University of Southampton

Faculty of Social Sciences
School of Mathematical Sciences

Homotopy Theory of Polyhedral Products

by

George Joshua Harry Simmons
ORCiD: 0000-0002-9076-4973

A thesis for the degree of
Doctor of Philosophy

18th April 2023

University of Southampton

Abstract

Faculty of Social Sciences
School of Mathematical Sciences

Doctor of Philosophy

Homotopy Theory of Polyhedral Products

by George Joshua Harry Simmons

In this thesis we use homotopy-theoretic techniques to establish a range of combinatorially-governed relations in the algebraic invariants of polyhedral product spaces.

First, for a flag simplicial complex $\mathcal{K}$, we specify a necessary and sufficient combinatorial condition for the commutator subgroup $RC'_{\mathcal{K}}$ of a right-angled Coxeter group, which is the fundamental group of the real moment-angle complex $\mathcal{R}_{\mathcal{K}}$, to be a one-relator group; and for the loop homology algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ of the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ to be a one-relator algebra. This moreover establishes a combinatorial link between distinct concepts of geometric group theory and homotopy theory.

Second, we give a substantial generalisation of the Whitehead product to a construction called the higher Whitehead map, which takes maps from homotopy sets of the form $[\Sigma X, Y]$ to a new map in homotopy sets related to polyhedral products. We analyse these maps systematically via the combinatorial structure underlying the polyhedral products involved, and derive combinatorial conditions describing when these maps are non-trivial. Moreover, we establish non-trivial relations between higher Whitehead maps which are governed combinatorially. These relations greatly generalise the Jacobi identity for Whitehead products, and results of Hardie on relations among exterior Whitehead products.

Contents

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as: J. Grbić, M. Ilyasova, T. E. Panov, and G. Simmons, *One-relator groups and algebras related to polyhedral products*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **152** (2022), no. 1, 128—147.

Signed:.. Date:..................

Acknowledgements

First and foremost, I would like to thank Professor Jelena Grbić for her unwavering devotion and support. Her supervision allowed me to work to my individual strengths, and build countless other invaluable skills. Moreover, her patience and understanding throughout the COVID-19 pandemic was unparalleled, and for this I am truly grateful.

I was joined on this journey by Dr Matthew Staniforth. It quickly transpired our skills and mindsets were highly complementary, and a collaboration with Matthew forms part of this thesis. Thank you for the countless conversations, challenges, and chess tuition.

I would like to mention and thank my other collaborators Professor Taras Panov and Marina Ilyasova for deep and informative discussions to shape and widen the scope of my results. Further mention goes to Professor Stephen Theriault, my second supervisor, along with Professor Ian Leary and Professor Peter Kropholler for discussions, questions and comments during talks, progression reviews and other impromptu meetings.

My thanks also goes to my thesis examiners Professor Sarah Whitehouse and Dr Nansen Petrosyan for their helpful comments and engaging questions in my viva examination.

I am very grateful to the Fields Institute and the organisers of the Thematic Program on Toric Topology and Polyhedral Products for the opportunity to visit Toronto in early 2020 and meet, learn from, and connect with some of the biggest names in my field.

This journey would not have been possible without all my fellow students and the memories they have made. A special mention goes to my "office bubble" Lily, Tom and Sam, who for one day a week eased the monotony of COVID lockdowns. Thanks also go to my academic siblings Abi, Xin and Briony; my fellow topologists Guy, Holly, Megan, Seb and Simon; and to Geraint, Ingrid, Karl, Laura, Laurie, Matthew B., Matthew C., Moteijus, Naomi, Pete, Ruth, Vlad and Will, who made the office feel like a second home.

And last, but by no means least, I thank my family. Thank you to Amelia for her encouragement and support, especially when we lived apart, and for the adventures we've shared over the last four years. Thank you to my mum Deborah, my dad Gary and my sister Rosina for the endless support and interest, and the occasional, very welcome free meal! Finally, thank you to my grandfathers Les and Richard, my uncle Mark, and my beloved cat Malcolm, who could not be here to see the conclusion of my Mathematical journey. You are all deeply and dearly missed.

Chapter 1

Introduction

Algebraic topology is the study of topological spaces via the assignment of algebraic invariants, which are easier to work with than the geometry of the space itself. Common examples are the fundamental group, homotopy groups, homology and cohomology groups, the cohomology ring, and the loop homology algebra. Such invariants are used, for example, to distinguish two spaces up to homotopy, but there are many deeper and more varied applications.

For example, the homotopy groups of a topological space X can be endowed with an operation called the Whitehead product, taking elements $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ to an element $[\alpha, \beta] \in \pi_{p+q-1}(X)$. The Whitehead product was introduced by J. H. C. Whitehead in 1941 [Whi41], who established that it is a bilinear operation which is graded symmetric, that is, $[\alpha, \beta]$ and $[\beta, \alpha]$ are identified up to a sign. Later it was established by Nakaoka–Toda [NT54] and Uehara–Massey [UM57] that for a further element $\gamma \in \pi_r(X)$ and $p, q, r \geqslant 2$, the iterated Whitehead products $[[\alpha, \beta], \gamma]$, $[[\beta, \gamma], \alpha]$ and $[[\gamma, \alpha], \beta]$ satisfy a linear dependence relation known as the Jacobi identity. This gives the homotopy groups of X a Lie algebra structure. An understanding of this structure leads to further understanding of X itself. For example, the Hilton–Milnor theorem describes the homotopy groups of wedges of spheres in terms of this Lie algebra structure, identifying generators as iterated Whitehead products, and relations between them coming from the Jacobi identity.

Finding generators and relations of algebraic objects, like those in the above Lie algebra, is a key way to understand its structure and compare it to other objects. This is often a difficult problem, and there is no single established method that is guaranteed to work. Therefore there is a constant development of new and varied methods which employ not only techniques from algebra, but from a wide range of different mathematical fields.

One approach is the following. Often, certain algebraic invariants of topological spaces give rise to algebraic objects which are of interest in their own right. Therefore not only can understanding these invariants enhance our understanding of a topological space, but the correspondence can go the other way, and features of the underlying topology can be used to identify interesting algebraic features. A key example that we will see in this thesis is the right-angled Coxeter group and its commutator subgroup both being identified as the fundamental groups of topological spaces called polyhedral products. In the case of the commutator subgroup, this description allows us to specify both generating sets and relations between these generators in certain cases, a task that is very difficult in a purely algebraic setting.

Polyhedral products are spaces constructed according to the combinatorial information contained in a simplicial complex $\mathcal{K}$. This combinatorial structure is invaluable in studying not only the geometry of the space itself, but also its algebraic invariants. The polyhedral product arose out of the field of toric topology, which is at its base the study of spaces with a torus action. A key player in toric topology is the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$. Over time, the merging of the methods and ideas of symplectic and algebraic geometry with toric topology naturally led to the introduction of combinatorial objects, at first in the form of simple polytopes, and more recently simplicial complexes, to decode the rich and varied information involved.

The polyhedral product was introduced to unify the topological approaches coming from the geometric, algebraic and combinatorial viewpoints of toric topology. In turn, this provided a base for generalisation and the use of purely homotopy-theoretic techniques to analyse polyhedral products. These spaces have strong functoriality properties, for example the preservation of fibrations of pairs, and the study of these spaces topologically has become a flourishing area in its own right. The polyhedral product expresses the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ as a union of products of discs D^2 and circles S^1 according to how the simplices of $\mathcal{K}$ intersect.

In this thesis we develop homotopy-theoretic techniques to detect novel, combinatorially-governed relations in the algebraic invariants of polyhedral products. First, we study the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ and its lower-dimensional counterpart, the real moment-angle complex $\mathcal{R}_{\mathcal{K}}$ for a class of simplicial complexes known as flag complexes, which are complexes completely determined by a graph. We study topological information which is propagated through the homology of the loop spaces $\Omega\mathcal{Z}_{\mathcal{K}}$ and $\Omega\mathcal{R}_{\mathcal{K}}$. In the latter case, the assumption of $\mathcal{K}$ being a flag complex implies that $\mathcal{R}_{\mathcal{K}}$ is a finite-dimensional aspherical space, and therefore this information is captured entirely in the fundamental group. Moreover, this fundamental group is identified as the commutator subgroup $RC'_{\mathcal{K}}$ of the right-angled Coxeter group $RC_{\mathcal{K}}$ associated to $\mathcal{K}$.

In Chapter 3 we build on existing work of Panov and Veryovkin [PV16] and also Grbić, Panov, Theriault and Wu [GPTW12] which identifies when the group $RC'_{\mathcal{K}}$ and the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ are free. In particular, we characterise when these algebraic objects have exactly one relation in terms of the same purely combinatorial condition as follows.

Theorem 3.1.1. *Let $\mathcal{K}$ be a flag simplicial complex. The following are equivalent:*

(i) $RC'_{\mathcal{K}} = \pi_1(\mathcal{R}_{\mathcal{K}})$ *is a one-relator group;*

(ii) $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ *is a one-relator algebra;*

(iii) $\mathcal{K}$ *has the form*

$$\mathcal{K} = C_p \text{ or } \mathcal{K} = C_p * \Delta^q \quad \text{for } p \geqslant 4,\ q \geqslant 0$$

where C_p is a p-cycle, Δ^q is a q-simplex and $*$ denotes the join of simplicial complexes.

The equivalence of statements (i) and (iii) was established by Ilyasova and Panov, whose work is summarised in in Sections 3.2.1 and 3.2.2. The full statement of Theorem 3.1.1 was presented in [GIPS22] as joint work with the two aforementioned authors, myself, and Jelena Grbić. This work is presented in Section 3.2.

We extend our results by giving further equivalent algebraic and homotopy-theoretic statements to the above one-relator properties. The first condition is that the simplicial complex $\mathcal{K}$, up to joining with a simplex, is minimally non-Golod, a notion in combinatorial algebra introduced by Burglund and Jöllenbeck [BJ07] in studying the algebra $\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$, where $\mathbb{Z}[m]$ is the polynomial ring on m generators and $\mathbb{Z}[\mathcal{K}]$ is a quotient of $\mathbb{Z}[m]$ determined by $\mathcal{K}$ known as the face ring. Another condition is that the spaces $\mathcal{R}_{\mathcal{K}}$ and $\mathcal{Z}_{\mathcal{K}}$ in this case are homotopy equivalent to connected sums of products of spheres, with two spheres in each product. In the case of $\mathcal{R}_{\mathcal{K}}$, this identifies it with a closed orientable surface of positive genus. After establishing Theorem 3.1.1, we consider the case that $\mathcal{K}$ is not assumed to be a flag complex. We construct a series of examples to highlight the key differences between the concepts of $\mathcal{K}$ being minimally non-Golod, $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ being a one-relator algebra, and $\mathcal{Z}_{\mathcal{K}}$ being a connected sum of sphere products, with two spheres in each product.

In Chapter 4, we give a substantial generalisation of the Whitehead product and the Lie algebra structure it induces on the homotopy groups of a space. We define the higher Whitehead map, an element of a higher Whitehead product in the sense of Porter [Por65], which associates to maps $f_i \in [\Sigma X_i, Y_i]$ an element $h_w(f_1, \ldots, f_m)$ of the homotopy set $[X_1 * \cdots * X_m, FW(Y_1, \ldots, Y_m)]$, where $*$ denotes the topological join of spaces and $FW(Y_1, \ldots, Y_m)$ is the subspace of $Y_1 \times \cdots \times Y_m$ with at least one coordinate the basepoint.

We build on existing work of Abramyan and Panov [AP19], who studied a spherical version of the higher Whitehead map in the case that $f_i \colon S^2 \longrightarrow \mathbb{C}P^\infty$ is the inclusion of the bottom cell. In particular we develop a method of determining combinatorial criteria identifying when the higher Whitehead map is non-trivial. Our main result then establishes non-trivial relations between higher Whitehead maps which are controlled combinatorially. A partition $\Pi = \{I_1, \ldots, I_k\}$ of a set of vertices $[m] = \{1, \ldots, m\}$ is called a k-partition. Given such a partition Π, we construct a simplicial complex $\mathcal{K}_\Pi$, called an identity complex, and show the following result which gives relations in the homotopy groups of the polyhedral product $(\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$ associated to $\mathcal{K}_\Pi$.

Theorem 4.3.7. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \ldots, m$. Let $\Pi = \{I_1, \ldots, I_k\}$ be a k-partition of $[m]$ for $k \geqslant 3$ and denote $I_i = \{i_1, \ldots, i_{l_i}\}$ and $J_i = [m] - I_i = \{j_1, \ldots, j_{r_j}\}$. Then if X_i is a suspension for each $i = 1, \ldots, m$,*

$$\sum_{i=1}^{k} h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_{j_1}, \ldots, f_{j_{r_i}}\right), f_{i_1}, \ldots, f_{i_{n_i}}\right) \circ \sigma_i = 0$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}\right]$, where

$$\sigma_i\colon \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m \longrightarrow \Sigma^{n_i}(\Sigma^{r_i - 2} X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}) \wedge X_{i_1} \wedge \cdots \wedge X_{i_{n_i}}$$

is the restriction of the coordinate permutation

$$CX_1 \times \cdots \times CX_m \longrightarrow CX_{j_1} \times \cdots \times CX_{j_{r_i}} \times CX_{i_1} \times \cdots \times CX_{i_{n_i}}.$$

Our result generalises one of Hardie [Har61], who developed a relation between higher Whitehead maps of the form $h_w(h_w(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m), f_i)$ when the f_i are spherical. Moreover we generalise the work of Cohen [Coh57], who defined the Whitehead product in the case that the f_i are not assumed to be spherical and gave an appropriate Jacobi identity. These relations also imply that when the maps f_i are spherical, the homotopy groups of the polyhedral product $(\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$ have the structure of an L_∞ algebra, also known as a homotopy Lie algebra, which extends the graded quasi-Lie algebra structure given by the Whitehead product.

We extend our results by considering a novel approach to derive relations between Whitehead products with some maps repeated. We define a folded higher Whitehead map by composing the higher Whitehead map with a map from $(\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$ induced by an H-space structure on some of the Y_i, which we call a fold map. We establish that such fold maps are induced on polyhedral products by simplicial maps. Therefore composing the relations of Theorem 4.3.7 with fold maps provides relations among folded higher Whitehead maps whose form is again governed purely combinatorially by the complex obtained by identifying certain subsets of vertices of the complex $\mathcal{K}_\Pi$.

The material in Chapter 4 was jointly produced by myself and Matthew Staniforth, under the supervision of Jelena Grbić. The development of the necessary tools to prove Theorem 4.3.7 is my own work. The techniques used to define and analyse the triviality of higher Whitehead maps and their folded versions were developed by Matthew Staniforth.

Chapter 2

Background

The aim of this thesis is to decode algebraic relations in the invariants of topological spaces in terms of combinatorial properties of a simplicial complex via a functorial construction called the polyhedral product. In this Chapter, we build up to the definition of the polyhedral product and summarise necessary constructions and existing results in homotopy theory, algebra, geometric group theory and combinatorics, on which we will build.

2.1 Homotopy theory

Throughout this thesis, we assume that all topological spaces are based CW-complexes and that all maps are continuous and basepoint-preserving. The category of all such spaces and maps is denoted TOP. For a space X we denote its basepoint by $*_X$, or simply $*$ if the context is clear.

Let $I \subseteq \mathbb{R}$ be the unit interval. The *cone* on a space X, denoted CX, is the quotient of the product $X \times I$ identifying $(x, 1)$ with $(*, t)$ for all $x \in X$ and $t \in I$. The *path space* of X, denoted PX, is the space of all maps $\omega\colon I \longrightarrow X$ such that $\omega(0) = *$.

For a space X, its *suspension* ΣX is the pushout of the diagram

$$CX \longleftarrow X \longrightarrow *$$

where the left map is the inclusion $x \longmapsto (x, 0)$. Similarly, its *loop space* ΩX is the pullback of the diagram

$$PX \longrightarrow X \longleftarrow *$$

where the left map is the evaluation $\omega \longmapsto \omega(1)$. For a map $f\colon X \longrightarrow Y$, there are induced maps $\Sigma f\colon \Sigma X \longrightarrow \Sigma Y$ and $\Omega f\colon \Omega X \longrightarrow \Omega Y$ which give rise to covariant functors $\Sigma\colon \text{TOP} \longrightarrow \text{TOP}$ and $\Omega\colon \text{TOP} \longrightarrow \text{TOP}$.

2.1.1 Homotopy limits and colimits

Fibrations and cofibrations are maps which satisfy the homotopy lifting and extension properties, respectively, with respect to all CW-complexes. See [Ark11, Sections 3.2 & 3.3] for definitions. Typical examples of fibrations include the projection map $X \times Y \longrightarrow X$ and the evaluation map $PX \longrightarrow X$. On the other hand, any inclusion of a cellular subcomplex A of a space X is a cofibration. These fibrations and cofibrations, along with homotopy equivalences, provide a model category structure on TOP.

Unless otherwise stated, we work in the homotopy category HTOP, whose objects are spaces, morphisms are homotopy classes of maps between them, and homotopy equivalences are viewed as categorical isomorphisms. We review necessary constructions which allow us to view fibrations, cofibrations and other categorical constructions in HTOP.

Given two categories I and C, a *diagram* is a covariant functor $F\colon I \longrightarrow C$. The limit $\lim F$ and colimit $\operatorname{colim} F$ of the diagram F do not translate to HTOP since they do not preserve homotopy equivalences. More precisely, if $C = \text{TOP}$, then if $F'\colon I \longrightarrow C$ is another diagram with homotopy equivalences $F(i) \longrightarrow F'(i)$ for each $i \in I$, the induced maps $\lim F \longrightarrow \lim F'$ and $\operatorname{colim} F \longrightarrow \operatorname{colim} F'$ need not be homotopy equivalences.

Homotopy limits and colimits are variants of the constructions of limits and colimits, respectively, which preserve homotopy equivalences. Often, this property comes at the expense of the relevant universal property, since their construction depends on making choices of homotopies. The study of homotopy limits and colimits was initiated by Bousfield–Kan [BK72] and Vogt [Vog73]. In this thesis, we will only need to recognise when a certain limit or colimit is a homotopy limit or colimit, respectively. The following can be found in [DHKS04], see also [BP15, Corollary C.3.3].

Proposition 2.1.1. *Let I be a category with an initial object, that is, there is $a \in I$ such that there is a unique morphism $a \longrightarrow i$ for each $i \in I$. Let $F\colon I \longrightarrow \text{TOP}$ be a functor sending $i \in I$ to a space X_i and a morphism $i \longrightarrow j$ to a map $f_{ij}\colon X_i \longrightarrow X_j$. Then if the map $f_{ai}\colon X_a \longrightarrow X_i$ is a cofibration, the map*

$$\operatorname{colim} F \longrightarrow \operatorname{hocolim} F$$

is a homotopy equivalence. □

A dual result holds for the homotopy limit, replacing initial objects with terminal objects, that is objects $b \in I$ such that there is a unique morphism $i \longrightarrow b$ for each $i \in I$, and replacing cofibrations with fibrations. The most common examples of homotopy limits and colimits are homotopy pullbacks and pushouts, respectively, defined as follows.

Example 2.1.2. The *homotopy pullback* of the diagram

$$B \xrightarrow{f} A \xleftarrow{g} C$$

is the pullback of the diagram

$$E_f \xrightarrow{f'} A \xleftarrow{g'} E_g$$

where f' and g' are fibrant replacements for f and g. Concretely, for a map $f\colon B \longrightarrow A$, the *mapping path* E_f given by

$$E_f = \{(b, l) \in B \times A^I \mid f(b) = l(0)\}$$

is homotopy equivalent to B, and the projection $f'\colon E_f \longrightarrow A$ sending $(b, l) \longmapsto l(1)$ is a fibration. For more details, and also for the dual notion of cofibrant replacements used in the following, see [Ark11, Section 3.5].

Dually, the *homotopy pushout* of the diagram

$$B \xleftarrow{f} A \xrightarrow{g} C$$

is the pushout of the diagram

$$M_f \xleftarrow{f'} A \xrightarrow{g'} M_g$$

where f' and g' are cofibrant replacements for f and g, respectively. Here, the *mapping cylinder* M_f is the quotient space $(A \times I \sqcup B)/\sim$, where $(a, 0) \sim f(a)$, and is homotopy equivalent to B. The inclusion $f'\colon A \longrightarrow M_f$ sending $a \longmapsto (a, 1)$ is a cofibration.

By [Ark11, Propositions 6.2.6 & 6.2.14], we only need to replace one of the maps f or g in the above to ensure the resulting pullbacks and pushouts are homotopy pullbacks and pushouts, respectively.

The *homotopy fibre* I_f of a map $f\colon X \longrightarrow Y$ is the homotopy pullback of the diagram

$$X \xrightarrow{f} Y \longleftarrow *.$$

The sequence of maps

$$I_f \longrightarrow X \xrightarrow{f} Y$$

is called a *homotopy fibration sequence.*

Dually, the *homotopy cofibre* C_f of a map $f\colon X \longrightarrow Y$ is the homotopy pushout of the diagram

$$Y \xleftarrow{f} X \longrightarrow *.$$

The sequence of maps

$$X \xrightarrow{f} Y \longrightarrow C_f$$

is a called a *homotopy cofibration sequence.*

2.1.2 Homotopy groups and exact sequences

For two spaces A and B we denote by $[A, B]$ the set of homotopy classes of maps $A \longrightarrow B$. If either A is a co-H space or B is an H space then there is an induced group structure on $[A, B]$. If both A is a co-H space and B is an H space then these group structures coincide, and moreover $[A, B]$ is abelian. The homotopy groups of a space are given by $\pi_n(B) = [S^n, B]$ for $n \geqslant 1$.

Proposition 2.1.3 ([Ark11]). *Let $f\colon X \longrightarrow Y$ be a map with homotopy fibre I_f and homotopy cofibre C_f.*

(i) There is a sequence

$$\cdots \longrightarrow \Omega^2 Y \longrightarrow \Omega I_f \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow I_f \longrightarrow X \longrightarrow Y$$

where each triple of consecutive spaces is a homotopy fibration sequence. Moreover, for any space W there is a long exact sequence

$$\cdots \longrightarrow [W, \Omega^n I_f] \longrightarrow [W, \Omega^n X] \longrightarrow [W, \Omega^n Y] \longrightarrow \left[W, \Omega^{n-1} I_f\right] \longrightarrow \cdots.$$

(ii) There is a sequence

$$X \longrightarrow Y \longrightarrow C_f \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \Sigma^2 X \longrightarrow \cdots$$

where each triple of consecutive spaces is a homotopy cofibration sequence. Moreover, for any space Z there is a long exact sequence

$$\cdots \longrightarrow [\Sigma^n C_f, Z] \longrightarrow [\Sigma^n Y, Z] \longrightarrow [\Sigma^n X, Z] \longrightarrow \left[\Sigma^{n-1} C_f, Z\right] \longrightarrow \cdots.$$

□

2.1.3 Topological operations

The *fat wedge* of spaces $X_1, \ldots, X_m$, denoted $FW(X_1, \ldots, X_m)$, is the subspace of $\prod_{i=1}^m X_i$ given by

$$FW(X_1, \ldots, X_m) = \{(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m \mid x_i = *_{X_i} \text{ for some } i = 1, \ldots, m\}.$$

In the case that $m = 2$, the space $FW(X_1, X_2)$ is called the *wedge* of X_1 and X_2 and is denoted $X_1 \vee X_2$.

The *smash product* of X and Y, denoted $X \wedge Y$, is the quotient space

$$(X \times Y)/(X \vee Y).$$

The m-fold smash product $X_1 \wedge \cdots \wedge X_m$ is defined inductively. Equivalently, it is the quotient space

$$(X_1 \times \cdots \times X_m)/FW(X_1, \ldots, X_m).$$

The *left half-smash* $X \ltimes Y$ and *right half-smash* $X \rtimes Y$ are the quotient spaces $(X \times Y)/(X \times *_Y)$ and $(X \times Y)/(*_X \times Y)$, respectively. Taking the further quotient of $X \ltimes Y$ by $(*_X \times Y)$ gives the smash $X \wedge Y$, and similarly for $X \rtimes Y$.

The *join* of two spaces X and Y is denoted $X * Y$ and is defined by

$$X * Y = CX \times Y \cup X \times CY$$

where the union is taken over $X \times Y$. Equivalently, $X * Y$ is the homotopy pushout of the diagram

$$X \longleftarrow X \times Y \longrightarrow Y.$$

We recall the following well-known homotopy equivalences, see for example [Sel97].

Proposition 2.1.4. *There are homotopy equivalences*

(i) $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$*;*

(ii) $\Sigma(X \ltimes Y) \simeq X \ltimes (\Sigma Y)$*;*

(iii) if Y *is co-H then* $X \ltimes Y \simeq Y \vee X \wedge Y$*;*

(iv) $X * Y \simeq \Sigma X \wedge Y$.

Combining Proposition 2.1.4(iv) with the definition of the join there are homotopy equivalences

$$\bigcup_{i=1}^{m} CX_1 \times \cdots \times X_i \times \cdots \times CX_m \simeq X_1 * \cdots * X_m \simeq \Sigma^{m-1} X_1 \wedge \cdots \wedge X_m.$$

Finally, let M and N be two n-manifolds. Let $\overline{M}$ and $\overline{N}$ be obtained from M and N, respectively, by removing an open n-ball from each. The *connected sum* $M \# N$ of M and N is given by $\overline{M} \cup \overline{N}$, with the union taken by identifying the boundary spheres of the removed n-balls via a homeomorphism. For any n-manifold M the connected sum $M \# S^n$ is homeomorphic to M.

2.1.4 Whitehead products

The Whitehead product is an operation on homotopy groups introduced by J.H.C. Whitehead [Whi41]. Originally given as an operation $\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)$ for

$p, q \geqslant 1$, the Whitehead product has undergone multiple generalisations and extensions. We study the generalised Whitehead product, first introduced by Cohen [Coh57] and studied in detail by Arkowitz [Ark62].

Definition 2.1.5. Let $f\colon \Sigma A \longrightarrow X$ and $g\colon \Sigma B \longrightarrow Y$ be maps. The *Whitehead product* of f and g, denoted $[f, g]$ is the homotopy class of the map

$$[f, g]\colon A * B = CA \times B \cup A \times CB \longrightarrow \Sigma A \vee \Sigma B \longrightarrow X \vee Y$$

where the first map is the restriction of the map $CA \times CB \longrightarrow \Sigma A \times \Sigma B$ to $A * B$, and the second map is $f \vee g$.

The Whitehead product is uniquely determined up to homotopy by the homotopy classes of f and g, and therefore defines an operation of homotopy groups

$$[\Sigma A, X] \times [\Sigma B, Y] \longrightarrow [A * B, X \vee Y].$$

In the specific case that $A = S^{p-1}$, $B = S^{q-1}$ and $X = Y$, composing the Whitehead product $[f, g]$ with the fold map $X \vee X \longrightarrow X$, which sends $(x, *) \longmapsto x$ and $(*, x) \longmapsto x$, gives a map $S^{p+q-1} = S^{p-1} * S^{q-1} \longrightarrow X$, which we also call the Whitehead product of f and g. In this case the Whitehead product defines a map

$$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)$$

which moreover is a bilinear map satisfying *graded symmetry*, that is $[\beta, \alpha] = (-1)^{pq}[\alpha, \beta]$ for all $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ with $p, q \geqslant 2$, and the *graded Jacobi identity*, that is

$$[[\alpha, \beta], \gamma] + (-1)^{pq}[[\beta, \gamma], \alpha] + (-1)^{qr}[[\gamma, \alpha], \beta] = 0$$

for all $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ and $\gamma \in \pi_r(X)$ with $p, q, r \geqslant 2$. These properties equip the homotopy groups $\pi_*(X) = \bigoplus_{n \geqslant 1} \pi_n(X)$ of a space with a graded quasi-Lie algebra structure, if $\pi_n(X)$ is given a degree of $n - 1$.

While the properties of bilinearity and graded symmetry were given by Whitehead with the original definition [Whi41], the Jacobi identity is a non-trivial result which attracted many classical and varied proofs in the early 1950s such as [Whi54, Hil55, Suz54]. Of particular interest are the proofs due to Uehara–Massey [UM57], which was one of the first applications of the triple Massey product, and of Nakaoka–Toda [NT54]. One of the main results of this thesis, Theorem 4.3.7, is a large generalisation of the Jacobi identity to a combinatorially-controlled class of maps called higher Whitehead maps, which generalise the Whitehead product. Our proof employs the core techniques of Nakaoka–Toda.

Aside from the algebraic operation on homotopy groups, the Whitehead product has deep geometric properties.

Proposition 2.1.6. *Let $f\colon \Sigma X \longrightarrow \Sigma X$ and $g\colon \Sigma Y \longrightarrow \Sigma Y$ be identity mappings. Then the cofibre of the Whitehead product $[f,g]\colon X * Y \longrightarrow \Sigma X \vee \Sigma Y$ is $\Sigma X \times \Sigma Y$.*

Proof. The square

$$\begin{tikzcd} X * Y \arrow[r] \arrow[d] & \Sigma X \vee \Sigma Y \arrow[d] \\ C(X * Y) \arrow[r] & \Sigma X \times \Sigma Y \end{tikzcd}$$

is homotopy commutative by definition of the Whitehead product. Since the vertical cofibres are both $\Sigma X * Y$, this a homotopy pushout square. ☐

Therefore the Whitehead product $[f,g]$ is precisely the map required to attach $X * Y$ to $\Sigma X \vee \Sigma Y$ to form $\Sigma X \times \Sigma Y$. In particular, if $X = S^p$ and $Y = S^q$ are spheres, then the Whitehead product is the cellular attaching map for the top cell D^{p+q} of the product $S^p \times S^q$.

