

GLUCK TWISTS ON CONCORDANT OR HOMOTOPIC SPHERES

DANIEL KASPROWSKI, MARK POWELL, AND ARUNIMA RAY

ABSTRACT. Let M be a compact 4-manifold and let S and T be embedded 2-spheres in M , both with trivial normal bundle. We write M_S and M_T for the 4-manifolds obtained by the Gluck twist operation on M along S and T respectively. We show that if S and T are concordant, then M_S and M_T are s -cobordant, and so if $\pi_1(M)$ is good, then M_S and M_T are homeomorphic. Similarly, if S and T are homotopic then we show that M_S and M_T are simple homotopy equivalent. Under some further assumptions, we deduce that M_S and M_T are homeomorphic. We show that additional assumptions are necessary by giving an example where S and T are homotopic but M_S and M_T are not homeomorphic. We also give an example where S and T are homotopic and M_S and M_T are homeomorphic but not diffeomorphic.

1. INTRODUCTION

Given a compact, smooth 4-manifold M and a smooth embedding of a 2-sphere in the interior of M with image S , such that the submanifold S has trivial normal bundle, we may obtain a 4-manifold M_S as follows. Fix an identification of a closed tubular neighbourhood of S as $\bar{\nu}S \cong D^2 \times S^2$ and let $\partial_S(M \setminus \nu S) \subseteq \partial(M \setminus \nu S)$ denote the boundary component identified with $\partial \bar{\nu}S \cong S^1 \times S^2$. Let $R: S^1 \rightarrow SO(3)$ be a smooth map that represents a generator of $\pi_1(SO(3))$. The *Gluck twist* on $S^1 \times S^2$, named after Herman Gluck [Glu62], is the diffeomorphism

$$G: S^1 \times S^2 \rightarrow S^1 \times S^2 \\ (x, y) \mapsto (x, R(x) \cdot y),$$

obtained by restricting the canonical action of $SO(3)$ to $S^2 \subseteq \mathbb{R}^3$. By definition the *Gluck twist operation* on M along S consists of replacing M by the smooth 4-manifold

$$M_S := (M \setminus \nu S) \cup_G (D^2 \times S^2).$$

Here we identify using $\partial(D^2 \times S^2) = S^1 \times S^2 \xrightarrow{G} S^1 \times S^2 \cong \partial_S(M \setminus \nu S)$. The Gluck twist on a null-homologous 2-sphere in a simply connected 4-manifold does not change the homotopy type, but this is not the case in general. Nevertheless, $\pi_1(M_S) \cong \pi_1(M)$, as we show in Lemma 2.1. There is also a canonical identification $\partial M_S = \partial M$. The construction can be carried through in exactly the same way in the topological category, provided S is a topological submanifold of M , i.e. the image of a locally flat embedding with trivial normal bundle. Note that we allow nonorientable 4-manifolds, as well as those with nonempty boundary.

A classical question compares the diffeomorphism type of M and M_S for a 2-knot $S \subseteq M$, most commonly in the case of $M = S^4$. Previous work on Gluck twists in 4-manifolds other than S^4 includes [Akb88, AY13].

We consider the effect of the Gluck twisting on pairs $\{S, T\}$ of 2-knots in some M , and compare the homeomorphism type and homotopy type of M_S and M_T . One hope is that by generalising the question, we might inspire readers to find new exotic 4-manifolds using Gluck twists. First we consider the case of concordant 2-spheres S and T in a 4-manifold M , i.e. when there is a proper, locally flat embedding $c: S^2 \times [0, 1] \rightarrow M \times [0, 1]$ with $c(S^2 \times \{0\}) = S \subseteq M \times \{0\}$ and $c(S^2 \times \{1\}) = T \subseteq M \times \{1\}$.

Theorem 1.1. *The following statement holds in both the smooth category and the topological category with locally flat embeddings. Let M be a compact 4-manifold, and let $S, T \subseteq M$ be embedded, concordant 2-spheres, both with trivial normal bundle. Then M_S and M_T are s -cobordant.*

If $\pi_1(M)$ is a good group, it follows that M_S and M_T are homeomorphic.

2020 *Mathematics Subject Classification.* 57K40, 57N70, 57R80.

Key words and phrases. Gluck twist, 4-manifolds.

We will prove Theorem 1.1 in Section 2. For the deduction that M_S and M_T are homeomorphic in Theorem 1.1, we use the s -cobordism theorem of Freedman and Quinn [FQ90, Theorem 7.1A], which states that every 5-dimensional compact s -cobordism with good fundamental group is homeomorphic to a product. A group is said to be *good*, in the sense of Freedman, if it satisfies the π_1 -null disc property [FT95a]. At present, the class of good groups is known to contain groups of subexponential growth [FT95b, KQ00] and to be closed under subgroups, quotients, extensions, and colimits. As concrete examples, the class includes finite groups and elementary amenable groups, and therefore in particular all solvable groups. It is not known to include nonabelian free groups. See [KOPR21] for a comprehensive survey on good groups.

For our second result, we consider the case of homotopic 2-knots in a given manifold. Note that concordant 2-knots are homotopic.

Theorem 1.2. *Let M be a compact 4-manifold, and let $S, T \subseteq M$ be locally flat, embedded, and homotopic 2-spheres, both with trivial normal bundle. Then M_S and M_T are simple homotopy equivalent via a simple homotopy equivalence that restricts to the identity on the canonically identified boundaries $\partial M_S = \partial M_T$.*

We will prove Theorem 1.2 in Section 3. For certain fundamental groups, simple homotopy equivalence can be upgraded to homeomorphism. This includes some fundamental groups for which there is an explicit homeomorphism classification [Fre82, FQ90, HK88, Wan95, HKT09]. However, the class of known good groups is much larger than the class of groups for which explicit classification results are available, and for many good groups, even in the absence of a known homeomorphism classification, it is possible to use the surgery exact sequence to conclude that simple homotopy equivalent 4-manifolds are homeomorphic. We describe the outcome of both methods in the following corollary.

Recall that an identification of the fundamental group $\pi_1(M)$ of a 4-manifold M with a group π determines a map $c_M: M \rightarrow B\pi$, up to homotopy, classifying the universal cover, where $B\pi \simeq K(\pi, 1)$ is the classifying space.

Corollary 1.3. *Let M be a compact 4-manifold, and let $S, T \subseteq M$ be locally flat, embedded, and homotopic 2-spheres, both with trivial normal bundle. Let $\pi := \pi_1(M)$ and let $w: \pi \rightarrow C_2 = \{\pm 1\}$ be the orientation character. Suppose in addition that one of the following holds.*

- (i) M is closed, orientable, and π is cyclic.
- (ii) The following four conditions hold.
 - (G) π is good;
 - (A4) the assembly map $A_4: H_4(\pi; \mathbb{L}\langle 1 \rangle^w) \rightarrow L_4^s(\mathbb{Z}\pi, w)$ is injective;
 - (A5) the assembly map $A_5: H_5(\pi; \mathbb{L}\langle 1 \rangle^w) \rightarrow L_5^s(\mathbb{Z}\pi, w)$ is surjective;
 - (H) the map $(c_M)_*: H_3(M; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$ is surjective, for some choice of classifying map c_M .

Then M_S and M_T are homeomorphic.

The conditions of Corollary 1.3 are in fact conditions under which simple homotopy equivalent 4-manifolds with equal Kirby–Siebenmann invariants are homeomorphic, irrespective of whether they arise via Gluck twisting. We explain the conditions in Corollary 1.3 (ii) further in Remark 4.1. For now we limit ourselves to pointing out that Condition (ii) is satisfied for the solvable Baumslag–Solitar groups $\langle a, b \mid aba^{-1} = b^k \rangle \cong \mathbb{Z}[\frac{1}{k}] \rtimes \mathbb{Z}$, including $\mathbb{Z} \oplus \mathbb{Z}$ when $k = 1$. Condition (H) is not satisfied for $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Details on these examples, and more, can also be found in Remark 4.1.

The assumptions in Theorems 1.1 and 1.2, that S and T are concordant/homotopic, are in general necessary, since Gluck twisting along spheres that are not homotopic can result in homotopy inequivalent 4-manifolds. For example, Gluck twisting along $S^2 \times \{p\} \subseteq S^2 \times S^2$ results in the twisted fibre bundle $S^2 \tilde{\times} S^2$, whereas Gluck twisting along an unknotted S^2 returns $S^2 \times S^2$ again (see Proposition 1.5 below). These 4-manifolds are not homotopy equivalent and so are certainly not s -cobordant. For non-homotopic 2-spheres, Gluck twisting can result in homotopy equivalent manifolds with different homeomorphism types. Fan [Fan15, Theorem 4.5] gave an example of a simply connected 4-manifold with nonempty boundary, together with a homotopically essential embedded 2-sphere, such that the Gluck twist preserves the homotopy type rel. boundary but changes the homeomorphism type. See Table 1 for a summary of these, and other, examples.

