \mathring{h}^{\mathrm{R2}'}_{\mu\nu} + \mathring{h}^{\mathrm{S2}'}_{\mu\nu}. \tag{3.39}$$

The regular field

$$\mathring{h}^{\mathrm{R2}'}_{\mu\nu} = {}^0h^{\mathrm{R2}'}_{\mu\nu} + {}^1h^{\mathrm{R1}'}_{\mu\nu} + {}^2g_{\mu\nu} \tag{3.40}$$

is given by

$$\mathring{h}^{\mathrm{R2}'}_{\mu\nu} = \mathcal{O}\Big(r^2\Big). \tag{3.41}$$

The singular field,

$$\mathring{h}^{\mathrm{S2}'}_{\mu\nu} = {}^0h^{\mathrm{S2}'}_{\mu\nu} + {}^1h^{\mathrm{S1}'}_{\mu\nu}, \tag{3.42}$$

is split into two pieces,

$$\mathring{h}^{\mathrm{S2}'}_{\mu\nu} = \mathring{h}^{\mathrm{SS}'}_{\mu\nu} + \mathring{h}^{\mathrm{SR}'}_{\mu\nu}, \tag{3.43}$$

where $\mathring{h}^{\mathrm{SS}'}_{\mu\nu}$ is the ‘singular times singular’ piece containing all terms proportional to m^2 and $\mathring{h}^{\mathrm{SR}'}_{\mu\nu}$ is the ‘singular times regular’ piece featuring all terms with the form $m\delta\mathcal{E}_{ab}$

and $m\delta\mathcal{B}_{ab}$. Individually, these are

$$\mathring{h}^{\mathrm{SR}'}_{tt} = \frac{11}{3}mr\delta\mathcal{E}_{ab}\hat{n}^{ab}, \tag{3.44a}$$

$$\mathring{h}^{\mathrm{SR}'}_{ta} = \frac{2}{15}mr\Big(11\delta\mathcal{E}_{ab}\hat{n}^{b} + 10\delta\mathcal{B}^{bc}\epsilon_{acd}\hat{n}_{b}{}^{d} + 15\delta\mathcal{E}_{bc}\hat{n}_{a}{}^{bc}\Big), \tag{3.44b}$$

$$\begin{aligned}\mathring{h}^{\mathrm{SR}'}_{ab} = \frac{1}{315}mr\Big(&154\delta\mathcal{E}_{ab} - 168\delta\mathcal{B}^{d}{}_{(a}\epsilon_{b)cd}\hat{n}^{c} + 480\delta\mathcal{E}^{c}_{(a}\hat{n}_{b)c} + 15\delta\mathcal{E}_{cd}\delta_{ab}\hat{n}^{cd}\\ &+ 840\delta\mathcal{B}^{cd}\epsilon_{c}{}^{i}{}_{(a}\hat{n}_{b)di} + 105\delta\mathcal{E}_{cd}\hat{n}_{ab}{}^{cd}\Big),\end{aligned} \tag{3.44c}$$

and[2]

$$\mathring{h}^{\mathrm{SS}'}_{tt} = -4m^2\Big[\mathcal{E}_{ab}\hat{n}^{ab} + r\Big(\frac{1}{3}\dot{\mathcal{E}}_{ab}\hat{n}^{ab}\Big\{11 - 6\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{2}{3}\mathcal{E}_{abc}\hat{n}^{abc}\Big)\Big] + \mathcal{O}\Big(r^2\Big), \tag{3.45a}$$

$$\begin{aligned}\mathring{h}^{\mathrm{SS}'}_{ta} = -4m^2\Big[&\frac{2}{5}\mathcal{E}_{ab}\hat{n}^{b} + \mathcal{E}_{bc}\hat{n}_{a}{}^{bc} + r\Big(\frac{6}{5}\dot{\mathcal{E}}_{ab}\hat{n}^{b}\Big\{2 - \log\Big(\frac{2m}{r}\Big)\Big\} + \frac{8}{21}\mathcal{E}_{abc}\hat{n}^{bc}\\ &+ \frac{2}{9}\dot{\mathcal{B}}^{bc}\epsilon_{acd}\hat{n}_{b}{}^{d}\Big\{4 - \log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{9}\dot{\mathcal{E}}_{bc}\hat{n}_{a}{}^{bc}\Big\{19 - 12\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{2}\mathcal{E}_{bcd}\hat{n}_{a}{}^{bcd}\\ &+ \frac{2}{9}\mathcal{B}^{bcd}\epsilon_{ab}{}^{i}\hat{n}_{cdi}\Big)\Big] + \mathcal{O}\Big(r^2\Big),\end{aligned} \tag{3.45b}$$

$$\begin{aligned}\mathring{h}^{\mathrm{SS}'}_{ab} = -4m^2\Big[&\frac{4}{5}\mathcal{B}^{d}{}_{(a}\epsilon_{b)cd}\hat{n}^{c} + \frac{8}{7}\mathcal{E}^{c}{}_{(a}\hat{n}_{b)c} - \frac{1}{21}\mathcal{E}_{cd}\delta_{ab}\hat{n}^{cd} + \mathcal{B}^{cd}\epsilon_{c}{}^{i}{}_{(a}\hat{n}_{b)di} + \frac{5}{6}\mathcal{E}_{cd}\hat{n}_{ab}{}^{cd}\\ &+ r\Big(\frac{2}{45}\dot{\mathcal{E}}_{ab}\Big\{31 - 12\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{4}{21}\mathcal{E}_{abc}\hat{n}^{c} - \frac{4}{45}\dot{\mathcal{B}}^{d}{}_{(a}\epsilon_{b)cd}\hat{n}^{c}\Big\{4 - 3\log\Big(\frac{2m}{r}\Big)\Big\}\\ &+ \frac{4}{7}\dot{\mathcal{E}}^{c}{}_{(a}\hat{n}_{b)c}\Big\{4 - 3\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{63}\dot{\mathcal{E}}_{cd}\delta_{ab}\hat{n}^{cd}\Big\{29 - 6\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{5}{9}\mathcal{E}_{cd(a}\hat{n}_{b)}{}^{cd}\\ &- \frac{8}{63}\mathcal{B}_{c}{}^{i}{}_{(a}\epsilon_{b)di}\hat{n}^{cd} + \frac{1}{27}\mathcal{E}_{cdi}\delta_{ab}\hat{n}^{cdi} + \frac{4}{9}\mathcal{B}^{cdi}\epsilon_{c}{}^{j}{}_{(a}\hat{n}_{b)dij} + \frac{1}{3}\mathcal{E}^{cdi}\hat{n}_{abcdi}\\ &+ \frac{4}{9}\dot{\mathcal{B}}^{cd}\epsilon_{c}{}^{i}{}_{(a}\hat{n}_{b)di}\Big\{4 - 3\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{2}{9}\dot{\mathcal{E}}^{cd}\hat{n}_{abcd}\Big\{4 - 3\log\Big(\frac{2m}{r}\Big)\Big\}\Big)\Big] + \mathcal{O}\Big(r^2\Big).\end{aligned} \tag{3.45c}$$

This split can be extended to any order in r by including all explicitly m-dependent terms in the singular fields and leaving the regular field to include all terms featuring tidal moments and no explicit m dependence. The regular field is then a smooth solution to the vacuum Einstein equations,

$$\delta\mathring{G}_{\mu\nu}[\mathring{h}^{\mathrm{R}1'}] = 0, \tag{3.46}$$

$$\delta\mathring{G}^{\mu\nu}[\mathring{h}^{\mathrm{R}2'}] = -\delta^2\mathring{G}_{\mu\nu}[\mathring{h}^{\mathrm{R}1'}], \tag{3.47}$$

which are the linearised and second-order Einstein operators constructed from $\mathring{g}_{\mu\nu}$. When combined with $\mathring{g}_{\mu\nu}$, the regular field forms an effective metric,

$$\tilde{g}'_{\mu\nu} = \mathring{g}_{\mu\nu} + \epsilon\mathring{h}^{\mathrm{R}1'}_{\mu\nu} + \epsilon^2\mathring{h}^{\mathrm{R}2'}_{\mu\nu} + \mathcal{O}\Big(\epsilon^3\Big), \tag{3.48}$$

as in Eq. (3.7), which is a vacuum metric, and in which the small object follows a geodesic.

[2] All $\mathcal{O}\big(m^0\big)$ terms in Eq. (116) in Ref. [81] have been corrected to be $\mathcal{O}\big(m^2\big)$.

Note that by inspection of the metric perturbations, we immediately see that in this gauge we $h^2_{\mu\nu} \sim r^0$ instead of the generic behaviour $\sim 1/r^2$. One final step remains, however, and that is to perform one final gauge transformation to get this form of the metric into a "practical" gauge with specific, useful properties.

3.5 Highly regular gauge

As discussed in Ch. 1.5, the highly regular gauge was introduced by Pound [81] to ameliorate problems that appear in the Lorenz gauge due to divergent terms and to be a more practical alternative to the lightcone gauge in Ch. 3.4. This chapter outlines the method used in Ref. [81] to derive the leading-order form of the second-order singular field in the highly regular gauge. We will then use this method in Ch. 4 to derive the full second-order singular field needed to correctly calculate the second-order self-force.

The lightcone rest gauge from Ch. 3.4 completely eliminates the $\sim 1/r^2$ pieces of $h^2_{\mu\nu}$ that appear in the Lorenz gauge. However, the "rest gauge" aspect forces the regular field to behave as $\sim r^2$, meaning that the regular field and its first derivative vanish when evaluated on the worldline. There is no obvious way to solve the field equations numerically in this gauge so another gauge transformation is required to get the metric into a more amenable form.

As in the previous section, the discussion in this section follows that originally presented in Ref. [81]. Generically, to perform a gauge transformation we follow the method as discussed in Ch. 2.1.3. Following Eqs. (2.43) and (2.44), we know that the perturbations transform as

$$h^1_{\mu\nu} \to h^1_{\mu\nu} + \mathcal{L}_{\xi_1} g_{\mu\nu}, \tag{3.49}$$

$$h^2_{\mu\nu} \to h^2_{\mu\nu} + \frac{1}{2}\Big(\mathcal{L}^2_{\xi_1} + 2\mathcal{L}_{\xi_2}\Big) g_{\mu\nu} + \mathcal{L}_{\xi_1} h^1_{\mu\nu}. \tag{3.50}$$

where $\xi^\mu = \epsilon\xi^\mu_1 + \epsilon^2\xi^\mu_2 + \mathcal{O}(\epsilon^3)$ is a smooth gauge vector. However, we want to consider how the regular and singular fields transform, and we must account for the fact that we have written our perturbations as perturbations of $\mathring{g}_{\mu\nu}$ and not $g_{\mu\nu}$. With a split in to regular and singular fields, our gauge transformation rules become

$$\mathring{h}^{\rm R1}_{\mu\nu} = \mathring{h}^{\rm R1'}_{\mu\nu} + \mathcal{L}_{\xi_1}\mathring{g}_{\mu\nu}, \tag{3.51}$$

$$\mathring{h}^{\rm S1}_{\mu\nu} = \mathring{h}^{\rm S1'}_{\mu\nu}, \tag{3.52}$$

$$\mathring{h}^{\rm R2}_{\mu\nu} = \mathring{h}^{\rm R2'}_{\mu\nu} + \mathcal{L}_{\xi_2}\mathring{g}_{\mu\nu} + \frac{1}{2}\mathcal{L}^2_{\xi_1}\mathring{g}_{\mu\nu} + \mathcal{L}_{\xi_1}\mathring{h}^{\rm R1'}_{\mu\nu}, \tag{3.53}$$

$$\mathring{h}^{\rm S2}_{\mu\nu} = \mathring{h}^{\rm S2'}_{\mu\nu} + \mathcal{L}_{\xi_1}\mathring{h}^{\rm S1'}_{\mu\nu}, \tag{3.54}$$

where we define our new regular/singular fields such that they contain the old regular/singular fields and Lie derivatives acting upon them. Doing so ensures that the effective metric, $\tilde{g}_{\mu\nu}$, transforms as any smooth vacuum metric would under a gauge transformation, meaning that

$$\tilde{g}_{\mu\nu} = \mathring{g}_{\mu\nu} + \epsilon \mathring{h}^{\rm R1}_{\mu\nu} + \epsilon^2 \mathring{h}^{\rm R2}_{\mu\nu} + \mathcal{O}\Big(\epsilon^3\Big), \tag{3.55}$$

remains a vacuum metric and any geodesics remain geodesics after performing the gauge transformation. Apart from smoothness, we only impose one other condition, that of the transformation being *worldline preserving*,

$$\xi^n_a\Big|_\gamma = 0. \tag{3.56}$$

This ensures that the worldline of the small object after the gauge transformation is identical to the one in the rest gauge. An alternative way to say this is that we ensure that no mass dipoles (or corrections to mass dipoles) are introduced as a result of our transformation.

Due to our gauge transformation, the highly regular gauge inherits the properties of the lightcone rest gauge presented in the previous chapter but only on the singular field. That is, the singular field satisfies the gauge condition from Eq. (3.22). Covariantly and in the outer expansion, the highly regular gauge condition can be written as

$$h^{\rm S}_{\mu\nu} k^\mu = 0, \tag{3.57}$$

where k^μ is a future directed null vector that emanates from the worldline, γ, and is tangent to the lightcone of the small object along radially outgoing curves. The original lightcone metric is also trace-free over the unit sphere, so that

$$h^{\rm S}_{\mu\nu} e^\mu_A e^\nu_B \Omega^{AB} = 0, \tag{3.58}$$

where Ω_{AB} is the metric on surfaces of constant luminosity distance and an upper case Latin letter indicates a quantity defined on those surfaces. The final quantity is the basis vector, $e^\mu_A := \partial x^\mu / \partial \theta^A$, where x^μ are coordinates in the full spacetime and θ^A are coordinates on the surface of constant luminosity distance. As discussed under Eq. (3.22), these gauge conditions ensure that the local background lightcone structure is preserved in the perturbed spacetime and that the background luminosity distance (given previously by r) is the distance from the small object and an affine parameter on the null rays that generate the lightcones as well. An image showing the geometric construction is given in Fig. 3.2.

As in the first calculation for the form of the highly regular gauge [81], we use the approach suggested in Ref. [94] and solve for ξ^μ in terms of the regular fields, $h^{\rm R}_{\mu\nu}$. Here, we allow the regular fields to be in any arbitrary gauge, and solve Eqs. (3.51) and (3.53) for ξ^μ_n in terms of $h^{{\rm R}n}_{\mu\nu}$. One can then choose their preferred gauge for $h^{{\rm R}n}_{\mu\nu}$ and impose the gauge conditions on the resulting form of the singular field.

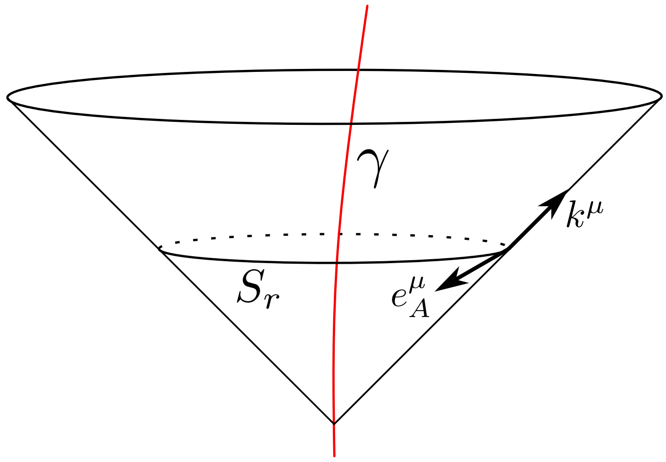

FIGURE 3.2: Geometric picture of the gauge conditions for the highly regular gauge. The image features a lightcone emanating from the worldline, γ. The null vector, k^μ, is tangent to the lightcone along radially outgoing curves and the basis vector, e^μ_A, is tangent to the lightcone along spheres of constant luminosity distance, S_r. Based on Fig. 16 from Ref. [175].

After finding ξ^μ_1, we can calculate the second-order singular field in the new gauge via Eq. (3.54). Despite the gauge transformation being smooth, it introduces an unbounded term into $h^{\rm S2}_{\mu\nu}$: $\mathcal{L}_{\xi_1} h^{\rm S1'}_{\mu\nu}$, which behaves as $\sim 1/r$. This is more divergent than the singular field in the original lightcone rest gauge but, as we shall demonstrate in Ch. 5, a 'singular times regular' term is much easier to deal with than a 'singular times singular' field even if it appears that the 'singular times regular' term is more divergent in r.

Not only do we determine the gauge vector in terms of the regular field, we also fully determine the other functions in the effective metric in terms of $h^{\rm R1}_{\mu\nu}$. These are the form of the acceleration vectors f^μ_n along with the perturbed tidal moments $\delta\mathcal{E}_{ab}$ and $\delta\mathcal{B}_{ab}$. After making the gauge transformation, our full metric has the form

$$\mathsf{g}_{\mu\nu} = \mathring{g}_{\mu\nu} + \epsilon \overbrace{\left({}^0h^1_{\mu\nu} + {}^1g_{\mu\nu}\right)}^{\mathring{h}^1_{\mu\nu}} + \epsilon^2 \overbrace{\left({}^0h^2_{\mu\nu} + {}^1h^1_{\mu\nu} + {}^2g_{\mu\nu}\right)}^{\mathring{h}^2_{\mu\nu}} + \mathcal{O}\left(\epsilon^3\right). \tag{3.59}$$

We then define the regular/singular split to be[3]

$$\mathring{h}^{\rm R1}_{\mu\nu} := {}^0h^{\rm R1}_{\mu\nu} + {}^1g_{\mu\nu}, \tag{3.60}$$
$$\mathring{h}^{\rm S1}_{\mu\nu} := {}^0h^{\rm S1}_{\mu\nu}, \tag{3.61}$$
$$\mathring{h}^{\rm R2}_{\mu\nu} := {}^0h^{\rm R2}_{\mu\nu} + {}^1h^{\rm R1}_{\mu\nu} + {}^2g_{\mu\nu}, \tag{3.62}$$
$$\mathring{h}^{\rm S2}_{\mu\nu} := {}^0h^{\rm S2}_{\mu\nu} + {}^1h^{\rm S1}_{\mu\nu}. \tag{3.63}$$

We wish to extract the perturbations that 'live' on the $g_{\mu\nu}$ background, *not* those on the background with no acceleration, $\mathring{g}_{\mu\nu}$. Thus, we wish to express ξ^μ_n in terms of $h^{\mathrm{R}n}_{\mu\nu}$, not

[3]The ${}^0g_{\mu\nu}$ term in Eq. (125) in Ref. [81] should read ${}^1g_{\mu\nu}$.

in terms of $\mathring{h}^{\mathrm{R}n}_{\mu\nu}$. To do so, we rewrite Eqs. (3.51) and (3.53) as

$$ {}^{0}h^{\mathrm{R1}}_{\mu\nu} = \mathring{h}^{\mathrm{R1}}_{\mu\nu} + \mathcal{L}_{\xi_1}\mathring{g}_{\mu\nu} - {}^{1}g_{\mu\nu}, \tag{3.64} $$

$$ {}^{0}h^{\mathrm{R2}}_{\mu\nu} + {}^{1}h^{\mathrm{R1}}_{\mu\nu} = \mathring{h}^{\mathrm{R2}'}_{\mu\nu} + \mathcal{L}_{\xi_2}\mathring{g}_{\mu\nu} - {}^{2}g_{\mu\nu} + \frac{1}{2}\mathcal{L}^{2}_{\xi_1}\mathring{g}_{\mu\nu} + \mathcal{L}_{\xi_1}\mathring{h}^{\mathrm{R1}'}_{\mu\nu}. \tag{3.65} $$

Here we have grouped all the unknowns (ξ^{μ}_{n}, f^{μ}_{n}, $\delta\mathcal{E}_{ab}$, and $\delta\mathcal{B}_{ab}$) on the right side of the equations.

We know that we must calculate $h^{\mathrm{S2}}_{\mu\nu}$ to order r as required to calculate the second-order self-force. This follows from Eq. (3.15), where we impose that the first derivative of the puncture must match the first derivative of the singular field. We already have $h^{\mathrm{S2}'}_{\mu\nu}$ from Eqs. (3.43)–(3.45) and $\mathcal{L}_{\xi_1}h^{\mathrm{S1'}}_{\mu\nu}$ was previously given in Ref. [81] to order $1/r$. In the following chapter we expand the calculation for $\mathcal{L}_{\xi_1}h^{\mathrm{S1'}}_{\mu\nu}$ up to and including order r by first calculating ξ^{μ}_{1} through order r^2. We then present the full second-order perturbations in the highly regular gauge.

Chapter 4

Transformation to the highly regular gauge from the lightcone rest gauge

In this chapter, we will present the full calculation for the second-order singular field in the highly regular gauge. As discussed in Ch. 1.5, the highly regular gauge introduced by Pound [81] features a weaker singularity structure than any other previously considered gauge in the gravitational self-force literature. It eliminates the most singular part of the second-order source that causes problems when solving the Einstein equations numerically near to the worldline of the small object. Not only that, by increasing the regularity of the second-order source, one can write down well-defined field equations that are valid as distributions everywhere in spacetime; the consequences of which will be explored further in Ch. 5.

In order to use the highly regular gauge in solving the field equations, such as in the puncture scheme detailed in Ch. 3.3, one requires the second-order singular field through order r. This ensures that one can accurately construct a residual field that mimics the regular field to a sufficiently high order, allowing one to construct an equation of motion that accurately tracks the small object. In the original paper introducing the highly regular gauge, Pound [81] presented the leading-order, $1/r$ part of the perturbations. In this chapter, we extend this calculation to the required order in r by using the scheme presented in Ch. 3.5.

This chapter is organised as follows: Ch. 4.1 describes the decomposition of the gauge vector and regular field into irreducible STF tensors, and Ch. 4.2 describes the process of solving for the gauge vector order-by-order in r. The final form of the second-order singular field is then presented in Ch. 4.3. The material in this chapter was published in Ref. [140].

The results in this chapter act as input for the calculations in Ch. 6. There, we convert the final expressions for the second-order singular field from the Fermi–Walker coordinates they are presented in in this chapter into both covariant form and into a generic coordinate expansion.

4.1 STF decomposition of the gauge vector and the regular field

To solve Eq. (3.64) for the gauge vector, we begin by expanding both ξ^1_μ (with index down) and ${}^0h^{\rm R1}_{\mu\nu}$ in irreducible STF form using App. A of Ref. [176] and App. B of Ref. [63].

4.1.1 Gauge vector

The gauge vector is decomposed as

$$\xi^1_\mu = \sum_{p,l\geq 0} r^p \xi^{(p,l)}_{\mu L}(t)\hat{n}^L, \tag{4.1}$$

where the t and a components are, respectively, given by

$$\xi^{(p,l)}_{t\langle L\rangle} = \hat{T}^{(p,l)}_L, \tag{4.2a}$$

$$\xi^{(p,l)}_{a\langle L\rangle} = \hat{X}^{(p,l)}_{aL} + \epsilon^j{}_{a\langle i_l}\hat{Y}^{(p,l)}_{L-1\rangle j} + \delta_{a\langle i_l}\hat{Z}^{(p,l)}_{L-1\rangle}, \tag{4.2b}$$

with the hat indicating that these are STF tensors. Each term in this decomposition is linearly independent from the others. The quantities $\hat{n}^L$ form a complete basis, equivalent to scalar spherical harmonics, for scalar fields on the unit sphere, and the further decomposition of Cartesian 3-vectors and 3-tensors into irreducible STF pieces is equivalent to a decomposition into spin-weighted or tensor spherical harmonics.

As mentioned we only impose two conditions on ξ^1_μ: Firstly, that ξ^1_μ is smooth so that our two gauges are smoothly related and secondly, that ξ^1_μ is *worldline preserving*, satisfying Eq. (3.56). These conditions imply that the expansion (4.1) must be equivalent to a Taylor series

$$\xi^1_\mu = \sum_{k\geq 0}\frac{1}{k!}\partial_K\xi^1_\mu(t,0)x^K \tag{4.3}$$

with $\xi^1_a(t,0) = 0$. Here $x^K = x^{i_1}\cdots x^{i_k}$. When written as a sum of STF quantities,

$$x^K = r^k[\hat{n}^K + c_1\delta^{(a_1a_2}\hat{n}^{K-2)} + c_2\delta^{(a_1a_2}\delta^{a_3a_4}\hat{n}^{K-4)} + \ldots] \tag{4.4}$$

for some some numerical coefficients c_n. Hence, our conditions on the gauge vector impose

$$\xi^1_t = \hat{T}^{(0,0)} + r\hat{n}^a\,\hat{T}^{(1,1)}_a + r^2\Big(\hat{T}^{(2,0)} + \hat{T}^{(2,2)}_{ab}\hat{n}^{ab}\Big) + r^3\Big(\hat{T}^{(3,1)}_a\hat{n}^a + \hat{T}^{(3,3)}_{abc}\hat{n}^{abc}\Big) + \mathcal{O}\Big(r^4\Big), \tag{4.5a}$$

$$\begin{aligned}\xi^1_a = {} & r\hat{n}^b\Big(\hat{X}^{(1,1)}_{ab} + \epsilon^j{}_{ab}\hat{Y}^{(1,1)}_j + \delta_{ab}\hat{Z}^{(1,1)}\Big)\\ & + r^2\Big[\hat{X}^{(2,0)}_a + \hat{n}^{bc}\Big(\hat{X}^{(2,2)}_{abc} + \epsilon^j{}_{ab}\hat{Y}^{(2,2)}_{cj} + \delta_{ab}\hat{Z}^{(2,2)}_c\Big)\Big]\\ & + r^3\Big[\hat{n}^b\Big(\hat{X}^{(3,1)}_{ab} + \epsilon^j{}_{ab}\hat{Y}^{(3,1)}_j + \delta_{ab}\hat{Z}^{(3,1)}\Big) + \hat{n}^{bcd}\Big(\hat{X}^{(3,3)}_{abcd} + \epsilon^j{}_{ab}\hat{Y}^{(3,3)}_{cdj} + \delta_{ab}\hat{Z}^{(3,3)}_{cd}\Big)\Big]\\ & + \mathcal{O}\Big(r^4\Big).\end{aligned} \tag{4.5b}$$

It is necessary to carry this expansion to order r^3 because the Lie derivative and the singular form of $\mathring{h}^{\mathrm{S1}'}_{\mu\nu}$ in Eq. (3.54) each reduce the order in r by one. Thus, order r^3 in the gauge vector is required for accuracy through order r in $\mathring{h}^{\mathrm{S2}}_{\mu\nu}$.

4.1.2 Regular field

We perform a similar decomposition for the regular field, so that

$${}^0h^{\mathrm{R1}}_{\mu\nu} = \sum_{p,l\geq 0} r^p\,{}^0h^{\mathrm{R1}(p,l)}_{\mu\nu L}(t)\hat{n}^L. \tag{4.6}$$

The tt, ta, and ab components are given by

$${}^0h^{\mathrm{R1}(p,l)}_{tt\langle L\rangle} = \hat{A}^{(p,l)}_L, \tag{4.7a}$$

$${}^0h^{\mathrm{R1}(p,l)}_{ta\langle L\rangle} = \hat{B}^{(p,l)}_{aL} + \epsilon^j{}_{a\langle i_l}\hat{C}^{(p,l)}_{L-1\rangle j} + \delta_{a\langle i_l}\hat{D}^{(p,l)}_{L-1\rangle}, \tag{4.7b}$$

$$\begin{aligned}{}^0h^{\mathrm{R1}(p,l)}_{ab\langle L\rangle} = {} & \hat{E}^{(p,l)}_{abL} + \delta_{ab}\hat{K}^{(p,l)}_L + \operatorname*{STF}_L\operatorname*{STF}_{ab}\Big(\epsilon^j{}_{ai_l}\hat{F}^{(p,l)}_{bjL-1} + \delta_{ai_l}\hat{G}^{(p,l)}_{bL-1}\\ & + \delta_{ai_l}\epsilon^j{}_{bi_{l-1}}\hat{H}^{(p,l)}_{jL-2} + \delta_{ai_l}\delta_{bi_{l-1}}\hat{I}^{(p,l)}_{L-2}\Big).\end{aligned} \tag{4.7c}$$

Since ${}^0h^{\mathrm{R1}}_{\mu\nu}$ is smooth, we require this expansion to be equivalent to a Taylor series in x^a. This leaves us with the expansion

$${}^0h^{\mathrm{R1}}_{tt} = \hat{A}^{(0,0)} + r\hat{A}^{(1,1)}_i\hat{n}^i + r^2\Big(\hat{A}^{(2,0)} + \hat{n}^{ij}\hat{A}^{(2,2)}_{ij}\Big) + \mathcal{O}\Big(r^3\Big), \tag{4.8a}$$

$$\begin{aligned}{}^0h^{\mathrm{R1}}_{ta} = {} & \hat{B}^{(0,0)}_a + r\hat{n}^i\Big(\hat{B}^{(1,1)}_{ai} + \epsilon^j{}_{ai}\hat{C}^{(1,1)}_j + \delta_{ai}\hat{D}^{(1,1)}\Big) + r^2\Big[\hat{B}^{(2,0)}_a\\ & + \hat{n}^{ij}\Big(\hat{B}^{(2,2)}_{aij} + \epsilon^k{}_{ai}\hat{C}^{(2,2)}_{jk} + \delta_{ai}\hat{D}^{(2,2)}_j\Big)\Big] + \mathcal{O}\Big(r^3\Big),\end{aligned} \tag{4.8b}$$

$$
\begin{aligned}
{}^0h^{\rm R1}_{ab} = {}& \hat{E}^{(0,0)}_{ab} + \delta_{ab}\hat{K}^{(0,0)} + r\hat{n}^i\Big[\hat{E}^{(1,1)}_{abi} + \delta_{ab}\hat{K}^{(1,1)}_i + \mathop{\rm STF}_{ab}\Big(\epsilon^k{}_{ai}\hat{F}^{(1,1)}_{bk} + \delta_{ai}\hat{G}^{(1,1)}_b\Big)\Big] \\
& + r^2\Big[\hat{E}^{(2,0)}_{ab} + \delta_{ab}\hat{K}^{(2,0)} + \hat{n}^{ij}\Big(\hat{E}^{(2,2)}_{abij} + \delta_{ab}\hat{K}^{(2,2)}_{ij} + \mathop{\rm STF}_{ab}\Big\{\epsilon^k{}_{ai}\hat{F}^{(2,2)}_{bkj} + \delta_{ai}\hat{G}^{(2,2)}_{bj} \\
& + \delta_{ai}\epsilon^k{}_{bj}\hat{H}^{(2,2)}_k + \delta_{ai}\delta_{bj}\hat{I}^{(2,2)}\Big\}\Big)\Big] + \mathcal{O}\Big(r^3\Big).
\end{aligned} \tag{4.8c}
$$

App. A gives the relation between the individual STF tensors and derivatives of the regular field evaluated on the worldline.

Additionally, we use constraints from the linearised vacuum Einstein equations

$$
\delta\mathring{G}_{\mu\nu}[{}^0h^{\rm R1}] = 0. \tag{4.9}
$$

Note that $\delta\mathring{G}_{\mu\nu}[\mathring{h}^{\rm R1}] = \delta\mathring{G}_{\mu\nu}[{}^0h^{\rm R1}]$ because ${}^1g_{\mu\nu}$ is a linear vacuum perturbation of $\mathring{g}_{\mu\nu}$.

The tt and ta components of Eq. (4.9) give

$$
\hat{I}^{(2,2)} = \frac{1}{5}\mathcal{E}^{ab}\hat{E}^{(0,0)}_{ab} + \frac{6}{5}\hat{K}^{(2,0)}, \tag{4.10a}
$$

$$
\hat{D}^{(2,2)}_a = \frac{6}{5}\hat{B}^{(2,0)}_a + \frac{3}{5}\hat{B}^{(0,0)}_b\mathcal{E}_a{}^b + \frac{3}{5}\mathcal{B}^{bc}\epsilon_{ac}{}^d\hat{E}^{(0,0)}_{bd} - \frac{1}{2}\frac{d}{dt}\hat{G}^{(1,1)}_a + \frac{3}{5}\frac{d}{dt}\hat{K}^{(1,1)}_a. \tag{4.10b}
$$

We use these equations to eliminate $\hat{I}^{(2,2)}$ and $\hat{D}^{(2,2)}_a$, but the choice is arbitrary; we could have easily chosen two other STF tensors to remove.

From the ab component of Eq. (4.9) we get two restrictions, one at $l = 0$ and one at $l = 2$. These are

$$
\hat{A}^{(2,0)} = -\frac{1}{3}\mathcal{E}^{ab}\hat{E}^{(0,0)}_{ab} + \frac{d}{dt}\hat{D}^{(1,1)} - \frac{1}{2}\frac{d^2}{dt^2}\hat{K}^{(0,0)}, \tag{4.11a}
$$

$$
\begin{aligned}
\hat{A}^{(2,2)}_{ab} = {}& \hat{A}^{(0,0)}\mathcal{E}_{ab} - 2\hat{B}^{(0,0)}_c\mathcal{B}_{d(a}\epsilon_{b)}{}^{cd} + \hat{E}^{(2,0)}_{ab} - 2\mathcal{E}^c{}_{\langle a}\hat{E}^{(0,0)}_{b\rangle c} - \frac{7}{6}\hat{G}^{(2,2)}_{ab} + \mathcal{E}_{ab}\hat{K}^{(0,0)} \\
& + \hat{K}^{(2,2)}_{ab} + \frac{d}{dt}\hat{B}^{(1,1)}_{ab} - \frac{1}{2}\frac{d^2}{dt^2}\hat{E}^{(0,0)}_{ab},
\end{aligned} \tag{4.11b}
$$

where the constraints from the tt and ta components have been used to simplify these expressions.

Combining Eqs. (4.8), (4.10), and (4.11) gives us the final expression for the components of ${}^0h^{\rm R1}_{\mu\nu}$:

$$
\begin{aligned}
{}^0h^{\rm R1}_{tt} = {}& \hat{A}^{(0,0)} + r\hat{A}^{(1,1)}_i\hat{n}^i + r^2\Big[-\frac{1}{3}\mathcal{E}^{ab}\hat{E}^{(0,0)}_{ab} + \frac{d}{dt}\hat{D}^{(1,1)} - \frac{1}{2}\frac{d^2}{dt^2}\hat{K}^{(0,0)} + \hat{n}^{ij}\Big(\hat{A}^{(0,0)}\mathcal{E}_{ij} \\
& - 2\hat{B}^{(0,0)}_c\mathcal{B}_{di}\epsilon_j{}^{cd} + \hat{E}^{(2,0)}_{ij} - 2\mathcal{E}^c{}_i\hat{E}^{(0,0)}_{jc} - \frac{7}{6}\hat{G}^{(2,2)}_{ij} + \mathcal{E}_{ij}\hat{K}^{(0,0)} + \hat{K}^{(2,2)}_{ij} + \frac{d}{dt}\hat{B}^{(1,1)}_{ij} \\
& - \frac{1}{2}\frac{d^2}{dt^2}\hat{E}^{(0,0)}_{ij}\Big)\Big] + \mathcal{O}\Big(r^3\Big),
\end{aligned} \tag{4.12a}
$$

$$
\begin{aligned}
{}^0h^{\rm R1}_{ta} = {}& \hat{B}^{(0,0)}_a + r\hat{n}^i\Big(\hat{B}^{(1,1)}_{ai} + \epsilon^j{}_{ai}\hat{C}^{(1,1)}_j + \delta_{ai}\hat{D}^{(1,1)}\Big) + r^2\Big[\hat{B}^{(2,0)}_a + \hat{n}^{ij}\Big(\hat{B}^{(2,2)}_{aij} \\
&+ \epsilon^k{}_{ai}\hat{C}^{(2,2)}_{jk} + \delta_{ai}\Big\{\frac{6}{5}\hat{B}^{(2,0)}_j + \frac{3}{5}\hat{B}^{(0,0)}_b\mathcal{E}_j{}^b + \frac{3}{5}\mathcal{B}^{bc}\epsilon_{jc}{}^d\hat{E}^{(0,0)}_{bd} - \frac{1}{2}\frac{d}{dt}\hat{G}^{(1,1)}_j \\
&+ \frac{3}{5}\frac{d}{dt}\hat{K}^{(1,1)}_j\Big\}\Big)\Big] + \mathcal{O}\Big(r^3\Big),
\end{aligned} \tag{4.12b}
$$

$$
\begin{aligned}
{}^0h^{\rm R1}_{ab} = {}& \hat{E}^{(0,0)}_{ab} + \delta_{ab}\hat{K}^{(0,0)} + r\hat{n}^i\Big[\hat{E}^{(1,1)}_{abi} + \delta_{ab}\hat{K}^{(1,1)}_i + \mathop{\rm STF}_{ab}\Big(\epsilon^k{}_{ai}\hat{F}^{(1,1)}_{bk} + \delta_{ai}\hat{G}^{(1,1)}_b\Big)\Big] \\
&+ r^2\Big[\hat{E}^{(2,0)}_{ab} + \delta_{ab}\hat{K}^{(2,0)} + \hat{n}^{ij}\Big(\hat{E}^{(2,2)}_{abij} + \delta_{ab}\hat{K}^{(2,2)}_{ij} + \mathop{\rm STF}_{ab}\Big\{\epsilon^k{}_{ai}\hat{F}^{(2,2)}_{bkj} + \delta_{ai}\hat{G}^{(2,2)}_{bj} \\
&+ \delta_{ai}\epsilon^k{}_{bj}\hat{H}^{(2,2)}_k + \delta_{ai}\delta_{bj}\Big(\frac{1}{5}\mathcal{E}^{cd}\hat{E}^{(0,0)}_{cd} + \frac{6}{5}\hat{K}^{(2,0)}\Big)\Big\}\Big)\Big] + \mathcal{O}\Big(r^3\Big).
\end{aligned} \tag{4.12c}
$$

This form is particularly advantageous as it automatically includes any constraints that would be imposed by the Einstein equations onto the form of our regular field.

4.2 Solving for ξ_1^μ

We now return to Eq. (3.64), where, recall, $\mathring{g}_{\mu\nu}$ is given by Eq. (3.24), ${}^1g_{\mu\nu}$ by Eq. (3.28), and $\mathring{h}^{\rm R1'}_{\mu\nu}$ by Eq. (3.36). To solve for the gauge vector, we substitute the expansions (4.5) and (4.12) and then work order by order in r and $\hat{n}^L$. This is possible because $\hat{n}^L$ forms an orthogonal basis, implying $A_{P\langle L\rangle}\hat{n}^L = B_{P\langle L\rangle}\hat{n}^L \implies A_{P\langle L\rangle} = B_{P\langle L\rangle}$. As a result, Eq. (3.64) reduces to a hierarchical set of equations for the STF tensors $\hat{T}^{(p,l)}_L$, $\hat{X}^{(p,l)}_{L+1}$, $\hat{Y}^{(p,l)}_L$, and $\hat{Z}^{(p,l)}_{L-1}$.

Rather than belabouring the technical details of the calculation, which are largely mechanical, we state the results that follow from each order in r in Eq. (3.64).

Note that in the equations that follow, ${}^0h^{\rm R1}_{\mu\nu}$ and its derivatives are always evaluated on the worldline, but we omit the notation $|_\gamma$ for brevity. Additionally, we define ${}^0h^{\rm R1} := {}^0h^{{\rm R1}a}_a := \delta^{ab}\,{}^0h^{\rm R1}_{ab}$.

4.2.1 Order r^0

Starting at the lowest order in the expansion of Eq. (3.64), we immediately discover rules for four of our gauge vector components. These are

$$\hat{T}^{(0,0)} = \frac{1}{2}\int \hat{A}^{(0,0)}\,dt\,, \tag{4.13}$$

$$\hat{T}^{(1,1)}_a = \hat{B}^{(0,0)}_a, \tag{4.14}$$

$$\hat{X}^{(1,1)}_{ab} = \frac{1}{2}\hat{E}^{(0,0)}_{ab}, \tag{4.15}$$

$$\hat{Z}^{(1,1)} = \frac{1}{2}\hat{K}^{(0,0)}. \tag{4.16}$$

In Ref. [81], the relations in Eqs. (129)–(131) were given in terms of the full gauge vector, ξ^1_μ. To compare, we perform equivalent operations but now on our expansion of ξ^1_μ, substituting our values for the STF tensors from Eqs. (4.13)–(4.16) and using App. A to relate the STF tensors to derivatives of the regular field. The results are

$$\begin{aligned}\frac{d}{dt}\xi^1_t &= \frac{d}{dt}\hat{T}^{(0,0)} = \frac{1}{2}\hat{A}^{(0,0)} \\ &= \frac{1}{2}\,{}^0h^{\rm R1}_{tt}, \qquad (4.17)\\ \xi^1_{t,a} &= \hat{T}^{(1,1)}_a = \hat{B}^{(0,0)}_a \\ &= {}^0h^{\rm R1}_{ta}, \qquad (4.18)\\ \xi^1_{(a,b)} &= \hat{X}^{(1,1)}_{ab} + \delta_{ab}\hat{Z}^{(1,1)} \\ &= \frac{1}{2}\hat{E}^{(0,0)}_{ab} + \frac{1}{2}\delta_{ab}\hat{K}^{(0,0)} \\ &= \frac{1}{2}\,{}^0h^{\rm R1}_{ab}, \qquad (4.19)\end{aligned}$$

which exactly match the expressions in Ref. [81], as expected. The value of $\xi^1_{[a,b]}$ is also given in Ref. [81] but relies on $\hat{Y}^{(1,1)}_c$, which is found at order r.

4.2.2 Order r

Having correctly reproduced the leading expressions from Ref. [81], we can confidently move on to higher orders. We continue our procedure, but now we find our higher-order STF tensors in terms of not just the STF tensors in ${}^0h^{\rm R1}_{\mu\nu}$ but also the tidal moments.

From the tt component of Eq. (3.64), we obtain an expression for the first-order self-force,

$$f^1_a = \frac{1}{2}\hat{A}^{(1,1)}_a - \frac{d}{dt}\hat{B}^{(0,0)}_a. \qquad (4.20)$$

When rewritten in terms of ${}^0h^{\rm R1}_{\mu\nu}$, this gives,

$$f^1_a = \frac{1}{2}\,{}^0h^{\rm R1}_{tt,a} - \frac{d}{dt}\,{}^0h^{\rm R1}_{ta}, \qquad (4.21)$$

which is the standard result for the first-order self-force when written in component form [81].

The ta component gives

$$\hat{T}^{(2,0)} = \frac{1}{2}\hat{D}^{(1,1)} - \frac{1}{4}\frac{d}{dt}\hat{K}^{(0,0)}, \qquad (4.22)$$

$$\hat{T}^{(2,2)}_{ab} = \frac{1}{2}\hat{B}^{(2,2)}_{ab} + \frac{1}{2}\mathcal{E}_{ab}\int \hat{A}^{(0,0)}\,dt - \frac{1}{4}\frac{d}{dt}\hat{E}^{(0,0)}_{ab}, \qquad (4.23)$$

$$\hat{Y}^{(1,1)}_a = \int \hat{C}^{(1,1)}_a\,dt\,. \qquad (4.24)$$

Using the value of $\hat{Y}_a^{(1,1)}$, we can now compare to the result from Ref. [81] for the antisymmetric part of the spatial derivative of the gauge vector. This gives

$$\begin{aligned}\xi^1_{[a,b]} &= \epsilon_{ab}{}^c \hat{Y}_c^{(1,1)} = \epsilon_{ab}{}^c \int \hat{C}_c^{(1,1)}\, dt \\ &= \int {}^0h^{\rm R1}_{t[a,b]}\, dt\,, \end{aligned} \tag{4.25}$$

which matches Eq. (133) from Ref. [81].

Finally, the ab component of Eq. (3.64) gives

$$\hat{X}_a^{(2,0)} = \frac{5}{18}\hat{G}_a^{(1,1)} - \frac{1}{12}\hat{K}_a^{(1,1)}, \tag{4.26}$$

$$\hat{X}_{abc}^{(2,2)} = \frac{1}{4}\hat{E}_{abc}^{(1,1)}, \tag{4.27}$$

$$\hat{Y}_{ab}^{(2,2)} = \frac{1}{2}\hat{F}_{ab}^{(1,1)} - \frac{1}{3}\mathcal{B}_{ab}\int \hat{A}^{(0,0)}\, dt\,, \tag{4.28}$$

$$\hat{Z}_a^{(2,2)} = \frac{1}{2}\hat{K}_a^{(1,1)} - \frac{1}{6}\hat{G}_a^{(1,1)}. \tag{4.29}$$

4.2.3 Order r^2

At the final order, not only do we find the last components of the gauge vector but we also fix the forms of $\delta\mathcal{E}_{ab}$ and $\delta\mathcal{B}_{ab}$ that appear in Eq. (3.36). They are

$$\begin{aligned}\delta\mathcal{E}_{ab} &= 2\mathcal{B}^d{}_{(a}\hat{B}^{(0,0)}_{|c|}\epsilon_{b)}{}^c{}_d + \mathcal{E}^c{}_{\langle a}\hat{E}^{(0,0)}_{b\rangle c} - \hat{E}^{(2,0)}_{ab} + \frac{7}{6}\hat{G}^{(2,2)}_{ab} - 2\mathcal{E}_{ab}\hat{K}^{(0,0)} - \hat{K}^{(2,2)}_{ab} \\ &\quad + \frac{1}{2}\dot{\mathcal{E}}_{ab}\int \hat{A}^{(0,0)}\, dt - 2\mathcal{E}^d{}_{(a}\epsilon_{b)}{}^c{}_d \int \hat{C}_c^{(1,1)}\, dt\,, \end{aligned} \tag{4.30}$$

$$\begin{aligned}\delta\mathcal{B}_{ab} &= \frac{1}{2}\hat{A}^{(0,0)}\mathcal{B}_{ab} + \frac{3}{2}\hat{C}^{(2,2)}_{ab} - \hat{B}^{(0,0)}_c\mathcal{E}^d{}_{(a}\epsilon_{b)}{}^c{}_d + \frac{1}{2}\dot{\mathcal{B}}_{ab}\int \hat{A}^{(0,0)}\, dt - \frac{3}{2}\mathcal{B}_{ab}\hat{K}^{(0,0)} \\ &\quad - \frac{3}{4}\frac{d}{dt}\hat{F}^{(1,1)}_{ab} - 2\mathcal{B}^d{}_{(a}\epsilon_{b)}{}^c{}_d \int \hat{C}_c^{(1,1)}\, dt\,. \end{aligned} \tag{4.31}$$

When the values of the STF tensors are substituted, however, these become[1]

$$\begin{aligned}\delta\mathcal{E}_{ab} &= \mathcal{E}_{ab}\,{}^0h^{\rm R1}_{tt} - \mathcal{E}_{\langle a}{}^c\,{}^0h^{\rm R1}_{b\rangle c} - \frac{1}{2}\,{}^0h^{\rm R1}_{tt,\langle ab\rangle} + \frac{d}{dt}\,{}^0h^{\rm R1}_{t\langle a,b\rangle} + \frac{1}{2}\dot{\mathcal{E}}_{ab}\int {}^0h^{\rm R1}_{tt}\, dt - \frac{1}{2}\frac{d^2}{dt^2}\,{}^0h^{\rm R1}_{\langle ab\rangle} \\ &\quad + 2\,\underset{ab}{\rm STF}\,\mathcal{E}_a{}^c\int {}^0h^{\rm R1}_{t[b,c]}\, dt\,, \end{aligned} \tag{4.32}$$

$$\begin{aligned}\delta\mathcal{B}_{ab} = &-\frac{1}{2}\mathcal{B}_{ab}\,{}^0h^{\rm R1} + \mathcal{E}_{(a}{}^c\epsilon_{b)c}{}^d\,{}^0h^{\rm R1}_{td} + \frac{1}{2}\mathcal{B}_{ab}\,{}^0h^{\rm R1}_{tt} + \frac{1}{2}\epsilon^{cd}{}_{(a}\,{}^0h^{\rm R1}_{|tc|,b)d} + \frac{1}{2}\dot{\mathcal{B}}_{ab}\int {}^0h^{\rm R1}_{tt}\, dt \\ &+ 2\,\underset{ab}{\rm STF}\,\mathcal{B}_a{}^c\int {}^0h^{\rm R1}_{t[b,c]}\, dt - \frac{1}{2}\epsilon^{cd}{}_{(a}\frac{d}{dt}\,{}^0h^{\rm R1}_{b)c,d}. \end{aligned} \tag{4.33}$$

[1] While we do not manipulate $\delta\mathcal{B}_{ab}$ after substitution, we do manipulate $\delta\mathcal{E}_{ab}$. Arriving at our second expression for $\delta\mathcal{E}_{ab}$ necessitates rewriting the Einstein field equation's condition for $\hat{A}^{(2,2)}_{ab}$ from Eq. (4.11b) in terms of $\hat{B}^{(0,0)}_c\mathcal{B}_{d(a}\epsilon_{b)}{}^{cd}$ and then substituting it into our initial expression for $\delta\mathcal{E}_{ab}$.

These expressions match those found for the transformation from the rest gauge to the Lorenz gauge in Ref. [81] but with the omission of the term $\propto m$. As in Ref. [81], we can also write the perturbations of the tidal moments as

$$\delta\mathcal{E}_{ab} = \delta\mathring{R}_{tatb}[{}^0h^{\mathrm{R1}} - \mathcal{L}_{\xi_1}\mathring{g}], \tag{4.34}$$

$$\delta\mathcal{B}_{ab} = \frac{1}{2}\epsilon^{pq}{}_{(a}\delta\mathring{R}_{b)tpq}[{}^0h^{\mathrm{R1}} - \mathcal{L}_{\xi_1}\mathring{g}], \tag{4.35}$$

in agreement with analogous results in Ref. [94]. These forms of $\delta\mathcal{E}_{ab}$ and $\delta\mathcal{B}_{ab}$ let us interpret them as the tidal moments of ${}^0h^{\mathrm{R1}}_{\mu\nu}$ (up to a gauge transformation).