Whitehead products also appear in homotopy fibration sequences. The following result is due to Ganea [Gan67].

Theorem 2.1.7. *There is a homotopy fibration sequence*

$$\Omega X * \Omega Y \longrightarrow X \vee Y \longrightarrow X \times Y.$$

Moreover, the map $\Omega X * \Omega Y \longrightarrow X \vee Y$ is the Whitehead product $[ev_X, ev_Y]$, where $ev_A\colon \Sigma\Omega A \longrightarrow A$ is the adjoint to the identity $\Omega A \longrightarrow \Omega A$. ☐

2.2 Homological algebra

We assume the definitions of graded algebras and coalgebras are known, as well as definitions and basic properties of homology and cohomology. All homology and cohomology groups are assumed to have coefficients in $\mathbb{Z}$, unless otherwise stated.

2.2.1 The Künneth theorem

For A-modules M and N the group $\mathrm{Tor}^i_A(M,N)$ is the ith homology of the sequence

$$\cdots \longrightarrow R^i \otimes_A N \longrightarrow \cdots \longrightarrow R^1 \otimes_A N \longrightarrow R^0 \otimes_A N \longrightarrow 0$$

where

$$\cdots \longrightarrow R^i \longrightarrow \cdots \longrightarrow R^1 \longrightarrow R^0 \longrightarrow M \longrightarrow 0$$

is a projective resolution of M.

The Künneth theorem describes the homology groups of a Cartesian product space in terms of Tor groups as follows. Suppose that X and Y are spaces. Then there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}^1_{\mathbb{Z}}(H_i(X), H_j(Y)) \longrightarrow 0 \tag{2.1}$$

which splits, but not naturally. When both X and Y have torsion-free homology, the group $\mathrm{Tor}^1_{\mathbb{Z}}(H_i(X), H_j(Y))$ vanishes, giving an isomorphism

$$H_*(X \times Y) \cong H_*(X) \otimes H_*(Y). \tag{2.2}$$

A reduced version of the Künneth theorem, replacing all homology groups in the second and fourth terms of (2.1) with their reduced versions, describes the reduced homology groups of the smash product $X \wedge Y$. When X and Y have torsion-free homology, this reduces to

$$\widetilde{H}_*(X \wedge Y) \cong \widetilde{H}_*(X) \otimes \widetilde{H}_*(Y). \tag{2.3}$$

2.2.2 Free associative algebras

The *free associative algebra* on a graded $\mathbb{Z}$-module M is given by

$$T(M) = \bigoplus_{n \geqslant 0} M^{\otimes n}.$$

where $M^{\otimes n}$ is the n-fold tensor product over $\mathbb{Z}$ of M with itself. We denote by $T_k(M) = M^{\otimes k}$.

The multiplicative structure in $T(M)$ given by concatenation is associative but not commutative in general. The *graded commutator* of a and b, denoted $[a, b]$, is given by

$$[a, b] = a \cdot b - (-1)^{\deg a \deg b} b \cdot a. \tag{2.4}$$

If $\{a_1, \ldots, a_k\}$ is a finite generating set for M then we use $T(a_1, \ldots, a_k)$ to denote the algebra $T(M)$.

2.2.3 Differential graded algebras and coalgebras

A *differential graded algebra*, or *dg-algebra* for short, is a pair (A, d) consisting of a graded algebra A together with a map $d\colon A \longrightarrow A$ of degree 1 or -1 which satisfies $d^2 = 0$ and the graded Leibniz rule

$$d(ab) = d(a)b + (-1)^{\deg a} a d(b) \tag{2.5}$$

for each $a, b \in A$. The condition $d^2 = 0$ makes a *dg*-algebra a chain complex and we can define the homology $H(A)$. A morphism of *dg*-algebras is is a graded algebra homomorphism which respects the map d. The category of *dg*-algebras is denoted DGA.

Example 2.2.1. The free associative algebra $T(M)$ is given the structure of a *dg*-algebra by specifying a differential $d\colon T_k(M) \longrightarrow T_{k-1}(M)$ as follows. If $\{a_1, \ldots, a_n\}$ is a basis of M, set $d(a_i) = 1$. Using the graded Leibniz rule (2.5), this extends to a differential

$$d(a_{i_1} \otimes \cdots \otimes a_{i_k}) = \sum_{j=1}^{k} (-1)^{j-1} a_{i_1} \otimes \cdots \otimes a_{i_{j-1}} \otimes a_{i_{j+1}} \otimes \cdots \otimes a_{i_k}.$$

A *differential graded coalgebra*, or *dg-coalgebra* is a triple (C, ∂, Δ) consisting of a graded coalgebra (C, Δ) together with a map $\partial\colon C \longrightarrow C$ of degree 1 or -1 which satisfies $\partial^2 = 0$ and

$$\Delta\partial(c) = (1 \otimes \partial + \tau(1 \otimes \partial)\tau)\Delta(c)$$

where $\tau(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a$. The category of *dg*-coalgebras is denoted DGC.

Example 2.2.2. Any singular chain complex $C_*(X)$ of a space X is a *dg*-coalgebra where the diagonal map $\Delta\colon C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$ is induced by the map $X \longrightarrow X \times X$, $x \longmapsto (x, x)$.

For a cellular chain complex $C_*(X)$, the diagonal map $X \longrightarrow X \times X$ does not in general induce a map $C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$. The diagonal map, however, is always homotopic to a cellular map $\widetilde{\Delta}$ which does induce a map $\widetilde{\Delta}\colon C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$. Therefore a cellular chain complex is a *dg*-coalgebra with diagonal map induced by a cellular approximation to $X \longrightarrow X \times X$.

A *differential graded Hopf algebra* H is a $\mathbb{Z}$-module which is simultaneously a *dg*-algebra (H, d) and *dg*-coalgebra (H, d, Δ) such that $\Delta\colon A \longrightarrow A \otimes A$ is an algebra homomorphism, that is,

$$\Delta(a \otimes b) = \Delta a \otimes \Delta b. \tag{2.6}$$

Algebraically, a Hopf algebra requires the further definition of an antipode map. This condition is automatically satisfied when considering graded connected Hopf algebras. Graded Hopf algebras appear topologically as the homology of certain loop spaces ΩX, for example when X is a suspension with torsion-free homology. The algebra structure is induced by the multiplication on loop spaces. We study such algebras in detail in Section 2.3.

The quotient of a graded Hopf algebra H by an ideal invariant under both the algebraic and coalgebraic structures remains a Hopf algebra, with grading induced by that of H.

Example 2.2.3. The free associative algebra has two coalgebra structures, one compatible with the multiplication defining a Hopf algebra, and one not.

Given $a_1 \cdots a_k \in T_k(M)$, we define

$$\Delta(a_1 \cdots a_k) = \sum_{i=0}^{k} (a_0 \cdots a_i) \otimes (a_{i+1} \cdots a_{k+1})$$

where $a_0 = a_{k+1} = 1$. This extends to a coalgebra structure on $T(M)$, but is not compatible with the algebra structure since the compatibility condition (2.6) is not satisfied. For example, consider $ab \in T_2(M)$. Then $\Delta(ab) = 1 \otimes ab + a \otimes b + ab \otimes 1$ whereas

$$\begin{aligned} \Delta(a) \otimes \Delta(b) &= (1 \otimes a + a \otimes 1) \otimes (1 \otimes b + b \otimes 1) \\ &= 1 \otimes ab + a \otimes b + (-1)^{\deg a \deg b} b \otimes a + ab \otimes 1. \end{aligned}$$

Instead, to define a Hopf algebra structure on $T(M)$ we start with condition (2.6) and define

$$\Delta(a_1 \cdots a_k) = \Delta(a_1) \otimes \cdots \otimes \Delta(a_k)$$

which can be written in the following form which we will utilise later on. We have

$$\Delta(a_1 \cdots a_k) = \sum_{\sigma} \epsilon(\sigma)(a_{j_1} \cdots a_{j_i}) \otimes (a_{j_{i+1}} \cdots a_{j_k}) \tag{2.7}$$

where σ is the permutation such that $\sigma(i) = j_i$, $\epsilon(\sigma)$ has a factor $(-1)^{\deg a_i \deg a_j}$ for every transposition (i, j) of σ, and the sum is taken over all $(i, k - i)$ shuffles σ, or more concretely over all disjoint partitions $\{j_1, \ldots, j_i\} \sqcup \{j_{i+1}, \ldots, j_k\}$ of $\{1, \ldots, k\}$, with one side potentially empty, with $j_1 < \cdots < j_i$ and $j_{i+1} < \cdots < j_k$.

2.3 The loop homology algebra

Given a space X, recall that the loop space ΩX is an H space equipped with a multiplication $\mu \colon \Omega X \times \Omega X \longrightarrow \Omega X$ given by concatenation of loops. There is an induced map $\mu_* \colon H_*(\Omega X \times \Omega X) \longrightarrow H_*(\Omega X)$ and therefore a product

$$H_*(\Omega X) \otimes H_*(\Omega X) \xrightarrow{\times} H_*(\Omega X \times \Omega X) \xrightarrow{\mu_*} H_*(\Omega X)$$

where the first map is the cross product in homology. This is known as the *Pontryagin product* and equips the homology groups of a loop space with the structure of an algebra. We call the algebra $H_*(\Omega X)$ the *loop homology algebra* of X.

The loop homology algebra captures homotopy-theoretic structure of X which is not seen by homology or cohomology. For example all Whitehead products are trivial in homology since their suspension is nullhomotopic [Por65]. On the other hand, let $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ and define the map θ as the composite of the adjunction isomorphism $\pi_*(X) \longrightarrow \pi_{*-1}(\Omega X)$ with the Hurewicz map $\pi_{*-1}(\Omega X) \longrightarrow H_{*-1}(\Omega X)$. Then it was shown by Samelson [Sam53] that

$$\theta([\alpha,\beta]) = (-1)^p\left(\theta\alpha\cdot\theta\beta - (-1)^{pq}\theta\beta\cdot\theta\alpha\right) = (-1)^p[\theta\alpha,\theta\beta] \tag{2.8}$$

where $[x,y]$ is the commutator (2.4) in the algebra $H_*(\Omega X)$.

Relation (2.8) is helpful in determining some structure of $H_*(\Omega X)$, for example if a space is formed by attaching a cell via a Whitehead product, then the corresponding commutator under (2.8) becomes trivial. We elaborate on this method further in Section 3.2.3.

In general the full computation of the algebra $H_*(\Omega X)$ is difficult for an arbitrary space X. We review some methods of computation. The first, the Bott–Samelson theorem, identifies $H_*(\Omega X)$ as a free associative algebra when X is a suspension with torsion-free homology. For simply-connected spaces, the Cobar construction gives a chain complex for ΩX from a simply-connected chain coalgebra for X.

2.3.1 The Bott–Samelson Theorem

Let X be a connected space. Then there is a homotopy equivalence

$$\Sigma\Omega\Sigma X \simeq \bigvee_{n\geqslant 1}\Sigma X^{\wedge n} \tag{2.9}$$

known as the James splitting [Jam55], where $X^{\wedge n}$ is the n-fold smash product of X with itself. In particular, there is a homology isomorphism

$$H_k(\Omega\Sigma X) \cong H_{k+1}(\Sigma\Omega\Sigma X) \cong H_{k+1}\left(\bigvee_{n\geqslant 1}\Sigma X^{\wedge n}\right) \cong \bigoplus_{n\geqslant 1} H_k(X^{\wedge n}) \tag{2.10}$$

for all $k\geqslant 0$.

Suppose that X has torsion-free homology. The Künneth formula (2.3) gives an isomorphism $\widetilde{H}_*(X^{\wedge n}) \cong \bigotimes_{i=1}^n \widetilde{H}_*(X)$. Therefore there is an isomorphism of graded groups

$$H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X)).$$

The Bott–Samelson Theorem establishes that this isomorphism is one of algebras.

Theorem 2.3.1 ([BS53]). *Suppose that X is a connected space such that $H_*(X)$ is torsion-free. Then there is an algebra isomorphism*

$$H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$$

induced by the map $X \longrightarrow \Omega\Sigma X$ adjoint to the identity $\Sigma X \longrightarrow \Sigma X$. □

Example 2.3.2. (i) Let $X = S^{n-1}$ for $n \geqslant 2$. Then since $H_*(S^{n-1})$ is torsion-free there is an algebra isomorphism

$$H_*(\Omega S^n) = H_*(\Omega\Sigma S^{n-1}) \cong T(x)$$

where $x \in H_{n-1}(S^{n-1})$ is a generator. Therefore $H_{k(n-1)}(\Omega S^n) = \mathbb{Z}$ is generated by x^k for $k \geqslant 1$ and $H_j(\Omega S^n) = 0$ otherwise.

(ii) Let $X = \bigvee_{i=1}^m S^{n_i-1}$ where $n_i \geqslant 2$ for $i = 1, \ldots, m$. Then $H_*(X)$ is freely generated by elements $x_1, \ldots, x_m$ with $\deg x_i = n_i - 1$ for each $i = 1, \ldots, m$. Therefore there is an algebra isomorphism

$$H_*\left(\Omega \bigvee_{i=1}^m S^{n_i}\right) = H_*\left(\Omega\Sigma \bigvee_{i=1}^m S^{n_i-1}\right) = T(x_1, \ldots, x_m).$$

(iii) Let $X = \mathbb{R}P^2$. Theorem 2.3.1 applies when $H_*(X; \underline{k})$ is torsion-free over some coefficient ring $\underline{k}$. If $\underline{k} = \mathbb{Z}_2$ we obtain that $H_*(\Omega\Sigma X; \mathbb{Z}_2) \cong T_{\mathbb{Z}_2}(x)$ where x generates $H_1(\mathbb{R}P^2)$. When $\underline{k} = \mathbb{Z}$, $H_*(X)$ is not torsion-free, so we cannot apply Theorem 2.3.1. In this case, homology isomorphism (2.10) can be used to show that

$$H_k(\Omega\Sigma\mathbb{R}P^2) = \bigoplus_{i=1}^{F(k+1)} \mathbb{Z}_2$$

for all $k \geqslant 0$, where $F(k)$ is the kth Fibonacci number such that $F(0) = 0$ and $F(1) = 1$. In particular, the number of summands in $H_k(\Omega\Sigma\mathbb{R}P^2)$ grows much faster than for $H_k(\Omega\Sigma S^n)$. In Example 2.3.4(iii) we will establish that $T(\widetilde{H}_*(\mathbb{R}P^2)) \cong T(x)/\langle 2x\rangle$ appears only as a subalgebra of $H_*(\Omega\Sigma\mathbb{R}P^2)$, which has countably many other generators.

2.3.2 Adams–Hilton Models

Adams and Hilton [AH56] developed a method of deriving a chain complex for the loop space ΩX of a simply-connected space X from the singular chain complex of X itself. The chain complex for ΩX naturally has the structure of a *dg*-algebra, which in turn derives the algebra structure of $H_*(\Omega X)$. Subsequently, Adams [Ada56] refined the construction to give a functor

$$\mathrm{Cobar}\colon \textsc{dgc}_1 \longrightarrow \textsc{dga}$$

from the category $\textsc{dgc}_1$ of simply-connected *dg*-coalgebras to *dg*-algebras, known as the *Cobar construction.* If $C = (C, \partial, \Delta)$ is a chain complex for X with $C_0 = \mathbb{Z}$ which is simply-connected, that is, $C_1 = 0$, with diagonal Δ, the Cobar construction assigns to C the *dg*-algebra

$$\mathrm{Cobar}_* C = (F(C), d)$$

which is a chain complex for ΩX. Here, $F(C) = T(s^{-1}\overline{C})$ is the free associative algebra on the desuspended module $\overline{C} = C/C_0$, and the differential d is given by

$$dc = -\partial c + \sum_{i=1}^{p-2} (-1)^i \Delta_{i,p-i} c \tag{2.11}$$

where $c \in s^{-1}\overline{C}_p$ has comultiplication $\Delta c = 1 \otimes c + c \otimes 1 + \sum_{i=1}^{p-2} \Delta_{i,p-i} c$. We summarise the main properties of the Cobar construction in the following.

Theorem 2.3.3 ([AH56, Ada56]). *Let $C = (C, \partial, \Delta)$ be a simply-connected chain complex for a simply-connected space X. Then:*

(i) there is a natural isomorphism of algebras $H_*(\Omega X) \cong H(\mathrm{Cobar}_* C)$;

(ii) if C' is a simply-connected quasi-isomorphic chain complex for X then $\mathrm{Cobar}_* C$ and $\mathrm{Cobar}_* C'$ are also quasi-isomorphic chain complexes for ΩX. □

A special case of statement (ii) above is when C' is chosen to be a cellular chain complex for X with the diagonal Δ a cellular approximation to the diagonal $\Delta\colon X \longrightarrow X \times X$. In this case the Cobar construction $\mathrm{Cobar}_* C'$ is known as the *Adams–Hilton model*, denoted $AH_*(X)$. Adams–Hilton models are highly effective for the computation of $H_*(\Omega X)$ in specific examples.

Example 2.3.4. (i) Let $X = S^p \times S^q$ for $p, q \geqslant 2$. Then the reduced cellular chain complex $\overline{C}_*(X)$ is generated by cells e^p, e^q and e^{p+q} of degrees p, q and $p+q$, respectively. The differential ∂ is trivial and the comultiplication is given by $\Delta e^p = \Delta e^q = 0$ and

$$\Delta e^{p+q} = 1 \otimes e^{p+q} + e^{p+q} \otimes 1 + e^p \otimes e^q + (-1)^{pq} e^q \otimes e^p.$$

Therefore $AH_*(X)$ is the free associative algebra $T(x, y, z)$ with $\deg x = p - 1$, $\deg y = q - 1$ and $\deg z = p + q - 1$ with differential given by $dx = dy = 0$ and

$$dz = (-1)^p xy + (-1)^{pq+q} yx = (-1)^p (xy - (-1)^{(p-1)(q-1)} yx) = (-1)^p [x, y]$$

where $[x, y]$ is the commutator (2.4). It follows that

$$H_*(\Omega X) \cong T(x, y)/\langle [x, y] \rangle$$

is the free commutative algebra on two generators. In Proposition 3.2.6 we give a generalisation to the case that X is a simply-connected connected sum of sphere products of the form

$$X = \#_{i=1}^k \left(S^{d_i} \times S^{d-d_i} \right)$$

and establish that the above algebra isomorphism is one of Hopf algebras.

(ii) Let $X = \mathbb{C}P^n$ for $n \geqslant 1$. Then X has a cell structure consisting of a single cell e^{2k} for each $1 \leqslant k \leqslant n$ of dimension $2k$. The differential is trivial and the diagonal is given by

$$\Delta e^{2k} = 1 \otimes e^{2k} + e^{2k} \otimes 1 + \sum_{i+j=k} e^{2i} \otimes e^{2j}.$$

Therefore $AH_*(\mathbb{C}P^n)$ is the differential graded algebra $(T(a_1, \dots, a_n), d)$ where $a_k = s^{-1}e^{2k}$ and $da_k = \sum_{i+j=k} a_i a_j$. In the case that $n = 1$ the Adams–Hilton model agrees with that of S^2, while for $n = 2$,

$$H_*(\Omega\mathbb{C}P^2) = H(AH_*(\mathbb{C}P^2)) = \frac{T(a_1, x)}{\langle a_1^2, [a_1, x]\rangle}$$

where $x = [a_1, a_2]$.

(iii) Let $X = \Sigma\mathbb{R}P^2$. We return to Example 2.3.2(iv) to conclude the computation of $H_*(\Omega X) = H_*(\Omega\Sigma\mathbb{R}P^2)$. A reduced cellular complex $\overline{C}_*(X)$ is generated by cells e^2 and e^3 of dimensions 2 and 3, respectively. The differential ∂ is given by $\partial e^2 = 0$ and $\partial e^3 = 2e^2$. The reduced diagonal is trivial. Therefore $AH_*(X)$ is the free associative algebra $T(x, y)$ with $\deg x = 1$ and $\deg y = 2$ and differential given by $dx = 0$ and $dy = -2x$. We see that $d(xy + yx) = 2x - 2x = 0$ is a cycle, and that $dy^2 = -2(xy + yx)$. Therefore $z_3 = xy + yx$ generates a $\mathbb{Z}_2$ summand $H_3(\Omega X)$. More generally,

$$z_{2j+1} = xy^j + yxy^{j-1} + \cdots + y^{j-1}xy + y^jx$$

generates a $\mathbb{Z}_2$ summand in $H_{2j+1}(\Omega X)$ for $j \geqslant 1$. Note that, for example, while $x^2y + yx^2$ also generates a homology class, this can be constructed algebraically as the commutator $[z_3, x]$. Relabelling x to z_1, we consider the algebra $T(z_1, z_3, z_5, \dots)/\langle 2z_1, 2z_3, 2z_5, \dots\rangle$. A counting argument shows that the number of degree k summands on the right-hand side is the same as that for $H_*(\Omega X)$ described in Example 2.3.2(iv). We therefore obtain an isomorphism of algebras

$$H_*(\Omega X) \cong T(z_1, z_3, z_5, \dots)/\langle 2z_1, 2z_3, 2z_5, \dots\rangle$$

with $\deg z_i = i$.

2.4 Simplicial complexes

We begin by defining simplicial complexes and giving associated constructions. A vertex set $[m]$ is a finite ordered set consisting of m elements. Where it does not create confusion, we denote the elements of $[m]$ by $\{1, \dots, m\}$.

Definition 2.4.1. A *simplicial complex* $\mathcal{K}$ on vertex set $[m]$ is a collection of subsets $\sigma \subseteq [m]$ such that if $\sigma \in \mathcal{K}$ and $\tau \subseteq \sigma$ then $\tau \in \mathcal{K}$. We always assume that $\varnothing \in \mathcal{K}$.

An element $\sigma \in \mathcal{K}$ is called a *simplex*. A map $\mathcal{K} \longrightarrow \mathcal{L}$ of simplicial complexes is a map ψ between their vertex sets such that whenever $\{i_1, \ldots, i_k\}$ is a simplex of $\mathcal{K}$, the image $\{\psi(i_1), \ldots, \psi(i_k)\}$ is a simplex of $\mathcal{L}$. The category of simplicial complexes and maps between them is denoted $\textsc{sc}$.

The *dimension* of a simplex $\dim \sigma$ is given by $|\sigma| - 1$. A simplex of dimension k is called a k-simplex. The *dimension* of $\mathcal{K}$ is given by

$$\dim \mathcal{K} = \max_{\sigma \in \mathcal{K}} \dim \sigma.$$

A simplex $\sigma \in \mathcal{K}$ is *maximal* if $\dim \sigma = \dim \mathcal{K}$. A simplicial complex on a given vertex set is completely determined by its set of maximal simplices.

Elements of $[m]$ which are not vertices of $\mathcal{K}$ are called *ghost vertices*. Unless otherwise stated, we assume that all simplicial complexes have no ghost vertices.

A *subcomplex* of $\mathcal{K}$ is collection of simplices of $\mathcal{K}$ which is itself a simplicial complex.

We introduce some common simplicial complexes and fix some notation.

Example 2.4.2. (i) We refer to the simplicial complex $\mathcal{K}$ with maximal simplices $\{\{1\}, \ldots, \{m\}\}$ as the complex consisting of m disjoint points or vertices. We write $\mathcal{K} = \bullet_{[m]}$. The complex consisting of a single point is denoted $\bullet$.

(ii) The simplex Δ^m is the simplicial complex on $[m]$ with maximal face $\{1, \ldots, m\}$.

(iii) The boundary of a simplex $\partial \Delta^{m-1}$ is the simplicial complex with maximal faces

$$\{\{1, \ldots, \hat{\imath}, \ldots, m\} \mid i = 1, \ldots, m\}$$

where $\hat{\imath}$ denotes the omission of the element i from the given set. The complex $\partial \Delta^2$ is shown in Figure 2.1(a).

(iv) The k-skeleton of a simplex $\operatorname{sk}^k \Delta^{m-1}$ is the subcomplex of Δ^{m-1} consisting of all simplices of dimension at most k. As a special case, $\partial \Delta^{m-1} = \operatorname{sk}^{m-2} \Delta^{m-1}$. The complex $\operatorname{sk}^1 \Delta^3$ is shown in Figure 2.1(b).

(v) For $p \geqslant 4$ the simplicial complex with maximal simplices

$$\{\{1, 2\}, \{2, 3\}, \ldots, \{p-1, p\}, \{p, 1\}\}$$

is called a p-gon or p-cycle and is denoted C_p. The complex C_5 is shown in Figure 2.1(c).

(vi) To specify certain subcomplexes more easily we list their vertex sets. For example if $\mathcal{K}$ is the complex $\operatorname{sk}^1 \Delta^3$ shown in Figure 2.1(b), then $\partial \Delta[1, 2, 3]$ and $\partial \Delta[1, 2, 4]$ denote the subcomplexes $\partial \Delta^2$ on vertex sets $\{1, 2, 3\}$ and $\{1, 2, 4\}$, respectively. We use $\bullet_i$ instead of $\bullet[i]$ to denote ith vertex of $\mathcal{K}$, to avoid confusion with the notation $\bullet_{[m]}$ for m disjoint points.

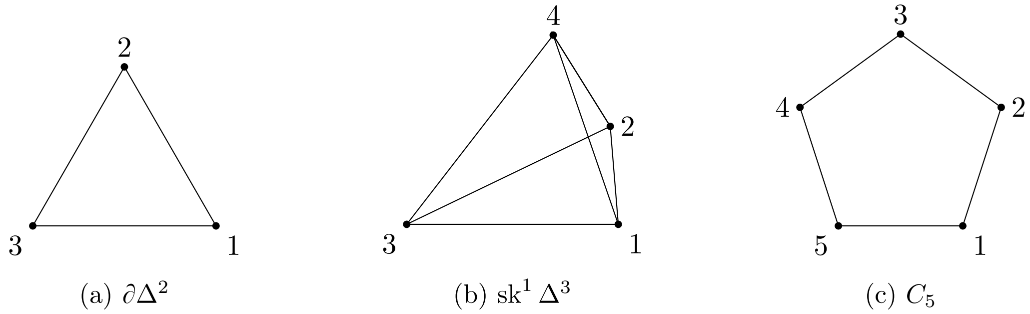


Figure 2.1: Some common simplicial complexes.

Definition 2.4.3. Given a subset $J \subseteq [m]$, the *full subcomplex* of $\mathcal{K}$ corresponding to J is denoted $\mathcal{K}_J$ and is given by

$$\mathcal{K}_J = \{\sigma \cap J \mid \sigma \in \mathcal{K}\}.$$

Any full subcomplex of $\mathcal{K}$ is a subcomplex.

Example 2.4.4. The full subcomplex of Δ^{m-1} for any $J \subseteq [m]$ is the simplex $\Delta[J]$. The full subcomplex of $\mathrm{sk}^{m-3}\,\Delta^{m-1}$ for $|J| = m-2$ is the complex $\partial\Delta[J]$.

Definition 2.4.5. The set of minimal missing faces of $\mathcal{K}$, denoted $MF(\mathcal{K})$ is the set of all σ such that $\sigma \notin \mathcal{K}$, but $\tau \in \mathcal{K}$ for every $\tau \subsetneq \sigma$.

Given a simplicial complex $\mathcal{K}$ on $[m]$, the *face ring*, also called the Stanley-Reisner ring, is the quotient of the polynomial ring $\mathbb{Z}[v_1, \ldots, v_m]$ by the ideal $I_{\mathcal{K}}$ generated by monomials $v_{i_1} \cdots v_{i_k}$ such that $(i_1, \ldots, i_k)$ is a minimal missing face of $\mathcal{K}$. We denote the face ring of $\mathcal{K}$ by $\mathbb{Z}[\mathcal{K}]$.

Example 2.4.6. Let $\mathcal{K} = C_5$ be the 5-gon shown in Figure 2.1(c). Then

$$MF(\mathcal{K}) = \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\}.$$

The face ring for $\mathcal{K}$ is

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, v_2, v_3, v_4, v_5]/\langle v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5\rangle.$$

In this case, the face ring is identified with the group ring $\mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z}[v_1]/\langle v_1^2\rangle$. To see this, we use the relations in $\mathbb{Z}[\mathcal{K}]$ to rewrite

$$v_1 = v_3^{-1} = v_5 = v_2^{-1} = v_4 = v_1^{-1}.$$

Therefore $v_1^2 = 1$, and each v_j for $j = 2, 3, 4, 5$ is identified with v_1.

The study of face ring $\mathbb{Z}[\mathcal{K}]$ originated in combinatorial algebra, but has become a key tool in the study of polyhedral products as it appears as the cohomology ring of the Davis–Januszkiewicz space, and in turn plays an important part in computing the cohomology of moment-angle complexes. We will return to such applications in Section 2.6.

Definition 2.4.7. The *face category* $\textsc{cat}(\mathcal{K})$ is the category with objects $\sigma \in \mathcal{K}$ and morphisms $\tau \longrightarrow \sigma$ whenever $\tau \subseteq \sigma$.