	S, T homotopic	M orientable	M closed	$M_S \simeq_s M_T$	$M_S \cong_{\text{Top}} M_T$	$M_S \cong_{\text{Diff}} M_T$
$M = S^2 \times S^2, S = S^2 \times \{*\},$ $T = \text{unknot}.$	✗	✓	✓	✗	✗	✗
$M = (S^1 \widetilde{\times} S^3) \# (S^2 \times S^2),$ $T = \text{unknot}$ [Akb88] cf. [Tor17].	✗	✗	✓	✓	✓	✗
Examples by Fan [Fan15].	✗	✓	✗	✓	✗	✗
Examples from Prop. 1.4.	✓	✓	✓	✓	✗	✗
Examples from Prop. 1.6.	✓	✗	✓	✓	✓	✗

TABLE 1. The properties of some interesting examples of Gluck twists. The symbol for the equivalence relations in the top row are defined in Section 1.2.

As a counterpoint to Corollary 1.3, we give new examples in this direction, with the additional properties that the spheres are homotopic and the ambient manifold is closed, as follows.

Proposition 1.4. *There exists a closed, orientable 4-manifold M and locally flat, embedded, and homotopic 2-spheres $S, T \subseteq M$, both with trivial normal bundle, such that M_S and M_T are simple homotopy equivalent but not homeomorphic.*

1.1. Effect on diffeomorphism type. The following proposition is well-known, but important to observe. It shows that the classical idea of comparing M_S and M is a special case of comparing M_S and M_T , by taking T to be an unknotted 2-sphere.

Proposition 1.5. *Let M be a compact, smooth 4-manifold. If $T \subseteq M$ is an unknotted smooth 2-knot, then M and M_T are diffeomorphic.*

Proof. First let $M = S^4$. Then $M_T \cong M$ because the Gluck twist G extends to a diffeomorphism of $S^1 \times D^3$, the complement in S^4 of the trivial 2-knot S . Since every orientation-preserving diffeomorphism can be isotoped to fix a ball, this implies that for an unknotted 2-sphere T we have a rel. boundary diffeomorphism $D_T^4 \cong D^4$. Now for a general M , every trivial 2-knot T is contained in a 4-ball, and the result follows. \square

Gluck twisting along nontrivial, smooth 2-knots in S^4 is a well-known method to produce smooth homotopy 4-spheres, which could potentially be counterexamples to the smooth 4-dimensional Poincaré conjecture. Progress thus far has been in the opposite direction: there are several families of nontrivial 2-knots in S^4 for which it has been proven that Gluck twisting along them yields S^4 again [Glu62, Gor76, Pao78, Lit79, NS12, NS20].

The topological 4-dimensional Poincaré conjecture or the homeomorphism classification of simply connected 4-manifolds is usually cited to prove that Gluck twisting on S^4 preserves homeomorphism type. Since every 2-knot in S^4 is concordant to the trivial 2-knot [Ker65], the combination of Theorem 1.1 and Proposition 1.5 gives a different route. This method, and also that of Corollary 1.3, both have the advantage that they can be applied to the study of Gluck twisting on 2-knots in more general non-simply connected 4-manifolds. We hope that readers will be motivated to investigate this.

For non-homotopic 2-spheres, Gluck twisting can change the smooth structure of a fixed 4-manifold. Akbulut [Akb88] constructed an example of a homotopically essential 2-knot S in $P := (S^1 \widetilde{\times} S^3) \# (S^2 \times S^2)$ such that Gluck twisting changes the smooth structure. That is $Q := P_S$ is homeomorphic [Wan95] but not diffeomorphic to P . Torres [Tor17] extended this construction to other nonorientable 4-manifolds.

The following proposition sharpens this result by exhibiting two homotopic spheres S and T in a closed, nonorientable 4-manifold M such that Gluck twisting gives rise to an exotic pair M_S and M_T . Our construction makes use of the Cappell–Shaneson exotic $\mathbb{R}P^4$ [CS76a, CS76b], which we denote by R . Recall that two 4-manifolds M and N are said to be *stably diffeomorphic* if there exist n, n' such that $M \# n(S^2 \times S^2) \cong N \# n'(S^2 \times S^2)$.

Proposition 1.6. *There exists a pair of homotopic smooth 2-knots S and T in $M := \mathbb{RP}^4 \# S^2 \tilde{\times} S^2$, both with trivial normal bundle, such that $M_S \cong \mathbb{RP}^4 \# (S^2 \times S^2)$ and $M_T \cong R \# (S^2 \times S^2)$. In particular M_S and M_T are homeomorphic but not stably diffeomorphic.*

The proof of this proposition uses neither Theorem 1.1 nor Corollary 1.3. We also remark that we do not produce a new example of an exotic pair of 4-manifolds, but rather the proof uses the fact that \mathbb{RP}^4 and R are known to not be stably diffeomorphic.

At present, there is no known example of the Gluck twisting operation changing the smooth structure of an orientable 4-manifold.

Question 1.7. Does there exist a smooth, orientable 4-manifold M and a pair of smooth 2-knots S and T in M with trivial normal bundle such that M_S and M_T are homeomorphic but not diffeomorphic?

Akbulut and Yasui give a simple criterion when a Gluck twist on a smooth 4-manifold with odd intersection form does not change its diffeomorphism type [AY13].

Theorem 1.1 and Corollary 1.3 provide methods to potentially show that M_S and M_T are homeomorphic, in the case that S and T are either homotopic or concordant. We hope that they will be useful to help answer Question 1.7.

1.2. Conventions. All 4-manifolds are assumed to be connected and based, with a local orientation at the basepoint. Given a pair of manifolds, we use the symbol \simeq for homotopy equivalence, \simeq_s for simple homotopy equivalence, \cong_{Top} for homeomorphism, and \cong_{Diff} for diffeomorphism. When clear from context, we use \cong instead of \cong_{Top} and \cong_{Diff} . We use C_2 to denote the multiplicative group $\{\pm 1\}$.

Outline. Theorem 1.1 is proven in Section 2, and Theorem 1.2 in Section 3. In Section 4 we prove Corollary 1.3. Propositions 1.4 and 1.6 are proven in Section 5.

Acknowledgements. Thanks to Selman Akbulut for helping to clarify the state of the art with regards to Gluck twisting. We thank the Max Planck Institute for Mathematics in Bonn and Durham University for hospitality.

DK was supported by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813. MP was partially supported by EPSRC New Investigator grant EP/T028335/1 and EPSRC New Horizons grant EP/V04821X/1.

2. s -COBORDISM

In this section we prove Theorem 1.1, that for S and T concordant we have an s -cobordism between M_S and M_T . We start with a straightforward lemma.

Lemma 2.1. *Let M be a compact 4-manifold, and let $S \subseteq M$ be a locally flat embedded 2-sphere with trivial normal bundle. Then $\pi_1(M) \cong \pi_1(M_S)$.*

Proof. The operation of attaching $D^2 \times S^2$ to a 4-manifold along $S^1 \times S^2$ can be decomposed into a 2-handle attachment followed by a 4-handle attachment. Both M and M_S are obtained from $M \setminus \nu S$ by gluing in a copy of $D^2 \times S^2$, and in both cases the 2-handle is attached along a meridian of S in $\partial \bar{\nu} S$. In the case of M_S , we can use the meridian at a fixed point of the rotation $R: S^1 \rightarrow SO(3)$. Then both $\pi_1(M)$ and $\pi_1(M_S)$ are isomorphic to the quotient of $\pi_1(M \setminus \nu S)$ by the normal subgroup generated by a meridian of S , and so in particular are isomorphic to each other. \square

Having dealt with the fundamental group, we give the proof of Theorem 1.1. The proof in the smooth category uses handle decompositions and we do this case first. The relevant tools, such as topological transversality, the existence of normal bundles for locally flat, embedded submanifolds of 4-manifolds, and the existence of topological handle decompositions for 5-manifolds, are also available in the topological category, due to Quinn [Qui82] (see also [KS77, FQ90, PR21]). The end of the upcoming proof explains how to use these tools to reduce the proof in the topological category to that in the smooth category.

Proof of Theorem 1.1. Suppose that S and T are concordant 2-knots in M , and let C be a concordance between them. Recall this means that C is the image of a proper embedding $c: S^2 \times [0, 1] \hookrightarrow M \times [0, 1]$ with $c(S^2 \times [0, 1]) \cap (M \times \{0\}) = c(S^2 \times \{0\}) = S$ and $c(S^2 \times [0, 1]) \cap (M \times \{1\}) = c(S^2 \times \{1\}) = T$. We first assume that M and C are smooth.