The rest of the STF tensors are found to be

$$\begin{aligned}
\hat{T}^{(3,1)}_a &= \frac{3}{5}\hat{B}^{(2,0)}_a + \frac{2}{5}\mathcal{E}_a{}^b\hat{B}^{(0,0)}_b + \frac{1}{5}\mathcal{B}^{bc}\epsilon_{ac}{}^d\hat{E}^{(0,0)}_{bd} + \frac{2}{5}\mathcal{B}_a{}^b\int\hat{C}^{(1,1)}_b\,dt - \frac{1}{6}\frac{d}{dt}\hat{G}^{(1,1)}_a \\
&\quad + \frac{1}{20}\frac{d}{dt}\hat{K}^{(1,1)}_a, \qquad (4.36)\\
\hat{T}^{(3,3)}_{abc} &= \hat{B}^{(2,2)}_{abc} + \frac{1}{6}\mathcal{E}_{abc}\int\hat{A}^{(0,0)}\,dt - \frac{1}{12}\frac{d}{dt}\hat{E}^{(1,1)}_{abc} + \underset{abc}{\mathrm{STF}}\bigg(\frac{2}{3}\hat{B}^{(0,0)}_a\mathcal{E}_{bc} + \frac{1}{3}\mathcal{B}^d{}_a\epsilon_{bd}{}^i\hat{E}^{(0,0)}_{ci} \\
&\quad - \frac{2}{9}\mathcal{B}_{ab}\int\hat{C}^{(1,1)}_c\,dt\bigg), \qquad (4.37)\\
\hat{X}^{(3,1)}_{ab} &= \frac{4}{15}\hat{E}^{(2,0)}_{ab} + \frac{7}{180}\hat{G}^{(2,2)}_{ab} - \frac{1}{30}\hat{K}^{(2,2)}_{ab} - \frac{1}{15}\mathcal{B}^d{}_{(a}\hat{B}^{(0,0)}_{|c|}\epsilon_{b)}{}^c{}_d - \frac{1}{10}\mathcal{E}^c{}_{\langle a}\hat{E}^{(0,0)}_{b\rangle c} \\
&\quad - \frac{1}{40}\dot{\mathcal{E}}_{ab}\int\hat{A}^{(0,0)}\,dt - \frac{1}{15}\mathcal{E}^d{}_{(a}\epsilon_{b)}{}^c{}_d\int\hat{C}^{(1,1)}_c\,dt\,, \qquad (4.38)\\
\hat{X}^{(3,3)}_{abcd} &= \frac{1}{6}\hat{E}^{(2,2)}_{abcd}\,, \qquad (4.39)\\
\hat{Y}^{(3,1)}_a &= -\frac{1}{10}\mathcal{B}_a{}^b\hat{B}^{(0,0)}_b - \frac{1}{60}\mathcal{E}^{bc}\epsilon_{ac}{}^d\hat{E}^{(0,0)}_{bd} + \frac{1}{4}\hat{H}^{(2,2)}_a + \frac{7}{30}\mathcal{E}_a{}^b\int\hat{C}^{(1,1)}_b\,dt\,, \qquad (4.40)\\
\hat{Y}^{(3,3)}_{abc} &= \frac{1}{4}\hat{F}^{(2,2)}_{abc} - \frac{1}{6}\mathcal{B}_{abc}\int\hat{A}^{(0,0)}\,dt + \underset{abc}{\mathrm{STF}}\bigg(\frac{1}{6}\mathcal{E}^d{}_a\epsilon_{bd}{}^i\hat{E}^{(0,0)}_{ci} - \frac{1}{3}\mathcal{B}_{ab}\hat{B}^{(0,0)}_c \\
&\quad - \frac{1}{3}\mathcal{E}_{ab}\int\hat{C}^{(1,1)}_c\,dt\bigg), \qquad (4.41)\\
\hat{Z}^{(3,1)} &= \frac{1}{45}\mathcal{E}^{ab}\hat{E}^{(0,0)}_{ab} + \frac{3}{10}\hat{K}^{(2,0)}, \qquad (4.42)\\
\hat{Z}^{(3,3)}_{ab} &= \frac{1}{9}\mathcal{B}^d{}_{(a}\epsilon_{b)}{}^c{}_d\hat{B}^{(0,0)}_c - \frac{1}{6}\hat{E}^{(2,0)}_{ab} + \frac{1}{36}\hat{G}^{(2,2)}_{ab} + \frac{1}{6}\mathcal{E}^c{}_{\langle a}\hat{E}^{(0,0)}_{b\rangle c} + \frac{1}{3}\hat{K}^{(2,2)}_{ab} \\
&\quad + \frac{1}{24}\dot{\mathcal{E}}_{ab}\int\hat{A}^{(0,0)}\,dt + \frac{1}{9}\mathcal{E}^d{}_{(a}\epsilon_{b)}{}^c{}_d\int\hat{C}^{(1,1)}_c\,dt\,. \qquad (4.43)
\end{aligned}$$

4.2.4 Final result for ξ_1^μ

Substituting the above results for the STF tensors into Eq. (4.5), we obtain our final form of the gauge vector required to transform from the rest gauge into the practical highly regular gauge. The components are given by[2]

$$\begin{aligned}
\xi^1_t ={}& \frac{1}{2}\int {}^0h^{\rm R1}_{tt}\,dt + r\hat{n}^a\,{}^0h^{\rm R1}_{ta} + \frac{r^2}{12}\bigg[2\,{}^0h^{\rm R1}_{ta,}{}^{a} - \frac{d}{dt}\,{}^0h^{\rm R1} + 3\hat{n}^{ab}\bigg(2\,{}^0h^{\rm R1}_{ta,b} - \frac{d}{dt}\,{}^0h^{\rm R1}_{ab} \\
&+ 2\mathcal{E}_{ab}\int {}^0h^{\rm R1}_{tt}\,dt\bigg)\bigg] + \frac{r^3}{60}\bigg[3\hat{n}^a\bigg(4\mathcal{B}^{bc}\epsilon_{ac}{}^{d}\,{}^0h^{\rm R1}_{bd} + 8\mathcal{E}_a{}^b\,{}^0h^{\rm R1}_{tb} + 2\,{}^0h^{\rm R1}_{ta,b}{}^{b} \\
&+ 4\mathcal{B}_{ab}\epsilon^{bcd}\int {}^0h^{\rm R1}_{tc,d}\,dt - 2\frac{d}{dt}\,{}^0h^{\rm R1}_{ab,}{}^{b} + \frac{d}{dt}\,{}^0h^{\rm R1}_{,a}\bigg) + 5\hat{n}^{abc}\bigg(8\mathcal{E}_{ab}\,{}^0h^{\rm R1}_{tc} \\
&+ 2\mathcal{E}_{abc}\int {}^0h^{\rm R1}_{tt}\,dt + 2\,{}^0h^{\rm R1}_{ta,bc} + 4\mathcal{B}_a{}^d\epsilon_{cd}{}^i\,{}^0h^{\rm R1}_{bi} - 4\mathcal{B}_{ab}\epsilon_c{}^{di}\int {}^0h^{\rm R1}_{td,i}\,dt - \frac{d}{dt}\,{}^0h^{\rm R1}_{ab,c}\bigg)\bigg] \\
&+ \mathcal{O}\big(r^4\big), \qquad (4.44a)
\end{aligned}$$

$$\begin{aligned}
\xi^1_a ={}& \frac{r\hat{n}^b}{2}\bigg[{}^0h^{\rm R1}_{ab} + 2\int {}^0h^{\rm R1}_{t[a,b]}\,dt\bigg] + \frac{r^2}{12}\bigg[2\,{}^0h^{\rm R1}_{ab,}{}^{b} - {}^0h^{\rm R1}_{,a} + \hat{n}^{bc}\bigg(6\,{}^0h^{\rm R1}_{ab,c} - 3\,{}^0h^{\rm R1}_{bc,a} \\
&- 4\mathcal{B}_b{}^d\epsilon_{acd}\int {}^0h^{\rm R1}_{tt}\,dt\bigg)\bigg] + \frac{r^3}{360}\bigg[12\mathcal{E}^{bc}\,{}^0h^{\rm R1}_{bc}\hat{n}_a - 3\hat{n}^b\bigg(4\mathcal{E}_b{}^c\,{}^0h^{\rm R1}_{ac} + 8\mathcal{E}_a{}^c\,{}^0h^{\rm R1}_{bc} \\
&+ 12\mathcal{B}^{cd}\epsilon_{abd}\,{}^0h^{\rm R1}_{tc} - 4\mathcal{E}_{ab}\,{}^0h^{\rm R1} - 8\mathcal{B}^c{}_{(a}\epsilon_{b)c}{}^d\,{}^0h^{\rm R1}_{td} - 8\,{}^0h^{\rm R1}_{b[a,c]}{}^{c} - 8\,{}^0h^{\rm R1}_{ac,b}{}^{c} + 2\,{}^0h^{\rm R1}_{,ab} \\
&+ 3\dot{\mathcal{E}}_{ab}\int {}^0h^{\rm R1}_{tt}\,dt + 24\mathcal{E}_b{}^c\int {}^0h^{\rm R1}_{t[a,c]}\,dt - 32\mathcal{E}_a{}^c\int {}^0h^{\rm R1}_{t[b,c]}\,dt\bigg) + 5\hat{n}_a{}^{bc}\bigg(12\mathcal{E}_b{}^d\,{}^0h^{\rm R1}_{cd} \\
&- 8\mathcal{B}_b{}^d\epsilon_{cd}{}^i\,{}^0h^{\rm R1}_{ti} - 24\mathcal{E}_b{}^d\int {}^0h^{\rm R1}_{t[c,d]}\,dt + 3\dot{\mathcal{E}}_{bc}\,{}^0h^{\rm R1}_{tt}\bigg) - 10\hat{n}^{bcd}\bigg(12\mathcal{E}_{b[c}\,{}^0h^{\rm R1}_{a]d} \\
&- 6\,{}^0h^{\rm R1}_{ab,cd} + 8\mathcal{B}_b{}^i\epsilon_{adi}\,{}^0h^{\rm R1}_{tc} + 4\mathcal{B}_{bc}\epsilon_{ad}{}^i\,{}^0h^{\rm R1}_{ti} + 3\,{}^0h^{\rm R1}_{bc,ad} + 12\mathcal{E}_{bc}\int {}^0h^{\rm R1}_{t[a,d]}\,dt \\
&+ 6\mathcal{B}_{bc}{}^i\epsilon_{adi}\int {}^0h^{\rm R1}_{tt}\,dt\bigg)\bigg] + \mathcal{O}\big(r^4\big). \qquad (4.44b)
\end{aligned}$$

The order-r^0 and -r terms match those found previously in Eqs. (129)–(131) and (133) of Ref. [81].

4.3 Second-order singular field

With the gauge vector determined through order r^3, we can now take the Lie derivative of $\mathring{h}^{\rm S1\prime}_{\mu\nu}$ as required to determine $h^{\rm S2}_{\mu\nu}$. Following Ref. [81], to more explicitly reveal the structure of the singular field, we perform an SS/SR split as in Eq. (3.43) so that

$$\mathring{h}^{\rm S2}_{\mu\nu} = \mathring{h}^{\rm SS}_{\mu\nu} + \mathring{h}^{\rm SR}_{\mu\nu} \qquad (4.45)$$

with

$$\mathring{h}^{\rm SR}_{\mu\nu} = \mathring{h}^{\rm SR\prime}_{\mu\nu} + \mathcal{L}_{\xi_1}\mathring{h}^{\rm S1\prime}_{\mu\nu}, \qquad (4.46)$$

$$\mathring{h}^{\rm SS}_{\mu\nu} = \mathring{h}^{\rm SS\prime}_{\mu\nu}. \qquad (4.47)$$

[2] ξ^1_a was additionally simplified using the constraint in Eq. (4.10a) from the Einstein field equations in terms of ${}^0h^{\rm R1}_{\mu\nu}$.

This means that $\mathring{h}^{\rm SR}_{\mu\nu}$ comprises terms $\sim m\,{}^0h^{\rm R1}_{\mu\nu}$, and $\mathring{h}^{\rm SS}_{\mu\nu}$ features all terms $\propto m^2$.

Calculating $\mathcal{L}_{\xi_1}\mathring{h}^{\rm S1'}_{\mu\nu}$ and combining it with $\mathring{h}^{\rm SR'}_{\mu\nu}$ in Eqs. (3.44), we find the first three orders of $\mathring{h}^{\rm SR}_{\mu\nu}$ are

$$\begin{aligned}
\mathring{h}^{\rm SR}_{tt} =\; & -2m\Bigg[\frac{1}{r}\left({}^0h^{\rm R1}_{tt}+\frac{1}{2}\,{}^0h^{\rm R1}_{ab}n^{ab}\right)+\left(\frac{1}{4}\,{}^0h^{\rm R1}_{ab,c}n^{abc}-n^{ab}\frac{d}{dt}\,{}^0h^{\rm R1}_{ab}+2n^{a}\frac{d}{dt}\,{}^0h^{\rm R1}_{ta}\right)\\
&+r\Bigg(\frac{11}{6}\mathcal{E}^{ab}\,{}^0h^{\rm R1}_{ab}+n^{ab}\Bigg\{-\frac{11}{3}\mathcal{E}_a{}^c\,{}^0h^{\rm R1}_{bc}+\frac{11}{6}\mathcal{E}_{ab}\delta^{ij}\,{}^0h^{\rm R1}_{ij}+\frac{11}{3}\mathcal{B}_a{}^c\epsilon_{bc}{}^d\,{}^0h^{\rm R1}_{td}\\
&+\frac{11}{6}\mathcal{E}_{ab}\,{}^0h^{\rm R1}_{tt}+\frac{11}{12}\,{}^0h^{\rm R1}_{ab,c}{}^c-\frac{11}{6}\,{}^0h^{\rm R1}_{ac,b}{}^c+\frac{11}{12}\delta^{ij}\,{}^0h^{\rm R1}_{ij,ab}+\frac{d}{dt}\,{}^0h^{\rm R1}_{ta,b}-\frac{1}{2}\frac{d^2}{dt^2}\,{}^0h^{\rm R1}_{ab}\Bigg\}\\
&-\frac{1}{2}n^{abc}\frac{d}{dt}\,{}^0h^{\rm R1}_{ab,c}+\frac{1}{12}n^{abcd}\Big\{11\mathcal{E}_{ab}\,{}^0h^{\rm R1}_{cd}+{}^0h^{\rm R1}_{ab,cd}\Big\}\Bigg)\Bigg]+\mathcal{O}\big(r^2\big), \qquad (4.48a)\\
\mathring{h}^{\rm SR}_{ta} =\; & -2m\Bigg[\frac{1}{r}\left({}^0h^{\rm R1}_{ta}+\frac{1}{2}\,{}^0h^{\rm R1}_{tt}n_a-{}^0h^{\rm R1}_{ab}n^b+{}^0h^{\rm R1}_{bc}n_a{}^{bc}\right)+\Bigg(n^b\Big\{{}^0h^{\rm R1}_{t[a,b]}-\frac{1}{2}\frac{d}{dt}\,{}^0h^{\rm R1}_{ab}\Big\}\\
&+n_a{}^b\frac{d}{dt}\,{}^0h^{\rm R1}_{tb}-\frac{1}{4}n^{bc}\Big\{2\,{}^0h^{\rm R1}_{ab,c}+{}^0h^{\rm R1}_{bc,a}\Big\}-\frac{1}{2}n_a{}^{bc}\frac{d}{dt}\,{}^0h^{\rm R1}_{bc}\Bigg)+r\Bigg(\frac{1}{3}\mathcal{B}^{bc}\epsilon_{ac}{}^d\,{}^0h^{\rm R1}_{bd}\\
&+\frac{4}{3}n_a\mathcal{E}^{bc}\,{}^0h^{\rm R1}_{bc}+\frac{1}{6}n^b\Big\{-4\mathcal{E}^c{}_{(a}\,{}^0h^{\rm R1}_{b)c}+2\mathcal{E}_{ab}\delta^{ij}\,{}^0h^{\rm R1}_{ij}+4\mathcal{B}^c{}_{(a}\epsilon_{b)c}{}^d\,{}^0h^{\rm R1}_{td}+\mathcal{E}_{ab}\,{}^0h^{\rm R1}_{tt}\\
&+2\,{}^0h^{\rm R1}_{a[b,c]}{}^c-2\,{}^0h^{\rm R1\,c}_{[b,c]a}\Big\}-\frac{1}{3}n_a{}^b\mathcal{B}^{cd}\epsilon_{bd}{}^i\,{}^0h^{\rm R1}_{ci}+\frac{1}{12}n^{bc}\Big\{-8\mathcal{B}_b{}^d\epsilon_d{}^i{}_{[a}\,{}^0h^{\rm R1}_{c]i}\\
&+10\mathcal{E}_{bc}\,{}^0h^{\rm R1}_{ta}+\mathcal{B}^{di}\epsilon_{aci}\,{}^0h^{\rm R1}_{bd}+6\,{}^0h^{\rm R1}_{ta,bc}-6\frac{d}{dt}\,{}^0h^{\rm R1}_{ab,c}+3\frac{d}{dt}\,{}^0h^{\rm R1}_{bc,a}\Big\}\\
&+n_a{}^{bc}\Big\{2\mathcal{E}_b{}^d\,{}^0h^{\rm R1}_{cd}+\mathcal{E}_{bc}\delta^{ij}\,{}^0h^{\rm R1}_{ij}+2\mathcal{B}_b{}^d\epsilon_{cd}{}^i\,{}^0h^{\rm R1}_{ti}+\frac{1}{2}\mathcal{E}_{bc}\,{}^0h^{\rm R1}_{tt}+\frac{1}{2}\,{}^0h^{\rm R1}_{bc,d}{}^d\\
&-{}^0h^{\rm R1}_{bd,c}{}^d+\frac{1}{2}\delta^{ij}\,{}^0h^{\rm R1}_{ij,bc}+\frac{1}{2}\frac{d}{dt}\,{}^0h^{\rm R1}_{tb,c}-\frac{1}{4}\frac{d^2}{dt^2}\,{}^0h^{\rm R1}_{bc}\Big\}-\frac{1}{6}n^{bcd}\Big\{6\mathcal{E}_{bc}\,{}^0h^{\rm R1}_{ad}\\
&+2\,{}^0h^{\rm R1}_{b(a,c)d}\Big\}-\frac{1}{4}n_a{}^{bcd}\frac{d}{dt}\,{}^0h^{\rm R1}_{bc,d}-\frac{1}{3}n^{bcdi}\mathcal{B}_b{}^j\epsilon_{aij}\,{}^0h^{\rm R1}_{cd}+n_a{}^{bcdi}\Big\{\mathcal{E}_{bc}\,{}^0h^{\rm R1}_{di}\\
&+\frac{1}{6}\,{}^0h^{\rm R1}_{bc,di}\Big\}\Bigg)\Bigg]+\mathcal{O}\big(r^2\big), \qquad (4.48b)
\end{aligned}$$

$$\begin{aligned}
\mathring{h}^{\rm SR}_{ab} =&\ -2m\bigg[\frac{1}{r}\Big(2\,{}^0h^{\rm R1}_{t(a}n_{b)} - 2\,{}^0h^{\rm R1}_{c(a}n_{b)}{}^{c} + \frac{3}{2}\,{}^0h^{\rm R1}_{cd}\,n_{ab}{}^{cd}\Big) + \Big({}^0h^{\rm R1}_{t(a,|c|}n_{b)}{}^{c} + {}^0h^{\rm R1}_{tc,(a}n_{b)}{}^{c} \\
&- {}^0h^{\rm R1}_{c(a,|d|}n_{b)}{}^{cd} - \frac{1}{2}\,{}^0h^{\rm R1}_{cd,(a}n_{b)}{}^{cd} + \frac{3}{4}\,{}^0h^{\rm R1}_{cd,i}n_{ab}{}^{cdi} - n^{c}{}_{(a}\frac{d}{dt}\,{}^0h^{\rm R1}_{b)c}\Big) + r\Big(\frac{2}{3}n^{c}\mathcal{E}_{c(a}\,{}^0h^{\rm R1}_{b)t} \\
&+ \frac{2}{3}n_{(a}\epsilon_{b)d}{}^{i}\mathcal{B}^{cd}\,{}^0h^{\rm R1}_{ci} + \frac{5}{6}n_{ab}\mathcal{E}^{cd}\,{}^0h^{\rm R1}_{cd} + \frac{2}{3}\mathop{\rm Sym}_{ab} n_a{}^{c}\Big\{\mathcal{E}_{bc}\delta^{ij}\,{}^0h^{\rm R1}_{ij} - 2\mathcal{E}_{(b}{}^{d}\,{}^0h^{\rm R1}_{c)d} \\
&+ 2\mathcal{B}_{(b}{}^{d}\epsilon_{c)d}{}^{i}\,{}^0h^{\rm R1}_{ti} + {}^0h^{\rm R1}_{b[c,d]}{}^{d} - {}^0h^{{\rm R1}\,d}_{[c\ \ ,d]b}\Big\} - \frac{2}{3}n^{cd}\Big\{\mathcal{E}_{c(a}\,{}^0h^{\rm R1}_{b)d} + 2\mathcal{B}_c{}^{i}\epsilon_{di(a}\,{}^0h^{\rm R1}_{b)t}\Big\} \\
&- \frac{2}{3}n_{ab}{}^{c}\mathcal{B}^{di}\epsilon_{ci}{}^{j}\,{}^0h^{\rm R1}_{dj} + \frac{1}{6}\mathop{\rm Sym}_{ab} n_a{}^{cd}\Big\{8\mathcal{B}_c{}^{i}\epsilon_i{}^{j}{}_{[b}\,{}^0h^{\rm R1}_{d]j} + 4\mathcal{B}^{ij}\epsilon_{bdj}\,{}^0h^{\rm R1}_{cj} \\
&- \mathcal{B}_c{}^{i}\epsilon_{bdi}\delta^{jk}\,{}^0h^{\rm R1}_{jk} + 4\mathcal{B}_c{}^{i}\epsilon_{bdi}\,{}^0h^{\rm R1}_{tt} + 6\,{}^0h^{\rm R1}_{tb,cd} - 6\frac{d}{dt}\,{}^0h^{\rm R1}_{bc,d} + 3\frac{d}{dt}\,{}^0h^{\rm R1}_{cd,b}\Big\} \\
&+ \frac{4}{3}n^{cdi}\mathcal{B}_c{}^{j}\epsilon_{ij(a}\,{}^0h^{\rm R1}_{b)d} + \frac{1}{12}n_{ab}{}^{cd}\Big\{2\mathcal{E}_{cd}\delta^{ij}\,{}^0h^{\rm R1}_{ij} - 4\mathcal{E}_c{}^{i}\,{}^0h^{\rm R1}_{di} + 4\mathcal{B}_c{}^{i}\epsilon_{di}{}^{j}\,{}^0h^{\rm R1}_{tj} \\
&+ {}^0h^{\rm R1}_{cd,i}{}^{i} - 2\,{}^0h^{\rm R1}_{ci,d}{}^{i} + \delta^{ij}\,{}^0h^{\rm R1}_{ij,cd}\Big\} - \frac{2}{3}\mathop{\rm Sym}_{ab} n_a{}^{cdi}\Big\{\mathcal{E}_{c[d}\,{}^0h^{\rm R1}_{b]i} + {}^0h^{\rm R1}_{c(b,d)i}\Big\} \\
&- \frac{4}{3}n^{cdij}{}_{(a}\epsilon_{b)jk}\mathcal{B}_c{}^{k}\,{}^0h^{\rm R1}_{di} + \frac{1}{4}n_{ab}{}^{cdij}\Big\{\mathcal{E}_{cd}\,{}^0h^{\rm R1}_{ij} + {}^0h^{\rm R1}_{cd,ij}\Big\}\Big)\bigg] + \mathcal{O}\big(r^2\big). \qquad (4.48c)
\end{aligned}$$

Here the first two orders, $\sim 1/r$ and $\sim r^0$, arise purely from $\mathcal{L}_{\xi_1}\mathring{h}^{\rm S1'}_{\mu\nu}$, while the linear-in-$r$ terms contain contributions from both $\mathcal{L}_{\xi_1}\mathring{h}^{\rm S1'}_{\mu\nu}$ and $\mathring{h}^{\rm SR'}_{\mu\nu}$.

The ‘singular times singular’ piece of the perturbation is given in Eq. (3.45), which we rewrite here as

$$\mathring{h}^{\rm SS}_{tt} = -4m^2\bigg[\mathcal{E}_{ab}n^{ab} + r\Big(\frac{1}{3}\dot{\mathcal{E}}_{ab}n^{ab}\Big\{11 - 6\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{2}{3}\mathcal{E}_{abc}n^{abc}\Big)\bigg] + \mathcal{O}\big(r^2\big), \qquad (4.49a)$$

$$\begin{aligned}
\mathring{h}^{\rm SS}_{ta} =&\ -4m^2\bigg[\mathcal{E}_{bc}n_a{}^{bc} + r\Big(\frac{2}{9}\dot{\mathcal{E}}_{ab}n^{b}\Big\{7 - 3\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{6}\mathcal{E}_{abc}n^{bc} - \frac{2}{9}\dot{\mathcal{B}}_b{}^{d}\epsilon_{acd}n^{bc} \\
&\times\Big\{4 - 3\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{9}\dot{\mathcal{E}}_{bc}n_a{}^{bc}\Big\{19 - 12\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{2}\mathcal{E}_{bcd}n_a{}^{bcd} \\
&- \frac{2}{9}\mathcal{B}_{bc}{}^{i}\epsilon_{adi}n^{bcd}\Big)\bigg] + \mathcal{O}\big(r^2\big), \qquad (4.49b)
\end{aligned}$$

$$\begin{aligned}
\mathring{h}^{\rm SS}_{ab} =&\ -4m^2\bigg[-\frac{1}{3}\mathcal{E}_{ab} + \mathcal{B}_{(a}{}^{d}\epsilon_{b)cd}n^{c} + \frac{2}{3}\mathcal{E}_{c(a}n_{b)}{}^{c} - \frac{1}{6}\mathcal{E}_{cd}\delta_{ab}n^{cd} - \mathcal{B}_c{}^{i}\epsilon_{di(a}n_{b)}{}^{cd} \\
&+ \frac{5}{6}\mathcal{E}_{cd}n_{ab}{}^{cd} + r\Big(\frac{2}{3}\dot{\mathcal{E}}_{ab} + \frac{4}{9}\dot{\mathcal{E}}_{c(a}n_{b)}{}^{c}\Big\{4 - 3\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{3}\dot{\mathcal{E}}_{cd}\delta_{ab}n^{cd} \\
&+ \frac{1}{9}n^{cd}{}_{(a}\Big\{3\mathcal{E}_{b)cd} - 4\dot{\mathcal{B}}_{|c|}{}^{i}\epsilon_{b)di}\Big[4 - 3\log\Big(\frac{2m}{r}\Big)\Big]\Big\} - \frac{4}{9}\mathcal{B}_{cd}{}^{j}\epsilon_{ij(a}n_{b)}{}^{cdi} \\
&+ \frac{2}{9}\dot{\mathcal{E}}_{cd}n_{ab}{}^{cd}\Big\{4 - 3\log\Big(\frac{2m}{r}\Big)\Big\} + \frac{1}{3}\mathcal{E}_{cdi}n_{ab}{}^{cdi}\Big)\bigg] + \mathcal{O}\big(r^2\big). \qquad (4.49c)
\end{aligned}$$

In $\mathring{h}^{\rm SS}_{\mu\nu}$ we have simply rewritten $\mathring{h}^{\rm SS'}_{\mu\nu}$, as given in Eq. (3.45), in terms of $n^L := n^{i_1}\cdots n^{i_l}$ instead of $\hat{n}^L = n^{\langle L\rangle}$. This will simplify the conversion to fully covariant form, as required for use in a puncture scheme; such a conversion can be done following the method in Ref. [96] and is performed in Ch. 6.

$\mathring{h}^{\rm SS}_{\mu\nu}$ and the leading, $1/r$ terms in $\mathring{h}^{\rm SR}_{\mu\nu}$ were given previously in Ref. [81].[3] The $\sim r^0$ and linear-in-r terms in $\mathring{h}^{\rm SR}_{\mu\nu}$ appear here for the first time. We also provide our full results for the singular field in a user-ready MATHEMATICA form in the Additional Material [141].

This completes our calculation of the second-order singular field. In the next chapter, we turn to the skeleton stress-energy that this field is associated with.

[3]The ${}^0h^{\rm R1}_{ab}$ term in Eq. (134c) of Ref. [81] has a typo and has been corrected in Eq. (4.48c) to ${}^0h^{\rm R1}_{cd}\,n_{ab}{}^{cd}$.

Chapter 5

Second-order stress-energy tensor

In this chapter, we provide the method and calculations to rigorously derive the second-order stress-energy tensor for the small object. We find this first in the lightcone rest gauge (Ch. 5.1.2) before finding the transformation to a generic highly regular gauge (Ch. 5.1.3) and the Lorenz gauge (Ch. 5.2). We demonstrate that the second-order stress-energy tensor has the same functional form in both the highly regular and Lorenz gauges. This form is compatible with the result we obtain for the lightcone rest gauge. The stress-energy tensor is found to be that of a point mass moving in the effective spacetime, $\tilde{g}_{\mu\nu}$, (Ch. 5.1.4) and has an invariant form under any smooth gauge transformation (Ch. 5.1.7). We derive additional useful properties along the way in Chs 5.1.5–5.1.6.

After calculating the second-order stress-energy tensor in both the highly regular and Lorenz gauges, we proceed to calculate the delta function content of the sources for the second-order Einstein equations and the second-order Teukolsky equation in Ch. 5.3. Material up to (but not including) Ch. 5.3 was previously published in Ref. [140].

5.1 The Detweiler stress-energy: derivation and properties in highly regular gauges

5.1.1 Stress-energy tensors in self-force theory

In self-force theory, where one uses the principle of matched asymptotic expansions to derive the form of the metric, we do not have the freedom to choose the form of the stress-energy tensor. We require a stress-energy tensor source that gives us the local form of the metric that was derived through matched asymptotic expansions. This contrasts the usual approach to working in general relativity where a stress-energy tensor is *prescribed*, based on some desired physical characteristics, and the metric that sources it is solved for.

There is a great deal of subtlety involved when dealing with stress-energy tensors in self-force theory which we illustrate here. Say we were using an ordinary Taylor expansion of the metric perturbations instead of the ϵ-dependent self-consistent scheme. We would then write our stress-energy tensor as

$$T^{\mu\nu}(x,\epsilon) = \epsilon T_1^{\mu\nu}(x) + \epsilon^2 T_2^{\mu\nu}(x) + \mathcal{O}\Big(\epsilon^3\Big), \tag{5.1}$$

which we could define, order-by-order, by

$$8\pi T_1^{\mu\nu} := \delta G^{\mu\nu}[h^1], \tag{5.2}$$

$$8\pi T_2^{\mu\nu} := \delta G^{\mu\nu}[h^2] + \delta^2 G^{\mu\nu}[h^1, h^1]. \tag{5.3}$$

This is not the procedure that we used when deriving the perturbations in the highly regular gauge in the previous chapter but we could recover this series by re-expanding the worldline γ around a geodesic in the background spacetime. It was previously shown in Ref. [139] that this would introduce a mass dipole moment into $h^2_{\mu\nu}$ which would be accounted for via the introduction of a mass-dipole stress-energy tensor, $T^{\mu\nu}_{\delta m}$, at second order.

In the self-consistent scheme, the equations must be written in combined form instead of splitting order-by-order in this framework as $\delta G^{\mu\nu}[h^1] \neq 0$ off the worldline, in fact

$$\epsilon\delta G^{\mu\nu}[h^1] = \mathcal{O}\Big(\epsilon^2\Big), \quad x^\alpha \notin \gamma, \tag{5.4}$$

so that for all points

$$\epsilon\delta G^{\mu\nu}[h^1] = 8\pi\epsilon T_1^{\mu\nu} + \mathcal{O}\Big(\epsilon^2\Big). \tag{5.5}$$

To see why, note that if we took Eq. (5.2) to be true for the self-consistent scheme then, by the conservation of energy, $\nabla_\nu T_1^{\mu\nu} = \nabla_\nu \delta G^{\mu\nu}[h^1] = 0$. This implies that the worldline is a geodesic in the background spacetime which is a contradiction as in the self-consistent scheme we have an accelerated worldline. Thus, we have Eq. (5.5) where $T_1^{\mu\nu}$ is the stress-energy tensor of a point mass on an accelerated worldline. This suggests that we should work with total quantities, such as

$$8\pi T^{\mu\nu} := \delta G^{\mu\nu}[\epsilon h^1 + \epsilon^2 h^2] + \epsilon^2 \delta^2 G^{\mu\nu}[h^1, h^1] + \mathcal{O}\Big(\epsilon^3\Big). \tag{5.6}$$

However, there exists another issue that we must deal with before continuing. When calculating the metric perturbations in the highly regular gauge, we did not actually calculate the $h_{\mu\nu}$ that appear in the self-consistent scheme; instead we calculated $\mathring{h}_{\mu\nu}$. The difference between the two is subtle but important: the $h_{\mu\nu}$ in the self-consistent scheme are defined in an ϵ-independent coordinate system whereas the values we have calculated in the highly regular gauge use a coordinate system that depends on ϵ. If we expand Eq. (5.6) around $\mathring{g}_{\mu\nu}$, we get

$$8\pi T^{\mu\nu} = \delta\mathring{G}^{\mu\nu}[\epsilon\,\mathring{h}^1 + \epsilon^2\mathring{h}^2] + \epsilon^2\,\delta^2\mathring{G}^{\mu\nu}[\mathring{h}^1, \mathring{h}^1] + \mathcal{O}\Big(\epsilon^3\Big), \tag{5.7}$$

where $\delta\mathring{G}^{\mu\nu}$ are $\delta^2\mathring{G}^{\mu\nu}$ are defined with respect to $\mathring{g}_{\mu\nu}$. This leads us to define the stress-energy tensors based on these quantities as

$$8\pi T_1^{\mu\nu} := \delta\mathring{G}^{\mu\nu}[\mathring{h}^1], \tag{5.8}$$

$$8\pi T_2^{\mu\nu} := \delta\mathring{G}^{\mu\nu}[\mathring{h}^2] + \delta^2\mathring{G}^{\mu\nu}[\mathring{h}^1, \mathring{h}^1]. \tag{5.9}$$

It follows from Eq. (5.8) that $T_1^{\mu\nu}$ is gauge invariant as $\delta\mathring{G}^{\mu\nu}$ is gauge invariant on a vacuum background [153]. Thus, $T_1^{\mu\nu}$ is the same in any gauge and is given by the stress-energy of a point mass on γ, given by

$$T_1^{\mu\nu} = m\int_\gamma u^\mu u^\nu \frac{\delta^4(x-z)}{\sqrt{-g}}\,d\tau\,. \tag{5.10}$$

which is Eq. (1.14) but with indices up. This is not the case at second order with $T_2^{\mu\nu}$. In fact, as mentioned Ch. 1.4.1, it is not even clear whether $T_2^{\mu\nu}$ is a well-defined quantity when in a gauge compatible with matched asymptotic expansions. While $\mathring{h}^1_{\mu\nu}$ is a locally integrable function valid as a distribution, when acted on by the second-order Einstein tensor, it becomes a product of distributions as $\delta^2\mathring{G}^{\mu\nu}[\mathring{h}, \mathring{h}] \sim \mathring{h}\nabla^2\mathring{h} + \nabla\mathring{h}\nabla\mathring{h}$. A product of distributions is (generally) not a well-defined distributional quantity. By construction, the total quantity $\delta\mathring{G}^{\mu\nu}[\mathring{h}^2] + \delta^2\mathring{G}^{\mu\nu}[\mathring{h}^1, \mathring{h}^1]$ vanishes for $r > 0$ but there is no unique way to promote this to a distribution on $r \geq 0$. If we were able to define each term individually as distributions, then we could write

$$\delta\mathring{G}^{\mu\nu}[\mathring{h}^2] = 8\pi T_2^{\mu\nu} - \delta^2\mathring{G}^{\mu\nu}[\mathring{h}^1, \mathring{h}^1], \tag{5.11}$$

allowing us to directly solve for the physical field, as is done at first order. This final problem will be considered in Ch. 5.3.1.

To expand upon the argument some more, we split $\mathring{h}^1_{\mu\nu}$ into singular and regular fields, so that

$$\delta^2\mathring{G}^{\mu\nu}[h^1, h^1] = \delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm S1}, \mathring{h}^{\rm S1}] + 2\delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm S1}, \mathring{h}^{\rm R1}] + \delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm R1}, \mathring{h}^{\rm R1}]. \tag{5.12}$$

We define

$$\mathring{Q}_R^{\mu\nu}[h] := \delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm R1}, h] \tag{5.13}$$

which is a smooth linear operator because $\mathring{h}^{\rm R1}_{\mu\nu}$ is smooth. This is then well-defined in the distributional sense when acting on the integrable function $\mathring{h}^{\rm S1}_{\mu\nu}$. The final term, $\delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm R1}, \mathring{h}^{\rm R1}] \sim r^0$ as $\mathring{h}^{\rm R1}_{\mu\nu} \sim r^0$ is a smooth field defined at all points in the spacetime and, as such, is well-defined distributionally. Therefore, the problematic part of the second-order Einstein tensor is the 'singular times singular' piece, $\delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm S1},\mathring{h}^{\rm S1}]$. Ref. [81] argued that, in a highly regular gauge, this is in fact an integrable function on the entire spacetime. We follow this argument in the next sections where we use it to derive the second-order stress energy tensor in a generic highly regular gauge.

5.1.2 Stress-energy in the lightcone rest gauge

We start by considering the distributional nature of the individual terms in a generic highly regular gauge. While it is not immediately obvious that $\delta^2\mathring{G}[\mathring{h}^{\rm S1},\mathring{h}^{\rm S1}]$ is well defined as a distribution in these, we note that because $\mathring{h}^{\rm R1}_{\mu\nu}$ features no terms with explicit factors of m and $\mathring{h}^{\rm S1}_{\mu\nu}$ features terms with an explicit factor of m, then $\mathring{h}^{\rm S1}_{\mu\nu}$ must be the source for $\mathring{h}^{\rm SS}_{\mu\nu}$ in $\mathring{h}^{2}_{\mu\nu}$ as it features terms with the factor m^2. This implies that

$$\delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm S1}] = -\delta\mathring{G}^{\mu\nu}[\mathring{h}^{\rm SS}], \quad r>0. \tag{5.14}$$

The previous relation is of course true in any gauge as we are free to choose the split of $\mathring{h}^{2}_{\mu\nu}$ so that is satisfies this equality. However, in a generic highly regular gauge, the RHS of Eq. (5.14) behaves as $\sim 1/r^2$ because $\mathring{h}^{\rm SS}_{\mu\nu}\sim r^0$. As such, it is a locally integrable function across the entire space $r\geq 0$, so we can write

$$\delta^2 G^{\mu\nu}[h^{\rm S1}] = -\delta G^{\mu\nu}[h^{\rm SS}], \quad \forall r. \tag{5.15}$$

Specialising to a rest gauge and evaluating the definition for $T^{\mu\nu}_2$ from Eq. (5.9), we find that

$$8\pi T^{\mu\nu}_2 = \delta\mathring{G}^{\mu\nu}[\mathring{h}^{\rm SR'}] + \delta\mathring{G}^{\mu\nu}[\mathring{h}^{\rm SS}] + 2\delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm S1'},\mathring{h}^{\rm R1'}] + \delta^2\mathring{G}^{\mu\nu}[\mathring{h}^{\rm S1'},\mathring{h}^{\rm S1}], \tag{5.16}$$

where we have eliminated the terms that only depend on the regular field as, by definition, they are vacuum solutions to the field equations. Following the argument in this chapter, the right-hand side is zero for $r>0$ and are all ordinary integrable functions. Therefore this vanishes when integrating against a test function, meaning

$$T^{\mu\nu}_{2'} = 0. \tag{5.17}$$

The physical interpretation is as follows: when in the local rest gauge for a non-spinning object, its stress-energy tensor is that of a point mass on the external background through order ϵ^2.

5.1.3 Stress-energy in a generic highly regular gauge

We now find $T^{\mu\nu}_2$ by finding how the stress-energy transforms under a gauge transformation from the rest gauge. We use Eqs. (3.51)–(3.54) to write the $\mathring{h}^{n}_{\mu\nu}$'s in terms of the rest gauge quantities. Additionally, we require the identities (C2)–(C4) from Ref. [139], which are

$$\mathcal{L}_\xi A[g] = \delta A[\mathcal{L}_\xi g], \tag{5.18}$$
$$\mathcal{L}_\xi^2 A[g] = \delta A[\mathcal{L}_\xi^2 g] + 2\delta^2 A[\mathcal{L}_\xi g, \mathcal{L}_\xi g], \tag{5.19}$$
$$\mathcal{L}_\xi \delta A[h] = \delta A[\mathcal{L}_\xi h] + 2\delta^2 A[\mathcal{L}_\xi g, h], \tag{5.20}$$

where A is a tensor of arbitrary rank which is constructed from a metric g. The first of these reduces to the invariance of the linearised Einstein tensor, $\delta G^{\mu\nu}[\mathcal{L}_\xi g] = 0$, when the background is vacuum.

Together, the above replacements and identities gives

$$\begin{aligned}
8\pi T_2^{\mu\nu} &= \delta \mathring{G}^{\mu\nu}[\mathring{h}^{2'} + \mathcal{L}_{\xi_1}\mathring{h}^{1'} + \tfrac{1}{2}\mathcal{L}_{\xi_1}^2 \mathring{g} + \mathcal{L}_{\xi_2}\mathring{g}] + \delta^2 \mathring{G}^{\mu\nu}[\mathring{h}^{1'} + \mathcal{L}_{\xi_1}\mathring{g}, \mathring{h}^{1'} + \mathcal{L}_{\xi_1}\mathring{g}] \\
&= \delta \mathring{G}^{\mu\nu}[\mathring{h}^{2'}] + \delta^2 \mathring{G}^{\mu\nu}[\mathring{h}^{1'}, \mathring{h}^{1'}] + \delta \mathring{G}^{\mu\nu}[\mathcal{L}_{\xi_1}\mathring{h}^{1'}] + \tfrac{1}{2}\delta \mathring{G}^{\mu\nu}[\mathcal{L}_{\xi_1}^2 \mathring{g}] \\
&\quad + 2\delta^2 \mathring{G}^{\mu\nu}[\mathring{h}^{1'}, \mathcal{L}_{\xi_1}\mathring{g}] + \delta^2 \mathring{G}^{\mu\nu}[\mathcal{L}_{\xi_1}\mathring{g}, \mathcal{L}_{\xi_1}\mathring{g}] \\
&= 8\pi T_{2'}^{\mu\nu} + \mathcal{L}_{\xi_1}\delta \mathring{G}^{\mu\nu}[\mathring{h}^{1'}] + \tfrac{1}{2}\mathcal{L}_{\xi_1}^2 G^{\mu\nu}[\mathring{g}] \\
&= 8\pi T_{2'}^{\mu\nu} + 8\pi \mathcal{L}_{\xi_1} T_1^{\mu\nu}.
\end{aligned} \tag{5.21}$$

In the first line, we have substituted Eqs. (3.51)–(3.54) into the right-hand side of the definition (5.9). In the third equality we have appealed to Eqs. (5.19) and (5.20). In the fourth, we have appealed to $G^{\mu\nu}[\mathring{g}] = 0$ and $\delta \mathring{G}^{\mu\nu}[\mathring{h}^{1'}] = 8\pi T_{1'}^{\mu\nu} = 8\pi T_1^{\mu\nu}$.

Equation (5.21) tells us we can write

$$T_2^{\mu\nu} = T_{2'}^{\mu\nu} + \mathcal{L}_{\xi_1} T_1^{\mu\nu}. \tag{5.22}$$

This is not too surprising as it is just the transformation law for a second-order tensor when the background tensor vanishes [153]. In our case, we have effectively defined $T_2^{\mu\nu}$ as the second-order term in an expansion of the Einstein tensor. However, note that the steps involved in Eq. (5.21) rely on the properties of the highly regular gauges; we have not established Eq. (5.22) for the transformation between any two generic gauges.

Next, since $T_{2'}^{\mu\nu} = 0$, Eq. (5.22) becomes

$$T_2^{\mu\nu} = \mathcal{L}_{\xi_1} T_1^{\mu\nu}. \tag{5.23}$$

The right-hand side was previously calculated in Eq. (D1) of Ref. [139] and is rederived in Eq. (B.11).[1] It reads

$$\mathcal{L}_{\xi_1} T_1^{\mu\nu} = -m \int_\gamma g^\mu_{\mu'} g^\nu_{\nu'} u^{\mu'} u^{\nu'} \left(\xi^\rho_{1;\rho} - \frac{d\xi^1_\parallel}{d\tau} \right) \delta^4(x,z)\, d\tau\,, \tag{5.24}$$

[1]The τ derivative term has a missing minus sign in Ref. [139], which has been added here.

where $\xi^1_\parallel := u_\rho \xi^\rho_1$ and we have removed the orthogonal parts of the gauge vector, $\xi^\mu_{1\perp} := P^\mu{}_\nu \xi^\nu_1$, as the worldline-preserving condition sets them to zero. We detail the derivation of Eq. (5.24) and various related results in App. B.

By taking ξ^ρ_1 to be the gauge vector from Eq. (4.44) and the proper time to be t, we find

$$\left.\frac{d\xi^1_\parallel}{d\tau}\right|_\gamma = -\left.\frac{d\xi^t_1}{dt}\right|_\gamma = \frac{1}{2}\,{}^0h^{\rm R1}_{tt}\Big|_\gamma = \frac{1}{2}u^\mu u^\nu\,{}^0h^{\rm R1}_{\mu\nu}\Big|_\gamma \tag{5.25}$$

and

$$\begin{aligned}\xi^\rho_{1;\rho}\Big|_\gamma &= \left(\partial_t \xi^t_1 + \partial_a \xi^a_1\right)\Big|_\gamma \\ &= \frac{1}{2}\left(-\,{}^0h^{\rm R1}_{tt} + \delta^{ab}\,{}^0h^{\rm R1}_{ab}\right)\Big|_\gamma \\ &= \frac{1}{2}g^{\alpha\beta}\,{}^0h^{\rm R1}_{\alpha\beta}\Big|_\gamma .\end{aligned} \tag{5.26}$$

Thus, the second-order stress-energy tensor in the highly regular gauge is given by

$$T^{\mu\nu}_2 = -\frac{m}{2}\int u^\mu u^\nu \left(g^{\alpha\beta} - u^\alpha u^\beta\right)\,{}^0h^{\rm R1}_{\alpha\beta}\delta^4(x,z)\,d\tau\,. \tag{5.27}$$

5.1.4 Point mass in the effective spacetime

With a short calculation, we can show the total stress-energy $\epsilon T^{\mu\nu}_1 + \epsilon^2 T^{\mu\nu}_2$ derived above is exactly equal, through order ϵ^2, to the stress-energy tensor of a point mass in the effective spacetime $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\rm R}_{\mu\nu}$. That stress-energy tensor is given by

$$\tilde{T}^{\mu\nu} = \epsilon m \int_\gamma \tilde{u}^\mu \tilde{u}^\nu \frac{\delta^4(x-z)}{\sqrt{-\tilde{g}}}\,d\tilde{\tau}\,. \tag{5.28}$$

Expanding this for small $h^{\rm R}_{\mu\nu}$, we see that

$$\begin{aligned}\tilde{T}^{\mu\nu} &= \epsilon m \int_\gamma \frac{d\tau}{d\tilde{\tau}} u^\mu u^\nu \delta^4(x,z)\left(1 - \frac{\epsilon}{2}g^{\alpha\beta}h^{\rm R1}_{\alpha\beta}\right)d\tau + \mathcal{O}\left(\epsilon^3\right) \\ &= \epsilon m \int_\gamma u^\mu u^\nu \delta^4(x,z)\left(1 - \frac{\epsilon}{2}\left[g^{\alpha\beta} - u^\alpha u^\beta\right]h^{\rm R1}_{\alpha\beta}\right)d\tau + \mathcal{O}\left(\epsilon^3\right),\end{aligned} \tag{5.29}$$

where we have used the standard expansion of a determinant and expanded $d\tau/d\tilde{\tau}$ using

$$\frac{d\tau}{d\tilde{\tau}} = \frac{1}{\sqrt{1 - h^{\rm R}_{\mu\nu}u^\mu u^\nu}} = 1 + \frac{\epsilon}{2}h^{\rm R1}_{\mu\nu}u^\mu u^\nu + \mathcal{O}\left(\epsilon^2\right), \tag{5.30}$$

which follows from

$$-1 = \tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = \left(g_{\mu\nu} + h^{\rm R}_{\mu\nu}\right)\left(\frac{d\tau}{d\tilde{\tau}}\right)^2 u^\mu u^\nu. \tag{5.31}$$

Comparing Eqs. (5.27) and (5.29), we see that

$$\epsilon T_1^{\mu\nu} + \epsilon^2 T_2^{\mu\nu} = \tilde{T}^{\mu\nu} + \mathcal{O}\Big(\epsilon^3\Big). \tag{5.32}$$

This confirms Detweiler's postulate in Ref. [93].

As Detweiler also noted, we can use this to write the field equations in a more transparent form. Eq. (5.15), together with $G_{\mu\nu}[\tilde{g}] = 0$, implies that

$$\begin{aligned} G^{\mu\nu}[\mathsf{g}] &= \epsilon\delta G^{\mu\nu}[h^{\mathrm{S1}}] + \epsilon^2\delta G^{\mu\nu}[h^{\mathrm{SR}}] + 2\epsilon^2\delta^2 G^{\mu\nu}[h^{\mathrm{S1}}, h^{\mathrm{R1}}] + \mathcal{O}\Big(\epsilon^3\Big) \\ &= \delta\tilde{G}^{\mu\nu}[\epsilon h^{\mathrm{S1}} + \epsilon^2 h^{\mathrm{SR}}] + \mathcal{O}\Big(\epsilon^3\Big), \end{aligned} \tag{5.33}$$

where "$G^{\mu\nu}[\mathsf{g}]$" is to be understood as the expansion of the Einstein tensor through order ϵ^2, and $\delta\tilde{G}^{\mu\nu}$ is the linearised Einstein tensor constructed from the effective metric, $\tilde{g}_{\mu\nu}$. In words, the Einstein curvature of the physical spacetime (extended to all $r > 0$ from outside the body) is identical to the linearised Einstein curvature of the perturbation $\epsilon h^{\mathrm{S1}}_{\mu\nu} + \epsilon^2 h^{\mathrm{SR}}_{\mu\nu}$ atop the effective background $\tilde{g}_{\mu\nu}$. Combining this with Eq. (5.32) allows us to write the field equations in the form of a point mass sourcing a linear perturbation of an effective background:

$$\delta\tilde{G}^{\mu\nu}[\epsilon h^{\mathrm{S1}} + \epsilon^2 h^{\mathrm{SR}}] = 8\pi\tilde{T}^{\mu\nu} + \mathcal{O}\Big(\epsilon^3\Big). \tag{5.34}$$

In the remainder of the section, we derive several useful properties of this stress-energy. In all cases, the properties further show that the Detweiler stress-energy behaves as an ordinary stress-energy tensor in the effective metric, even as it behaves strikingly *unlike* an ordinary stress-energy in the physical spacetime.

5.1.5 Raising and lowering indices

If our stress-energy tensor was an ordinary tensor in the physical spacetime, then we could write its expansion as

$$\mathsf{T}^{\mu\nu} = \epsilon T_1^{\mu\nu} + \epsilon^2 T_2^{\mu\nu} + \mathcal{O}\Big(\epsilon^3\Big), \tag{5.35}$$

which would have indices raised and lowered with the full metric, $\mathsf{g}_{\mu\nu}$. This would mean that, say,

$$\begin{aligned} \mathsf{T}_\mu{}^\nu &= \mathsf{g}_{\mu\rho}\mathsf{T}^{\rho\nu}, \\ &= \epsilon T_\mu{}^\nu + \epsilon^2 (T_\mu^{2\nu} + h^1_{\mu\rho} T_1^{\rho\nu}) + \mathcal{O}\Big(\epsilon^3\Big). \end{aligned} \tag{5.36}$$

However, this would mean that $h^1_{\mu\rho}T_1^{\rho\nu} \sim \delta^4(x)/r$ as $h^1_{\mu\nu} \sim 1/r$ and $T_1^{\mu\nu}$ features a delta function. This is an ill-defined quantity meaning that the full metric cannot be used to raise and lower indices on this stress-energy tensor.

In fact, the indices on the stress-energy tensor are raised and lowered with the effective metric from the previous section. To see this, if we define

$$8\pi\tilde{T}_\mu{}^\nu := \delta G_\mu{}^\nu[\epsilon h^1 + \epsilon^2 h^2] + \epsilon^2\delta^2(g_{\mu\rho}G^{\rho\nu})[h^1, h^1] + \mathcal{O}\Big(\epsilon^3\Big), \tag{5.37}$$

$$8\pi\tilde{T}_{\mu\nu} := \delta G_{\mu\nu}[\epsilon h^1 + \epsilon^2 h^2] + \epsilon^2\delta^2(g_{\mu\rho}g_{\nu\sigma}G^{\rho\sigma})[h^1, h^1] + \mathcal{O}\Big(\epsilon^3\Big), \tag{5.38}$$

in analogy with Eq. (5.6), then

$$\begin{aligned}\tilde{T}_\mu{}^\nu &= \tilde{g}_{\mu\alpha}\tilde{T}^{\alpha\nu} \\ &= \epsilon T^{1\nu}_\mu + \epsilon^2(T^{2\nu}_\mu + {}^0h^{\mathrm{R}1}_{\mu\alpha}T_1^{\alpha\nu}) + \mathcal{O}\Big(\epsilon^3\Big),\end{aligned} \tag{5.39}$$

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \tilde{g}_{\mu\alpha}\tilde{g}_{\nu\beta}\tilde{T}^{\alpha\beta} \\ &= \epsilon T^1_{\mu\nu} + \epsilon^2(T^2_{\mu\nu} + 2\,{}^0h^{\mathrm{R}1}_{\alpha(\mu}g_{\nu)\beta}T_1^{\alpha\beta}) + \mathcal{O}\Big(\epsilon^3\Big).\end{aligned} \tag{5.40}$$

The right-hand sides of Eqs. (5.37) and (5.38) are the expansions of the Einstein tensor with mixed indices and both indices down, as given in Eqs. (2.29)–(2.34).