2.4.1 Combinatorial operations

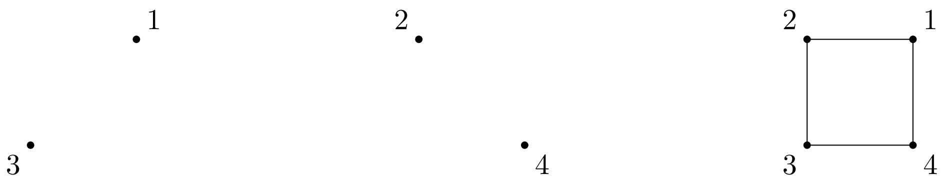
The *pushout*, or amalgamated union, of two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ on $[m]$ over a common subcomplex $\mathcal{M}$ is denoted by $\mathcal{K} \cup_{\mathcal{M}} \mathcal{L}$ is computed as the standard set-wise union and is always a simplicial complex. When $\mathcal{M} = \varnothing$, the resulting pushout is the coproduct in the category $\textsc{sc}$. On the other hand, the product in $\textsc{sc}$ is not the Cartesian product, since this does not preserve simplicial complexes. Instead, the product in $\textsc{sc}$ is the join, defined as follows.

Definition 2.4.8. Given simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ on $[m]$ and $[n]$, respectively, the *join* of $\mathcal{K}$ and $\mathcal{L}$, denoted $\mathcal{K} * \mathcal{L}$ is the simplicial complex on $[m] \sqcup [n]$ given by

$$\mathcal{K} * \mathcal{L} = \{\sigma \sqcup \tau \mid \sigma \in \mathcal{K}, \tau \in \mathcal{L}\}.$$

Here, $[m] \sqcup [n]$ denotes the vertex set $[m+n]$ with ordering inherited from the orderings on $[m]$ and $[n]$, and all elements of $[n]$ ordered after those of $[m]$.

Example 2.4.9. Let $\mathcal{K} = \{\varnothing, \{1\}, \{3\}\}$ and $\mathcal{L} = \{\varnothing, \{2\}, \{4\}\}$ each consist of two disjoint points. Then $\mathcal{K} * \mathcal{L}$ has maximal simplices $(1,2)$, $(2,3)$, $(3,4)$ and $(1,4)$. Geometrically, $\mathcal{K} * \mathcal{L}$ is a square, as Figure 2.2 shows.



(a) $\mathcal{K}$ (b) $\mathcal{L}$ (c) $\mathcal{K} * \mathcal{L}$

Figure 2.2: The join of simplicial complexes.

For a simplicial complex K, the *link* $\mathrm{lk}_{\mathcal{K}}(\sigma)$ of a simplex $\sigma \in K$ is given by

$$\mathrm{lk}_{\mathcal{K}}(\sigma) = \{\tau \in \mathcal{K} \mid \sigma \cup \tau \in \mathcal{K}, \sigma \cap \tau = \varnothing\}$$

while the *star* $\mathrm{st}_{\mathcal{K}}(\sigma)$ of σ is given by

$$\mathrm{st}_{\mathcal{K}}(\sigma) = \{\tau \in \mathcal{K} \mid \sigma \cup \tau \in \mathcal{K}\}.$$

Define also the *boundary of the star* to be the subcomplex

$$\partial\,\mathrm{st}_{\mathcal{K}}(\sigma) = \{\tau \in \mathcal{K} \mid \sigma \cup \tau \in \mathcal{K}, \sigma \not\subset \tau\}.$$

For any simplicial complex $\mathcal{K}$ and vertex $j \in \mathcal{K}$ there is a decomposition

$$\mathcal{K} = \mathrm{st}_{\mathcal{K}}(j) \cup_{\mathrm{lk}_{\mathcal{K}}(j)} \mathcal{K}_{[m]-\{j\}}. \tag{2.12}$$

which we call the *link-star decomposition* of $\mathcal{K}$ at vertex j, see [GT07].

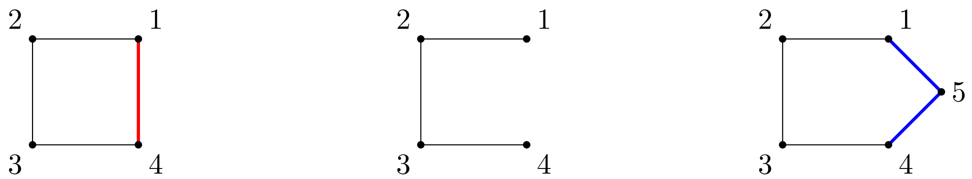
Example 2.4.10. Let $\mathcal{K}$ be the complex shown in Figure 2.2(c). Then $\mathrm{lk}_{\mathcal{K}}(1)$ consists of the two disjoint points $\{2, 4\}$ while $\mathrm{st}_{\mathcal{K}}(1)$ has simplices $(1, 2)$ and $(1, 4)$ and $\partial\,\mathrm{st}_{\mathcal{K}}(1)$ again consists of the disjoint points $\{2, 4\}$. Since $\mathcal{K}_{2,3,4}$ has maximal simplices $(2, 3)$ and $(3, 4)$ then we obtain the decomposition (2.12) in this case.

Definition 2.4.11. The *stellar subdivision* $\mathrm{sts}_{\mathcal{K}}(\sigma)$ of $\mathcal{K}$ at a simplex σ is the simplicial complex on $[m] \sqcup \{j\}$ given by

$$\mathrm{sts}_{\mathcal{K}}(\sigma) = (\mathcal{K} - \mathrm{st}_{\mathcal{K}}(\sigma)) \cup (\partial\,\mathrm{st}_{\mathcal{K}}(\sigma) * j)$$

where j is a new vertex not in $\mathcal{K}$.

Example 2.4.12. Let $\mathcal{K}$ be the complex shown in Figure 2.2(c) and let $\sigma = (1, 4)$. Then $\mathrm{lk}_{\mathcal{K}}(\sigma) = \varnothing$, while $\mathrm{st}_{\mathcal{K}}(\sigma) = \sigma$ and $\partial\,\mathrm{st}_{\mathcal{K}}(\sigma) = \{1, 4\}$. Then $\mathcal{L} = \mathrm{sts}_{\mathcal{K}}(\sigma)$ has maximal simplices $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 5)$ and $(1, 5)$. Therefore $\mathcal{L} = C_5$, a 5-gon. Figure 2.3 shows the process of stellar subdivision.



(a) $\sigma = (1, 4)$, coloured red (b) Remove $\mathrm{st}_{\mathcal{K}}(\sigma)$ (c) Attach $\partial\,\mathrm{st}_{\mathcal{K}}(\sigma) * 5$

Figure 2.3: Stellar subdivision of a square at $\sigma = (1, 4)$.

2.5 Polyhedral products

In this section we define the polyhedral product, the main object of study in this thesis. We give examples of the computation of the homotopy type of polyhedral products, before establishing the functoriality properties that allow us to study polyhedral products using homotopy-theoretic techniques.

A topological pair of spaces (X, A), hereon called a pair, is a space X and a subspace A together with an inclusion $A \longrightarrow X$. A map of pairs $f\colon (X, A) \longrightarrow (Y, B)$ consists of a map $f\colon X \longrightarrow Y$ such that $f(A) \subseteq B$.

Definition 2.5.1. Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let

$$(\underline{X}, \underline{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$$

be an m-tuple of pairs. The polyhedral product $(\underline{X}, \underline{A})^{\mathcal{K}}$ is the subspace of the Cartesian product $X_1 \times \cdots \times X_m$ defined as

$$(\underline{X}, \underline{A})^{\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}} \prod_{i=1}^{m} Y_i, \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases} \tag{2.13}$$

If $(X_i, A_i) = (X, A)$ for each $i = 1, \dots, m$ then the polyhedral product $(\underline{X}, \underline{A})^{\mathcal{K}}$ is denoted $(X, A)^{\mathcal{K}}$.

The polyhedral product in this general form appeared in [GT07] as a generalisation of functorial constructions of the moment-angle complex in [BP00] and the Davis–Januszkiewicz space [DJ91].

Example 2.5.2. 1. Let $(X_i, A_i) = (D^2, S^1)$ for $i = 1, \dots, m$. The polyhedral product $(D^2, S^1)^{\mathcal{K}}$ is the *moment-angle complex*, and is denoted $\mathcal{Z}_{\mathcal{K}}$. Moment-angle complexes have emerged as a key tool in linking combinatorics with many areas such as algebra, homotopy theory and symplectic geometry. As such there are many equivalent formulations and methods to study them. In this thesis we will be mainly concerned with the homotopy–theoretic features of moment-angle complexes, which are readily studied from its definition as a polyhedral product.

2. Let $(X_i, A_i) = (\mathbb{C}P^\infty, *)$ for $i = 1, \dots, m$. Then the polyhedral product $(\mathbb{C}P^\infty, *)^{\mathcal{K}}$ is the *Davis–Januszkiewicz space*, and is denoted $DJ_{\mathcal{K}}$. The Davis–Januszkiewicz space is an important intermediary in the study of moment-angle complexes since both appear in a single homotopy fibration sequence. In Section 2.6 we study the moment-angle complex and Davis–Januszkiewicz spaces in detail.

3. Let $(X_i, A_i) = (D^1, S^0)$ for $i = 1, \dots, m$. The polyhedral product $(D^1, S^0)^{\mathcal{K}}$ is the *real moment-angle complex*, and is denoted $\mathcal{R}_{\mathcal{K}}$. If we take D^1 to be the closed interval $[-1, 1] \subseteq \mathbb{R}$ and S^0 its boundary $\{-1, 1\}$, then $\mathcal{R}_{\mathcal{K}}$ is a cubical subcomplex of the cube $[-1, 1]^m \subseteq \mathbb{R}^m$. In Section 2.7 we study the real moment-angle complex and its links to geometric group theory in more detail.

Let $(\underline{Y}, \underline{B}) = \{(Y_1, B_1), \dots, (Y_m, B_m)\}$ be another m-tuple of pairs. A map of m-tuples of pairs $f\colon (\underline{X}, \underline{A}) \longrightarrow (\underline{Y}, \underline{B})$ is a collection of maps of pairs $f_i\colon (X_i, A_i) \longrightarrow (Y_i, B_i)$ for $i = 1, \dots, m$.

The polyhedral product is functorial with respect to both maps of m-tuples of pairs and inclusions of simplicial complexes. The following properties are immediate from the definition of polyhedral products.

Proposition 2.5.3. *Let $(\underline{X}, \underline{A})$ and $(\underline{Y}, \underline{B})$ be m-tuples of pairs and let $\mathcal{K}$ be a simplicial complex on $[m]$. Then*

(i) a map $f\colon (\underline{X}, \underline{A}) \longrightarrow (\underline{Y}, \underline{B})$ induces a map of polyhedral products $(\underline{X}, \underline{A})^{\mathcal{K}} \longrightarrow (\underline{Y}, \underline{B})^{\mathcal{K}}$;

(ii) if $\mathcal{L}$ is a simplicial complex such that $\mathcal{K} \subseteq \mathcal{L}$ then the inclusion $\mathcal{K} \longrightarrow \mathcal{L}$ induces a map of polyhedral products $(\underline{X}, \underline{A})^{\mathcal{K}} \longrightarrow (\underline{X}, \underline{A})^{\mathcal{L}}$. □

Slightly less immediate is the following. Let $\mathcal{K}_J$ be a full subcomplex of $\mathcal{K}$. Then the restriction $\prod_{i=1}^{m} X_i \longrightarrow \prod_{i\in J} X_i$ induces a map $(\underline{X}, \underline{A})^{\mathcal{K}} \longrightarrow (\underline{X}, \underline{A})^{\mathcal{K}_J}$. Furthermore, since $\prod_{i\in J} X_i$ is a retract of $\prod_{i=1}^{m} X_i$, we have the following very useful properties.

Proposition 2.5.4. *(i) Suppose that the pair (X_i, A_i) is a retract of (Y_i, B_i) for $i = 1, \dots, m$. Then the polyhedral product $(\underline{X}, \underline{A})^{\mathcal{K}}$ is a retract of $(\underline{Y}, \underline{B})^{\mathcal{K}}$.*

(ii) Suppose that $\mathcal{K}_J$ is a full subcomplex of $\mathcal{K}$. Then the polyhedral product $(\underline{X}, \underline{A})^{\mathcal{K}_J}$ is a retract of $(\underline{X}, \underline{A})^{\mathcal{K}}$. □

2.5.1 Homotopy type of polyhedral products

Let $\mathcal{K}$ be a simplicial complex on $[m]$. In general, the homotopy type of the polyhedral product $(\underline{X}, \underline{A})^{\mathcal{K}}$ is highly non-trivial. We summarise some common methods of computation and families of polyhedral products whose homotopy type is known.

Example 2.5.5. Suppose that $A_i = *$ for each $i = 1, \dots, m$. Then:

(i) if $\mathcal{K}$ consists of m disjoint points then $(\underline{X}, \underline{*})^{\mathcal{K}} = X_1 \vee \cdots \vee X_m$;

(ii) if $\mathcal{K} = \partial\Delta^{m-1}$ then $(\underline{X}, \underline{*})^{\mathcal{K}} = FW(X_1, \dots, X_m)$;

(iii) if $\mathcal{K} = \Delta^{m-1}$ then $(\underline{X}, \underline{*})^{\mathcal{K}} = X_1 \times \cdots \times X_m$.

In particular, the polyhedral product $(\underline{X}, \underline{*})^{\mathcal{K}}$ interpolates between the wedge $X_1 \vee \cdots \vee X_m$ and the product $X_1 \times \cdots \times X_m$.

Example 2.5.6. Suppose that $X_i = CA_i$ for each $i = 1, \dots, m$. If $\mathcal{K} = \Delta^{m-1}$, then $(\underline{CA}, \underline{A})^{\mathcal{K}} = CA_1 \times \cdots \times CA_m$. If $\mathcal{K} = \partial\Delta^{m-1}$ then

$$\begin{aligned}(\underline{CA}, \underline{A})^{\mathcal{K}} &= \bigcup_{i=1}^{m} CA_1 \times \cdots \times CA_{i-1} \times A_i \times CA_{i+1} \times \cdots \times CA_m \\ &= A_1 * \cdots * A_m.\end{aligned}$$

As a special case, when $\mathcal{K} = \partial\Delta^{m-1}$ we have $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}} = S^1 * \cdots * S^1 = S^{2m-1}$.

The above building blocks can be propagated to compute more complex polyhedral products via the following result.

Proposition 2.5.7 ([BP15, Proposition 4.2.5])**.** *Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes. Then $(\underline{X}, \underline{A})^{\mathcal{K}*\mathcal{L}} = (\underline{X}, \underline{A})^{\mathcal{K}} \times (\underline{X}, \underline{A})^{\mathcal{L}}$.* □

Example 2.5.8. Let $\mathcal{K}$ and $\mathcal{L}$ be the simplicial complexes from Example 2.4.9, so that $\mathcal{K} * \mathcal{L}$ is a square. Then

$$(\underline{X}, \underline{A})^{\mathcal{K}*\mathcal{L}} = (\underline{X}, \underline{A})^{\mathcal{K}} \times (\underline{X}, \underline{A})^{\mathcal{L}} = (A_1 * A_3) \times (A_2 * A_4).$$

In particular, if $\mathcal{K}$ is a square then $\mathcal{Z}_{\mathcal{K}} = S^3 \times S^3$.

Let $\mathcal{K}$ consist of $[m]$ disjoint points for $m \geqslant 3$. The homotopy type of the polyhedral product $(\underline{CA}, \underline{A})^{\mathcal{K}}$ is not easily computable directly from the definition. Instead, the combinatorial structure of $\mathcal{K}$ can be combined with homotopy-theoretic techniques to determine homotopy type. We will require the following.

Lemma 2.5.9 ([GT16, Lemma 6.11])**.** *Suppose that $A \longrightarrow C$ is a nullhomotopic map. Then for any space A the homotopy pushout of the diagram*

$$A \longleftarrow A \times B \longrightarrow C \times B$$

is $A * B \vee (C \rtimes B)$. □

Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $j \in \mathcal{K}$ be a vertex. Then the link-star decomposition (2.12) of $\mathcal{K}$ gives a pushout of simplicial complexes

$$\begin{tikzcd}
\mathrm{lk}_{\mathcal{K}}(j) \arrow[r] \arrow[d] & \mathrm{lk}_{\mathcal{K}}(j) * \{j\} \arrow[d] \\
\mathcal{K} - \{j\} \arrow[r] & \mathcal{K}
\end{tikzcd}$$

and therefore a homotopy pushout of polyhedral products

$$\begin{tikzcd}
(\underline{X}, \underline{A})^{\mathrm{lk}_{\mathcal{K}}(j)} \times A_j \arrow[r] \arrow[d] & (\underline{X}, \underline{A})^{\mathrm{lk}_{\mathcal{K}}(j)} \times X_j \arrow[d] \\
(\underline{X}, \underline{A})^{\mathcal{K}-\{j\}} \times A_j \arrow[r] & (\underline{X}, \underline{A})^{\mathcal{K}}.
\end{tikzcd}$$

Therefore if the map $(\underline{X}, \underline{A})^{\mathrm{lk}_{\mathcal{K}}(j)} \longrightarrow (\underline{X}, \underline{A})^{\mathcal{K}-\{j\}}$ is nullhomotopic and X_j is contractible then by Lemma 2.5.9 there is a homotopy equivalence

$$(\underline{X}, \underline{A})^{\mathcal{K}} \simeq (\underline{X}, \underline{A})^{\mathrm{lk}_{\mathcal{K}}(j)} * A_j \vee (\underline{X}, \underline{A})^{\mathcal{K}-\{j\}} \rtimes A_j). \tag{2.14}$$

Example 2.5.10. Let $\mathcal{K}$ consist of m disjoint points. Then applying (2.14) inductively one obtains that

$$\mathcal{Z}_{\mathcal{K}} \simeq \bigvee_{k=2}^{n} \left(S^{k+1}\right)^{\vee(k-1)\binom{n}{k}}$$

where $A^{\vee m}$ denotes the m-fold wedge of A with itself. For example, for $m = 3$, using parts (i), (iii) and (iv) of Proposition 2.1.4 we obtain that

$$\mathcal{Z}_{\mathcal{K}} \simeq (S^1 \times S^1) * S^1 \vee S^3 \rtimes S^1 \simeq (S^3 \vee S^3 \vee S^4) \vee (S^3 \vee S^4).$$

The study of those simplicial complexes for which $\mathcal{Z}_{\mathcal{K}}$ has the homotopy type of a wedge of spheres is very well established. By [BBCG10, Corollary 2.23] there is a homotopy equivalence

$$\Sigma\mathcal{Z}_{\mathcal{K}} \simeq \bigvee_{J \notin \mathcal{K}} \Sigma^{2+|J|}|\mathcal{K}_J|$$

so an equivalent question is to determine for which $\mathcal{K}$ the left-hand side is a wedge of spheres, and moreover the homotopy equivalence desuspends. One of the most general classes for which this is the case come from homology fillable complexes, see [IK19]. This is a homological generalisation of a fillable complex, a complex for which adding some subset of its minimal missing faces makes it contractible. Iriye and Kishimoto [IK19, Corollary 7.12] show that if every full subcomplex $\mathcal{K}_J$ of $\mathcal{K}$ for $J \neq \varnothing$ is homology fillable, then $\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to a wedge of spheres.

2.5.2 Homotopy theory of polyhedral products

Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $(\underline{X}, \underline{A})$ be an m-tuple of pairs. Recall the definition of the face category $\textsc{cat}(\mathcal{K})$ of $\mathcal{K}$ from Definition 2.4.7. The polyhedral product $(\underline{X}, \underline{A})^{\mathcal{K}}$ is a colimit of a diagram over $\textsc{cat}(\mathcal{K})$ given as follows. Define a diagram

$$\mathcal{D}_{\mathcal{K}} \colon \textsc{cat}(\mathcal{K}) \longrightarrow \textsc{top}$$

which sends $\sigma \longmapsto (\underline{X}, \underline{A})^{\sigma}$, where

$$(\underline{X}, \underline{A})^{\sigma} = \prod_{i=1}^{m} Y_i \qquad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma \end{cases}$$

and sends the morphism $\sigma \longrightarrow \tau$ to the inclusion of spaces $(\underline{X}, \underline{A})^{\sigma} \longrightarrow (\underline{X}, \underline{A})^{\tau}$. Then

$$(\underline{X}, \underline{A})^{\mathcal{K}} = \operatorname{colim} \mathcal{D}_{\mathcal{K}}(\underline{X}, \underline{A}) = \operatorname{colim}_{\sigma \in \mathcal{K}}(\underline{X}, \underline{A})^{\sigma}.$$

Observe that since $\varnothing \in \textsc{cat}(\mathcal{K})$ and $\varnothing \subseteq \sigma$ for any $\sigma \in \textsc{cat}(\mathcal{K})$, then $\varnothing$ is an initial object in $\textsc{cat}(\mathcal{K})$. Moreover, since $A_i \longrightarrow X_i$ is an inclusion for each $i = 1, \ldots, m$, the map

$$(\underline{X},\underline{A})^{\varnothing} \longrightarrow (\underline{X},\underline{A})^{\sigma}$$

is also an inclusion. Therefore

$$\mathrm{hocolim}_{\sigma\in\mathcal{K}}(\underline{X},\underline{A})^{\sigma} \simeq \mathrm{colim}_{\sigma\in\mathcal{K}}(\underline{X},\underline{A})^{\sigma}$$

by Proposition 2.1.1. In particular, we obtain that the polyhedral product $(\underline{X},\underline{A})^{\mathcal{K}}$ is unchanged, up to homotopy, by replacing a pair (X_i, A_i) with a homotopy equivalent pair (X_i', A_i') for some $i = 1,\dots,m$. We refer to this property as homotopy invariance of the polyhedral product.

Example 2.5.11. The infinite-dimensional complex sphere S^∞ is the colimit of the sequence of cellular inclusions $S^n \longrightarrow S^{n+1}$ as $n \longrightarrow \infty$. Since S^{n+1} is n-connected, the sphere S^∞ is ∞-connected, and therefore contractible. Moreover, there is a natural cellular inclusion $S^1 \longrightarrow S^\infty$. Therefore the pair (S^∞, S^1) is homotopy equivalent to (D^2, S^1). The form $\mathcal{Z}_{\mathcal{K}} = (S^\infty, S^1)^{\mathcal{K}}$ arises naturally when viewing $\mathcal{Z}_{\mathcal{K}}$ as having a torus action, since S^∞ admits a free action of S^1.

An important feature of polyhedral products is that they preserve fibrations. Precisely, let $(\underline{E},\underline{E}') \longrightarrow (\underline{B},\underline{B}')$ be a map of m-tuples of pairs such that $(E_i, E_i') \longrightarrow (B_i, B_i')$ is a fibration of pairs for $i = 1,\dots,m$, and let $(\underline{F},\underline{F}')$ be the m-tuple of pairs where for each $i = 1,\dots,m$ the fibre of $E_i \longrightarrow B_i$ is F_i and the fibre of $E_i' \longrightarrow B_i'$ is F_i'. Then for any simplicial complex $\mathcal{K}$ on $[m]$ there is a fibration

$$(\underline{F},\underline{F}')^{\mathcal{K}} \longrightarrow (\underline{E},\underline{E}')^{\mathcal{K}} \longrightarrow (\underline{B},\underline{B}')^{\mathcal{K}}$$

which was established by Denham and Siciu [DS07, Lemma 2.3.1]. In particular, for any spaces $X_1,\dots,X_m$ and simplicial complex $\mathcal{K}$ on $[m]$ there is a homotopy fibration

$$(\underline{C\Omega X},\underline{\Omega X})^{\mathcal{K}} \longrightarrow (\underline{X},\underline{*})^{\mathcal{K}} \xrightarrow{\ i\ } \textstyle\prod_{i=1}^m X_i \tag{2.15}$$

where $i\colon (\underline{X},\underline{*})^{\mathcal{K}} \longrightarrow \prod_{i=1}^m X_i$ is the coordinate-wise inclusion, obtained by considering the fibrations $(\underline{X},\underline{*}) \simeq (\underline{X\times PX},\underline{PX}) \longrightarrow (\underline{X},\underline{X})$ and using homotopy invariance of the polyhedral product. When $\mathcal{K}$ consists of two disjoint points, we recover the Theorem of Ganea recovering the homotopy fibre of the inclusion $X_1 \vee X_2 \longrightarrow X_1 \times X_2$, see Theorem 2.1.7.

The inclusion $\prod_{i=1}^m \Omega X_i \longrightarrow (\underline{C\Omega X},\underline{\Omega X})^{\mathcal{K}}$ is null-homotopic since it factors successively through the spaces $\Omega X_1 \times \cdots \times C\Omega X_i \times \cdots \times \Omega X_m$ for $i = 1,\dots,m$. Therefore the homotopy fibration sequence induced by (2.15) splits after looping, giving a homotopy equivalence

$$\Omega(\underline{X},\underline{*})^{\mathcal{K}} \simeq \Omega(\underline{C\Omega X},\underline{\Omega X})^{\mathcal{K}} \times \prod_{i=1}^m \Omega X_i. \tag{2.16}$$

In general, this splitting is not one of H spaces, as the following example shows.

Example 2.5.12. The moment-angle complex $\mathcal{Z}_\mathcal{K} = (D^2, S^1)^\mathcal{K}$ and the Davis–Januszkiewicz space $DJ_\mathcal{K} = (\mathbb{C}P^\infty, *)^\mathcal{K}$ are related using fibration (2.15) as follows. First, the homotopy fibre of the inclusion

$$DJ_\mathcal{K} \longrightarrow \prod_{i=1}^{m} \mathbb{C}P^\infty$$

is the polyhedral product $(C\Omega\mathbb{C}P^\infty, \Omega\mathbb{C}P^\infty)^\mathcal{K}$. Then since there is a homotopy equivalence $\Omega\mathbb{C}P^\infty \simeq S^1$ there is a homotopy equivalence of pairs

$$(C\Omega\mathbb{C}P^\infty, \Omega\mathbb{C}P^\infty) \simeq (CS^1, S^1) \simeq (D^2, S^1).$$

Therefore by homotopy invariance of the polyhedral product, there is a homotopy fibration

$$\mathcal{Z}_\mathcal{K} \longrightarrow DJ_\mathcal{K} \longrightarrow \textstyle\prod_{i=1}^{m} \mathbb{C}P^\infty \tag{2.17}$$

for any simplicial complex $\mathcal{K}$. Furthermore, after looping there is a homotopy equivalence

$$\Omega DJ_\mathcal{K} \simeq \Omega\mathcal{Z}_\mathcal{K} \times T^m \tag{2.18}$$

where T^m is the m-fold torus $\prod_{i=1}^{m} S^1$.

Now let $\mathcal{K}$ be the simplicial complex consisting of 2 disjoint points. Then splitting (2.18) gives

$$\Omega(\mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \simeq \Omega S^3 \times T^2.$$

The right-hand side is homotopy commutative since S^3 is an H-space, while the left-hand side is not homotopy commutative. Therefore in general splitting (2.16) is not a splitting of H-spaces. We return to this observation when computing the algebra structure in $H_*(\Omega\mathcal{Z}_\mathcal{K})$.

2.6 Moment-angle complexes

Moment-angle complexes are central objects in toric topology, a field which interfaces between topology, algebraic and symplectic geometry and combinatorics. At its base, toric topology is the study of spaces with a torus action, and a moment-angle complex $\mathcal{Z}_\mathcal{K}$ can be viewed as a space whose orbit under a torus action is $\mathcal{K}$. Moment-angle complexes in the form of polyhedral products first appeared in the work of Davis–Januszkiewicz [DJ91], who also defined the space $DJ_\mathcal{K}$ as the homotopy orbit space of $\mathcal{Z}_\mathcal{K}$ under the same torus action.

More recently, the moment-angle complex has been studied as a purely topological object with many interesting geometric and algebraic features. These features can moreover be decoded as combinatorial properties of the underlying simplicial complex $\mathcal{K}$, and this is the viewpoint we take in this thesis. Introducing the moment-angle complex as a polyhedral product gives us many tools to study these spaces topologically, such as functoriality and homotopy invariance. It also makes some key aspects of the homotopy theory of $\mathcal{Z}_\mathcal{K}$ more accessible. Firstly, the decomposition of $\mathcal{Z}_\mathcal{K}$ as a polyhedral product gives rise to a coordinate-wise cell structure controlled by the information in $\mathcal{K}$. In turn, the cohomology ring $H^*(\mathcal{Z}_\mathcal{K})$ decomposes as a direct sum of cohomology groups of full subcomplexes of $\mathcal{K}$, with the cup product being induced by simplicial maps.

Secondly, the link with the Davis–Januszkiewicz space remains even without the knowledge of torus actions. In particular, homotopy fibration (2.17) provides a homotopy-theoretic link to the Davis–Januszkiewicz space $DJ_\mathcal{K}$. As such, many homotopical constructions on $\mathcal{Z}_\mathcal{K}$ can be understood by first analysing equivalent properties in $DJ_\mathcal{K}$, which are often more accessible.

2.6.1 Cell structure

Viewing the moment-angle complex as a coordinate-wise subcomplex of the product $(D^2)^m$ gives a cellular structure for $\mathcal{Z}_\mathcal{K}$ as follows.

Construction 2.6.1 ([BP15, Section 4.4])**.** We equip the disc D^2 with a cellular structure consisting of one 0-cell $\bullet$, a 1-cell S with its boundary collapsed to $\bullet$, and a 2-cell D attached to S via a homeomorphism of its boundary.