For a submanifold $Y \subseteq Z$ with an open tubular neighbourhood νY , denote the exterior of Y by E_Y . So in particular $E_S := M \setminus \nu S$, $E_T := M \setminus \nu T$, and $E_C := (M \times [0, 1]) \setminus \nu C$. Identify a closed tubular neighbourhood $\bar{\nu}C$ of C with $S^2 \times D^2 \times [0, 1]$, and define $W := (M \times [0, 1])_C$, the result of gluing $S^2 \times D^2 \times [0, 1]$ to E_C via

$$G \times \text{Id}_{[0,1]}: S^1 \times S^2 \times [0, 1] \rightarrow \partial \bar{\nu}C \subseteq \partial E_C.$$

We will show that W is a rel. boundary s -cobordism between M_S and M_T . By definition, this means that W is a manifold with corners, with $\partial W \cong M_S \cup (\partial M \times [0, 1]) \cup M_T$, and the inclusion maps $M_S \hookrightarrow W$ and $M_T \hookrightarrow W$ are simple homotopy equivalences. By symmetry it suffices to show that $M_S \hookrightarrow W$ is a simple homotopy equivalence; the proof for $M_T \hookrightarrow W$ is identical with T replacing S . To prove this, we will show that the inclusion induced map $\pi_1(M_S) \rightarrow \pi_1(W)$ is an isomorphism, and that the relative chain complex $C_*(W, M_S; \mathbb{Z}[\pi_1(W)])$ is acyclic and has trivial Whitehead torsion.

First note that by Lemma 2.1, $\pi_1(M) \cong \pi_1(M_S) =: \pi$. We assert that $\pi_1(W) \cong \pi$ and that the inclusion $M_S \hookrightarrow W$ induces an isomorphism. Consider the following commutative diagram, where all the maps are induced by inclusion:

$$\begin{array}{ccccc} \pi_1(M_S) & \longleftarrow & \pi_1(E_S) & \longrightarrow & \pi_1(M) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \pi_1(W) & \longleftarrow & \pi_1(E_C) & \longrightarrow & \pi_1(M \times [0, 1]). \end{array} \quad (2.1)$$

Since the horizontal maps are surjective, the left and right vertical maps are those induced by the inclusion $E_S \rightarrow E_C$. Let $K_S := \ker(\pi_1(E_S) \rightarrow \pi_1(M))$ and $K_C := \ker(\pi_1(E_C) \rightarrow \pi_1(M \times [0, 1]))$. Then we have an induced map $\iota: \pi_1(E_S)/K_S \rightarrow \pi_1(E_C)/K_C$, and since ι is identified via the inclusion maps with the right vertical map $\pi_1(M) \xrightarrow{\cong} \pi_1(M \times [0, 1])$ in (2.1), we learn that ι is an isomorphism. The kernels of each of the horizontal maps in (2.1) are the normal subgroups generated by a meridian of S or C , as appropriate, and as described in the proof of Lemma 2.1. Thus $\ker(\pi_1(E_S) \rightarrow \pi_1(M_S)) = K_S$ and $\ker(\pi_1(E_C) \rightarrow \pi_1(W)) = K_C$. We deduce from the latter that $\pi_1(M \times [0, 1]) \cong \pi_1(W)$, and moreover that ι can also be identified via the inclusion maps with the left vertical map $\pi_1(M_S) \rightarrow \pi_1(W)$. So this map is an isomorphism as asserted.

Next we build a convenient handle decomposition of $M \times [0, 1]$. Consider the concordance exterior E_C as a manifold with corners, where the boundary decomposes as

$$\partial E_C = E_S \cup (C \times S^1) \cup (\partial M \times [0, 1]) \cup E_T.$$

That is, we consider E_C as a cobordism rel. boundary between E_S and E_T . Choose a handle decomposition of E_C relative to E_S . Note that ‘relative to E_S ’ means that no handles are attached to $E_T \cup (\partial M \times [0, 1]) \cup (C \times S^1)$. Glue on the product $C \times D^2$ using the identity map on $C \times S^1$ to obtain a handle decomposition of $M \times [0, 1]$ relative to M . Similarly glue $C \times D^2$ using $G \times \text{Id}_{[0,1]}$ to obtain a handle decomposition of W relative to M_S . In both cases, all the handles are located within the E_C piece and are attached away from $E_T \cup (\partial M \times [0, 1]) \cup (C \times S^1)$.

Corresponding to the handle decompositions constructed above we have relative chain complexes $C_*(M \times [0, 1], M; \mathbb{Z}\pi)$ and $C_*(W, M_S; \mathbb{Z}\pi)$. The chain complex $C_*(M \times [0, 1], M; \mathbb{Z}\pi)$ is acyclic, and has trivial Whitehead torsion, since $M \times [0, 1]$ is a product. By construction, in particular because the handles are attached away from $E_T \cup (\partial M \times [0, 1]) \cup (C \times S^1)$, the chain complex $C_*(W, M_S; \mathbb{Z}\pi)$ has the same handle attaching maps as $C_*(M \times [0, 1], M; \mathbb{Z}\pi)$ and therefore is also acyclic and has trivial Whitehead torsion. We have now shown that $M_S \hookrightarrow W$ is a simple homotopy equivalence, and therefore that W is a rel. boundary smooth s -cobordism between M_S and M_T .

As mentioned in the introduction, to deduce that for $\pi_1(M)$ a good group M_S and M_T are homeomorphic, we apply the s -cobordism theorem [FQ90, Theorem 7.1A], which completes the proof in the smooth category.

Next we extend the proof to the case where M need not be smooth, and C is only locally flat. The only salient difference is in the construction of an appropriate handle decomposition for

the claimed s -cobordism. The concordance C has a normal bundle νC in $M \times [0, 1]$ by [KS75], extending normal bundles for S and T in $M \times \{0, 1\}$, which exist by work of Quinn [Qui82; FQ90, Chapter 9]. As before write $E_S := M \setminus \nu S$, $E_T := M \setminus \nu T$, and $E_C := (M \times [0, 1]) \setminus \nu C$. Identify $\bar{\nu}C$ with $S^2 \times D^2 \times [0, 1]$ and define $W := S^2 \times D^2 \times [0, 1] \cup_{G \times \text{Id}_{[0,1]}} E_C$. Consider E_C as a cobordism rel. boundary between E_S and E_T . By [FQ90, Theorem 9.1], there is a topological handle decomposition of E_C relative to E_S . The rest of the proof is directly analogous to the smooth case. \square

Remark 2.2. The proof of Theorem 1.1 extends to other settings. For example two logarithmic transforms [GS99, Sections 3.3 and 8.3] on concordant embedded tori $T_1, T_2 \subseteq M$ with trivial normal bundle, and equal gluing coefficients, give rise to s -cobordant 4-manifolds. Similarly Price twists [Pri77] on concordant embeddings of $\mathbb{R}\mathbb{P}^2$ give rise to s -cobordant 4-manifolds. We leave the details of these extensions to the reader.

3. SIMPLE HOMOTOPY EQUIVALENCE

In this section we prove Theorem 1.2. The theorem states that for S and T homotopic, the manifolds M_S and M_T are simple homotopy equivalent. We begin with a lemma, which contains the key observation. In the smooth category, the diffeomorphism of the ambient manifolds $M_S \# \mathbb{C}\mathbb{P}^2 \cong M \# \mathbb{C}\mathbb{P}^2$ in the statement is well-known (see e.g. [GS99, Exercise 5.2.7 (b)]) but we give a proof for the reader's convenience, to verify the additional claim about the sphere $S \# \mathbb{C}\mathbb{P}^1$, and to give the argument in the topological category.

Recall that a homotopy equivalence f between CW complexes is said to be a simple homotopy equivalence if the Whitehead torsion $\tau(f)$ vanishes. While it is open whether every topological 4-manifold is homeomorphic to a 4-dimensional CW complex, it is still possible to define the notion of simple homotopy equivalence for topological 4-manifolds by embedding them in high-dimensional Euclidean space. For more details on this, see [KS77, Essay III, Section 4; KPR22, Section 2].

Lemma 3.1. *Let M be a compact 4-manifold and let $S \subseteq M$ be a locally flat embedded 2-sphere with trivial normal bundle. There is a homeomorphism of pairs*

$$\Psi_S: (M_S, \emptyset) \# (\mathbb{C}\mathbb{P}^2, \mathbb{C}\mathbb{P}^1) \xrightarrow{\cong} (M, S) \# (\mathbb{C}\mathbb{P}^2, \mathbb{C}\mathbb{P}^1),$$

that restricts to the identity on the naturally identified boundaries $\partial M = \partial M_S$.