To derive these expressions, we use exactly the same method as for $T_2^{\mu\nu}$. Performing Eq. (5.21) again but for the different index position, we find

$$\tilde{T}_\mu{}^\nu = \epsilon T^{1\nu}_\mu + \epsilon^2\mathcal{L}_{\xi_1}T^{1\nu}_\mu, \tag{5.41}$$

$$\tilde{T}_{\mu\nu} = \epsilon T^1_{\mu\nu} + \epsilon^2\mathcal{L}_{\xi_1}T^1_{\mu\nu}, \tag{5.42}$$

where the Lie derivatives are given in Eqs. (B.24)–(B.23). By substituting in the values of the gauge vector from Eq. (4.44) and converting to Fermi–Walker coordinates, we see that the individual components for both indices down are given by

$$\mathcal{L}_{\xi_1}T^1_{tt} = -\frac{m}{2}\int\Big(2\,{}^0h^{\mathrm{R}1}_{tt} + \delta^{ab}\,{}^0h^{\mathrm{R}1}_{ab}\Big)\delta^4(x,z)\,dt\,, \tag{5.43a}$$

$$\mathcal{L}_{\xi_1}T^1_{ta} = -m\int {}^0h^{\mathrm{R}1}_{ta}\delta^4(x,z)\,dt\,, \tag{5.43b}$$

$$\mathcal{L}_{\xi_1}T^1_{ab} = 0 \tag{5.43c}$$

and for one up and one down by

$$\mathcal{L}_{\xi_1}T^{1t}_t = \frac{m}{2}\int \delta^{ab}\,{}^0h^{\mathrm{R}1}_{ab}\delta^4(x,z)\,dt\,, \tag{5.44a}$$

$$\mathcal{L}_{\xi_1}T^{1a}_t = 0, \tag{5.44b}$$

$$\mathcal{L}_{\xi_1}T^{1t}_a = m\int {}^0h^{\mathrm{R}1}_{ta}\delta^4(x,z)\,dt\,, \tag{5.44c}$$

$$\mathcal{L}_{\xi_1}T^{1b}_a = 0. \tag{5.44d}$$

In covariant form, these become

$$\mathcal{L}_{\xi_1} T^{1\nu}_{\mu} = -\frac{m}{2}\int_\gamma \Big[\Big(g^{\alpha\beta} - u^\alpha u^\beta\Big)\, {}^0h^{\rm R1}_{\alpha\beta} u_\mu - 2\, {}^0h^{\rm R1}_{\mu\alpha} u^\alpha\Big] u^\nu \delta^4(x,z)\, d\tau \,. \tag{5.45}$$

$$\mathcal{L}_{\xi_1} T^{1}_{\mu\nu} = -\frac{m}{2}\int_\gamma \Big[\Big(g^{\alpha\beta} - u^\alpha u^\beta\Big)\, {}^0h^{\rm R1}_{\alpha\beta} u_\mu u_\nu - 4u^\alpha u_{(\mu}\, {}^0h^{\rm R1}_{\nu)\alpha}\Big] \delta^4(x,z)\, d\tau \,, \tag{5.46}$$

We see by comparison with Eq. (5.27) that these agree with the order-ϵ^2 terms in Eqs. (5.39)–(5.40).

5.1.6 Conservation of stress-energy

In this section, we will demonstrate that the Detweiler stress-energy tensor is conserved in the effective spacetime, $\tilde{g}_{\mu\nu}$, and not the full spacetime, $\mathsf{g}_{\mu\nu}$, as one might expect. It is a standard result that the stress-energy of a point-mass in a metric $g_{\mu\nu}$ is conserved if and only if the mass moves on a geodesic of that metric. As $\tilde{T}^{\mu\nu}$ is a point mass in the effective spacetime $\tilde{g}_{\mu\nu}$, this means that

$$\tilde{\nabla}_\nu \tilde{T}^{\mu\nu} = 0. \tag{5.47}$$

However, if our stress-energy acted as a normal tensor, as in Eq. (5.35), then it would be conserved in $\mathsf{g}_{\mu\nu}$. This would mean that

$$\epsilon \nabla_\nu T^{\mu\nu}_1 + \epsilon^2(\nabla_\nu T^{\mu\nu}_2 + \delta\Gamma^\mu_{\rho\nu} T^{\rho\nu}_1 + \delta\Gamma^\nu_{\rho\nu} T^{\mu\rho}_1) = \mathcal{O}\Big(\epsilon^3\Big), \tag{5.48}$$

where $\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(2h^1_{\rho(\nu;\sigma)} - h^1_{\rho\nu;\sigma})$ is the order-ϵ term that comes from expanding the metric perturbations in Eq. (2.12). As with the argument below Eq. (5.36), this would be ill-defined for our stress-energy tensor as $(\partial_\alpha h^1_{\beta\gamma}) T^{\mu\nu}_1 \sim \delta^4/r^2$, which is an ill-defined term.

5.1.7 Gauge invariance under smooth transformations

The results we have so far for the stress-energy tensor are valid for any generic highly regular gauge that is related by a worldline-preserving transformation. However, it is interesting to consider what happens if we do not impose this condition on our gauge transformation. Our metrics perturbations and worldline are now related by [139]

$$h^{\rm R1\ddagger}_{\mu\nu} = h^{\rm R1}_{\mu\nu} + \mathcal{L}_{\xi_1} g_{\mu\nu}, \tag{5.49}$$

$$z^\mu_\ddagger = z^\mu - \epsilon \xi^\mu_1 + \mathcal{O}\Big(\epsilon^2\Big), \tag{5.50}$$

where the new gauge is denoted with a double dagger. We perform the calculation in Eq. (5.21) again, accounting for the shift in the worldline from Eq. (5.50), to find that

$$T^{\mu\nu}_{2\ddagger} = T^{\mu\nu}_{2} + (\mathcal{L}_{\xi_1} + \pounds_{\xi_1})T^{\mu\nu}_{1}. \tag{5.51}$$

Here $\pounds_{\xi_1}$ acts on $T^{\mu\nu}_1$'s dependence on z^μ; see Ref. [139] for a thorough description of this type of transformation.

Eq. (B.16) gives the action of the Lie derivatives on $T^{\mu\nu}_1$. One finds the gauge vector by solving $\mathcal{L}_{\xi_1} g_{\mu\nu} = \Delta h^{\rm R1}_{\mu\nu}$, without the gauge condition $\xi^a_1|_\gamma = 0$ The result is that the terms involving gauge vectors are again given by Eqs. (5.25) and (5.26) but replacing $h^{\rm R1}_{\mu\nu}$ with $\Delta h^{\rm R1}_{\mu\nu}$. Performing these substitutions, we obtain

$$(\mathcal{L}_{\xi_1} + \pounds_{\xi_1})\, T^{\mu\nu}_{1} = -\frac{m}{2}\int u^\mu u^\nu \left(g^{\alpha\beta} - u^\alpha u^\beta\right) \Delta h^{\rm R1}_{\alpha\beta}\delta^4(x,z)\, d\tau\,. \tag{5.52}$$

Eqs. (5.25) and (5.26) are specialised for the transformation from a rest gauge, where $h^{\rm R1'}_{\mu\nu} = 0$, to a non-rest gauge resulting in $\Delta h^{\rm R1}_{\mu\nu} = h^{\rm R1}_{\mu\nu}$. However, this is not always the case, so Eq. (5.52) is the general form of the transformation.

The stress-energy tensor in the new gauge is then given by

$$T^{\mu\nu}_{2\ddagger} = -\frac{m}{2}\int u^\mu u^\nu \left(g^{\alpha\beta} - u^\alpha u^\beta\right) h^{\rm R1\ddagger}_{\alpha\beta}\delta^4(x,z)\, d\tau\,, \tag{5.53}$$

which demonstrates that the functional form of Eq. (5.27) is always valid for a smoothly related gauge but with a regular field specific to the new gauge. Note that this is also consistent with the value of zero in the rest gauge. In the rest gauge, $h^{\rm R}_{\mu\nu}\Big|_\gamma = 0$, leading to a vanishing $T^{\mu\nu}_{2'}$.

5.2 The Detweiler stress-energy in the Lorenz gauge

One of the most widely used gauges in self-force calculations is the Lorenz gauge [15], and, as such, it is natural to look at the form of $T^{\mu\nu}_2$ in this gauge. The Lorenz gauge satisfies the level of regularity assumed by matched asymptotic expansions: $h^n_{\mu\nu} \sim m^n/r^2$ with $h^n_{\mu\nu}$ being a smooth field away from the worldline.

Unfortunately, it is not possible to perform the same treatment in the Lorenz gauge as in the highly regular gauge due to the non-distributional nature of some of the terms, as discussed at the beginning of the chapter. Additionally, the gauge transformation between the Lorenz gauge and the highly regular gauge is not smooth as the Lie derivative is singular on the worldline and the argument from Ch. 5.1.7 cannot be used.

Instead, we perform a full analysis of Eq. (5.6) in the Lorenz gauge. This involves explicitly calculating the right-hand side of the equation using a distributional definition for $\delta^2 G^{\mu\nu}[h^1, h^1]$, which we call the Detweiler canonical definition. The choice of this definition allows us to recover the Detweiler stress-energy tensor.

5.2.1 Lorenz gauge field equations and metric perturbations

To calculate the form of the perturbations in the self-consistent Lorenz-gauge scheme [72, 87, 95], one imposes the standard Lorenz-gauge gauge-condition (given in Eq. (1.18)) that the divergence of the trace-reversed metric perturbation vanishes, on the full $h_{\mu\nu}$ instead of the $h^n_{\mu\nu}$'s that appear at each order in ϵ. After imposing the gauge condition on the field equations, the individual $h^n_{\mu\nu}$'s satisfy

$$E^{\mu\nu}[\bar{h}^{1*}] = 0 \qquad \text{for} \quad x \notin \gamma, \tag{5.54}$$

$$E^{\mu\nu}[\bar{h}^{2*}] = -\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}] \qquad \text{for} \quad x \notin \gamma, \tag{5.55}$$

where a star denotes a quantity in the Lorenz gauge and $E^{\mu\nu}$ is the linearised Einstein tensor in the Lorenz gauge given by Eq. (1.19).

In the Lorenz gauge, we may use the full $h^n_{\mu\nu}$ instead of the $\mathring{h}^n_{\mu\nu}$ as used in the highly regular gauge [72, 87, 96]. The first-order singular field takes the form

$$h^{S1*}_{\mu\nu} = \frac{2m}{r}(g_{\mu\nu} + 2u_\mu u_\nu) + \mathcal{O}\Big(r^0\Big), \tag{5.56}$$

where $u^\alpha = (1, 0, 0, 0)$ so that $u^\alpha n_\alpha = 0$. The second-order singular field is split in to three pieces

$$h^{S2*}_{\mu\nu} = h^{SS*}_{\mu\nu} + h^{SR*}_{\mu\nu} + h^{\delta m*}_{\mu\nu}, \tag{5.57}$$

satisfying

$$E^{\mu\nu}[\bar{h}^{SS*}] = -\delta^2 G^{\mu\nu}[h^{S1*}, h^{S1*}] \qquad \text{for} \quad x \notin \gamma, \tag{5.58}$$

$$E^{\mu\nu}[\bar{h}^{SR*}] = -2\delta^2 G^{\mu\nu}[h^{R1*}, h^{S1*}] \qquad \text{for} \quad x \notin \gamma, \tag{5.59}$$

$$E^{\mu\nu}[\bar{h}^{\delta m*}] = 0 \qquad \text{for} \quad x \notin \gamma. \tag{5.60}$$

Here, $h^{SS*}_{\mu\nu}$ has the same structure as in the highly-regular gauge; that is, it contains all terms $\sim m^2$ but the leading term is now $\sim 1/r^2$. An explicit expression for $h^{SS*}_{\mu\nu}$ is not required for this calculation but will be needed for one in Ch. 5.3 and is given by

$$h^{SS*}_{tt} = -\frac{2m^2}{r^2} + \mathcal{O}(1/r), \tag{5.61a}$$

$$h^{SS*}_{ta} = \mathcal{O}\Big(r^0\Big), \tag{5.61b}$$

$$h^{SS*}_{ab} = \frac{m^2}{r^2}\left(\frac{8}{3}\delta_{ab} - 7\hat{n}_{ab}\right) + \mathcal{O}(1/r). \tag{5.61c}$$

The sum of the terms $h^{\mathrm{SR}*}_{\mu\nu}$ and $h^{\delta m*}_{\mu\nu}$ is analogous to the quantity $h^{\mathrm{SR}}_{\mu\nu}$ in the highly regular gauge wherein they contain terms proportional to products of m and $h^{\mathrm{R1}}_{\mu\nu}$. The components of the 'singular times regular' pieces are given by

$$h^{\mathrm{SR}*}_{tt} = -\frac{m}{r}h^{\mathrm{R1}*}_{ab}\hat{n}^{ab} + \mathcal{O}\Big(r^0\Big), \tag{5.62a}$$

$$h^{\mathrm{SR}*}_{ta} = -\frac{m}{r}h^{\mathrm{R1}*}_{tb}\hat{n}_a{}^b + \mathcal{O}\Big(r^0\Big), \tag{5.62b}$$

$$h^{\mathrm{SR}*}_{ab} = \frac{m}{r}\Big[2\hat{n}^c{}_{(a}h^{\mathrm{R1}*}_{b)a} - \delta_{ab}h^{\mathrm{R1}*}_{cd}\hat{n}^{cd} - \Big(h^{\mathrm{R1}*}_{ij}\delta^{ij} + h^{\mathrm{R1}*}_{tt}\Big)\hat{n}_{ab}\Big] + \mathcal{O}\Big(r^0\Big), \tag{5.62c}$$

and

$$h^{\delta m*}_{tt} = -\frac{m}{3r}\Big(h^{\mathrm{R1}*}_{ab}\delta^{ab} + 6h^{\mathrm{R1}*}_{tt}\Big) + \mathcal{O}\Big(r^0\Big), \tag{5.63a}$$

$$h^{\delta m*}_{ta} = -\frac{4m}{3r}h^{\mathrm{R1}*}_{ta} + \mathcal{O}\Big(r^0\Big), \tag{5.63b}$$

$$h^{\delta m*}_{ab} = \frac{m}{3r}\Big(2h^{\mathrm{R1}*}_{ab} + \delta_{ab}\delta^{cd}h^{\mathrm{R1}*}_{cd} + 2\delta_{ab}h^{\mathrm{R1}*}_{tt}\Big) + \mathcal{O}\Big(r^0\Big). \tag{5.63c}$$

Motivated by the results earlier in this chapter for the highly regular gauge, we use the definition of $T^{\mu\nu}$ from Eq. (5.6). As in the highly regular gauge, the regular fields are defined to be solutions of the vacuum Einstein equations and their Einstein tensor does not contribute to $T^{\mu\nu}$, leaving us with the analogue of Eq. (5.16),

$$\begin{aligned}8\pi T^{\mu\nu} = {}& \delta G^{\mu\nu}[\epsilon h^{\mathrm{S1}*} + \epsilon^2 h^{\mathrm{SS}*} + \epsilon^2 h^{\mathrm{SR}*} + \epsilon^2 h^{\delta m*}] + 2\epsilon^2\delta^2 G^{\mu\nu}[h^{\mathrm{S1}*}, h^{\mathrm{R1}*}] \\ & + \epsilon^2\delta^2 G^{\mu\nu}[h^{\mathrm{S1}*}, h^{\mathrm{S1}*}] + \mathcal{O}\Big(\epsilon^3\Big).\end{aligned} \tag{5.64}$$

To proceed from here we must choose a distributional definition of $\delta^2 G^{\mu\nu}[h^{\mathrm{S1}*}, h^{\mathrm{S1}*}]$. Using the gauge condition, $\epsilon^2\nabla_\nu \bar{h}^{\mu\nu}_{\mathrm{SS}*} = \mathcal{O}(\epsilon^3)$, we can rewrite Eq. (5.58) as

$$\delta G^{\mu\nu}[\bar{h}^{\mathrm{SS}*}] = -\delta^2 G^{\mu\nu}[h^{\mathrm{S1}*}, h^{\mathrm{S1}*}] + \mathcal{O}\Big(\epsilon^3\Big) \quad \text{for} \quad x \notin \gamma. \tag{5.65}$$

While in the highly regular gauge we get 'for free' that Eq. (5.15) is true and the most singular parts can be cancelled, in the Lorenz gauge we can only cancel the singular parts off of the worldline. However, as we know from the highly regular gauge that we can associate $\delta^2 G^{\mu\nu}[h^{\mathrm{S1}}]$ and $\delta G^{\mu\nu}[h^{\mathrm{SS}}]$ at all points in spacetime, we define this to be true, distributionally, in the Lorenz gauge, i.e.

$$\delta^2 G^{\mu\nu}[h^{\mathrm{S1}*}] := -\delta G^{\mu\nu}[h^{\mathrm{SS}*}], \quad \forall r. \tag{5.66}$$

As $\delta G^{\mu\nu}[h^{\mathrm{SS}*}]$ is a linear operator acting on an integrable function, this means that both sides of Eq. (5.66) are well defined as distributions for all r.

There is one subtlety that needs to be addressed before proceeding. The forms of the perturbations presented in this section are only defined as local expansions around the worldline and, as such, Eq. (5.66) is only valid in an infinitesimal region around γ. To incorporate this into our definitions, we say that

$$\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}] := \lim_{s\to 0} \delta^2 G_s^{\mu\nu}[h^{1*}, h^{1*}], \tag{5.67}$$

where

$$\begin{aligned}\delta^2 G_s^{\mu\nu}[h^{1*}, h^{1*}] := \Big(&-\delta G^{\mu\nu}[h^{\mathrm{SS}*}] + 2\delta^2 G^{\mu\nu}[h^{\mathrm{S1}*}, h^{\mathrm{R1}*}] + \delta^2 G^{\mu\nu}[h^{\mathrm{R1}*}, h^{\mathrm{R1}*}]\Big)\theta(s-r)\\ &+ \delta^2 G^{\mu\nu}[h^{1*}, h^{1*}]\theta(r-s),\end{aligned} \tag{5.68}$$

with θ being the Heaviside function. This localises Eq. (5.66) as required: for $r > s$, $\delta^2 G_s^{\mu\nu}[h^{1*}, h^{1*}]$ is the smooth function $\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}]$ and for $r < s$, $h^{1*}_{\mu\nu}$ is split into regular and singular fields and we apply Eq. (5.66). Eq. (5.68) implies that as a distribution, $\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}]$ acts on test fields $\phi_{\mu\nu}$, as[2]

$$\begin{aligned}\int \phi_{\mu\nu}\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}]\, dV := \lim_{s\to 0}\bigg\{&\int \phi_{\mu\nu}\Big(-\delta G^{\mu\nu}[h^{\mathrm{SS}*}] + 2\delta^2 G[h^{\mathrm{S1}*}, h^{\mathrm{R1}*}]\\ &+ \delta^2 G^{\mu\nu}[h^{\mathrm{R1}*}, h^{\mathrm{R1}*}]\Big)\theta(s-r)\, dV\\ &+ \int \phi_{\mu\nu}\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}]\theta(r-s)\, dV\bigg\}.\end{aligned} \tag{5.70}$$

In Ref. [93], Detweiler takes Eq. (5.65) to be valid distributionally on the region $r \geq 0$, and so we refer to Eq. (5.67) as the *Detweiler canonical definition* of $\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}]$. This results in our Einstein equations taking the form

$$8\pi T^{\mu\nu} = \epsilon\delta G^{\mu\nu}[h^{\mathrm{S1}*}] + \epsilon^2(\delta G^{\mu\nu}[h^{\mathrm{SR}*}] + \delta G^{\mu\nu}[h^{\delta m*}] + 2Q_{\mathrm{R}}^{\mu\nu}[h^{\mathrm{S1}*}]) + \mathcal{O}\big(\epsilon^3\big), \tag{5.71}$$

where $Q_{\mathrm{R}}^{\mu\nu}[h] := \delta^2 G^{\mu\nu}[h^{\mathrm{R1}*}, h]$, in analogy with Eq. (5.13). When calculating $T^{\mu\nu}$ in Eq. (5.71), we may use the locally defined fields from Eqs. (5.56) and (5.62)–(5.63) as the total Einstein equation vanishes off the worldline. If writing the field equations to solve for $h^{2*}_{\mu\nu}$ globally, we would use Eq. (5.67). This setup, where one solves for the whole of the second-order metric perturbation, will be explored further in Ch. 5.3.1.

5.2.2 Distributional analysis

To determine the distribution $T^{\mu\nu}$, we integrate the right-hand side of Eq. (5.71) against a test function.

[2] Here, the second integral, for $r > s$, is an ordinary integral of smooth functions, whereas the first one, for $r < s$ is defined distributionally (see Ch. 5.2.2) as

$$\int \phi_{\mu\nu}\delta G^{\mu\nu}[h^{\mathrm{SS}*}]\theta(s-r)\, dV := \int \delta G^{\mu\nu}[\theta(s-r)\phi]h^{\mathrm{SS}*}_{\mu\nu}. \tag{5.69}$$

Both integrals individually diverge as $1/s$ when taking the limit $s \to 0$ but exactly cancel one another. This will be explored further in Ch. 5.3.1 when looking at the source for the second-order Einstein equation.

Doing so requires the adjoints of our operators $\delta G^{\mu\nu}$ and $Q^{\mu\nu}_{\rm R}$. Here the adjoint of a linear operator $D^{\mu\nu}$ is defined by

$$\phi_{\mu\nu}D^{\mu\nu}[\psi] - D^{\dagger\mu\nu}[\phi]\psi_{\mu\nu} = \nabla_\mu K^\mu_D, \tag{5.72}$$

where $\phi_{\mu\nu}$ and $\psi_{\mu\nu}$ are arbitrary smooth fields and $K^\mu_D = K^\mu_D(\phi,\psi)$. If $\psi_{\mu\nu}$ is a distribution, then we define the integral of $D^{\mu\nu}[h]$ against a test field $\phi_{\mu\nu}$ as

$$\int \phi_{\mu\nu}D^{\mu\nu}[\psi]\, dV := \int D^{\dagger\mu\nu}[\phi]\psi_{\mu\nu}\, dV\,. \tag{5.73}$$

The linearised Einstein operator is self-adjoint [152]; that is, $\delta G^{\dagger\mu\nu}[h] = \delta G^{\mu\nu}[h]$. $Q^{\dagger\mu\nu}_R$ is given in Eq. (2.28) with $h^\flat_{\mu\nu} = h^{\rm R1*}_{\mu\nu}$.

We now evaluate the integral of Eq. (5.71) against a test field $\phi_{\mu\nu}$,

$$\begin{aligned}8\pi \int \phi_{\mu\nu}T^{\mu\nu}dV = \int \phi_{\mu\nu}\Big\{&\epsilon\delta G^{\mu\nu}[h^{\rm S1*}] + \epsilon^2(\delta G^{\mu\nu}[h^{\rm SR*}] \\ &+ \delta G^{\mu\nu}[h^{\delta m*}] + 2Q^{\mu\nu}_{\rm R}[h^{\rm S1*}])\Big\}\, dV\,. \end{aligned}\tag{5.74}$$

We then move the operators $\delta G^{\mu\nu}$ and $Q^{\mu\nu}_{\rm R}$ onto the test tensor using Eq. (5.73), so that the right-hand side of Eq. (5.74) becomes

$$\begin{aligned}&\int \Big(\delta G^{\mu\nu}[\phi]\Big\{\epsilon h^{\rm S1*}_{\mu\nu} + \epsilon^2(h^{\rm SR*}_{\mu\nu} + h^{\delta m*}_{\mu\nu})\Big\} + 2\epsilon^2 Q^{\dagger\mu\nu}_{\rm R}[\phi]h^{\rm S1*}_{\mu\nu}\Big)\, dV \\ &= \lim_{R\to 0}\int_{r>R}\Big(\delta G^{\mu\nu}[\phi]\Big\{\epsilon h^{\rm S1*}_{\mu\nu} + \epsilon^2(h^{\rm SR*}_{\mu\nu} + h^{\delta m*}_{\mu\nu})\Big\} + \epsilon^2 2Q^{\dagger\mu\nu}_{\rm R}[\phi]h^{\rm S1*}_{\mu\nu}\Big)\, dV \\ &= \lim_{R\to 0}\Big[\int_{r>R}\Big\{\phi_{\mu\nu}\delta G^{\mu\nu}[\epsilon h^{\rm S1*} + \epsilon^2(h^{\rm SR*} + h^{\delta m*})] + 2\epsilon^2\phi_{\mu\nu}Q^{\mu\nu}_{\rm R}[h^{\rm S1*}]\Big\}\, dV \\ &\quad - \int_{r=R}\Big\{K^{\delta G}_\alpha[\epsilon h^{\rm S1*} + \epsilon^2(h^{\rm SR*} + h^{\delta m*})] + 2\epsilon^2 K^Q_\alpha[h^{\rm S1*}]\Big\}\, dS^\alpha\Big], \end{aligned}\tag{5.75}$$

where K^D_α denotes the boundary term for the operator D. In the first equality we note that as the integral is now over ordinary integrable functions instead of distributions, we can remove the region $r < R$ and then take the limit as R goes to 0. Following that, in the second equality, we integrate by parts using Stokes' theorem to move the operators back onto the metric perturbations. The values of K^D_α are given by

$$\begin{aligned}K^{\delta G}_\alpha[h] = &\frac{1}{2}\phi^{\beta\mu}h_{\beta\mu;\alpha} - \frac{1}{2}h^{\beta\mu}\phi_{\beta\mu;\alpha} + \phi^\beta{}_\beta h^\mu{}_{[\alpha;\mu]} + h^\beta{}_\beta\phi^\mu{}_{[\mu;\alpha]} + \frac{1}{2}\phi_\alpha{}^\beta h^\mu{}_{\mu;\beta} \\ &- \frac{1}{2}h_\alpha{}^\beta\phi^\mu{}_{\mu;\beta} + h^{\beta\mu}\phi_{\alpha\beta;\mu} - \phi^{\beta\mu}h_{\alpha\beta;\mu}. \end{aligned}\tag{5.76}$$

and

$$
\begin{aligned}
K^{Q}_{\alpha}[h] = \frac{1}{8}\Big[& h^{\beta\gamma}\{\phi^{\zeta}{}_{\zeta} h^{\mathrm{R1}*}_{\beta\gamma;\alpha} + \phi_{\beta\gamma} h^{\mathrm{R1}*\zeta}_{\zeta}{}_{;\alpha} - 4\phi_{\alpha}{}^{\zeta} h^{\mathrm{R1}*}_{\beta\zeta;\gamma} - 2\phi_{\alpha\beta} h^{\mathrm{R1}*\zeta}_{\zeta}{}_{;\gamma}\} \\
&- h^{\beta}{}_{\beta}\Big\{\phi^{\gamma}{}_{\gamma} h^{\mathrm{R1}*\zeta}_{\zeta}{}_{;\alpha} + \phi^{\gamma\zeta}(h^{\mathrm{R1}*}_{\gamma\zeta;\alpha} - 2h^{\mathrm{R1}*}_{\alpha\gamma;\zeta})\Big\} + 2\Big\{h_{\alpha}{}^{\beta}\Big(2\phi^{\gamma\zeta} h^{\mathrm{R1}*}_{\beta\gamma;\zeta} \\
&+ 2\phi^{\gamma}{}_{\gamma} h^{\mathrm{R1}*\zeta}_{[\zeta}{}_{;\beta]}\Big) + h^{\mathrm{R1}*\beta\gamma}\Big(\phi_{\beta\gamma} h^{\zeta}{}_{\zeta;\alpha} - h^{\zeta}{}_{\zeta}\phi_{\beta\gamma;\alpha} + 2h_{\beta}{}^{\zeta}\phi_{\gamma\zeta;\alpha} - h_{\beta\gamma}\phi^{\zeta}{}_{\zeta;\alpha} \\
&+ 2\phi^{\zeta}{}_{\zeta} h_{\beta[\gamma;\alpha]} - 2\phi_{\alpha\beta} h^{\zeta}{}_{\zeta;\gamma} - 2h_{\beta}{}^{\zeta}\phi_{\alpha\zeta;\gamma} + h_{\alpha\beta}\phi^{\zeta}{}_{\zeta;\gamma} + 2\phi_{\beta}{}^{\zeta}[h_{\alpha\zeta;\gamma} + h_{\alpha\gamma;\zeta} \\
&- h_{\gamma\zeta;\alpha}] - \phi_{\beta\gamma} h_{\alpha}{}^{\zeta}{}_{;\zeta} - \phi_{\alpha}{}^{\zeta} h_{\beta\gamma;\zeta} - 2h_{\beta}{}^{\zeta}\phi_{\alpha\gamma;\zeta} + h_{\beta\gamma}\phi_{\alpha}{}^{\zeta}{}_{;\zeta} + h_{\alpha}{}^{\zeta}\phi_{\beta\gamma;\zeta}\Big) \\
&+ h^{\mathrm{R1}*\beta}_{\alpha}\Big(h^{\gamma\zeta}\phi_{\gamma\zeta;\beta} - h^{\gamma}{}_{\gamma}\phi^{\zeta}{}_{\zeta;\beta} + h_{\beta}{}^{\gamma}\phi^{\zeta}{}_{\zeta;\gamma} + 2\phi^{\gamma}{}_{\gamma} h^{\zeta}{}_{[\zeta;\beta]} - \phi^{\gamma\zeta}[h_{\gamma\zeta;\beta} - 2h_{\beta\gamma;\zeta}] \\
&- 2h^{\gamma\zeta}\phi_{\beta\gamma;\zeta} + 2h^{\gamma}{}_{\gamma}\phi_{\beta}{}^{\zeta}{}_{;\zeta}\Big)\Big\}\Big],
\end{aligned} \tag{5.77}
$$

while the surface element in Fermi–Walker coordinates is given by

$$
dS^{\alpha} = -R^2 n^{\alpha}\, dt\, d\Omega + \mathcal{O}\Big(R^3\Big), \tag{5.78}
$$

where $n^{\alpha} = (0, n^i)$ and the minus sign comes from the orientation of the normal vector to the boundary of the region $r > R$.

To evaluate the volume integral, note that the integrand is order ϵ^3 off the worldline,

$$
\epsilon\delta G^{\mu\nu}[h^{\mathrm{S1}*}] + \epsilon^2(\delta G^{\mu\nu}[h^{\mathrm{SR}*} + h^{\delta m*}] + 2Q^{\mu\nu}_{\mathrm{R}}[h^{\mathrm{S1}*}]) = \mathcal{O}\Big(\epsilon^3\Big), \quad r > 0. \tag{5.79}
$$

So the volume integral contributes nothing to the final result and can be ignored, leaving only the boundary terms:

$$
\begin{aligned}
\int \phi_{\mu\nu} T^{\mu\nu}\, dV = -\frac{1}{8\pi}\lim_{R\to 0}\int_{r=R}\Big(&\epsilon K^{\delta G}_{\alpha}[h^{\mathrm{S1}*}] + \epsilon^2 K^{\delta G}_{\alpha}[h^{\mathrm{SR}*}] \\
&+ \epsilon^2 K^{\delta G}_{\alpha}[h^{\delta m*}] + 2\epsilon^2 K^{Q}_{\alpha}[h^{\mathrm{S1}*}]\Big)\, dS^{\alpha} + \mathcal{O}\Big(\epsilon^3\Big).
\end{aligned} \tag{5.80}
$$

5.2.3 Evaluation of boundary terms

For the rest of this section, all occurrences of $h^{\mathrm{R1}*}_{\mu\nu}$ and $\phi_{\mu\nu}$ are evaluated on the worldline, but we omit the notation for visual clarity. We substitute $h^{\mathrm{S1}*}_{\mu\nu}$ from (5.56), $h^{\mathrm{SR}*}_{\mu\nu}$ from (5.62) and $h^{\delta m*}_{\mu\nu}$ from (5.63) into Eq. (5.76), giving

$$
K^{\delta G}_{\alpha}[h^{\mathrm{S1}*}] = -\frac{2mn_{\beta}}{r^2}\Big(\phi^{\mu}{}_{\mu} u_{\alpha} u^{\beta} - 2\phi^{\beta}{}_{\mu} u_{\alpha} u^{\mu} + g_{\alpha}{}^{\beta}\phi_{\mu\nu} u^{\mu} u^{\nu}\Big) + \mathcal{O}(1/r), \tag{5.81}
$$

$$
\begin{aligned}
K^{\delta G}_{\alpha}[h^{\mathrm{SR}*}] = -\frac{mn_{\alpha}\hat{n}^{ab}}{2r^2}\Big(&2h^{\mathrm{R1}*}_{ac}\phi_{b}{}^{c} + 2h^{\mathrm{R1}*}_{ta}\phi_{tb} - h^{\mathrm{R1}*}_{tt}\phi_{ab} - \delta^{ij} h^{\mathrm{R1}*}_{ij}\phi_{ab} - h^{\mathrm{R1}*}_{ab}\phi^{c}{}_{c} \\
&- h^{\mathrm{R1}*}_{ab}\phi_{tt}\Big) + \mathcal{O}(1/r),
\end{aligned} \tag{5.82}
$$

$$
\begin{aligned}
K^{\delta G}_{\alpha}[h^{\delta m*}] = -\frac{m}{6r^2}\Big[&6\phi_{\alpha a} n^{a}\Big(\delta^{ij} h^{\mathrm{R1}*}_{ij} + 2h^{\mathrm{R1}*}_{tt}\Big) + n_{\alpha}\Big(2h^{\mathrm{R1}*}_{ab}\phi^{ab} + 8h^{\mathrm{R1}*}_{ta}\phi_{t}{}^{a} \\
&- 10h^{\mathrm{R1}*}_{tt}\phi^{a}{}_{a} - 5\delta^{ij} h^{\mathrm{R1}*}_{ij}\phi^{b}{}_{b} + 5\delta^{ij} h^{\mathrm{R1}*}_{ij}\phi_{tt} + 6h^{\mathrm{R1}*}_{tt}\phi_{tt}\Big)\Big] + \mathcal{O}(1/r).
\end{aligned} \tag{5.83}
$$

Note that we only require terms of order $1/r^n$ where $n \geq 2$ as all other terms will vanish after taking the limit $R \to 0$. We follow the same procedure for K^Q_α, substituting Eq. (5.56) into Eq. (5.77), to get

$$\begin{aligned}
K^Q_\alpha[h^{\rm S1*}] = \frac{m}{r^2}\Big[&n_\alpha\Big(h^{\rm R1*}_{tt}\phi^a{}_a - h^{\rm R1*}_{ab}\phi^{ab} - 2h^{\rm R1*}_{ta}\phi_t{}^a + \delta^{ij}h^{\rm R1*}_{ij}\phi^b{}_b - \delta^{ij}h^{\rm R1*}_{ij}\phi_{tt} \\
&+ 2h^{\rm R1*}_{tt}\phi_{tt}\Big) + n^a\Big(4h^{\rm R1*}_{\alpha b}\phi_a{}^b - h^{\rm R1*}_{\alpha a}\phi^b{}_b - h^{\rm R1*b}_{b}\phi_{\alpha a} - h^{\rm R1*}_{tt}\phi_{\alpha a} \\
&- 2h^{\rm R1*}_{ab}\phi_\alpha{}^b + 2h^{\rm R1*}_{ta}\phi_{t\alpha} - h^{\rm R1*}_{\alpha a}\phi_{tt} + 2u_\alpha\Big(2h^{\rm R1*}_{tb}\phi_a{}^b - 2h^{\rm R1*}_{tt}\phi_{ta} \\
&- h^{\rm R1*}_{ta}\phi^b{}_b + 2h^{\rm R1*}_{ab}\phi^b{}_t - h^{\rm R1*}_{ta}\phi_{tt}\Big)\Big)\Big] + \mathcal{O}(1/r).
\end{aligned} \tag{5.84}$$

We then integrate each of these quantities with the surface element from Eq. (5.78), noting that [176]

$$\int \hat{n}_L\, d\Omega = 0 \quad \text{for} \quad l \geq 1. \tag{5.85}$$

The first-order integral is given by

$$\lim_{R\to 0}\int_{r=R} K^{\delta G}_\alpha[h^{\rm S1*}]\, dS^\alpha = -8\pi m \int \phi_{tt}\, dt\,, \tag{5.86}$$

and the second-order ones by

$$\begin{aligned}
\lim_{R\to 0}\int_{r=R} K^{\delta G}_\alpha[h^{\rm SR*}]\, dS^\alpha = &-\frac{8\pi m}{9}\int\Big(h^{\rm R1*}_{ab}\phi^{ab} + 2h^{\rm R1*}_{ta}\phi_t{}^a + h^{\rm R1*}_{tt}\phi^a{}_a - \delta^{ij}h^{\rm R1*}_{ij}\phi_{tt} \\
&- 3h^{\rm R1*}_{tt}\phi_{tt}\Big)\, dt\,, \qquad (5.87)\\
\lim_{R\to 0}\int_{r=R} K^{\delta G}_\alpha[h^{\delta m*}]\, dS^\alpha = &-\frac{4\pi m}{9}\int\Big(h^{\rm R1*}_{ab}\phi^{ab} + 8h^{\rm R1*}_{ta}\phi_t{}^a - 8h^{\rm R1*}_{tt}\phi^a{}_a - 3\delta^{ij}h^{\rm R1*}_{ij}\phi^b{}_b \\
&+ 5\delta^{ij}h^{\rm R1*}_{ij}\phi_{tt} + 6h^{\rm R1*}_{tt}\phi_{tt}\Big)\, dt\,, \qquad (5.88)\\
\lim_{R\to 0}\int_{r=R} K^{Q}_\alpha[h^{\rm S1*}]\, dS^\alpha = &\frac{4\pi m}{3}\int\Big(h^{\rm R1*}_{ab}\phi^{ab} + 4h^{\rm R1*}_{ta}\phi^a{}_t - 2h^{\rm R1*}_{tt}\phi^a{}_a - \delta^{ij}h^{\rm R1*}_{ij}\phi^b{}_b \\
&+ 4\delta^{ij}h^{\rm R1*}_{ij}\phi_{tt} - 6h^{\rm R1*}_{tt}\phi_{tt}\Big)\, dt\,. \qquad (5.89)
\end{aligned}$$

5.2.4 Recovering the Detweiler stress-energy

As we explained in Ch. 5.1.1, if we were to define $8\pi T^{\mu\nu}_1 := \delta G^{\mu\nu}[h^{1*}]$ in the self-consistent expansion, then we would find $T^{\mu\nu}_1$ contains a subdominant correction that is extended away from γ. That prompted us to define the total $T^{\mu\nu}$ in Eq. (5.6), rather than defining each $T^{\mu\nu}_n$ separately. However, our formula (5.80) now provides an unambiguous split:

$$\begin{aligned}
\int \phi_{\mu\nu}T^{\mu\nu}_1\, dV &= -\frac{1}{8\pi}\lim_{R\to 0}\int_{r=R} K^{\delta G}_\alpha[h^{\rm S1*}]\, dS^\alpha\,, \qquad (5.90)\\
\int \phi_{\mu\nu}T^{\mu\nu}_{2*}\, dV &= -\frac{1}{8\pi}\lim_{R\to 0}\int_{r=R}\Big(K^{\delta G}_\alpha[h^{\rm SR*}] + K^{\delta G}_\alpha[h^{\delta m*}] + 2K^Q_\alpha[h^{\rm S1*}]\Big)\, dS^\alpha\,. \qquad (5.91)
\end{aligned}$$

These are equivalent to the definitions (5.8) and (5.9).

At first order, we can immediately see from Eq. (5.86) that Eq. (5.90) can be written as

$$\int \phi_{\mu\nu} T_1^{\mu\nu}\, dV = m \iint \phi_{\mu\nu} u^\mu u^\nu \delta^4(x,z)\, d\tau\, dV. \tag{5.92}$$

Since this holds for an arbitrary test field $\phi_{\mu\nu}$, we infer that $T_1^{\mu\nu}$ is the point-mass stress-energy in Eq. (5.10), as expected; a nearly identical derivation appears in Ref. [86].

Moving to second order, we sum the boundary terms to obtain

$$\begin{aligned} \lim_{R\to 0} \int_{r=R} \Big(K^{\delta G}_\alpha[h^{\mathrm{SR}*}] + K^{\delta G}_\alpha[h^{\delta m*}] + K^{Q}_\alpha[h^{\mathrm{S1}*}] \Big)\, dS^\alpha \\ = 4\pi m \int (2h^{\mathrm{R1}*}_{tt} - \delta^{ij} h^{\mathrm{R1}*}_{ij}) \phi_{tt}\, dt\,. \end{aligned} \tag{5.93}$$

We can therefore write Eq. (5.91) as

$$\int \phi_{\mu\nu} T_{2*}^{\mu\nu}\, dV = \frac{m}{2} \iint \phi_{\mu\nu} u^\mu u^\nu (u^\alpha u^\beta - g^{\alpha\beta}) h^{\mathrm{R1}*}_{\alpha\beta}\, d\tau\, dV\,. \tag{5.94}$$

This implies that, given Detweiler's canonical definition of $\delta^2 G^{\mu\nu}[h^{1*}, h^{1*}]$, $T_{2*}^{\mu\nu}$ in the Lorenz gauge has the same functional form as the $T_2^{\mu\nu}$ found in the highly regular gauge in Eq. (5.27). Additionally, using the methods and arguments outlined in this section, we can show that the functional forms of $T^{2\nu}_\mu$ and $T^2_{\mu\nu}$ in the Lorenz gauge match the ones found in the highly regular gauge, as is to be expected.

5.3 Distributional sources for field equations

With our distributional definitions for the second-order stress-energy tensor, it is possible to formulate field equations that directly solve for the second-order metric perturbation. To do so first requires us to calculate the source of these equations. In this section we consider two such methods: firstly, directly using the Einstein field equations and secondly, using the Teukolsky equation. The Teukolsky equation [177] rewrites the Einstein field equations as a single master equation that solves for certain scalar quantities. These scalar quantities can then be used to reconstruct the metric perturbations.

One complication that is encountered when calculating the sources for these equations comes from the non-compact nature of the second-order Einstein equation, given in Eq. (1.10). While the stress-energy tensor is a compact source only supported on the worldline, $\delta^2 G^{\mu\nu}[h^1, h^1]$ has support across the entire spacetime and, crucially, also features delta function content only supported on the worldline. This is unlike most distributional sources which tend to either have only have delta function content or feature no delta functions at all. When looking at the source for the second-order Einstein equation, we show that one must explicitly calculate this delta function content to ensure that the source is well-defined on the entire spacetime.

We explain how one can calculate the source for the Einstein equation in Ch. 5.3.1 and for the Teukolsky equation in Ch. 5.3.2.

5.3.1 Source for the second-order Einstein equation

With our distributional definitions for the second-order stress-energy tensor, we consider the source for the second-order Einstein field equations when in the Lorenz gauge,

$$\delta G_{\mu\nu}[h^{2*}] = 8\pi T^{2*}_{\mu\nu} - \delta^2 G_{\mu\nu}[h^{1*}, h^{1*}], \tag{5.95}$$

where $T^{2*}_{\mu\nu}$ is given by the order-ϵ^2 term in Eq. (5.46) but with ${}^0h^{\rm R1}_{\mu\nu} \to h^{\rm R1*}_{\mu\nu}$ and $\delta^2 G_{\mu\nu}$ uses the distributional definition from Eq. (5.66).

Eq. (5.95) is a well-defined equation following our distributional definitions. However, it is not immediately obvious how one would practically use our definition for $\delta^2 G_{\mu\nu}[h^{1*}, h^{1*}]$ to numerically solve for the physical field with this distributional source. The aim of this section is to reformulate our distributional definition for $\delta^2 G_{\mu\nu}[h^{1*}, h^{1*}]$ in a more practical form.

In the text around Eqs. (5.69)–(5.70) we mentioned that that the canonical definition did feature quantities that would diverge individually in the $s \to 0$ limit but that they exactly cancelled one another when explicitly calculated. Looking at Eq. (5.70), when solving Eq. (5.95) numerically in four dimensions or in some mode basis, it is the second integral for $r > s$ that we would be calculating, neglecting the distributional first term. By incorporating the distributional first term, we can provide a counter term that exactly cancels the divergent behaviour in the second integral. For the calculation of the stress-energy tensor, we never needed to calculate these terms as $\delta^2 G_{\mu\nu}[h^{\rm S1*}, h^{\rm S1*}]$ was exactly cancelled by $\delta G_{\mu\nu}[h^{\rm SS*}_{\mu\nu}]$. However, moving to Eq. (5.95), we now do need to explicitly calculate these terms to ensure our equation is well defined.

To calculate the distributional nature of $h^{\rm SS*}_{\mu\nu}$, we follow the same procedure as in Ch. 5.2.2 when determining the form of the stress-energy tensor. Defined distributionally, the integral of $\delta G_{\mu\nu}[h^{\rm SS*}]$ against a tensor field $\phi^{\mu\nu}$ is

$$\int \phi^{\mu\nu} \delta G_{\mu\nu}[h^{\rm SS*}] \theta(s-r)\, dV := \int \delta G_{\mu\nu}[\theta(s-r)\phi] h^{\mu\nu}_{\rm SS*}\, dV\,, \tag{5.96}$$

where we have used that both $\delta G_{\mu\nu}$ and the Heaviside function are self adjoint. This has the same form as Eq. (5.69) but defined for the linearised Einstein tensor with indices down. Integrating the right-hand side by parts and including the minus sign factor from

Eq. (5.70), we see that

$$-\int \delta G_{\mu\nu}[\theta(s-r)\phi]h^{\mu\nu}_{\mathrm{SS}*}\,dV = \lim_{R\to 0}\biggl(-\int_{r>R}\delta G_{\mu\nu}[h^{\mathrm{SS}*}]\phi^{\mu\nu}\theta(s-r)\,dV + \int_{r=R}K^{\delta G}_{\alpha}[\theta(s-r)\phi, h^{\mathrm{SS}*}]\,dS^{\alpha}\biggr). \tag{5.97}$$

We already know the form of $K^{\delta G}_{\alpha}[\phi, h]$ from Eq. (5.76) so we can simply make the replacements $\phi_{\mu\nu}\to\phi_{\mu\nu}\theta(r-s)$ and $h_{\mu\nu}\to h^{\mathrm{SS}*}_{\mu\nu}$, where $h^{\mathrm{SS}*}_{\mu\nu}$ is given by Eq. (5.61). However, unlike in Eq. (5.79) of Ch. 5.2.2, the volume integral in the above equation does not vanish as the integrand is non-zero off the worldline. This means we must explicitly calculate its contribution to the final result.

Calculating the boundary term first, we substitute in $h^{\mathrm{SS}*}_{\mu\nu}$ to find that

$$\begin{aligned}K^{\delta G}_{\alpha}[h^{\mathrm{SS}*}] = &-\frac{m^2}{r^3}\theta(s-r)\phi^{\beta\mu}\Bigl(6n_{\beta}u_{\alpha}u_{\mu} - n_{\alpha}(14n_{\beta\mu} - 7P_{\beta\mu} + 3u_{\beta}u_{\mu})\Bigr)\\ &+\frac{m^2}{2r^2}\theta(s-r)\phi^{\beta\mu}{}_{\gamma}\Bigl(P_{\alpha}{}^{\gamma}(3u_{\beta}u_{\mu} - 7n_{\beta\mu}) + n^{\gamma}[2n_{\beta}(7P_{\alpha\mu} - 13u_{\alpha}u_{\mu})\\ &+ n_{\alpha}(28n_{\beta\mu} - 21P_{\beta\mu} + 13u_{\beta}u_{\mu})] + u^{\gamma}(7n_{\beta\mu}u_{\alpha} + 3P_{\beta\mu}u_{\alpha} - 6P_{\alpha\mu}u_{\beta})\Bigr)\\ &+\mathcal{O}(1/r),\end{aligned} \tag{5.98}$$

where $\phi^{\mu\nu}$ is evaluated on the worldline and $\phi^{\mu\nu}{}_{\alpha} := \partial_{\alpha}\phi^{\mu\nu}|_{\gamma}$. Contracting with the Fermi–Walker surface element from Eq. (2.130) and integrating over the angles gives

$$\lim_{R\to 0}\int_{r=R}K^{\delta G}_{\alpha}[h^{\mathrm{SS}*}]\,dS^{\alpha} = \frac{4\pi m^2}{3}\lim_{R\to 0}\int \phi^{\mu\nu}\frac{\theta(s-R)}{R}(7P_{\mu\nu} - 9u_{\mu}u_{\nu})\,dt\,, \tag{5.99}$$

where we have distributed the limit to remove terms of order R and higher.

Returning to the volume integral in Eq. (5.97), we see

$$\begin{aligned}&-\lim_{R\to 0}\int_{r>R}\delta G_{\mu\nu}[h^{\mathrm{SS}*}]\phi^{\mu\nu}\theta(s-r)\,dV\\ &= \lim_{R\to 0}\int_{r>R}\biggl[\frac{m^2}{r^4}\theta(s-r)\phi^{\mu\nu}\Bigl(3u_{\mu}u_{\nu} + 14n_{\mu\nu} - 7P_{\mu\nu}\Bigr)\\ &\quad + \frac{m^2}{r^3}\theta(s-r)\phi^{\mu\nu}{}_{\gamma}n^{\gamma}\Bigl(3u_{\mu}u_{\nu} + 14n_{\mu\nu} - 7P_{\mu\nu}\Bigr) + \mathcal{O}\Bigl(1/r^2\Bigr)\biggr]\,dV\\ &= \frac{4m^2\pi}{3}\lim_{R\to 0}\int_{r>R}\biggl[\frac{\phi^{\mu\nu}}{r^2}\theta(s-r)\Bigl(9u_{\mu}u_{\nu} - 7P_{\mu\nu}\Bigr) + \mathcal{O}\Bigl(r^0\Bigr)\biggr]\,dt\,dr\\ &= \frac{4m^2\pi}{3}\lim_{R\to 0}\frac{s-R}{Rs}\theta(s-R)\int(9u_{\mu}u_{\nu} - 7P_{\mu\nu})\phi^{\mu\nu}\,dt\,,\end{aligned} \tag{5.100}$$

where the order-r^{-3} term vanishes going from the first to second equality due to the integral over the angles and we use

$$\int_{R}^{\infty}\frac{\theta(s-r)}{r^2}\,dr = \frac{s-R}{Rs}\theta(s-R), \tag{5.101}$$

to go to the final line. We have eliminated the higher-order terms in the final line by noting that they vanish when taking the limit as $R \to 0$.

Summing up Eqs. (5.99)–(5.100), we find that our counter term is given by

$$\lim_{R\to 0}\left(-\int_{r>R} \delta G_{\mu\nu}[h^{\mathrm{SS}*}]\phi^{\mu\nu}\theta(s-r)\,dV + \int_{r=R} K^{\delta G}_{\alpha}[\theta(s-r)\phi, h^{\mathrm{SS}*}]\,dS^{\alpha}\right)$$
$$= \frac{4m^2\pi}{3s}\int (7g_{\mu\nu} - 2u_\mu u_\nu)\phi^{\mu\nu}\,dt\,, \quad (5.102)$$

where we have written the projection operator in terms of the metric and four velocity, and we have eliminated the Heaviside function by noting that, in the area we are interested in, s will always be greater than R. This has the knock on effect of eliminating the limit as $R \to 0$ as we have eliminated all the R dependence in the equation.