Given subsets $I, J \subseteq [m]$ such that $I \cap J = \varnothing$, define the product cell $\chi(I, J)$ by

$$\chi(I,J) = E_1 \times \cdots \times E_m, \qquad \text{where } E_i = \begin{cases} D & \text{if } i \in I \\ S & \text{if } i \in J \\ \bullet & \text{otherwise.} \end{cases}$$

Then, viewing the moment-angle complex as a polyhedral product, we equip it with a cellular structure consisting of cells $\chi(I, J)$ such that $I \in \mathcal{K}$. The cell $\chi(I, J)$ has degree $2|I| + |J|$, and also has a naturally defined bidegree given by $(-|J|, 2|I| + 2|J|)$.

We obtain a cellular chain complex $C_*(\mathcal{Z}_\mathcal{K})$ generated by cells $\chi(I, J)$ for $I \in \mathcal{K}$ with differential d induced by $d(S) = d(\bullet) = 0$, $d(D) = S$ and the Leibniz rule

$$d(a \times b) = da \times b + (-1)^{\deg a} a \times db.$$

Example 2.6.2. Let $\mathcal{K}$ be the simplicial complex consisting of two disjoint points. Then $C_*(\mathcal{Z}_\mathcal{K})$ is generated by cells

$$\begin{array}{llll} \chi(\varnothing,\varnothing) = \bullet & \chi(\varnothing,\{1\}) = S_1 & \chi(\varnothing,\{2\}) = S_2 & \chi(\varnothing,\{1,2\}) = S_1S_2 \\ \chi(\{1\},\varnothing) = D_1 & \chi(\{2\},\varnothing) = D_2 & \chi(\{1\},\{2\}) = D_1S_2 & \chi(\{2\},\{1\}) = S_1D_2 \end{array}$$

where we drop the product symbol $\times$ when referring to product cells, and the subscript notation is used to keep track of the coordinates of the cells within $(D^2)^m$. The differential d is given by $d(D_i) = S_i$, $d(D_1S_2) = S_1S_2 = -d(S_1D_2)$ and is zero otherwise. Therefore $H_3(\mathcal{Z}_\mathcal{K}) = \mathbb{Z}$, generated by $D_1S_2 + S_1D_2$ and $H_n(\mathcal{Z}_\mathcal{K}) = 0$ for all other $n > 0$.

2.6.2 Cohomology Ring

The cellular structure of $\mathcal{Z}_\mathcal{K}$ also defines a cochain complex $C^*(\mathcal{Z}_\mathcal{K})$ generated by cochains $\chi(I, J)^*$ for $I \in \mathcal{K}$ dual to the cells $\chi(I, J)$. The complex $C^*(\mathcal{Z}_\mathcal{K})$ is given a bigrading by setting $\operatorname{bideg}\chi(I, J)^* = (-|J|, 2|I| + 2|J|)$. The differential is induced by $dS^* = D^*$, $dD^* = 0$ and the Leibniz rule. The cohomology groups $H^*(\mathcal{Z}_\mathcal{K})$ therefore have a natural bigrading arising from the bigrading in the CW-structure of $\mathcal{Z}_K$, that is

$$H^k(\mathcal{Z}_K) \cong \bigoplus_{-i+2j=k} H^{-i,2j}(\mathcal{Z}_K). \tag{2.19}$$

Algebraic models for $C^*(\mathcal{Z}_\mathcal{K})$ are obtained from the following setting.

The Eilenberg–Moore spectral sequence for the homotopy fibration (2.17) collapses on the second page giving an isomorphism of graded modules

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \operatorname{Tor}_{H^*((\mathbb{C}P^\infty)^m)}(H^*(DJ_\mathcal{K}), \mathbb{Z}) \tag{2.20}$$

where the Tor groups on the right-hand side are defined in Section 2.2.1. By the Künneth formula (2.2) we have $H^*((\mathbb{C}P^\infty)^m) = \mathbb{Z}[v_1, \ldots, v_m]$ with $\deg v_i = 2$, which we denote by $\mathbb{Z}[m]$. Moreover, we have the following. Recall that for a simplicial complex $\mathcal{K}$ on $[m]$, the face ring $\mathbb{Z}[\mathcal{K}]$ is the quotient of $\mathbb{Z}[m]$ by the ideal generated by monomials $v_{i_1} \cdots v_{i_k}$ such that $(i_1, \ldots, i_k) \in MF(\mathcal{K})$.

Proposition 2.6.3 ([BP15, Proposition 4.3.1])**.** *Let $\mathcal{K}$ be a simplicial complex on $[m]$. Then $H^*(DJ_\mathcal{K}) \cong \mathbb{Z}[\mathcal{K}]$.* □

Therefore the isomorphism (2.20) becomes

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \operatorname{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}). \tag{2.21}$$

There are two main projective resolutions associated to $\mathcal{Z}_\mathcal{K}$, one of $\mathbb{Z}$ as a $\mathbb{Z}[m]$-module, known as the Koszul resolution, and the second of $\mathbb{Z}[\mathcal{K}]$ as a $\mathbb{Z}[m]$-module, known as the Taylor resolution. These both define *dg*-algebras with homology $\operatorname{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$ known as the Koszul and Taylor complexes, respectively. In this thesis, we will only make use of the Taylor complex, which was developed in the Ph.D. thesis of Taylor [Tay66]. Details of the Koszul resolution are covered extensively in [BP15, Chapter 4].

The advantage of the Taylor complex is that resolving $\mathbb{Z}[\mathcal{K}]$ gives a closer connection to the combinatorial data in $\mathcal{K}$, since the resolution is defined in terms of minimal missing faces of $\mathcal{K}$. Therefore the Taylor complex is very useful for detecting topological data encoded in missing faces, for example Whitehead products, see [AP19] for a detailed analysis, including details of the following construction.

Construction 2.6.4 (Taylor Complex)**.** Let $\mathcal{M} = \{\sigma_J \mid J \in MF(\mathcal{K})\}$ and consider the exterior algebra $\Lambda(\mathcal{M})$. The Taylor complex is the *dg*-algebra with ith graded component $\Lambda^i(\mathcal{M})$, which is generated as $\mathbb{Z}$-module by finite formal products $\sigma_{J_1}\cdots\sigma_{J_i}$ for $\sigma_{J_1},\ldots,\sigma_{J_i} \in \mathcal{M}$. The differential is given by

$$d(\sigma_{J_1}\cdots\sigma_{J_i}) = \sum_k (-1)^{k-1}\sigma_{J_1}\cdots\sigma_{J_{k-1}}\sigma_{J_{k+1}}\cdots\sigma_{J_i}$$

where the summation is taken over all $1 \leqslant k \leqslant i$ such that $J_1 \cup \cdots \cup J_i = J_1 \cup \cdots \cup J_{k-1} \cup J_{k+1} \cup \cdots \cup J_i$ and the algebra product is given by

$$(\sigma_{J_1}\cdots\sigma_{J_i})\cdot(\sigma_{J'_1}\cdots\sigma_{J'_i}) = \sigma_{J_1}\cdots\sigma_{J_i}\sigma_{J'_1}\cdots\sigma_{J'_i}$$

if $(J_1\cup\cdots\cup J_i)\cap(J'_1\cup\cdots\cup J'_i) = \varnothing$ and is zero otherwise. The Taylor complex is enhanced to have the structure of a bigraded differential algebra by setting $\operatorname{bideg}(\sigma_{J_1}\cdots\sigma_{J_i}) = (-i, 2|J_1 \cup \cdots \cup J_i|)$.

Example 2.6.5. Let $\mathcal{K} = C_4$ be a square with $\mathcal{M} = MF(\mathcal{K}) = \{(1,3),(2,4)\}$. Then $\Lambda^1(\mathcal{M})$ is generated by σ_{13} and σ_{24}, each with bidegree $(-1,4)$ and total degree 3, while $\Lambda^2(\mathcal{M})$ is generated by $\sigma_{13}\sigma_{24} = \sigma_{13}\cdot\sigma_{24}$ with bidegree $(-2,8)$ and total degree 6. The differential is trivial. This agrees with the topological observation of Proposition 2.5.7 that $\mathcal{Z}_\mathcal{K} = S^3 \times S^3$.

2.6.3 Homology

There are dual versions of the Koszul and Taylor complexes which compute the homology $H_*(\mathcal{Z}_\mathcal{K})$ as bigraded groups. A full treatment is given in [AP19], and we give only the main construction of the homological Taylor complex.

The dual to the polynomial ring $\mathbb{Z}[v_1,\ldots,v_m]$ is the cocommutative coalgebra $\mathbb{Z}\langle v_1,\ldots,v_m\rangle$, denoted $\mathbb{Z}\langle m\rangle$, which is generated by all monomials v_σ corresponding to multisets

$$\sigma = \{\underbrace{1,\ldots,1}_{k_1},\underbrace{2,\ldots,2}_{k_2},\ldots,\underbrace{m,\ldots,m}_{k_m}\}.$$

The face coalgebra of a simplicial complex $\mathcal{K}$ is the subcoalgebra of $\mathbb{Z}\langle m\rangle$ generated by all monomials v_σ such that $\{i \in [m] \mid k_i \neq 0\}$ is a simplex of $\mathcal{K}$.

Dual to the cohomology case, there is an isomorphism of bigraded modules

$$H_{*,*}(\mathcal{Z}_{\mathcal{K}}) \cong \mathrm{Cotor}^{\mathbb{Z}\langle m\rangle}(\mathbb{Z}\langle\mathcal{K}\rangle, \mathbb{Z}).$$

The homological Taylor complex is a chain complex for $\mathrm{Cotor}^{\mathbb{Z}\langle m\rangle}(\mathbb{Z}\langle\mathcal{K}\rangle, \mathbb{Z})$ which is derived from an injective resolution of $\mathbb{Z}\langle\mathcal{K}\rangle$ as a $\mathbb{Z}\langle m\rangle$-comodule in a way dual to the Taylor resolution.

Construction 2.6.6 (Homological Taylor Complex)**.** For $s > 0$, the graded component I_{-s} is the free $\mathbb{Z}$-module with basis exterior monomials $w_{J_1} \wedge \cdots \wedge w_{J_s}$, where $J_1, \dots, J_s$ are distinct minimal missing faces of $\mathcal{K}$. The differential is given by

$$\partial(w_{J_1} \wedge \cdots \wedge w_{J_s}) = \sum_{J \subset J_1 \cup \cdots \cup J_s} w_J \wedge w_{J_1} \wedge \cdots \wedge w_{J_s}$$

where the summation is taken over missing faces $J \subset J_1 \cup \cdots \cup J_s$ different from any of the $J_1, \dots, J_s$. The Taylor complex is bigraded by setting $\mathrm{bideg}(\omega_{J_1} \wedge \cdots \wedge \omega_{J_s}) = (-s, 2|J_1 \cup \cdots \cup J_s|)$.

Example 2.6.7. Let $\mathcal{K} = C_4$ be a square. Then I_{-1} is generated by ω_{13} and ω_{24}, and I_{-2} is generated by $\omega_{13}\omega_{24} = -\omega_{24}\omega_{13}$. The differential is trivial, recovering the homology groups of $\mathcal{Z}_{\mathcal{K}} = S^3 \times S^3$.

In Section 3.3.2 we will give the homological Taylor complex the structure of a bigraded differential coalgebra by specifying a comultiplication dual to the multiplication in the Taylor complex described in Construction 2.6.4.

2.6.4 Hochster's Theorem

Hochster [Hoc77] showed the bigraded components $\mathrm{Tor}^{-i,2|J|}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$ decompose into reduced simplicial homology groups of full subcomplexes $\mathcal{K}_J$ of $\mathcal{K}$ as

$$\mathrm{Tor}^{-i,2j}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}) \cong \bigoplus_{|J|=j} \widetilde{H}^{j-i-1}(\mathcal{K}_J). \tag{2.22}$$

Hochster's result significantly predates the appearance of the Tor algebra in moment-angle complexes, and is itself an important tool transitioning between algebraic and combinatorial problems. Combining (2.22) with (2.19) and (2.21) we obtain isomorphisms

$$H^k(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{-i+2j=k} \bigoplus_{|J|=j} \widetilde{H}^{j-i-1}(\mathcal{K}_J) = \bigoplus_{J\subseteq[m]} \widetilde{H}^{i-|J|-1}(\mathcal{K}_J). \tag{2.23}$$

Baskakov [Bas02] showed that the right-hand side has a multiplicative structure given by canonical maps

$$H^{k-|I|-1}(\mathcal{K}_I) \otimes H^{l-|J|-1}(\mathcal{K}_J) \to H^{k+l-|I|-|J|-1}(\mathcal{K}_{I\cup J}) \tag{2.24}$$

which are induced by simplicial maps $K_{I\cup J} \to K_I * K_J$ if $I \cap J = \varnothing$ and are zero otherwise. This product structure agrees with the product structure on $\mathrm{Tor}^{-i,2|J|}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}],\mathbb{Z})$. Therefore the group isomorphisms (2.22) combine to give a ring isomorphism

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\mathcal{K}_J), \qquad H^k(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^{k-|J|-1}(\mathcal{K}_J). \tag{2.25}$$

Equivalent decompositions for $H_*(\mathcal{Z}_\mathcal{K})$ hold replacing cohomology for homology in the group isomorphisms (2.22) and (2.23).

Decompositions of the homology and cohomology groups of general polyhedral products $(\underline{X},\underline{A})^\mathcal{K}$ follow from the work of Bahri, Bendersky, Cohen and Gitler. For example, if each X_i is contractible, by [BBCG10, Theorem 2.21] there is a homotopy equivalence

$$\Sigma(\underline{X},\underline{A})^\mathcal{K} \simeq \Sigma\left(\bigvee_{J\notin\mathcal{K}} |\mathcal{K}_J| * \bigwedge_{i\in J} A_i\right). \tag{2.26}$$

When $(X_i, A_i) = (D^2, S^1)$ for $i = 1,\dots,m$, taking cohomology on each side of (2.26) recovers decomposition (2.23) since $\mathcal{K}_J$ is contractible whenever $J \in \mathcal{K}$.

Example 2.6.8. Let $\mathcal{K}$ be a square. Then the full subcomplexes $\mathcal{K}_{13}$ and $\mathcal{K}_{24}$ have $\widetilde{H}^0(\mathcal{K}_{13}) = \widetilde{H}^0(\mathcal{K}_{24}) = \mathbb{Z}$ while $\widetilde{H}^1(\mathcal{K}_{1234}) = \mathbb{Z}$. All other full subcomplexes are contractible. Therefore we have

$$H^3(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^0(\mathcal{K}_J) = \widetilde{H}^0(\mathcal{K}_{13}) \oplus \widetilde{H}^0(\mathcal{K}_{24}) = \mathbb{Z}^2$$
$$H^6(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^1(\mathcal{K}_J) = \widetilde{H}^1(\mathcal{K}_{1234}) = \mathbb{Z}$$

and all other cohomology groups are trivial. Moreover, the simplicial map $\mathcal{K}_{1234} \longrightarrow \mathcal{K}_{13} * \mathcal{K}_{24}$ is the identity, giving a non-trivial map

$$\widetilde{H}^0(\mathcal{K}_{13}) \otimes \widetilde{H}^0(\mathcal{K}_{24}) \longrightarrow \widetilde{H}^1(\mathcal{K}_{1234})$$

and therefore the non-trivial cup product $H^3(\mathcal{Z}_\mathcal{K}) \otimes H^3(\mathcal{Z}_\mathcal{K}) \longrightarrow H^6(\mathcal{Z}_\mathcal{K})$.

2.6.5 Loop homology of Davis–Januszkiewicz spaces

Let $\mathcal{K}$ be a simplicial complex on $[m]$ and for $i = 1,\dots,m$ let μ_i be the inclusion

$$\mu_i\colon S^2 \longrightarrow \mathbb{C}P^\infty \longrightarrow DJ_\mathcal{K}$$

where the second map is induced by the inclusion $\{i\} \longrightarrow \mathcal{K}$. We consider the Whitehead product $[\mu_i,\mu_j] \in \pi_3(DJ_\mathcal{K})$. By (2.8), if θ is the composite of the adjoint map

$\pi_n(DJ_{\mathcal{K}}) \longrightarrow \pi_{n-1}(\Omega DJ_{\mathcal{K}})$ with the Hurewicz map $\pi_{n-1}(\Omega DJ_{\mathcal{K}}) \longrightarrow H_{n-1}(\Omega DJ_{\mathcal{K}})$ then

$$\theta([\mu_i, \mu_j]) = -(u_i u_j + u_j u_i) = -[u_i, u_j]$$

where $u_k = \theta(\mu_k)$ and $[x, y] = xy - (-1)^{\deg x \deg y} yx$ is the algebraic commutator (2.4). The following result therefore identifies some relations in $H_*(\Omega DJ_{\mathcal{K}})$.

Lemma 2.6.9. *For $i \neq j$, the Whitehead product $[\mu_i, \mu_j] \in \pi_3(DJ_{\mathcal{K}})$ is trivial if and only if $(i, j) \in \mathcal{K}$.*

Proof. By Proposition 2.1.6, the product $S^2 \times S^2$ is the homotopy cofibre of the map $S^3 \longrightarrow S^2 \vee S^2$. Therefore $[\mu_i, \mu_j]$ is trivial if and only if the map $\mu_i \vee \mu_j \colon S^2 \vee S^2 \longrightarrow DJ_{\mathcal{K}}$ extends to a map $S^2 \times S^2 \longrightarrow DJ_{\mathcal{K}}$. If $(i, j) \in \mathcal{K}$, then $DJ_{\mathcal{K}}$ contains the subspace $\mathbb{C}P_i^\infty \times \mathbb{C}P_j^\infty$ and such an extension is given by $\mu_i \times \mu_j$. Conversely, any extension of $\mu_i \vee \mu_j$ to $S^2 \times S^2$ is given by $\mu_i \times \mu_j$, which implies that $\mathbb{C}P_i^\infty \times \mathbb{C}P_j^\infty$ is a coordinate subspace of $DJ_{\mathcal{K}}$, and hence that $(i, j) \in \mathcal{K}$. □

Proposition 2.6.10 ([BP15, Corollary 8.4.3])**.** *If $\underline{k}$ is a field with characteristic different from 2, the loop homology algebra $H_*(\Omega DJ_{\mathcal{K}}; \underline{k})$ contains the subalgebra*

$$\frac{T(u_1, \ldots, u_m)}{\langle u_i^2, [u_i, u_j] \text{ if } (i, j) \in \mathcal{K}\rangle}. \tag{2.27}$$

Proof. It remains to show that $u_i^2 = 0$ for each $i = 1, \ldots, m$. The Whitehead square $[\mu_i, \mu_i] \in \pi_3(DJ_{\mathcal{K}})$ is trivial since it factors through a map $S^3 \longrightarrow \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$. Therefore $2u_i^2 = 0$ in $H_*(\Omega DJ_{\mathcal{K}}; \underline{k})$ and the result follows. □

The result of Proposition 2.6.10 remains true when $\underline{k} = \mathbb{Z}$. The difficulty is finding an alternative way to show that $u_i^2 = 0$, rather than a 2-torsion element.

We use the Cobar construction to create an algebraic model for $H_*(\Omega DJ_{\mathcal{K}})$. Recall from Section 2.6.3 that the face coalgebra $\mathbb{Z}\langle\mathcal{K}\rangle$ of a simplicial complex $\mathcal{K}$ is the subcoalgebra of $\mathbb{Z}\langle m\rangle$ generated by all monomials v_σ corresponding to multisets

$$\sigma = \{\underbrace{1, \ldots, 1}_{k_1}, \underbrace{2, \ldots, 2}_{k_2}, \ldots, \underbrace{m, \ldots, m}_{k_m}\}.$$

such that $\{i \in [m] \mid k_i \neq 0\}$ is a simplex of $\mathcal{K}$.

Dual to Proposition 2.6.3, the face coalgebra $\mathbb{Z}\langle\mathcal{K}\rangle$ is a model for the cellular chain complex of $DJ_{\mathcal{K}}$. Since $\mathbb{Z}\langle\mathcal{K}\rangle$ is a simply-connected dg-coalgebra in which the differential is zero, we apply the Adams–Hilton model, Theorem 2.3.3, to obtain an isomorphism of algebras

$$H_*(\Omega DJ_{\mathcal{K}}) \cong H(AH_*(\mathbb{Z}\langle\mathcal{K}\rangle))$$

where $AH_*(\mathbb{Z}\langle\mathcal{K}\rangle)$ is the free associative algebra $T(s^{-1}\overline{(\mathbb{Z}\langle\mathcal{K}\rangle})$ on the desuspended module $\overline{(\mathbb{Z}\langle\mathcal{K}\rangle)} = \mathbb{Z}\langle\mathcal{K}\rangle/((\mathbb{Z}\langle\mathcal{K}\rangle)_0$ and the differential is given by

$$d(s^{-1}v_\sigma) = \sum_{\sigma=\tau\sqcup\tau'} s^{-1}v_\tau \otimes s^{-1}v_{\tau'}$$

where the sum is taken over all proper partitions of σ into submultisets τ and τ' with $\tau, \tau' \neq \varnothing$. For an element $v_\sigma \in \mathbb{Z}\langle\mathcal{K}\rangle$ we denote by $\chi_\sigma = s^{-1}v_\sigma$.

Example 2.6.11. As a first example, we establish Proposition 2.6.10 for integral coefficients. Let u_i denote the homology class of χ_i. Then $d\chi_i = 0$ and $d\chi_{ii} = \chi_i^2$, which establishes the relation $u_i^2 = 1$.

Example 2.6.12. Let $\mathcal{K} = \partial\Delta^{m-1}$. Let u_i be the homology class of χ_i for $i = 1, \ldots, m$. If $m \geqslant 3$ then since $d\chi_{ii} = \chi_i^2$ and $d\chi_{ij} = \chi_i\chi_j + \chi_j\chi_i$, we deduce that $u_1, \ldots, u_m$ generate a subalgebra of $H_*(\Omega DJ_\mathcal{K})$ subject to the exterior relations $u_i^2 = [u_i, u_j] = 1$. For $m = 2$, the element χ_{12} is not in $AH_*(\mathbb{Z}\langle\mathcal{K}\rangle)$. In this case we obtain that $H_*(\Omega DJ_\mathcal{K}) \cong T(u_1, u_2)/\langle u_1^2, u_2^2\rangle$.

Returning to the case $m \geqslant 3$, consider the element $\chi_{12\cdots m} \in AH_*(\mathbb{Z}\langle\Delta^{m-1}\rangle)$. Then $\psi = d(\chi_{12\cdots m}) \in AH_*(\mathbb{Z}\langle\mathcal{K}\rangle)$ is a cycle since $d\psi = d^2(\chi_{12\cdots m}) = 0$ which is not a boundary since $(1, \ldots, m)$ is not a simplex of $\mathcal{K}$. Therefore w, the homology class of ψ also generates a class of $H_*(\Omega DJ_\mathcal{K})$. Moreover, w commutes with each u_i. To see this, in $AH_*(\mathbb{Z}\langle\Delta^{m-1}\rangle)$ consider

$$0 = d^2(\chi_{1\cdots ii\cdots m}) = d(\chi_i\chi_{1\cdots m} + \chi_{1\cdots m}\chi_i + \beta) = -\chi_i\psi + \psi\chi_i + d\beta$$

where $\beta \in AH_*(\mathbb{Z}\langle\mathcal{K}\rangle)$. Therefore we obtain that $[\chi_i, \psi] = d\beta$, that is, $[u_i, w] = 0$ in homology for each $i = 1, \ldots, m$. Therefore there is an algebra isomorphism

$$H_*(\Omega DJ_\mathcal{K}) \cong \mathbb{Z}[w] \otimes \Lambda[u_1, \ldots, u_m].$$

The element w can be viewed as a 'higher commutator' of $u_1, \ldots, u_m$. Concretely, w is the image under $\theta\colon \pi_*(DJ_\mathcal{K}) \longrightarrow H_{*-1}(\Omega DJ_\mathcal{K})$ of a certain element of a higher Whitehead product, studied by Abramyan–Panov [AP19]. Further example computations of $H_*(\Omega\mathcal{Z}_\mathcal{K})$ are studied in [BP15, Example 8.4.15]. In particular, for the complex $\mathcal{K} = \operatorname{sk}^1 \Delta^3$ there is a relation

$$[u_1, w_{234}] + [u_2, w_{134}] + [u_3, w_{124}] + [u_4, w_{123}] = 0$$

between higher commutators w_{ijk} coming from the previous example via the inclusions $\partial\Delta[i, j, k] \longrightarrow \mathcal{K}$. Such a relation is a higher version of the algebraic Jacobi identity. In Chapter 4 we study a generalisation of the maps of [AP19] and prove that relations such as above are true on a geometric level, and not just on passing to loop homology. This significantly expands the work of Hardie [Har61].

2.6.6 Loop homology of moment-angle complexes

Recall the homotopy fibration (2.17)

$$\mathcal{Z}_{\mathcal{K}} \longrightarrow DJ_{\mathcal{K}} \longrightarrow \prod_{i=1}^{m} \mathbb{C}P^{\infty}$$

which splits after looping to give a homotopy equivalence $\Omega DJ_{\mathcal{K}} \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times T^m$. There is therefore an isomorphism of $\mathbb{Z}$-modules

$$H_*(\Omega DJ_{\mathcal{K}}) \cong H_*(\Omega\mathcal{Z}_{\mathcal{K}}) \otimes \Lambda[u_1, \ldots, u_m]$$

which is not an isomorphism of algebras since the splitting $\Omega DJ_{\mathcal{K}} \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times T^m$ is not an H-splitting, see Example 2.5.12. We see this explicitly when $\mathcal{K}$ consists of two disjoint points. In this case, in Example 2.6.12, we computed that $H_*(\Omega DJ_{\mathcal{K}}) = T(u_1, u_2)/\langle u_1^2, u_2^2\rangle$, while since $\mathcal{Z}_{\mathcal{K}} \simeq S^3$, we have $H_*(\Omega\mathcal{Z}_{\mathcal{K}})\otimes\Lambda[u_1, u_2] = \mathbb{Z}[w]\otimes\Lambda(u_1, u_2)$, where $\deg w = 2$. Despite this, there is always a short exact sequence of algebras

$$1 \longrightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}) \longrightarrow H_*(\Omega DJ_{\mathcal{K}}) \longrightarrow \Lambda[u_1, \ldots, u_m] \longrightarrow 1.$$

Recall that a simplicial complex is flag if its minimal missing faces are 1-dimensional. In the case that $\mathcal{K}$ is flag, by [BP15, Theorem 8.5.2] the algebra (2.27) is the whole loop homology algebra $H_*(\Omega DJ_{\mathcal{K}})$. Therefore the map $H_*(\Omega DJ_{\mathcal{K}}) \longrightarrow \Lambda[u_1, \ldots, u_m]$ can be viewed as the algebraic abelianisation map, and so the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ can be viewed as the commutator subalgebra of $H_*(\Omega DJ_{\mathcal{K}})$.

It follows that $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ has a generating set consisting of commutators and iterated commutators of the elements u_i. Concretely, the following result was established by Grbić, Panov, Theriault and Wu [GPTW12].

Theorem 2.6.13 ([GPTW12, Theorem 4.2])**.** *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$. Then $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ has a minimal generating set of iterated commutators given by*

$$[u_j, u_i], \qquad [u_{k_1}, [u_j, u_i]], \qquad \cdots \qquad [u_{k_1}, [u_{k_2}, \ldots, [u_{k_{m-2}}, [u_j, u_i]] \cdots]] \tag{2.28}$$

where $k_1 < k_2 < \cdots < k_{m-2} < j > i$ are distinct and i is the smallest vertex in a connected component of $\mathcal{K}_{\{k_1,k_2,\ldots,k_{m-2},j,i\}}$ not containing j.

Example 2.6.14. Let $\mathcal{K}$ have maximal simplices $\{\{1,2\},\{3\}\}$. Then the full subcomplexes $\mathcal{K}_{\{1,3\}}$ and $\mathcal{K}_{\{2,3\}}$ consist of two disjoint points, and therefore $[u_3, u_1]$ and $[u_3, u_2]$ are multiplicative generators of $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$. On the other hand the full subcomplex $\mathcal{K}_{\{1,2\}}$ is a 1-simplex. Therefore the vertices 1 and 2 are in the same connected component of $\mathcal{K}_{\{1,2\}}$, and so $[u_2, u_1]$ is not a generator. This can be seen since $[u_2, u_1] = 0$ in $H_*(\Omega DJ_{\mathcal{K}})$ by Proposition 2.6.10. Finally, considering $\mathcal{K} = \mathcal{K}_{\{1,2,3\}}$, we obtain a generator $[[u_2, [u_3, u_1]]$.

In this case it can be shown that there are no relations between the generators $[u_3, u_1]$, $[u_3, u_2]$ and $[u_2, [u_3, u_1]]$, that is, $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is the free associative algebra on this generating set. This agrees with the observation that $\mathcal{Z}_\mathcal{K} \simeq S^3 \vee (S^3 \rtimes S^1) \simeq S^3 \vee S^3 \vee S^4$ obtained from formula (2.14).

The observation that $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is free when $\mathcal{Z}_\mathcal{K}$ has the homotopy type of a wedge of spheres turns out to be an equivalence in the case that $\mathcal{K}$ is flag. Moreover, these properties are related to the following cohomological condition.

Definition 2.6.15. A simplicial complex $\mathcal{K}$ is Golod over a ring $\underline{k}$ if the multiplication and all higher Massey products are trivial in $H^*(\mathcal{Z}_\mathcal{K}; \underline{k})$.