Proof. Choosing a trivialisation of the normal bundle of S gives rise to an identification of a closed tubular neighbourhood $\bar{\nu}S$ with $S^2 \times D^2$. The standard smooth structure on $S^2 \times D^2$ induces a smooth structure on $\bar{\nu}S$. Let $E_S := M \setminus \nu S$ be the exterior of S . We may and shall assume that the connected sum operation has been performed within $\bar{\nu}S = S^2 \times D^2$, but away from S , to yield $M \# \mathbb{C}\mathbb{P}^2 \cong E_S \cup_{S^2 \times S^1} ((S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2)$. We will work inside $(S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2$, using its smooth structure, and therefore we may apply the methods of handle calculus. Figure 1 (a) shows a handle diagram for $(S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2$ relative to the boundary $S^2 \times S^1$, using the notation of [GS99, Section 5.5]. That is, we start with $S^2 \times S^1 \times [0, 1]$, attach two 2-handles to $S^2 \times S^1 \times \{1\}$, and then a single 4-handle.

Ignoring the grey arrow, Figure 1 (b) shows a handle diagram for the same manifold, but where the parametrisation of the boundary has been modified by the inverse of the Gluck twist G^{-1} . To see this, note that the diagram consisting of the unknot labelled with $\langle 0 \rangle$ determines an implicit parametrisation of $S^1 \times S^2$. The rotation in the definition of G^{-1} changes that parametrisation in such a way that the initial 0-framing on the left hand circle in Figure 1 (a) becomes a +1-framing in Figure 1 (b). Thus the 4-manifolds shown in (a) and (b) are diffeomorphic, via a diffeomorphism $\psi: (S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2 \xrightarrow{\cong} (S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2$ which, using the given parametrisations of the boundary, restricts on $S^2 \times S^1$ to G^{-1} .

Next we perform two handle slides, starting with the diagram in (b). The first slide is indicated by the arrow in Figure 1 (b), and yields Figure 1 (c). The second slide is indicated by the arrow in Figure 1 (c), and produces the diagram in Figure 1 (a). This sequence of handle slides gives rise to a rel. boundary diffeomorphism ϕ from the 4-manifold presented in Figure 1 (b) to the 4-manifold presented in Figure 1 (a). To see how handle sliding a 2-handle h gives rise to a diffeomorphism, let V denote $S^2 \times S^1 \times [0, 1]$ union the 2-handles other than h , and let $C \cong \partial V \times [0, 1]$ be a collar neighbourhood of ∂V . Write h' for the slid 2-handle, i.e. the 2-handle h after sliding. Note

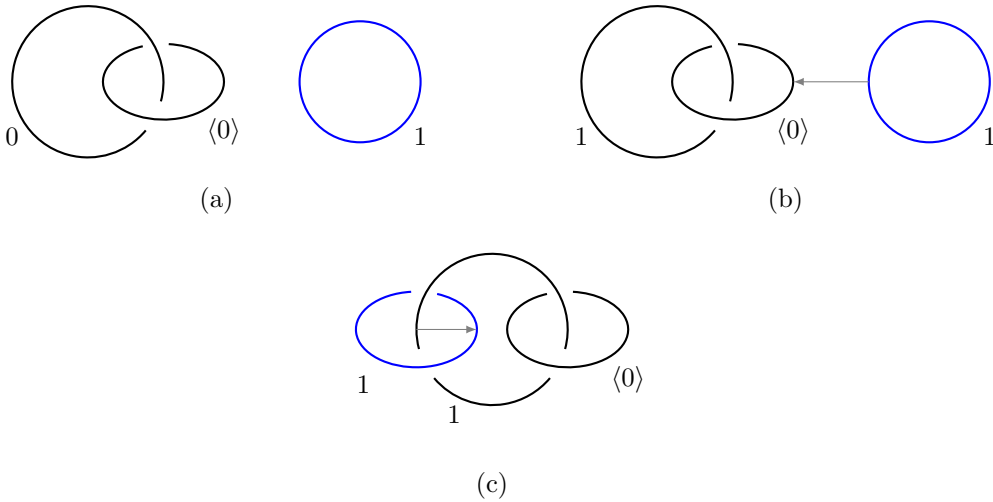


FIGURE 1. (a) A handle diagram for $(S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2$, relative to the boundary $S^2 \times S^1$. The 4-handle is not shown. The 2-handle corresponding to $\mathbb{C}\mathbb{P}^1 \subseteq \mathbb{C}\mathbb{P}^2$ is shown in blue. (b) A handle diagram for the same manifold, but relative to a different parametrisation of the boundary, where the parametrisation has been altered by the Gluck twist. The grey arrow indicates a handle slide which produces the bottom figure. (c) The outcome of the handle slide shown in (b). The grey arrow indicates a further handle slide, which recovers the picture in (a).

that a handle slide of h can be achieved by an ambient isotopy $\vartheta: \partial V \times [0, 1] \rightarrow \partial V$. Define a diffeomorphism $V \cup h \rightarrow V \cup h'$ to be the identity on $V \setminus C$, to be the ambient isotopy ϑ on C , and to be the natural extension $h \rightarrow h'$ on the 2-handle. This extends over the 4-handle to give the desired diffeomorphism. The extension over the 4-handle is not necessarily uniquely determined, not even up to isotopy, since it is unknown whether the smooth mapping class group of the 4-ball rel. boundary is trivial. This will not be important since we will only need to control the diffeomorphism on the 2-skeleton.

Composition of the diffeomorphisms discussed in the previous two paragraphs gives rise to a diffeomorphism $\Phi := \phi \circ \psi: (S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2 \xrightarrow{\cong} (S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2$ that restricts on the boundary to the inverse of the Gluck twist $G^{-1}: S^2 \times S^1 \xrightarrow{\cong} S^2 \times S^1$. Since the 2-handle corresponding to the +1-framed unknot in Figure 1 (b) was slid over the S^2 -factor in $S^2 \times S^1$ (corresponding to the curve labelled $\langle 0 \rangle$), and was involved in no other handle slides, Φ sends $\mathbb{C}\mathbb{P}^1$ to a sphere ambiently isotopic to $S \# \mathbb{C}\mathbb{P}^1$. Now extend the diffeomorphism Φ by the identity over E_S to obtain the desired homeomorphism

$$\Psi_S := \Phi \cup \text{Id}: M_S \# \mathbb{C}\mathbb{P}^2 = ((S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2) \cup_G E_S \rightarrow M \# \mathbb{C}\mathbb{P}^2 = ((S^2 \times D^2) \# \mathbb{C}\mathbb{P}^2) \cup_{\text{Id}} E_S.$$

This is well-defined on $S^2 \times S^1$ because $\Phi|_{S^2 \times S^1} = G^{-1}$. Since Φ sends $\mathbb{C}\mathbb{P}^1$ to a sphere ambiently isotopic to $S \# \mathbb{C}\mathbb{P}^1$, after an isotopy we may assume that $\Psi_S(\mathbb{C}\mathbb{P}^1) = S \# \mathbb{C}\mathbb{P}^1$, as asserted. \square

The next lemma concerns the result of blowing down a 4-manifold along homotopic 2-spheres, both with normal bundle having euler number +1, i.e. each sphere has a pushoff which intersects it precisely once. A similar argument applies to such spheres with normal bundle having euler number -1 . The combination of Lemmas 3.1 and 3.2 gives the proof of Theorem 1.2, as we show at the end of the section.

Lemma 3.2. *Let N be a compact 4-manifold. Let $A, B \subseteq N$ be locally flat, embedded, and homotopic 2-spheres, both with normal bundle having euler number +1. Let N^A and N^B denote the result of blowing down N along A and B respectively. Then N^A and N^B are simple homotopy equivalent.*

Proof. The upcoming proof is inspired by an argument due to Stong [Sto94b]. For an arbitrary 4-manifold W and $\beta \in \pi_2(W)$, let $\text{cap}(W, \beta)$ denote the result $W \cup_\beta D^3$ of adding a 3-cell to W along β . By the definition of the blow down operation, there are canonical homeomorphisms

$N^A \# \mathbb{C}\mathbb{P}^2 \cong N$ and $N^B \# \mathbb{C}\mathbb{P}^2 \cong N$ sending $\mathbb{C}\mathbb{P}^1$ to A and B respectively. Let $\sigma \in \pi_2(\mathbb{C}\mathbb{P}^2) \cong \mathbb{Z}$ denote a generator, which we consider as an element in both $\pi_2(N^A \# \mathbb{C}\mathbb{P}^2)$ and $\pi_2(N^B \# \mathbb{C}\mathbb{P}^2)$, via the inclusion induced maps. Note that

$$\text{cap}(\mathbb{C}\mathbb{P}^2, \sigma) \simeq_s S^4, \quad (3.1)$$

where \simeq_s denotes simple homotopy equivalence. To see this extend the standard degree one map $\mathbb{C}\mathbb{P}^2 \rightarrow S^4$ over $\text{cap}(\mathbb{C}\mathbb{P}^2, \sigma)$ and observe that both spaces are simply connected and that the map induces an isomorphism on all homology groups. Therefore $\text{cap}(\mathbb{C}\mathbb{P}^2, \sigma) \simeq S^4$ by the Whitehead theorem. Since the Whitehead group of the trivial group is trivial, we deduce that (3.1) holds. Now we make the following computation:

$$\begin{aligned} N^B &\cong N^B \# S^4 \simeq_s N^B \# \text{cap}(\mathbb{C}\mathbb{P}^2, \sigma) \cong \text{cap}(N^B \# \mathbb{C}\mathbb{P}^2, \sigma) \\ &\cong \text{cap}(N, B) \simeq_s \text{cap}(N, A) \cong \text{cap}(N^A \# \mathbb{C}\mathbb{P}^2, \sigma) \cong N^A \# \text{cap}(\mathbb{C}\mathbb{P}^2, \sigma) \\ &\simeq_s N^A \# S^4 \cong N^A. \end{aligned}$$