As Eq. (5.102) has compact support (recall $\phi_{\mu\nu}$ is evaluated on the worldline), we can think of it as some effective stress-energy tensor, $T^{\mathrm{counter}}_{\mu\nu}$, which we define as

$$\int \phi^{\mu\nu}T^{\mathrm{counter}}_{\mu\nu}\,dV := \frac{1}{8\pi}\lim_{R\to 0}\left(-\int_{r>R} \delta G_{\mu\nu}[h^{\mathrm{SS}*}]\phi^{\mu\nu}\theta(s-r)\,dV\right.$$
$$\left. + \int_{r=R} K^{\delta G}_{\alpha}[\theta(s-r)\phi, h^{\mathrm{SS}*}]\,dS^{\alpha}\right), \quad (5.103)$$

We can use the fact that $\phi^{\mu\nu}$ is a test field and combine Eqs. (5.102) and (5.103), to write the stress-energy tensor explicitly as

$$T^{\mathrm{counter}}_{\mu\nu} = \frac{m^2}{6s}\int (7g_{\mu\nu} - 2u_\mu u_\nu)\delta^4(x,z)\,d\tau\,. \quad (5.104)$$

With the $h^{\mathrm{SS}*}_{\mu\nu}$ calculated, we turn our attention to the other pieces in Eq. (5.68). By construction, the $\delta^2 G_{\mu\nu}[h^{\mathrm{R1}*}, h^{\mathrm{R1}*}]$ piece does not contribute anything as $h^{\mathrm{R1}*}_{\mu\nu}$ is a smooth field everywhere. Moving to the $Q^{\mathrm{R}}_{\mu\nu}[h^{\mathrm{S1}*}]$ term, we write it as in Eq. (5.75), to see that

$$\lim_{R\to 0}\int_{r>R} Q^{\mathrm{R}\dagger}_{\mu\nu}[\phi\theta(s-r)]h^{\mathrm{S1}*}_{\mu\nu}\,dV = \lim_{R\to 0}\left[\int_{r>R}\phi^{\mu\nu}\theta(s-r)Q^{\mathrm{R}}_{\mu\nu}[h^{\mathrm{S1}*}]\,dV\right.$$
$$\left. - \int_{r=R} K^{Q\flat\flat}_{\alpha}[h^{\mathrm{S1}*}, \phi\theta(s-r)]\,dS^{\alpha}\right]. \quad (5.105)$$

We note that $K^{Q\flat\flat}_{\alpha}$ is not the same operator as K^{Q}_{α}, given in Eq. (5.77). The operator in Eq. (5.77) was constructed from $Q^{\mu\nu}_{\mathrm{R}}$ whereas $K^{Q\flat\flat}_{\alpha}$ is constructed from $Q^{\mathrm{R}}_{\mu\nu}$ and is given by

$$K^{Q\flat\flat}_\alpha[h,\phi] = \frac{1}{4}\Big[h^{\beta\gamma}\Big(2\phi_{\alpha\beta}h^{\mathrm{R1}\zeta}_{\zeta}{}_{;\gamma} - \phi^\zeta{}_\zeta h^{\mathrm{R1}}_{\beta\gamma;\alpha} - \phi_{\beta\gamma}h^{\mathrm{R1}\zeta}_{\zeta}{}_{;\alpha} - 4\phi_\beta{}^\zeta(h^{\mathrm{R1}}_{\alpha\zeta;\gamma} - h^{\mathrm{R1}}_{\gamma\zeta;\alpha})\Big)$$
$$+ h^\beta{}_\beta\Big(\phi^\gamma{}_\gamma h^{\mathrm{R1}\zeta}_{\zeta}{}_{;\alpha} + \phi^{\gamma\zeta}(2h^{\mathrm{R1}}_{\alpha\gamma;\zeta} - 3h^{\mathrm{R1}}_{\gamma\zeta;\alpha})\Big) + 2\Big[h_\alpha{}^\beta\Big(2\phi^{\gamma\zeta}(h^{\mathrm{R1}}_{\gamma\zeta;\beta} - h^{\mathrm{R1}}_{\beta\gamma;\zeta})$$
$$+ \phi^\gamma{}_\gamma(-h^{\mathrm{R1}\zeta}_{\zeta}{}_{;\beta} + h^{\mathrm{R1}\zeta}_{\beta}{}_{;\zeta})\Big) + h^{\mathrm{R1}\beta}_\alpha\Big(h^\gamma{}_\gamma\phi^\zeta{}_{\zeta;\beta} - h^{\gamma\zeta}\phi_{\gamma\zeta;\beta} - h_\beta{}^\gamma\phi^\zeta{}_{\zeta;\gamma}$$
$$+ \phi^{\gamma\zeta}(h_{\gamma\zeta;\beta} - 2h_{\beta\gamma;\zeta}) + 2\phi^\gamma{}_\gamma h^\zeta{}_{[\beta;\zeta]}\Big)\Big] + 2h^{\mathrm{R1}\beta\gamma}\Big(2\phi^\zeta{}_\zeta h_{\beta[\alpha;\gamma]} - h^\zeta{}_\zeta\phi_{\beta\gamma;\alpha}$$
$$+ 2h_\beta{}^\zeta\phi_{\alpha\zeta;\gamma} - h_{\alpha\beta}\phi^\zeta{}_{\zeta;\gamma} + 2\phi_{\beta\gamma}h^\zeta{}_{[\zeta;\alpha]} + \phi_\alpha{}^\zeta h_{\beta\gamma;\zeta} + 2h_{\beta\gamma}\phi^\zeta{}_{[\zeta;\alpha]}$$
$$+ h_\alpha{}^\zeta\phi_{\beta\gamma;\zeta}\Big)\Big]. \tag{5.106}$$

In the original calculation of the Detweiler stress-energy tensor that, as we used the total quantity $\delta G_{\mu\nu}[h^{2*}] + \delta^2 G_{\mu\nu}[h^{1*}, h^{1*}]$, the volume integral would be identically zero as the integrand itself was order ϵ^3 when evaluated off the worldline. We can not make that argument now when looking at constituent parts of the field equations and must explicitly evaluate the volume integral in Eq. (5.105) to see if there are any s-dependent pieces. After calculating the integral, we find that it is identically zero after performing the angular integration and, as such, contributes no s-dependent terms to the final sum. However, as shown in Eq. (5.89), the boundary term does contribute delta function content. Performing the same calculation but for $K^{Q\flat\flat}_\alpha$ from Eq. (5.106), we find that

$$\lim_{R\to 0}\int_{r=R} K^{Q\flat\flat}_\alpha[h^{\mathrm{S1}*}]\,dS^\alpha = -\frac{4\pi m}{3}\int h^{\mathrm{R1}}_{\mu\nu}\phi^{\alpha\beta}\Big(P^\mu{}_\alpha P^\nu{}_\beta - 8P^\mu{}_\alpha u^\nu u_\beta - 2u^\mu u^\nu P_{\alpha\beta}$$
$$- P^{\mu\nu}P_{\alpha\beta} + 4P_{\mu\nu}u_\alpha u_\beta + 6u^\mu u^\nu u_\alpha u_\beta\Big)\,dt\,. \tag{5.107}$$

This has the same structural form as K^Q_α but with different numerical coefficients on some of the terms. As we did for $K^{\delta G}_\alpha[h^{\mathrm{SS}*}]$ in Eqs. (5.104), we can think of this as a stress-energy tensor, defined as

$$\int \phi^{\mu\nu}T^{Q\flat\flat}_{\mu\nu}\,dV := -\frac{1}{8\pi}\lim_{R\to 0}\int_{r=R} 2K^{Q\flat\flat}_\alpha[h^{\mathrm{S1}*}]\,dS^\alpha$$
$$= \frac{m}{3}\int h^{\mathrm{R1}*}_{\mu\nu}\phi^{\alpha\beta}\Big(P^\mu{}_\alpha P^\nu{}_\beta - 8P^\mu{}_\alpha u^\nu u_\beta - 2u^\mu u^\nu P_{\alpha\beta} - P^{\mu\nu}P_{\alpha\beta}$$
$$+ 4P_{\mu\nu}u_\alpha u_\beta + 6u^\mu u^\nu u_\alpha u_\beta\Big)\,dt\,. \tag{5.108}$$

This can then be explicitly written as a stress-energy tensor,

$$T^{Q\flat\flat}_{\alpha\beta} = \frac{m}{3}\int h^{\mathrm{R1}*}_{\mu\nu}\Big(P^\mu{}_\alpha P^\nu{}_\beta - 8P^\mu{}_\alpha u^\nu u_\beta - 2u^\mu u^\nu P_{\alpha\beta} - P^{\mu\nu}P_{\alpha\beta}$$
$$+ 4P_{\mu\nu}u_\alpha u_\beta + 6u^\mu u^\nu u_\alpha u_\beta\Big)\,dt\,. \tag{5.109}$$

With our stress-energy counter terms calculated, we can rewrite Eq. (5.95) to take this into account as

$$\delta G_{\mu\nu}[h^{2*}] = 8\pi(T^{2*}_{\mu\nu} - T^{Q\flat\flat}_{\mu\nu}) + \lim_{s\to 0}\left\{8\pi T^{\text{counter}}_{\mu\nu} - \theta(r-s)\delta^2 G_{\mu\nu}[h^{1*}, h^{1*}]\right\}. \quad (5.110)$$

The terms on the RHS after $T^{2*}_{\mu\nu}$ are exactly the source given in Eq. (5.67) with $T^{\text{counter}}_{\mu\nu}/T^{Q\flat\flat}_{\mu\nu}$ being the first/second terms in Eq. (5.68) re-expressed as delta functions. We have not explicitly checked but it is likely that the difference $T^{2*}_{\mu\nu} - T^{Q\flat\flat}_{\mu\nu}$ is related to the stress-energy sourced by $h^{\delta m}_{\mu\nu}$ previously calculated by Pound [72, Eq. (133)]. This can be seen from Eq. (5.91) where the introduction of $T^{Q\flat\flat}_{\mu\nu}$ cancels the $K^{Q\flat\flat}_{\alpha}[h^{\text{S1}}]$ part of $T^{2*}_{\mu\nu}$.

As an example of how this could be implemented in a four-dimensional setup when solving numerically, one could institute a small-s cut-off for the right-hand side when integrating over it. After doing this for a sequence of decreasing values of s, one could then numerically take the limit as $s \to 0$. This would, in theory, provide a finite answer but it would rely on delicate numerical cancellations between the second-order Einstein tensor and our new stress-energy term. It may also be possible to implement Eq. (5.110) in a mode decomposition but we leave investigations of this and the numerical implementation to future work.

5.3.2 Source for the Teukolsky equation

In the previous section, we considered the source for the Einstein field equations to solve for the metric perturbations. However, one alternative is to use the Teukolsky equation [177]. As mentioned in the introduction to this chapter, the Teukolsky equation reformulates the Einstein field equations in terms of scalar quantities. These are the Weyl scalars, Ψ_i, given in Eq. (C.11).

Teukolsky demonstrated that the equations for the perturbations of each of the Ψ_i decouple so one may solve for each individually. We denote the nth order perturbation of the Weyl scalar as $\Psi_i^{(n)}$. Not only do the equations for each of the Ψ_i decouple, they also become separable in Kerr and Schwarzschild. Decoupled, separable equations had previously been calculated for perturbations in Schwarzschild [178–180] but not for perturbations in Kerr. This separability and decoupling allows one to write equations for each of the perturbations of the Weyl scalars, individually, as a collection of ordinary differential equations. Of particular interest are perturbations of Ψ_4 as they encode all of the information about outgoing gravitational waves. They also contain the entire information about the metric perturbations. With the perturbations to the Weyl scalar calculated, one may then use metric reconstruction to recover the metric perturbations [154, 181, 182].

The Teukolsky equation is written in the Kerr spacetime which, in standard Boyer–Lindquist coordinates (t, r, θ, ϕ), has the metric [183]

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2(\theta)}{\Sigma}\, dt\, d\phi + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 \\ + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2(\theta)}{\Sigma}\right)\sin^2(\theta)\, d\phi^2\,, \quad (5.111)$$

with $\Sigma := r^2 + a^2\cos^2(\theta)$, $\Delta := r^2 - 2Mr + a^2$ and a being the spin parameter. The Teukolsky equation takes advantage of the geometric structure of the Kerr spacetime where all of the background Weyl scalars, except Ψ_2, vanish. The ultimate goal of EMRI modelling is to generate waveforms on a Kerr background which motivates the use of the Teukolsky equation in Kerr. One may also use the Teukolsky equation with the Schwarzschild background by setting the spin parameter, a, to zero.

The Teukolsky equation is often written in terms of the Newman–Penrose (NP) formalism (see App. C) which, when solving for $\Psi_4^{(1)}$, is given by[3]

$$[(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho) \\ - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\Psi_2]\Psi_4^{(1)} = 4\pi T_4 \quad (5.112)$$

where the source is given by

$$T_4 = (\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})[(\bar{\delta} - 2\bar{\tau} + 2\alpha)T_{n\bar{m}} - (\Delta + 2\gamma - 2\bar{\gamma} + \bar{\mu})T_{\bar{m}\bar{m}}] \\ + (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)[(\Delta + 2\gamma + 2\bar{\mu})T_{n\bar{m}} - (\bar{\delta} - \bar{\tau} + 2\bar{\beta} + 2\alpha)T_{nn}] \quad (5.113)$$

All quantities in these equations (except $T_{\mu\nu}$) are Newman–Penrose scalars or derivatives and are defined in Eqs. (C.8)–(C.9).

In operator form, this can be compactly written as [152]

$$\hat{\mathcal{S}}\hat{\mathcal{E}}[h^1] = \hat{\mathcal{O}}\hat{\mathcal{T}}[h^1], \quad (5.114)$$

where $\hat{\mathcal{O}}$ is defined as the term inside the square brackets on the left-hand side of Eq. (5.112),

$$\hat{\mathcal{O}} := (\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho) - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\Psi_2, \quad (5.115)$$

[3] On the RHS of this equation, π refers to the numerical constant whereas on the LHS, it refers to the NP quantity given in Eq. (C.8e).

$\hat{\mathcal{S}}$ is defined as the expression acting on $T_{\mu\nu}$ in Eq. (5.113),

$$\begin{aligned}\hat{\mathcal{S}} := &(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})[(\bar{\delta} - 2\bar{\tau} + 2\alpha)n^\mu \bar{m}^\nu - (\Delta + 2\gamma - 2\bar{\gamma} + \bar{\mu})\bar{m}^\mu \bar{m}^\nu] \\ &+ (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)[(\Delta + 2\gamma + 2\bar{\mu})n^\mu \bar{m}^\nu - (\bar{\delta} - \bar{\tau} + 2\bar{\beta} + 2\alpha)n^\mu n^\nu], \quad (5.116)\end{aligned}$$

and $\hat{\mathcal{E}}[h]$ represents the linearised Einstein tensor. Finally, at linear order,

$$\hat{\mathcal{T}}[h^1] = \Psi_4^{(1)}, \tag{5.117}$$

where the superscript refers to the perturbative order.

At second order, the Teukolsky equation takes a slightly more complicated form and was reformulated by Spiers et al. [184] to take advantage of the distributional definitions presented earlier in this chapter. The operator $\hat{\mathcal{S}}$ now acts on $\delta^2 G_{\mu\nu}$, and $\Psi_4^{(2)}$ is composed of a linear and a quadratic piece,

$$\Psi_4^{(2)} = \Psi_{4L}^{(2)} + \Psi_{4Q}^{(2)}, \tag{5.118}$$

where

$$\Psi_{4L}^{(2)} = \hat{\mathcal{T}}[h^2], \tag{5.119}$$

$$\Psi_{4Q}^{(2)} = \delta^2 \Psi_4[h^1, h^1], \tag{5.120}$$

with $\delta^2\Psi_4$ being the quadratic Weyl scalar operator. The second-order Teukolsky equation is then given by

$$\hat{\mathcal{S}}\hat{\mathcal{E}}[h^2] = \hat{\mathcal{O}}\hat{\mathcal{T}}[h^2], \tag{5.121}$$

which, when expanded, gives

$$\hat{\mathcal{O}}[\Psi_{4L}^{(2)}] = \hat{\mathcal{S}}\left[8\pi T_{\mu\nu}^2 - \delta^2 G_{\mu\nu}[h^1, h^1]\right]. \tag{5.122}$$

The first term is trivially given by substituting $T_{\mu\nu}^2$ into Eq. (5.113), so that

$$\begin{aligned}\hat{\mathcal{S}}[T_{\mu\nu}^2] = &(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})[(\bar{\delta} - 2\bar{\tau} + 2\alpha)T_{n\bar{m}}^2 - (\Delta + 2\gamma - 2\bar{\gamma} + \bar{\mu})T_{\bar{m}\bar{m}}^2] \\ &+ (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)[(\Delta + 2\gamma + 2\bar{\mu})T_{n\bar{m}}^2 - (\bar{\delta} - \bar{\tau} + 2\bar{\beta} + 2\alpha)T_{nn}^2],\end{aligned} \tag{5.123}$$

where $T_{\mu\nu}^2$ is the stress-energy previously derived and is given by Eq. (5.46). The second term is more complicated and requires a distributional treatment, similar to Ch. 5.2. We follow a similar procedure as before to calculate the delta content but with the inclusion of the $\hat{\mathcal{S}}$ operator. Due to the extra derivatives, this increases the singular nature of the integrands, requiring higher-order expansions of the metric perturbations to be used.

Moving to the term including the second-order Einstein tensor and specialising to the Lorenz gauge, we substitute in Detweiler's canonical definition (5.67) to get

$$\hat{\mathcal{S}}\left[\delta^2 G_{\mu\nu}[h^{1*},h^{1*}]\right] = \hat{\mathcal{S}}\Big[\lim_{s\to 0}\Big\{\Big(-\delta G_{\mu\nu}[h^{\rm SS*}] + 2Q^{\rm R}_{\mu\nu}[h^{\rm S1*},h^{\rm R1*}] + \delta^2 G_{\mu\nu}[h^{\rm R1*},h^{\rm R1*}]\Big)\theta(s-r) + \delta^2 G_{\mu\nu}[h^{1*},h^{1*}]\theta(r-s)\Big\}\Big], \quad (5.124)$$

where $Q_{\mu\nu}$ is given by Eq. (5.13) but defined with indices down. As before, we can ignore the term quadratic in $h^{\rm R1*}_{\mu\nu}$ as, while it contributes to the source, it does not feature any delta content.

To find the distributional content in the $h^{\rm SS*}_{\mu\nu}$ term, we proceed in a similar fashion to the previous chapter and write it as an integral against a test function, ${}_2\phi$, so that

$$\int {}_2\phi S^{\mu\nu}\delta G_{\mu\nu}[h^{\rm SS*}]\theta_s\, dV := \int h^{\mu\nu}_{\rm SS*}\delta G_{\mu\nu}[S^\dagger[\theta_s {}_2\phi]]\, dV\,. \quad (5.125)$$

Here, $S^{\mu\nu} := \hat{\mathcal{S}}$ from Eq. (5.116) with indices restored, $\theta_s := \theta(s-r)$, and ${}_2\phi$ is a spin-weighted test function to account for the spin weight of $S^{\mu\nu}$ to ensure that the integrand has spin weight zero. Then, using our canonical definition for the relation between $\delta^2 G^{\mu\nu}[h^{\rm S1*},h^{\rm S1*}]$ and $\delta G^{\mu\nu}[h^{\rm SS*}]$ from Eq. (5.66), we write

$$\begin{aligned}
&\lim_{s\to 0}\int h^{\mu\nu}_{\rm SS*}\delta G_{\mu\nu}[S^\dagger[\theta_s {}_2\phi]]\, dV \\
&\quad = \lim_{s\to 0}\lim_{R\to 0}\int_{r>R} h^{\mu\nu}_{\rm SS*}\delta G_{\mu\nu}[S^\dagger[\theta_s {}_2\phi]]\, dV \\
&\quad = \lim_{s\to 0}\lim_{R\to 0}\Big[\int_{r>R} {}_2\phi\theta_s S^{\mu\nu}[\delta G[h^{\rm SS*}]]\, dV \\
&\qquad - \int_{r=R}\Big(K^S_\alpha[\delta G[h^{\rm SS*}], {}_2\phi\theta_s] + K^{\delta G}_\alpha[h^{\rm SS*}, S^\dagger[{}_2\phi\theta_s]]\Big)\, dS^\alpha\Big]
\end{aligned} \quad (5.126)$$

In the first line, we use the fact that this is now an ordinary integral as the integrand multiplied by the volume element is of order r^0 and in the final equality, we have integrated by parts to recover the original integrand while keeping the boundary terms, K^α_P, that appear for the specific operators, P.

We may now show that this integral vanishes without explicitly calculating the individual terms of the expression. To do so, we first note that, in the Lorenz gauge, $h^{\rm SS*}_{\mu\nu}$ has alternate but definite parity at each order in r [96]. In Fermi–Walker coordinates, recalling that each power of $\hat{n}^\alpha$ is equivalent to increasing ℓ by 1,

$$h^{\rm SS*}_{\mu\nu} \sim \frac{\hat{n}^2}{r^2} + \frac{\hat{n}^3}{r} + \hat{n}^4 r^0 + \hat{n}^5 r + \mathcal{O}\big(r^2\big), \quad (5.127)$$

where the notation $\hat{n}^\ell$, for some integer ℓ, means terms of the form $\hat{n}^L, \hat{n}^{L-2}, \dots, n$ for odd ℓ and $\hat{n}^L, \hat{n}^{L-2}, \dots, 1$ for even ℓ. Therefore, even (and odd) powers of r only feature even (and odd) powers of ℓ. Inspecting each of the boundary terms, we find that

$$K^{\alpha}_{S}[\delta G[h^{\mathrm{SS}*}], {}_2\phi] \sim \nabla^3 h^{\mathrm{SS}*}_{\mu\nu} \sim \frac{\hat{n}^5}{r^5} + \frac{\hat{n}^6}{r^4} + \frac{\hat{n}^7}{r^3} + \frac{\hat{n}^8}{r^2} + \mathcal{O}\Big(r^{-1}\Big), \tag{5.128}$$

$$K^{\alpha}_{\delta G}[h^{\mathrm{SS}*}, S^{\dagger}[{}_2\phi]] \sim \nabla h^{\mathrm{SS}*}_{\mu\nu} \sim \frac{\hat{n}^3}{r^3} + \frac{\hat{n}^4}{r^2} + \mathcal{O}\Big(r^{-1}\Big), \tag{5.129}$$

as $K^{\alpha}_{P} \sim \nabla$ and $\delta G_{\mu\nu} \sim \nabla^2$. Firstly, we note that any terms r^n with $n \geq -1$ will go to 0 when taking the limit $\mathrm{R} \to 0$ as dS_α features a factor of R^2 (5.78). Secondly, as dS_α has a factor of n_α, both terms $\sim r^{-2}$ will integrate to 0 as they originally feature even powers of $\hat{n}$. To see this, we note that Eq. (5.85) states that the angular integral of $\hat{n}^L$ evaluates to 0 for $l \geq 1$. As the power of $\hat{n}$ in the R^0 term is now always odd, this means that any power of $\hat{n}$ appearing is also odd, meaning that at least one $\hat{n}$ appears in every term. This would then evaluate to 0 when integrated.

Finally, we know that Eq. (5.126) is a well-defined integral of an ordinary function and therefore can not diverge. As stated before, the integrand multiplied by the volume element gives a quantity of order r^0. This will be multiplied by a Heaviside function (or an nth-order derivative of one) which, when integrated for $r > R$ will give a finite result. Thus, all of the divergent terms that appear from taking the derivatives in Eqs. (5.128)–(5.129) and in the volume integral must cancel to ensure the integral is well defined. While it's true that we have no small R divergence, it's likely that we will have a small s divergence as we found in case of the source for the Einstein equation. Unfortunately, we have been unable to calculate this quantity due to the complexity of taking so many derivatives of $h^{\mathrm{SS}*}_{\mu\nu}$. We leave the derivation of the (potential) stress-energy counter term to future work.

Moving to the final term, we proceed in a similar way to before by defining

$$\int {}_2\phi S^{\mu\nu} Q_{\mu\nu}[h^{\mathrm{S1}*}]\theta_s \, dV := \int Q^{\dagger}_{\mu\nu}[S^{\dagger}[{}_2\phi\theta_s]] h^{\mu\nu}_{\mathrm{S1}*} \, dV \,. \tag{5.130}$$

Integrating by parts to move the operators onto $h^{\mathrm{S1}*}_{\mu\nu}$, we get

$$\begin{aligned} \int Q^{\dagger}_{\mu\nu}[S^{\dagger}[{}_2\phi\theta_s]] h^{\mu\nu}_{\mathrm{S1}*} \, dV = & \lim_{s\to 0} \lim_{R\to 0} \bigg[\int_{r>R} S^{\mu\nu}_{\dagger}[{}_2\phi\theta_s] Q_{\mu\nu}[h^{\mathrm{S1}*}] \, dV \\ & - \int_{r=R} K^{Q}_{\mu}[S^{\dagger}[{}_2\phi\theta_s], h^{\mathrm{S1}*}] \, dS^{\mu} \bigg] \\ = & \lim_{s\to 0} \lim_{R\to 0} \bigg[\int_{r>R} {}_2\phi\theta_s S_{\mu\nu} Q^{\mu\nu}[h^{\mathrm{S1}*}] \, dV \\ & - \int_{r=R} (K^{Q}_{\mu}[S^{\dagger}[{}_2\phi\theta_s], h^{\mathrm{S1}*}] + K^{S}_{\mu}[Q[h^{\mathrm{S1}*}], {}_2\phi\theta_s]) \, dS^{\mu} \bigg] \end{aligned} \tag{5.131}$$

The form of K^{α}_{Q} was previously derived in Eq. (5.89) and, as $S^{\dagger}_{\mu\nu}$ is regular, we can make the substitution $\phi_{\mu\nu} \to S^{\dagger}_{\mu\nu}[{}_2\phi\theta_s]$ in the previous expression, which, when written covariantly, gives

$$
\begin{aligned}
&\lim_{s\to 0}\lim_{R\to 0}\int_{r=R} K^{Q}_{\alpha}[S^{\dagger}[{}_2\phi\theta_s]]\,dS^{\alpha} \\
&\quad= -\frac{4\pi m}{3}\lim_{s\to 0}\int h^{\mathrm{R1}*}_{\alpha'\beta'}S^{\mu'\nu'}_{\dagger}[{}_2\phi\theta(s)]\Big(P^{\alpha'\beta'}P_{\mu'\nu'}+2P_{\mu'\nu'}u^{\alpha'}u^{\beta'}-4P^{\alpha'\beta'}u_{\mu'}u_{\nu'} \\
&\qquad -6u^{\alpha'}u^{\beta'}u_{\mu'}u_{\nu'}-P^{(\alpha'}{}_{(\mu'}P_{\nu')}{}^{\beta')}+8P^{(\alpha'}{}_{(\mu'}u_{\nu')}u^{\beta')}\Big)\,d\tau' \\
&\quad= \lim_{s\to 0}\int k_{\mu'\nu'}S^{\mu'\nu'}_{\dagger}[{}_2\phi\theta(s)]\,d\tau' \\
&\quad= \lim_{s\to 0}\iint k_{\mu'\nu'}g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}S^{\mu\nu}_{\dagger}[{}_2\phi\theta(s)]\delta^4(x,z)\,dV\,d\tau' \\
&\quad= \iint {}_2\phi k_{\mu'\nu'}S^{\mu\nu}[g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}\delta^4(x,z)]\,dV\,d\tau'\,,
\end{aligned}
\tag{5.132}
$$

where $k_{\mu'\nu'}$ is defined by the second equality as

$$
\begin{aligned}
k_{\mu'\nu'} := -\frac{4\pi m}{3}h^{\mathrm{R1}*}_{\alpha'\beta'}\Big(&P^{\alpha'\beta'}P_{\mu'\nu'}+2P_{\mu'\nu'}u^{\alpha'}u^{\beta'}-4P^{\alpha'\beta'}u_{\mu'}u_{\nu'}-6u^{\alpha'}u^{\beta'}u_{\mu'}u_{\nu'} \\
&-P^{(\alpha'}{}_{(\mu'}P_{\nu')}{}^{\beta')}+8P^{(\alpha'}{}_{(\mu'}u_{\nu')}u^{\beta')}\Big),
\end{aligned}
\tag{5.133}
$$

and use that $\lim_{s\to 0}\theta(s)=1$ in the final line. The components of $k_{\mu\nu}$ in the NP basis can be written as

$$k_{ll} = \frac{4\pi m}{3}(h^{\mathrm{R1}*}_{ll}-6h^{\mathrm{R1}*}_{lu}u_l+3g^{\mu\nu}h^{\mathrm{R1}*}_{\mu\nu}u_lu_l), \tag{5.134a}$$

$$k_{ln} = \frac{4\pi m}{3}(g^{\mu\nu}h^{\mathrm{R1}*}_{\mu\nu}+h^{\mathrm{R1}*}_{ln}+3h^{\mathrm{R1}*}_{uu}-6h^{\mathrm{R1}*}_{u(l}u_{n)}+3g^{\mu\nu}h^{\mathrm{R1}*}_{\mu\nu}u_lu_n), \tag{5.134b}$$

$$k_{m\bar{m}} = -\frac{4\pi m}{3}(g^{\mu\nu}h^{\mathrm{R1}*}_{\mu\nu}-h^{\mathrm{R1}*}_{m\bar{m}}+3h^{\mathrm{R1}*}_{uu}+6h^{\mathrm{R1}*}_{u(m}u_{\bar{m})}-3g^{\mu\nu}h^{\mathrm{R1}*}_{\mu\nu}u_mu_{\bar{m}}), \tag{5.134c}$$

with the other components obtained by switching the labels between the different basis vectors.

To write the final source, we need to know how $S^{\mu\nu}[g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}\delta^4(x,z)]$ acts distributionally. When expanding $S^{\mu\nu}$, we get terms containing either zero, one or two derivatives. With no derivatives, we just recover the standard delta function in the source. For one derivative, we use the identity from Eq. (13.3) from Ref. [63] to write

$$(g^{\alpha}{}_{\alpha'}(x,z)\delta^4(x,z))_{;\alpha} = -\partial_{\alpha'}\delta^4(x,z). \tag{5.135}$$

We can find the distributional identity for the double covariant derivative by writing

$$
\begin{aligned}
&\int \phi\nabla_{\beta}\nabla_{\alpha}(g^{\alpha}{}_{\alpha'}g^{\beta}{}_{\beta'}\delta^4(x,z))\,dV \\
&= \int \phi_{;\beta\alpha}g^{\alpha}{}_{\alpha'}g^{\beta}{}_{\beta'}\delta^4(x,z)\,dV+\int\Big(\phi(g^{\alpha}{}_{\alpha'}g^{\beta}{}_{\beta'}\delta^4(x,z))_{;\alpha}-\phi_{;\alpha}g^{\beta}{}_{\alpha'}g^{\alpha}{}_{\beta'}\delta^4(x,z)\Big)\,dS_{\beta} \\
&= [\phi_{;\beta\alpha}g^{\alpha}{}_{\alpha'}g^{\beta}{}_{\beta'}] \\
&= \phi_{;\beta\alpha}.
\end{aligned}
\tag{5.136}
$$

where the boundary terms vanish as ϕ has compact support, meaning that it and its derivatives vanish on the boundary. Therefore, we can write this as the distributional identity,

$$(g^{\alpha}{}_{\alpha'}g^{\beta}{}_{\beta'}\delta^4(x,z))_{;\alpha\beta} = \nabla_{\beta'}\nabla_{\alpha'}\delta^4(x,z). \tag{5.137}$$

This will then allow us to write the source from Eq. (5.132) in terms of a delta function, a derivative of a delta function and a second derivative of a delta function, each multiplied by some coefficient.

With the K^Q_α term calculated, we now turn to K^S_α from Eq. (5.131). This is found in the usual way, by integrating Eq. (5.131) by parts to move $S^{\mu\nu}_\dagger$ onto $Q_{\mu\nu}$ and applying Stokes' theorem to find the boundary terms to give

$$\begin{aligned} K^S_\alpha[Q[h], {}_2\phi] = \bar{m}_\alpha\Big[&\Big(\bar{\delta}[Q_{nn}] - \Delta[Q_{\bar{m}n}[h]]\Big)\phi - Q_{nn}[h]\Big(\bar{\delta}[\phi] - \phi(6\alpha + 2\bar{\beta} + 3\pi - \bar{\tau})\Big) \\ &+ Q_{\bar{m}n}[h]\Big(\Delta[\phi] - \phi(6\gamma + 3\mu + 2\bar{\mu})\Big)\Big] \\ &+ n_\alpha\Big[\Big(\Delta[Q_{\bar{m}\bar{m}}[h]] - \bar{\delta}[Q_{\bar{m}n}[h]]\Big)\phi + Q_{\bar{m}n}[h]\Big(\bar{\delta}[\phi] + \phi(2\bar{\tau} - 6\alpha - 3\pi)\Big) \\ &- Q_{\bar{m}\bar{m}}[h]\Big(\Delta[\phi] - \phi(6\gamma - 2\bar{\gamma} + 3\mu + \bar{\mu})\Big)\Big], \end{aligned} \tag{5.138}$$

where $\Delta[f]$ and $\bar{\delta}[f]$ are NP derivatives, given in Eqs. (C.9b) and (C.9d). In earlier chapters, we had used notation for contractions, such as $R_{u\sigma u\sigma;\sigma} := R_{\alpha\beta\gamma\delta;\mu}u^\alpha\sigma^\beta u^\gamma\sigma^\delta\sigma^\mu$, where we ignored the action of the derivative when contracting vectors into tensors. However, when considering NP quantities, this is no longer true. As an example, $\bar{\delta}[Q_{\bar{m}n}[h]] \neq Q_{\mu\nu;\rho}\bar{m}^\mu n^\nu \bar{m}^\rho$, instead, $\bar{\delta}[Q_{\bar{m}n}[h]] := \bar{m}^\rho\nabla_\rho(\bar{m}^\mu n^\nu Q_{\mu\nu})$.

After substituting in the expansions for $Q_{\mu\nu}$ and $\nabla_\alpha Q_{\mu\nu}$, Taylor expansions for the tetrad legs and NP spin coefficients, and the surface element from Eq. (2.130) we may proceed with our surface integral. We find the integrand combined with the surface element has the form

$$K^S_\alpha[Q[h^{\mathrm{S1}*}], {}_2\phi\theta_s]\, dS^\alpha \sim \frac{1}{R^2} + \frac{1}{R} + R^0 + \mathcal{O}(R). \tag{5.139}$$

This initially seems problematic as we have that the leading two orders diverge when taking the limit as $R \to 0$. However, these two orders vanish when performing the angular integral. This is to be expected as we have a well-defined integral and, as such, all terms that would to diverge when taking the limit $R \to 0$ must either cancel or evaluate to 0 when performing the angular integration.

The resulting expression can be written as terms proportional to ${}_2\phi$, $h^{\mathrm{R1}*}_{\mu\nu}$ and their derivatives as

$$\begin{aligned} &\lim_{R\to 0}\int_{r=R} K^S_\alpha[\phi]\, dS^\alpha \\ &= \int\Big({}_2\phi_{;\mu'\nu'}A^{\alpha'\beta'\mu'\nu'}h^{\mathrm{R1}*}_{\alpha'\beta'} + {}_2\phi_{;\mu'}(B^{\alpha'\beta'\mu'}h^{\mathrm{R1}*}_{\alpha'\beta'} + C^{\alpha'\beta'\nu'\mu'}h^{\mathrm{R1}*}_{\alpha'\beta';\nu'}) \\ &\quad + {}_2\phi(D^{\alpha'\beta'\mu'\nu'}h^{\mathrm{R1}*}_{\alpha'\beta';\mu'\nu'} + E^{\alpha'\beta'\mu'}h^{\mathrm{R1}*}_{\alpha'\beta';\mu'} + F^{\alpha'\beta'}h^{\mathrm{R1}*}_{\alpha'\beta'})\Big)\, d\tau' \end{aligned}$$

$$= \iint \Big[{}_2\phi_{;\mu\nu} g^{\mu}{}_{\mu'} g^{\nu}{}_{\nu'} A^{\alpha'\beta'\mu'\nu'} h^{\mathrm{R1}*}_{\alpha'\beta'} + {}_2\phi_{;\mu} g^{\mu}{}_{\mu'} (B^{\alpha'\beta'\mu'} h^{\mathrm{R1}*}_{\alpha'\beta'} + C^{\alpha'\beta'\nu'\mu'} h^{\mathrm{R1}*}_{\alpha'\beta';\nu'}) + {}_2\phi (D^{\alpha'\beta'\mu'\nu'} h^{\mathrm{R1}*}_{\alpha'\beta';\mu'\nu'} + E^{\alpha'\beta'\mu'} h^{\mathrm{R1}*}_{\alpha'\beta';\mu'} + F^{\alpha'\beta'} h^{\mathrm{R1}*}_{\alpha'\beta'}) \Big) \Big] \delta^4(x,z)\, dV\, d\tau' \,. \tag{5.140}$$

Most of our newly defined tensors are too long to include here but are included in the Additional Material [141]. However, as an example of the structure, we display the shortest one here,

$$\begin{aligned}
A^{\alpha'\beta'\mu'\nu'} = & -\frac{4\pi m}{105} \bar{m}^{\gamma'} \bar{m}^{\zeta'} n^{\iota'} n^{\kappa'} \Big[70 P^{(\alpha'}{}_{\gamma'} P_{\zeta'}{}^{\beta')} P_{\iota'}{}^{(\mu'} P^{\nu')}{}_{\kappa'} - 140 P^{(\alpha'}{}_{\gamma'} P_{\iota'}{}^{\beta')} P_{\zeta'}{}^{(\mu'} P^{\nu')}{}_{\kappa'} \\
& + 70 P^{(\alpha'}{}_{\iota'} P_{\kappa'}{}^{\beta')} P_{\gamma'}{}^{(\mu'} P^{\nu')}{}_{\zeta'} + 120 P_{\gamma'}{}^{(\alpha'} P^{\beta')(\mu'} P^{\nu')}{}_{\zeta'} u_{\iota'} u_{\kappa'} \\
& - 120 P_{\zeta'}{}^{(\alpha'} P^{\beta')(\mu'} P^{\nu')}{}_{\kappa'} u_{\gamma'} u_{\iota'} - 120 P_{\zeta'}{}^{(\mu'} P^{\nu')(\alpha'} P^{\beta')}{}_{\kappa'} u_{\gamma'} u_{\iota'} \\
& + 120 P_{\iota'}{}^{(\alpha'} P^{\beta')(\mu'} P^{\nu')}{}_{\kappa'} u_{\gamma'} u_{\zeta'} - 49 P^{(\alpha'}{}_{\gamma'} P_{\zeta'}{}^{\beta')} P_{\kappa'}{}^{(\mu'} u^{\nu')} u_{\iota'} \\
& + 49 P^{(\alpha'}{}_{\gamma'} P_{\kappa'}{}^{\beta')} P_{\zeta'}{}^{(\mu'} u^{\nu')} u_{\iota'} + 49 P^{(\alpha'}{}_{\zeta'} P_{\iota'}{}^{\beta')} P_{\kappa'}{}^{(\mu'} u^{\nu')} u_{\gamma'} \\
& - 49 P^{(\alpha'}{}_{\iota'} P_{\kappa'}{}^{\beta')} P_{\zeta'}{}^{(\mu'} u^{\nu')} u_{\gamma'} + 70 P_{\gamma'}{}^{(\mu'} P^{\nu')}{}_{\zeta'} P_{\kappa'}{}^{(\alpha'} u^{\beta')} u_{\iota'} \\
& - 70 P_{\zeta'}{}^{(\mu'} P^{\nu')}{}_{\kappa'} P_{\gamma'}{}^{(\alpha'} u^{\beta')} u_{\iota'} - 70 P_{\zeta'}{}^{(\mu'} P^{\nu')}{}_{\kappa'} P_{\iota'}{}^{(\alpha'} u^{\beta')} u_{\gamma'} \\
& + 70 P_{\iota'}{}^{(\mu'} P^{\nu')}{}_{\kappa'} P_{\zeta'}{}^{(\alpha'} u^{\beta')} u_{\gamma'} - 12 P^{(\alpha'}{}_{\gamma'} P^{\beta')}{}_{\zeta'} P^{\mu'\nu'} u_{\iota'} u_{\kappa'} \\
& + 24 P^{(\alpha'}{}_{\zeta'} P^{\beta')}{}_{\kappa'} P^{\mu'\nu'} u_{\gamma'} u_{\iota'} - 12 P^{(\alpha'}{}_{\iota'} P^{\beta')}{}_{\kappa'} P^{\mu'\nu'} u_{\gamma'} u_{\zeta'} \\
& - 75 P_{\gamma'}{}^{(\mu'} P_{\zeta'}{}^{\nu')} P^{\alpha'\beta'} u_{\iota'} u_{\kappa'} + 150 P_{\zeta'}{}^{(\mu'} P_{\kappa'}{}^{\nu')} P^{\alpha'\beta'} u_{\gamma'} u_{\iota'} \\
& - 75 P_{\iota'}{}^{(\mu'} P_{\kappa'}{}^{\nu')} P^{\alpha'\beta'} u_{\gamma'} u_{\zeta'} - 7 P_{\gamma'}{}^{(\alpha'} u^{\beta')} P_{\zeta'}{}^{(\mu'} u^{\nu')} u_{\iota'} u_{\kappa'} \\
& + 7 P_{\zeta'}{}^{(\alpha'} u^{\beta')} P_{\kappa'}{}^{(\mu'} u^{\nu')} u_{\gamma'} u_{\iota'} + 7 P_{\zeta'}{}^{(\mu'} u^{\nu')} P_{\kappa'}{}^{(\alpha'} u^{\beta')} u_{\gamma'} u_{\iota'} \\
& - 7 P_{\iota'}{}^{(\alpha'} u^{\beta')} P_{\kappa'}{}^{(\mu'} u^{\nu')} u_{\gamma'} u_{\zeta'} - 35 P_{\gamma'}{}^{(\mu'} P_{\zeta'}{}^{\nu')} u^{\alpha'} u^{\beta'} u_{\iota'} u_{\kappa'} \\
& + 70 P_{\zeta'}{}^{(\mu'} P_{\kappa'}{}^{\nu')} u^{\alpha'} u^{\beta'} u_{\gamma'} u_{\iota'} - 35 P_{\iota'}{}^{(\mu'} P_{\kappa'}{}^{\nu')} u^{\alpha'} u^{\beta'} u_{\gamma'} u_{\zeta'} \Big] .
\end{aligned} \tag{5.141}$$

Each of the terms in Eq. (5.140) features two covariant derivatives acting on either ${}_2\phi$, $h^{\mathrm{R1}*}_{\mu'\nu'}$ or in a term contained in one of our newly defined tensors. We can see this in the case of $A^{\alpha'\beta'\mu'\nu'}$: with the two derivatives taken, the only terms that are remaining that can be used to construct the expression are the tetrad legs, the projection operator and the four velocity. This is the same case in $C^{\alpha'\beta'\nu'\mu'}$ and $D^{\alpha'\beta'\mu'\nu'}$. When our newly defined tensor can feature exactly one derivative, as is the case in $B^{\alpha'\beta'\mu'}$ and $E^{\alpha'\beta'\mu'}$, then we find terms that are proportional to one of the spin coefficients. In the final term, $F^{\alpha'\beta'}$, where we have terms constructed from two derivatives, we find terms proportional to the Riemann tensor, derivatives of the spin coefficients or two spin coefficients multiplied together.

We then move the derivatives onto the parallel propagators and delta function using the same process as in Eq. (5.132) and then use the distributional identities from Eqs. (5.135) and (5.139), to write the source as

$$T_S = \int A^{\alpha'\beta'\mu'\nu'} h^{\mathrm{R}1*}_{\alpha'\beta'} \nabla_{\nu'} \nabla_{\mu'} \delta^4(x,z) - \nabla_{\mu'} \delta^4(x,z) (B^{\alpha'\beta'\mu'} h^{\mathrm{R}1*}_{\alpha'\beta'} + C^{\alpha'\beta'\nu'\mu'} h^{\mathrm{R}1*}_{\alpha'\beta';\nu'})$$
$$+ \delta^4(x,z) (D^{\alpha'\beta'\mu'\nu'} h^{\mathrm{R}1*}_{\alpha'\beta';\mu'\nu'} + E^{\alpha'\beta'\mu'} h^{\mathrm{R}1*}_{\alpha'\beta';\mu'} + F^{\alpha'\beta'} h^{\mathrm{R}1*}_{\alpha'\beta'})\, d\tau' \,. \quad (5.142)$$

Combining the above equation with Eq. (5.132) (after expanding $\hat{\mathcal{S}}$ acting on the delta function) and Eq. (5.123) gives us the delta content of the source for the second-order Teukolsky equation. While we have the delta function content, we have not yet calculated the counter terms that may appear, as in the case of the Einstein equation in the previous section, but we leave this to future work. With the delta content of the source written in fully covariant form, we can easily write it in a specific coordinate system for either Kerr or Schwarzschild, depending on the problem being investigated. When in the Schwarzschild spacetime, a great number of the spin coefficients vanish which will dramatically simplify the $A^{\alpha'\beta'\mu'\nu'}$–$F^{\alpha'\beta'}$ tensors in Eq. (5.142). One can then combine the delta function content with a calculation for the parts of the source that are not supported on the worldline to give the full source for the second-order Teukolsky equation.

Chapter 6

Covariant and coordinate punctures in a highly regular gauge

In Ch. 4, we presented the full form of the singular field perturbations in the highly regular gauge in Fermi–Walker coordinates. While this has been sufficient for our analysis of the second-order stress-energy tensor in Ch. 5, if we are interested in implementing the highly regular gauge in self-force calculations, for example in a puncture scheme, then we need the ability to write the perturbations in an arbitrary coordinate system. This can then be specified later and tailored to the specific problem being investigated.

To avoid the requirement to perform a (potentially) complicated coordinate transformation from Fermi–Walker coordinates to the preferred coordinate system, in this chapter we will provide covariant expressions and coordinate expansions that can then be projected into any desired coordinate system. The methods in this section were originally designed to generate a second-order Lorenz gauge puncture [96] but can easily be applied to constructing a puncture in the highly regular gauge.

In Ch. 6.1, we describe the method used in Ref. [96] to create the Lorenz gauge punctures before moving on to apply it to our Fermi–Walker highly-regular gauge expressions in Ch. 6.2 to generate covariant punctures. Finally, in Ch. 6.3, we take our covariant highly-regular gauge punctures and perform a coordinate expansion, as in Ch. 2.2.1.2, to put them in a form where they can then be expressed in any chosen coordinate system.

6.1 Converting Fermi–Walker coordinates to covariant form

In this section, we outline the method as used in Ref. [96] to create the Lorenz gauge puncture. While the full technical details containing derivations of the various quantities

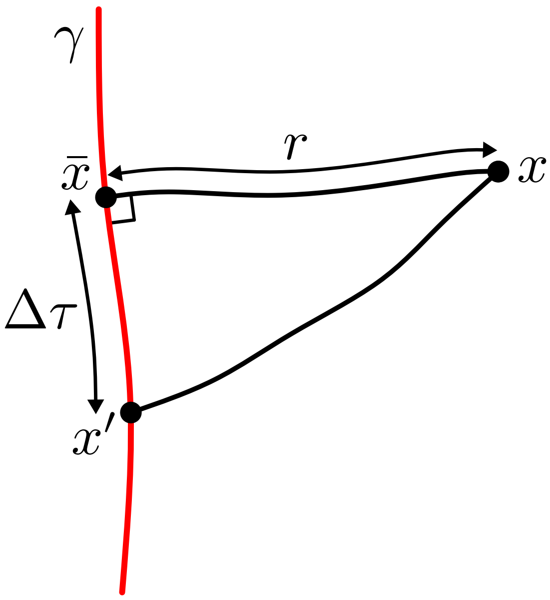

FIGURE 6.1: Diagram illustrating the relationship between x, x' and $\bar{x}$. The two points x' and $\bar{x}$ are points on the worldline, γ, separated by $\Delta\tau$ while $\bar{x}$ and x are connected by the geodesic that intersects γ orthogonally. Based on Fig. 1 from Ref. [96].

are contained within that paper, we reproduce the essential results that we will need to produce the highly regular gauge puncture. The final results will be covariant quantities expressed entirely in terms of parallel propagators, the four-velocity, Riemann tensors and Synge's world function. We reviewed the definitions and properties of parallel propagators and Synge's world function in Ch. 2.2.1.1 when discussing covariant expansion methods.

The idea behind the method from Ref. [96] is to express the field at a point x in terms of an arbitrary nearby point on the worldline, $x' = z(\tau')$. This is done through an intermediary point, $\bar{x} = z(\bar{\tau})$, which lies on γ and is separated from x' by the difference in proper time

$$\Delta\tau := \bar{\tau} - \tau'. \tag{6.1}$$

The intermediary point, $\bar{x}$, is then connected to x by the unique geodesic that intersects the worldline orthogonally. A visual representation is provided in Fig. 6.1.

As Fermi–Walker coordinates are constructed geometrically, see Ch. 2.2.2, there is a very straightforward way to convert them into covariant form. We know from Eqs. (2.117)–(2.119), that there is a simple correspondence between FW coordinates and covariant quantities, which we give again as

$$x^0 = t, \tag{6.2}$$

$$x^a = -e^a_{\bar{\alpha}}(\bar{x})\sigma^{\bar{\alpha}}(x,\bar{x}), \tag{6.3}$$

$$\sigma_{\bar{\alpha}}(x,\bar{x})u^{\bar{\alpha}}(\bar{x}) = 0, \tag{6.4}$$

where the barred indices on Synge's world function refer to derivatives taken with respect to $\bar{x}$. As stated previously, Synge's world function gives half the geodesic distance squared between two points (up to a minus sign) meaning that a derivative gives the geodesic distance. This quantity is then contracted with the spatial Fermi–Walker tetrad leg, $e^a_{\bar{\alpha}}$, to give the Fermi–Walker spatial distance, x^a. The third equation ensures that $\sigma_{\bar{\alpha}}$ is always orthogonal to the worldline. As we saw in the text below Eq. (2.119), we can write the Fermi–Walker radial distance in terms of covariant quantities with

$$r := \sqrt{\delta_{ab}x^a x^b} = \sqrt{P_{\bar{\alpha}\bar{\beta}}\sigma^{\bar{\alpha}}\sigma^{\bar{\beta}}} = \sqrt{2\bar{\sigma}}, \tag{6.5}$$

where

$$\bar{\sigma} := \sigma(x, \bar{x}). \tag{6.6}$$

We have added an extra step in Eq. (6.5), where we have rewritten the flat-space metric in terms of the projection operator,

$$e^\alpha_a e^{a\beta} = P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta, \tag{6.7}$$

which immediately follows from Eq. (2.116). The radial unit vector is then given by

$$n^a = \frac{x^a}{r} = \frac{-e^a_{\bar{\alpha}}\sigma^{\bar{\alpha}}}{\sqrt{2\bar{\sigma}}}. \tag{6.8}$$

Additionally, we must replace the Fermi–Walker basis one-forms, as when written explicitly, the singular field has the standard form

$$h^{\mathrm{S}}_{\mu\nu}\, dx^\mu\, dx^\nu = h^{\mathrm{S}}_{tt}\, dt\, dt + 2h^{\mathrm{S}}_{ta}\, dt\, dx^a + h^{\mathrm{S}}_{ab}\, dx^a\, dx^b\,. \tag{6.9}$$

These are given by [96, Eqs. (82)–(84)]

$$dt = \mu\sigma_{\bar{\alpha}\alpha}u^{\bar{\alpha}}\, dx^\alpha\,, \tag{6.10}$$

$$dx^a = -\, e^a_{\bar{\alpha}}(\sigma^{\bar{\alpha}}{}_\alpha + \mu\sigma^{\bar{\alpha}}{}_{\bar{\beta}}u^{\bar{\beta}}\sigma_{\alpha\bar{\gamma}}u^{\bar{\gamma}}), \tag{6.11}$$

where

$$\mu = -(\sigma_{\bar{\alpha}\bar{\beta}}u^{\bar{\alpha}}u^{\bar{\beta}} + \sigma_{\bar{\alpha}}a^{\bar{\alpha}})^{-1}. \tag{6.12}$$

Finally, the second-order singular field, $h^{\rm SR}_{\mu\nu}$, features derivatives of the first-order regular field, $h^{\rm R1}_{\mu\nu}$. These can be written as [96, Eqs. (122)–(123)]

$$\partial_t h^{\rm R1}_{\mu\nu} = h^{\rm R1}_{\bar\mu\bar\nu|\bar\alpha} u^{\bar\alpha} + \mathcal{O}(a^\mu), \tag{6.13}$$
$$\partial_a h^{\rm R1}_{\mu\nu} = h^{\rm R1}_{\bar\mu\bar\nu|\bar\alpha} e^{\bar\alpha}_a + \mathcal{O}(a^\mu), \tag{6.14}$$
$$\partial_t\partial_t h^{\rm R1}_{\mu\nu} = h^{\rm R1}_{\bar\mu\bar\nu|\bar\alpha\bar\beta} u^{\bar\alpha} u^{\bar\beta} + \mathcal{O}(a^\mu), \tag{6.15}$$
$$\partial_t\partial_a h^{\rm R1}_{\mu\nu} = h^{\rm R1}_{\bar\mu\bar\nu|\bar\alpha\bar\beta} e^{\bar\alpha}_a u^{\bar\beta} + \mathcal{O}(a^\mu), \tag{6.16}$$
$$\partial_a\partial_b h^{\rm R1}_{\mu\nu} = h^{\rm R1}_{\bar\mu\bar\nu|\bar\alpha\bar\beta} e^{\bar\alpha}_a e^{\bar\beta}_b + 2R^{\bar\mu}{}_{b0a} u_{(\bar\alpha} h^{\rm R1}_{\bar\beta)\bar\mu} - \frac{4}{3} R^{\bar\mu}{}_{(b\bar\nu)a} P^{\bar\nu}{}_{(\bar\alpha} h^{\rm R1}_{\bar\beta)\bar\mu} + \mathcal{O}(a^\mu), \tag{6.17}$$

where the bar, |, indicates a covariant derivative at $x^{\bar\alpha}$ and any acceleration terms can be ignored as they would belong to the third-order singular field. These expressions can be derived by taking covariant derivatives of $h^{\rm R1}_{\bar\alpha\bar\beta}$ and calculating the Christoffel symbols constructed from the FW background metric in Eq. (2.120).