The following was proved in [GPTW12] in the case that coefficients are taken in a field.

Theorem 2.6.16 ([GPTW12, Theorem 4.6])**.** *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$ and let $\underline{k}$ be a field. Then the following are equivalent:*

(i) the simplicial complex $\mathcal{K}$ is Golod;

(ii) the multiplication in $H^*(\mathcal{Z}_\mathcal{K})$ is trivial;

(iii) the 1-skeleton of $\mathcal{K}$ is a chordal graph, that is, every set of 4 or more vertices which form a cycle has a chord, that is, an edge between two non-adjacent vertices in a cycle;

(iv) $\mathcal{Z}_\mathcal{K}$ has the homotopy type of a wedge of spheres;

(v) $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a free associative algebra. □

In Theorem 3.2.13 we will establish this result when $\underline{k} = \mathbb{Z}$.

2.7 Real moment-angle complexes and right-angled Coxeter groups

Right-angled Coxeter groups are classical objects in geometric group theory coming from the study of reflectional symmetries across orthogonal planes. In this section we study these groups topologically. In particular, we show that the right-angled Coxeter group associated to a simplicial complex $\mathcal{K}$ is the fundamental group of the polyhedral product $(\mathbb{R}P^\infty, *)^\mathcal{K}$, a real analogue of the Davis–Januszkiewicz space.

Of particular interest are the commutator subgroups of right-angled Coxeter groups. Aside from purely algebraic interest, our topological approach reveals that such commutator subgroups are the fundamental groups of the real moment-angle complex. This correspondence between geometric group theory and homotopy theory goes both ways, and we present equivalent statements about algebraic properties of the right-angled Coxeter group, and homotopy-theoretic properties of the real moment-angle complex. We extend these results during the course of this thesis.

While our main focus is on the right-angled Coxeter group, we present the more general class of groups called graph product groups, of which the equally important right-angled Artin groups are also an example. We build up the theory of the classifying spaces of graph products, before relating them and their commutator subgroups to polyhedral products. We finally specialise to the real moment-angle complex and results on the commutator subgroup of the right-angled Coxeter group.

2.7.1 Group actions and classifying spaces

Given a group G, there exists a contractible space EG on which G has a free action. Define BG to be the orbit space EG/G. Then there is a fibration

$$G \longrightarrow EG \longrightarrow BG. \tag{2.29}$$

Both EG and BG are determined uniquely up to homotopy by G. We call EG the *total space* of G, and BG the *classifying space* of G.

Example 2.7.1. (i) Let $G = \mathbb{Z}$. Then there is a free action of $\mathbb{Z}$ on $\mathbb{R}$ given by translation, that is, $(n, x) \longmapsto x + n$. This is a free action whose orbit space is homotopy equivalent to S^1. Therefore $E\mathbb{Z} = \mathbb{R}$ and $B\mathbb{Z} = S^1$.

(ii) Let $G = \mathbb{Z}_2$. Then $\mathbb{Z}_2$ acts freely on the sphere S^k for every $k \geqslant 0$ by the antipodal action. In the limit, there is a free action on the contractible space S^∞, the infinite sphere. The orbit space is the infinite real projective space $\mathbb{R}P^\infty$.

Looping fibration (2.29) gives a homotopy equivalence $\Omega BG \simeq G$, since EG is contractible. Therefore there is an isomorphism of homotopy groups

$$\pi_n(G) \cong \pi_n(\Omega BG) \cong \pi_{n+1}(BG).$$

In particular if G is discrete, then $\pi_0(G)$ consists of the path components of G, and is therefore homotopy equivalent to G, while $\pi_n(G) = 0$ for $n > 0$. Therefore the space BG encodes all of the information of G within its fundamental group. Equivalently, BG is the Eilenberg–MacLane space $K(G, 1)$.

2.7.2 Graph products of groups

Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $\underline{G} = (G_1, \ldots, G_m)$ be an m-tuple of groups. The *graph product* $\underline{G}^{\mathcal{K}}$ of $G_1, \ldots, G_m$ over $\mathcal{K}$ is the quotient of the free product

$G_1 * \cdots * G_k$ by the relations $(g_i, g_j) = 1$ for all $g_i \in G_i$ and $g_j \in G_j$ whenever $(i,j) \in \mathcal{K}$, where $(a,b) = aba^{-1}b^{-1}$.

The graph product $\underline{G}^{\mathcal{K}}$ is fully determined by the 1-skeleton $\mathcal{K}^1$ of $\mathcal{K}$, a graph. In what follows, however, it simplifies matters to refer to a graph product associated to a simplicial complex, rather than a graph.

The graph product interpolates between the free product $G_1 * \cdots * G_m$ when $\mathcal{K}$ has no edges, and the Cartesian product $G_1 \times \cdots \times G_m$ when $\mathcal{K}$ has a full 1-skeleton. Graph products play a key role in combinatorial group theory since they construct a wide variety of groups whose group–theoretic properties are encoded by the underlying graph or simplicial complex. Of particular classical importance are the following.

Example 2.7.2. (i) Suppose that $G_i = \mathbb{Z}$ for $i = 1, \ldots, m$. Then the corresponding graph product $\underline{G}^{\mathcal{K}}$ is called the *right-angled Artin group* corresponding to $\mathcal{K}$ and is denoted $RA_{\mathcal{K}}$. Right-angled Artin groups have attracted much recent interest in geometric group theory due to their actions on CAT(0) cube complexes.

(ii) Suppose instead that $G_i = \mathbb{Z}_2$ for $i = 1, \ldots, m$. Then the graph product $\underline{G}^{\mathcal{K}}$ is called the *right-angled Coxeter group* corresponding to $\mathcal{K}$ and is denoted $RC_{\mathcal{K}}$.

Suppose that each $G_1, \ldots, G_m$ is a discrete group, so that $\pi_1(BG_i) = G_i$ for each $i = 1, \ldots, m$. Let $\mathcal{K}$ be the simplicial complex consisting of $[m]$ disjoint points. Then by definition $\underline{G}^{\mathcal{K}}$ is the free product $G_1 * \cdots * G_m$. On the other hand, $G_1 * \cdots * G_m$ is the fundamental group of the wedge $BG_1 \vee \cdots \vee BG_m$, which is the polyhedral product $(\underline{BG}, \underline{*})^{\mathcal{K}}$.

Now form a new complex $\mathcal{K}'$ from $\mathcal{K}$ by adding the edge (i,j). On the one hand, by definition of the polyhedral product

$$(\underline{BG}, \underline{*})^{\mathcal{K}'} = (\underline{BG}, \underline{*})^{\mathcal{K}} \cup (BG_i \times BG_j)$$

where the union is taken over $BG_i \vee BG_j$. On the other hand, by van Kampen's Theorem $\pi_1\left((\underline{BG}, \underline{*})^{\mathcal{K}'}\right)$ is obtained from $\pi_1\left((\underline{BG}, \underline{*})^{\mathcal{K}}\right)$ by adding relations $g_i g_j = g_j g_i$ for all $g_i \in G_i$ and $g_j \in G_j$. Continuing inductively obtains the following.

Proposition 2.7.3 ([PV16, Theorem 3.2])**.** *Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $G_1, \ldots, G_m$ be discrete groups. Then $\pi_1\left((\underline{BG}, \underline{*})^{\mathcal{K}}\right) = \underline{G}^{\mathcal{K}}$.* □

We therefore obtain our first result linking the group-theoretical object $\underline{G}^{\mathcal{K}}$ with the purely topological object $(\underline{BG}, \underline{*})^{\mathcal{K}}$, with the link established via the combinatorial information in $\mathcal{K}$.

Example 2.7.4. (i) The right-angled Artin group $RA_{\mathcal{K}}$ is the fundamental group of the polyhedral product $(S^1, *)^{\mathcal{K}}$, which is known as the Salvetti complex.

(ii) The right-angled Coxeter group $RC_{\mathcal{K}}$ is the fundamental group of the polyhedral product $(\mathbb{R}P^{\infty}, *)^{\mathcal{K}}$.

2.7.3 Commutator subgroups and polyhedral products

Commutator subgroups of right-angled Coxeter groups are of important interest in geometric group theory, and from a topological perspective in low-dimensional topology where examples include surface groups and 3-manifold groups. We begin with some examples of these subgroups.

Example 2.7.5. (i) Let $\mathcal{K}$ have maximal simplices $\{1, 2\}$ and $\{3\}$. Then the right-angled Coxeter group $RC_{\mathcal{K}}$ is given by

$$RC_{\mathcal{K}} = \frac{\langle g_1, g_2, g_3 \rangle}{\langle g_1^2, g_2^2, g_3^2, (g_1, g_2) \rangle}.$$

Therefore the commutator subgroup $RC'_{\mathcal{K}}$ is generated by the commutators (g_1, g_3), (g_2, g_3) and $((g_1, g_3), g_2)$, and furthermore there are no relations between these generators.

(ii) Let $\mathcal{K} = C_4$. Then

$$RC_{\mathcal{K}} = \frac{\langle g_1, g_2, g_3, g_4 \rangle}{\langle g_1^2, g_2^2, g_3^2, g_4^2, (g_1, g_2), (g_2, g_3), (g_3, g_4), (g_4, g_1) \rangle}.$$

Therefore the commutator subgroup $RC'_{\mathcal{K}}$ is generated by the two commutators (g_1, g_3) and (g_2, g_4), and moreover a direct calculation using the relations in $RC'_{\mathcal{K}}$ shows that $((g_1, g_3), (g_2, g_4)) = 1$. Therefore $RC'_{\mathcal{K}}$ is isomorphic to a free abelian group on two generators.

The above examples demonstrate that even for small simplicial complexes, the computation of $RC'_{\mathcal{K}}$ is complex, requiring both the calculation of a generating set and any relations between those generators. Polyhedral products provide a unifying framework for the study of these commutator subgroups.

Using fibration (2.29), the homotopy fibration sequence (2.15) for $X_i = BG_i$ gives a homotopy fibration

$$(\underline{EG}, \underline{G})^{\mathcal{K}} \longrightarrow (\underline{BG}, \underline{*})^{\mathcal{K}} \longrightarrow \textstyle\prod_{i=1}^{m} BG_i \tag{2.30}$$

since EG is a contractible space containing G. This fibration was studied by Panov and Veryovkin [PV16] who obtained the following. Recall that a simplicial complex $\mathcal{K}$ is flag if $\mathcal{K}$ has no minimal missing n-faces for $n \geqslant 2$.

Theorem 2.7.6 ([PV16, Theorem 3.2])**.** *Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $G_1, \ldots, G_m$ be discrete groups. Then* *(i) both* $(\underline{EG}, \underline{G})^{\mathcal{K}}$ *and* $(\underline{BG}, \underline{*})^{\mathcal{K}}$ *are aspherical if and only if* $\mathcal{K}$ *is flag;*

(ii) there are isomorphisms $\pi_i\left((\underline{EG}, \underline{G})^{\mathcal{K}}\right) \cong \pi_i\left((\underline{BG}, \underline{*})^{\mathcal{K}}\right)$ *for* $i \geqslant 2$*;*

(iii) the fundamental group $\pi_1\left((\underline{EG}, \underline{G})^{\mathcal{K}}\right)$ *is isomorphic to the kernel of the projection* $\underline{G}^{\mathcal{K}} \longrightarrow \prod_{i=1}^m G_i$*.* □

Combining (i) with Proposition 2.7.3 we see that $B\left(\underline{G}^{\mathcal{K}}\right) = (\underline{BG}, \underline{*})^{\mathcal{K}}$. While the proof of (i) is technically involved, statements (ii) and (iii) follow quickly from (i) and the long exact sequence in homotopy of the homotopy fibration (2.30).

When each G_i is abelian for $i = 1, \ldots, m$, the projection $\underline{G}^{\mathcal{K}} \longrightarrow \prod_{i=1}^m G_i$ coincides with the abelianisation map. Therefore in this case $\pi_1\left((\underline{EG}, \underline{G})^{\mathcal{K}}\right)$ is the commutator subgroup $\left(\underline{G}^{\mathcal{K}}\right)'$ of $\underline{G}^{\mathcal{K}}$.

Example 2.7.7. (i) Suppose that $G_i = \mathbb{Z}$ for $i = 1, \ldots, m$. Then since $\mathbb{Z}$ is discrete and abelian, combining Theorem 2.7.6(i) with Examples 2.7.1(i) and 2.7.2(i) gives

$$\pi_1\left((\mathbb{R}, \mathbb{Z})^{\mathcal{K}}\right) \cong RA'_{\mathcal{K}}$$

and that $(\mathbb{R}, \mathbb{Z})^{\mathcal{K}}$ is a classifying space for $RA'_{\mathcal{K}}$ if and only if $\mathcal{K}$ is flag.

(ii) Similarly if $G_i = \mathbb{Z}_2$ for $i = 1, \ldots, m$ and using the homotopy equivalence of pairs $(C\mathbb{Z}_2, \mathbb{Z}_2) \simeq (D^1, S^0)$ we obtain that

$$\pi_1\left((D^1, S^0)^{\mathcal{K}}\right) \cong RC'_{\mathcal{K}}$$

and that $(D^1, S^0)^{\mathcal{K}}$ is a classifying space for $RC'_{\mathcal{K}}$ if and only if $\mathcal{K}$ is flag.

Before focusing on the commutator subgroup $RC'_{\mathcal{K}}$, we end our discussion of commutator subgroups of graph products with the following result, which demonstrates the powerful role that the polyhedral product plays in unifying group theory, homotopy theory and combinatorics. We recall the definition of a chordal graph from Theorem 2.6.16.

Theorem 2.7.8 ([PV16, Theorem 4.3])**.** *Let* $\mathcal{K}$ *be a flag simplicial complex on* $[m]$ *and let* $G_1, \ldots, G_m$ *be discrete groups. Then the following are equivalent:*

(i) the kernel of the projection

$$\underline{G}^{\mathcal{K}} \longrightarrow \prod_{i=1}^m G_i$$

is a free group;

(ii) the polyhedral product $(\underline{EG}, \underline{G})^{\mathcal{K}}$ *is homotopy equivalent to a wedge of circles;*

(iii) the 1*-skeleton of* $\mathcal{K}$ *is a chordal graph.* □

We state the specific case of Theorems 2.7.6 and 2.7.8 when $G_i = \mathbb{Z}_2$ for $i = 1, \ldots, m$.

Theorem 2.7.9. *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$. Then the group $\pi_1(\mathcal{R}_\mathcal{K}) = RC'_\mathcal{K}$ is free if and only if the 1-skeleton of $\mathcal{K}$ is a chordal graph.* □

We compare this to the statement of Theorem 2.6.16. In particular we see that in the flag case, the polyhedral product $\mathcal{R}_\mathcal{K}$ being a wedge of circles and the moment-angle complex $\mathcal{Z}_\mathcal{K}$ being a wedge of spheres, and moreover both $RC'_\mathcal{K} = \pi_1(\mathcal{R}_\mathcal{K})$ and $H_*(\Omega\mathcal{Z}_\mathcal{K})$ being free, are all classified in the same, completely combinatorial way, namely by $\mathcal{K}^1$ being chordal. One of the main results of this thesis, Theorem 3.1.1, studies the related case where $RC'_\mathcal{K} = \pi_1(\mathcal{R}_\mathcal{K})$ and $H_*(\Omega\mathcal{Z}_\mathcal{K})$ are one-relator as groups and algebras, respectively, and we give equivalent topological and combinatorial statements.

2.7.4 Real moment-angle complexes

Similar to the case for the moment-angle complex $\mathcal{Z}_\mathcal{K}$, there has been significant study into determining the homotopy type of $\mathcal{R}_\mathcal{K}$ for various simplicial complexes. As a result, the fundamental group $\pi_1(\mathcal{R}_\mathcal{K})$, and therefore the commutator subgroup $RC_\mathcal{K}$ can be computed in a wide variety of cases.

Example 2.7.10. (i) Let $\mathcal{K} = C_4$ be a square. Then $\mathcal{R}_\mathcal{K} = S^1 \times S^1$ by Proposition 2.5.7. It follows that $RC'_\mathcal{K} = \pi_1(\mathcal{R}_\mathcal{K})$ is the one-relator surface group $\mathbb{Z}^2$.

(ii) More generally, let $\mathcal{K} = C_p$ be a p-cycle for $p \geqslant 4$. Then it is known that $\mathcal{R}_\mathcal{K}$ is homeomorphic to a surface of genus $(p-4)2^{p-3}+1$, see [PV16, Example 3.5]. In particular, $RC'_\mathcal{K} = \pi_1(\mathcal{R}_\mathcal{K})$ is again a one-relator surface group, a calculation which is very difficult in a purely algebraic setting.

(iii) Let $\mathcal{K}$ be obtained from $\partial\Delta[1,2,3] * \partial\Delta[4,5]$ by performing a stellar subdivision at the simplex $(2,3,5)$. Equivalently, $\mathcal{K}$ is the simplicial complex on vertex set $[6]$ with minimal missing faces

$$MF(\mathcal{K}) = \{(1,2,3), (1,6), (2,3,5), (4,5), (4,6)\}.$$

By a result of McGavran [McG79], see [GdM13, Theorem 2.1],

$$\mathcal{R}_\mathcal{K} = (S^1 \times S^2)\#(S^1 \times S^2)\#(S^1 \times S^2)\#(S^1 \times S^2)\#(S^1 \times S^2)$$

from which it follows that $RC'_\mathcal{K} = \pi_1(\mathcal{R}_\mathcal{K}) = *_{i=1}^5 \mathbb{Z}$. Since $\mathcal{R}_\mathcal{K}$ is a 3-manifold, this is an example of a 3-manifold group.

In general, the group $\pi_1\left((\underline{EG}, \underline{G})^\mathcal{K}\right)$ is not finitely generated. In fact, it is known that the commutator subgroup $RA'_\mathcal{K}$ of the right-angled Artin group is finitely generated if and only if $\mathcal{K}$ contains a full graph as its 1-skeleton. A special feature of the real moment-angle complex is that its fundamental group is always finitely generated. Moreover, a minimal generating set can be written down for any simplicial complex $\mathcal{K}$, as follows.

Theorem 2.7.11 ([PV16, Theorem 4.5])**.** *Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $RC'_{\mathcal{K}}$ be the commutator subgroup of the right-angled Coxeter group on $\mathcal{K}$. Then $RC'_{\mathcal{K}}$ is finitely generated, with a minimal generating set given by*

$$(g_j, g_i), \qquad (g_{k_1}, (g_j, g_i)), \qquad \cdots \qquad (g_{k_1}, (g_{k_2}, \dots, (g_{k_{m-2}}, (g_j, g_i)) \cdots)) \tag{2.31}$$

where $k_1 < k_2 < \cdots < k_{m-2} < j > i$ are all distinct and i is the smallest vertex in a connected component not containing j of $\mathcal{K}_{\{k_1,k_2,\dots,k_{m-2},j,i\}}$. $\square$

We compare this to the statement of Theorem 2.6.13. In particular we observe that in the flag case, the minimal generating sets of $\pi_1(\mathcal{R}_{\mathcal{K}}) = RC'_{\mathcal{K}}$ as a group, and $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ as an algebra, are both given by iterated commutators of the same form.

Example 2.7.12. Let $\mathcal{K} = C_5$ be a 5-gon. Then $\pi_1(\mathcal{R}_{\mathcal{K}}) = RC'_{\mathcal{K}}$ has a minimal generating set consisting of 10 generators

$$\begin{array}{ccccc} & & & & \\ (g_3, g_1), & (g_4, g_1), & (g_4, g_2), & (g_5, g_2), & (g_5, g_3), \\ (g_2, (g_4, g_1)), & (g_3, (g_5, g_2)), & (g_3, (g_4, g_1)), & (g_4, (g_5, g_2)), & (g_1, (g_5, g_3)). \end{array}$$

By Example 2.7.10(ii), this is not a free generating set since $RC'_{\mathcal{K}}$ is a one-relator group. If we number those generators on the first row above by $a_1, \dots, a_5$ and those on the second row by $b_1, \dots, b_5$, it can be shown that the relation is given by

$$(a_1, b_4)(a_2, b_2)(a_3, b_5)(a_4, b_3)(a_5, b_1) = 1.$$

On the other hand, $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ has a minimal generating set

$$\begin{array}{ccccc} [u_3, u_1], & [u_4, u_1], & [u_4, u_2], & [u_5, u_2], & [u_5, u_3], \\ [u_2, [u_4, u_1]], & [u_3, [u_5, u_2]], & [u_3, [u_4, u_1]], & [u_4, [u_5, u_2]], & [u_1, [u_5, u_3]]. \end{array}$$

In [GPTW12] a relation was established between the above commutators. Again naming the generators on the first row $a_1, \dots, a_5$ and those on the second row $b_1, \dots, b_5$, the relation is given by

$$[a_1, b_4] + [a_2, b_2] + [a_3, b_5] + [a_4, b_3] + [a_5, b_1] = 0.$$

Chapter 3

One-relator groups and algebras related to polyhedral products

3.1 Introduction

Let $\mathcal{K}$ be a flag simplicial complex on $[m]$ and let $RC_{\mathcal{K}}$ be the corresponding right-angled Coxeter group. By Theorem 2.7.6, the commutator subgroup $RC'_{\mathcal{K}}$ of a right-angled Coxeter group $RC_{\mathcal{K}}$ is the fundamental group of the real moment-angle complex $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$, which is a finite-dimensional aspherical space. Moreover, by Theorem 2.7.9, the group $RC'_{\mathcal{K}} = \pi_1(\mathcal{R}_{\mathcal{K}})$ is free if and only if $\mathcal{K}^1$, the 1-skeleton of $\mathcal{K}$, is a chordal graph.

Similarly, by Theorem 2.6.16, when coefficients are in a field $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is a graded free associative algebra if and only if the 1-skeleton $\mathcal{K}^1$ is a chordal graph. Therefore, for both $\mathcal{R}_{\mathcal{K}}$ and $\mathcal{Z}_{\mathcal{K}}$ the algebraic freeness property, that is, that $\pi_1(\mathcal{R}_{\mathcal{K}})$ and $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ are free as groups and algebras, respectively, is described in the same, completely combinatorial way. Precisely, they are free if and only if the 1-skeleton $\mathcal{K}^1$ of the simplicial complex $\mathcal{K}$ is a chordal graph.

We study other properties of objects naturally arising in geometric group theory and homotopy theory that have the same combinatorial characterisation. In particular, we describe a combinatorial condition on a flag complex $\mathcal{K}$ under which $\pi_1(\mathcal{R}_{\mathcal{K}}) = RC'_{\mathcal{K}}$ is a one-relator group, and $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is a one-relator algebra. For $p \geqslant 4$ recall that the simplicial complex C_p is a p-cycle. In [GPTW12] it was shown that when $\mathcal{K} = C_5$ then there is only one relation between the 10 multiplicative generators of $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$; while in [Ver16], a single relation was again found between the 34 multiplicative generators of $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ when $\mathcal{K} = C_6$. Similarly, in [PV16], it was shown that if $\mathcal{K} = C_p$ for $p \geqslant 4$, then $\pi_1(\mathcal{R}_{\mathcal{K}})$ is a one-relator group, see Example 2.7.10(ii). The one-relator condition places strong restrictions on the form of $\mathcal{K}$, and our main combinatorial characterisation is the following.

Theorem 3.1.1. *Let $\mathcal{K}$ be a flag simplicial complex. The following are equivalent:*

(i) $RC'_{\mathcal{K}} = \pi_1(\mathcal{R}_{\mathcal{K}})$ *is a one-relator group;*

(ii) $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ *is a one-relator algebra;*

(iii) $\mathcal{K}$ *has the form*

$$\mathcal{K} = C_p \text{ or } \mathcal{K} = C_p * \Delta^q \quad \text{for } p \geqslant 4,\ q \geqslant 0 \tag{3.1}$$

where C_p is a p-cycle, Δ^q is a q-simplex and $*$ denotes the join of simplicial complexes.

These results hold integrally. In particular, a consequence of our result shows that $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is free with integral coefficients if and only if $\mathcal{K}^1$ is chordal. This is proved in Theorem 3.2.13.

We move on to consider algebraic one-relator conditions in the case that $\mathcal{K}$ is not assumed to be flag. In this case, the real moment-angle complex $\mathcal{R}_{\mathcal{K}}$ is not aspherical, so its topology is not determined by its fundamental group. In particular, any further study of $\mathcal{R}_{\mathcal{K}}$ does not lie entirely within geometric group theory. Therefore we focus on the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$.

When the combinatorial condition (3.1) is satisfied, the moment-angle complex is a connected sum of sphere products, with two spheres in each product. More generally, infinite families of simplicial complexes for which $\mathcal{Z}_{\mathcal{K}}$ is homeomorphic to a connected sum of sphere products have been studied by Bosio and Meersseman [BM04], and in more detail by Gitler and López de Medrano [GdM13]. In proving Theorem 3.1.1, we show that for all such $\mathcal{Z}_{\mathcal{K}}$, the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is one-relator. Moreover, by a result of Amelotte [Ame20], for every such $\mathcal{Z}_{\mathcal{K}}$, the complex $\mathcal{K}$ satisfies the property of being *minimally non-Golod*, see [Lim15]. In fact, our main results show that, when $\mathcal{K}$ is flag, the properties that $\mathcal{K}$ is minimally non-Golod, or such a complex joined with a simplex, and $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ being a one-relator algebra coincide.

In general, the concepts of minimally non-Golod complexes and one-relator algebras differ fundamentally, and should be analysed using different techniques. In Section 3.3 we demonstrate the differences by presenting a simplicial complex $\mathcal{K}$ for which $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is a one-relator algebra, but $\mathcal{K}$ is not minimally non-Golod, or such a complex joined with a simplex. Moreover, we give a minimally non-Golod complex $\mathcal{K}$ for which $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ has at least two relations. To provide these examples we develop a new way of computing $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ using an Adams–Hilton model applied the chain complex of $\mathcal{Z}_{\mathcal{K}}$ given by the homological Taylor complex, see Construction 2.6.6.

In the case that $\mathcal{Z}_{\mathcal{K}}$ is a manifold, or equivalently by a result of Cai [Cai17] that $\mathcal{K}$ is a generalised homology sphere, a homological generalisation of a simplicial sphere, the minimally non-Golod condition on $\mathcal{K}$ can be viewed as an algebraic approximation to $\mathcal{Z}_{\mathcal{K}}$ being a connected sum of sphere products, since it implies there is a single cup product class in the cohomology of $\mathcal{Z}_{\mathcal{K}}$. When $\mathcal{K}$ is a 2-dimensional simplicial sphere, Bosio and Meersseman [BM04] show that $\mathcal{K}$ being minimally non-Golod is equivalent to $\mathcal{Z}_{\mathcal{K}}$ being a connected sum of sphere products. For $\mathcal{K}$ a simplicial sphere of dimension above 3, the relationship between these two concepts is not known. We make some initial progress on this front. First, we construct a simplicial sphere $\mathcal{K}$ for which $H^*(\mathcal{Z}_{\mathcal{K}})$ is isomorphic to the cohomology ring of a connected sum of sphere products, while this isomorphism is not due to a homotopy equivalence. This answers negatively questions of [BM04, GdM13] about whether $\mathcal{Z}_{\mathcal{K}}$ being a connected sum of sphere products can be identified by looking at the cohomology ring. Moreover, we construct a minimally non-Golod simplicial sphere $\mathcal{L}$ for which $\mathcal{Z}_{\mathcal{L}}$ is not a connected sum of sphere products due to the existence of torsion in its homology. This answers negatively another question of [GdM13] about whether connected sums can be identified solely from a combinatorial connectedness assumption.

Acknowledgement and declaration

The material in this chapter was jointly produced by myself, under the supervision of Jelena Grbić, with collaboration from Marina Ilyasova and Taras Panov, who developed the work presented in Sections 3.2.1 and 3.2.2. All other work in this chapter is my own. The material of Section 3.2 is a published work of all mentioned authors [GIPS22]. The presentation and contextualisation of all results included is my own.

3.2 One-relator groups and algebras for flag complexes

We split the proof of Theorem 3.1.1 into two separate results: for $\mathcal{R}_{\mathcal{K}}$ this is Theorem 3.2.4 and for $\mathcal{Z}_{\mathcal{K}}$ this is Theorem 3.2.8.

The proofs of Theorem 3.2.4 and Theorem 3.2.8 are completely different in character. For Theorem 3.2.4, the key argument comes from geometric group theory. When $\mathcal{K} = C_p$ or $\mathcal{K} = C_p * \Delta^q$ for $q \geqslant 0$, the space $\mathcal{R}_{\mathcal{K}}$ is homeomorphic to the product $S_g \times D^{q+1}$, where S_g is a closed orientable surface of genus $g = (p-4)2^{p-3} + 1$ and D^{q+1} is a $(q+1)$-dimensional disc, and therefore its fundamental group is a one-relator surface group. The converse statement is proved using the Lyndon Identity Theorem [Lyn50], since the group $\pi_1(\mathcal{R}_{\mathcal{K}}) = RC'_{\mathcal{K}}$ is torsion-free.