Since the computation is symmetrical in A and B it suffices to justify this as far as the $\text{cap}(N, A)$ term. The first equivalence is immediate. The second equivalence uses (3.1). The third equivalence uses that we may perform the connected sum away from σ . The fourth equivalence uses the canonical homeomorphism $N^B \# \mathbb{C}\mathbb{P}^2 \cong N$ discussed above. The fifth, central simple homotopy equivalence uses that A and B are homotopic. The remaining equivalences can be justified by switching the rôles of A and B , and applying the previous justifications in reverse order. This gives the desired simple homotopy equivalence $N^A \simeq_s N^B$. The boundaries of N , N^A , and N^B are naturally identified, and by inspection our simple homotopy equivalence restricts to the identity on the boundary. This completes the proof of Lemma 3.2. \square

Proof of Theorem 1.2. By Lemma 3.1, M_S and M_T are obtained as blow downs of $N := M \# \mathbb{C}\mathbb{P}^2$ along the homotopic spheres $A := S \# \mathbb{C}\mathbb{P}^1$ and $B := T \# \mathbb{C}\mathbb{P}^1$. By Lemma 3.2 it follows that M_S and M_T are simple homotopy equivalent. \square

4. HOMEOMORPHISM

In this section we prove Corollary 1.3, which upgrades simple homotopy equivalence to homeomorphism for certain fundamental groups. We prove the two items in the corollary separately.

Proof of Corollary 1.3(i). Suppose that M is closed and orientable, and that $\pi_1(M)$ is cyclic. Since S and T are homotopic, the manifolds M_S and M_T are simple homotopy equivalent by Theorem 1.2. For every cyclic group G , one of Freedman [Fre82], Freedman–Quinn [FQ90], or Hambleton–Kreck [HK88, HK93] showed that given closed 4-manifolds X and Y with $X \simeq_s Y$ and $\pi_1(X) \cong \pi_1(Y) \cong G$, we have that X and Y are homeomorphic if and only if $\text{ks}(X) = \text{ks}(Y)$. The Kirby–Siebenmann invariant ks is additive with respect to gluing along part of the boundary [FQ90, Section 10.2B; FNOP19, Theorem 8.2(5)]. This additivity is indifferent to the precise gluing homeomorphism used, so we have

$$\text{ks}(M) = \text{ks}(E_S) + \text{ks}(S^2 \times D^2) = \text{ks}(M_S), \quad (4.1)$$

and similarly $\text{ks}(M) = \text{ks}(M_T)$. Therefore M_S and M_T are homeomorphic, as desired. \square

The proof of Corollary 1.3(ii) will use the simple surgery exact sequence. For details and definitions of the latter, we refer the reader to [Wal99; FQ90, Chapter 11; OPR21]. Before giving the proof we discuss the conditions in Corollary 1.3(ii).

Remark 4.1.

- (a) Observe that Corollary 1.3(ii) imposes no restrictions on w nor on ∂M .
- (b) The symbol \mathbb{L} denotes the L -theory spectrum of the integers, whose homotopy groups are $L_*(\mathbb{Z})$, and $\mathbb{L}\langle 1 \rangle$ denotes the 1-connective cover. The generalised homology $H_*(\pi; \mathbb{L}\langle 1 \rangle^w)$ denotes the w -twisted $\mathbb{L}\langle 1 \rangle$ -homology [Ran92, Appendix A], see also [KL22, Remark 2.1]. The groups $L_*^s(\mathbb{Z}\pi, w)$ are the simple L -groups of the ring with involution $\mathbb{Z}\pi$, with involution determined by $\bar{g} = w(g)g^{-1}$.

- (c) For $r \in \{4, 5\}$, the assembly map A_r factors as

$$H_r(\pi; \mathbb{L}\langle 1 \rangle^w) \rightarrow H_r(\pi; \mathbb{L}^w) \xrightarrow{FJ} L_r^s(\mathbb{Z}\pi, w).$$

The Farrell–Jones conjecture for torsion-free groups and coefficient ring \mathbb{Z} predicts that the map FJ is an isomorphism. This conjecture holds for many families of groups, including hyperbolic groups, CAT(0) groups, 3-manifold groups, and solvable groups [BLR08, BL12, BFL14, Weg15]. In particular, by comparing the Atiyah–Hirzebruch spectral sequences for $H_r(\pi; \mathbb{L}\langle 1 \rangle^w) \rightarrow H_r(\pi; \mathbb{L}^w)$, using the naturality with respect to change in generalised homology theory, conditions (A4) and (A5) are satisfied for all groups of dimension at most 4 that satisfy the Farrell–Jones conjecture [KL22, Lemma 2.3].

- (d) Condition (ii) holds for the solvable Baumslag–Solitar groups $B(k)$, for $k \in \mathbb{Z}$, with any orientation character w ; cf. [HKT09]. These groups have presentation $B(k) \cong \langle a, b \mid aba^{-1} = b^k \rangle \cong \mathbb{Z}[\frac{1}{k}] \rtimes \mathbb{Z}$, and are 2-dimensional. For these groups one can compute directly that the presentation complex for the above presentation is aspherical. In fact the presentation complex is aspherical for all torsion-free one relator groups [Lyn50]. Note that $B(1) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- (e) Condition (ii) also holds for \mathbb{Z} with both orientation characters $w: \mathbb{Z} \rightarrow C_2$. For closed M the result can be deduced from the classification theorems in [FQ90, Theorem 10.7A] and [Wan95].
- (f) For closed M , let $[M]$ denote the $\mathbb{Z}/2$ -fundamental class and consider the diagram

$$\begin{array}{ccc} H^1(M; \mathbb{Z}/2) & \xleftarrow[\cong]{(c_M)^*} & H^1(\pi; \mathbb{Z}/2) \\ \cong \downarrow -\cap[M] & & \downarrow -\cap(c_M)_*[M] \\ H_3(M; \mathbb{Z}/2) & \xrightarrow{(c_M)_*} & H_3(\pi; \mathbb{Z}/2) \end{array}$$

The bottom horizontal map is surjective if and only if

$$-\cap(c_M)_*[M]: H^1(\pi; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$$

is surjective, so we can replace condition (H) with the above condition. Note this means condition (H) does not hold for closed M and 3-dimensional π (e.g. \mathbb{Z}^3), because in that case $(c_M)_*[M] \in H_4(\pi; \mathbb{Z}/2) = 0$.

- (g) Suppose that π is a good group that satisfies the Farrell–Jones conjecture. Also suppose that π is a 4-dimensional Poincaré duality group with respect to w . If M is closed and $(c_M)_*[M] \neq 0 \in H_4(\pi; \mathbb{Z}/2)$, then using the reformulation of condition (H) from (f), and using (c) to obtain conditions (A4) and (A5), it follows that condition (ii) is satisfied. This applies in particular when $\pi = \mathbb{Z}^4$, the manifold M is closed, and $(c_M)_*[M] \neq 0 \in H_4(\pi; \mathbb{Z}/2)$.

Proof of Corollary 1.3 (ii). Since S and T are homotopic, by Theorem 1.2 the 4-manifolds M_S and M_T are simple homotopy equivalent. Recall from Lemma 2.1 that $\pi_1(M_S) \cong \pi_1(M) =: \pi$. We will use the simple surgery sequence for M_S :

$$\mathcal{N}(M_S \times [0, 1], \partial) \xrightarrow{\sigma_5} L_5^s(\mathbb{Z}\pi, w) \longrightarrow \mathcal{S}^s(M_S, \partial M_S) \xrightarrow{\eta} \mathcal{N}(M_S, \partial M_S) \xrightarrow{\sigma_4} L_4^s(\mathbb{Z}\pi, w),$$

where the symbol ∂ is shorthand for $\partial := \partial(M_S \times [0, 1]) = (M_S \times \{0, 1\}) \cup (\partial M_S \times [0, 1])$ and $w: \pi \rightarrow C_2$ is the orientation character. Since we assume that π is a good group (Condition (G)), the sequence is exact [FQ90, Chapter 11; OPR21]. By Theorem 1.2 there is a simple homotopy equivalence $f: M_T \rightarrow M_S$ restricting to a homeomorphism $\partial M_T \cong \partial M_S$. We will show that $[f] = [g: M_S \rightarrow M_S]$ in the simple structure set $\mathcal{S}^s(M_S, \partial M_S)$, for some self homotopy equivalence g of M_S . By definition of the simple structure set, this means that there is an s -cobordism rel. boundary between M_T and M_S , which implies that M_T and M_S are homeomorphic rel. boundary by the s -cobordism theorem [FQ90, Theorem 7.1A]. In fact we will show that every element $[f': M' \rightarrow M_S]$ of $\mathcal{S}^s(M_S, \partial M_S)$ with $\text{ks}(M') = \text{ks}(M_S)$ is represented by a self homotopy equivalence of M_S . Since $\text{ks}(M_T) = \text{ks}(M_S)$, as argued in (4.1), this will complete the proof that M_S and M_T are homeomorphic.