After rewriting all quantities in terms of $\bar x$, we then re-expand them in powers of $\Delta\tau$, the time difference given in Eq. (6.1). For example,

$$h_{tt}(x,\bar x) = \sum_{n=0}^{\infty} \Delta\tau^n \frac{d^n}{d\tau'^n} h_{tt}(x,x'), \tag{6.18}$$

where $d/d\tau' = u^{\alpha'}\nabla_{\alpha'}$ and the expansion in distance of the difference in proper time is given by

$$\Delta\tau = \lambda \boldsymbol{r} + \lambda^2 \boldsymbol{r} a_\sigma + \mathcal{O}\Big(\lambda^3\Big), \tag{6.19}$$

originally from Eqs. (97)–(98) in Ref. [96]. Here, λ is our formal order counting parameter from Ch. 2.2.1.2, and we have reintroduced the quantity $\boldsymbol{r}$ from Eq. (2.97). Below we will also use the quantity $\boldsymbol{\rho}$ from Eq. (2.98). We note that we expand all quantities through four total orders but we only display the leading two orders here to indicate the forms of the expressions; the full expansions can be found in the original paper [96]. We may do our series expansions as a normal power series as all the Fermi–Walker quantities (including one-forms) are scalars at $\bar x$. The expansion of Synge's world function is given by [96, Eqs. (99)–(101)]

$$\begin{aligned}\sigma(x,\bar x) &= \sigma(x,x') + \frac{d\sigma}{d\tau'}\Delta\tau + \frac{1}{2}\frac{d^2\sigma}{d\tau'^2}\Delta\tau^2 + \frac{1}{6}\frac{d^3\sigma}{d\tau'^3}\Delta\tau^3 + \mathcal{O}\Big(\lambda^4\Big)\\ &= \frac{1}{2}\Big[\lambda^2\boldsymbol{\rho}^2 + \lambda^3\boldsymbol{r}^2 a_\sigma\Big] + \mathcal{O}\Big(\lambda^4\Big),\end{aligned} \tag{6.20}$$

and expansions of the Fermi–Walker basis one-forms are given by [96, Eqs. (103)–(106)]

$$dt = -\, g^{\alpha'}_\mu \Big[\lambda^0 u_{\alpha'} + \lambda(\boldsymbol{r} a_{\alpha'} + a_\sigma u_{\alpha'}) + \mathcal{O}\Big(\lambda^2\Big)\Big]\, dx^\mu\,, \tag{6.21}$$
$$dx^a = g^{\alpha'}_\mu \Big[\lambda^0 e^a_{\alpha'} + \lambda(e^{a\beta'}\boldsymbol{r} u_{\alpha'} a_{\beta'}) + \mathcal{O}\Big(\lambda^2\Big)\Big]\, dx^\mu\,. \tag{6.22}$$

In the above expressions, we see that acceleration terms have appeared. This is a result of taking the derivatives with respect to τ'. As stated, $d/d\tau' = u^{\alpha'}\nabla_{\alpha'}$, so taking multiple τ' derivatives results in us taking derivatives of $u^{\alpha'}$ along the worldline, providing us with acceleration terms. These can then be differentiated along the worldline, giving us terms like $\dot{a}^{\alpha'}$, where a dot indicates a time derivative in the usual manner.

When accounting for these terms, at first order, we split up $h^{\rm S1}_{\mu\nu}$ into an acceleration-independent and a linear-in-acceleration piece:

$$h^{\rm S1}_{\mu\nu} = h^{{\rm S1}\not{a}}_{\mu\nu} + h^{{\rm S1}\boldsymbol{a}}_{\mu\nu} + \mathcal{O}\Big(a^2\Big). \tag{6.23}$$

As each acceleration term carries an ϵ, this effectively makes $h^{{\rm S1}\boldsymbol{a}}_{\mu\nu}$ a second-order term and allows us to ignore any non-linear acceleration terms that appear in the expansion of $h^{\rm S1}_{\mu\nu}$. Additionally, we can ignore any explicit acceleration terms that appear in both $h^{\rm SR}_{\mu\nu}$ and $h^{\rm SS}_{\mu\nu}$ as these would become third-order terms.

6.2 Creating the covariant puncture

With the methods from Ref. [96] recapped, we can now proceed to use them to generate our covariant puncture for the singular field in the highly regular gauge. Ch. 6.2.1 will provide the components of the highly regular gauge singular field when evaluated at $\bar{x}$ with each being written in covariant form. We then move to Ch. 6.2.2, which provides the components evaluated at x' before combining this with one-form expansions to find the final, fully covariant form in Ch. 6.2.3.

6.2.1 Perturbation components at $\bar{x}$

We begin by calculating the form of the components of the first-order singular field, $h^{\rm S1}_{\mu\nu}$, when evaluated at $\bar{x}^\alpha$. As discussed in Ch. 3.5, $h^{\rm S1}_{\mu\nu}$ in the highly regular gauge is the same as in the light-cone rest gauge, given in Eq. (3.38). Therefore, the first-order singular field in the highly regular gauge evaluated at $\bar{x}^\alpha$ is merely given by substituting the appropriate expressions from Ch. 6.1 into Eq. (3.38). These are then given by

$$\begin{aligned} h^{\rm S1}_{tt} = {}& \frac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}} + \frac{11m\lambda}{3\sqrt{2\bar{\sigma}}}R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}} + \frac{m\lambda^2}{24}\Big[\dot{R}_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}}\Big\{80 - 48\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\} - 19\sqrt{\frac{2}{\bar{\sigma}}}R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\sigma}}\Big] \\ & + \mathcal{O}\Big(\lambda^3\Big), \end{aligned} \tag{6.24a}$$

$$
\begin{aligned}
h^{\mathrm{S1}}_{ta} = \frac{m e^{\bar{\alpha}}_a}{36\bar{\sigma}} \Bigg(& -\frac{36\sigma_{\bar{\alpha}}}{\lambda} + 12\lambda\Big[2\sqrt{2}R_{\bar{\alpha}\bar{\sigma}\bar{u}\bar{\sigma}}\sqrt{\bar{\sigma}} - 2R_{\bar{\alpha}\bar{u}\bar{\sigma}\bar{u}}\bar{\sigma} - 3R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}}\sigma_{\bar{\alpha}}\Big] \\
& + \lambda^2\Big[\bar{\sigma}\Big(\dot{R}_{\bar{\alpha}\bar{\sigma}\bar{u}\bar{\sigma}}\Big\{52 - 48\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\} + 7\Big\{2R_{\bar{\alpha}\bar{u}\bar{\sigma}\bar{u}|\bar{\sigma}} + R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\alpha}}\Big\}\Big) \\
& - 16\sqrt{2\bar{\sigma}^3}\dot{R}_{\bar{\alpha}\bar{u}\bar{\sigma}\bar{u}}\Big\{5 - 3\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\} + 12R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\sigma}}\sigma_{\bar{\alpha}} - \sqrt{2\bar{\sigma}} \\
& \times\Big(15R_{\bar{\alpha}\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\sigma}} + 4\dot{R}_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}}\sigma_{\bar{\alpha}}\Big\{5 - 3\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\}\Big)\Big]\Bigg) + \mathcal{O}\big(\lambda^3\big),
\end{aligned} \tag{6.24b}
$$

$$
\begin{aligned}
h^{\mathrm{S1}}_{ab} = \frac{m e^{\bar{\alpha}}_a e^{\bar{\beta}}_b}{144\bar{\sigma}^{3/2}} \Bigg(& \frac{72\sqrt{2}\sigma_{\bar{\alpha}}\sigma_{\bar{\beta}}}{\lambda} + 12\lambda\Big[\sqrt{2}R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}}\sigma_{\bar{\alpha}}\sigma_{\bar{\beta}} - 16\sqrt{\bar{\sigma}}\sigma_{(\bar{\alpha}}R_{\bar{\beta})\bar{\sigma}\bar{u}\bar{\sigma}} + 8\sqrt{2}\bar{\sigma}\sigma_{(\bar{\alpha}}R_{\bar{\beta})\bar{u}\bar{\sigma}\bar{u}}\Big] \\
& + \lambda^2\Big[-32\sqrt{2}\bar{\sigma}^2\Big(\dot{R}_{\bar{u}(\bar{\alpha}\bar{\beta})\bar{\sigma}}\Big\{5 - 6\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\} + R_{\bar{\alpha}\bar{u}\bar{\beta}\bar{u}|\bar{\sigma}} + 2R_{\bar{u}\bar{\sigma}\bar{u}(\bar{\alpha}|\bar{\beta})}\Big) \\
& + 192\bar{\sigma}^{5/2}\dot{R}_{\bar{\alpha}\bar{u}\bar{\beta}\bar{u}}\Big\{3 - 2\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\} - 3\sqrt{2}\sigma_{\bar{\alpha}}\sigma_{\bar{\beta}}R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\sigma}} + 72\sqrt{\bar{\sigma}}\sigma_{(\bar{\alpha}}R_{\bar{\beta})\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\sigma}} \\
& - 4\sqrt{2}\bar{\sigma}\Big(g_{\bar{\alpha}\bar{\beta}}R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\sigma}} + 8\sigma_{(\bar{\alpha}}\dot{R}_{\bar{\beta})\bar{\sigma}\bar{u}\bar{\sigma}}\Big\{4 - 3\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\} - 2\sigma_{(\bar{\alpha}}R_{\bar{\beta})\bar{u}\bar{\sigma}\bar{u}|\bar{\sigma}} \\
& - R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}|(\bar{\alpha}}\sigma_{\bar{\beta})}\Big) + 32\bar{\sigma}^{3/2}\Big(\dot{R}_{\bar{\alpha}\bar{\sigma}\bar{\beta}\bar{\sigma}} + 3R_{\bar{u}(\bar{\alpha}\bar{\beta})\bar{\sigma}|\bar{\sigma}} + g_{\bar{\alpha}\bar{\beta}}\dot{R}_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}}\Big\{4 - 3\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big)\Big\} \\
& + 2\sigma_{(\bar{\alpha}}\dot{R}_{\bar{\beta})\bar{u}\bar{\sigma}\bar{u}}\Big)\Big]\Bigg) + \mathcal{O}\big(\lambda^3\big).
\end{aligned} \tag{6.24c}
$$

This can then be continued at second order for the singular fields $h^{\mathrm{SR}}_{\mu\nu}$ (4.48) and $h^{\mathrm{SS}}_{\mu\nu}$ (4.49). The 'singular times regular' piece is given by

$$
h^{\mathrm{SR}}_{tt} = -\frac{m}{4\sqrt{2}\bar{\sigma}^{3/2}}\left[\frac{2}{\lambda}(h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}} + 4h^{\mathrm{R1}}_{\bar{u}\bar{u}}\bar{\sigma}) - \lambda^0\Big(16\bar{\sigma}\dot{h}^{\mathrm{R1}}_{\bar{\sigma}\bar{u}} + h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}|\bar{\sigma}} + 4\sqrt{2}\bar{\sigma}^{1/2}\dot{h}^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}}\Big)\right] + \mathcal{O}(\lambda), \tag{6.25a}
$$

$$
\begin{aligned}
h^{\mathrm{SR}}_{ta} = -\frac{m e^{\bar{\alpha}}_a}{4\bar{\sigma}^2}\Bigg[& \frac{2}{\lambda}\Big(2h^{\mathrm{R1}}_{\bar{\alpha}\bar{\sigma}}\bar{\sigma} + 2\sqrt{2}h^{\mathrm{R1}}_{\bar{\alpha}\bar{u}}\bar{\sigma}^{3/2} - h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}}\sigma_{\bar{\alpha}} - h^{\mathrm{R1}}_{\bar{u}\bar{u}}\bar{\sigma}\sigma_{\bar{\alpha}}\Big) + \lambda^0\Big(\bar{\sigma}_{\bar{\alpha}}h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}|\bar{\sigma}} \\
& - 2\sqrt{2}\bar{\sigma}^{3/2}(h^{\mathrm{R1}}_{\bar{\sigma}\bar{u}|\bar{\alpha}} + h^{\mathrm{R1}}_{\bar{\alpha}\bar{u}|\bar{\sigma}} - \dot{h}^{\mathrm{R1}}_{\bar{\alpha}\bar{\sigma}}) + \sqrt{2\bar{\sigma}}\sigma_{\bar{\alpha}}\dot{h}^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}} - \bar{\sigma}(h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}|\bar{\alpha}} + 2h^{\mathrm{R1}}_{\bar{\alpha}\bar{\sigma}|\bar{\sigma}} - 4\sigma_{\bar{\alpha}}\dot{h}^{\mathrm{R1}}_{\bar{\sigma}\bar{u}})\Big)\Bigg] \\
& + \mathcal{O}(\lambda)
\end{aligned} \tag{6.25b}
$$

$$
\begin{aligned}
h^{\mathrm{SR}}_{ab} = -\frac{m e^{\bar{\alpha}}_a e^{\bar{\beta}}_b}{16\bar{\sigma}^{5/2}}\Bigg[& \frac{2}{\lambda}\Big(3\sqrt{2}h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}}\sigma_{\bar{\alpha}}\sigma_{\bar{\beta}} - 8\sqrt{2}\bar{\sigma}h^{\mathrm{R1}}_{\bar{\sigma}(\bar{\alpha}}\sigma_{\bar{\beta})} - 16\bar{\sigma}^{3/2}h^{\mathrm{R1}}_{\bar{u}(\bar{\alpha}}\sigma_{\bar{\beta})}\Big) \\
& + \lambda^0\Big(4\sqrt{2}\bar{\sigma}(h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}|(\bar{\alpha}}\sigma_{\bar{\beta})} + 2\sigma_{(\bar{\alpha}}h^{\mathrm{R1}}_{\bar{\beta})\bar{\sigma}|\bar{\sigma}}) - 3\sqrt{2}\sigma_{\bar{\alpha}}\sigma_{\bar{\beta}}h^{\mathrm{R1}}_{\bar{\sigma}\bar{\sigma}|\bar{\sigma}} + 16\bar{\sigma}^{3/2}(h^{\mathrm{R1}}_{\bar{\sigma}\bar{u}|(\bar{\alpha}}\sigma_{\bar{\beta})} \\
& + \sigma_{(\bar{\alpha}}h^{\mathrm{R1}}_{\bar{\beta})u|\bar{\sigma}} - \sigma_{(\bar{\alpha}}\dot{h}^{\mathrm{R1}}_{\bar{\beta})\bar{\sigma}}\Big)\Bigg] + \mathcal{O}(\lambda).
\end{aligned} \tag{6.25c}
$$

We have omitted the highest-order piece of $h^{\mathrm{SR}}_{\bar{\mu}\bar{\nu}}$ due to its length but it will be used to calculate the covariant punctures. Finally, the 'singular times singular' piece is given by

$$
h^{\mathrm{SS}}_{tt} = -\frac{2m^2\lambda^0}{\bar{\sigma}} + \frac{2m^2\lambda}{3\bar{\sigma}}\Big[2R_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}|\bar{\sigma}} - \sqrt{2\bar{\sigma}}\dot{R}_{\bar{u}\bar{\sigma}\bar{u}\bar{\sigma}}(11 - 6\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar{\sigma}}}\Big))\Big] + \mathcal{O}\big(\lambda^2\big), \tag{6.26a}
$$

$$h^{\rm SS}_{ta} = \frac{m^2 e^{\bar\alpha}_a}{18\bar\sigma^{3/2}}\Bigg[18\sqrt{2}\lambda^0 R_{\bar u\bar\sigma\bar u\bar\sigma}\sigma_{\bar\alpha} + \lambda\Bigg(-2\sqrt{2}\bar\sigma\Big[2R_{\bar\alpha\bar u\bar\sigma\bar u|\sigma} + R_{\bar u\bar\sigma\bar u\bar\sigma|\bar\alpha} + 4\dot R_{\bar\alpha\sigma\bar u\bar\sigma}$$
$$\times\Big(4 - 3\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar\sigma}}\Big)\Big)\Big] + 4\dot R_{\bar\alpha\bar u\bar\sigma\bar u}\bar\sigma^{3/2}\Big(29 - 12\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar\sigma}}\Big)\Big) - 9\sqrt{2}R_{\bar u\bar\sigma\bar u\bar\sigma|\sigma}\sigma_{\bar\alpha}$$
$$+ \sqrt{\bar\sigma}\Big[6R_{\bar\alpha\sigma\bar u\bar\sigma|\sigma} + 2\dot R_{\bar u\bar\sigma\bar u\bar\sigma}\Big(37 - 24\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar\sigma}}\Big)\Big)\sigma_{\bar\alpha}\Big]\Bigg)\Bigg] + \mathcal{O}\Big(\lambda^2\Big), \tag{6.26b}$$

$$h^{\rm SS}_{ab} = \frac{m^2 e^{\bar\alpha}_a e^{\bar\beta}_b}{18\bar\sigma^2}\Bigg[3\lambda^0\Big(2R_{\bar u\bar\sigma\bar u\bar\sigma}g_{\bar\alpha\bar\beta}\bar\sigma + 8R_{\bar\alpha\bar u\bar\beta\bar u}\bar\sigma^2 - 5R_{\bar u\bar\sigma\bar u\bar\sigma}\sigma_{\bar\alpha}\sigma_{\bar\beta} - 12\sqrt{2}\bar\sigma^{3/2}R_{\bar u(\bar\alpha\bar\beta)\sigma}$$
$$+ 6\sqrt{2}\bar\sigma^{1/2}R_{\sigma\bar u\bar\sigma(\bar\alpha}\sigma_{\bar\beta)} - 8\bar\sigma R_{\bar u\bar\sigma\bar u(\bar\alpha}\sigma_{\bar\beta)}\Big) + \lambda\Bigg(6R_{\bar u\bar\sigma\bar u\bar\sigma|\sigma}\sigma_{\bar\alpha}\sigma_{\bar\beta} - 48\sqrt{2}\dot R_{\bar\alpha\bar u\bar\beta\bar u}\bar\sigma^{5/2}$$
$$- \bar\sigma^{1/2}\Big[2\sqrt{2}\dot R_{\bar u\bar\sigma\bar u\bar\sigma}\Big(7 - 6\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar\sigma}}\Big)\Big)\sigma_{\bar\alpha}\sigma_{\bar\beta} + 6\sqrt{2}\sigma_{(\bar\alpha}R_{\bar\beta)\sigma\bar u\bar\sigma|\sigma}\Big]$$
$$+ \bar\sigma\Big[\Big(64 - 48\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar\sigma}}\Big)\Big)\dot R_{\sigma\bar u\bar\sigma(\bar\alpha}\sigma_{\bar\beta)} + 4\Big(2\sigma_{(\bar\alpha}R_{\bar\beta)\bar u\bar\sigma\bar u|\sigma} + R_{\bar u\bar\sigma\bar u\bar\sigma|(\bar\alpha}\sigma_{\bar\beta)}\Big)\Big]$$
$$- 4\sqrt{2}\bar\sigma^{3/2}\Big[3\dot R_{\bar u\bar\sigma\bar u\bar\sigma}g_{\bar\alpha\bar\beta} + \Big(17 - 12\log\Big(\tfrac{\sqrt{2}m}{\lambda\sqrt{\bar\sigma}}\Big)\Big)\sigma_{(\bar\alpha}\dot R_{\bar\beta)\bar u\bar\sigma\bar u}\Big]\Bigg)\Bigg] + \mathcal{O}\Big(\lambda^2\Big). \tag{6.26c}$$

6.2.2 Expansion at x'

Accounting for the introduction of acceleration terms and splitting up $h^{\rm S1}_{\mu\nu}$ as in Eq. (6.23), we find that the components of $h^{\rm S1\not a}_{\mu\nu}$ when expanded at x'^α are given by

$$h^{\rm S1\not a}_{tt} = \frac{2m}{\lambda\boldsymbol\rho} + \frac{m\lambda}{3\boldsymbol\rho^3}R_{u\sigma u\sigma}(\boldsymbol r^2 + 11\boldsymbol\rho^2) - \frac{m\lambda^2}{12\boldsymbol\rho^3}\Big(24R_{u\sigma u\sigma;u}\log\Big(\tfrac{2m}{\lambda\boldsymbol\rho}\Big)\boldsymbol\rho^3$$
$$+ R_{u\sigma u\sigma;\sigma}(\boldsymbol r^2 + 19\boldsymbol\rho^2) - \dot R_{u\sigma u\sigma}(\boldsymbol r^3 + 25\boldsymbol r\boldsymbol\rho^2 + 40\boldsymbol\rho^3)\Big) + \mathcal{O}\Big(\lambda^3\Big), \tag{6.27a}$$

$$h^{\rm S1\not a}_{ta} = -\frac{m e^{\alpha'}_a}{36\boldsymbol\rho^4}\Bigg[\frac{72\boldsymbol\rho^2\sigma_{\alpha'}}{\lambda} + \lambda\Bigg(12\boldsymbol\rho^2\Big(R_{\alpha' u\sigma u}(\boldsymbol r^2 + 4\boldsymbol r\boldsymbol\rho + 2\boldsymbol\rho^2) - 2R_{\alpha'\sigma u\sigma}(\boldsymbol r + 2\boldsymbol\rho)\Big)$$
$$+ 24R_{u\sigma u\sigma}(\boldsymbol r^2 + 3\boldsymbol\rho^2)\sigma_{\alpha'}\Bigg) + \lambda^2\Bigg(\boldsymbol\rho^2\Bigg(-3\boldsymbol r^2(3\dot R_{\alpha'\sigma u\sigma} + R_{\alpha' u\sigma u;\sigma} - 2\dot R_{\alpha' u\sigma u}\boldsymbol r)$$
$$- 6\boldsymbol r(3\dot R_{\alpha'\sigma u\sigma} + 5R_{\alpha' u\sigma u;\sigma} - 3\dot R_{\alpha' u\sigma u}\boldsymbol r)\boldsymbol\rho - \Big[7(2R_{\alpha' u\sigma u;\sigma} + R_{u\sigma u\sigma;\alpha'})$$
$$+ \dot R_{\alpha'\sigma u\sigma}\Big(52 - 48\log\Big(\tfrac{2m}{\lambda\boldsymbol\rho}\Big)\Big) - 2\dot R_{\alpha' u\sigma u}\Big(31 - 24\log\Big(\tfrac{2m}{\lambda\boldsymbol\rho}\Big)\Big)\boldsymbol r\Big]\boldsymbol\rho^2$$
$$+ 16\dot R_{\alpha' u\sigma u}\Big(5 - 3\log\Big(\frac{2m}{\lambda\boldsymbol\rho}\Big)\Big)\boldsymbol\rho^3 + 6R_{\alpha'\sigma u\sigma;\sigma}(\boldsymbol r + 5\boldsymbol\rho)\Bigg)$$
$$+ \Bigg(6\boldsymbol r^2(\dot R_{u\sigma u\sigma}\boldsymbol r - R_{u\sigma u\sigma;\sigma}) - 24(R_{u\sigma u\sigma;\sigma} - 2\dot R_{u\sigma u\sigma}\boldsymbol r)\boldsymbol\rho^2$$
$$+ 8\dot R_{u\sigma u\sigma}\Big(5 - 3\log\Big(\tfrac{2m}{\lambda\boldsymbol\rho}\Big)\Big)\boldsymbol\rho^3\Bigg)\sigma_{\alpha'}\Bigg)\Bigg] + \mathcal{O}\Big(\lambda^3\Big), \tag{6.27b}$$

$$
\begin{aligned}
h^{\mathrm{S1}\not{a}}_{ab} = & -\frac{m e^{\alpha'}_a e^{\beta'}_b}{36\boldsymbol{\rho}^5}\Bigg[-\frac{72\boldsymbol{\rho}^2\sigma_{\alpha'}\sigma_{\beta'}}{\lambda} - 12\lambda\Big(R_{u\sigma u\sigma}(3\boldsymbol{r}^2+\boldsymbol{\rho}^2)\sigma_{\alpha'}\sigma_{\beta'} - 2\boldsymbol{\rho}^2\Big[2(\boldsymbol{r}+2\boldsymbol{\rho}) \\
& \times \sigma_{(\alpha'}R_{\beta')\sigma u\sigma} - (\boldsymbol{r}^2+4\boldsymbol{r}\boldsymbol{\rho}+2\boldsymbol{\rho}^2)\sigma_{(\alpha'}R_{\beta')u\sigma u}\Big]\Big) + \lambda^2\Big(4g_{\alpha'\beta'}\boldsymbol{\rho}^4(3R_{u\sigma u\sigma;\sigma} \\
& + 3\dot{R}_{u\sigma u\sigma}\boldsymbol{r} - 8\dot{R}_{u\sigma u\sigma}\boldsymbol{\rho} + 6\dot{R}_{u\sigma u\sigma}\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}) - \Big[9\dot{R}_{u\sigma u\sigma}\boldsymbol{r}(\boldsymbol{r}^2+\boldsymbol{\rho}^2) \\
& - 3R_{u\sigma u\sigma;\sigma}(3\boldsymbol{r}^2+\boldsymbol{\rho}^2)\Big]\sigma_{\alpha'}\sigma_{\beta'}\Big) + 8\boldsymbol{\rho}^5\Big(R_{\alpha' u\beta' u;\sigma}(3\boldsymbol{r}+\boldsymbol{\rho}) - \dot{R}_{\alpha'\sigma\beta'\sigma} + \dot{R}_{\alpha' u\beta' u} \\
& \times \Big(2\boldsymbol{r}(\boldsymbol{r}-2\boldsymbol{\rho}) - 9\boldsymbol{\rho}^2 + 6\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\Big) - 3R_{u(\alpha'\beta')\sigma;\sigma} - \dot{R}_{u(\alpha'\beta')\sigma} \\
& \times \Big(\boldsymbol{r} - 5\boldsymbol{\rho} + 6\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}\Big) + 2\boldsymbol{\rho}R_{u\sigma u(\alpha';\beta')}\Big) - 2\boldsymbol{\rho}^2\sigma_{(\alpha'}\Big(3\boldsymbol{r}^2(2\dot{R}_{\beta')u\sigma u}\boldsymbol{r} \\
& - 3\dot{R}_{\beta')\sigma u\sigma} - R_{\beta')u\sigma u;\sigma}) + 6\boldsymbol{r}(-5\dot{R}_{\beta')\sigma u\sigma} - 3R_{\beta')u\sigma u;\sigma} + 5\dot{R}_{\beta')u\sigma u}\boldsymbol{r})\boldsymbol{\rho} \\
& + \Big(2R_{\beta')u\sigma u;\sigma} + R_{|u\sigma u\sigma|;\beta')} - 8\dot{R}_{\beta')\sigma u\sigma}\Big(4 - 3\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\Big) + 2\dot{R}_{\beta')u\sigma u} \\
& \times \Big(29 - 12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\Big)\boldsymbol{r}\Big)\boldsymbol{\rho}^2 + 8\dot{R}_{\beta')u\sigma u}\boldsymbol{\rho}^3 + 6R_{\beta')\sigma u\sigma;\sigma}(\boldsymbol{r}+3\boldsymbol{\rho})\Big)\Bigg] + \mathcal{O}\big(\lambda^3\big).
\end{aligned}
\tag{6.27c}
$$

The acceleration terms that appear as a result of our expansion of the first-order singular field are

$$
h^{\mathrm{S1}\boldsymbol{a}}_{tt} = -\frac{m\lambda^0 a_\sigma \boldsymbol{r}^2}{\boldsymbol{\rho}^3} - \frac{m\lambda\dot{a}_\sigma\boldsymbol{r}^3}{3\boldsymbol{\rho}^3} + \mathcal{O}\big(\lambda^2\big), \tag{6.28a}
$$

$$
h^{\mathrm{S1}\boldsymbol{a}}_{ta} = -\frac{m e^{\alpha'}_a \boldsymbol{r}}{3\boldsymbol{\rho}^6}\Big[3\lambda^0\boldsymbol{r}\boldsymbol{\rho}^2(a_{\alpha'}\boldsymbol{\rho}^2 - 2a_\sigma\sigma_{\alpha'}) + \lambda\boldsymbol{r}^2\boldsymbol{\rho}^2(\dot{a}_{\alpha'}\boldsymbol{\rho}^2 - 2\dot{a}_\sigma\sigma_{\alpha'})\Big] + \mathcal{O}\big(\lambda^2\big), \tag{6.28b}
$$

$$
\begin{aligned}
h^{\mathrm{S1}\boldsymbol{a}}_{ab} = & -\frac{m e^{\alpha'}_a e^{\beta'}_b \boldsymbol{r}^2}{3\boldsymbol{\rho}^5}\Big[3\lambda^0(3a_\sigma\sigma_{\alpha'}\sigma_{\beta'} - 2\boldsymbol{\rho}^2 a_{(\alpha'}\sigma_{\beta')}) + \lambda\boldsymbol{r}(3\dot{a}_\sigma\sigma_{\alpha'}\sigma_{\beta'} - 2\boldsymbol{\rho}^2\dot{a}_{(\alpha'}\sigma_{\beta')})\Big] \\
& + \mathcal{O}\big(\lambda^2\big).
\end{aligned}
\tag{6.28c}
$$

As $h^{\mathrm{S1}\boldsymbol{a}}_{\mu\nu}$ is a second-order term, we can neglect any terms of order-λ^2 and higher to match the orders required for $h^{\mathrm{SR}}_{\mu\nu}$ and $h^{\mathrm{SS}}_{\mu\nu}$.

Moving to the second-order field, we calculate the SR components to be

$$
\begin{aligned}
h^{\mathrm{SR}}_{tt} = & -\frac{m}{2\boldsymbol{\rho}^3}\Bigg[\frac{2}{\lambda}\Big(h^{\mathrm{R1}}_{\sigma\sigma} + 2h^{\mathrm{R1}}_{\sigma u}\boldsymbol{r} + h^{\mathrm{R1}}_{uu}(\boldsymbol{r}^2+2\boldsymbol{\rho}^2)\Big) - \lambda^0\Big(\boldsymbol{r}(\boldsymbol{r}h^{\mathrm{R1}}_{uu;\sigma} + 2h^{\mathrm{R1}}_{\sigma u;\sigma}) + h^{\mathrm{R1}}_{\sigma\sigma;\sigma} \\
& - (\boldsymbol{r}-4\boldsymbol{\rho})h^{\mathrm{R1}}_{\sigma\sigma;u} - 2(\boldsymbol{r}^2-4\boldsymbol{r}\boldsymbol{\rho}-4\boldsymbol{\rho}^2)h^{\mathrm{R1}}_{\sigma u;u} - \boldsymbol{r}(\boldsymbol{r}^2-4\boldsymbol{r}\boldsymbol{\rho}-4\boldsymbol{\rho}^2)h^{\mathrm{R1}}_{uu;u}\Big)\Bigg] \\
& + \mathcal{O}(\lambda),
\end{aligned}
\tag{6.29a}
$$

$$
\begin{aligned}
h^{\rm SR}_{ta} = &- \frac{m e_a{}^{\alpha'}}{2\boldsymbol{\rho}^4}\Bigg[\frac{2}{\lambda}\Big(2h^{\rm R1}_{\alpha'\sigma}\boldsymbol{\rho}^2 + 2h^{\rm R1}_{\alpha' u}\boldsymbol{\rho}^2(\boldsymbol{r}+\boldsymbol{\rho}) - \Big(2h^{\rm R1}_{\sigma\sigma} + 4h^{\rm R1}_{\sigma u}\boldsymbol{r} + h^{\rm R1}_{uu}(2\boldsymbol{r}^2+\boldsymbol{\rho}^2)\Big) \\
&\cdot \sigma_{\alpha'}\Big) - \lambda^0\Big[\boldsymbol{\rho}^2\Big(h^{\rm R1}_{\sigma\sigma;\alpha'} + 2(\boldsymbol{r}+\boldsymbol{\rho})h^{\rm R1}_{\sigma u;\alpha'} + \boldsymbol{r}(\boldsymbol{r}+2\boldsymbol{\rho})h^{\rm R1}_{uu;\alpha'} + 2\Big(h^{\rm R1}_{\alpha'\sigma;\sigma} \\
&+ (\boldsymbol{r}+\boldsymbol{\rho})(h^{\rm R1}_{\alpha' u;\sigma} - h^{\rm R1}_{\alpha'\sigma;u}) - \boldsymbol{r}(\boldsymbol{r}+2\boldsymbol{\rho})h^{\rm R1}_{\alpha' u;u}\Big)\Big) + \sigma_{\alpha'}\Big(-2(\boldsymbol{r}^2 h^{\rm R1}_{uu;\sigma} \\
&+ 2\boldsymbol{r}h^{\rm R1}_{\sigma u;\sigma} + h^{\rm R1}_{\sigma\sigma;\sigma}) + 2(\boldsymbol{r}-\boldsymbol{\rho})h^{\rm R1}_{\sigma\sigma;u} + 4(\boldsymbol{r}^2 - \boldsymbol{r}\boldsymbol{\rho} - \boldsymbol{\rho}^2)h^{\rm R1}_{\sigma u;u} \\
&+ 2\boldsymbol{r}(\boldsymbol{r}^2 - \boldsymbol{r}\boldsymbol{\rho} - \boldsymbol{\rho}^2)h^{\rm R1}_{uu;u}\Big)\Big]\Bigg] + \mathcal{O}(\lambda), \qquad (6.29\text{b})
\end{aligned}
$$

$$
\begin{aligned}
h^{\rm SR}_{ab} = &- \frac{m e_a^{\alpha'} e_b^{\beta'}}{2\boldsymbol{\rho}^5}\Bigg[\frac{2}{\lambda}\Big[3\Big(h^{\rm R1}_{\sigma\sigma} + \boldsymbol{r}(2h^{\rm R1}_{\sigma u} + h^{\rm R1}_{uu}\boldsymbol{r})\Big)\sigma_{\alpha'}\sigma_{\beta'} - 4\boldsymbol{\rho}^2 h^{\rm R1}_{(\alpha'|\sigma|}\sigma_{\beta')} \\
&- 4\boldsymbol{\rho}^2(\boldsymbol{r}+\boldsymbol{\rho})h^{\rm R1}_{(\alpha'|u|}\sigma_{\beta')}\Big] - \lambda^0\Big[3\sigma_{\alpha'}\sigma_{\beta'}\Big(h^{\rm R1}_{\sigma\sigma;\sigma} - \boldsymbol{r}\Big(h^{\rm R1}_{\sigma\sigma;u} - 2h^{\rm R1}_{\sigma u;\sigma} \\
&+ \boldsymbol{r}(2h^{\rm R1}_{\sigma u;u} - h^{\rm R1}_{uu;\sigma} + \boldsymbol{r}h^{\rm R1}_{uu;u})\Big)\Big) - 2\boldsymbol{\rho}^2\Big(\sigma_{(\alpha'}h^{\rm R1}_{|\sigma\sigma|;\beta')} + 2(\boldsymbol{r}+\boldsymbol{\rho})\sigma_{(\alpha'}h^{\rm R1}_{|\sigma u|;\beta')} \\
&+ \boldsymbol{r}(\boldsymbol{r}+2\boldsymbol{\rho})\sigma_{(\alpha'}h^{\rm R1}_{|uu|;\beta')} + 2\Big(\sigma_{(\alpha'}h^{\rm R1}_{\beta')\sigma;\sigma} + (\boldsymbol{r}+\boldsymbol{\rho})(\sigma_{(\alpha'}h^{\rm R1}_{\beta')u;\sigma} - \sigma_{(\alpha'}h^{\rm R1}_{\beta')\sigma;u}) \\
&- \boldsymbol{r}(\boldsymbol{r}+2\boldsymbol{\rho})\sigma_{(\alpha'}h^{\rm R1}_{\beta')u;u}\Big)\Big)\Big]\Bigg] + \mathcal{O}(\lambda), \qquad (6.29\text{c})
\end{aligned}
$$

where, again, we have omitted the highest order term. The SS components are calculated to be

$$
h^{\rm SS}_{tt} = -\frac{4m^2}{3\boldsymbol{\rho}^2}\Big[3\lambda^0 R_{u\sigma u\sigma} - \lambda\Big(2R_{u\sigma u\sigma;\sigma} - \dot{R}_{u\sigma u\sigma}\Big[\boldsymbol{r} + 11\boldsymbol{\rho} - 6\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}\Big]\Big)\Big] + \mathcal{O}\Big(\lambda^2\Big), \qquad (6.30\text{a})
$$

$$
\begin{aligned}
h^{\rm SS}_{ta} = &\frac{2m^2 e_a^{\alpha'}}{9\boldsymbol{\rho}^3}\Big[18\lambda^0 R_{u\sigma u\sigma}\sigma_{\alpha'} + \lambda\Big(\boldsymbol{\rho}\Big[3R_{\alpha'\sigma u\sigma;\sigma} - R_{u\sigma u\sigma;\alpha'}\boldsymbol{\rho} - R_{\alpha' u\sigma u;\sigma}(3\boldsymbol{r}+2\boldsymbol{\rho}) \\
&+ \dot{R}_{\alpha'\sigma u\sigma}\Big(3\boldsymbol{r} - 4\Big(4 - 3\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\Big)\boldsymbol{\rho}\Big) - \dot{R}_{\alpha' u\sigma u}\Big(3\boldsymbol{r}^2 - 14\boldsymbol{r}\boldsymbol{\rho} - 29\boldsymbol{\rho}^2 + 12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big) \\
&\times \boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\Big)\Big] - \sigma_{\alpha'}\Big[9R_{u\sigma u\sigma;\sigma} - \dot{R}_{u\sigma u\sigma}\Big(9\boldsymbol{r} + 37\boldsymbol{\rho} - 24\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}\Big)\Big]\Big)\Big] + \mathcal{O}\Big(\lambda^2\Big), \qquad (6.30\text{b})
\end{aligned}
$$

$$
\begin{aligned}
h^{\rm SS}_{ab} = &\frac{2m^2 e_a^{\alpha'} e_b^{\beta'}}{9\boldsymbol{\rho}^4}\Big[3\lambda^0\Big(\boldsymbol{\rho}^2\Big[R_{u\sigma u\sigma}g_{\alpha'\beta'} + 2R_{\alpha' u\beta' u}\boldsymbol{\rho}(3\boldsymbol{r}+\boldsymbol{\rho}) - 6\boldsymbol{\rho}R_{u(\alpha'\beta')\sigma}\Big] \\
&+ 2\boldsymbol{\rho}\sigma_{(\alpha'}\Big[3R_{\beta')\sigma u\sigma} - R_{\beta')u\sigma u}(3\boldsymbol{r}+2\boldsymbol{\rho})\Big] - 5R_{u\sigma u\sigma}\sigma_{\alpha'}\sigma_{\beta'}\Big) \\
&+ \lambda\Big(3\boldsymbol{\rho}^2\Big[\dot{R}_{u\sigma u\sigma}g_{\alpha'\beta'}(\boldsymbol{r}-2\boldsymbol{\rho}) + 2\dot{R}_{\alpha' u\beta' u}(3\boldsymbol{r}-2\boldsymbol{\rho})\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho}) - 6\boldsymbol{r}\boldsymbol{\rho}\dot{R}_{u(\alpha'\beta')\sigma}\Big] \\
&+ \sigma_{\alpha'}\sigma_{\beta'}\Big[6R_{u\sigma u\sigma;\sigma} - \dot{R}_{u\sigma u\sigma}\Big(9\boldsymbol{r} + 14\boldsymbol{\rho} - 12\boldsymbol{\rho}\log\Big(\tfrac{2m}{\boldsymbol{\rho}\lambda}\Big)\Big)\Big] - 2\boldsymbol{\rho}\Big[3\sigma_{(\alpha'}R_{\beta')\sigma u\sigma;\sigma} \\
&- 2\Big(3\boldsymbol{r} + 8\boldsymbol{\rho} - 6\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}\Big)\dot{R}_{\sigma u\sigma(\alpha'}\sigma_{\beta')} - (3\boldsymbol{r}+2\boldsymbol{\rho})\sigma_{(\alpha'}R_{\beta')u\sigma u;\sigma} \\
&+ \Big(6\boldsymbol{r}^2 + 20\boldsymbol{r}\boldsymbol{\rho} + 17\boldsymbol{\rho}^2 - 12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\Big)\dot{R}_{u\sigma u(\alpha'}\sigma_{\beta')} \\
&- \boldsymbol{\rho}R_{u\sigma u\sigma;(\alpha'}\sigma_{\beta')}\Big]\Big)\Big] + \mathcal{O}\Big(\lambda^2\Big). \qquad (6.30\text{c})
\end{aligned}
$$

6.2.3 Final expressions for the covariant punctures

With all of the individual components of the singular field now expressed as functions of x'^{α}, we now combine with the expansions of dt and dx^a, given in Eqs. (6.21)–(6.22) to find the final form of the covariant punctures. After contracting with the basis vectors, we obtain the covariant form of $h^{\mathrm{S}}_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$, as in Eq. (6.9). We then read off the coefficients of $dx^{\mu}\, dx^{\nu}$ to obtain $h^{\mathrm{S}}_{\mu\nu}$.

The first-order singular field is given by

$$
\begin{aligned}
h^{\mathrm{S1}\not{d}}_{\alpha\beta} = &- \frac{m g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta}}{36\boldsymbol{\rho}^5}\Bigg[\frac{-72\boldsymbol{\rho}^2}{\lambda}\Big(\sigma_{\alpha'} + (\boldsymbol{r}+\boldsymbol{\rho})u_{\alpha'}\Big)\Big(\sigma_{\beta'} + (\boldsymbol{r}+\boldsymbol{\rho})u_{\beta'}\Big) \\
&- 12\lambda\Big(R_{u\sigma u\sigma}(3\boldsymbol{r}^2+\boldsymbol{\rho}^2)\sigma_{\alpha'}\sigma_{\beta'} + 2R_{u\sigma u\sigma}\boldsymbol{r}(3\boldsymbol{r}-\boldsymbol{\rho})(\boldsymbol{r}+\boldsymbol{\rho})\sigma_{(\alpha'}u_{\beta')} \\
&+ R_{u\sigma u\sigma}(\boldsymbol{r}-\boldsymbol{\rho})(\boldsymbol{r}+\boldsymbol{\rho})^2(3\boldsymbol{r}+\boldsymbol{\rho})u_{\alpha'}u_{\beta'} + 2\boldsymbol{\rho}^2\Big(\sigma_{(\alpha'} + (\boldsymbol{r}+\boldsymbol{\rho})u_{(\alpha'}\Big) \\
&\cdot\Big(R_{\beta')\sigma u\sigma}(\boldsymbol{r}-3\boldsymbol{\rho}) + 2R_{\beta')u\sigma u}(\boldsymbol{\rho}^2-\boldsymbol{r}^2)\Big)\Big) + \lambda^2\Big(3\Big(-3\dot{R}_{u\sigma u\sigma}\boldsymbol{r}(\boldsymbol{r}^2+\boldsymbol{\rho}^2) \\
&+ R_{u\sigma u\sigma;\sigma}(3\boldsymbol{r}^2+\boldsymbol{\rho}^2)\Big)\sigma_{\alpha'}\sigma_{\beta'} + 6\Big(R_{u\sigma u\sigma;\sigma}\boldsymbol{r}(3\boldsymbol{r}-\boldsymbol{\rho})(\boldsymbol{r}+\boldsymbol{\rho}) - \dot{R}_{u\sigma u\sigma} \\
&\times(3\boldsymbol{r}^4 + 2\boldsymbol{r}^3\boldsymbol{\rho} + 3\boldsymbol{\rho}^4)\Big)\sigma_{(\alpha'}u_{\beta')} + 3\Big(R_{u\sigma u\sigma;\sigma}(\boldsymbol{r}-\boldsymbol{\rho})(\boldsymbol{r}+\boldsymbol{\rho})^2(3\boldsymbol{r}+\boldsymbol{\rho}) \\
&- \dot{R}_{u\sigma u\sigma}(3\boldsymbol{r}^5 + 4\boldsymbol{r}^4\boldsymbol{\rho} - 2\boldsymbol{r}^3\boldsymbol{\rho}^2 - 6\boldsymbol{r}^2\boldsymbol{\rho}^3 + 3\boldsymbol{r}\boldsymbol{\rho}^4 + 14\boldsymbol{\rho}^5)\Big)u_{\alpha'}u_{\beta'} \\
&+ 4g_{\alpha'\beta'}\boldsymbol{\rho}^4(3R_{u\sigma u\sigma;\sigma} + 3\dot{R}_{u\sigma u\sigma}\boldsymbol{r} - 8\dot{R}_{u\sigma u\sigma}\boldsymbol{\rho}) - 8\boldsymbol{\rho}^5\Big(\dot{R}_{\alpha'\sigma\beta'\sigma} - R_{\alpha' u\beta' u;\sigma} \\
&\times(3\boldsymbol{r}+\boldsymbol{\rho}) + \dot{R}_{\alpha' u\beta' u}(-2\boldsymbol{r}^2 + 4\boldsymbol{r}\boldsymbol{\rho} + 9\boldsymbol{\rho}^2) + 3R_{u(\alpha'\beta')\sigma;\sigma} + (\boldsymbol{r}-5\boldsymbol{\rho})\dot{R}_{u(\alpha'\beta')\sigma} \\
&+ 2\boldsymbol{\rho}R_{\sigma uu(\alpha';\beta')}\Big) - 2\boldsymbol{\rho}^2\sigma_{(\beta'}\Big(6(2\boldsymbol{\rho}-\boldsymbol{r})R_{\alpha')\sigma u\sigma;\sigma} + (3\boldsymbol{r}^2 - 24\boldsymbol{r}\boldsymbol{\rho} - 32\boldsymbol{\rho}^2)\dot{R}_{\alpha')\sigma u\sigma} \\
&+ 9\boldsymbol{r}^2R_{\alpha')u\sigma u;\sigma} + 2\boldsymbol{\rho}^2R_{\alpha')u\sigma u;\sigma} - 6\boldsymbol{r}^3\dot{R}_{\alpha')u\sigma u} + 12\boldsymbol{r}^2\boldsymbol{\rho}\dot{R}_{\alpha')u\sigma u} \\
&+ 58\boldsymbol{r}\boldsymbol{\rho}^2\dot{R}_{\alpha')u\sigma u} + 8\boldsymbol{\rho}^3\dot{R}_{\alpha')u\sigma u} + \boldsymbol{\rho}^2R_{|u\sigma u\sigma|;\alpha')}\Big) - 2\boldsymbol{\rho}^2u_{(\beta'}\Big(6(4\boldsymbol{\rho}^2 + \boldsymbol{r}\boldsymbol{\rho} - \boldsymbol{r}^2) \\
&\times R_{\alpha')\sigma u\sigma;\sigma} + (3\boldsymbol{r}^3 - 21\boldsymbol{r}^2\boldsymbol{\rho} - 44\boldsymbol{r}\boldsymbol{\rho}^2 - 44\boldsymbol{\rho}^3)\dot{R}_{\alpha')\sigma u\sigma} + 9\boldsymbol{r}^3R_{\alpha')u\sigma u;\sigma} \\
&+ 9\boldsymbol{r}^2\boldsymbol{\rho}R_{\alpha')u\sigma u;\sigma} - 10\boldsymbol{r}\boldsymbol{\rho}^2R_{\alpha')u\sigma u;\sigma} - 2\boldsymbol{\rho}^3R_{\alpha')u\sigma u;\sigma} - 6\boldsymbol{r}^4\dot{R}_{\alpha')u\sigma u} \\
&+ 6\boldsymbol{r}^3\boldsymbol{\rho}\dot{R}_{\alpha')u\sigma u} + 58\boldsymbol{r}^2\boldsymbol{\rho}^2\dot{R}_{\alpha')u\sigma u} + 74\boldsymbol{r}\boldsymbol{\rho}^3\dot{R}_{\alpha')u\sigma u} + 52\boldsymbol{\rho}^4\dot{R}_{\alpha')u\sigma u} \\
&+ (\boldsymbol{r}-7\boldsymbol{\rho})\boldsymbol{\rho}^2R_{|u\sigma u\sigma|;\alpha')}\Big) + 24\dot{R}_{u\sigma u\sigma}\log\Big(\frac{2m}{\lambda\boldsymbol{\rho}}\Big)g_{\alpha'\beta'}\boldsymbol{\rho}^5 + 48\log\Big(\frac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}^4 \\
&\times\Big(\dot{R}_{\alpha' u\beta' u}\boldsymbol{\rho}^2(\boldsymbol{r}+\boldsymbol{\rho}) - \boldsymbol{\rho}^2\dot{R}_{u(\alpha'\beta')\sigma} - \sigma_{(\alpha'}\dot{R}_{\beta')\sigma u\sigma} - u_{(\alpha'}\boldsymbol{r}\dot{R}_{\beta')\sigma u\sigma} \\
&- 2\boldsymbol{\rho}u_{(\alpha'}\dot{R}_{\beta')\sigma u\sigma} + \boldsymbol{r}\sigma_{(\alpha'}\dot{R}_{\beta')u\sigma u} + (\boldsymbol{r}+\boldsymbol{\rho})^2u_{(\alpha'}\dot{R}_{\beta')u\sigma u}\Big)\Big)\Bigg] + \mathcal{O}\Big(\lambda^3\Big). \qquad (6.31)
\end{aligned}
$$

We have confirmed that this satisfies the Einstein field equations to the appropriate order, i.e.

$$
\delta G^{\mu\nu}[h^{\mathrm{S1}\not{d}}] = \mathcal{O}(\lambda), \quad x \notin \gamma. \qquad (6.32)
$$