To prove Theorem 3.2.8, we study the simply-connected space $\Omega\mathcal{Z}_\mathcal{K}$ using homotopy-theoretic methods. When $\mathcal{K} = C_p$ or $\mathcal{K} = C_p * \Delta^q$ for $q \geqslant 0$, by a result of McGavran [McG79], there is a homotopy equivalence

$$\mathcal{Z}_\mathcal{K} \simeq \#_{k=3}^{p-1}(S^k \times S^{p+2-k})^{\#(k-2)\binom{p-2}{k-1}} \tag{3.2}$$

where $M^{\#n}$ denotes the n-fold connected sum of a manifold M with itself. Beben and Wu [BW15] computed the algebra $H_*(\Omega X;\mathbb{Z}_p)$, p prime, where X is a highly-connected manifold obtained by attaching a single cell to a space Y which has the homotopy type of a double suspension. This implies that $H^*(Y)$ has no non-trivial cup products, which places sufficient restrictions on $H^*(X)$ so that $H_*(\Omega X)$ can be studied via a homology Serre spectral sequence. We adapt the Beben–Wu method to study the Pontryagin algebra of an arbitrary connected sum of sphere products

$$M = \#_{i=1}^{k}(S^{d_i} \times S^{d-d_i}) \tag{3.3}$$

where $d_i \geqslant 2$ and $d \geqslant 4$. In this case, the Beben–Wu method reduces to the Adams–Hilton model and the highly-connectedness assumption can be dropped. In Proposition 3.2.6, we prove that $H_*(\Omega M)$ is isomorphic as a Hopf algebra to the quotient of a graded free associative algebra by a single relation. Proposition 3.2.6 implies that when condition (3.1) holds, $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a one-relator algebra. We also compute the Poincaré series $P(H_*(\Omega\mathcal{Z}_K);t)$ explicitly in Proposition 3.2.7, mirroring a result given in [GPTW12] in the case that $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is free.

We extend the equivalences of Theorem 3.1.1 by determining equivalent homological criteria on $\mathcal{R}_\mathcal{K}$ and $\mathcal{Z}_\mathcal{K}$. For $\mathcal{R}_\mathcal{K}$, the combinatorial condition (3.1) is equivalent to the homological condition $H_2(\mathcal{R}_\mathcal{K}) = \mathbb{Z}$, and this is proved in Theorem 3.2.4. The homology groups of $\mathcal{Z}_\mathcal{K}$ have a natural bigrading, obtained in the same way as the cohomological bigrading (2.19). The combinatorial condition (3.1) is also equivalent to the homological condition

$$H_{2-j,2j}(\mathcal{Z}_\mathcal{K}) = \begin{cases} \mathbb{Z} & \text{if } j = p \\ 0 & \text{otherwise.} \end{cases}$$

This is proved in Theorem 3.2.8.

3.2.1 One-relator groups

A group G is called a *one-relator group* if G is not a free group and can be written $G = F/R$, where F is a free group and R is the smallest normal subgroup in F generated by a single element $r \in F$.

Let $G = F/R$ be a one-relator group and consider the space

$$Y(G) = \left(\bigvee_{i=1}^{l} S_i^1 \right) \cup_{\overline{r}} e^2 \tag{3.4}$$

obtained by attaching a 2-cell to a wedge of circles via the map $\overline{r}\colon S^1 \to \bigvee S_i^1$ described by the element $r \in F$. The homology groups of $Y(G)$ are given as follows.

Proposition 3.2.1. *$H_k(Y(G)) = 0$ for $k \geqslant 3$, $H_1(Y(G)) = \mathbb{Z}^l$ and*

$$H_2(Y(G)) = \begin{cases} \mathbb{Z} & \text{if } r \in F', \\ 0 & \text{otherwise} \end{cases}$$

where F' is the commutator subgroup of F.

Proof. The claims for $H_k(Y(G))$ when $k \neq 2$ follow immediately from the cellular structure of $Y(G)$. When $k = 2$, the map induced by $\overline{r}$ in the cellular chain complex of $Y(G)$ is trivial if and only if its image under the abelianisation map is trivial, that is, if and only if $r \in F'$. Therefore if $r \in F'$ then $d(e^2) = 0$ and $H_2(Y(G)) = \mathbb{Z}$, and otherwise $H_2(Y(G)) = 0$. □

Lyndon [Lyn50] studied the cohomology theory of one-relator groups by considering the corresponding space $Y(G)$ defined in (3.4). Dyer and Vasquez [DV73] gave the following formulation of a result known as the Lyndon Identity Theorem.

Theorem 3.2.2 ([DV73, Theorem 2.1])**.** *Let $G = F/R$ be a one-relator group where $R = \langle r \rangle$ for some $r \in F$. Suppose that r is not a proper power, that is, $r \neq u^n$ for any $u \in F$ or $n > 1$. Then $Y(G)$ is a $K(G,1)$ space.* □

We give some examples of Proposition 3.2.1 and Theorem 3.2.2.

Example 3.2.3. Let $G = F/R$ be a one-relator group, where $F = \langle a, b\rangle$ and $R = \langle r \rangle$ for some $r \in F$. We consider the space $Y(G)$ for different $r \in \langle a, b \rangle$.

(i) Let $r = aba^{-1}b^{-1}$. Then G is the abelian group $\mathbb{Z}^2$, and $Y(G)$ defines the torus $T^2 = S^1 \times S^1$. Since r is the commutator of a and b then $r \in F'$, and it follows that $H_2(Y(G)) = \mathbb{Z}$. Furthermore, we verify directly that $\pi_1(Y(G)) = \pi_1(S^1 \times S^1) = \mathbb{Z}^2 = G$, and that $\pi_k(Y(G)) = 0$ if $k > 1$.

(ii) Let $r = a^2$. Then G is the free product $\mathbb{Z}_2 * \mathbb{Z}$, and the space $Y(G)$ is the wedge $\mathbb{R}P^2 \vee S^1$. Then while $\pi_1(Y(G)) = G$, the inclusion $\mathbb{R}P^2 \longrightarrow Y(G)$ induces retractions $\pi_k(S^2) \cong \pi_k(\mathbb{R}P^2) \longrightarrow \pi_k(Y(G))$ for $k > 1$, since the universal cover of $\mathbb{R}P^2$ is S^2. Therefore $Y(G)$ is not a $K(G,1)$-space.

(iii) Let $r = aba^{-1}b$. Then G is the abelian group $\mathbb{Z}_2 \times \mathbb{Z}$, and $Y(G)$ is the Klein bottle. Then $\pi_1(Y(G)) = G$, while $\pi_k(Y(G)) = 0$ for $k > 1$ since the universal cover of the Klein bottle is $\mathbb{R}^2$. Furthermore, since $r \notin F'$, we have that $H_2(Y(G)) = 0$.

Under the conditions of the Lyndon Identity Theorem, we have $H_k(G;\mathbb{Z}) = H_k(Y(G);\mathbb{Z})$, that is, the homological dimension of G is at most 2.

3.2.2 The commutator subgroup of the right-angled Coxeter group

We now return to the real moment-angle complex and the commutator subgroup $RC'_{\mathcal{K}}$ of the right-angled Coxeter group. We aim to give a combinatorial condition on $\mathcal{K}$ which is equivalent to $RC'_{\mathcal{K}}$ being a one-relator group.

Let $\mathcal{K} = C_p$ be a p-cycle for $p \geqslant 4$. Then the real moment-angle complex $\mathcal{R}_{\mathcal{K}}$ is homeomorphic to the closed orientable surface of genus $(p-4)2^{p-3}+1$, see [PV16, Example 3.5]. Therefore $RC'_{\mathcal{K}} = \pi_1(\mathcal{R}_{\mathcal{K}})$ is a one-relator surface group. Similar to Theorem 2.7.9 that says $RC'_{\mathcal{K}}$ is free if and only if the graph $\mathcal{K}^1$ is chordal, the algebraic property that $RC'_{\mathcal{K}}$ is a one-relator group also has the following combinatorial classification.

Theorem 3.2.4. *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$. Then the following are equivalent:*

(i) $\pi_1(\mathcal{R}_{\mathcal{K}}) = RC'_{\mathcal{K}}$ *is a one-relator group;*

(ii) $H_2(\mathcal{R}_{\mathcal{K}}) = \mathbb{Z}$*;*

(iii) $\mathcal{K} = C_p$ *or* $\mathcal{K} = C_p * \Delta^q$ *for* $p \geqslant 4$ *and* $q \geqslant 0$*, where* C_p *is a* p*-cycle,* Δ^q *is a* q*-simplex, and* $*$ *denotes the join of simplicial complexes.*

If any one of these conditions is met, we have $H_k(\mathcal{R}_{\mathcal{K}}) = 0$ *for* $k \geqslant 3$.

Condition (ii) in the statement of Theorem 3.2.4 is not only a convenient technical tool to pass from the algebraic condition (i) to combinatorial condition (iii), but also says that the homological dimension of the group $RC'_{\mathcal{K}}$ is 2.

The homology groups $H_*(\mathcal{R}_{\mathcal{K}})$ decompose via a Hochster decomposition obtained by applying decomposition (2.26) to $\mathcal{R}_{\mathcal{K}}$, that is, for every $k \geqslant 1$,

$$H_k(\mathcal{R}_{\mathcal{K}}) \cong \bigoplus_{J \subseteq [m]} \widetilde{H}_{k-1}(\mathcal{K}_J). \tag{3.5}$$

Proof of Theorem 3.2.4. (i) $\Longrightarrow$ (ii). Since $\mathcal{R}_{\mathcal{K}}$ is an aspherical finite cell complex, the group $\pi_1(\mathcal{R}_{\mathcal{K}})$ is torsion-free, see, for example [Hat02, Proposition 2.45]. Therefore if

$\pi_1(\mathcal{R}_\mathcal{K}) = F/R$ is a one-relator group with a relation $R = \langle r \rangle$, then r is not a proper power u^n for $n > 1$. In particular, if $RC'_\mathcal{K}$ is a one-relator group, then it is torsion-free.

Consider the space $Y(RC'_\mathcal{K})$, as constructed in (3.4). By Theorem 3.2.2, the space $Y(RC'_\mathcal{K})$ is homotopy equivalent to $K(RC'_\mathcal{K}, 1)$. Additionally, since $\mathcal{K}$ is flag, then $\mathcal{R}_\mathcal{K}$ is also homotopy equivalent to $K(RC'_\mathcal{K}, 1)$ by Theorem 2.7.6. In particular, the homology groups of $Y(RC'_\mathcal{K})$ coincide with those of $\mathcal{R}_\mathcal{K}$.

Proposition 3.2.1 then implies that $H_2(\mathcal{R}_\mathcal{K})$ is either $\mathbb{Z}$ or trivial. By Theorem 2.7.9, since the group $RC'_\mathcal{K}$ is not free, the graph $\mathcal{K}^1$ is not chordal, that is, there exists a chordless cycle on I of length $p \geqslant 4$. Therefore, one of the summands on the right hand side of (3.5) is equal to $\mathbb{Z} = \widetilde{H}_1(\mathcal{K}_I)$. It follows that $H_2(\mathcal{R}_\mathcal{K}) = \mathbb{Z}$.

(ii) $\Longrightarrow$ (iii). Suppose $H_2(\mathcal{R}_\mathcal{K}) = \mathbb{Z}$. Then only one summand on the right of (3.5) is $\mathbb{Z}$, and all other summands are zero. Since $\mathcal{K}$ is a flag complex, this implies that there exists a set of vertices $I = \{i_1, \dots, i_p\}$ such that $\mathcal{K}_I$ is a p-cycle with $p \geqslant 4$. Since $\widetilde{H}_1(\mathcal{K}_J) = 0$ for any proper subset $J \subseteq I$, any two vertices which are not adjacent in the p-cycle are not connected by an edge. If there exists a vertex $j \notin I$ in the complex $\mathcal{K}$, then $\widetilde{H}_1(\mathcal{K}_{I\cup\{j\}}) = 0$ implies that the vertex j is connected to each vertex in the p-cycle I. If $\mathcal{K}$ has two vertices $j_1, j_2 \notin I$ which are not connected by an edge, then the subcomplex $\mathcal{K}_{\{i_1,i_3\}\cup\{j_1,j_2\}}$ is a 4-cycle and $\widetilde{H}_1(\mathcal{K}_{\{i_1,i_3\}\cup\{j_1,j_2\}}) = \mathbb{Z}$, which contradicts the assumption. Hence, all vertices of $\mathcal{K}$ which are not in the set I are connected to each other and to all vertices of I. Since $\mathcal{K}$ is a flag complex, we obtain $\mathcal{K} = C_p * \Delta^q$ for some $p \geqslant 4$ and $q \geqslant 0$.

(iii) $\Longrightarrow$ (i). If $\mathcal{K}$ is a p-cycle then $RC'_\mathcal{K} = \pi_1(\mathcal{R}_\mathcal{K})$ is a one-relator group by the remarks before the statement of Theorem 3.2.4. It remains to deal with the case that $\mathcal{K}$ is of the form $C_p * \Delta^q$ for $p \geqslant 4$ and $q \geqslant 0$. In this case $\mathcal{R}_\mathcal{K} = \mathcal{R}_{C_p} \times D^{q+1}$ by Proposition 2.5.7 and so

$$RC'_\mathcal{K} = \pi_1(\mathcal{R}_\mathcal{K}) = \pi_1(\mathcal{R}_{C_p}) = RC'_{C_p}$$

is a one-relator group.

It remains to prove that (iii) implies that $H_k(\mathcal{R}_\mathcal{K}) = 0$ for $k \geqslant 3$. Considering the Hochster decomposition (3.5), we claim that all summands with $k \geqslant 3$ on the right hand side are trivial. Indeed, let $I = \{i_1, \dots, i_p\}$ be the set of vertices of $\mathcal{K}$ forming a p-cycle. Then $\widetilde{H}_{k-1}(\mathcal{K}_I) = 0$ for $k \geqslant 3$. Since any full subcomplex $\mathcal{K}_J$ with $J \neq I$ is contractible, we get $\widetilde{H}_{k-1}(\mathcal{K}_J) = 0$. Hence, $H_k(\mathcal{R}_\mathcal{K}) = 0$ for $k \geqslant 3$. □

Example 3.2.5. Consider the following simplicial complexes on vertex set [5].

(i) Let $\mathcal{K}$ be the flag complex in Figure 3.1(a), which is not of the form C_p or $C_p * \Delta^q$ for $p \geqslant 4$ and $q \geqslant 0$. Generator set (2.31) for the commutator subgroup $RC'_\mathcal{K}$ is

$$(g_3, g_1),\ (g_4, g_2),\ (g_5, g_4),\ (g_2, (g_5, g_4)).$$

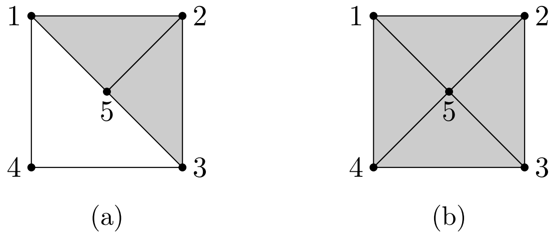


Figure 3.1: Similar simplicial complexes can give rise to different algebraic structures.

These satisfy the relations

$$(g_3, g_1)^{-1}(g_4, g_2)^{-1}(g_3, g_1)(g_4, g_2) = 1, \quad (g_3, g_1)^{-1}(g_5, g_4)^{-1}(g_3, g_1)(g_5, g_4) = 1$$

and

$$(g_3, g_1)^{-1}(g_2, (g_5, g_4))^{-1}(g_3, g_1)(g_2, (g_5, g_4)) = 1.$$

Indeed, since each of g_1 and g_3 commutes with each of g_2 and g_4, the commutators $(g_4, g_2)^{-1}$ and (g_3, g_1) commute too. We therefore obtain

$$(g_3, g_1)^{-1}(g_4, g_2)^{-1}(g_3, g_1)(g_4, g_2) = (g_3, g_1)^{-1}(g_3, g_1)(g_4, g_2)^{-1}(g_4, g_2) = 1.$$

The other two relations are proved similarly. In particular, $RC'_{\mathcal{K}}$ is not a one-relator group. Using homology decomposition (3.5), we see that $H_2(\mathcal{R}_{\mathcal{K}}) = \mathbb{Z}^3$, and therefore condition (ii) of Theorem 3.2.4 is not satisfied.

(ii) Let $\mathcal{K}$ be the flag complex in Figure 3.1(b). Generator set (2.31) for $RC'_{\mathcal{K}}$ is

$$(g_3, g_1), \ (g_4, g_2)$$

which satisfy a single relation $(g_3, g_1)^{-1}(g_4, g_2)^{-1}(g_3, g_1)(g_4, g_2) = 1$. Therefore $RC'_{\mathcal{K}}$ is a one-relator group and moreover by homology decomposition (3.5) we get that $H_2(\mathcal{R}_{\mathcal{K}}) = \mathbb{Z}$.

3.2.3 Connected sums of sphere products

Let $M = \#_{i=1}^{k}\left(S^{d_i} \times S^{d-d_i}\right)$, where $d_i \geqslant 2$, $d \geqslant 4$ and $\#$ denotes the connected sum operation on manifolds. Topologically, such connected sums are obtained by attaching a single cell to a wedge of spheres, that is, there is a cofibration sequence

$$S^{d-1} \xrightarrow{\ w\ } \bigvee_{i=1}^{k} S^{d_i} \vee S^{d-d_i} \xrightarrow{\ i\ } \#_{i=1}^{k}(S^{d_i} \times S^{d-d_i}) \tag{3.6}$$

where w is the sum of Whitehead products $w_i = [\iota_{d_i}, \iota_{d-d_i}]\colon S^{d-1} \to S^{d_i} \vee S^{d-d_i}$ of identity maps $\iota_k\colon S^k \longrightarrow S^k$. Denote by $\overline{M}$ the wedge $\bigvee_{i=1}^{d} S^{d_i} \vee S^{d-d_i}$. Then by the Bott–Samelson Theorem (Theorem 2.3.1), $H_*(\Omega\overline{M}) \cong T(a_1, b_1, \ldots, a_k, b_k)$, where

$\deg(a_i) = d_i - 1$ and $\deg(b_i) = d - d_i - 1$. The looped inclusion $\Omega i\colon \Omega\overline{M} \to \Omega M$ induces a map of algebras

$$(\Omega i)_*\colon T(a_1, b_1, \ldots, a_k, b_k) \longrightarrow H_*(\Omega M).$$

The adjoint $\overline{w}\colon S^{d-2} \to \Omega\left(\overline{M}\right)$ of the sum of Whitehead products w induces a map $\overline{w}_*\colon H_{d-2}(S^{d-2}) \to H_{d-2}(\Omega\overline{M})$, which by a result of Samelson [Sam53] (see relation (2.8)) sends the canonical generator to the element $\chi = (-1)^{d_1}[a_1, b_1] + \cdots + (-1)^{d_k}[a_k, b_k]$. In particular, χ is primitive and $(\Omega i)_*(\chi) = 0$ in $H_*(\Omega M)$. Then the algebra

$$\frac{T(a_1, b_1, \ldots, a_k, b_k)}{\langle(-1)^{d_1}[a_1, b_1] + \cdots + (-1)^{d_k}[a_k, b_k]\rangle} \tag{3.7}$$

is a primitively generated Hopf algebra, where the quotient ideal is two-sided, and the algebra map $(\Omega i)_*$ factors as a map of Hopf algebras

$$\begin{tikzcd} T(a_1, b_1, \ldots, a_k, b_k) \arrow[r, "(\Omega i)_*"] \arrow[d] & H_*(\Omega M) \\ \dfrac{T(a_1, b_1, \ldots, a_k, b_k)}{\langle(-1)^{d_1}[a_1, b_1] + \cdots + (-1)^{d_k}[a_k, b_k]\rangle} \arrow[ur, "\theta"'] & \end{tikzcd} \tag{3.8}$$

defining the map θ.

We recall the Cobar construction and Adams–Hilton models from Section 2.3.2. The following statement generalises [AH56, Corollary 2.4] and Example 2.3.4(i).

Proposition 3.2.6. *For $d_i \geqslant 2$ and $d \geqslant 4$, there is an isomorphism of Hopf algebras*

$$H_*\left(\Omega\left(\#_{i=1}^k S^{d_i} \times S^{d-d_i}\right)\right) \cong \frac{T(a_1, b_1, \ldots, a_k, b_k)}{\langle(-1)^{d_1}[a_1, b_1] + \cdots + (-1)^{d_k}[a_k, b_k]\rangle}$$

where $\deg a_i = d_i - 1$, $\deg b_i = d - d_i - 1$, and $[a_i, b_i] = a_i b_i - (-1)^{\deg a_i \deg b_i} b_i a_i$ is the graded commutator.

Proof. We consider the Adams–Hilton model of $M = \#_{i=1}^k S^{d_i} \times S^{d-d_i}$. The cofibration sequence (3.6) gives a CW-structure on M consisting of cells $e^0, e_i^{d_i}, e_i^{d-d_i}$ for $i = 1, \ldots, k$, each attached trivially, and a single cell e^d attached by the sum of Whitehead products $w_i\colon S^{d-1} \to S^{d_i} \vee S^{d-d_i}$. The cellular chain complex has a coalgebra structure in which $e_i^{d_i}$ and $e_i^{d-d_i}$ are primitives, and

$$\Delta e^d = e^d \otimes 1 + 1 \otimes e^d + \sum_{i=1}^k \left(e_i^{d_i} \otimes e_i^{d-d_i} + (-1)^{d_i(d-d_i)} e_i^{d-d_i} \otimes e_i^{d_i}\right). \tag{3.9}$$

The Adams–Hilton model is therefore

$$AH_*(M) = \big(T(a_1, b_1, \ldots, a_k, b_k, z), d\big)$$

where $a_i = s^{-1}e_i^{d_i}$, $b_i = s^{-1}e_i^{d-d_i}$, $z = s^{-1}e^d$ and $\deg a_i = d_i - 1$, $\deg b_i = d - d_i - 1$ and $\deg z = d - 1$.

The differential is given by $d(a_i) = d(b_i) = 0$ and

$$\begin{aligned} d(z) &= \sum_{i=1}^{k} \Big((-1)^{d_i} a_i b_i + (-1)^{d-d_i}(-1)^{d_i(d-d_i)} b_i a_i\Big) \\ &= \sum_{i=1}^{k} (-1)^{d_i} \Big(a_i b_i - (-1)^{(d_i-1)(d-d_i-1)} b_i a_i\Big) = \sum_{i=1}^{k} (-1)^{d_i} [a_i, b_i]. \end{aligned}$$

A nonzero $x \in AH_*(M)$ is a cycle if and only if x is not in the two-sided ideal $\langle z\rangle$, and x is a boundary if and only if $x \in \langle d(z)\rangle$. Therefore, homology of ΩM is as stated. $\square$

For a graded vector space V, denote by $P(V;t)$ the Poincaré series of V. The following result mirrors [GPTW12, Theorem 3.2].

Proposition 3.2.7. *There is the following identity for the Poincaré series*

$$P\left(H_*\left(\Omega\left(\#_{i=1}^k S^{d_i} \times S^{d-d_i}\right); t\right)\right) = \frac{1}{1 - \sum_{i=1}^k (t^{d_i-1} + t^{d-d_i-1}) + t^{d-2}}.$$

Proof. Let $A = H_*\left(\Omega\left(\#_{i=1}^k S^{d_i} \times S^{d-d_i}\right)\right)$. By Proposition 3.2.6, A is the quotient of the free associative algebra on the graded set $S = \{a_1, b_1, \ldots, a_k, b_k\}$, where $\deg a_i = d_i - 1$ and $\deg b_i = d - d_i - 1$, by the two-sided ideal generated by the element

$$\chi = \sum_{i=1}^{k} (-1)^{d_i}[a_i, b_i] = (-1)^{d_1}(a_1 b_1 - (-1)^{(d_i-1)(d-d_i-1)} b_1 a_1) + \sum_{i=2}^{k} (-1)^{d_i}[a_i, b_i].$$

Let B be the graded free monoid on S. Then B_n, the nth graded component of B, is a generating set for A_n, the nth graded component of A. For any monomial $x \in A - \{1\}$, write $x = sy$ for some unique $s \in S$ and $y \in B_{n-\deg s}$. If $x = a_1 b_1 y'$ then using relation χ we rewrite

$$x = \left((-1)^{(d_1-1)(d-d_1-1)} b_1 a_1 - \sum_{i=2}^{k} (-1)^{d_i - d_1}[a_i, b_i]\right) y'.$$

Let B'_n be the set of all elements in B_n which do not start with $a_1 b_1$. By induction, B'_n is a minimal generating set for A_n. Define $c_n = |B'_n| = \operatorname{rank} A_n$ for $n \geqslant 1$, $c_n = 0$ for $n < 0$, and $c_0 = 1$. From the above description, c_n satisfies the recurrence formula

$$c_n = \sum_{i=1}^{k} (c_{n-d_i+1} + c_{n-d+d_i+1}) - c_{n-d+2}$$

for $n \geqslant 1$.

Multiplying by t^n and summing over $n > 0$ gives

$$\begin{aligned}
P(A;t) - 1 &= \sum_{n=1}^{\infty} c_n t^n \\
&= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k} (c_{n-d_i+1} + c_{n-d+d_i+1}) - c_{n-d+2} \right) t^n \\
&= \sum_{i=1}^{k} \sum_{n=2-d_i}^{\infty} c_n t^{n+d_i-1} + \sum_{i=1}^{k} \sum_{n=2-d+d_i}^{\infty} c_n t^{n+d-d_i-1} - \sum_{n=3-d}^{\infty} c_n t^{n+d-2} \\
&= \left(\sum_{i=1}^{k} (t^{d_i-1} + t^{d-d_i-1}) - t^{d-2} \right) \sum_{n=0}^{\infty} c_n t^n \\
&= \left(\sum_{i=1}^{k} (t^{d_i-1} + t^{d-d_i-1}) - t^{d-2} \right) P(A;t)
\end{aligned}$$

which is rearranged to give the claimed identity. □

3.2.4 The loop homology algebra of $\mathcal{Z}_{\mathcal{K}}$

An algebra is a *one-relator algebra* if it is not free and can be written as the quotient of a free associative algebra by a two-sided ideal generated by a single element.

We recall the homology of the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ has a natural bigrading coming from the bigrading of its cellular structure, see Construction 2.6.1. Moreover, the Hochster decomposition describes the bigraded homology groups in terms of the homology of full subcomplexes of $\mathcal{K}$ as

$$H_{-i,2j}(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{J \subseteq [m],\, |J|=j} \widetilde{H}_{j-i-1}(\mathcal{K}_J) \tag{3.10}$$

which is dual to the cohomology decomposition (2.22).

Theorem 3.2.8. *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$. The following conditions are equivalent:*

(i) $H_*(\Omega \mathcal{Z}_{\mathcal{K}})$ *is a one-relator algebra;*

(ii) $H_{2-j,2j}(\mathcal{Z}_{\mathcal{K}}) = \begin{cases} \mathbb{Z} & \text{if } j = p \text{ for some } p,\ 4 \leqslant p \leqslant m \\ 0 & \text{otherwise;} \end{cases}$

(iii) $\mathcal{K} = C_p$ *or* $\mathcal{K} = C_p * \Delta^q$ *for* $p \geqslant 4$ *and* $q \geqslant 0$*, where* C_p *is a* p*-cycle and* Δ^q *is a* q*-simplex.*

If any one of these conditions is met, we have $H_{-i,2j}(\mathcal{Z}_{\mathcal{K}}) = 0$ *for* $j - i \geqslant 3$.

We start by showing that if $\mathcal{K}$ is a flag complex which is not of the form given in (iii), then either $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is free, or it has at least two relations. The following result gives a condition for $H_*(\Omega\mathcal{Z}_\mathcal{K})$ to have at least two relations.

Lemma 3.2.9. *Let $\mathcal{K}$ be a simplicial complex and suppose that $\mathcal{K}_I$ and $\mathcal{K}_J$ are distinct full subcomplexes of $\mathcal{K}$ such that both $H_*(\Omega\mathcal{Z}_{\mathcal{K}_I})$ and $H_*(\Omega\mathcal{Z}_{\mathcal{K}_J})$ have at least one relation. Then $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is not a one-relator algebra.*

Proof. By Proposition 2.5.4, both of $\mathcal{Z}_{\mathcal{K}_I}$ and $\mathcal{Z}_{\mathcal{K}_J}$ retract off $\mathcal{Z}_\mathcal{K}$ since $\mathcal{K}_I$ and $\mathcal{K}_J$ are full subcomplexes of $\mathcal{K}$. Therefore each of $\Omega\mathcal{Z}_{\mathcal{K}_I}$ and $\Omega\mathcal{Z}_{\mathcal{K}_J}$ retracts off $\Omega\mathcal{Z}_\mathcal{K}$ and we obtain a commutative diagram of algebras

$$\begin{tikzcd}
H_*(\Omega\mathcal{Z}_{\mathcal{K}_I}) \arrow[r] \arrow[dr, equal] & H_*(\Omega\mathcal{Z}_\mathcal{K}) \arrow[d] \\
& H_*(\Omega\mathcal{Z}_{\mathcal{K}_I})
\end{tikzcd}$$

and similarly for $\mathcal{K}_J$. In particular, each relation of $H_*(\Omega\mathcal{Z}_{\mathcal{K}_I})$ and $H_*(\Omega\mathcal{Z}_{\mathcal{K}_J})$ appears as a relation of $H_*(\Omega\mathcal{Z}_\mathcal{K})$ under the induced inclusion map, and the induced relations are distinct since $\mathcal{K}_I$ and $\mathcal{K}_J$ are. $\square$

Let $\mathcal{K}$ be a simplicial complex on $[m]$. The following is a consequence of decomposition (2.14), see also of [GT07, Lemma 3.3].