Using the identification

$$\mathcal{N}(M_S, \partial M_S) \cong [(M_S, \partial M_S), (G/TOP, *)] \cong H^0(M_S, \partial M_S; \mathbb{L}\langle 1 \rangle) \cong H_4(M_S; \mathbb{L}\langle 1 \rangle^w),$$

and the similar identification

$$\mathcal{N}(M_S \times [0, 1], \partial) \cong H_5(M_S \times [0, 1]; \mathbb{L}\langle 1 \rangle^w) \cong H_5(M_S; \mathbb{L}\langle 1 \rangle^w),$$

we identify the surgery obstruction maps σ_r , for $r \in \{4, 5\}$, with the compositions

$$H_r(M_S; \mathbb{L}\langle 1 \rangle^w) \xrightarrow{(c_{M_S})_r} H_r(\pi; \mathbb{L}\langle 1 \rangle^w) \xrightarrow{A_r} L_r^s(\mathbb{Z}\pi, w); \quad (4.2)$$

see [Ran92, B9, p. 324]. By assumption (Condition (A5)), the map A_5 is surjective. We want to show that $(c_{M_S})_5$ is surjective. Recall that

$$\pi_n(\mathbb{L}\langle 1 \rangle) = \begin{cases} \mathbb{Z} & n = 4k > 1 \\ \mathbb{Z}/2 & n = 4k + 2 > 1 \\ 0 & \text{otherwise.} \end{cases}$$

The (generalised) homology groups $H_5(M_S; \mathbb{L}\langle 1 \rangle^w)$ and $H_5(\pi; \mathbb{L}\langle 1 \rangle^w)$ can be computed using the Atiyah–Hirzebruch spectral sequence, and the map c_{M_S} induces maps between the corresponding E^2 and E^∞ pages. The only relevant maps between the E^2 pages for the computation of $(c_{M_S})_5$ from (4.2) are the maps $(c_{M_S})_1: H_1(M_S; \mathbb{Z}^w) \rightarrow H_1(\pi; \mathbb{Z}^w)$ and $(c_{M_S})_3: H_3(M_S; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$. It will suffice to show that each is surjective. The first map is an isomorphism because $c_{M_S}: M_S \rightarrow B\pi$ is 2-connected. For the second map, we use the following diagram.

$$\begin{array}{ccc} H_3(M_S \# \mathbb{C}\mathbb{P}^2; \mathbb{Z}/2) & \xrightarrow{\cong} & H_3(M \# \mathbb{C}\mathbb{P}^2; \mathbb{Z}/2) \\ \downarrow \cong & & \downarrow \cong \\ H_3(M_S; \mathbb{Z}/2) & \xrightarrow{(c_{M_S})_3} H_3(\pi; \mathbb{Z}/2) \xleftarrow{(c_M)_3} & H_3(M; \mathbb{Z}/2). \end{array}$$

The top horizontal map above is induced by the diffeomorphism $\Psi_S: M_S \# \mathbb{C}\mathbb{P}^2 \xrightarrow{\cong} M \# \mathbb{C}\mathbb{P}^2$ from Lemma 3.1. The vertical maps are induced by the standard degree one maps $M_S \# \mathbb{C}\mathbb{P}^2 \rightarrow M_S$ and $M \# \mathbb{C}\mathbb{P}^2 \rightarrow M$. The diagram commutes for some choice of classifying maps because both of the maps $M_S \# \mathbb{C}\mathbb{P}^2 \rightarrow B\pi$, corresponding to the two routes around the diagram for the associated maps of spaces, are classifying maps for the universal cover. It then follows from the diagram, and our assumption (Condition (H)) that $H_3(M; \mathbb{Z}/2) \xrightarrow{(c_M)_3} H_3(\pi; \mathbb{Z}/2)$ is surjective, that $(c_{M_S})_3: H_3(M_S; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$ is also surjective, as desired. This completes the argument that the map $(c_{M_S})_5: H_5(M_S; \mathbb{L}\langle 1 \rangle^w) \rightarrow H_5(\pi; \mathbb{L}\langle 1 \rangle^w)$ in (4.2) is surjective. Hence, using Condition (A5), the surgery obstruction map $\sigma_5 = A_5 \circ (c_{M_S})_5$ is surjective, as needed. It follows that $\eta: \mathcal{S}^s(M_S, \partial M_S) \rightarrow \mathcal{N}(M_S, \partial M_S)$ is injective.

We therefore need to investigate $\text{Im } \eta = \ker \sigma_4$. By assumption (Condition (A4)), the assembly map A_4 is injective, so $\ker \sigma_4 = \ker (c_{M_S})_4$. We will describe a map inducing an isomorphism $\ker (c_{M_S})_4 \cong \ker (c_{M_S})_2$. To prove this, we consider the following diagram, arising from comparing the Atiyah–Hirzebruch spectral sequences computing $H_4(M_S; \mathbb{L}\langle 1 \rangle^w)$ and $H_4(B\pi; \mathbb{L}\langle 1 \rangle^w)$.

$$\begin{array}{ccccccc} H_3(M_S; \mathbb{Z}/2) & \xrightarrow{d_3} & H_0(M_S; \mathbb{Z}^w) & \longrightarrow & H_4(M_S; \mathbb{L}\langle 1 \rangle^w) & \xrightarrow{h} & H_2(M_S; \mathbb{Z}/2) \longrightarrow 0 \\ \downarrow (c_{M_S})_3 & & \cong \downarrow (c_{M_S})_0 & & \downarrow (c_{M_S})_4 & & \downarrow (c_{M_S})_2 \\ H_3(\pi; \mathbb{Z}/2) & \xrightarrow{d_3} & H_0(\pi; \mathbb{Z}^w) & \longrightarrow & H_4(\pi; \mathbb{L}\langle 1 \rangle^w) & \longrightarrow & H_2(\pi; \mathbb{Z}/2) \longrightarrow 0. \end{array}$$

The rows are exact and the diagram commutes. There are two nonzero terms on the 4-line of the E^2 pages, corresponding to the second and fourth terms of each row. There are no differentials interacting with the $H_2(-; \mathbb{Z}/2)$ terms, and there is one potential d_3 differential, as indicated. The map $(c_{M_S})_0$ is an isomorphism. The d_3 maps are trivial when $w = 0$, and when $w \neq 0$ we use that $(c_{M_S})_3$ is surjective to deduce that the images of d_3 are identified under $(c_{M_S})_0$. It follows that we have a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_0(M_S; \mathbb{Z}^w) / \text{Im}(d_3) & \longrightarrow & H_4(M_S; \mathbb{L}\langle 1 \rangle^w) & \xrightarrow{h} & H_2(M_S; \mathbb{Z}/2) \longrightarrow 0 \\ & & \cong \downarrow (c_{M_S})_0 & & \downarrow (c_{M_S})_4 & & \downarrow (c_{M_S})_2 \\ 0 & \longrightarrow & H_0(\pi; \mathbb{Z}^w) / \text{Im}(d_3) & \longrightarrow & H_4(\pi; \mathbb{L}\langle 1 \rangle^w) & \longrightarrow & H_2(\pi; \mathbb{Z}/2) \longrightarrow 0. \end{array}$$

By the snake lemma, and since the kernel of the left vertical map is trivial, the map h induces an isomorphism $\ker (c_{M_S})_4 \cong \ker (c_{M_S})_2$, as asserted. We therefore investigate the kernel of the map $(c_{M_S})_2: H_2(M_S; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2)$.

Claim 4.2. Every element of $\ker (H_2(M_S; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2))$ can be represented by a map of a 2-sphere in M_S .