At second-order, the SS piece of the singular field is given by

$$
\begin{aligned}
h^{\mathrm{SS}}_{\alpha\beta} = &-\frac{2m^2 g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta}}{9\boldsymbol{\rho}^4}\Big[3\lambda^0\Big(R_{u\sigma u\sigma}(5\boldsymbol{r}^2+6\boldsymbol{r}\boldsymbol{\rho}+5\boldsymbol{\rho}^2)u_{\alpha'}u_{\beta'}+5R_{u\sigma u\sigma}\sigma_{\alpha'}\sigma_{\beta'}\\
&+2R_{u\sigma u\sigma}(5\boldsymbol{r}+3\boldsymbol{\rho})\sigma_{(\alpha'}u_{\beta')}-2\boldsymbol{\rho}u_{(\alpha'}\Big(3R_{\beta')\sigma u\sigma}\boldsymbol{r}-R_{\beta')u\sigma u}(3\boldsymbol{r}^2+2\boldsymbol{r}\boldsymbol{\rho}+3\boldsymbol{\rho}^2)\Big)\\
&-2\boldsymbol{\rho}\sigma_{(\alpha'}\Big(3R_{\beta')\sigma u\sigma}-R_{\beta')u\sigma u}(3\boldsymbol{r}+2\boldsymbol{\rho})\Big)-\boldsymbol{\rho}^2\Big(R_{u\sigma u\sigma}g_{\alpha'\beta'}+2R_{\alpha' u\beta' u}\boldsymbol{\rho}(3\boldsymbol{r}+\boldsymbol{\rho})\\
&-6\boldsymbol{\rho}R_{u(\alpha'\beta')\sigma}\Big)\Big)+\lambda\Big(u_{\alpha'}u_{\beta'}\Big[\dot{R}_{u\sigma u\sigma}(\boldsymbol{r}+\boldsymbol{\rho})\Big(9\boldsymbol{r}^2+11\boldsymbol{r}\boldsymbol{\rho}+38\boldsymbol{\rho}^2\\
&-12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})-6R_{u\sigma u\sigma;\sigma}(\boldsymbol{r}+\boldsymbol{\rho})^2\Big)\Big]+\sigma_{\alpha'}\sigma_{\beta'}\Big[\dot{R}_{u\sigma u\sigma}\Big(9\boldsymbol{r}+14\boldsymbol{\rho}\\
&-12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}\Big)-6R_{u\sigma u\sigma;\sigma}\Big]+2u_{(\alpha'}\sigma_{\beta')}\Big[\dot{R}_{u\sigma u\sigma}\Big(9\boldsymbol{r}^2+17\boldsymbol{r}\boldsymbol{\rho}+20\boldsymbol{\rho}^2\\
&-12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\Big)-6R_{u\sigma u\sigma;\sigma}(\boldsymbol{r}+\boldsymbol{\rho})\Big]-2\boldsymbol{\rho}u_{(\alpha'}\Big[\dot{R}_{\beta')\sigma u\sigma}\Big(6\boldsymbol{r}^2+13\boldsymbol{r}\boldsymbol{\rho}\\
&+16\boldsymbol{\rho}^2-12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\Big)-3R_{\beta')\sigma u\sigma;\sigma}(\boldsymbol{r}+\boldsymbol{\rho})+(\boldsymbol{r}+\boldsymbol{\rho})\Big(R_{|u\sigma u\sigma|;\beta')}\boldsymbol{\rho}\\
&+R_{\beta')u\sigma u;\sigma}(3\boldsymbol{r}+2\boldsymbol{\rho})-\dot{R}_{\beta')u\sigma u}\Big[6\boldsymbol{r}^2+11\boldsymbol{r}\boldsymbol{\rho}+29\boldsymbol{\rho}^2-12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\Big]\Big)\Big]\\
&-2\boldsymbol{\rho}\sigma_{(\alpha'}\Big[R_{|u\sigma u\sigma|;\beta')}\boldsymbol{\rho}-3R_{\beta')\sigma u\sigma;\sigma}+R_{\beta')u\sigma u;\sigma}(3\boldsymbol{r}+2\boldsymbol{\rho})+2\dot{R}_{\beta')\sigma u\sigma}\\
&\times\Big(3\boldsymbol{r}+8\boldsymbol{\rho}-6\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\boldsymbol{\rho}\Big)-\dot{R}_{\beta')u\sigma u}\Big(6\boldsymbol{r}^2+20\boldsymbol{r}\boldsymbol{\rho}+17\boldsymbol{\rho}^2-12\log\Big(\tfrac{2m}{\lambda\boldsymbol{\rho}}\Big)\\
&\times\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\Big)\Big]-3\boldsymbol{\rho}^2\Big[\dot{R}_{u\sigma u\sigma}g_{\alpha'\beta'}(\boldsymbol{r}-2\boldsymbol{\rho})+2\dot{R}_{\alpha' u\beta' u}(3\boldsymbol{r}-2\boldsymbol{\rho})\boldsymbol{\rho}(\boldsymbol{r}+\boldsymbol{\rho})\\
&-6\boldsymbol{r}\boldsymbol{\rho}\dot{R}_{u(\alpha'\beta')\sigma}\Big]\Big)\Big]+\mathcal{O}\Big(\lambda^2\Big). \qquad (6.33)
\end{aligned}
$$

This again satisfies the appropriate Einstein field equations,

$$
\delta G^{\mu\nu}[h^{\mathrm{SS}}]+\delta^2 G^{\mu\nu}[h^{\mathrm{S1}\not{a}},h^{\mathrm{S1}\not{a}}]=\mathcal{O}\Big(\lambda^0\Big),\quad x\notin\gamma. \qquad (6.34)
$$

The first-order singular field with linear acceleration terms is

$$
\begin{aligned}
h^{\mathrm{S1}\boldsymbol{a}}_{\alpha\beta} = &\frac{g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta}}{3\boldsymbol{\rho}^5}\Big[3\lambda^0\Big(2\boldsymbol{r}\boldsymbol{\rho}^2(\boldsymbol{r}+2\boldsymbol{\rho})a_{(\alpha'}\Big(\sigma_{\beta')}+(\boldsymbol{r}+\boldsymbol{\rho})u_{\beta')}\Big)+a_\sigma\Big(-3\boldsymbol{r}^2\sigma_{\alpha'}\sigma_{\beta'}\\
&+2(-3\boldsymbol{r}^3-2\boldsymbol{r}^2\boldsymbol{\rho}+2\boldsymbol{r}\boldsymbol{\rho}^2+2\boldsymbol{\rho}^3)\sigma_{(\alpha'}u_{\beta')}-(\boldsymbol{r}+\boldsymbol{\rho})(3\boldsymbol{r}^3+\boldsymbol{r}^2\boldsymbol{\rho}-4\boldsymbol{r}\boldsymbol{\rho}^2-4\boldsymbol{\rho}^3)\\
&\cdot u_{\alpha'}u_{\beta'}\Big)\Big)+\lambda\Big(2\dot{a}_{(\alpha'}\boldsymbol{r}^2\boldsymbol{\rho}^2(\boldsymbol{r}+3\boldsymbol{\rho})\Big(\sigma_{\beta')}+(\boldsymbol{r}+\boldsymbol{\rho})u_{\beta)'}\Big)+\dot{a}_\sigma\boldsymbol{r}\Big(-3\boldsymbol{r}^2\sigma_{\alpha'}\sigma_{\beta'}\\
&+2(-3\boldsymbol{r}^3-2\boldsymbol{r}^2\boldsymbol{\rho}+3\boldsymbol{r}\boldsymbol{\rho}^2+6\boldsymbol{\rho}^3)\sigma_{(\alpha'}u_{\beta')}-(\boldsymbol{r}+\boldsymbol{\rho})(3\boldsymbol{r}^3+\boldsymbol{r}^2\boldsymbol{\rho}-6\boldsymbol{r}\boldsymbol{\rho}^2-12\boldsymbol{\rho}^3)\\
&\cdot u_{\alpha'}u_{\beta'}\Big)\Big)\Big]+\mathcal{O}\Big(\lambda^2\Big), \qquad (6.35)
\end{aligned}
$$

while the SR piece of the second-order singular field is

$$\begin{aligned}
h^{\rm SR}_{\alpha\beta} = &-\frac{m g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta}}{2\boldsymbol{\rho}^5}\Bigg[\frac{2}{\lambda}\Bigg[4\boldsymbol{\rho}^2\Big(h^{\rm R1}_{(\alpha'|\sigma|}\sigma_{\beta')} + (\boldsymbol{r}+\boldsymbol{\rho})\Big(h^{\rm R1}_{(\alpha'|u|}\sigma_{\beta')} + h^{\rm R1}_{(\alpha'|\sigma|}u_{\beta')}\\
&+ (\boldsymbol{r}+\boldsymbol{\rho})h^{\rm R1}_{(\alpha'|u|}u_{\beta')}\Big)\Big) - h^{\rm R1}_{\sigma\sigma}\Big(3\sigma_{\alpha'}\sigma_{\beta'} + (\boldsymbol{r}+\boldsymbol{\rho})(3\boldsymbol{r}+\boldsymbol{\rho})u_{\alpha'}u_{\beta'}\\
&+ 2(3\boldsymbol{r}+2\boldsymbol{\rho})\sigma_{(\alpha'}u_{\beta')}\Big) - h^{\rm R1}_{\sigma u}\Big(6\boldsymbol{r}\sigma_{\alpha'}\sigma_{\beta'} + 2(\boldsymbol{r}+\boldsymbol{\rho})\Big((3\boldsymbol{r}-2\boldsymbol{\rho})(\boldsymbol{r}+\boldsymbol{\rho})\\
&\cdot u_{\alpha'}u_{\beta'} + 2(3\boldsymbol{r}-\boldsymbol{\rho})\sigma_{(\alpha'}u_{\beta')}\Big)\Big) - h^{\rm R1}_{uu}\Big(3\boldsymbol{r}^2\sigma_{\alpha'}\sigma_{\beta'} + (\boldsymbol{r}+\boldsymbol{\rho})\Big((\boldsymbol{r}+\boldsymbol{\rho})\\
&\times(3\boldsymbol{r}^2-2\boldsymbol{r}\boldsymbol{\rho}-2\boldsymbol{\rho}^2)u_{\alpha'}u_{\beta'} + 2(3\boldsymbol{r}^2-\boldsymbol{r}\boldsymbol{\rho}-\boldsymbol{\rho}^2)\sigma_{(\alpha'}u_{\beta')}\Big)\Big)\Bigg]\\
&+\lambda^0\Bigg[h^{\rm R1}_{\sigma\sigma;\sigma}\Big(-3\sigma_{\alpha'}\sigma_{\beta'} - (\boldsymbol{r}+\boldsymbol{\rho})(3\boldsymbol{r}+\boldsymbol{\rho})u_{\alpha'}u_{\beta'} - 2(3\boldsymbol{r}+2\boldsymbol{\rho})\sigma_{(\alpha'}u_{\beta')}\Big)\\
&+\boldsymbol{r}h^{\rm R1}_{uu;u}\Big(3\boldsymbol{r}^2\sigma_{\alpha'}\sigma_{\beta'} + (\boldsymbol{r}+\boldsymbol{\rho})(3\boldsymbol{r}^3+\boldsymbol{r}^2\boldsymbol{\rho}-6\boldsymbol{r}\boldsymbol{\rho}^2-8\boldsymbol{\rho}^3)u_{\alpha'}u_{\beta'}\\
&+2(3\boldsymbol{r}^3+2\boldsymbol{r}^2\boldsymbol{\rho}-3\boldsymbol{r}\boldsymbol{\rho}^2-4\boldsymbol{\rho}^3)\sigma_{(\alpha'}u_{\beta')}\Big) + h^{\rm R1}_{\sigma u;u}\Big(6\boldsymbol{r}^2\sigma_{\alpha'}\sigma_{\beta'} + 2(\boldsymbol{r}+\boldsymbol{\rho})\\
&\times(3\boldsymbol{r}^3+\boldsymbol{r}^2\boldsymbol{\rho}-4\boldsymbol{r}\boldsymbol{\rho}^2-4\boldsymbol{\rho}^3)u_{\alpha'}u_{\beta'} + 4(3\boldsymbol{r}^3+2\boldsymbol{r}^2\boldsymbol{\rho}-2\boldsymbol{r}\boldsymbol{\rho}^2-2\boldsymbol{\rho}^3)\sigma_{(\alpha'}u_{\beta')}\Big)\\
&+h^{\rm R1}_{uu;\sigma}\Big(-3\boldsymbol{r}^2\sigma_{\alpha'}\sigma_{\beta'} - (\boldsymbol{r}+\boldsymbol{\rho})(3\boldsymbol{r}^3+\boldsymbol{r}^2\boldsymbol{\rho}-4\boldsymbol{r}\boldsymbol{\rho}^2-4\boldsymbol{\rho}^3)u_{\alpha'}u_{\beta'}\\
&-2(3\boldsymbol{r}^3+2\boldsymbol{r}^2\boldsymbol{\rho}-2\boldsymbol{r}\boldsymbol{\rho}^2-2\boldsymbol{\rho}^3)\sigma_{(\alpha'}u_{\beta')}\Big) - 2h^{\rm R1}_{\sigma u;\sigma}\Big(3\boldsymbol{r}\sigma_{\alpha'}\sigma_{\beta'} + (\boldsymbol{r}+\boldsymbol{\rho})\\
&\times\Big((3\boldsymbol{r}-2\boldsymbol{\rho})(\boldsymbol{r}+\boldsymbol{\rho})u_{\alpha'}u_{\beta'} + 2(3\boldsymbol{r}-\boldsymbol{\rho})\sigma_{(\alpha'}u_{\beta')}\Big)\Big) + h^{\rm R1}_{\sigma\sigma;u}\Big(3\boldsymbol{r}\sigma_{\alpha'}\sigma_{\beta'}\\
&+(\boldsymbol{r}+\boldsymbol{\rho})\Big((3\boldsymbol{r}-2\boldsymbol{\rho})(\boldsymbol{r}+\boldsymbol{\rho})u_{\alpha'}u_{\beta'} + 2(3\boldsymbol{r}-\boldsymbol{\rho})\sigma_{(\alpha'}u_{\beta')}\Big)\Big)\\
&+2\boldsymbol{\rho}^2\Big(\sigma_{(\alpha'}h^{\rm R1}_{|\sigma\sigma|;\beta')} + 2(\boldsymbol{r}+\boldsymbol{\rho})\sigma_{(\alpha'}h^{\rm R1}_{|\sigma u|;\beta')} + \boldsymbol{r}(\boldsymbol{r}+2\boldsymbol{\rho})\sigma_{(\alpha'}h^{\rm R1}_{|uu|;\beta')}\\
&+2\sigma_{(\alpha'}h^{\rm R1}_{\beta')\sigma;\sigma} + 2\boldsymbol{r}\sigma_{(\alpha'}h^{\rm R1}_{\beta')u;\sigma} + 2\boldsymbol{\rho}\sigma_{(\alpha'}h^{\rm R1}_{\beta')u;\sigma} - 2\boldsymbol{r}\sigma_{(\alpha'}h^{\rm R1}_{\beta')\sigma;u} - 2\boldsymbol{\rho}\sigma_{(\alpha'}h^{\rm R1}_{\beta')\sigma;u}\\
&-2\boldsymbol{r}^2\sigma_{(\alpha'}h^{\rm R1}_{\beta')u;u} - 4\boldsymbol{r}\boldsymbol{\rho}\sigma_{(\alpha'}h^{\rm R1}_{\beta')u;u} + \boldsymbol{r}u_{(\alpha'}h^{\rm R1}_{|\sigma\sigma|;\beta')} + \boldsymbol{\rho}u_{(\alpha'}h^{\rm R1}_{|\sigma\sigma|;\beta')}\\
&+2\boldsymbol{r}^2u_{(\alpha'}h^{\rm R1}_{|\sigma u|;\beta')} + 4\boldsymbol{r}\boldsymbol{\rho}u_{(\alpha'}h^{\rm R1}_{|\sigma u|;\beta')} + 2\boldsymbol{\rho}^2u_{(\alpha'}h^{\rm R1}_{|\sigma u|;\beta')} + \boldsymbol{r}^3u_{(\alpha'}h^{\rm R1}_{|uu|;\beta')}\\
&+3\boldsymbol{r}^2\boldsymbol{\rho}u_{(\alpha'}h^{\rm R1}_{|uu|;\beta')} + 2\boldsymbol{r}\boldsymbol{\rho}^2u_{(\alpha'}h^{\rm R1}_{|uu|;\beta')} + 2\boldsymbol{r}u_{(\alpha'}h^{\rm R1}_{\beta')\sigma;\sigma} + 2\boldsymbol{\rho}u_{(\alpha'}h^{\rm R1}_{\beta')\sigma;\sigma}\\
&+2\boldsymbol{r}^2u_{(\alpha'}h^{\rm R1}_{\beta')u;\sigma} + 4\boldsymbol{r}\boldsymbol{\rho}u_{(\alpha'}h^{\rm R1}_{\beta')u;\sigma} + 2\boldsymbol{\rho}^2u_{(\alpha'}h^{\rm R1}_{\beta')u;\sigma} - 2\boldsymbol{r}^2u_{(\alpha'}h^{\rm R1}_{\beta')\sigma;u}\\
&-4\boldsymbol{r}\boldsymbol{\rho}u_{(\alpha'}h^{\rm R1}_{\beta')\sigma;u} - 2\boldsymbol{\rho}^2u_{(\alpha'}h^{\rm R1}_{\beta')\sigma;u} - 2\boldsymbol{r}(\boldsymbol{r}+\boldsymbol{\rho})(\boldsymbol{r}+2\boldsymbol{\rho})u_{(\alpha'}h^{\rm R1}_{\beta')u;u}\Big)\Bigg]\Bigg] + \mathcal{O}(\lambda).
\end{aligned}\tag{6.36}$$

These need to satisfy

$$\delta G^{\mu\nu}[h^{\rm SR}] + \delta G^{\mu\nu}[h^{{\rm S1}\boldsymbol{a}}] + 2\delta^2 G^{\mu\nu}[h^{\rm R1}, h^{{\rm S1}\not{\boldsymbol{a}}}] = \mathcal{O}\Big(\lambda^0\Big), \quad x\notin\gamma. \tag{6.37}$$

We have successfully checked that the covariant punctures for $h^{\rm SR}_{\mu\nu}$ and $h^{{\rm S1},\boldsymbol{a}}_{\mu\nu}$ satisfy Eq. (6.37) through the leading two orders, λ^{-3} and λ^{-2}. However, we have not been able to check this at the highest order we have calculated, order λ^{-1}. This is due to the complexity and length of the expressions when taking multiple different combinations of derivatives. Despite this, we provide all orders of the covariant punctures for the different singular field terms in a Mathematica notebook in the Additional Material [141].

Comparing the covariant puncture for $h^{\rm S1}_{\mu\nu}$ from Eq. (6.31) to the Lorenz gauge version of the puncture from Eq. (127) of Ref. [96],

$$h^{\mathrm{S1}\not{d},\mathrm{Lor}}_{\alpha\beta} = \frac{2m}{\lambda\boldsymbol{\rho}} g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta}\left(g_{\alpha'\beta'} + 2u_{\alpha'}u_{\beta'}\right) + \mathcal{O}(\lambda), \tag{6.38}$$

we see that the highly regular gauge puncture has a more complicated form. This continues at higher order with the Lorenz gauge puncture being substantially simpler and shorter at all orders. The more complex form results from the highly regular gauge conditions that seek to preserve the background lightcone structure emanating from the worldline in the perturbed spacetime; see Ch. 7.1 for further discussion. This has the knock-on effect that the coordinate expansion in the highly regular gauge will be much more complicated than the Lorenz gauge one as we are introducing more and more terms, and more quantities will need to be expanded. Thus, if we wanted to perform a mode decomposition of the singular field in the highly regular gauge, we would find that the process is likely to be more complicated than in the Lorenz gauge due to an increase in the number of quantities that need to be decomposed into modes. However, we believe that the benefits of the highly regular gauge outweigh any disadvantages that may come from the metric perturbations having a more complicated structure. Merely eliminating the two leading orders of $h^{\rm SS}_{\mu\nu}$ in Eq. (6.33) has dramatic consequences as it alleviates the problem of infinite mode coupling [132] that was discussed in the introduction. This should allow one to much more efficiently calculate modes of the second-order source for whatever calculation is being performed.

6.3 Coordinate expansion of covariant puncture

With our covariant punctures derived, we can proceed to write them as a generic coordinate expansion using the techniques discussed for the singular scalar field in Ch. 2.2.1.2. This will allow them to be easily written in any desired coordinate system.

To do so, we substitute our coordinate expansion for $\sigma_{\alpha'}$ from Eq. (2.95)–(2.96), $\boldsymbol{\rho}$ from Eqs. (2.102)–(2.104) and $g^{\mu'}{}_{\mu}$ from Eqs. (2.106)–(2.107) into the expression for $h^{\rm S}_{\mu\nu}$ from Ch. 6.2.3. Doing so results in expressions that are written in terms of the coordinate difference, $\Delta x^{\mu'}$ and the four-velocity, $u^{\mu'}$ along with $h^{\rm R1}_{\mu\nu}$, $\Gamma^{\mu}_{\nu\rho}$, and $R_{\alpha'\beta'\mu'\nu'}$ and their respective derivatives. The final expressions are incredibly long and, as such, we only display them through order λ^0 (except for $h^{\rm SR}_{\mu\nu}$, for which we just display the leading-order term). The higher order terms are available in the Additional Material in a Mathematica notebook [141].

The coordinate expansion of the first-order singular field, with no acceleration, in the highly regular gauge is given by

$$\begin{aligned}h^{\mathrm{S1}\not{a}}_{\mu\nu} = {}& \frac{2m}{\lambda\boldsymbol{\rho}_0^3}\Big(\Delta x_{\mu'} + u_{\mu'}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\Big)\Big(\Delta x_{\nu'} + u_{\nu'}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\Big) - \frac{m\lambda^0}{\boldsymbol{\rho}_0^5}\Big[u_{\mu'}u_{\nu'}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\\
&\times\Big(3\boldsymbol{r}_0(\Gamma^{\Delta}_{\Delta\Delta}+\Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0) - (\Gamma^{\Delta}_{\Delta\Delta}+\Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0)\boldsymbol{\rho}_0 - 2\Gamma^{u}_{\Delta\Delta}\boldsymbol{\rho}_0^2\Big) + 3\Delta x_{\mu'}\Delta x_{\nu'}\\
&\times(\Gamma^{\Delta}_{\Delta\Delta}+\Gamma^{u}_{\Delta\Delta}\boldsymbol{r}_0) + 2u_{(\mu'}\Delta x_{\nu')}\Big(\Gamma^{\Delta}_{\Delta\Delta}(3\boldsymbol{r}_0-2\boldsymbol{\rho}_0) + \Gamma^{u}_{\Delta\Delta}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)(3\boldsymbol{r}_0+\boldsymbol{\rho}_0)\Big)\\
&- 2u_{(\mu'}\Big(2\Gamma^{\Delta}_{\nu')\Delta} + g_{\nu')\alpha'}\Gamma^{\alpha'}_{\Delta\Delta} + 2\Gamma^{u}_{\nu')\Delta}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\Big)(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\boldsymbol{\rho}_0^2\\
&- 2\Delta x_{(\mu'}\boldsymbol{\rho}_0^2\Big(2\Gamma^{\Delta}_{\nu')\Delta} + g_{\nu')\alpha'}\Gamma^{\alpha'}_{\Delta\Delta} + 2\Gamma^{u}_{\nu')\Delta}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\Big)\Big] + \mathcal{O}(\lambda).\end{aligned}\tag{6.39}$$

Moving to second order, the first-order singular field with acceleration is

$$\begin{aligned}h^{\mathrm{S1}\boldsymbol{a}}_{\mu\nu} = {}& -\frac{m\lambda^0}{\boldsymbol{\rho}_0^5}\Big[a_\Delta u_{\mu'}u_{\nu'}(4\boldsymbol{r}_0{}^3\boldsymbol{\rho}_0 + 3\boldsymbol{r}_0{}^2\boldsymbol{\rho}_0^2 - 8\boldsymbol{r}_0\boldsymbol{\rho}_0^3 + 4\boldsymbol{\rho}_0^4 - 3\boldsymbol{r}_0{}^4) - 3\Delta x_{\mu'}\Delta x_{\nu'}a_\Delta\boldsymbol{r}_0{}^2\\
&+ 2\Big(\boldsymbol{r}_0(\boldsymbol{r}_0-2\boldsymbol{\rho}_0)\boldsymbol{\rho}_0^2\Delta x_{(\mu'}a_{\nu')} + a_\Delta(2\boldsymbol{r}_0{}^2\boldsymbol{\rho}_0 + 2\boldsymbol{r}_0\boldsymbol{\rho}_0^2 - 2\boldsymbol{\rho}_0^3 - 3\boldsymbol{r}_0{}^3)\Delta x_{(\mu'}u_{\nu')}\\
&+ \boldsymbol{r}_0(\boldsymbol{r}_0-2\boldsymbol{\rho}_0)(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\boldsymbol{\rho}_0^2 a_{(\mu'}u_{\nu')}\Big)\Big] + \mathcal{O}(\lambda).\end{aligned}\tag{6.40}$$

The ‘singular times singular’ piece is given by

$$\begin{aligned}h^{\mathrm{SS}}_{\mu\nu} = {}& -\frac{2m^2\lambda^0}{3\boldsymbol{\rho}_0^4}\Big[6\boldsymbol{\rho}_0^3 R_{(\mu'|\Delta|\nu')u} + 2(3\boldsymbol{r}_0-\boldsymbol{\rho}_0)\boldsymbol{\rho}_0^3 R_{(\mu'|u|\nu')u} + R_{\Delta u\Delta u}\Big(5\Delta x_{\mu'}\Delta x_{\nu'}\\
&- g_{\mu'\nu'}\boldsymbol{\rho}_0^2 + u_{\mu'}u_{\nu'}(5\boldsymbol{r}_0{}^2 - 6\boldsymbol{r}_0\boldsymbol{\rho}_0 + 5\boldsymbol{\rho}_0^2) + (10\boldsymbol{r}_0 - 6\boldsymbol{\rho}_0)\Delta x_{(\mu'}u_{\nu')}\Big)\\
&- 6\boldsymbol{\rho}_0\Delta x_{(\mu'}R_{\nu')\Delta\Delta u} + (-6\boldsymbol{r}_0\boldsymbol{\rho}_0 + 4\boldsymbol{\rho}_0^2)\Delta x_{(\mu'}R_{\nu')u\Delta u} - 6\boldsymbol{r}_0\boldsymbol{\rho}_0 u_{(\mu'}R_{\nu')\Delta\Delta u}\\
&- 2\boldsymbol{\rho}_0(3\boldsymbol{r}_0{}^2 - 2\boldsymbol{r}_0\boldsymbol{\rho}_0 + 3\boldsymbol{\rho}_0^2)u_{(\mu'}R_{\nu')u\Delta u}\Big] + \mathcal{O}(\lambda).\end{aligned}\tag{6.41}$$

Finally, the ‘singular times regular’ piece is

$$\begin{aligned}h^{\mathrm{SR}}_{\mu\nu} = {}& \frac{m}{\lambda\boldsymbol{\rho}_0^5}\Big[4\boldsymbol{\rho}_0^2\Delta x_{(\mu'}h^{\mathrm{R1}}_{\nu')\Delta} + 4(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\boldsymbol{\rho}_0^2\Delta x_{(\mu'}h^{\mathrm{R1}}_{\nu')u} - h^{\mathrm{R1}}_{\Delta u}\Big(6\Delta x_{\mu'}\Delta x_{\nu'}\boldsymbol{r}_0\\
&+ 2u_{\mu'}u_{\nu'}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)^2(3\boldsymbol{r}_0+2\boldsymbol{\rho}_0) + 4(\boldsymbol{r}_0-\boldsymbol{\rho}_0)(3\boldsymbol{r}_0+\boldsymbol{\rho}_0)\Delta x_{(\mu'}u_{\nu')}\Big)\\
&- h^{\mathrm{R1}}_{\Delta\Delta}\Big(3\Delta x_{\mu'}\Delta x_{\nu'} + u_{\mu'}u_{\nu'}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)(3\boldsymbol{r}_0-\boldsymbol{\rho}_0) + (6\boldsymbol{r}_0-4\boldsymbol{\rho}_0)\Delta x_{(\mu'}u_{\nu')}\Big)\\
&- h^{\mathrm{R1}}_{uu}\Big(3\Delta x_{\mu'}\Delta x_{\nu'}\boldsymbol{r}_0{}^2 + u_{\mu'}u_{\nu'}(\boldsymbol{r}_0-\boldsymbol{\rho}_0)^2(3\boldsymbol{r}_0{}^2 + 2\boldsymbol{r}_0\boldsymbol{\rho}_0 - 2\boldsymbol{\rho}_0^2)\\
&+ 2(\boldsymbol{r}_0-\boldsymbol{\rho}_0)(3\boldsymbol{r}_0{}^2 + \boldsymbol{r}_0\boldsymbol{\rho}_0 - \boldsymbol{\rho}_0^2)\Delta x_{(\mu'}u_{\nu')}\Big) + 4(\boldsymbol{r}_0-\boldsymbol{\rho}_0)\boldsymbol{\rho}_0^2 h^{\mathrm{R1}}_{(\mu'|\Delta|}u_{\nu')}\\
&+ 4(\boldsymbol{r}_0-\boldsymbol{\rho}_0)^2\boldsymbol{\rho}_0^2 h^{\mathrm{R1}}_{(\mu'|u|}u_{\nu')}\Big] + \mathcal{O}\Big(\lambda^0\Big).\end{aligned}\tag{6.42}$$

Chapter 7

Local gauge transformation from the Lorenz gauge to the highly regular gauge for quasicircular orbits in Schwarzschild spacetime

This chapter deals with the transformation from the Lorenz gauge to the highly regular gauge. As has been previously discussed, Ref. [105] performed the first calculations of a second-order self-force quantity, that being the binding energy in a quasicircular orbit around a Schwarzschild black hole, using quantities defined in the Lorenz gauge. However, these calculations required significant computing time even for this simple case and would become likely impossible to compute when moving to the astrophysically realistic scenario of generic orbits in Kerr. We therefore wish to transform to the highly regular gauge and take advantage of the properties that have been demonstrated and discussed in this report so far. These calculations form part of a broader, international effort to solve the second-order self-force problem in time for LISA's launch in 2037. As data for quasicircular orbits in Schwarzschild already exists in the Lorenz gauge [105], we seek to find a gauge transformation to take us to the highly regular gauge. Specifically, we transform the first-order singular field to the highly regular gauge to take advantage of the weaker divergences as discussed in Ch. 5.

This chapter is organised as follows: firstly, we determine a covariant expression for the gauge vector to transform us between the two gauges in Ch. 7.2, which is then expanded in terms of coordinates in Ch. 7.3. After this, in Ch. 7.4, we determine the components of the gauge vector for the specific case of quasicircular orbits in Schwarzschild to allow direct comparison with the previously mentioned existing Lorenz gauge data. To do this, we introduce a rotated coordinate system in which the small object is instantaneously at the north pole. We then utilise the Newman–Penrose formalism and decompose the gauge vector into a spin-weighted spherical harmonic basis. The rotated coordinate system has the advantage that when we perform the decomposition in Ch. 7.5, only the lowest m mode is non-zero (we will find that we need to calculate some higher modes to take account of derivatives that appear). We then rotate back to the original coordinate system in Ch. 7.6 and construct the metric perturbations in Ch. 7.7.

7.1 Gauge conditions for the highly regular gauge

The gauge conditions for the highly regular gauge were previously given in Eqs. (3.57)–(3.58) but we repeat them here. There are two gauge conditions imposed on the singular field: firstly, that the contraction of the perturbation with a null vector is identically zero,

$$h^{\mathrm{HR,S}}_{\mu\nu} k^{\nu} = 0, \tag{7.1}$$

where k^{μ} is a future directed null vector that is tangent to the lightcone of the small object; and secondly, that the trace over the angular components of the perturbation vanishes,

$$h^{\mathrm{HR,S}}_{AB} \Omega^{AB} = 0, \tag{7.2}$$

where Ω_{AB} is the metric on a surface of constant luminosity distance and θ^{A} are coordinates on this surface. These gauge conditions have the geometrical interpretation of ensuring that the background luminosity distance is still an affine parameter along null rays emanating from the particle in the perturbed spacetime and that lightcones and the element surfaces of constant luminosity distance are the same in the background and perturbed spacetimes. Both of these gauge conditions are inherited from the lightcone rest gauge, defined in Ref. [174], that the highly regular gauge is built off of; see the discussion around Eq. (3.57). We emphasise again that we are only imposing the gauge conditions on the singular field.

As we are looking to make a local gauge transformation, we seek an expansion of the gauge conditions in terms of distance from the worldline, λ. To do so, we can use the covariant expansion methods detailed in Ch. 2.2.1.1.

To find the form of the null vector, k^{μ}, at x^{μ} we decompose it into pieces parallel and orthogonal to the worldline, so that

$$k^{\mu}(x) = g^{\mu}{}_{\mu'} \left(k_{\parallel} u^{\mu'} + k_{\perp}^{\nu'} P^{\mu'}{}_{\nu'} \right), \tag{7.3}$$

where we can expand

$$k_{\parallel} = k_{\parallel}^{(0)} + \lambda k_{\parallel}^{(1)} + \mathcal{O}\left(\lambda^{2}\right), \tag{7.4}$$

$$k_{\perp}^{\mu'} = k_{\perp}^{(0)\mu'} + \lambda k_{\perp}^{(1)\mu'} + \mathcal{O}\left(\lambda^{2}\right), \tag{7.5}$$

in terms of λ. We then contract it with itself to see

$$\begin{aligned} g_{\mu\nu}k^{\mu}k^{\nu} &= g_{\mu'\nu'}\left(k_{\parallel}u^{\mu'} + k_{\perp}^{\alpha'}P^{\mu'}{}_{\alpha'}\right)\left(k_{\parallel}u^{\nu'} + k_{\perp}^{\beta'}P^{\nu'}{}_{\beta'}\right) \\ &= -\left(k_{\parallel}\right)^2 + P_{\mu'\nu'}k_{\perp}^{\mu'}k_{\perp}^{\nu'} \\ &= -\lambda^0\Big[\left(k_{\parallel}^{(0)}\right)^2 - P_{\mu'\nu'}k_{\perp}^{(0)\mu'}k_{\perp}^{(0)\nu'}\Big] - 2\lambda\Big[k_{\parallel}^{(0)}k_{\parallel}^{(1)} - P_{\mu'\nu'}k_{\perp}^{(0)\mu'}k_{\perp}^{(1)\nu'}\Big] \\ &\quad + \mathcal{O}\left(\lambda^2\right). \end{aligned} \tag{7.6}$$

For this to be null, we require Eq. (7.6) to be 0. As null vectors have no unique length, we are free to choose $k_{\parallel}^{0} = 1$, so that

$$P_{\mu'\nu'}k_{\perp}^{(0)\mu'}k_{\perp}^{(0)\nu'} = 1, \tag{7.7}$$

which we can show has a solution

$$k_{\perp}^{(0)\mu'} = -\frac{P^{\mu'}{}_{\nu'}\sigma^{\nu'}}{\boldsymbol{\rho}}, \tag{7.8}$$

using $P_{\mu'\alpha'}P^{\alpha'}{}_{\nu'} = P_{\mu'\nu'}$ and the definition of $\boldsymbol{\rho}$ from Eq. (2.98).

Moving to the next order in λ, we arrive at the condition

$$k_{\parallel}^{(0)}k_{\parallel}^{(1)} = P_{\mu'\nu'}k_{\perp}^{(0)\mu'}k_{\perp}^{(1)\nu'}. \tag{7.9}$$

We immediately see that we have freedom to set both of the first-order terms to 0. This would then continue at each higher order in λ which means that all higher order terms are 0. This gives us an exact expression for our null vector,

$$k^{\mu} = g^{\mu}{}_{\mu'}\left(u^{\mu'} - \frac{P^{\mu'}{}_{\nu'}\sigma^{\nu'}}{\boldsymbol{\rho}}\right). \tag{7.10}$$

The first-order singular field in the highly regular gauge can be written in terms of a gauge transformation from the Lorenz gauge as

$$h_{\mu\nu}^{\mathrm{S1}} = h_{\mu\nu}^{\mathrm{S1}*} + \mathcal{L}_{\xi}g_{\mu\nu}, \tag{7.11}$$

where the first-order singular field in the Lorenz gauge can be written covariantly as [96, 138]

$$\begin{aligned} h_{\mu\nu}^{\mathrm{S1}*} = \frac{2m}{\lambda\boldsymbol{\rho}}g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}(g_{\mu'\nu'} + 2u_{\mu'}u_{\nu'}) + \frac{m\lambda^0}{\boldsymbol{\rho}^3}g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}[(\boldsymbol{\rho}^2 - \boldsymbol{r}^2)a_{\sigma}(g_{\mu'\nu'} + 2u_{\mu'}u_{\nu'}) \\ + 8\boldsymbol{r}\boldsymbol{\rho}^2 a_{(\mu'}u_{\nu')}] + \mathcal{O}(\lambda). \end{aligned} \tag{7.12}$$

Applying our gauge conditions from Eqs. (7.1) and (7.2) to Eq. (7.11), we see that

$$(h^{\mathrm{S1}*}_{\mu\nu} + 2\xi_{(\mu;\nu)})k^{\nu} = 0 \tag{7.13}$$

and

$$(h^{\mathrm{S1}*} + 2\xi_{(\mu;\nu)})e^{\mu}_{A}e^{\nu}_{B}\Omega^{AB} = 0 \tag{7.14}$$

respectively. We must then solve for the gauge vector that enforces these conditions.

7.2 Covariant expression for the gauge vector

To ensure that the highly regular gauge condition is satisfied on the worldline, we solve for the gauge vector through order λ by using the ansatz

$$\xi_{\mu} = g_{\mu}{}^{\mu'}\left[\ln(\boldsymbol{\rho})X^{0}_{\mu'} + Y^{0}_{\mu'} + \frac{Z^{0}_{\mu'\nu'}\sigma^{\nu'}}{\boldsymbol{\rho}} + \lambda\left(\boldsymbol{\rho}\ln(\lambda\boldsymbol{\rho})X^{1}_{\mu'} + \boldsymbol{\rho}Y^{1}_{\mu'} + Z^{1}_{\mu'\nu'}\sigma^{\nu'}\right)\right] + \mathcal{O}\left(\lambda^{2}\right), \tag{7.15}$$

where the numerical superscript indicates the order in λ that the terms appear at. This method was previously used by Spiers [142] to find part of the leading-order piece of the gauge vector. Note, for generality, we do not assume that $Z^{n}_{\mu'\nu'}$ is symmetric. However, as the singular field of the Lorenz gauge only features acceleration terms (which when expanded become order-ϵ^2 terms) at order λ^0, we can immediately set

$$X^{1}_{\mu'} = Y^{1}_{\mu'} = Z^{1}_{\mu'\nu'} = 0, \tag{7.16}$$

leaving us with

$$\xi_{\mu} = g_{\mu}{}^{\mu'}\left(\ln(\boldsymbol{\rho})X^{0}_{\mu'} + Y^{0}_{\mu'} + \frac{Z^{0}_{\mu'\nu'}\sigma^{\nu'}}{\boldsymbol{\rho}}\right) + \mathcal{O}\left(\lambda^{2}\right). \tag{7.17}$$

When differentiating this expression, we arrive at

$$\xi_{\mu;\nu} = -\frac{g_{\mu}{}^{\mu'}g_{\nu}{}^{\nu'}}{\lambda\boldsymbol{\rho}}\left(Z^{0}_{\mu'\nu'} + \frac{1}{\boldsymbol{\rho}}X^{0}_{\mu'}P_{\alpha'\nu'}\sigma^{\alpha'} - \frac{1}{\boldsymbol{\rho}^{2}}Z^{0}_{\mu'\alpha'}P_{\beta'\nu'}\sigma^{\alpha'}\sigma^{\beta'}\right) + \mathcal{O}(\lambda). \tag{7.18}$$

Eq. (7.13) becomes

$$\begin{aligned}0 = &- g_{\mu}{}^{\mu'}\Bigg[\frac{1}{\boldsymbol{\rho}}\left(2mu_{\mu'} - X^{0}_{\mu'} + 2u^{\alpha'}Z^{0}_{(\mu'\alpha')}\right) + \frac{1}{\boldsymbol{\rho}^{2}}(P_{\alpha'\mu'}\sigma^{\alpha'}\left\{2m + u^{\beta'}X^{0}_{\beta'}\right\} \\ &+ Z^{0}_{\mu'\alpha'}\sigma^{\alpha'} - 2P_{\beta'}{}^{\nu'}\sigma^{\beta'}Z^{0}_{(\mu'\nu')}) - \frac{1}{\boldsymbol{\rho}^{3}}\left(P_{\mu'\beta'}\sigma^{\beta'}\sigma^{\alpha'}\left\{P_{\alpha'}{}^{\nu'}X^{0}_{\nu'} + u^{\nu'}Z^{0}_{\nu'\alpha'}\right\}\right) \\ &+ \frac{1}{\boldsymbol{\rho}^{4}}P_{\mu'\nu'}P_{\beta'}{}^{\iota'}\sigma^{\alpha'}\sigma^{\beta'}\sigma^{\nu'}Z^{0}_{\iota'\alpha'}\Bigg],\end{aligned} \tag{7.19}$$

where all terms are order λ^{-1}. By inspection, we see that this can be solved by setting

$$X^0_{\mu'} = 2mu_{\mu'}, \tag{7.20}$$
$$Z^0_{\mu'\nu'} = AP_{\mu'\nu'}, \tag{7.21}$$

where A is an unknown constant.

We now move onto the angular traceless condition from Eq. (7.2). To simplify this calculation, we rewrite the inverse of the four dimensional metric in terms of a null basis

$$g^{\mu\nu} = -N^\mu k^\nu - k^\mu N^\nu + \Omega^{AB} e^\mu_A e^\nu_B, \tag{7.22}$$

where k^μ is the same as the one given in Eq. (7.10) and N^μ is an undetermined null vector with conditions $N_\mu e^\mu_A = 0$ and $N^\mu k_\mu = -1$. Rearranging, we see that

$$\Omega^{AB} e^\mu_A e^\nu_B = g^{\mu\nu} + N^\mu k^\nu + k^\mu N^\nu. \tag{7.23}$$

We then combine this with the trace condition from Eq. (7.2) and the null vector condition from Eq. (7.1) to see that

$$h^{\rm HR}_{\mu\nu} e^\mu_A e^\nu_B \Omega^{AB} = h^{\rm HR}_{\mu\nu} g^{\mu\nu} = 0. \tag{7.24}$$

Therefore we can simply trace over Eq. (7.11) using the four dimensional metric to impose the final gauge condition. By explicitly expanding the projection operators in terms of the metric and four velocity, we find that

$$A = m. \tag{7.25}$$

Substituting our values for the previously undetermined coefficients into the gauge vector from Eq. (7.15), we get

$$\xi_\mu = g_\mu{}^{\mu'}\left[2m\ln(\boldsymbol{\rho})u_{\mu'} + Y^0_{\mu'} + \frac{m}{\boldsymbol{\rho}} P_{\mu'\nu'}\sigma^{\nu'}\right] + \mathcal{O}\left(\lambda^2\right). \tag{7.26}$$

Note, however, that $Y^0_{\mu'}$ is completely undetermined by the gauge condition. Therefore we are free to set it to zero. Our final gauge vector is then given by

$$\xi_\mu = g_\mu{}^{\mu'}\left[2m\ln(\boldsymbol{\rho})u_{\mu'} + \frac{m}{\boldsymbol{\rho}} P_{\mu'\nu'}\sigma^{\nu'}\right] + \mathcal{O}\left(\lambda^2\right). \tag{7.27}$$

The logarithm term in the gauge vector at leading order was originally found by Spiers [142], where $\boldsymbol{\rho} \sim \boldsymbol{\rho}_0$, $\sigma^{\alpha'} \sim \Delta x^{\alpha'}$ and $g_\nu{}^{\mu'} \sim \delta^{\mu'}_{\nu'}$. In this section, we have calculated the fully covariant form of the gauge vector through two orders and will expand it into a generic coordinate expansion in the next section.

7.3 Coordinate expansion of gauge vector

With our covariant expression for the gauge vector found, we now wish to convert it to coordinates that can be used in the problem at hand. To do so requires us to re-expand Eq. (7.27) in terms of the coordinate difference. This can be done using the methods from Ch. 2.2.1.2. Using the expansion for $\boldsymbol{\rho}$ derived in Eqs. (2.104a)–(2.104d), we can expand the functions of $\boldsymbol{\rho}$ that appear in the gauge vector as

$$\frac{1}{\boldsymbol{\rho}} = \frac{1}{\lambda\boldsymbol{\rho}_0}\left(1 - \frac{\lambda}{2\boldsymbol{\rho}_0^2}\Gamma^{\alpha'}_{\mu'\nu'}P_{\alpha'\beta'}\Delta x^{\mu'}\Delta x^{\nu'}\Delta x^{\beta'}\right) + \mathcal{O}(\lambda) \tag{7.28}$$

and

$$\ln(\boldsymbol{\rho}) = \ln(\boldsymbol{\rho}_0) + \frac{\lambda}{2\boldsymbol{\rho}_0^2}\Gamma^{\alpha'}_{\mu'\nu'}P_{\alpha'\beta'}\Delta x^{\mu'}\Delta x^{\nu'}\Delta x^{\beta'} + \mathcal{O}\Big(\lambda^2\Big). \tag{7.29}$$

Using the previous two expressions along with the expansions of Synge's world function, Eqs. (2.96a)–(2.96d), and the parallel propagator, Eqs. (2.107a)–(2.107c), the coordinate expansion of the gauge vector is given by

$$\begin{aligned}\xi_\mu = &\left(2m\ln(\boldsymbol{\rho}_0)u_{\mu'} - \frac{m}{\boldsymbol{\rho}_0}P_{\mu'\nu'}\Delta x^{\nu'}\right) + \lambda\Bigg(2m\ln(\boldsymbol{\rho}_0)\Gamma^{\alpha'}_{\mu'\nu'}\Delta x^{\nu'}u_{\alpha'} \\ &- \frac{m}{\boldsymbol{\rho}_0}\Gamma^{\alpha'}_{\mu'\beta'}\Delta x^{\beta'}\Delta x^{\nu'}P_{\alpha'\nu'} + \frac{m}{2\boldsymbol{\rho}_0^3}\Gamma^{\gamma'}_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\nu'}\Delta x^{\iota'}P_{\mu'\nu'}P_{\gamma'\iota'} \\ &+ \frac{m}{\boldsymbol{\rho}_0^2}\Gamma^{\iota'}_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\nu'}P_{\iota'\nu'}u_{\mu'} - \frac{m}{2\boldsymbol{\rho}_0}\Gamma^{\nu'}_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}P_{\mu'\nu'}\Bigg) + \mathcal{O}\Big(\lambda^2\Big)\end{aligned} \tag{7.30}$$

7.4 Gauge vector components in Schwarzschild spacetime

In this section, we detail the process of calculating the gauge vector's components in the Schwarzschild spacetime. We follow the methods previously presented in Refs. [137, 138, 143]. Before proceeding, as we are going to introduce a number of coordinate systems, we must ensure that our notation is clear. As we have completed the calculation for the form of the vector, we can dispense with the old notation that an unprimed index refers to a point in the field and a primed index refers to a point on the worldline. We proceed with the understanding that all quantities are defined on the worldline except for Δx^μ which refers to a coordinate difference. We will introduce three coordinate systems that only differ in the angular section: (θ, ϕ) coordinates, (α, β) coordinates and (w_1, w_2) coordinates. These three bases will be denoted with unprimed, primed and double primed indices, respectively.

The Schwarzschild solution [185, 186] in (t, r, θ, ϕ) coordinates is given by

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\,d\Omega^2 \tag{7.31}$$

where

$$f(r) := \frac{r-2M}{r} \tag{7.32}$$

and

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2\,. \tag{7.33}$$

The Schwarzschild manifold can be written as the Cartesian product, $\mathcal{M} = \mathcal{M}^2 \times S^2$, between the (t, r) plane, $\mathcal{M}^2$, and the unit two-sphere, S^2.

Following Refs. [137, 143], when considering a circular orbit in Schwarzschild, we can write the four-velocity in terms of the specific energy,

$$E_0 = f_0\sqrt{\frac{r_0}{r_0-3M}}, \tag{7.34}$$

and angular momentum,

$$L_0 = r_0\sqrt{\frac{M}{r_0-3M}}, \tag{7.35}$$

as

$$u_\mu = (-E_0, 0, 0, L_0), \tag{7.36}$$

where $f_0 := f(r_0)$ and r_0 refers to r evaluated at the location of the small object. Note that due to the spherical symmetry of the Schwarzschild spacetime, we can consider the orbit to be equatorial without any loss of generality. The azimuthal frequency of the orbit is given by

$$\Omega_\phi = \frac{d\phi_p}{dt} = \sqrt{\frac{M}{r_0^3}}. \tag{7.37}$$

7.4.1 Coordinate systems

As discussed in the introduction to this chapter, we want to introduce a coordinate system that instantaneously places the small object at the north pole. This method has previously been used in, e.g. Refs. [187, 188], as it allows us to only calculate a small number of m modes instead of the potentially hundreds that would be required for a non-rotated system. By placing the particle at the north pole, we see from Eq. (D.23) that all but the lowest m modes vanish (the exact one depends on the spin-weight being considered). The new (α, β) coordinate system is related to the standard Schwarzschild coordinates through the relations

$$\sin\theta\cos(\phi-\phi_p) = \cos\alpha, \tag{7.38a}$$

$$\sin\theta\sin(\phi-\phi_p) = \sin\alpha\cos\beta, \tag{7.38b}$$

$$\cos\theta = \sin\alpha\sin\beta. \tag{7.38c}$$

This transformation does not change the structural form of metric; i.e. we still have the metric in the form of Eq. (7.31) but with (α, β) replacing (θ, ϕ), but has the effect of moving the particle to the north pole. This is equivalent to an Euler angle rotation of $(\phi_p, \pi/2, \pi/2)$ (using the $z-y-z$ convention) [143].

Near the north pole, $\alpha = 0$, we follow the method detailed in Refs. [137, 138, 143] and adopt quasi-Cartesian Riemann normal coordinates on S^2,

$$w_1 = 2\sin\left(\frac{\alpha}{2}\right)\cos\beta, \tag{7.39a}$$

$$w_2 = 2\sin\left(\frac{\alpha}{2}\right)\sin\beta. \tag{7.39b}$$

which transform the metric to have the form

$$ds^2 = -\left(\frac{r-2M}{r}\right)dt^2 + \left(\frac{r}{r-2M}\right)dr^2 + r^2\left\{\left[\frac{16 - w_2^2(8 - w_1^2 - w_2^2)}{4(4 - w_1^2 - w_2^2)}\right]dw_1^2 + 2\left[\frac{w_1 w_2(8 - w_1^2 - w_2^2)}{4(4 - w_1^2 - w_2^2)}\right]dw_1\,dw_2 + \left[\frac{16 - w_1^2(8 - w_1^2 - w_2^2)}{4(4 - w_1^2 - w_2^2)}\right]dw_2^2\right\}. \tag{7.40}$$

Note that (w_1, w_2) are Riemann normal coordinates on S^2 but (t, r, w_1, w_2) are *not* Riemann normal coordinates on $\mathcal{M}$. We perform our calculations in this coordinate system to avoid the situation where the (α, β) coordinates become ill-defined on the worldline of the small object. The (w_1, w_2) coordinate differences feature trigonometric functions of α and β as well, simplifying the form of the integrals required in the mode decomposition. Geometrically, when we are close to the particle, we can think of (w_1, w_2) as coordinates on a two-dimensional Cartesian plane orthogonal to a sphere of constant Schwarzschild radius. To see this, note that for small α,

$$w_1 \sim \alpha\cos\beta, \tag{7.41a}$$

$$w_2 \sim \alpha\sin\beta, \tag{7.41b}$$

so α acts as a radial distance away from the particle with β acting as a polar angle coordinate. Therefore, changes in (w_1, w_2) refer to movements around this plane whereas a change in r gives a distance closer or further away to the central black hole.

Expressing the coordinate form of the gauge vector (7.30) in this basis requires us to calculate the form of the metric, the four-velocity and the coordinate difference in the (w_1, w_2) coordinate system. When evaluated on the particle, $r = r_0$ and $\alpha = 0$, the metric takes the form

$$ds^2|_p = -\left(\frac{r_0 - 2M}{r_0}\right)dt^2 + \left(\frac{r_0}{r_0 - 2M}\right)dr^2 + r_0^2(dw_1^2 + dw_2^2), \tag{7.42}$$

while the four-velocity in the (w_1, w_2) basis, denoted with a double prime, is given by

$$u_{\mu''} = \lim_{\alpha\to 0} e^{\mu}_{\mu''} u_\mu = (-E_0, 0, L_0, 0). \tag{7.43}$$

These two quantities then allow us to construct the projection operator, $P_{\mu''\nu''} = g_{\mu''\nu''} + u_{\mu''}u_{\nu''}$, and the Christoffel symbols, $\Gamma^{\alpha''}_{\beta''\gamma''}$, on the particle. The non-zero components of the Christoffel symbols on the particle are given by

$$\Gamma^t_{tr}|_p = \frac{M}{r_0(r_0 - 2m)}, \tag{7.44}$$

$$\Gamma^r_{\mu''\nu''}|_p = \mathrm{diag}\left(\frac{M(r_0-2M)}{r_0^3}, -\frac{M}{r_0(r_0-2M)}, -(r_0-2M), -(r_0-2M)\right), \tag{7.45}$$

$$\Gamma^{w_1}_{rw_1}|_p = \Gamma^{w_2}_{rw_2}|_p = \frac{1}{r_0}. \tag{7.46}$$

The coordinate difference is straightforwardly given by

$$\begin{aligned} \Delta x^{\mu''} &= (0, \Delta r, \Delta w_1, \Delta w_2) \\ &= \left(0, \Delta r, 2\sin\left(\frac{\alpha}{2}\right)\cos\beta, 2\sin\left(\frac{\alpha}{2}\right)\sin\beta\right), \end{aligned} \tag{7.47}$$

where we have written this in terms of α and β but, crucially, still in the (w_1, w_2) basis.