Lemma 3.2.10. *Let $j \in \mathcal{K}$ be a vertex and suppose that the map $i\colon \mathcal{Z}_{\mathrm{lk}_\mathcal{K}(j)} \to \mathcal{Z}_{\mathcal{K}_{[m]-\{j\}}}$ is nullhomotopic, where $\mathrm{lk}_\mathcal{K}(j)$ is viewed as a simplicial complex on vertex set $[m]-\{j\}$. Then there is a homotopy equivalence $\mathcal{Z}_K \simeq \Sigma^2\mathcal{Z}_{\mathrm{lk}_K(j)} \vee (\mathcal{Z}_{\mathcal{K}_{[m]-\{j\}}} \rtimes S^1)$.* $\square$

The following result shows that if $\mathcal{Z}_{\mathcal{K}_{[m]-\{j\}}}$ has the homotopy type of a connected sum of sphere products then $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is not a one-relator algebra.

Lemma 3.2.11. *Suppose that $M = \#_{i=1}^k \left(S^{d_i} \times S^{d-d_i}\right)$ where $d_i \geqslant 2$ and $d \geqslant 4$. Then $H_*(\Omega(M \rtimes S^1))$ is not a one-relator algebra.*

Proof. As in Proposition 3.2.6, we apply the Adams–Hilton model. A cell structure on $M \rtimes S^1$ is given by the image under the quotient map $M \times S^1 \to M \rtimes S^1$, and therefore consists of cells e^0, $e_i^{d_i}, e_i^{d-d_i}, e_i^{d_i+1}, e_i^{d-d_i+1}$ for $i = 1, \dots, k$, along with two cells e^d and e^{d+1}. The cellular chain complex has trivial differential and a coalgebra structure in which $e_i^{d_i}, e_i^{d-d_i}, e_i^{d_i+1}, e_i^{d-d_i+1}$ are primitives, Δe^d is given by (3.9) and

$$\Delta e^{d+1} = e^{d+1} \otimes 1 + 1 \otimes e^{d+1} + \sum_{i=1}^k \Big(e_i^{d_i} \otimes e_i^{d-d_i+1} + (-1)^{d_i(d-d_i+1)} e_i^{d-d_i+1} \otimes e_i^{d_i} \\ + e_i^{d_i+1} \otimes e_i^{d-d_i} + (-1)^{(d_i+1)(d-d_i)} e_i^{d-d_i} \otimes e_i^{d_i+1}\Big).$$

The Adams–Hilton model is therefore given by

$$AH_*(M \rtimes S^1) = (T(a_1, b_1, x_1, y_1, \ldots, a_k, b_k, x_k, y_k, z, w), d)$$

where $a_i = s^{-1}e_i^{d_i}$, $b_i = s^{-1}e_i^{d-d_i}$, $x_i = s^{-1}e_i^{d_i+1}$, $y_i = s^{-1}e_i^{d-d_i+1}$, $z = s^{-1}e^d$ and $w = s^{-1}e^{d+1}$, with $\deg a_i = d_i - 1$, $\deg b_i = d - d_i - 1$, $\deg x_i = d_i$, $\deg y_i = d - d_i$, $\deg z = d - 1$ and $\deg w = d$. The differential is given by $d(a_i) = d(b_i) = d(x_i) = d(y_i) = 0$,

$$d(z) = \sum_{i=1}^{k} (-1)^{d_i}[a_i, b_i]$$

as in Proposition 3.2.6, and

$$\begin{aligned} d(w) &= \sum_{i=1}^{k} \Big((-1)^{d_i} a_i y_i + (-1)^{d-d_i+1}(-1)^{d_i(d-d_i+1)} y_i a_i \\ &\qquad\qquad + (-1)^{d_i+1} x_i b_i + (-1)^{d-d_i}(-1)^{(d_i+1)(d-d_i)} b_i x_i \Big) \\ &= \sum_{i=1}^{k} \Big((-1)^{d_i} \left(a_i y_i - (-1)^{(d_i-1)(d-d_i)} y_i a_i \right) + (-1)^{d_i+1} \left(x_i b_i - (-1)^{d_i(d-d_i-1)} b_i y_i \right) \Big) \\ &= \sum_{i=1}^{k} (-1)^{d_i} \left([a_i, y_i] - [x_i, b_i] \right). \end{aligned}$$

Therefore any element in $\langle d(z) \rangle$ or $\langle d(w) \rangle$ is trivial in homology since it is a boundary. This induces two independent relations in $H_*(\Omega(M \rtimes S^1))$, as claimed. □

We now have all the tools required to prove Theorem 3.2.8.

Proof of Theorem 3.2.8. (iii) $\Longrightarrow$ (i). Suppose that $\mathcal{K} = C_p$ or $\mathcal{K} = C_p * \Delta^q$ for $p \geqslant 4$, $q \geqslant 0$. Since $\mathcal{Z}_\mathcal{K}$ is homotopy equivalent to the connected sum of sphere products (3.2), the implication follows from Proposition 3.2.6.

(i) $\Longrightarrow$ (iii). Suppose that $\mathcal{K}$ is a flag complex on $[m]$ such that $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a one-relator algebra. If $\mathcal{K}^1$ is a chordal graph, then $\mathcal{Z}_\mathcal{K}$ has the homotopy type of a wedge of spheres [GPTW12, Theorem 4.6], and thus $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a graded free associative algebra, which is a contradiction.

Therefore assume that $\mathcal{K}^1$ is not chordal. In particular, there exists a set of vertices $I \subseteq [m]$ such that the full subcomplex $\mathcal{K}_I$ is a p-cycle, and we enumerate $I = \{b_1, b_2, \ldots, b_p\}$. If $I = [m]$, that is $\mathcal{K} = C_p$, then $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a one-relator algebra by Proposition 3.2.6.

Assume that $[m] - I \neq \varnothing$. First, we show that each $j \in [m] - I$ is connected to each vertex in I. Consider the full subcomplex $\mathcal{K}_{I \cup j}$ of $\mathcal{K}$, and observe that by the link-star decomposition (2.12) we have

$$\mathcal{K}_{I\cup j} = \mathcal{K}_I \cup_{\mathrm{lk}_{I\cup j}(j)} \mathrm{st}_{I\cup j}(j).$$

Suppose that $\mathcal{K}_{I\cup j} \neq \mathcal{K}_I * j$. Since $\mathcal{K}$ is flag, there exists $b_l \in I$ such that there is no edge from j to b_l. Form the sequence of adjacent vertices $b_{l+1}, b_{l+2}, \ldots, b_{l+n_1}$, with the convention that $b_{p+1} = b_1$, where $n_1 \geqslant 1$ is the smallest index such that there is an edge from j to b_{l+n_1}. Similarly, form the sequence of adjacent vertices $b_{l-1}, b_{l-2}, \ldots, b_{l-n_2}$, where $n_2 \geqslant 1$ is again the smallest index such that there is an edge from j to b_{l-n_2}. We consider four cases.

(1) Assume that there are no indices n_1 and n_2 as described above. In this case, there are no edges between j and any vertex in I, and so $\mathrm{lk}_{I\cup j}(j) = \varnothing$. Then the map $\mathcal{Z}_{\mathrm{lk}_{I\cup j}(j)} = T^p \longrightarrow \mathcal{Z}_{\mathcal{K}_I}$ is nullhomotopic and therefore $\mathcal{Z}_{\mathcal{K}_{I\cup j}} \simeq \Sigma^2 T^p \vee (\mathcal{Z}_{\mathcal{K}_I} \rtimes S^1)$ by Lemma 3.2.10. Since $\mathcal{Z}_{\mathcal{K}_I}$ is homeomorphic to a connected sum of sphere products, Lemma 3.2.11 gives that $H_*(\Omega(\mathcal{Z}_{\mathcal{K}_I} \rtimes S^1))$ is not a one-relator algebra, and hence neither is $H_*\left(\Omega\mathcal{Z}_{\mathcal{K}_{I\cup j}}\right)$.

(2) If $b_{l+n_1} = b_{l-n_2}$, then $\mathrm{lk}_{I\cup j}(j) = b_{l+n_1}$, and $\mathcal{Z}_{\mathcal{K}_{I\cup j}} \simeq \Sigma^2 T^{p-1} \vee (Z_{\mathcal{K}_I} \rtimes S^1)$ by Lemma 3.2.10. Thus $H_*\left(\Omega\mathcal{Z}_{\mathcal{K}_{I\cup j}}\right)$ is not a one-relator algebra.

(3) When b_{l+n_1} and b_{l-n_2} are adjacent in $\mathcal{K}_I$, the link $\mathrm{lk}_{I\cup j}(j) = \{(b_{l+n_1}, b_{l-n_2})\}$, and so $\mathcal{Z}_{\mathcal{K}_{I\cup j}} \simeq \Sigma^2 T^{p-2} \vee (Z_{\mathcal{K}_I} \rtimes S^1)$ by Lemma 3.2.10. Thus $H_*\left(\Omega\mathcal{Z}_{\mathcal{K}_{I\cup j}}\right)$ is not a one-relator algebra.

(4) Finally, let b_{l+n_1} and b_{l-n_2} be distinct and not adjacent in $\mathcal{K}_I$. Then by construction the full subcomplex $\mathcal{K}_{\{j, b_{l-n_2}, \ldots, b_{l-1}, b_l, b_{l+1}, \ldots, b_{l+n_1}\}}$ of $\mathcal{K}$ is a (n_1+n_2+2)-cycle, which is distinct from $\mathcal{K}_I$. Therefore by Lemma 3.2.9, $H_*\left(\Omega\mathcal{Z}_{\mathcal{K}_{I\cup j}}\right)$ is not a one-relator algebra.

In all of the above cases, since the full subcomplex $\mathcal{K}_{I\cup j}$ retracts off $\mathcal{K}$ and $H_*\left(\Omega\mathcal{Z}_{\mathcal{K}_{I\cup j}}\right)$ is not a one-relator algebra, then neither is $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$. This is a contradiction. We therefore conclude that j is connected to each vertex in $\mathcal{K}_I$ and therefore $\mathcal{K}_{I\cup j} = \mathcal{K}_I * j$.

Second, we show that if $j_1, j_2 \in [m] - I$, then j_1 and j_2 are connected by an edge. If not, since both j_1 and j_2 are connected to each vertex in I, then the full subcomplex $K_{\{j_1, b_{i_1}, j_2, b_{i_3}\}}$ is a 4-cycle distinct from the p-cycle $\mathcal{K}_I$. Therefore, Lemma 3.2.9 implies that since $H_*\left(\Omega\mathcal{Z}_{I\cup\{j_1,j_2\}}\right)$ is not a one-relator algebra, neither is $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$, which is a contradiction.

Therefore any vertex in $[m] - I$ is connected to every vertex in I and to every other vertex in $[m] - I$. Since $\mathcal{K}$ is flag, $\mathcal{K} = \mathcal{K}_I * \Delta^q$ for some $q \geqslant 0$.

(iii) $\Longrightarrow$ (ii). Suppose that $\mathcal{K} = C_p$ or $\mathcal{K} = C_p * \Delta^q$ for $p \geqslant 4$ and $q \geqslant 0$. Let the p-cycle C_p of $\mathcal{K}$ be supported on the set of vertices $I = \{b_1, \ldots, b_p\}$. By the Hochster decomposition (3.10) we have

$$H_{2-j,2j}(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subseteq[m],\,|J|=j} \widetilde{H}_1(\mathcal{K}_J). \tag{3.11}$$

Since $\mathcal{K}_I$ is a p-cycle, we have $\widetilde{H}_1(\mathcal{K}_I) = \mathbb{Z}$. Because any subcomplex $\mathcal{K}_J$ with $J \neq I$ is contractible, $\widetilde{H}_1(\mathcal{K}_J) = 0$ for $J \neq I$. It follows that $H_{2-p,2p}(\mathcal{Z}_\mathcal{K}) = \mathbb{Z}$ and $H_{2-j,2j}(\mathcal{Z}_\mathcal{K}) = 0$ for $j \neq p$.

(ii) $\Longrightarrow$ (iii). Suppose that $H_{2-j,2j}(\mathcal{Z}_\mathcal{K})$ is as described in (ii). Then only one summand on the right hand side of (3.11) is $\mathbb{Z}$, and all other summands are zero. The same argument as in the proof of implication (ii) $\Rightarrow$ (iii) of Theorem 3.2.4 shows that $\mathcal{K} = C_p * \Delta^q$ for some $p \geqslant 4$ and $q \geqslant 0$.

It remains to prove that if $\mathcal{K} = C_p$ or $\mathcal{K} = C_p * \Delta^q$ for $q \geqslant 0$ then $H_{-i,2j}(\mathcal{Z}_\mathcal{K}) = 0$ for $j - i \geqslant 3$. Considering the bigraded Hochster decomposition (3.10), we claim that all summands on the right hand side with $j - i \geqslant 3$ disappear. Indeed, let $I = \{b_1, \ldots, b_p\}$ be the set of vertices of $\mathcal{K}$ forming a p-cycle, $p \geqslant 4$. Then $\widetilde{H}_{j-i-1}(\mathcal{K}_I) = 0$ for $j - i \geqslant 3$. Since any full subcomplex $\mathcal{K}_J$ with $J \neq I$ is contractible, we get $\widetilde{H}_{j-i-1}(\mathcal{K}_J) = 0$. Hence, $H_{-i,2j}(\mathcal{Z}_\mathcal{K}) = 0$ for $j - i \geqslant 3$. □

Example 3.2.12. We consider the same examples as in Example 3.2.5.

(i) Let $\mathcal{K}$ be the flag complex in Figure 3.1(a). Generator set (2.28) for $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is

$$[u_3, u_1],\ [u_4, u_2],\ [u_5, u_4],\ [u_2, [u_5, u_4]].$$

These satisfy the relations

$$[u_3, u_1][u_4, u_2] - [u_4, u_2][u_3, u_1] = 0, \quad [u_3, u_1][u_5, u_4] - [u_5, u_4][u_3, u_1] = 0$$

and

$$[u_3, u_1][u_2, [u_5, u_4]] - [u_2, [u_5, u_4]][u_3, u_1] = 0$$

which are derived by using the commutativity relations given in (2.27). By formula (3.10) we obtain $H_{-2,8}(\mathcal{Z}_\mathcal{K}) = \mathbb{Z}^2$ and $H_{-3,10}(\mathcal{Z}_\mathcal{K}) = \mathbb{Z}$. Hence, the homological condition of Theorem 3.2.8(ii) is not satisfied.

(ii) Let $\mathcal{K}$ be the flag complex in Figure 3.1(b). Generator set (2.28) for $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is

$$[u_3, u_1],\ [u_4, u_2]$$

with a single relation $[u_3, u_1][u_4, u_2] - [u_3, u_1][u_4, u_2] = 0$. Here $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a one-relator algebra, and formula (3.10) gives $H_{-2,8}(\mathcal{Z}_\mathcal{K}) = \mathbb{Z}$.

3.2.5 Golod and minimally non-Golod flag complexes

We conclude this section by giving further topological and combinatorial equivalences to $H_*(\Omega\mathcal{Z}_\mathcal{K})$ being a free associative algebra or a one-relator algebra. In the next section we will investigate the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ in more detail, using these results as a starting point.

In the case that $\mathcal{K}$ is flag and $\underline{k}$ is a field, it was shown in [GPTW12] that $H_*(\Omega\mathcal{Z}_\mathcal{K};\underline{k})$ is a free associative algebra if and only if $\mathcal{K}$ is Golod over $\underline{k}$, that is, the multiplication and all higher Massey products are trivial in $H^*(\mathcal{Z}_\mathcal{K};\underline{k})$, see Theorem 2.6.16. We first establish this equivalence when $\underline{k} = \mathbb{Z}$.

Theorem 3.2.13. *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$. Then the following are equivalent:*

(i) $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a free associative algebra;

(ii) $\mathcal{Z}_\mathcal{K}$ is homotopy equivalent to a wedge of spheres;

(iii) $\mathcal{K}$ is Golod.

Proof. Since $\mathcal{K}$ is flag, by Theorem 2.6.16 it is Golod if and only if the 1-skeleton $\mathcal{K}^1$ of $\mathcal{K}$ is a chordal graph. Therefore if $\mathcal{K}$ is not Golod, then there is a p-cycle C_p contained in $\mathcal{K}^1$ for some $p \geqslant 4$. Then the inclusion of the full subcomplex C_p into $\mathcal{K}$ induces an inclusion of algebras $H_*(\Omega\mathcal{Z}_{C_p}) \longrightarrow H_*(\Omega\mathcal{Z}_\mathcal{K})$. Therefore $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is not free by Theorem 3.2.8. □

It follows that when $\mathcal{K}$ is flag, the property of $H_*(\Omega\mathcal{Z}_\mathcal{K})$ being free can be analysed by looking at the cohomology ring and Massey products in $\mathcal{Z}_\mathcal{K}$.

An analogous result is true for identifying when $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a one-relator algebra. A simplicial complex is *minimally non-Golod* if it is not Golod, but $K - v$ is Golod for any vertex $v \in \mathcal{K}$. For example, if $\mathcal{K} = C_p$ for $p \geqslant 4$, then $\mathcal{K}$ is not Golod because $\mathcal{Z}_\mathcal{K}$ is a connected sum of sphere products, which has a non-trivial cup product, but removing any vertex gives a Golod complex since $\mathcal{K} - v$ is flag and chordal. The property of $\mathcal{K}$ being minimally non-Golod was introduced by Berglund and Jöllenbeck [BJ07] as a property of the algebra $\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}],\mathbb{Z})$, and studied by Limonchenko [Lim15] in the context of moment-angle complexes.

Theorem 3.2.14. *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$. Then the following are equivalent:*

(i) $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a one-relator algebra;

(ii) $\mathcal{Z}_{\mathcal{K}}$ *is homotopy equivalent to a connected sum of sphere products, with two spheres in each product;*

(iii) Either $\mathcal{K}$ *is minimally non-Golod, or* $\mathcal{K}$ *is the join of a minimally non-Golod complex and a simplex.*

Proof. (ii) $\implies$ (i). This is Proposition 3.2.6.

(i) $\implies$ (iii). By Theorem 3.2.8, if $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is a one-relator algebra, then either $\mathcal{K} = C_p$ or $\mathcal{K} = C_p * \Delta^q$ for some $p \geqslant 4$ and $q \geqslant 0$. Since the complex C_p is minimally non-Golod, the implication follows.

(iii) $\implies$ (ii). By [GPTW12, Theorem 4.8], any flag simplicial complex $\mathcal{K}$ which is minimally non-Golod is equal to C_p for some $p \geqslant 4$. Therefore $\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to the connected sum of sphere products (3.2). □

3.3 Loop homology in the non-flag case

Theorems 3.2.13 and 3.2.14 establish combinatorial and topological equivalences for the conditions for the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ to be a free associative algebra or a one-relator algebra in the case that $\mathcal{K}$ is a flag complex. For general simplicial complexes, the corresponding moment-angle complexes have considerably deeper and more complex homotopy theory. Little is known about what information the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ sees in general. This section is a preliminary investigation into this question, with a focus on examples. We begin by summarising some known results.

3.3.1 Free and one-relator loop homology algebras

For a general simplicial complex $\mathcal{K}$, if $\mathcal{Z}_{\mathcal{K}}$ has the homotopy type of a wedge of spheres then both $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is a free algebra, by the Bott–Samelson Theorem, and $\mathcal{K}$ is Golod. In general, however, these properties are no longer equivalent. The following example was presented in [GPTW12].

Example 3.3.1. Let $\mathcal{K}$ be the 6-vertex triangulation of $\mathbb{R}P^2$ shown in Figure 3.2. It was shown in [GPTW12] that there is a homotopy equivalence

$$\mathcal{Z}_{\mathcal{K}} \simeq (S^5)^{\vee 10} \vee (S^6)^{\vee 15} \vee (S^7)^{\vee 6} \vee \Sigma^7\mathbb{R}P^2.$$

In particular, $\mathcal{Z}_{\mathcal{K}}$ is not homotopy equivalent to a wedge of spheres. On the other hand, all multiplication and higher Massey products vanish in $H^*(\mathcal{Z}_{\mathcal{K}})$ for dimensional reasons, so $\mathcal{K}$ is Golod. Therefore in general, the class of Golod complexes $\mathcal{K}$ is larger than the class for which $\mathcal{Z}_{\mathcal{K}}$ is a wedge of spheres.

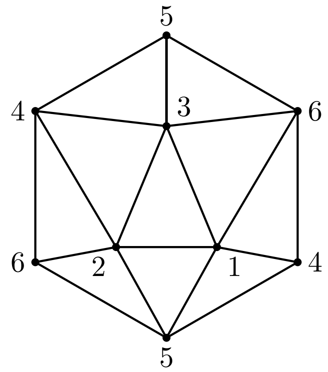


Figure 3.2: A 6-vertex Golod triangulation of $\mathbb{R}P^2$.

Likewise, there are complexes $\mathcal{K}$ for which $\mathcal{Z}_\mathcal{K}$ is not a wedge of spheres, but $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is free. An example simplicial complex $\mathcal{K}$ was given by Limonchenko [Lim15] who gave a 9-vertex triangulation of $\mathbb{C}P^2$ for which

$$\mathcal{Z}_\mathcal{K} \simeq (S^7)^{\vee 36} \vee (S^8)^{\vee 90} \vee (S^9)^{\vee 84} \vee (S^{10})^{\vee 36} \vee (S^{11})^{\vee 9} \vee \Sigma^{10}\mathbb{C}P^2.$$

Since $\mathcal{Z}_\mathcal{K}$ is a suspension space and $H_*(\mathcal{Z}_\mathcal{K})$ is torsion-free, it follows that $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a free associative algebra by the Bott–Samelson Theorem.

On the other hand, the relationship between $\mathcal{K}$ being Golod and $H_*(\Omega\mathcal{Z}_\mathcal{K})$ being free is much closer.

Theorem 3.3.2 ([LP22])**.** *When $\underline{k}$ is a field, the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\underline{k})$ is free if and only if $\mathcal{K}$ is Golod over $\underline{k}$.* □

When $\underline{k} = \mathbb{Z}$, this equivalence is not true due to the presence of torsion in $H_*(\Omega\mathcal{Z}_\mathcal{K})$. Using an argument similar to Example 2.3.4(iii), the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ for $\mathcal{K}$ the triangulation of $\mathbb{R}P^2$ in Example 3.3.1 is not free over $\mathbb{Z}$.

While Theorem 3.3.2 implies that when $\mathcal{K}$ is not Golod, the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is not free, little is known about what algebraic relations appear beyond those specified in Proposition 3.2.6 and Lemma 3.2.11. In Example 3.3.6 we give a modification of an example in [LP22] which gives a simplicial complex $\mathcal{K}$ such that $H^*(\mathcal{Z}_\mathcal{K})$ has trivial multiplication, but contains a non-trivial triple Massey product. We show that the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ has at least two algebraic relations.

We perform the equivalent analysis of the properties of $\mathcal{K}$ being minimally non-Golod, $\mathcal{Z}_\mathcal{K}$ being a connected sum of sphere products and $H_*(\Omega\mathcal{Z}_\mathcal{K})$ being a one-relator algebra in the non-flag case. For any moment-angle complex homotopy equivalent to a connected sum of sphere products, the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is one-relator by Proposition 3.2.6. Moreover, Amelotte [Ame20], showed that $\mathcal{K}$ is a minimally non-Golod complex, or such a complex joined with a simplex, affirmatively answering a question in [GPTW12, Question 3.5].

On the other hand, Limonchenko [Lim15] constructs minimally non-Golod complexes which do not have the homotopy type of a connected sum of sphere products. Moreover, in Example 3.3.5 we construct a simplicial complex $\mathcal{K}$ for which $H_*(\Omega\mathcal{Z}_\mathcal{K})$ is a one-relator algebra, but $\mathcal{Z}_\mathcal{K}$ is not a connected sum of sphere products.

The relationship between $\mathcal{K}$ being minimally non-Golod and $H_*(\Omega\mathcal{Z}_\mathcal{K})$ being one-relator is further from the relationship between $\mathcal{K}$ being Golod and $H_*(\Omega\mathcal{Z}_\mathcal{K})$ being free. The example constructed in Example 3.3.5 is a complex that is not of the form a minimally non-Golod complex joined with a simplex. Nevertheless it does contain a minimally non-Golod complex as a full subcomplex. The complex $\mathcal{K}$ constructed in Example 3.3.6 is minimally non-Golod, but $H_*(\Omega\mathcal{Z}_\mathcal{K})$ has at least two relations.

3.3.2 A chain complex for $\Omega\mathcal{Z}_\mathcal{K}$

Known examples of loop homology algebras $H_*(\Omega\mathcal{Z}_\mathcal{K})$ have mainly been calculated in two ways, either by knowing the homotopy type and applying results such as the Bott–Samelson Theorem or the Cobar construction, as we did in Proposition 3.2.6; or in the case of [GPTW12] by algebraically identifying $H_*(\Omega\mathcal{Z}_\mathcal{K})$ as the subalgebra of $H_*(\Omega DJ_\mathcal{K})$ generated by iterated commutators. In general, however, the algebraic structure of $H_*(\Omega DJ_\mathcal{K})$ does not determine the algebraic structure of $H_*(\Omega\mathcal{Z}_\mathcal{K})$, see Section 2.6.6, so this second method does not readily generalise.

Since determining the homotopy type of $\mathcal{Z}_\mathcal{K}$ is a hard question, we seek a more direct approach. We recall the homological Taylor complex for the face coalgebra from Construction 2.6.6 computing

$$H_*(\mathcal{Z}_\mathcal{K}) \cong \mathrm{Cotor}^{\mathbb{Z}\langle m\rangle}(\mathbb{Z}\langle\mathcal{K}\rangle, \mathbb{Z})$$

which we state again for completeness.

For $s > 0$, the graded component I_{-s} is the free $\mathbb{Z}$-module with basis exterior monomials $w_{J_1} \wedge \cdots \wedge w_{J_s}$, where $J_1, \ldots, J_s$ are distinct minimal missing faces of $\mathcal{K}$. The differential is given by

$$\partial(w_{J_1} \wedge \cdots \wedge w_{J_s}) = \sum_{J \subset J_1 \cup \cdots \cup J_s} w_J \wedge w_{J_1} \wedge \cdots \wedge w_{J_s}$$

where the summation is taken over missing faces $J \subset J_1 \cup \cdots \cup J_s$ different from any of the $J_1, \ldots, J_s$.

We give the resolution (I_*, d) the structure of a differential bigraded coalgebra by setting $\mathrm{bideg}(w_{J_1} \wedge \cdots \wedge w_{J_s}) = (-s, 2|J_1 \cup \cdots \cup J_s|)$ and defining comultiplication using formula (2.7). Specifically, we set

$$\Delta(w_{J_1} \wedge \cdots \wedge w_{J_s}) = \sum (-1)^{\mathrm{sign}\,\sigma} \left(w_{J_{i_1}} \wedge \cdots \wedge w_{J_{i_k}}\right) \otimes \left(w_{J_{i_{k+1}}} \wedge \cdots \wedge w_{J_{i_s}}\right)$$

where the summation is taken over all partitions $\{J_1, \ldots, J_s\} = \{J_{i_1}, \ldots, J_{i_k}\} \sqcup \{J_{i_{k+1}}, \ldots, J_{i_s}\}$, with one side potentially empty, such that $(J_{i_1} \cup \cdots \cup J_{i_k}) \cap (J_{i_{k+1}} \cup \cdots \cup J_{i_s}) = \varnothing$ and both $i_1 < \cdots < i_k$ and $i_{k+1} < \cdots < i_s$. The permutation σ is given by sending j to i_j for each $j = 1, \ldots, s$. The reduced comultiplication $\overline{\Delta}$ is given by partitions with $0 < k \leqslant s$.

Example 3.3.3. Let $\mathcal{K} = C_4$ be a square. Then I_{-1} is generated by ω_{13} and ω_{24}, and I_{-2} is generated by $\omega_{13}\omega_{24}$. The reduced comultiplication is given by $\overline{\Delta}\omega_{13} = \overline{\Delta}\omega_{24} = 0$ and

$$\overline{\Delta}(\omega_{13}\omega_{24}) = \omega_{13} \otimes \omega_{24} - \omega_{24} \otimes \omega_{13}.$$

We apply the Cobar construction of Section 2.3.2 to the Taylor complex to obtain a chain complex for the space $\Omega\mathcal{Z}_{\mathcal{K}}$.

Proposition 3.3.4. *Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let (I_*, ∂, Δ) be the homological Taylor complex associated to $\mathcal{K}$. Then there is an isomorphism of graded algebras*

$$H_*(\Omega\mathcal{Z}_{\mathcal{K}}) \cong H(\mathrm{Cobar}_*(I))$$

where $(\mathrm{Cobar}_*(I), d)$ *is a dg-algebra with* $\mathrm{Cobar}_*(I)$ *the free associative algebra on the desuspended module I and d is given by formula* (2.11).

Proof. Since I_* is a simply-connected dg-coalgebra quasi-isomorphic to a cellular chain complex for $\mathcal{Z}_{\mathcal{K}}$, this follows from Theorem 2.3.3(ii). ☐

We demonstrate this construction on a series of examples. We first construct a simplicial complex $\mathcal{K}$ which is not of the form of a minimally non-Golod complex joined with a simplex, but for which $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is a one-relator algebra.

Example 3.3.5. Let $\mathcal{K}$ be the simplicial complex formed by taking the cone over the square C_4 with cone vertex (5), and then removing the face $(2, 4, 5)$. The complex $\mathcal{K}$ is shown in Figure 3.3.