Proof of Claim 4.2. By considering the Leray–Serre spectral sequence for the fibration $\widetilde{M}_S \rightarrow M_S \rightarrow B\pi$ with homology theory $H_*(-; \mathbb{Z}/2)$, where \widetilde{M}_S denotes the universal cover of M_S , we can obtain a $\mathbb{Z}/2$ version of the Hopf exact sequence:

$$H_0(\pi; H_2(\widetilde{M}_S; \mathbb{Z}/2)) \rightarrow H_2(M_S; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2) \rightarrow 0.$$

The first term is isomorphic to $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(M_S)$. Therefore every element of $\ker (H_2(M_S; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2))$ can be represented by a sum of spheres, and so ultimately by a single sphere, in M_S . \square

By the proof of [CH90, Theorem 5.1], for every element ι of $H_2(M_S; \mathbb{Z}/2)$ which is represented by an immersed sphere with $w_2(\iota) = 0$, there exists a self homotopy equivalence g of M_S with $\eta(g) = \iota$. Since w_2 is additive, the quotient of $\mathcal{S}^s(M_S, \partial M_S)$ by self homotopy equivalences has at most 2 elements. If $w_2(\widetilde{M}_S) = 0$, then w_2 is zero on all spheres, and so all elements of $\mathcal{S}^s(M_S, \partial M_S)$ are represented by self homotopy equivalences. If $w_2(\widetilde{M}_S) \neq 0$, there exists a star partner $*M_S$ of M_S with $\text{ks}(*M_S) \neq \text{ks}(M_S)$ and a simple homotopy equivalence $*M_S \rightarrow M_S$, by [KPR22, Proposition 5.8 and Lemma 5.6], cf. [FQ90, Chapter 10.4; Sto94b, Section 2]. Since the Kirby–Siebenmann invariant ks is a homeomorphism invariant, the maps $\text{Id}: M_S \rightarrow M_S$ and $*M_S \rightarrow M_S$ represent two distinct elements in the quotient of $\mathcal{S}^s(M_S, \partial M_S)$ by self homotopy equivalences. It follows that every element $[f': M' \rightarrow M_S]$ of $\mathcal{S}^s(M_S, \partial M_S)$ with $\text{ks}(M') = \text{ks}(M_S)$ is represented by a self homotopy equivalence, as claimed. This completes the proof that the simple homotopy equivalence $M_T \rightarrow M_S$ lies in the class of Id_{M_S} in $\mathcal{S}^s(M_S, \partial M_S)$, and therefore M_S and M_T are homeomorphic. \square

5. FAKE AND EXOTIC MANIFOLDS FROM GLUCK TWISTS ON HOMOTOPIC SPHERES

In this section we prove Propositions 1.4 and 1.6. Let E denote the total space of the unique S^2 -fibre bundle over $\mathbb{R}\mathbb{P}^2$ that has orientable but not spin total space. This is a smooth, closed, orientable 4-manifold with fundamental group $\mathbb{Z}/2$. Proposition 1.4 is a consequence of the following result. Recall that two 4-manifolds M and N are said to be *stably homeomorphic* if there exist n, n' such that $M \# n(S^2 \times S^2) \cong N \# n'(S^2 \times S^2)$.

Proposition 5.1. *There exists a pair of locally flat, embedded, and homotopic 2-spheres S and T in $M := E \# E \# \mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$, both with trivial normal bundle, such that M_S and M_T are simple homotopy equivalent but not stably homeomorphic.*

For the proof we need some results on star partners of manifolds which we recall next. A *star partner* of a 4-manifold M is a manifold $*M$ such that there exists a homeomorphism $M \# *M \cong *M \# M$ preserving the decomposition on π_2 , where $*M$ is the Chern manifold constructed by Freedman [Fre82]. Notably, $*M$ is (simple) homotopy equivalent, but not homeomorphic, to M , and we know that $\text{ks}(*M) = 1$. The 4-manifolds M and $*M$ are simple homotopy equivalent but have opposite Kirby–Siebenmann invariants. The manifold E has a star partner $*E$ [Tei97, Proposition 1; Sto94a]. By [Tei97, Theorem 1] the star partner $*E$ of E is unique up to homeomorphism.

We will also need the following observation to ensure that the constructed spheres are indeed homotopic.

Remark 5.2. Note that the map $\mathbb{C}^3 \rightarrow \mathbb{C}^3$ sending (x, y, z) to $(\bar{x}, \bar{y}, \bar{z})$ induces an orientation preserving self-diffeomorphism of $\mathbb{C}\mathbb{P}^2$ that restricts to an orientation reversing self-diffeomorphism of $\mathbb{C}\mathbb{P}^1$.

Proof of Proposition 5.1. By [KPR22, Proposition 5.9], we know that E is also a star partner of $*E$, i.e. there is a homeomorphism $*E \# *E \cong E \# E$ preserving the decomposition on π_2 . Hence there exists a homeomorphism $\psi: *E \# *E \# \mathbb{C}\mathbb{P}^2 \rightarrow E \# E \# \mathbb{C}\mathbb{P}^2$ preserving the decomposition on π_2 , obtained as the composition

$$E \# (E \# \mathbb{C}\mathbb{P}^2) \cong E \# (*E \# \mathbb{C}\mathbb{P}^2) = *E \# (E \# \mathbb{C}\mathbb{P}^2) \cong *E \# (*E \# \mathbb{C}\mathbb{P}^2).$$

Let $C \subseteq E\#E\#\mathbb{C}\mathbb{P}^2$ be the image of $\mathbb{C}\mathbb{P}^1 \subseteq \mathbb{C}\mathbb{P}^2$ under ψ . Using the diffeomorphism from Remark 5.2 to change orientations if necessary, we can assume that the locally flat embedded sphere C is homotopic to $\mathbb{C}\mathbb{P}^1 \subseteq E\#E\#\mathbb{C}\mathbb{P}^2$. Consider the locally flat embedded spheres

$$S := \mathbb{C}\mathbb{P}^1\#\overline{\mathbb{C}\mathbb{P}^1} \text{ and } T := C\#\overline{\mathbb{C}\mathbb{P}^1}$$

in the manifold $M := E\#E\#\mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$. The Gluck twist on $\mathbb{C}\mathbb{P}^1\#\overline{\mathbb{C}\mathbb{P}^1}$ in $\mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$ yields $S^2 \times S^2$. It is the inverse to the Gluck twist on $S^2 \times \{p\} \subseteq S^2 \times S^2$ that yields $S^2 \tilde{\times} S^2 \cong \mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$. It follows that M_S is diffeomorphic to $E\#E\#(S^2 \times S^2)$.

On the other hand, M_T is homeomorphic to the result of the Gluck twist on $\mathbb{C}\mathbb{P}^1\#\overline{\mathbb{C}\mathbb{P}^1}$ in $*E\#*E\#\mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$ by construction of T . Hence M_T is homeomorphic to $*E\#*E\#(S^2 \times S^2)$. By [Tei97, Proposition 3] the manifolds $E\#E$ and $*E\#*E$ are not stably homeomorphic. Hence the manifolds M_S and M_T are not stably homeomorphic. \square

Lemma 5.3. *Let $N := \mathbb{R}\mathbb{P}^4\#\mathbb{C}\mathbb{P}^2$ and let λ_N be the equivariant intersection form. Let $x, y \in \pi_2(N)$ with $\lambda_N(x, x) = \lambda_N(y, y) = 1$. Then $x = \pm y$.*

Proof. We have $\pi_2(N) \cong \mathbb{Z}[\mathbb{Z}/2]$. By sesquilinearity of the intersection form, $\lambda_N(a+bT, a'+b'T) = aa' - bb' + (ba' - ab')T$, where $T \in \mathbb{Z}/2$ is the generator and noting that $w(T) = -1$, for the orientation character $w: \pi_1(N) \rightarrow C_2$. Hence $\lambda_N(a+bT, a+bT) = a^2 - b^2$. The only possible solutions in \mathbb{Z} for $1 = a^2 - b^2$ are $a = \pm 1$ and $b = 0$. Hence $x = \pm y$ as claimed. \square

We are now ready for the proof of Proposition 1.6. Recall that R denotes the Cappell–Shaneson exotic $\mathbb{R}\mathbb{P}^4$, i.e. R is homeomorphic not diffeomorphic to $\mathbb{R}\mathbb{P}^4$.