Finally, we can calculate the form of the $\boldsymbol{\rho}_0$ term that appears with the simple contraction

$$\begin{aligned} \boldsymbol{\rho}_0^2 &= P_{\mu''\nu''}\Delta x^{\mu''}\Delta x^{\nu''} \\ &= \frac{2\chi r_0^2(r_0-2M)}{r_0-3M}(\delta^2 + 1 - \cos\alpha), \end{aligned} \tag{7.48}$$

where

$$\delta^2 := \frac{\Delta r^2}{2\chi r_0}\frac{r_0-3M}{(r_0-2M)^2}, \tag{7.49}$$

and

$$\chi := 1 - \frac{M}{r_0-2M}\sin^2\beta. \tag{7.50}$$

This expression exactly matches equivalent expressions for $\boldsymbol{\rho}_0$ in Refs. [137, 143], as expected.[1]

7.4.2 Carter tetrad

Instead of writing our gauge vector components in a coordinate basis, we opt instead to write them in the Carter tetrad. This is a null basis that utilises the Newman–Penrose formalism (see App. C) to write quantities in terms of spin-weighted scalars. The Newman–Penrose formalism allows one to use spin-weighted spherical harmonics (see App. D.2) as the basis for the mode decomposition. This has the advantage of simplifying the integrals that need to be performed when computing the mode decomposition of the gauge vector. We will discuss this further in Ch. 7.5.

[1] Ref. [143] uses the notation ρ_0 and Ref. [137] uses ρ for what we refer to as $\boldsymbol{\rho}_0$.

In Schwarzschild, the Carter tetrad is, matching the convention of Ref. [135], given by

$$l^{\mu'} = \frac{1}{\sqrt{2f}}(1, f, 0, 0), \tag{7.51}$$

$$n^{\mu'} = \frac{1}{\sqrt{2f}}(1, -f, 0, 0), \tag{7.52}$$

$$m^{\mu'} = \frac{1}{\sqrt{2}r}(0, 0, 1, i\csc\alpha), \tag{7.53}$$

$$\bar{m}^{\mu'} = \frac{1}{\sqrt{2}r}(0, 0, 1, -i\csc\alpha), \tag{7.54}$$

which satisfies the Newman–Penrose formalism conditions from Eqs. (C.1)–(C.3). To avoid potential confusion, we will be using ℓ and $\boldsymbol{m}$ for harmonic mode numbers and l and m for two of the components of the Carter tetrad. Note that we have defined our Carter tetrad in the (α, β) coordinates as it is the properties of this coordinate basis (where all but the lowest few $\boldsymbol{m}$ modes vanish) that we want to take advantage of when decomposing into spin-weighted spherical harmonics. This is consistent with the description of the method in Ref. [65, Ch 7.2] although using slightly different steps. In Ref. [65, Ch 7.2], the Carter tetrad is defined in the (θ, ϕ) basis and coordinates and then transformed to the (w_1, w_2) basis. A factor of $e^{-is(\beta+\pi/2)}$ is then inserted to account for the rotation of the frame between (θ, ϕ) and (α, β). Through a quick calculation, one can show that the resulting vector is exactly the same as if one defined the tetrad in the (α, β) basis and transformed to the (w_1, w_2) basis, at least through the orders we are working to.

Transforming to the (w_1, w_2) coordinates and keeping the leading two orders in distance (which are now counted in powers of Δr and α, following Eq. (7.47)) from the small object, we get the Carter tetrad in our Riemann normal coordinates given by

$$l^{\mu''} = \frac{1}{\sqrt{2f_0}}\left(1 - \frac{\Delta r(1-f_0)}{2f_0 r_0}, f_0 + \frac{\Delta r(1-f_0)}{2r_0}, 0, 0\right) + \mathcal{O}\left(\lambda^2\right), \tag{7.55}$$

$$n^{\mu''} = \frac{1}{\sqrt{2f_0}}\left(1 - \frac{\Delta r(1-f_0)}{2f_0 r_0}, -f_0 + \frac{\Delta r(1-f_0)}{2r_0}, 0, 0\right) + \mathcal{O}\left(\lambda^2\right), \tag{7.56}$$

$$m^{\mu''} = \frac{1}{\sqrt{2}r_0^2}\left(0, 0, (r_0 - \Delta r)e^{-i\beta}, i(r_0 - \Delta r)e^{-i\beta}\right) + \mathcal{O}\left(\lambda^2\right), \tag{7.57}$$

$$\bar{m}^{\mu''} = \frac{1}{\sqrt{2}r_0^2}\left(0, 0, (r_0 - \Delta r)e^{i\beta}, -i(r_0 - \Delta r)e^{i\beta}\right) + \mathcal{O}\left(\lambda^2\right), \tag{7.58}$$

where, as before, we have left the angular dependence in terms of (α, β) coordinates but with the expression in the (w_1, w_2) basis. These satisfy the Newman–Penrose formalism conditions from Eqs. (C.1)–(C.3) through order λ.

7.4.3 Gauge vector in the Carter tetrad

We may now calculate the components of the gauge vector by contracting the Carter tetrad legs into the expression for the gauge vector from Eq. (7.30). The l leg of the gauge vector is given by

$$\begin{aligned}
\xi_l = & -\sqrt{\frac{2(r_0-2M)}{r_0-3M}}m\log(\boldsymbol{\rho}_0)\left(1-\sqrt{\frac{M}{r_0}}\cos\beta\sin\alpha\right) \\
& + \frac{m}{\boldsymbol{\rho}_0}\Bigg[\frac{M\sqrt{2r_0(r_0-2M)}\cos^2\beta}{3M-r_0} + \frac{\Delta r^2 M - 2\Delta r r_0(r_0-2M) - 2r_0(r_0-2M)^2}{2\sqrt{2r_0}(r_0-2M)^{3/2}} \\
& + \sqrt{\frac{r_0(r_0-2M)}{2}}\cos\alpha\left(1+\frac{2M\cos^2\beta}{r_0-3M}\right) - \frac{(\Delta r + r_0)\sqrt{2M(r_0-2M)}\cos\beta\sin\alpha}{2(r_0-3M)}\Bigg] \\
& + \frac{m}{\boldsymbol{\rho}_0^2}\Bigg[\frac{\Delta r\Big(\Delta r^2 M - 2r_0(r_0-2M)^2\Big)}{\sqrt{2(r_0-3M)}(r_0-2M)^{3/2}} - \frac{2\Delta r M r_0\sqrt{2(r_0-2M)}\cos^2\beta}{(r_0-3M)^{3/2}} \\
& + \frac{\Delta r r_0\sqrt{2(r_0-2M)}\cos\alpha(r_0-3M+2M\cos^2\beta)}{(r_0-3M)^{3/2}}\Bigg] \\
& + \frac{m}{\boldsymbol{\rho}_0^3}\Bigg[\frac{-\Delta r^2\sqrt{r_0}}{2\sqrt{2}(r_0-3M)(r_0-2M)^{5/2}}\Big[(r_0-3M)\Big(\Delta r^2 M - 2r_0(r_0-2M)^2\Big) \\
& - 4Mr_0(r_0-2M)^2\cos^2\beta\Big] - \frac{\Delta r^2 {r_0}^{3/2}\cos\alpha(r_0-3M+2M\cos^2\beta)}{\sqrt{2}(r_0-3M)\sqrt{r_0-2M}} \\
& + \frac{\Delta r r_0\cos\beta\sin\alpha}{2\sqrt{2}(r_0-3M)^2(r_0-2M)^{3/2}}\Big[\sqrt{M}(r_0-3M)\Big(\Delta r^2 M - 2r_0(r_0-2M)^2\Big) \\
& - 4M^{3/2}r_0(r_0-2M)^2\cos^2\beta\Big] + \frac{\Delta r\sqrt{M}{r_0}^2\sqrt{r_0-2M}\cos\alpha\cos\beta\sin\alpha}{\sqrt{2}(r_0-3M)^2} \\
& \times(r_0-3M+2M\cos^2\beta)\Bigg] + \mathcal{O}\Big(\lambda^2\Big),
\end{aligned} \tag{7.59}$$

and the n leg by

$$\begin{aligned}
\xi_n = & -\sqrt{\frac{2(r_0-2M)}{r_0-3M}}m\log(\boldsymbol{\rho}_0)\left(1+\sqrt{\frac{M}{r_0}}\cos\beta\sin\alpha\right) \\
& + \frac{m}{\boldsymbol{\rho}_0}\Bigg[\frac{M\sqrt{2r_0(r_0-2M)}\cos^2\beta}{r_0-3M} - \frac{\Delta r^2 M - 2\Delta r r_0(r_0-2M) - 2r_0(r_0-2M)^2}{2\sqrt{2r_0}(r_0-2M)^{3/2}} \\
& - \sqrt{\frac{r_0(r_0-2M)}{2}}\cos\alpha\left(1-\frac{2M\cos^2\beta}{r_0-3M}\right) - \frac{(\Delta r + r_0)\sqrt{2M(r_0-2M)}\cos\beta\sin\alpha}{2(r_0-3M)}\Bigg] \\
& + \frac{m}{\boldsymbol{\rho}_0^2}\Bigg[\frac{\Delta r\Big(\Delta r^2 M - 2r_0(r_0-2M)^2\Big)}{\sqrt{2(r_0-3M)}(r_0-2M)^{3/2}} - \frac{2\Delta r M r_0\sqrt{2(r_0-2M)}\cos^2\beta}{(r_0-3M)^{3/2}} \\
& + \frac{\Delta r r_0\sqrt{2(r_0-2M)}\cos\alpha(r_0-3M+2M\cos^2\beta)}{(r_0-3M)^{3/2}}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{m}{\boldsymbol{\rho}_0^3}\Bigg[\frac{\Delta r^2\sqrt{r_0}}{2\sqrt{2}(r_0-3M)(r_0-2M)^{5/2}}\Big[(r_0-3M)\Big(\Delta r^2 M-2r_0(r_0-2M)^2\Big)\\
&-4Mr_0(r_0-2M)^2\cos^2\beta\Big]+\frac{\Delta r^2 {r_0}^{3/2}\cos\alpha(r_0-3M+2M\cos^2\beta)}{\sqrt{2}(r_0-3M)\sqrt{r_0-2M}}\\
&+\frac{\Delta r r_0\cos\beta\sin\alpha}{2\sqrt{2}(r_0-3M)^2(r_0-2M)^{3/2}}\Big[\sqrt{M}(r_0-3M)\Big(\Delta r^2 M-2r_0(r_0-2M)^2\Big)\\
&-4M^{3/2}r_0(r_0-2M)^2\cos^2\beta\Big]+\frac{\Delta r\sqrt{M}{r_0}^2\sqrt{r_0-2M}\cos\alpha\cos\beta\sin\alpha}{\sqrt{2}(r_0-3M)^2}\\
&\times(r_0-3M+2M\cos^2\beta)\Bigg]+\mathcal{O}\Big(\lambda^2\Big),
\end{aligned}
\tag{7.60}
$$

The angular pieces are given by

$$
\begin{aligned}
\xi_m=\;&me^{-i\beta}\sqrt{\frac{2M}{r_0-3M}}\log\boldsymbol{\rho}_0-\frac{m\sin\alpha e^{-i\beta}}{\sqrt{2}\boldsymbol{\rho}_0(r_0-3M)}\Big[\Big(\Delta rM+r_0(r_0-2M)\Big)\cos\beta\\
&+i(r_0-3M)r_0\sin\beta\Big]-\frac{\Delta r\sqrt{M}me^{-i\beta}}{\sqrt{2}\boldsymbol{\rho}_0^2(r_0-3M)^{3/2}(r_0-2M)^2}\Big[(r_0-3M)\\
&\times\Big(\Delta r^2M-2r_0(r_0-2M)^2\Big)+2r_0(r_0-2M)^2\Big((r_0-3M)\cos\alpha\\
&+2M(\cos\alpha-1)\cos^2\beta\Big)\Big]-\frac{\Delta r m r_0}{\sqrt{2}\boldsymbol{\rho}_0^3(r_0-2M)^2(r_0-3M)^2}\Big[(e^{-2i\beta}-5)M+2r_0\Big]\\
&\times\Big[(r_0-3M)\Big(\Delta r^2M-2r_0(r_0-2M)^2+2r_0(r_0-2M)^2\cos\alpha\Big)\\
&+4Mr_0(r_0-2M)^2(\cos\alpha-1)\cos^2\beta\Big]\sin\alpha+\mathcal{O}\Big(\lambda^2\Big),
\end{aligned}
\tag{7.61}
$$

with $\xi_{\bar{m}}$ being the complex conjugate of the above.

7.5 Mode decomposition of the gauge vector

Having written the gauge vector in the Carter tetrad we can proceed with the mode decomposition. As mentioned previously, as all of the gauge vector tetrad legs are spin-weighted scalars, we may decompose each of them using spin-weighted spherical harmonics; see App. D for properties of the spin-weighted spherical harmonics. This method is described in Ch. 7.2 of Ref. [65] for the decomposition of the singular field but the same principles apply for the gauge vector, with one major difference that will be explained. We outline the method below before moving onto explicit calculations. We also note that we use $(\ell, \boldsymbol{m})$ as labels for modes in the (θ, ϕ) coordinates and $(\ell, \boldsymbol{m}')$ as labels for modes in the (α, β) coordinates.

7.5.1 Outline of method

To begin, we combine our expressions for the gauge vector tetrad components with the explicit representation of the spin-weighted spherical harmonics given in Eq. (D.23). Before performing the integral over α, there is a complication that we must overcome. This was previously discussed and a solution presented in Refs. [132, 137] and we summarise here. The problem appears as the quantity $\boldsymbol{\rho}_0$ from Eq. (7.48) features a non-physical, β dependent directional discontinuity at the south pole of our (α, β) coordinate system. For example,

$$\lim_{\beta\to 0}\lim_{\alpha\to\pi}\boldsymbol{\rho}_0 = \sqrt{\frac{4r_0^2(r_0-2M)}{r_0-3M}+\frac{r_0\Delta r^2}{r_0-2M}}, \tag{7.62a}$$

$$\lim_{\beta\to \frac{\pi}{2}}\lim_{\alpha\to\pi}\boldsymbol{\rho}_0 = \sqrt{r_0\left(\frac{\Delta r^2}{r_0-2M}+4r_0\right)}. \tag{7.62b}$$

This discontinuity has the effect of vastly slowing down the rate of convergence when summing over the ℓ modes. To handle this problem, Ref. [137] introduced a window function to 'smooth out' the directional discontinuity. This was then refined in Ref. [132] to be

$$\mathcal{W}^n_{m'}(\alpha) := 1 - \frac{n}{2}\binom{(m'+n-2)/2}{n/2} B\left(\frac{1-\cos\alpha}{2}; \frac{n}{2}, \frac{m'}{2}\right), \tag{7.63}$$

where $\binom{n}{k}$ is the binomial coefficient and

$$B(z;a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\,dt \tag{7.64}$$

is the incomplete Beta function [189, Ch. 8.17]. The window function behaves as $\mathcal{W}^n_{m'} = 1+\mathcal{O}(\alpha^n)$ at the north pole. As discussed in Refs. [132, 137], we do not wish the window function to affect our expressions for the gauge vector, we only wish it to cancel the non-physical behaviour at the south pole. Therefore, we choose $n=2$, to match the order in λ that we are discarding. Additionally, we enforce that m' is even so that for m' odd we choose the smallest even integer above the current value. This results in the window function taking the form

$$\mathcal{W}^2_{m'}(\alpha) = \cos^{2\lceil \frac{m'}{2}\rceil}\left(\frac{\alpha}{2}\right). \tag{7.65}$$

Continuing on, we find that performing the α integral results in an expression with the sum of two polynomials in δ^2, from Eq. (7.49), each multiplied by a δ^2-dependent prefactor. As an example, for a power of $\boldsymbol{\rho}_0$, p, with $s=m'=0$ and no trigonometric functions of α, we get an expression of the form [65, Eqs. (448)–(449)]

$$\delta^{p+2}(\delta^2+2)^{(p+2)/2}\sum_{i=0}^{\ell-(p+4)/2} a_i\delta^{2i} + \log\left(\frac{\delta^2+2}{\delta^2}\right)\sum_{i=0}^{\ell+(p+2)/2} b_i\delta^{2i}, \quad p \text{ even}, \tag{7.66}$$

$$(\delta^2+2)^{(n+2)/2}\sum_{i=0}^{\ell} c_i\delta^{2i} + |\delta|\delta^{n+1}\sum_{i=m'}^{\ell} d_i\delta^{2i}, \quad p \text{ odd}. \tag{7.67}$$

It is at this stage that the major difference between the method for the gauge vector and for the singular field from Ref. [65] appears. We first note that δ^2 counts powers of distance squared as $\delta^2 \sim \Delta r^2 \sim \lambda^2$. In the singular field case, one simply performs a series expansion in δ on Eq. (7.66) up to the required order in λ to match the order that the singular field goes up to. This can then be integrated over β resulting in either a power series, elliptic integral or derivative of a hypergeometric function with respect to one of its arguments, depending on whether we have an even power, odd power or logarithm of δ, respectively. When integrating over β, the integral is identically zero unless certain conditions are met: for $m'=0$, we require the powers of $\cos\alpha$ and $\sin\beta$ to both be even; for m' even and $\neq 0$, we require the powers of $\cos\alpha$ and $\sin\beta$ to either both be odd or both be even; for m' odd, we require one of the powers to be even and one to be odd. With this knowledge ahead of time, we can drastically reduce the amount of integrals that we need to perform over α and β.

Returning to the differences between the approaches, the approximation in the singular field case in δ^2 is no longer possible when decomposing the gauge vector. We found that doing this for the gauge vector introduced a large ℓ divergence when summing up the modes. To see why this happens, note that in Eq. (7.66), the upper limit on the sums depends on the value of ℓ. By truncating the series at some power in δ^2, we throw away some of the ℓ-dependent behaviour leading to the non-convergence when summing up the final (ℓ, m') modes. We believe this issue also appears in the singular field, but it is not directly observed as any poor large ℓ behaviour in the puncture field will be compensated for in the residual field, meaning that when recovering the physical field by summing the puncture and residual fields, the large ℓ divergence will not occur. In the gauge vector case, we cannot compensate for the large ℓ divergence in the same way, necessitating us to keep all of the ℓ dependence in Eq. (7.66) when integrating over β.

Inspecting the individual terms, it's not immediately clear how one would integrate a term like

$$\Delta r^{2n}\log\left(\frac{2+d^2\Delta r^2-2e\sin^2\beta}{d^2\Delta r^2}\right)\left(\frac{d}{\sqrt{1-e\sin^2\beta}}\right)^{2n}, \tag{7.68}$$

or

$$\Delta r^{2n}\left(\frac{d}{\sqrt{1-e\sin 2\beta}}\right)^{2n}\left(2+\frac{d^2\Delta r^2}{1-e\sin^2\beta}\right)^{2k}, \tag{7.69}$$

for some positive integers n and k, where we have substituted in the expressions for δ and χ from Eqs. (7.49)–(7.50) and let $d := \sqrt{(r_0 - 3M)/(2r_0(r_0 - 2M)^2)}$ and $e := M/(r_0 - 2M)$. However, we can perform an expansion of the non-ℓ dependent pieces of Eq. (7.66), allowing us to perform the integrals over β analytically. These can all be done in terms of hypergeometric functions or parameter derivatives of hypergeometric functions which reduce to the power series or elliptic integrals mentioned previously. The final expressions that one ends up with in the gauge vector case are substantially more complicated than the singular field case due to the higher powers of δ that appear. In Sec. 7.2.1 of Ref. [65], expressions for the mode-decomposed components of the punctures are provided as functions of ℓ. Unfortunately, we have been unable to find a general expression for the (ℓ, m') modes of the punctures due to the complexity of the final expressions that we find. Instead, we explicitly compute the mode coefficients up to $\ell = 20$ and for the specific m' values that we need.

7.5.2 Integrals over α

Beginning the mode decomposition, we start by noting that all of the individual terms in the gauge vectors have the form,

$$\sim \frac{\cos^{n_1}\alpha \sin^{n_2}\alpha \cos^{n_3}\beta \sin^{n_4}\beta}{\boldsymbol{\rho}_0^p}, \tag{7.70}$$

or

$$\sim \log(\boldsymbol{\rho}_0) \cos^{n_1}\alpha \sin^{n_2}\alpha \cos^{n_3}\beta \sin^{n_4}\beta. \tag{7.71}$$

From Eq. (D.23), we know we can write the spin-weighted spherical harmonics explicitly as

$${}_sY_{\ell m}(\arccos(x), \beta) = a_{s\ell m}(1-x)^{\ell} e^{im\beta} \sum_{r=0}^{\ell-s} b_{s\ell m r} \left(\sqrt{\frac{1+x}{1-x}}\right)^{2r+s-m}, \tag{7.72}$$

where we have made the substitution $x = \cos\alpha$, and defined

$$a_{s\ell m} := \frac{(-1)^m}{2^{\ell}} \sqrt{\frac{(\ell+m)!(\ell-m)!(2\ell+1)}{4\pi(l+s)!(l-s)!}}, \tag{7.73}$$

$$b_{s\ell m r} := (-1)^{\ell-r-s} \binom{\ell-s}{r} \binom{\ell+s}{r+s-m}, \tag{7.74}$$

with $\binom{p}{q}$ being the binomial coefficient. The reason that we have performed the change of variables is that we found the computer algebra package we used to perform the integrals (Mathematica [144]), calculated the integrals much more efficiently when not using trigonometric functions. Thus we can combine the expressions from Eqs. (7.70)–(7.71), the explicit expression for the spin-weighted spherical harmonics from Eq. (7.72) and the window function from Eq. (7.65) to construct the integrand for the decomposition coefficients. We can factor out the non-α dependence in $\boldsymbol{\rho}_0$ by writing

$$\boldsymbol{\rho}_0 = K\sqrt{\delta^2 + 1 - \cos\alpha}, \tag{7.75}$$

where

$$K := \sqrt{\frac{2\chi r_0^2(r_0 - 2M)}{r_0 - 3M}}, \tag{7.76}$$

so that $\boldsymbol{\rho}_0^n \sim (\delta^2 + 1 - \cos\alpha)^{n/2}$. Therefore, to calculate the integral against a spin-weighted spherical harmonic, it suffices to look at an integral of the form

$$\int_{-1}^{1} 2^{-m'/2} x^{n_1} (\delta^2 - x + 1)^{-p/2} (x+1)^{\frac{1}{2}(n_2+2r+s)} (1-x)^{\frac{1}{2}(2\ell+m'+n_2-2r-s)}\, dx\,. \tag{7.77}$$

Expressions of this form can readily be integrated by a computer algebra package, e.g. Mathematica [144].

As an example of how to calculate the decomposition, we start with the case where $p = 1$ and $n_i = 0$ in Eq. (7.70) and calculate the α integral when m' is even. In this case, Eq. (7.77) becomes

$$\begin{aligned}
&\int_{-1}^{1} \frac{2^{-m'/2}(x+1)^{r+\frac{s}{2}}(1-x)^{\frac{1}{2}(2\ell+m'-2r-s)}}{\sqrt{\delta^2 - x + 1}}\, dx \\
&\quad = i^{2\ell+3m'+2r+s} 2^{-m'/2}\sqrt{\pi}(2+\delta^2)^{r+s/2}|\delta|^{2\ell+m'-2r-s}\Bigg[\sqrt{2+\delta^2}\,\Gamma\Big(1+r+\frac{s}{2}\Big) \\
&\qquad \times {}_2\tilde{F}_1\Big(\frac{1}{2}, \frac{1}{2}(2r+s-2\ell-m'); \frac{1}{2}(3+2r+s); \frac{\delta^2+2}{\delta^2}\Big) \\
&\qquad - |\delta|\Gamma\Big(1+\ell+\frac{m'}{2}-r-\frac{s}{2}\Big){}_2\tilde{F}_1\Big(\frac{1}{2}, -r-\frac{s}{2}; \frac{1}{2}(3+2\ell+m'-2r-s); \frac{\delta^2}{2+\delta^2}\Big)\Bigg],
\end{aligned} \tag{7.78}$$

where $\Gamma(x)$ is the Gamma function [189, Ch. 5] and

$${}_2\tilde{F}_1(a,b;c;z) = \frac{{}_2F_1(a,b;c;z)}{\Gamma(c)}, \tag{7.79}$$

is the regularised hypergeometric function [189, Eq. (15.1.2)]. This is defined from the hypergeometric function which itself is given by [189, Eq. (15.2.1)],

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!}. \tag{7.80}$$

While it appears that Eq. (7.78) is (potentially) imaginary, this is merely an artefact of the way that Mathematica has written the output. When values for ℓ, m', s and r are substituted in then we recover a real function as expected. Combining Eq. (7.80) with

Eq. (7.72), we can write a sum for generic s, ℓ and even m' as

$$\begin{aligned}
&\int_{-1}^{1} \frac{\mathcal{W}^2_{m'}(\arccos x)}{\sqrt{\delta^2 - x + 1}} {}_sY^*_{\ell m}(\arccos x, 0)\, dx \\
&= a_{s\ell m'} \sum_{r=0}^{\ell - s} b_{s\ell m r} i^{2\ell + 3m' + 2r + s} 2^{-m'/2} \sqrt{\pi} (2 + \delta^2)^{r + s/2} |\delta|^{2\ell + m' - 2r - s} \\
&\quad \Big[\sqrt{2+\delta^2}\,\Gamma\Big(1 + r + \frac{s}{2}\Big) {}_2\tilde{F}_1\Big(\frac{1}{2}, \frac{1}{2}(2r + s - 2\ell - m'); \frac{1}{2}(3 + 2r + s); \frac{\delta^2 + 2}{\delta^2}\Big) \\
&\quad - |\delta| \Gamma\Big(1 + \ell + \frac{m'}{2} - r - \frac{s}{2}\Big) {}_2\tilde{F}_1\Big(\frac{1}{2}, -r - \frac{s}{2}; \frac{1}{2}(3 + 2\ell + m' - 2r - s); \frac{\delta^2}{2 + \delta^2}\Big)\Big].
\end{aligned} \tag{7.81}$$

The sum itself appears very complicated but it collapses to the form of Eq. (7.67) when the mode numbers are substituted in. For example, if we let $s = 0$, $m' = 0$ and $\ell = 6$, then we find that the sum in Eq. (7.81) evaluates to

$$\begin{aligned}
&\frac{1}{\sqrt{13\pi}} \Big[\sqrt{2+\delta^2}\Big(1 + 42\delta^2 + 280\delta^4 + 672\delta^6 + 720\delta^8 + 352\delta^{10} + 64\delta^{12}\Big) \\
&\qquad - |\delta|\Big(13 + 182\delta^2 + 728\delta^4 + 1248\delta^6 + 1040\delta^8 + 416\delta^{10} + 64\delta^{12}\Big)\Big],
\end{aligned} \tag{7.82}$$

which has exactly the form that we would expect from Eq. (7.67).

The advantage of performing the integrals in this way is that we avoid having to repeat the integrals multiple times for different values of (s, ℓ, m'), instead we may just perform the integral once and then use the sum to generate our final expression. After calculating the generating sum for the different cases, we then check that it gives the right result by numerically calculating the integral and comparing the answer to ensure that we have correctly performed the integral analytically.

The calculation of the integrals for different trigonometric factors and for odd p proceeds in the same manner as our example above. One finds linear combinations of different hypergeometric functions with some prefactor. These all collapse as expected to the expected form, as in Eq. (7.67).

The even powers of p have a somewhat different structure though. We find that it is not possible to compute the integral for generic (s, m'). Instead, we specialise to the required s and m' values and calculate the integral for generic ℓ. We illustrate this for the simplest case where $p = 2$, $n_i = 0$, $s = 0$ and $m' = 0$. Choosing these values, Eq. (7.77) becomes

$$\int_{-1}^{1} \frac{(1-x)^{\ell - r}(1+x)^r}{\delta^2 + 1 - x}\, dx\,. \tag{7.83}$$

As $\ell - r$ and r are always integers, we find it most convenient to use the binomial theorem to calculate the integral,

$$\begin{aligned}\int_{-1}^{1} \frac{(1-x)^{\ell-r}(1+x)^r}{\delta^2+1-x}\,dx &= \sum_{k_1=0}^{\ell-r}\sum_{k_2=0}^{r}\binom{\ell-r}{k_1}\binom{r}{k_2}\int_{-1}^{1}\frac{(-x)^{k_1}x^{k_2}}{\delta^2+1-x} \\ &= \sum_{k_1=0}^{\ell-r}\sum_{k_2=0}^{r}(-1)^{k_1}\binom{\ell-r}{k_1}\binom{r}{k_2}(1+\delta^2)^{k_1+k_2} \\ &\quad \times B\Big(-\frac{1}{1+\delta^2},\frac{1}{1+\delta^2};1+k_1+k_2,0\Big) \end{aligned} \tag{7.84}$$

where

$$B(z_0,z_1;a,b) = \int_{z_0}^{z_1} t^{a-1}(1-t)^{b-1}\,dt\,, \tag{7.85}$$

is the generalised incomplete beta function. This reduces to the incomplete beta function from Eq. (7.64) when $z_0 = 0$. We can do as in Eq. (7.81) and write the integral against the spin-weighted spherical harmonic for generic ℓ as

$$\begin{aligned}&\int_{-1}^{1}\frac{\mathcal{W}_0^2(\arccos x)}{\delta^2-x+1}Y^*_{\ell 0}(\arccos x,0)\,dx \\ &\quad = a_{0\ell 0}\sum_{r=0}^{\ell} b_{0\ell 0r}\sum_{k_1=0}^{\ell-r}\sum_{k_2=0}^{r}(-1)^{k_1}\binom{\ell-r}{k_1}\binom{r}{k_2}(1+\delta^2)^{k_1+k_2} \\ &\quad\quad \times B\Big(-\frac{1}{1+\delta^2},\frac{1}{1+\delta^2};1+k_1+k_2,0\Big). \end{aligned} \tag{7.86}$$

As in the previous case, we have checked that the sum collapses to the expected form from Eq. (7.66) and we have numerically evaluated the integral for specific values of ℓ and δ to ensure that we have calculated the generic ℓ form correctly. The integrations for different values of n_i, s and m' proceed in a similar fashion and feature linear combinations of the incomplete beta functions multiplied by some (n_i, s, ℓ, m') dependent prefactor.

With the terms featuring logarithms, we proceed in a similar way to the even p case and calculate an expression for generic ℓ but with specific s and m'. For example, for $s = m' = 0$, we have

$$\begin{aligned}&\frac{1}{2}\int_{-1}^{1}(1-x)^{\ell-r}(1+x)^r\log\Big(\delta^2+1-x\Big)\,dx \\ &\quad = 2^{\ell}\Gamma(2+\ell-r)\Big[\frac{r!\log(\delta^2+2)}{\Gamma(\ell+2)} - \frac{2\Gamma(r+2)}{\delta^2+2}\,{}_3\tilde{F}_2\Big(1,1,r+2;2,\ell+3,\frac{2}{\delta^2+2}\Big)\Big]. \end{aligned} \tag{7.87}$$

The quantity

$${}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \frac{{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)}{\Gamma(b_1)\cdots\Gamma(b_q)}, \tag{7.88}$$

is the regularised generalised hypergeometric function [189, Eq. (16.2.5)] constructed from the generalised hypergeometric function [189, Eq. (16.2.1)]

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{7.89} $$

where

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{7.90} $$

is Pochhammer's symbol [189, Eq. (5.2.5)]. These reduce to Eqs. (7.79)–(7.80) for $p = 2$ and $q = 1$. As before, we have numerically checked all of the integrals to ensure we have generated the correct analytic expressions. The form of the integrals for other n_i, s, ℓ and m' all feature a similar structure but different factors in the gamma functions and hypergeometric function depending on the specific value of (n_i, s, ℓ, m').

With all of the integrals calculated for generic ℓ, we can use our expression for the spin-weighted spherical harmonic expressed as a sum and calculate the explicit form for each ℓ, m' and s required. We find that all resulting expression is in the form of either Eq. (7.66) or Eq. (7.67), as expected. We then multiply by the appropriate factor of K from Eq. (7.76) to ensure we have the correct constant and β dependence. With this in hand, and with all the analytic expressions checked against a numerical integration of the same value, we can confidently proceed to performing the β integrals.

7.5.3 Integrals over β

As discussed in the text around Eqs. (7.68)–(7.69), it's not clear how one can analytically integrate terms of the form $\log\left(\frac{\delta^2+2}{\delta^2}\right)\delta^2 i$ or $(\delta^2+2)^{(n+2)/2}\delta^{2i}$ when the expressions are explicitly presented in terms of β. Instead, we choose to perform a series expansion in δ^2 of the non-δ^{2i} terms,

$$ \log\left(\frac{\delta^2+2}{\delta^2}\right) = \log\left(\frac{2}{\delta^2}\right) + \mathcal{O}\left(\delta^2\right), \tag{7.91} $$

$$ (\delta^2+2)^{n/2} = 2^{n/2} + \mathcal{O}\left(\delta^2\right). \tag{7.92} $$

We believe we are justified in performing this expansion as, even though we are throwing away ℓ dependent terms, they are all suppressed by a factor of λ^2 meaning that they are at a higher order than what the gauge vector was calculated through. If we find, at some later date, that we require a higher order approximation to these terms, it is fairly trivial with the code we have written to extend this to whatever order is required.

After performing the approximation, we find that all remaining terms can be integrated analytically. To do so, we take our mode coefficients and express them in terms of χ

using Eq. (7.49). This results in expressions of the form

$$\sum_{i>0} \frac{a_i}{\chi^{i/2}}, \quad p \text{ odd}, \tag{7.93}$$

$$\sum_{i\geq 0} \frac{b_i + c_i \log(\chi)}{\chi^i}, \quad p \text{ even or log}, \tag{7.94}$$

where the specific limits on the sums depend on ℓ, m' and s. These will potentially also be multiplied by a trigonometric factor coming from the gauge vector expressions. For $m' > 0$, we will also introduce exponential functions. We find the best way to handle these is to express them in terms of trigonometric functions so that our expressions just feature inverse powers of χ multiplied by either $\sin\beta$, $\cos\beta$, $\log\chi$ or some constant function.

To perform the integrals, for terms featuring no logs, we may use Eq. (3.681.1) of Ref. [190]

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu-1}(x)\cos^{2\nu-1}(x)}{(1-k^2\sin^2(x))^\rho}\,dx = \frac{1}{2}B(\mu,\nu)\,{}_2F_1(\mu,\rho;\mu+\nu;k^2), \qquad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0. \tag{7.95}$$

where $B(\mu,\nu)$ is the Beta function (Eq. (7.85) with $z_0 = 0$ and $z_1 = 1$) and the denominator is χ, given in Eq. (7.50), with $k^2 = M/(r_0 - 2M)$. One note is that the integration limits on Eq. (7.95) are between 0 and $\pi/2$. As stated previously, this integral vanishes unless both powers of $\sin\beta$ and $\cos\beta$ are even. This can be seen from a simple periodicity argument. In the case where the integrals do not vanish, we can simply multiply the final result from Eq. (7.95) by four to get the integration over $(0, 2\pi)$.

For the term featuring logs, we take a derivative of Eq. (7.95) with respect to ρ, so that

$$\int_0^{\pi/2} \frac{\sin^{2\mu-1}(x)\cos^{2\nu-1}(x)}{(1-k^2\sin^2(x))^\rho}\log\Big(1-k^2\sin^2 x\Big)\,dx = -\frac{1}{2}B(\mu,\nu){}_2F_1^{(0,1,0,0)}(\mu,\rho;\mu+\nu;k^2), \tag{7.96}$$

where

$${}_2F_1^{(0,1,0,0)}(\alpha,\beta;\gamma;\zeta) \coloneqq \partial_b\Big({}_2F_1(a,b;c;z)\Big)\Big|_{a=\alpha,b=\beta,c=\gamma,z=\zeta}, \tag{7.97}$$

is the derivative of the hypergeometric function with respect to its second parameter.

With the β integrals complete, we now have the modes of our gauge vector in the (ℓ, m') basis. We have checked that the analytic modes of the gauge vector that we have obtained are the same as those that would be obtained by numerically integrating the full expressions from Eqs. (7.59)–(7.61). It appears that the individual analytically calculated modes have a relative error of between 0.01% and 1% when compared to the numerically calculated modes for $M = 1$, $r_0 = 10$ and $\Delta r = 1/2$. If one desired a higher

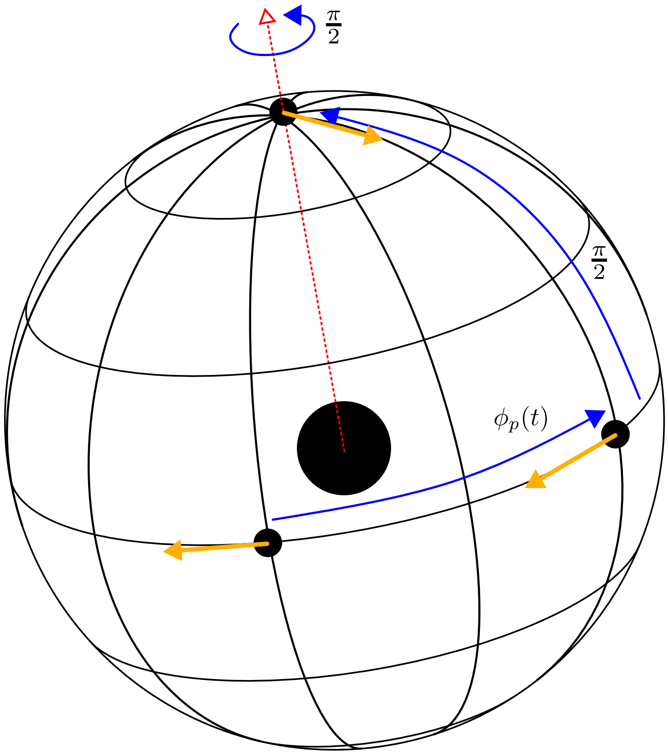

FIGURE 7.1: Figure illustrating the action of the Euler angle rotations in our spacetime. We rotate the small object to $\phi = 0$ before rotating it to the north pole. We then align the small object's tangent vector with $\hat{\beta}$.

accuracy then it is fairly trivial to do as one can perform higher order Taylor expansions of the quantities in Eq. (7.91). This will generate more complicated expressions but the code we have written is easily extendible to be able to handle this. We must now convert these into modes in the (ℓ, m) basis which we do so in the following section.

7.6 Rotation between (α, β) and (θ, ϕ) coordinates

With our gauge vector decomposed in the rotated coordinate system (primed indices), we can now rotate back to the original (θ, ϕ) system (unprimed indices) to sum the modes. To do this, we make use of the Wigner D-matrix [191, 192]. These are described in App. D.3 and allow one to related modes calculated in one basis to modes calculated in another if the two bases are related by a rotation using Euler angles. Euler angles represent rigid body rotations using a composition of three rotations around two of the coordinate axes.

In our case, the rotation to put the small object at the north pole is described by the Euler angles $(\phi_p(t), \pi/2, \pi/2)$ [143]. Fig. 7.1 provides a visual depiction of the rotation process. We first rotate around the z axis by $\phi_p(t)$ to move our small object to $\phi = 0$. The next step is to rotate our small object to the north pole; this is done using a rotation of $\pi/2$ around the y axis. The final step is to rotate around the z axis by $\pi/2$ to ensure that the tangent vector to the small object's worldline points in the direction of $\hat{\beta}$.

The mode coefficients of the tetrad legs in the (θ, ϕ) basis can be written in terms of those in the (α, β) basis using Eq. (D.27),

$$\xi_{\mathfrak{a}}^{\ell m} = \sum_{m'=-\ell}^{\ell} D^{\ell}_{mm'}(\phi_p(t), \pi/2, \pi/2)\xi_{\mathfrak{a}}^{\ell m'} \tag{7.98}$$

where we use $\mathfrak{a}$ to denote one of the NP tetrad legs, as in App. C. It may seem at first that we need to calculate an m' for every ℓ that we calculate, however, this is not the case. When summed against a spin-weighted spherical harmonic, the Wigner-D matrix, with our specific angles, picks out certain m' values. For example, say we want to evaluate a spin-weight zero function at $\theta = \pi/2$ and $\phi = 2$, we see that

$$\begin{aligned} f(\pi/2, 0) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell} D^{\ell}_{mm'}(\phi_p, \pi/2, \pi/2) f_{\ell m'} Y_{\ell m}(\pi/2, \phi_p) \\ &= \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} f_{\ell 0}, \end{aligned} \tag{7.99}$$

with the understanding that the mode labels refer to the (ℓ, m') basis. To calculate this, we explicitly calculate the form of the individual ℓ modes and find the sequence that it corresponds to. Thus we know that we can just calculate the $m' = 0$ mode to fully determine this function. This can be done for different spin weights, leading to similar results but with a different m' being required. One additional case appears, which is where we have a factor of m in the expression. This can occur after taking time derivatives of the gauge vector, as $\partial_t \xi_{\mathfrak{a}}^{\ell m'} \sim m \xi_{\mathfrak{a}}^{\ell m'}$; the reason why will be explained in the next section. Taking Eq. (7.99) and inserting a factor of m, we see that

$$\begin{aligned} &\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell} m D^{\ell}_{mm'}(\phi_p, \pi/2, \pi/2) f_{\ell m'} Y_{\ell m}(\pi/2, \phi_p) \\ &\qquad = \sum_{\ell=1}^{\infty} i \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{4\pi}} f_{\ell 1}, \end{aligned} \tag{7.100}$$

picking out a different value of m'. With this noted, we can repeat this procedure for different values of s to precalculate the action of summing over the Wigner-D matrix and the spin-weighted spherical harmonics, avoiding having to calculate modes that will be cancelled in the final result.

7.7 Perturbations in Carter tetrad

To see what the perturbation components are in terms of the gauge vector, we first decompose the perturbations into spin-weighted spherical harmonics [135, 193]

$$\Delta h^1_{ab} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \Delta h^{1\ell m}_{ab} Y_{\ell m}, \tag{7.101}$$

$$\Delta h^1_{aA} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{r\lambda_1}{\sqrt{2}}\Big[{}_{-1}Y_{\ell m} m_A\Big(\Delta h^{1\ell m}_{a+} - i\Delta h^{1\ell m}_{a-}\Big) - {}_{1}Y_{\ell m}\bar{m}_A\Big(\Delta h^{1\ell m}_{a+} + i\Delta h^{1\ell m}_{a-}\Big)\Big], \tag{7.102}$$

$$\Delta h^1_{AB} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \Big(\Delta h^{1\ell m}_{\circ}\Omega_{AB}Y_{\ell m} + \frac{r^2\lambda_2}{2}\Big[{}_{-2}Y_{\ell m} m_A m_B(\Delta h^{1\ell m}_{+} - i\Delta h^{1\ell m}_{-}) + {}_{2}Y_{\ell m}\bar{m}_A\bar{m}_B(\Delta h^{1\ell m}_{+} + i\Delta h^{1\ell m}_{-})\Big]\Big), \tag{7.103}$$

where we have converted the expressions, originally written in vector and tensor harmonics, into spin-weighted spherical harmonics, and λ_s is given by Eq. (D.12). Here we use the notation of Ref. [135], so that a lowercase Latin letter indicates a tensor on the manifold $\mathcal{M}^2$ and an uppercase Latin letter indicates a tensor on the manifold S^2.

It has been shown in Ref. [135] that components of the gauge perturbation's mode decomposition can be given in terms of the gauge vector by[2]

$$\Delta h^{1\ell m}_{ab} = 2\delta_{(a}\xi^{\ell m}_{b)}, \tag{7.104}$$

$$\Delta h^{1\ell m}_{a+} = \xi^{\ell m}_a + \delta_a\xi^{\ell m}_+ - 2r_a r^{-1}\xi^{\ell m}_+, \tag{7.105}$$

$$\Delta h^{1\ell m}_{a-} = \delta_a\xi^{\ell m}_- - 2r_a r^{-1}\xi^{\ell m}_-, \tag{7.106}$$

$$\Delta h^{1\ell m}_{\circ} = 2rr^a\xi^{\ell m}_a - \ell(\ell+1)\xi^{\ell m}_+, \tag{7.107}$$

$$\Delta h^{1\ell m}_{\pm} = 2\xi^{\ell m}_{\pm}, \tag{7.108}$$

where δ_a is the covariant derivative on $\mathcal{M}^2$ and $r_a := \partial_a r$. Here, $\xi^{\ell m}_{\pm}$ come from a vector harmonic decomposition of the gauge vector. By writing the vector harmonics in terms of spin-weighted spherical harmonics and contracting with m^A or $\bar{m}^A$, one can show that $\xi^{\ell m}_{\pm}$ can be written in terms of the Carter tetrad by

$$\xi^{\ell m}_+ = \frac{r}{\sqrt{2}\lambda_1}(\xi^{\ell m}_{\bar{m}} - \xi^{\ell m}_m), \tag{7.109}$$

$$\xi^{\ell m}_- = \frac{ir}{\sqrt{2}\lambda_1}(\xi^{\ell m}_m + \xi^{\ell m}_{\bar{m}}). \tag{7.110}$$

[2] Note that these expressions do not exactly match those of the original paper. This is due to the fact that the original paper expands ξ^μ whereas we are expanding ξ_μ. This results in extra factors of r^2 appearing which may end up differentiated, giving the expressions a slightly different structure.

Any radial derivatives that appear in Eqs. (7.104)–(7.106) are straightforward to compute as the only r dependence in the gauge vector decomposition is contained inside Δr. When performed in the (θ, ϕ) coordinate system, the t derivative has the effect of multiplying the original expression by $-im\dot{\phi}_p(t)$, e.g.

$$\partial_t \xi_t^{\ell m} = -im\dot{\phi}_p(t)\xi_t^{\ell m}. \tag{7.111}$$

To see why, we note that the only t dependency in the gauge vectors is through the Wigner D-matrix when it rotates the system by $\phi_p(t)$. Rewriting Eq. (D.25), we see that

$$\begin{aligned} D^{\ell}_{mm'}(\phi_p(t), \pi/2, \pi/2) &= i^{m'}\sqrt{\frac{4\pi}{2\ell+1}}\,{}_{-m'}Y^{*}_{\ell m}\left(\frac{\pi}{2}, \phi_p(t)\right) \\ &= i^{m'}\sqrt{\frac{4\pi}{2\ell+1}}\,{}_{-m'}Y^{*}_{\ell m}\left(\frac{\pi}{2}, 0\right)e^{-im\phi_p(t)}. \end{aligned} \tag{7.112}$$

Therefore, the t derivative has the effect of bringing down the factor of $-im\dot{\phi}_p(t)$ from the exponential contained in the spherical harmonic.

Evaluating the t derivatives is straightforward in the unrotated system but a subtlety appears if we wanted to take t derivatives in the rotated coordinate system. This results from the fact that in the (α, β) coordinate system, the particle is *always* at the north pole, thus there is no obvious time dependence in the system. We can see this from our expressions for the gauge vector as all the time dependence is contained within the Wigner D-matrices. However, a resolution was presented in App. A of Ref. [132]. This method expresses the time derivatives as a one-parameter family of rotations so that t is no longer a coordinate but a parameter of the rotation. The full method is described in Ref. [132] but it results in us being able to write time derivatives in the rotated system as

$$\partial_t f_{\ell m'} = \frac{1}{2}\sqrt{\frac{M}{r_0^3}}\left(\mu^{-}_{\ell m'} f_{\ell, m'+1} - \mu^{+}_{\ell m'} f_{\ell, m'-1}\right), \tag{7.113}$$

where

$$\mu^{\pm}_{\ell m'} \coloneqq \sqrt{(l \pm m')(l \mp m' + 1)}. \tag{7.114}$$

Returning to the Carter tetrad, we can then immediately find $\Delta h^{1\ell m}_{ll}$, $\Delta h^{1\ell m}_{ln}$ and $\Delta h^{1\ell m}_{nn}$ by contracting the relevant tetrad basis vector with Eq. (7.104). The various m and $\bar{m}$

components are given by [135]

$$\Delta h^{1\ell m}_{am} = -\frac{\lambda_1}{\sqrt{2}r}\left(\Delta h^{1\ell m}_{a+} + i\Delta h^{1\ell m}_{a-}\right) \tag{7.115}$$

$$\Delta h^{1\ell m}_{a\bar{m}} = \frac{\lambda_1}{\sqrt{2}r}\left(\Delta h^{1\ell m}_{a+} - i\Delta h^{1\ell m}_{a-}\right) \tag{7.116}$$

$$\Delta h^{1\ell m}_{mm} = -\frac{\lambda_2}{2r^2}\left(\Delta h^{1\ell m}_{+} + i\Delta h^{1\ell m}_{-}\right) \tag{7.117}$$

$$\Delta h^{1\ell m}_{\bar{m}\bar{m}} = \frac{\lambda_2}{2r^2}\left(\Delta h^{1\ell m}_{+} - i\Delta h^{1\ell m}_{-}\right) \tag{7.118}$$

$$\Delta h^{1\ell m}_{m\bar{m}} = \frac{1}{r^2}\Delta h^{1\ell m}_{\circ}. \tag{7.119}$$

Combining the previous results, expanding $r = r_0 + \Delta r$ and projecting onto the Carter tetrad, we see that the components in the tetrad are given by

$$\Delta h^{1\ell m}_{ll} = -2\xi^{\ell m}_{l}\left(2\epsilon + \frac{im\Omega_\phi}{\sqrt{2f}}\right) + \sqrt{2f}\xi^{\ell m}_{l,r}, \tag{7.120}$$

$$\Delta h^{1\ell m}_{nn} = 2\xi^{\ell m}_{n}\left(2\epsilon - \frac{im\Omega_\phi}{\sqrt{2f}}\right) - \sqrt{2f}\xi^{\ell m}_{n,r}, \tag{7.121}$$

$$\Delta h^{1\ell m}_{ln} = -\frac{im\Omega_\phi}{\sqrt{2f}}\left(\xi^{\ell m}_{l} + \xi^{\ell m}_{n}\right) + 2\epsilon\left(\xi^{\ell m}_{n} - \xi^{\ell m}_{l}\right) + \sqrt{\frac{f}{2}}\left(\xi^{\ell m}_{n,r} - \xi^{\ell m}_{l,r}\right), \tag{7.122}$$

$$\Delta h^{1\ell m}_{lm} = \frac{1}{\sqrt{2f}r}\left[fr\xi^{\ell m}_{m,r} - \xi^{\ell m}_{m}\left(f + im\Omega_\phi r\right) - \sqrt{f}\lambda_1\xi^{\ell m}_{l}\right], \tag{7.123}$$

$$\Delta h^{1\ell m}_{nm} = -\frac{1}{\sqrt{2f}r}\left[fr\xi^{\ell m}_{m,r} - \xi^{\ell m}_{m}\left(f - im\Omega_\phi r\right) + \sqrt{f}\lambda_1\xi^{\ell m}_{l}\right], \tag{7.124}$$

$$\Delta h^{1\ell m}_{l\bar{m}} = \frac{1}{\sqrt{2f}r}\left[fr\xi^{\ell m}_{\bar{m},r} - \xi^{\ell m}_{\bar{m}}\left(f - im\Omega_\phi r\right) + \sqrt{f}\lambda_1\xi^{\ell m}_{l}\right], \tag{7.125}$$

$$\Delta h^{1\ell m}_{n\bar{m}} = -\frac{1}{\sqrt{2f}r}\left[fr\xi^{\ell m}_{\bar{m},r} - \xi^{\ell m}_{\bar{m}}\left(f - im\Omega_\phi r\right) - \sqrt{f}\lambda_1\xi^{\ell m}_{l}\right], \tag{7.126}$$

$$\Delta h^{1\ell m}_{mm} = -\frac{\sqrt{2(\ell-1)(\ell+2)}}{r}\xi^{\ell m}_{m}, \tag{7.127}$$

$$\Delta h^{1\ell m}_{\bar{m}\bar{m}} = \frac{\sqrt{2(\ell-1)(\ell+2)}}{r}\xi^{\ell m}_{\bar{m}}, \tag{7.128}$$

$$\Delta h^{1\ell m}_{m\bar{m}} = \frac{\sqrt{2f}}{r}\left(\xi^{\ell m}_{l} - \xi^{\ell m}_{n}\right) + \frac{1}{r}\sqrt{\frac{\ell(\ell+1)}{2}}\left(\xi^{\ell m}_{m} - \xi^{\ell m}_{\bar{m}}\right), \tag{7.129}$$

where ϵ is the NP spin coefficient defined in Eq. (C.8i). For quasicircular orbits in Schwarzschild,

$$\epsilon = \frac{M[r_0(r_0 - 2M) - \Delta r(2r_0 - 3M)]}{2\sqrt{2}r_0^{5/2}(r_0 - 2M)^{3/2}}. \tag{7.130}$$

To sum up the modes to recover the full perturbation requires us to use the appropriate spin-weighted spherical harmonic. Each m tetrad leg that appears adds one to the spin weight required and each $\bar{m}$ tetrad leg removes one to the spin weight required. Therefore: $\Delta h^{1\ell m}_{ll}$, $\Delta h^{1\ell m}_{ln}$, $\Delta h^{1\ell m}_{nn}$ and $\Delta h^{1\ell m}_{m\bar{m}}$ each require spin weight zero (equivalent to a standard scalar spherical harmonic); $\Delta h^{1\ell m}_{lm}$ and $\Delta h^{1\ell m}_{nm}$ require spin weight one; $\Delta h^{1\ell m}_{l\bar{m}}$ and $\Delta h^{1\ell m}_{n\bar{m}}$ require spin weight minus one; $\Delta h^{1\ell m}_{mm}$ requires spin weight two and finally, $\Delta h^{1\ell m}_{\bar{m}\bar{m}}$ requires spin weight minus two.

With the full gauge transformations calculated in the Carter tetrad, one can then combine them with the expressions for the singular field in the Lorenz gauge from App. C of Ref. [137] to find the form of the perturbations in the highly regular gauge. The Lorenz gauge expressions are presented in the BLS basis but can be straightforwardly converted to the Carter tetrad using the relations from Ch. 4.2.1 of Ref. [65]. We provide the modes of the gauge vector in the Additional Material [141] so one may combine all these quantities to construct the metric perturbations in the highly regular gauge.

Chapter 8

Conclusions and summary

In this final chapter, we summarise the work presented in this thesis and give potential avenues for future research.

8.1 Summary of work presented in this thesis

There are five main results for this thesis. Firstly, in Ch. 4, we have derived the form of the metric perturbations in a class of highly regular gauges that feature a much weaker divergence than any gauge previously considered at second-order. This class of gauges was originally introduced by Pound [81] to help alleviate the problem of infinite mode coupling that appears when solving the second-order field equations. With the weaker divergence, the highly regular gauge should help with this issue. Only the leading order term of the metric perturbations was derived by Pound [81], we have derived the metric perturbations to all orders required to perform a numerical implementation in a puncture scheme and presented them in Eqs. (4.48)–(4.49). Secondly, in Ch. (6), we converted the highly regular gauge metric perturbations that were originally written in Fermi–Walker coordinates into a fully covariant form, in Ch. 6.2.3 and into a generic coordinate expansion, in Ch. 6.3. This will allow others to easily implement the highly regular gauge in any preferred coordinate system without having to calculate a potentially complicated gauge transformation from some starting gauge to the highly regular gauge. We were unable to successfully show that the highest-order pieces of the $h^{\mathrm{S1},\boldsymbol{a}}_{\mu\nu}$ term from Eq. (6.35) and $h^{\mathrm{SR}}_{\mu\nu}$ term from Eq. (6.36) in the covariant expansion satisfied the appropriate Einstein field equations but we have provided them (and all of the other covariant and coordinate punctures) in a Mathematica notebook in the Additional Material [141].

Thirdly, in Ch. 5.1, using the weaker divergence properties of the highly regular gauge, we rigorously derived the form of a unique second-order stress-energy tensor that describes the system, which we gave in Eq. (5.27). We found, in Ch. 5.1.4, that this is merely given by the stress-energy tensor of a point particle in an effective metric, $\tilde{g}_{\mu\nu}$, allowing one to rewrite the field equations in terms of effective quantities. This confirms a previous conjecture by Detweiler [93] and, as such, we name this effective stress-energy the Detweiler stress-energy tensor. The result demonstrates the validity of point masses beyond linear order in perturbation theory and provides the physical interpretation of a self-gravitating object both moving as and having the stress-energy of a test body in the effective spacetime.

We demonstrated in Ch. 5.1.7 that this result is true in any gauge smoothly related to the highly regular gauge and, in Ch. 5.2, it is also true in one of the most widely use gauges in self-force, the Lorenz gauge. It is likely that this is true in other gauges as well but will require the adoption of Detweiler's canonical definition, from Eq. (5.67), for the second-order Einstein tensor. In non-highly regular gauges, one is required to know the local form of the metric perturbations before performing the calculation of stress-energy tensor. This is in contrast to the highly regular gauge, where one can derive the result purely using its distributional nature that is a consequence of the weaker divergence at the worldline.

Our fourth result is the reformulation of the second-order Einstein equations, in Ch. 5.3.1, and the second-order Teukolsky equation, in Ch. 5.3.2, to account for Detweiler's canonical definition of the second-order Einstein tensor and to provide a practical way one could implement our distributional results in a numerical scheme. We had originally worked with total quantities $\delta G_{\mu\nu}[h^2] + \delta^2 G_{\mu\nu}[h^1, h^1] = T^2_{\mu\nu}$, meaning that the most singular parts of the left-hand side of the equation cancelled each other by the canonical definition. As stated in the text above Eq. (5.61), this meant we never had to calculate the distributional nature of $\delta^2 G_{\mu\nu}[h^{\mathrm{S}1}, h^{\mathrm{S}1}]$ as it was cancelled by $\delta G_{\mu\nu}[h^{\mathrm{SS}}]$. When moving $\delta^2 G_{\mu\nu}[h^{\mathrm{S}1}, h^{\mathrm{S}1}]$ to the right-hand side of the equation, we used Detweiler's canonical definition and derived the distributional nature of this term in Eq. (5.104). This was a divergent quantity but it acted as a counter term and exactly cancelled the divergent behaviour that one would encounter if numerically integrating the expression. We also calculated the delta function content in the source of the second-order Teukolsky equation in Eqs. (5.123), (5.132) and (5.140). Some of the resulting expressions were too length to include in the text of the thesis so have been included in the Additional Material [141]. It remains to calculate the form of any counter terms that may appear, as in the Einstein field equations, but this is an avenue for future research and our derivation provides the necessary steps for performing such a calculation.

The final result is the gauge transformation from the Lorenz gauge to the highly regular gauge for quasicircular orbits in Schwarzschild in Ch. 7. We derived the form of the gauge vector to all orders required, in Eq. (7.27). This built on work previously done by Spiers [142]. The covariant form of the gauge vector was expanded into a generic coordinate form in Eq. (7.30) and then written in the Newman–Penrose formalism in Eqs. (7.59)–(7.61).

In Ch. 7.5, we developed an algorithmic way to perform the mode decomposition. We began by calculating the polar integrals against spin-weighted spherical harmonics for generic ℓ (and sometime for generic s and m' as well). While the resulting expressions featured complicated sums, we found that these reduced to sums of polynomials in δ^2 multiplied by some δ-dependent prefactor, as expected from Ref. [65, Eqs. (448)–(449)]. We then performed an expansion of the non-ℓ dependent prefactor to the polynomials that matched the order of the gauge vector we had calculated through. This was required to analytically evaluate the integrals over β but we avoided truncating the polynomials in δ^2 as this would introduce poor behaviour at large values of ℓ. After performing the final integration to find the modes of the gauge vector, we checked our analytically derived modes for the gauge vector against a full numerical calculation of the modes and found a relative error of, at worst, 1% between the two. As mentioned, if we wished to reduce this error, we could just perform a higher order expansion of the δ dependent prefactors that appear after doing the integral over α. In Ch. 7.7, we then used expressions from Ref. [135] for the metric perturbations in terms of a gauge vector to construct the form of the change in the metric perturbations induced by the gauge transformation. These expressions were written in the Carter tetrad and were given in Eqs. (7.120)–(7.129). We have provided the modes of the gauge vector in the Additional Material [141].

8.2 Future research

In terms of future directions for research following from this thesis, one obvious case is to extend the highly regular gauge to include the case where the small object is spinning or is not spherically symmetric. Our derivation specialised to a spherically symmetric, non-spinning body and should be generalised to the more astrophysically relevant case. It would be interesting to see how this affects the results for the Detweiler stress-energy tensor as well. It may also be possible to make $h^{\rm SS}_{\mu\nu}$ more regular by removing the r^0 piece of the metric perturbation meaning that $h^{\rm SS}_{\mu\nu}|_\gamma = 0$. We have attempted this but so far been unsuccessful in our attempts to find the gauge transformation that performs this task. It may be that these pieces contain physically meaningful information that can not be gauged away. One could then generate covariant and coordinate punctures in this case as well.

Following from the gauge conditions of the highly regular gauge, given in Ch. 7.1, one can write the perturbations in terms of null vectors. For example, if one defines

$$\begin{aligned} k_\alpha &= \frac{g^{\alpha'}{}_\alpha}{\sqrt{2}}\left(u_{\alpha'} + \frac{P_{\alpha'\beta'}\sigma^{\beta'}}{\boldsymbol{\rho}}\right) \\ &= \frac{g^{\alpha'}{}_\alpha}{\sqrt{2}\boldsymbol{\rho}}\Big(\sigma_{\alpha'} + (\boldsymbol{r}+\boldsymbol{\rho})u_{\alpha'}\Big), \end{aligned} \tag{8.1}$$

so that $k^\alpha k_\alpha = 0$, one can write the the first-order singular field from Eq. (6.31) as

$$h^{\mathrm{S1}\not{d}}_{\alpha\beta} = \frac{4m}{\lambda\boldsymbol{\rho}} k_\alpha k_\beta + \mathcal{O}(\lambda). \tag{8.2}$$

One can then write Eq. (8.2) in terms of these null vectors as

$$\begin{aligned} \mathsf{g}_{\mu\nu} &= g_{\mu\nu} + \epsilon h^{\mathrm{S1}\not{d}}_{\mu\nu} + \mathcal{O}(\epsilon), \\ &= g_{\mu\nu} + 2\epsilon V k_\mu k_\nu + \mathcal{O}\Big(\epsilon\lambda, \epsilon^2\Big), \end{aligned} \tag{8.3}$$

where $V = 2m/(\lambda\boldsymbol{\rho})$, which has the form of a Kerr–Schild metric [194–196]. It would be interesting to further explore the connection between the highly regular gauge and Kerr–Schild gauges, potentially drawing on previous work by Harte [197] and Harte and Vines [198].

We mentioned in the body of the thesis that it so far has not been possible to perform mode-sum regularisation at second order due to the strength of the divergence encountered on the worldline of the small object. With the highly regular gauge, however, this may be possible. As stated in the introduction, the modes of the most singular part of the second-order source in the highly regular gauge behave, at worst, as $\sim \log(\Delta r)$. Because we can think of $\delta G^{\mu\nu}[h^2] \sim \partial^2 h^2$, this implies the the modes of $h^2_{\mu\nu}$ could behave, at worst, as $\sim \Delta r^2 \log(\Delta r)$. This would be smooth enough to use for mode-sum regularisation. With the knowledge of the second-order stress-energy tensor, one could directly solve for the modes of $h^2_{\mu\nu}$ and construct the modes of the regular field by subtracting the modes of the singular field from it. This may also be possible in the Lorenz gauge, using our formulation from Ch. 5.3.1, but it is not immediately clear exactly how one would implement this.

The weaker divergence would also be useful in a puncture scheme as well. As a first test, one could use the methods of Ref. [137] to decompose the highly regular gauge puncture into modes in the Schwarzschild spacetime. With the modes calculated, one could calculate $\delta^2 G_{i\ell m}[h^{\mathcal{P}1}, h^{\mathcal{P}1}]$ using the methods of Ref. [135] to see how the modes behave in this case. The best case result would be no longer encountering the infinite mode coupling problem. One would then not have to use the method developed in Ref. [132] to avoid this problem. As stated in the introduction, while the use of this method is necessary to compute the second-order source, it also accounts for the overwhelming majority of the computational cost of the calculation.

While not required for EMRI research, it would be conceptually interesting to see if the result for the Detweiler stress-energy tensor can be extended to higher order in ϵ. That is, does the point-mass view of the small object continue to remain valid at higher perturbative orders? Is it always possible to think of the small object as a point mass in an effective spacetime or is there some upper limit where this breaks down?

With regards to the gauge transformation in Schwarzschild, the next step would be to construct the modes of the change in the metric perturbation and test that, when combined with the perturbations in the Lorenz gauge, they satisfy the highly regular gauge conditions. We have had preliminary success with this as initial testing has shown that the trace test from Ch. 7.1 is satisfied. However, we have not been able to successfully show that the null vector test is satisfied. It is not entirely clear why this is the case as the analytic gauge vector shows good agreement with the numerically calculated one. This leads us to believe that this is an issue with the actual test itself or the way in which we have constructed the $\Delta h^1_{\mu\nu}$'s.

Finally, the work determining the delta content of the source for the Teukolsky equation in Ch. 5.3.2 contributes to an international collaboration working towards solving the Teukolsky equation in the Kerr spacetime. The first goal is to solve it in the case of quasicircular orbits in Schwarzschild [199]. To incorporate our work into this requires us to express the final source in Schwarzschild coordinates. Doing so, we explicitly write the covariant delta function in terms of coordinate quantities and calculate how the derivatives act upon them (if any do). This is done by writing the angular dependence as a sum over spin-weighted spherical harmonics and acting upon the harmonics with the angular derivatives. One also acts upon the (t, r) dependence in the delta functions by taking derivatives of them, leaving us with $\delta'(r - r_0)$, for example. Once in coordinates, we can evaluate the integrals over the time coordinate that appear and explicitly find the form of the delta content of the source. One final thing that must be done is to determine if any counter terms appear in these expressions and calculate them to ensure that any divergences cancel when numerically calculating the source. It would be interesting to see if the method of introducing a counter term is equivalent to other regularisation procedures that are performed in physics and mathematics, such as Hadamard or dimensional regularisation.

Appendix A

Correspondence between STF expansion of the regular field and derivatives of the regular field

This section details how to relate the STF tensors featured in the decomposition of the regular field in Sec. 4.1 to derivatives of the field evaluated on the worldline. The first two orders match those presented in App. B of Ref. [81] but with some STF labels switched.[1]

At order r^0

$$\hat{A}^{(0,0)} = {}^0h^{\mathrm{R1}}_{tt}\Big|_\gamma, \tag{A.1}$$

$$\hat{B}^{(0,0)}_a = {}^0h^{\mathrm{R1}}_{ta}\Big|_\gamma, \tag{A.2}$$

$$\hat{E}^{(0,0)}_{ab} = {}^0h^{\mathrm{R1}}_{\langle ab\rangle}\Big|_\gamma, \tag{A.3}$$

$$\hat{K}^{(0,0)} = \frac{1}{3}\delta^{ab}\,{}^0h^{\mathrm{R1}}_{ab}\Big|_\gamma. \tag{A.4}$$

At order r

$$\hat{A}^{(1,1)}_a = {}^0h^{\mathrm{R1}}_{tt,a}\Big|_\gamma, \tag{A.5}$$

$$\hat{B}^{(1,1)}_{ab} = {}^0h^{\mathrm{R1}}_{t\langle a,b\rangle}\Big|_\gamma, \tag{A.6}$$

$$\hat{C}^{(1,1)}_a = \frac{1}{2}\epsilon_a{}^{bc}\,{}^0h^{\mathrm{R1}}_{tb,c}\Big|_\gamma, \tag{A.7}$$

$$\hat{D}^{(1,1)} = \frac{1}{3}\,{}^0h^{\mathrm{R1}a}_{ta,}\Big|_\gamma, \tag{A.8}$$

$$\hat{E}^{(1,1)}_{abc} = {}^0h^{\mathrm{R1}}_{\langle ab,c\rangle}\Big|_\gamma, \tag{A.9}$$

[1] Eq. (B5h) in Ref. [81] has the prefactor 1/6 which has been corrected here in Eq. (A.12) to 1/3

$$\hat{F}^{(1,1)}_{ab} = \frac{2}{3}\epsilon^{cd}{}_{(a}\,{}^{0}h^{\mathrm{R1}}_{b)c,d}\Big|_{\gamma}, \tag{A.10}$$

$$\hat{G}^{(1,1)}_{a} = \frac{3}{5}\,{}^{0}h^{\mathrm{R1}}_{\langle ab\rangle,}{}^{b}\Big|_{\gamma}, \tag{A.11}$$

$$\hat{K}^{(1,1)}_{a} = \frac{1}{3}\delta^{bc}\,{}^{0}h^{\mathrm{R1}}_{bc,a}\Big|_{\gamma}. \tag{A.12}$$

Finally at order r^2

$$\hat{A}^{(2,0)} = \frac{1}{6}\,{}^{0}h^{\mathrm{R1}}_{tt,a}{}^{a}\Big|_{\gamma}, \tag{A.13}$$

$$\hat{A}^{(2,2)}_{ab} = \frac{1}{2}\,{}^{0}h^{\mathrm{R1}}_{tt,\langle ab\rangle}\Big|_{\gamma}, \tag{A.14}$$

$$\hat{B}^{(2,0)}_{a} = \frac{1}{6}\,{}^{0}h^{\mathrm{R1}}_{ta,b}{}^{b}\Big|_{\gamma}, \tag{A.15}$$

$$\hat{B}^{(2,2)}_{abc} = \frac{1}{2}\,{}^{0}h^{\mathrm{R1}}_{t\langle a,bc\rangle}\Big|_{\gamma}, \tag{A.16}$$

$$\hat{C}^{(2,2)}_{ab} = \frac{1}{3}\epsilon^{cd}{}_{(a}\,{}^{0}h^{\mathrm{R1}}_{|tc|,b)d}\Big|_{\gamma}, \tag{A.17}$$

$$\hat{D}^{(2,2)}_{a} = \frac{3}{10}\,{}^{0}h^{\mathrm{R1}b}_{t}{}_{,\langle ab\rangle}\Big|_{\gamma}, \tag{A.18}$$

$$\hat{E}^{(2,0)}_{ab} = \frac{1}{6}\,{}^{0}h^{\mathrm{R1}}_{\langle ab\rangle,c}{}^{c}\Big|_{\gamma}, \tag{A.19}$$

$$\hat{E}^{(2,2)}_{abcd} = \frac{1}{2}\,{}^{0}h^{\mathrm{R1}}_{\langle ab,cd\rangle}\Big|_{\gamma}, \tag{A.20}$$

$$\hat{F}^{(2,2)}_{abc} = \frac{1}{2}\,\underset{abc}{\mathrm{STF}}\Big(\epsilon_{a}{}^{pq}\,{}^{0}h^{\mathrm{R1}}_{\langle pb\rangle,qc}\Big|_{\gamma}\Big), \tag{A.21}$$

$$\hat{G}^{(2,2)}_{ab} = \frac{6}{7}\,\underset{ab}{\mathrm{STF}}\Big({}^{0}h^{\mathrm{R1}}_{\langle ja\rangle,}{}^{j}{}_{b}\Big|_{\gamma}\Big), \tag{A.22}$$

$$\hat{H}^{(2,2)}_{a} = \frac{1}{5}\epsilon_{a}{}^{cd}\,{}^{0}h^{\mathrm{R1}}_{bc,d}{}^{b}\Big|_{\gamma}, \tag{A.23}$$

$$\hat{I}^{(2,2)} = \frac{1}{10}\,{}^{0}h^{\mathrm{R1}}_{\langle ab\rangle,}{}^{ab}\Big|_{\gamma}, \tag{A.24}$$

$$\hat{K}^{(2,0)} = \frac{1}{18}\,{}^{0}h^{\mathrm{R1}a,b}_{a}{}_{b}\Big|_{\gamma}, \tag{A.25}$$

$$\hat{K}^{(2,2)}_{ab} = \frac{1}{6}\,{}^{0}h^{\mathrm{R1}c}_{c}{}_{,\langle ab\rangle}\Big|_{\gamma}. \tag{A.26}$$

Appendix B

Lie derivative of the first-order stress-energy tensor

For the transformation to the highly regular gauge, we are only concerned with worldline-preserving transformations where the flow orthogonal to the worldline vanishes on the worldline so that the position of the worldline is unchanged. However, it is interesting to consider non-worldline preserving gauge transformations as well. As discussed in Ref. [139], this necessitates the introduction of another Lie derivative, $\pounds$, which generates a flow by dragging points of the worldline, z^μ, relative to points of the field, x^μ. Instead of Eq. (5.22), the second-order stress-energy tensor now transforms as

$$T_2^{\mu\nu} = T_{2'}^{\mu\nu} + (\mathcal{L}_{\xi_1} + \pounds_{\xi_1})\, T_1^{\mu\nu}. \tag{B.1}$$

The Lie derivatives of $T_1^{\mu\nu}$ were previously presented in Ref. [139]. Here we reproduce (and correct a small error in) that result, and we derive analogous results for the Lie derivatives of $T^{1\nu}_{\mu}$ and $T^1_{\mu\nu}$.

B.1 Lie derivatives of $T_1^{\mu\nu}$

Eq. (5.10) may be written so that it is invariant under reparametrisation as [63]

$$T_1^{\mu\nu}(x;z) = m \int_\gamma g^\mu_{\mu'}(x,z) g^\nu_{\nu'}(x,z) \dot{z}^{\mu'} \dot{z}^{\nu'} \frac{\delta^4(x,z)}{\sqrt{-g_{\rho'\sigma'}(z)\dot{z}^{\rho'}\dot{z}^{\sigma'}}}\, ds\,, \tag{B.2}$$

where $g^\mu_{\mu'}(x,z)$ is a parallel propagator from $x^{\mu'} := z^\mu$ to x^μ, and $\dot{z}^{\mu'} := \frac{dz^{\mu'}}{ds}$. This form is particularly useful for our calculations of Lie derivatives of $T_1^{\mu\nu}$.

The ordinary Lie derivative is evaluated in the standard way, so

$$\mathcal{L}_{\xi_1} T_1^{\mu\nu} = m \int_\gamma \mathcal{L}_{\xi_1} \left(g^\mu_{\mu'}(x,z) g^\nu_{\nu'}(x,z) \dot{z}^{\mu'} \dot{z}^{\nu'} \frac{\delta^4(x,z)}{\sqrt{-g_{\rho'\sigma'}(z)\dot{z}^{\rho'}\dot{z}^{\sigma'}}} \right) ds\,. \tag{B.3}$$

The Lie derivative of the Dirac delta is found by integrating against a test function and is given by

$$\mathcal{L}_{\xi_1} \delta^4(x,z) = -\left(\xi_1^{\alpha'}{}_{;\alpha'} + \xi_1^{\alpha'} \nabla_{\alpha'} \right) \delta^4(x,z). \tag{B.4}$$

The other term in Eq. (B.3) is

$$\begin{aligned} \int_\gamma \mathcal{L}_{\xi_1} \left(W^{\mu\nu} \right) \delta^4(x,z)\, ds =& \int_\gamma \left(\xi_1^\rho W^{\mu\nu}{}_{;\rho} - 2 \xi_1^{(\mu}{}_{;\rho} W^{\nu)\gamma} \right) \delta^4(x,z)\, ds \\ =& -2 \int_\gamma \xi_1^{(\mu}{}_{;\rho} W^{\nu)\gamma} \delta^4(x,z)\, ds\,, \end{aligned} \tag{B.5}$$

where

$$W^{\mu\nu} := \frac{g^\mu_{\mu'} g^\nu_{\nu'} \dot{z}^{\mu'} \dot{z}^{\nu'}}{\sqrt{-g_{\rho'\sigma'}\dot{z}^{\rho'}\dot{z}^{\sigma'}}} \tag{B.6}$$

In the second line of Eq. (B.5), we have used the identity $g^\alpha_{\beta';\beta}\delta^4(x,z) = 0$ [63] to eliminate $W^{\mu\nu}{}_{;\rho}$. Taking our parameter s to be proper time, we see that

$$\begin{aligned} \int_\gamma \mathcal{L}_{\xi_1} \left(W^{\mu\nu} \right) \delta^4(x,z)\, ds =& -2 \int_\gamma g^{(\mu}_{\mu'} \xi_1^{\nu)}{}_{;\rho} g^\rho_{\nu'} u^{\mu'} u^{\nu'} \delta^4(x,z)\, d\tau \\ =& -2 \int_\gamma g^\mu_{\mu'} g^\nu_{\nu'} u^{(\mu'} \frac{D\xi_1^{\nu')}}{d\tau} \delta^4(x,z)\, d\tau\,. \end{aligned} \tag{B.7}$$

The final line is obtained by integrating the previous line against a test field $\phi_{\mu\nu}$:

$$\begin{aligned} \int \phi_{\mu\nu} \int_\gamma g^{(\mu}_{\mu'} \xi_1^{\nu)}{}_{;\rho} g^\rho_{\nu'} u^{\mu'} u^{\nu'} \delta^4(x,z)\, d\tau\, dV =& \int_\gamma \phi_{\mu'\nu'} u^{(\mu'} \xi_1^{\nu')}{}_{;\rho'} u^{\rho'}\, d\tau \\ =& \int \phi_{\mu\nu} \int_\gamma g^\mu_{\mu'} g^\nu_{\nu'} u^{(\mu'} \frac{D\xi_1^{\nu')}}{d\tau} \delta^4(x,z)\, d\tau\, dV\,. \end{aligned} \tag{B.8}$$

By combining Eqs. (B.4) and (B.7), we see

$$\begin{aligned} \int_\gamma \mathcal{L}_{\xi_1} \left(\frac{g^\mu_{\mu'} g^\nu_{\nu'} \dot{z}^{\mu'} \dot{z}^{\nu'}}{\sqrt{-g_{\rho'\sigma'}\dot{z}^{\rho'}\dot{z}^{\sigma'}}} \delta^4(x,z) \right) ds =& -\int_\gamma g^\mu_{\mu'} g^\nu_{\nu'} \Bigg[\left(2u^{(\mu'} \frac{D\xi_1^{\nu')}}{d\tau} + u^{\mu'} u^{\nu'} \xi_1^{\rho'}{}_{;\rho'} \right) \\ & \times \delta^4(x,z) + u^{\mu'} u^{\nu'} \xi_1^{\rho'} \nabla_{\rho'} \delta^4(x,z) \Bigg] d\tau\,. \end{aligned} \tag{B.9}$$

This can be simplified by decomposing $\xi_1^{\alpha'}$ into parallel and orthogonal parts,

$$\xi_1^{\alpha'} = -u^{\alpha'} \xi^1_{\|} + \xi_{1\perp}^{\alpha'}, \tag{B.10}$$

where $\xi_{1\perp}^{\alpha'} := P^{\alpha'}{}_{\beta'}\xi_1^{\beta'}$. With this decomposition, we obtain

$$\mathcal{L}_{\xi_1} T_1^{\mu\nu} = -m \int_\gamma g^\mu_{\mu'} g^\nu_{\nu'} \left[2u^{(\mu'} \frac{D\xi_{1\perp}^{\nu')}}{d\tau} \delta^4(x,z) + u^{\mu'}u^{\nu'} \left(\xi_1^{\rho'}{}_{;\rho'} - \frac{d\xi_\parallel^1}{d\tau} \right) \delta^4(x,z) + u^{\mu'}u^{\nu'} \xi_{1\perp}^{\rho'} \nabla_{\rho'} \delta^4(x,z) \right] d\tau + \mathcal{O}(\epsilon), \tag{B.11}$$

which agrees with Eq. (D1) in Ref. [139] (with the correction of the minus sign as discussed in footnote 1 of Ch. 5).

As discussed in Ref. [139], because $T_1^{\mu\nu}$ can be written in the form

$$A^{\mu\nu}(x;z) = \int_\gamma B^{\mu\nu}(x,z(s)) \sqrt{-g_{\mu'\nu'}\dot{z}^{\mu'}\dot{z}^{\nu'}}\, ds\,, \tag{B.12}$$

its Lie derivative with respect to the dependence on z^μ is given by

$$£_{\xi_1} A^{\mu\nu}(x;z) = \int_\gamma \xi_{1\perp}^{\rho'} \nabla_{\rho'} B^{\mu\nu}(x,z)\, d\tau\,. \tag{B.13}$$

For $T_1^{\mu\nu}$, we see that

$$B^{\mu\nu} = m \frac{g^\mu_{\mu'} g^\nu_{\nu'} \dot{z}^{\mu'}\dot{z}^{\nu'}}{-g_{\rho'\sigma'}\dot{z}^{\rho'}\dot{z}^{\sigma'}} \delta^4(x,z), \tag{B.14}$$

which implies

$$\begin{aligned} £_{\xi_1} T_1^{\mu\nu} &= m \int_\gamma \xi_{1\perp}^{\rho'} \nabla_{\rho'} \left(\frac{g^\mu_{\mu'} g^\nu_{\nu'} \dot{z}^{\mu'}\dot{z}^{\nu'}}{-g_{\rho'\sigma'}\dot{z}^{\rho'}\dot{z}^{\sigma'}} \delta^4(x,z) \right) ds \\ &= \int_\gamma g^\mu_{\mu'} g^\nu_{\nu'} \left(2u^{(\mu'} \dot{\xi}_{1\perp}^{\nu')} \delta^4(x,z) + u^{\mu'}u^{\nu'} \xi_{1\perp}^{\rho'} \nabla_{\rho'} \delta^4(x,z) \right) d\tau\,, \end{aligned} \tag{B.15}$$

where we have used $g^\alpha_{\beta';\gamma'}\delta^4(x,z) = 0$ and $\xi_{1\perp}^\nu \nabla_\nu \dot{z}^\mu = \dot{z}^\nu \nabla_\nu \xi_{1\perp}^\mu$. The latter identity follows from Eq. (B1) in Ref. [139].

Eqs. (B.11) and (B.15) sum to give

$$(\mathcal{L}_{\xi_1} + £_{\xi_1}) T_1^{\mu\nu} = -m \int_\gamma g^\mu_{\mu'} g^\nu_{\nu'} u^{\mu'}u^{\nu'} \delta^4(x,z) \left(\xi^\rho_{1;\rho} - \frac{d\xi_\parallel^1}{d\tau} \right) d\tau\,, \tag{B.16}$$

which matches Eq. (D2) from Ref. [139] as expected (again with the missing minus sign added). Note that this is also the same as Eq. (5.24) as $£_{\xi_1} T_1^{\mu\nu} = 0$ for that specific calculation.

B.2 Lie derivatives of $T^1_{\mu\nu}$ and $T^{1\nu}_{\mu}$

The first-order stress-energy tensor with both indices down is given by

$$T^1_{\mu\nu}(x;z) = m\int_\gamma g_{\mu\alpha}g_{\nu\beta}g^{\alpha}_{\mu'}(x,z)g^{\beta}_{\nu'}(x,z)\dot{z}^{\mu'}\dot{z}^{\nu'}\frac{\delta^4(x,z)}{\sqrt{-g_{\rho'\sigma'}(z)\dot{z}^{\rho'}\dot{z}^{\sigma'}}}\,ds \tag{B.17}$$

and with mixed indices by

$$T^{1\nu}_{\mu}(x;z) = m\int_\gamma g_{\mu\alpha}g^{\alpha}_{\mu'}(x,z)g^{\nu}_{\nu'}(x,z)\dot{z}^{\mu'}\dot{z}^{\nu'}\frac{\delta^4(x,z)}{\sqrt{-g_{\rho'\sigma'}(z)\dot{z}^{\rho'}\dot{z}^{\sigma'}}}\,ds\,. \tag{B.18}$$

To calculate the Lie derivatives of these quantities, we follow the same methods described above. The results are

$$\begin{aligned}\mathcal{L}_{\xi_1}T^1_{\mu\nu} &= m\int_\gamma g_{\mu\alpha}g_{\nu\beta}g^{\alpha}_{\alpha'}g^{\beta}_{\beta'}\Big(2\xi^1_{\rho';}{}^{(\alpha'}u^{\beta')}u^{\rho'} - u^{\alpha'}u^{\beta'}\Big[\xi_1^{\rho'}{}_{;\rho'} + \dot{\xi}^1_{\|} + \xi^{\rho'}_{1\perp}\nabla_{\rho'}\Big]\Big) \\ &\qquad\times\delta^4(x,z)\,d\tau\,, \qquad\qquad (B.19)\\ \mathcal{L}_{\xi_1}T^{1\nu}_{\mu} &= m\int_\gamma g_{\mu\alpha}g^{\alpha}_{\alpha'}g^{\nu}_{\nu'}\Big(\xi^1_{\rho';}{}^{\alpha'}u^{\nu'}u^{\rho'} - \xi_1^{\nu'}{}_{;\rho'}u^{\alpha'}u^{\rho'} - u^{\alpha'}u^{\nu'}\Big[\xi_1^{\rho'}{}_{;\rho'} + \dot{\xi}^1_{\|} + \xi^{\rho'}_{1\perp}\nabla_{\rho'}\Big]\Big) \\ &\qquad\times\delta^4(x,z)\,d\tau\,. \qquad\qquad (B.20)\end{aligned}$$

Here and below, an overdot denotes a derivative with respect to τ.

The Lie derivatives at z^μ follow trivially from Eq. (B.15). Since we can pass the contraction through the derivative, as in $g_{\mu\rho}\pounds_{\xi_1}T_1^{\rho\nu} = \pounds_{\xi_1}(g_{\mu\rho}T_1^{\rho\nu})$, we get

$$\pounds_{\xi_1}T^1_{\mu\nu} = \int_\gamma g_{\mu\alpha}g_{\nu\beta}g^{\alpha}_{\mu'}g^{\beta}_{\nu'}\Big(2u^{(\mu'}\dot{\xi}^{\nu')}_{1\perp}\delta^4(x,z) + u^{\mu'}u^{\nu'}\xi^{\rho'}_{1\perp}\nabla_{\rho'}\delta^4(x,z)\Big)\,d\tau\,, \tag{B.21}$$

$$\pounds_{\xi_1}T^{1\nu}_{\mu} = \int_\gamma g_{\mu\alpha}g^{\alpha}_{\mu'}g^{\beta}_{\nu'}\Big(2u^{(\mu'}\dot{\xi}^{\nu')}_{1\perp}\delta^4(x,z) + u^{\mu'}u^{\nu'}\xi^{\rho'}_{1\perp}\nabla_{\rho'}\delta^4(x,z)\Big)\,d\tau\,. \tag{B.22}$$

Combining these results, we find

$$\begin{aligned}\left(\mathcal{L}_{\xi_1}+\pounds_{\xi_1}\right)T^1_{\mu\nu} &= m\int_\gamma g_{\mu\alpha}g_{\nu\beta}g^{\alpha}_{\alpha'}g^{\beta}_{\beta'}\Big(2\xi^1_{\rho';}{}^{(\alpha'}u^{\beta')}u^{\rho'} + 2u^{(\alpha'}\dot{\xi}_1^{\beta')} + u^{\alpha'}u^{\beta'}\dot{\xi}^1_{\|} \\ &\qquad - u^{\alpha'}u^{\beta'}\xi^{\rho'}_{1;\rho'}\Big)\delta^4(x,z)\,d\tau\,, \qquad\qquad (B.23)\\ \left(\mathcal{L}_{\xi_1}+\pounds_{\xi_1}\right)T^{1\nu}_{\mu} &= m\int_\gamma g_{\mu\alpha}g^{\alpha}_{\alpha'}g^{\nu}_{\nu'}u^{\nu'}\Big(\xi^1_{\|;}{}^{\alpha'} - \xi^{\rho'}_{1;\rho'}u^{\alpha'} + \dot{\xi}_1^{\alpha'} + u^{\alpha'}\dot{\xi}^1_{\|}\Big)\delta^4(x,z)\,d\tau\,. \qquad (B.24)\end{aligned}$$

Appendix C

Newman–Penrose formalism

Introduced by Newman and Penrose [200], the eponymous Newman–Penrose (NP) formalism rewrites the equations and tools of general relativity in terms of spinor calculus through the introduction of a null tetrad basis, $e^{\mu}_{\mathfrak{a}} := (l^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu})_{\mathfrak{a}}$, where $\mathfrak{a} \in \{1, 2, 3, 4\}$. In this appendix, we outline the construction of the NP formalism and provide definitions for NP quantities that will be used in this thesis. We mainly follow the details from the original paper [200] with some additional information and identities provided by Refs. [201, 202].

In the NP formalism, l^{μ} and n^{μ} are both real valued vectors and m^{μ} is a complex valued vector with $\bar{m}^{\mu}$ being its complex conjugate. The vectors satisfy the orthogonality conditions[1]

$$l^{\mu}l_{\mu} = n^{\mu}n_{\mu} = m^{\mu}m_{\mu} = \bar{m}^{\mu}\bar{m}_{\mu} = 0, \tag{C.1}$$

$$l^{\mu}m_{\mu} = l^{\mu}\bar{m}_{\mu} = n^{\mu}m_{\mu} = n^{\mu}\bar{m}_{\mu} = 0, \tag{C.2}$$

$$l^{\mu}n_{\mu} = -m^{\mu}\bar{m}_{\mu} = -1. \tag{C.3}$$

It follows from these definitions that one may then write the metric as

$$g_{\mu\nu} = -l_{\mu}n_{\nu} - n_{\mu}l_{\nu} + m_{\mu}\bar{m}_{\nu} + \bar{m}_{\mu}m_{\nu}, \tag{C.4}$$

with indices down and

$$g^{\mu\nu} = -l^{\mu}n^{\nu} - n^{\mu}l^{\nu} + m^{\mu}\bar{m}^{\nu} + \bar{m}^{\mu}m^{\nu}, \tag{C.5}$$

with indices up.

[1] In Newman and Penrose's original paper [200], they used the mostly-minus metric signature, $(+, -, -, -)$. However, in this thesis, we use the mostly-plus metric signature, $(-, +, +, +)$, which has the effect of introducing an overall minus sign to some expressions. For example, in the original paper, $l^{\mu}n_{\mu} = -m^{\mu}\bar{m}_{\mu} = 1$, which has the sign swapped in Eq. (C.3).

The Ricci rotation coefficients (connection coefficients for orthonormal, non-holonomic bases) for the tetrad can then be written as

$$\gamma^{\mathfrak{abc}} = e^{\mathfrak{a}}_{\mu;\nu} e^{\mathfrak{b}\mu} e^{\mathfrak{c}\nu}. \tag{C.6}$$

These are complex valued and antisymmetric in the first two indices,

$$\gamma^{\mathfrak{abc}} = -\gamma^{\mathfrak{bac}}. \tag{C.7}$$

The different components of the Ricci rotation coefficients can then be expressed as 12 complex spin coefficients, defined as

$$\kappa := -m^\mu D l_\mu, \tag{C.8a}$$
$$\tau := -m^\mu \Delta l_\mu, \tag{C.8b}$$
$$\sigma := -m^\mu \delta l_\mu, \tag{C.8c}$$
$$\rho := -m^\mu \bar{\delta} l_\mu, \tag{C.8d}$$
$$\pi := \bar{m}^\mu D n_\mu, \tag{C.8e}$$
$$\nu := \bar{m}^\mu \Delta n_\mu, \tag{C.8f}$$
$$\mu := \bar{m}^\mu \delta n_\mu, \tag{C.8g}$$
$$\lambda := \bar{m}^\mu \bar{\delta} n_\mu, \tag{C.8h}$$
$$\epsilon := -\frac{1}{2}\Big(n^\mu D l_\mu - \bar{m}^\mu D m_\mu\Big), \tag{C.8i}$$
$$\gamma := -\frac{1}{2}\Big(n^\mu \Delta l_\mu - \bar{m}^\mu \Delta m_\mu\Big), \tag{C.8j}$$
$$\beta := -\frac{1}{2}\Big(n^\mu \delta l_\mu - \bar{m}^\mu \delta m_\mu\Big), \tag{C.8k}$$
$$\alpha := -\frac{1}{2}\Big(n^\mu \bar{\delta} l_\mu - \bar{m}^\mu \bar{\delta} m_\mu\Big), \tag{C.8l}$$

where we have introduced notation for the directional derivatives, given by

$$D\phi := l^\mu \nabla_\mu \phi, \tag{C.9a}$$
$$\Delta\phi := n^\mu \nabla_\mu \phi, \tag{C.9b}$$
$$\delta\phi := m^\mu \nabla_\mu \phi, \tag{C.9c}$$
$$\bar{\delta}\phi := \bar{m}^\mu \nabla_\mu \phi. \tag{C.9d}$$

Directional derivatives of the tetrad legs are then given by

$$D l^\mu = (\epsilon + \bar{\epsilon}) l^\mu - \bar{\kappa} m^\mu - \kappa \bar{m}^\mu, \tag{C.10a}$$
$$\Delta l^\mu = (\gamma + \bar{\gamma}) l^\mu - \bar{\tau} m^\mu - \tau \bar{m}^\mu, \tag{C.10b}$$
$$\delta l^\mu = (\bar{\alpha} + \beta) l^\mu - \bar{\rho} m^\mu - \sigma \bar{m}^\mu, \tag{C.10c}$$
$$\bar{\delta} l^\mu = (\alpha + \bar{\beta}) l^\mu - \bar{\sigma} m^\mu - \rho \bar{m}^\mu, \tag{C.10d}$$

$$D n^\mu = -(\epsilon+\bar\epsilon) n^\mu + \pi m^\mu + \bar\pi \bar m^\mu, \tag{C.10e}$$
$$\Delta n^\mu = -(\gamma+\bar\gamma) n^\mu + \nu m^\mu + \bar\nu \bar m^\mu, \tag{C.10f}$$
$$\delta n^\mu = -(\bar\alpha+\beta) n^\mu + \mu m^\mu + \bar\lambda \bar m^\mu, \tag{C.10g}$$
$$\bar\delta n^\mu = -(\alpha+\bar\beta) n^\mu + \lambda m^\mu + \bar\mu \bar m^\mu, \tag{C.10h}$$
$$D m^\mu = (\epsilon-\bar\epsilon) m^\mu + \bar\pi l^\mu - \kappa n^\mu, \tag{C.10i}$$
$$\Delta m^\mu = (\gamma-\bar\gamma) m^\mu + \bar\nu l^\mu - \tau n^\mu, \tag{C.10j}$$
$$\delta m^\mu = (\beta-\bar\alpha) m^\mu + \bar\lambda l^\mu - \sigma n^\mu, \tag{C.10k}$$
$$\bar\delta m^\mu = (\alpha-\bar\beta) m^\mu + \bar\mu l^\mu - \rho n^\mu, \tag{C.10l}$$
$$D \bar m^\mu = (\bar\epsilon-\epsilon) \bar m^\mu + \pi l^\mu - \bar\kappa n^\mu, \tag{C.10m}$$
$$\Delta \bar m^\mu = (\bar\gamma-\gamma) \bar m^\mu + \nu l^\mu - \bar\tau n^\mu, \tag{C.10n}$$
$$\delta \bar m^\mu = (\bar\alpha-\beta) \bar m^\mu + \mu l^\mu - \bar\rho n^\mu, \tag{C.10o}$$
$$\bar\delta \bar m^\mu = (\bar\beta-\alpha) \bar m^\mu + \lambda l^\mu - \bar\sigma n^\mu, \tag{C.10p}$$

One may also construct five complex-valued scalars, Ψ_n, from contractions of the tetrad legs with the Weyl tensor, $C_{\alpha\beta\mu\nu}$,

$$\Psi_0 \coloneqq C_{\alpha\beta\mu\nu} l^\alpha m^\beta l^\mu m^\nu, \tag{C.11a}$$
$$\Psi_1 \coloneqq C_{\alpha\beta\mu\nu} l^\alpha n^\beta l^\mu m^\nu, \tag{C.11b}$$
$$\Psi_2 \coloneqq C_{\alpha\beta\mu\nu} l^\alpha m^\beta \bar m^\mu n^\nu, \tag{C.11c}$$
$$\Psi_3 \coloneqq C_{\alpha\beta\mu\nu} n^\alpha l^\beta n^\mu \bar m^\nu, \tag{C.11d}$$
$$\Psi_4 \coloneqq C_{\alpha\beta\mu\nu} n^\alpha \bar m^\beta n^\mu \bar m^\nu. \tag{C.11e}$$

These are known as Weyl scalars as they contain the information about the ten independent components of the Weyl tensor. As we work in a vacuum background with $R_{\mu\nu} = 0$, the Weyl and Riemann tensors coincide and we are free to swap between them as required.

Appendix D

Spin-weighted spherical harmonics and Wigner-D matrices

This appendix details functions defined on the two-sphere, S^2. In particular, we outline the spherical harmonics, spin-weighted spherical harmonics and the Wigner-D matrix. These will be used extensively in Ch. 7 when decomposing the gauge vector for the transformation from the Lorenz gauge to the highly regular gauge, and in a number of other places in this thesis.

D.1 Spherical harmonics

The spherical harmonics are defined as [189, Ch. 14]

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_\ell^m(\cos\theta) e^{im\phi}, \tag{D.1}$$

where $\ell \leq m \leq \ell$. Here,

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_\ell(x)}{dx^m}, \quad m \geq 0 \tag{D.2}$$

are the associated Legendre polynomials, defined in terms of derivatives of the Legendre polynomials. The relation between positive and negative m values is given by

$$P_\ell^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_\ell^m(x). \tag{D.3}$$

Legendre polynomials themselves are solutions to the differential equation

$$(1-x^2)\frac{d^2 P_\ell(x)}{dx^2} - 2x\frac{dP_\ell(x)}{dx} + \ell(\ell+1)P_\ell(x) = 0. \tag{D.4}$$

The spherical harmonics are orthonormal on the unit two-sphere,

$$\int_{S^2} Y_{\ell m} Y^*_{\ell' m'}\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}, \tag{D.5}$$

where $d\Omega := \sin\theta\, d\theta\, d\phi$, satisfy the completeness relation

$$\sum_{\ell m} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) = \delta(\cos\theta - \cos\theta')\delta(\phi - \phi'). \tag{D.6}$$

and satisfy the symmetry relation

$$Y_{\ell,-m} = (-1)^m Y^*_{\ell m}, \tag{D.7}$$

where the asterisk in the previous three expressions denotes complex conjugation.

As the spherical harmonics form an orthonormal basis, we may write a function of (θ, ϕ) in terms of spherical harmonics as

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\theta, \phi), \tag{D.8}$$

where the angular dependence is solely contained in the spherical harmonics. To find the coefficients, we integrate against the complex conjugate of the spherical harmonics, so that

$$f_{\ell m} = \int_{S^2} f(\theta, \phi) Y^*_{\ell m}(\theta, \phi)\, d\Omega\,. \tag{D.9}$$

When evaluated at the north pole, all but the $m = 0$ modes vanish

$$Y_{\ell m}(0, \phi) = \begin{cases} \sqrt{\frac{2\ell+1}{4\pi}}, & m = 0, \\ 0, & |m| > 0. \end{cases} \tag{D.10}$$

D.2 Spin-weighted spherical harmonics

Spin-weighted spherical harmonics, ${}_sY_{\ell m}$, are generalisations of the spherical harmonics to objects with spin weight, s [203, 204]. An object, a, has spin weight, s, if it transforms as $a \to a e^{is\phi}$ under complex rotation of the basis, $m^\mu \to m^\mu e^{i\phi}$. In this appendix, we use sign conventions to match those of Ref. [135] when defining various quantities.

The spin-weight spherical harmonics can be defined by their relation to spherical harmonics by

$${}_sY_{\ell m} = \frac{1}{\lambda_s} \begin{cases} (-1)^s \eth_\uparrow^s Y_{\ell m}, & 0 \le s \le \ell, \\ \eth_\downarrow^{|s|} Y_{\ell m}, & -\ell \le s \le 0, \\ 0, & |s| > \ell, \end{cases} \tag{D.11}$$

where

$$\lambda_s := \sqrt{\frac{(\ell+|s|)!}{(\ell-|s|)!}}, \tag{D.12}$$

and

$$\eth_\uparrow a = (m^A D_A - s D_A m^A) a, \tag{D.13}$$

$$\eth_\downarrow a = (\bar{m}^A D_A + s D_A \bar{m}^A) a, \tag{D.14}$$

are the spin-raising/lowering operators, respectively, with A being an index on S_2 and D_A being the covariant derivative compatible with the metric on the unit sphere, $\Omega_{AB} = \mathrm{diag}(1, \sin^2(\theta))_{AB}$. They also have the properties that

$${}_sY^*_{\ell m} = (-1)^{m+s}{}_{-s}Y_{\ell,-m}, \tag{D.15}$$

$$\eth_\uparrow {}_sY_{\ell m} = -\sqrt{(\ell-s)(\ell+s+1)}{}_{s+1}Y_{\ell m}, \tag{D.16}$$

$$\eth_\downarrow {}_sY_{\ell m} = \sqrt{(\ell+s)(\ell-s+1)}{}_{s-1}Y_{\ell m}, \tag{D.17}$$

$$\eth_\downarrow \eth_\uparrow {}_sY_{\ell m} = -(\ell-s)(\ell+s+1){}_sY_{\ell m}, \tag{D.18}$$

are orthonormal on the unit two-sphere,

$$\int_{S^2} {}_sY^*_{\ell m}\, {}_sY_{\ell' m'}\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}, \tag{D.19}$$

and satisfy the completeness relation,

$$\sum_{\ell m} {}_sY^*_{\ell m}(\theta', \phi'){}_sY_{\ell m}(\theta, \phi) = \delta(\cos\theta - \cos\theta')\delta(\phi - \phi'). \tag{D.20}$$

As with the spherical harmonics, one can decompose a function of (θ, ϕ), with spin weight s as

$${}_sf(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}\, {}_sY_{\ell m}(\theta, \phi), \tag{D.21}$$

where the coefficients are given by

$$f_{\ell m} = \int_{S^2} f(\theta, \phi){}_sY^*_{\ell m}(\theta, \phi)\, d\Omega\,. \tag{D.22}$$

Finally, one can explicitly write the spin-weighted spherical harmonics as [203]

$$\begin{aligned} {}_sY_{\ell m}(\theta, \phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell+m)!(\ell-m)!}{(\ell+s)!(\ell-s)!}} \sin^{2\ell}\left(\frac{\theta}{2}\right) \\ \times \sum_{r=0}^{\ell-s} \binom{\ell-s}{r} \binom{\ell+s}{r+s-m} (-1)^{\ell-r-s} e^{im\phi} \cot^{2r+s-m}\left(\frac{\theta}{2}\right). \end{aligned} \tag{D.23}$$

This form is particularly useful when deriving the mode decomposition of the gauge vector in Ch. 7.

D.3 Wigner D-matrix

To rotate the mode coefficients between two coordinate systems, as is required in Ch. 7, one can make use of the Wigner D-matrix, $D^{\ell}_{mm'}(\alpha,\beta,\gamma)$ [191, 192]. The Wigner D-matrix encodes information from Euler angles, (α,β,γ), which represent rigid body rotations in three dimensional space: we rotate around the z axis by α, the y axis by β and then finally around z again by γ. There are a number of differing sign conventions for the Wigner D-matrices but we choose ours to be consistent with Ref. [137, 143] (which are themselves consistent with Ref. [205]), so that

$$D^{\ell}_{m_1 m_2}(\alpha,\beta,\gamma) = e^{-i(m_1\alpha + m_2\gamma)} D^{\ell}_{m_1 m_2}(0,\beta,0). \tag{D.24}$$

The Wigner D-matrix may also be written in terms of spin-weighted spherical harmonics as [203, 204]

$$D^{\ell}_{ms}(\alpha,\beta,\gamma) = (-1)^s \sqrt{\frac{4\pi}{2\ell+1}}\, {}_{-s}Y^*_{\ell m}(\beta,\alpha) e^{-is\gamma}, \tag{D.25}$$

which reduces to a relation in terms of regular spherical harmonics for $s = 0$,

$$D^{\ell}_{m0}(\alpha,\beta,0) = \sqrt{\frac{4\pi}{2\ell+1}} Y^*_{\ell m}(\beta,\alpha). \tag{D.26}$$

If we transform coordinates from (θ',ϕ') to (θ,ϕ), then the mode coefficients of a function, f, in the new coordinates, $f_{\ell m}$, can be written in terms of the mode coefficients of the old coordinates, $f_{\ell m'}$, using

$$f_{\ell m} = \sum_{m'=-\ell}^{\ell} D^{\ell}_{mm'}(\alpha,\beta,\gamma) f_{\ell m'}. \tag{D.27}$$

This statement is also true for the modes of spin-weighted functions as well [204]. Thus, if f has spin weight s, we can sum over spin-weighted spherical harmonics to recover the original function in terms of the mode coefficients in the old coordinates,

$$f(\theta,\phi) = \sum_{\ell m} \left(\sum_{m'=-\ell}^{\ell} D^{\ell}_{mm'}(\alpha,\beta,\gamma) f_{\ell m'} \right) {}_sY_{\ell m}(\theta,\phi) \tag{D.28}$$

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