Proof of Proposition 1.6. Recall that $S^2 \tilde{\times} S^2 \cong \mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$. We define $M := \mathbb{R}\mathbb{P}^4\#(S^2 \tilde{\times} S^2) \cong \mathbb{R}\mathbb{P}^4\#\mathbb{C}\mathbb{P}^2\#\overline{\mathbb{C}\mathbb{P}^2}$. By [Akb84, Theorem 2], $R\#\mathbb{C}\mathbb{P}^2$ and $\mathbb{R}\mathbb{P}^4\#\mathbb{C}\mathbb{P}^2$ are diffeomorphic. By Remark 5.2 and Lemma 5.3, there is a diffeomorphism sending $\mathbb{C}\mathbb{P}^1$ to a smooth sphere $U \subseteq \mathbb{R}\mathbb{P}^4\#\mathbb{C}\mathbb{P}^2$ homotopic to $\mathbb{C}\mathbb{P}^1 \subseteq \mathbb{C}\mathbb{P}^2$. Hence as in the proof of Proposition 5.1, for the spheres $S := \mathbb{C}\mathbb{P}^1\#\overline{\mathbb{C}\mathbb{P}^1}$ and $T := U\#\overline{\mathbb{C}\mathbb{P}^1}$ in M we get $M_S \cong \mathbb{R}\mathbb{P}^4\#(S^2 \times S^2)$ and $M_T \cong R\#(S^2 \times S^2)$. Since $\mathbb{R}\mathbb{P}^4$ and R are homeomorphic [HKT94] but not stably diffeomorphic [CS71, Theorem 2.4; CS76b, p. 61], the same applies to these manifolds. \square

REFERENCES

- [Akb84] S. Akbulut, *A fake 4-manifold*, Four-manifold theory (Durham, N.H., 1982), 1984, pp. 75–141.
- [Akb88] S. Akbulut, *Constructing a fake 4-manifold by Gluck construction to a standard 4-manifold*, *Topology* **27** (1988), no. 2, 239–243.
- [AY13] S. Akbulut and K. Yasui, *Gluck twisting 4-manifolds with odd intersection form*, *Math. Res. Lett.* **20** (2013), no. 2, 385–389.
- [BFL14] A. Bartels, F. T. Farrell, and W. Lück, *The Farrell–Jones conjecture for cocompact lattices in virtually connected Lie groups*, *J. Amer. Math. Soc.* **27** (2014), no. 2, 339–388.
- [BL12] A. Bartels and W. Lück, *The Borel conjecture for hyperbolic and CAT(0)-groups*, *Ann. of Math. (2)* **175** (2012), no. 2, 631–689.
- [BLR08] A. Bartels, W. Lück, and H. Reich, *The K-theoretic Farrell–Jones conjecture for hyperbolic groups*, *Invent. Math.* **172** (2008), no. 1, 29–70.
- [CH90] T. Cochran and N. Habegger, *On the homotopy theory of simply connected four manifolds*, *Topology* **29** (1990), no. 4, 419–440.
- [CS71] S. E. Cappell and J. L. Shaneson, *On four dimensional surgery and applications*, *Comment. Math. Helv.* **46** (1971), 500–528.
- [CS76a] S. E. Cappell and J. L. Shaneson, *Construction of some new four-dimensional manifolds*, *Bull. Am. Math. Soc.* **82** (1976), 69–70.
- [CS76b] S. E. Cappell and J. L. Shaneson, *Some new four-manifolds*, *Ann. Math. (2)* **104** (1976), 61–72.
- [Fan15] W. Fan, *Plugs in simply-connected 4 manifolds with boundaries*, ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–Michigan State University.
- [FNOP19] S. Friedl, M. Nagel, P. Orson, and M. Powell, *A survey of the foundations of four-manifold theory in the topological category*, 2019. Preprint, available at arXiv:1910.07372.
- [FQ90] M. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, 1990.
- [Fre82] M. Freedman, *The topology of four-dimensional manifolds*, *J. Differential Geom.* **17** (1982), no. 3, 357–453.
- [FT95a] M. H. Freedman and P. Teichner, *4-manifold topology. I. Subexponential groups*, *Invent. Math.* **122** (1995), no. 3, 509–529.

- [FT95b] M. H. Freedman and P. Teichner, *4-manifold topology. II. Dwyer's filtration and surgery kernels*, Invent. Math. **122** (1995), no. 3, 531–557.
- [Glu62] H. Gluck, *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc. **104** (1962), 308–333.
- [Gor76] C. McA. Gordon, *Knots in the 4-sphere*, Comment. Math. Helv. **51** (1976), no. 4, 585–596.
- [GS99] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999.
- [HK88] I. Hambleton and M. Kreck, *On the classification of topological 4-manifolds with finite fundamental group*, Math. Ann. **280** (1988), no. 1, 85–104.
- [HK93] I. Hambleton and M. Kreck, *Cancellation, elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math. **444** (1993), 79–100.
- [HKT09] I. Hambleton, M. Kreck, and P. Teichner, *Topological 4-manifolds with geometrically two-dimensional fundamental groups*, J. Topol. Anal. **1** (2009), no. 2, 123–151.
- [HKT94] I. Hambleton, M. Kreck, and P. Teichner, *Nonorientable 4-manifolds with fundamental group of order 2*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 649–665.
- [Ker65] M. A. Kervaire, *Les nœuds de dimensions supérieures*, Bull. Soc. Math. France **93** (1965), 225–271.
- [KL22] D. Kasprowski and M. Land, *Topological 4-manifolds with 4-dimensional fundamental group*, Glasg. Math. J. **64** (2022), no. 2, 454–461.
- [KOPR21] M. H. Kim, P. Orson, J. Park, and A. Ray, *Good groups, The disc embedding theorem*, 2021.
- [KPR22] D. Kasprowski, M. Powell, and A. Ray, *Counterexamples in 4-manifold topology*, Preprint, available at arXiv:2203.13332 (2022).
- [KQ00] V. S. Krushkal and F. Quinn, *Subexponential groups in 4-manifold topology*, Geom. Topol. **4** (2000), 407–430 (electronic).
- [KS75] R. C. Kirby and L. C. Siebenmann, *Normal bundles for codimension 2 locally flat imbeddings*, Geometric topology (Proc. Conf., Park City, Utah, 1974), 1975, pp. 310–324. Lecture Notes in Math., Vol. 438.
- [KS77] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J., 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.
- [Lit79] R. A. Litherland, *Deforming twist-spun knots*, Trans. Amer. Math. Soc. **250** (1979), 311–331.
- [Lyn50] R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. (2) **52** (1950), 650–665.
- [NS12] D. Nash and A. I. Stipsicz, *Gluck twist on a certain family of 2-knots*, Michigan Math. J. **61** (2012), no. 4, 703–713.
- [NS20] P. Naylor and H. Schwartz, *Gluck twisting roll spun knots*, 2020. Preprint, available at arXiv:2009.05703.
- [OPR21] P. Orson, M. Powell, and A. Ray, *Surgery theory and the classification of closed, simply connected 4-manifolds* (S. Behrens, B. Kalmár, M. H. Kim, M. Powell, and A. Ray, eds.), Oxford University Press, 2021.
- [Pao78] P. S. Pao, *Nonlinear circle actions on the 4-sphere and twisting spun knots*, Topology **17** (1978), no. 3, 291–296.
- [PR21] M. Powell and A. Ray, *The development of topological 4-manifold theory* (S. Behrens, B. Kalmár, M. H. Kim, M. Powell, and A. Ray, eds.), Oxford University Press, 2021.
- [Pri77] T. M. Price, *Homeomorphisms of quaternion space and projective planes in four space*, J. Austral. Math. Soc. Ser. A **23** (1977), no. 1, 112–128.
- [Qui82] F. Quinn, *Ends of maps. III. Dimensions 4 and 5*, J. Differential Geom. **17** (1982), no. 3, 503–521.
- [Ran92] A. A. Ranicki, *Algebraic L-theory and topological manifolds*, Cambridge Tracts in Mathematics, vol. 102, Cambridge University Press, Cambridge, 1992.
- [Sto94a] R. Stong, *Existence of π_1 -negligible embeddings in 4-manifolds. A correction to Theorem 10.5 of Freedman and Quinn*, Proc. Amer. Math. Soc. **120** (1994), no. 4, 1309–1314.
- [Sto94b] R. Stong, *Uniqueness of connected sum decompositions in dimension 4*, Topology and its applications **56** (1994), 277–291.
- [Tei97] P. Teichner, *On the star-construction for topological 4-manifolds*, Geometric topology (Athens, GA, 1993), 1997, pp. 300–312.
- [Tor17] R. Torres, *Smooth structures on nonorientable four-manifolds and free involutions*, J. Knot Theory Ramifications **26** (2017), no. 13, 1750085, 20.
- [Wal99] C. T. C. Wall, *Surgery on compact manifolds*, Second, Mathematical Surveys and Monographs, vol. 69, American Mathematical Society, Providence, RI, 1999. Edited and with a foreword by A. A. Ranicki.
- [Wan95] Z. Wang, *Classification of closed nonorientable 4-manifolds with infinite cyclic fundamental group*, Math. Res. Lett. **2** (1995), no. 3, 339–344.
- [Weg15] C. Wegner, *The Farrell-Jones conjecture for virtually solvable groups*, J. Topol. **8** (2015), no. 4, 975–1016.

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SOUTHAMPTON, UNITED KINGDOM
Email address: d.kasprowski@soton.ac.uk

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GLASGOW, UNITED KINGDOM
Email address: mark.powell@glasgow.ac.uk

MAX PLANCK INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY
Email address: aruray@mpim-bonn.mpg.de