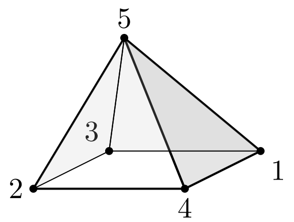


Figure 3.3: A simplicial complex K for which $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra.

The minimal missing faces are $(1,2)$, $(3,4)$ and $(2,4,5)$. We label the corresponding generators of the Taylor resolution by w_{12}, w_{34} and w_{245}, respectively. We have

$$\begin{aligned}
I_{-1} &= \langle w_{12}, w_{34}, w_{245} \rangle \\
I_{-2} &= \langle w_{12}w_{34}, w_{12}w_{245}, w_{34}w_{245} \rangle \\
I_{-3} &= \langle w_{12}w_{34}w_{245} \rangle.
\end{aligned}$$

All differentials are trivial and reduced comultiplication is trivial except

$$\overline{\Delta}(w_{12}w_{34}) = w_{12} \otimes w_{34} - w_{34} \otimes w_{12}.$$

Define generators with $\deg u_i = 2$, $\deg v = 4$, $\deg x = 5$, $\deg y_i = 5$, $\deg w = 6$ as

$$\begin{aligned}
u_1 &= s^{-1}w_{12}, & u_2 &= s^{-1}w_{34}, \\
v &= s^{-1}w_{245}, & x &= s^{-1}w_{12}w_{34}, \\
y_1 &= s^{-1}w_{12}w_{245}, & y_2 &= s^{-1}w_{34}w_{245}, \\
w &= s^{-1}w_{12}w_{34}w_{245}.
\end{aligned}$$

All differentials in the Cobar construction are zero except $d(x) = -u_1u_2 + u_2u_1$. It follows that

$$H_*(\Omega\mathcal{Z}_K) \cong H(\mathrm{Cobar}_*(I)) = \frac{T(u_1,u_2)}{\langle [u_1,u_2] \rangle} \otimes T(v, y_1, y_2, w)$$

is a one-relator algebra. Using the Hochster decomposition (2.25) we observe that the cohomology ring $H^*(\mathcal{Z}_K)$ is isomorphic to that of

$$H^*\left((S^3 \times S^3) \vee S^5 \vee S^6 \vee S^6 \vee S^7\right)$$

and therefore $\mathcal{Z}_{\mathcal{K}}$ does not have the homotopy type of a connected sum of sphere products. In this case, the fat wedge filtration of Iriye and Kishimoto [IK19] can be used to show the above cohomology isomorphism is due to a homotopy equivalence.

We now analyse a simplicial complex which is minimally non-Golod, and prove that $H_*(\Omega\mathcal{Z}_K)$ is neither free nor one-relator. This complex is a modification of one presented in [LP22], adjusted to make it easier to analyse using the Taylor resolution. These examples are realisations in moment-angle complexes of one presented by Katthän [Kat17] as a counterexample to the long-standing assertion of Berglund and Jöllenbeck [BJ07] that the triviality of the multiplication in $\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$ implied that all higher Massey products are trivial.

Example 3.3.6. Let $\mathcal{K}$ be the simplicial complex formed from the join $\partial\Delta[1,2,3] * \partial\Delta[4,5,6] * \partial\Delta[7,8,9]$ by removing the simplices $(1,4,7)$, $(2,3,5,6)$, $(1,2,7,8)$, $(4,5,8,9)$ and $(2,3,5,8,9)$.

The reduced cohomology groups and Hochster decomposition (2.25) of $\mathcal{Z}_K$ are given as follows.

n	$\widetilde{H}^n(\mathcal{Z}_K)$	J for which $\widetilde{H}(K_J)$ is a summand of $\widetilde{H}(\mathcal{Z}_K)$
$\leqslant 4$	0	
5	$\mathbb{Z}^4$	$\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\}$
6	0	
7	$\mathbb{Z}^3$	$\{2,3,5,6\},\{1,2,7,8\},\{4,5,8,9\}$
8	$\mathbb{Z}^{10}$	
9	$\mathbb{Z}^3$	$\{2,3,5,8,9\},\{1,2,4,7,8,9\},\{1,2,3,4,7,8\}$
10	$\mathbb{Z}^4$	$\{2,3,5,7,8,9\},\{2,3,4,5,8,9\},\{1,2,3,5,8,9\},\{2,3,5,6,8,9\}$
11	$\mathbb{Z}^6$	
12	0	
13	0	
14	$\mathbb{Z}$	$\{1,2,3,4,5,6,7,8,9\}$
$\geqslant 15$	0	

The entries left blank for $n = 8, 11$ are not relevant for what follows. We first show that all cup products in $H^*(\mathcal{Z}_\mathcal{K})$ vanish. For dimensional reasons, the only potential non-trivial cup products are of degree m and n classes for $(m, n) \in \{(5, 5), (5, 9), (7, 7)\}$. Using the vertex supports in the third column, we calculate that all these cup products vanish too. For example the product of the degree 5 class with support $\{1, 4, 7\}$ and the degree 9 class with support $\{2, 3, 5, 8, 9\}$ is trivial since the vertex 6 does not appear, whereas it does in the support of the degree 14 class. A similar calculation works for all other permutations of potential cup products.

We now show that there is a non-trivial triple Massey product in $H^*(\mathcal{Z}_\mathcal{K})$. Following [May69], if $\alpha_i \in H^{p_i}(X)$ are the cohomology classes of $a_i \in C^{p_i}(X)$ such that the cup products $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ are trivial, then there are cochains $a_{12} \in C^{p_1+p_2-1}(X)$ and $a_{23} \in C^{p_2+p_3-1}(X)$ such that $da_{12} = a_1a_2$ and $da_{23} = a_2a_3$. The triple Massey product $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is the set of cohomology classes of the cocycles $a_1a_{23} - a_{12}a_3$ for all such choices of a_{12} and a_{23}. The triple Massey product is non-trivial if it does not contain 0.

We now show that the triple Massey product $\langle\alpha_1, \alpha_2, \alpha_3\rangle$, where $\alpha_1, \alpha_2, \alpha_3$ correspond to the degree 5 classes with supports $\{4, 5, 6\}, \{1, 2, 3\}, \{7, 8, 9\}$, respectively, is defined and non-trivial. It is defined since the cup products $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ are trivial by the above.

To see it is non-trivial, we use the Taylor complex, see Construction 2.6.4. The generators of the Taylor complex correspond to the minimal missing faces of $\mathcal{K}$ and are given by σ_{123}, σ_{456}, σ_{789}, σ_{2356}, σ_{1278}, σ_{4589}, σ_{147} and σ_{23589}.

The Massey product $\langle\sigma_{456}, \sigma_{123}, \sigma_{789}\rangle$ is computed as follows. Let $a_1 = \sigma_{456}$, $a_2 = \sigma_{123}$ and $a_3 = \sigma_{789}$. Let $a_{12} = -\sigma_{123}\sigma_{456}\sigma_{2356}$ and $a_{23} = \sigma_{123}\sigma_{789}\sigma_{1278}$. Then $d(a_{12}) =$

$-\sigma_{123}\sigma_{456} = a_1a_2$ and $d(a_{23}) = \sigma_{123}\sigma_{789} = a_2a_3$. Note that a_{12} and a_{23} are the only cochains whose differential satisfies this property, since the 9th total degree of the Taylor resolution is generated by

$$\{\sigma_{123}\sigma_{456}\sigma_{2356}, \sigma_{123}\sigma_{789}\sigma_{1278}, \sigma_{456}\sigma_{789}\sigma_{4589}, \sigma_{123}\sigma_{1278}\sigma_{147}, \\ \sigma_{789}\sigma_{1278}\sigma_{147}, \sigma_{789}\sigma_{4589}\sigma_{147}, \sigma_{23589}\}.$$

Then $\langle\sigma_{456}, \sigma_{123}, \sigma_{789}\rangle$ is the singleton set containing

$$\sigma_{456}\cdot\sigma_{123}\sigma_{789}\sigma_{1278} - \sigma_{123}\sigma_{456}\sigma_{2356}\cdot\sigma_{789} = -\sigma_{123}\sigma_{456}\sigma_{789}\sigma_{1278} + \sigma_{123}\sigma_{456}\sigma_{789}\sigma_{2356}.$$

Subtracting from the boundary

$$d(\sigma_{123}\sigma_{456}\sigma_{789}\sigma_{2356}\sigma_{1278}) = \sigma_{456}\sigma_{789}\sigma_{2356}\sigma_{1278} - \sigma_{123}\sigma_{456}\sigma_{789}\sigma_{1278} + \sigma_{123}\sigma_{456}\sigma_{789}\sigma_{2356}$$

gives the cochain $\chi = \sigma_{456}\sigma_{789}\sigma_{2356}\sigma_{1278}$. Since $d\chi = 0$, it remains to show it is not a coboundary. Suppose it were, then there are coefficients such that

$$\begin{aligned}\chi &= d(\sigma_{456}\sigma_{789}\sigma_{2356}\sigma_{1278} \wedge (c_1\sigma_{123} + c_2\sigma_{147} + c_3\sigma_{4589} + c_4\sigma_{23589}))\\ &= c_1(\sigma_{456}\sigma_{789}\sigma_{1278}\sigma_{123} - \sigma_{456}\sigma_{789}\sigma_{2356}\sigma_{123} + \chi)\\ &\quad + c_2(\sigma_{789}\sigma_{2356}\sigma_{1278}\sigma_{147} - \sigma_{456}\sigma_{789}\sigma_{2356}\sigma_{147} + \chi)\\ &\quad + c_3(\sigma_{789}\sigma_{2356}\sigma_{1278}\sigma_{4589} - \sigma_{456}\sigma_{2356}\sigma_{1278}\sigma_{4589} + \chi)\\ &\quad + c_4(\sigma_{456}\sigma_{2356}\sigma_{1278}\sigma_{23589} - \sigma_{456}\sigma_{789}\sigma_{1278}\sigma_{23589} + \chi)\end{aligned}$$

for which there are no solutions. Therefore χ is a generator of $H^{14}(\mathcal{Z}_K)$. Therefore the Massey product $\langle\sigma_{456}, \sigma_{123}, \sigma_{789}\rangle$ is non-trivial.

It therefore follows that $\mathcal{K}$ fails to be Golod due to the existence of a non-trivial triple Massey product in $H^*(\mathcal{Z}_\mathcal{K})$, and is therefore minimally non-Golod. We now show there are at least two relations in the loop homology algebra $H_*(\Omega\mathcal{Z}_K)$.

Applying Proposition 3.3.4 for the homological Taylor complex gives a chain complex for $\Omega\mathcal{Z}_\mathcal{K}$ consisting of a free dg-algebra on 255 generators. It is therefore difficult to give a full computation, but nonetheless we are able to deduce some relations. For missing faces $I_1, \ldots, I_k$ of $\mathcal{K}$ we denote by $\tau_{I_1}\cdots\tau_{I_k} = s^{-1}\omega_{I_1}\cdots\omega_{I_k}$, which has degree $2|I_1 \cup \cdots \cup I_k| - s - 1$.

Consider the chain $\tau_{123}\tau_{456}\tau_{789}\tau_{2356}$ of degree 13. We have

$$\begin{aligned}d(\tau_{123}\tau_{456}\tau_{789}\tau_{2356}) = &-(\tau_{123}\tau_{456}\tau_{789}\tau_{2356}\tau_{1278} + \tau_{123}\tau_{456}\tau_{789}\tau_{2356}\tau_{4589}\\ &+ \tau_{123}\tau_{456}\tau_{789}\tau_{2356}\tau_{147} + \tau_{123}\tau_{456}\tau_{789}\tau_{2356}\tau_{23589})\\ &- (\tau_{789} \otimes \tau_{123}\tau_{456}\tau_{2356} - \tau_{123}\tau_{456}\tau_{2356} \otimes \tau_{789}).\end{aligned}$$

We first observe that the sum of the first four terms is trivial in homology. Indeed, a direct calculation shows that this sum is given by the boundary

$$\begin{aligned}d(&\tau_{456}\tau_{789}\tau_{1278}\tau_{2356}+\tau_{123}\tau_{789}\tau_{2356}\tau_{4589}+\tau_{123}\tau_{456}\tau_{789}\tau_{147}+\tau_{123}\tau_{456}\tau_{789}\tau_{23589}\\&-\tau_{123}\tau_{789}\tau_{4589}\tau_{23589}-\tau_{456}\tau_{789}\tau_{1278}\tau_{23589}+\tau_{123}\tau_{789}\tau_{147}\tau_{4589}+\tau_{456}\tau_{789}\tau_{147}\tau_{1278}\\&-\tau_{789}\tau_{1278}\tau_{2356}\tau_{4589}+\tau_{789}\tau_{1278}\tau_{4589}\tau_{23589}-\tau_{789}\tau_{147}\tau_{1278}\tau_{4589}).\end{aligned}$$

Therefore in homology we have that $[\tau_{789},\tau_{123}\tau_{456}\tau_{2356}]=0$. Next, we calculate that

$$d(\tau_{123}\tau_{456})=-\tau_{123}\tau_{456}\tau_{2356}-(\tau_{123}\otimes\tau_{456}-\tau_{456}\otimes\tau_{123}).$$

so that in homology we have $\tau_{123}\tau_{456}\tau_{2356}=-[\tau_{123},\tau_{456}]$. Since abc is a generator of $H_{12}(\Omega\mathcal{Z}_{\mathcal{K}})$ for $a,b,c\in\{\tau_{123},\tau_{456},\tau_{789}\}$, the ideal generated by the iterated commutator

$$[\tau_{789},[\tau_{123},\tau_{456}]]=-[\tau_{789},\tau_{123}\tau_{456}\tau_{2356}]$$

is trivial in $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$. Similarly there is a relation $[\tau_{123},[\tau_{456},\tau_{789}]]=0$. The symmetric relation $[\tau_{456},[\tau_{123},\tau_{789}]]=0$ is also true, but is accounted for algebraically by the Jacobi identity.

Example 3.3.6 demonstrates that in the non-flag case, the class of those complexes $\mathcal{K}$ which are minimally non-Golod is bigger than those for which $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ is a one-relator algebra. Examples providing minimally non-Golod complexes $\mathcal{K}$ for which $\mathcal{Z}_{\mathcal{K}}$ is not a connected sum of sphere products have already been provided by Limonchenko [Lim15]. This can also be seen from Example 3.3.6 since the cohomology groups of $\mathcal{Z}_{\mathcal{K}}$ do not have Poincaré duality, and so $\mathcal{Z}_{\mathcal{K}}$ does not have the homotopy type of a manifold. Therefore the equivalences of Theorem 3.2.14 do not hold in general.

Remark 3.3.7. The appearance of iterated commutators as relations in loop homology is reflective of topological structure that is seen by a triple Massey product. Let $X=S^p\vee S^q\vee S^r$ be a wedge of simply-connected spheres and let M be the space obtained by attaching a $(p+q+r-1)$-cell to X via the Whitehead product $[\alpha,[\beta,\gamma]]$, where $\alpha\in\pi_p(S^p)$, $\beta\in\pi_q(S^q)$ and $\gamma\in\pi_r(S^r)$ are identity maps. Let u, v and w be the cocycles corresponding to the cells S^p, S^q and S^r of M, respectively. Then by [UM57, Lemma 7] the triple Massey products $\langle u,v,w\rangle$ and $\langle v,w,u\rangle$ in $H^*(M)$ are singleton sets containing non-trivial elements, identifying, up to sign, the cocycle of the attached cell. On the other hand, the image in $H_*(\Omega X)$ of $[\alpha,[\beta,\gamma]]$ under the composite θ of the isomorphism $\pi_{*+1}(X)\longrightarrow\pi_*(\Omega X)$ and the Hurewicz map $\pi_*(\Omega X)\longrightarrow H_*(\Omega X)$ is, up to sign, the iterated commutator $[\theta\alpha,[\theta\beta,\theta\gamma]]$, which is trivialised under the map $H_*(\Omega X)\longrightarrow H_*(\Omega M)$. Therefore in this case the non-triviality of a triple Massey product in $H^*(M)$ corresponds to the triviality of an iterated commutator in $H_*(\Omega M)$. It is not known if the relations in $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ obtained in Example 3.3.6 are induced topologically in this way.

3.4 Moment-angle manifolds

A moment-angle complex which is also a manifold is called a moment-angle manifold. The study of moment-angle manifolds originated from the perspective of symplectic geometry and is one of the oldest forms of toric topology. By [BP15, Theorem 4.1.4], if $\mathcal{K}$ is a simplicial sphere, then $\mathcal{Z}_{\mathcal{K}}$ is a manifold. More generally, moment-angle complexes which are manifolds have a complete combinatorial characterisation, as follows.

Let $\mathcal{K}$ be a simplicial complex on $[m]$. We call $\mathcal{K}$ a *generalised homology* $(n-1)$*-sphere*, or a GHS^{n-1}, if $\mathcal{K}$ has the same homology as S^{n-1}, and for every non-empty $\sigma \in \mathcal{K}$, the complex $\mathcal{K} - \sigma$ is acyclic. By a result of Cai [Cai17], the moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is an $(m+n)$-manifold if and only if $\mathcal{K}$ is a GHS^{n-1}. In general, generalised homology spheres need not be simplicial spheres. In Example 3.4.10 we analyse a triangulation of the Poincaré sphere, which is a GHS^3 but not a simplicial sphere and determine the associated moment-angle manifold.

We study moment-angle manifolds which are connected sums of sphere products, of which there are numerous infinite families. McGavran [McG79] showed that if $\mathcal{K}$ is obtained from the boundary of a simplex by a positive number of iterated stellar subdivisions at maximal faces, then $\mathcal{Z}_{\mathcal{K}}$ is homeomorphic to a connected sum of sphere products. Gitler and López de Medrano [GdM13] improve this result by showing that iterated stellar subdivisions at both maximal and codimension 1 simplices on a complex $\mathcal{K}$ preserves $\mathcal{Z}_{\mathcal{K}}$ being a connected sum of sphere products, see [GdM13, Theorem 2.4].

A simplicial complex $\mathcal{K}$ on $[m]$ of dimension $2d$ is *neighbourly* if the $(d-1)$-skeleton of $\mathcal{K}$ is complete. In [GdM13, Theorem 1.3], Gitler and López de Medrano prove a conjecture of Bosio and Meersseman [BM04] that every even-dimensional neighbourly simplicial sphere has $\mathcal{Z}_{\mathcal{K}}$ homeomorphic to a connected sum of sphere products. They also prove a result for simplicial spheres of odd dimension, which we summarise as follows. For $x \in \mathbb{R}$ let $\lceil x \rceil$ be the unique integer n such that $x \leqslant n < x+1$.

Theorem 3.4.1. *Let $\mathcal{K}$ be a generalised homology $(n-1)$-sphere on $[m]$ for $n \geqslant 4$ which is not the boundary of a simplex. Suppose that the c-skeleton of $\mathcal{K}$ is a complete graph, where $c = \lceil \frac{n-3}{2} \rceil$. Then if n is even, or if n is odd and $\mathcal{Z}_{\mathcal{K}}$ has torsion-free homology, then $\mathcal{Z}_{\mathcal{K}}$ is homeomorphic to a connected sum of sphere products.*

Proof. Since the real moment-angle complex $\mathcal{R}_{\mathcal{K}}$ being c-connected is equivalent to $\mathcal{K}$ having full c-skeleton this follows from [GdM13, Theorem 1.3] for n even, and from [GdM13, Theorem 1.4] for n odd. □

By a result of Amelotte [Ame20], all of the above families give $\mathcal{K}$ to be minimally non-Golod. It is a natural question to ask whether there are minimally non-Golod simplicial spheres for which $\mathcal{Z}_{\mathcal{K}}$ is not a connected sum of sphere products. The aim of this section is to develop two examples of moment-angle manifolds which demonstrate the complex behaviour of minimally non-Golod simplicial spheres. In Proposition 3.4.4 we establish a condition on the skeleton of $\mathcal{K}$ which ensures it is minimally non-Golod, which is less strict than the condition of Theorem 3.4.1. Moreover, in Example 3.4.11 we construct a minimally non-Golod simplicial 4-sphere with full 1-skeleton which is not a minimally non-Golod complex. This example also answers negatively a question of [BM04] as to whether the torsion-free assumption is necessary when n is odd in Theorem 3.4.1.

A stronger question is asking if $\mathcal{Z}_{\mathcal{K}}$ being a connected sum of sphere products can be identified by its cohomology ring. Bosio and Meersseman [BM04, Proposition 11.6] showed that for simplicial spheres of dimension 2, every $\mathcal{K}$ for which $\mathcal{Z}_{\mathcal{K}}$ is homeomorphic to a connected sum of sphere products is obtained from $\partial\Delta^2$ by a positive number of iterated stellar subdivisions at maximal faces. This is proved by showing these conditions are equivalent to $\mathcal{Z}_{\mathcal{K}}$ having the same cohomology ring as a connected sum of sphere products. For dimensions above 2, an example of Allen and La Luz [ALL07] answering this question to the negative was shown to be incorrect in [GdM13]. In Example 3.4.14 we give a different example, again answering this question negatively.

3.4.1 Generalised homology spheres and simplicial operations

Let $\mathcal{K}$ be a generalised homology sphere, so that $\mathcal{Z}_{\mathcal{K}}$ is a manifold. One of the advantages to restricting our attention to moment-angle manifolds is that the structure of Poincaré duality on $\mathcal{Z}_{\mathcal{K}}$ places strong conditions on $\mathcal{K}$ itself.

Lemma 3.4.2. *Let $\mathcal{K}$ be a GHS^{n-1} on $[m]$. Then $\mathcal{K}$ is Golod if and only if $\mathcal{K} = \partial\Delta^{m-1}$.*

Proof. If $\mathcal{K} = \partial\Delta^{m-1}$, then $\mathcal{Z}_{\mathcal{K}} = S^{2m-1}$. Therefore $\mathcal{K}$ is Golod since all cup products and higher Massey products vanish for dimensional reasons.

Alternatively, if $\mathcal{K}$ is not the boundary of a simplex, there is a minimal missing face J of $\mathcal{K}$ such that $|J| < m$. Therefore $H^{2|J|-1}(\mathcal{Z}_{\mathcal{K}})$ contains a $\mathbb{Z}$ summand generated by α, say. Since $\mathcal{K}$ is a GHS^{n-1}, then $\mathcal{Z}_{\mathcal{K}}$ is a $(m+n)$-manifold. Therefore by Poincaré duality there is $\beta \in H^{m+n-2|J|+1}(\mathcal{Z}_{\mathcal{K}})$ generating a $\mathbb{Z}$ summand such that $\alpha\beta$ generates $H^{m+n}(\mathcal{Z}_{\mathcal{K}}) = \mathbb{Z}$. Therefore $H^*(\mathcal{Z}_{\mathcal{K}})$ has non-trivial multiplication, so $\mathcal{K}$ is not Golod. □

Along with Poincaré duality of the manifold $\mathcal{Z}_{\mathcal{K}}$ itself, the full subcomplexes of $\mathcal{K}$ itself also satisfy a form of duality which is sometimes called combinatorial Alexander duality.

Theorem 3.4.3 ([FW15, Theorem 3.4])**.** *A simplicial complex $\mathcal{K}$ on $[m]$ is a GHS^{n-1} if and only if*

$$\widetilde{H}^l(K_J) \cong \widetilde{H}_{n-l-2}(K_{[m]-J}). \tag{3.12}$$

for every $J \subseteq [m]$ and $0 \leqslant l \leqslant n-2$. □

This form of duality is useful since it explicitly encodes vertex supports of homology classes. The following application identifies when a generalised homology sphere $\mathcal{K}$ is minimally non-Golod with a condition on the skeleton in the same style as Theorem 3.4.1.

Proposition 3.4.4. *Let $\mathcal{K}$ be a generalised homology $(n-1)$-sphere on $[m]$ for $n \geqslant 4$ which is not the boundary of a simplex. Suppose that the c-skeleton of $\mathcal{K}$ is complete, where $c = \lceil \frac{n-2}{3} \rceil$. Then $\mathcal{K}$ is minimally non-Golod.*

Proof. By Lemma 3.4.2, the complex $\mathcal{K}$ is not Golod. We show that $(\mathcal{K} - v)$ is Golod for every $v \in [m]$, or equivalently, that $\mathcal{K}_J$ is Golod for any full subcomplex $\mathcal{K}_J$ with $|J| < m$.

We first check that all cup products are trivial. For $i \geqslant n-1$ and $J \subsetneq [m]$ we have

$$H^i(\mathcal{K}_J) \cong \widetilde{H}_{n-2-i}(\mathcal{K}_{[m]-J}) = 0$$

since $[m] - J$ is non-empty. Therefore by the Hochster decomposition (2.24), for $I \sqcup J \subsetneq [m]$, non-trivial cup products are described by non-trivial maps

$$H^k(\mathcal{K}_I) \otimes H^l(\mathcal{K}_J) \longrightarrow H^{k+l+1}(\mathcal{K}_{I \sqcup J}) \tag{3.13}$$

for $1 \leqslant k + l + 1 \leqslant n - 2$. Since the c-skeleton of $\mathcal{K}$ is complete, for any $0 \leqslant j \leqslant c-1$ and $J \subsetneq [m]$ we have $\widetilde{H}^j(\mathcal{K}_J) = 0$, with the same true for $\widetilde{H}_j(\mathcal{K}_J)$. Therefore we must have $k, l \geqslant c$ in (3.13). But then $H^{k+l+1}(\mathcal{K}_{I \sqcup J}) = 0$ since

$$H^{k+l+1}(\mathcal{K}_{I \sqcup J}) \cong \widetilde{H}_{n-3-k-l}(\mathcal{K}_{[m]-(I \sqcup J)}) = 0$$

and $n-3-k-l \leqslant n-3-2c \leqslant n-3-2\left(\frac{n-2}{3}\right) = \frac{n-5}{3} \leqslant c-1$, with the second inequality following since $c = \lceil \frac{n-2}{3} \rceil$ if and only if $\frac{n-2}{3} \leqslant c < \frac{n+1}{3}$. Therefore this product is trivial too for any $I, J \subsetneq [m]$.

Next, we check all higher Massey products are also trivial. By [GL21], higher Massey products in moment-angle complexes are associated to non-trivial maps

$$H^{p_1}(\mathcal{K}_{J_1}) \otimes \cdots \otimes H^{p_k}(\mathcal{K}_{J_k}) \longrightarrow H^{p_1+\cdots+p_k+1}(\mathcal{K}_{J_1 \sqcup \cdots \sqcup J_k}) \tag{3.14}$$

for $J_1 \sqcup \cdots \sqcup J_k \subsetneq [m]$. We show all such maps are trivial by showing that for $k \geqslant 3$ that $H^{p_1+\cdots+p_k+1}(\mathcal{K}_J) = 0$ for any $J \subsetneq [m]$. As for the cup product case we require $p_1, \ldots, p_k \geqslant c$. Then

$$p_1 + \cdots + p_k + 1 \geqslant kc + 1 \geqslant k\left(\frac{n-2}{3}\right) + 1 = \frac{kn - 2k + 3}{3} \geqslant n - 1$$

for all $k \geqslant 3$. Therefore all higher Massey products are trivial for dimensional reasons. Therefore $\mathcal{K}_J$ is Golod for every full subcomplex $\mathcal{K}_J$ of $\mathcal{K}$ with $J \subsetneq [m]$. Therefore $\mathcal{K}$ is minimally non-Golod. □

Comparing the statement of Proposition 3.4.4 with that of Theorem 3.4.1, we see that any generalised homology 3-sphere $\mathcal{K}$ which has full 1-skeleton is both minimally non-Golod and has $\mathcal{Z}_{\mathcal{K}}$ a connected sum of sphere products. In Example 3.4.10 we construct a generalised homology 3-sphere with complete 1-skeleton, which is not a triangulation of a sphere. For generalised homology 4-spheres, the conditions on the skeleta are the same, but we do not require torsion-free homology of $\mathcal{Z}_{\mathcal{K}}$ to conclude that $\mathcal{K}$ is minimally non-Golod. In Example 3.4.11 we construct a minimally non-Golod simplicial 4-sphere with complete 1-skeleton for which the associated moment-angle manifold is not a connected sum of sphere products.

3.4.2 Constructing moment-angle manifolds

Let $\mathcal{K}$ be a simplicial complex on $[m]$. The following construction of Bosio and Meersseman [BM04], later generalised by Li and Wang [LW19] constructs a simplicial sphere which has $\mathcal{K}$ as a full subcomplex.

Construction 3.4.5. Let $\mathcal{S}$ be a simplicial sphere with $\mathcal{K}$ as a subcomplex. Let $\mathcal{M} = MF(\mathcal{K})$ be the set of minimal missing faces and write $\mathcal{M} = \{J_1, \ldots, J_k\}$.

Iteratively form the complexes $\mathcal{S}_i = \mathrm{sts}_{\mathcal{S}_{i-1}}(J_i)$ for $i = 1, \ldots, k$, where $\mathcal{S}_0 = \mathcal{S}$. Then $\mathcal{L} = \mathcal{S}_k$ is a simplicial complex which contains $\mathcal{K}$ as a full subcomplex. Since stellar subdivision preserves the PL-homeomorphism type of a simplicial complex, then $\mathcal{L}$ is PL-homeomorphic to $\mathcal{S}$, that is, $\mathcal{L}$ is a simplicial sphere.

Bosio and Meersseman use this construction to provide the first example of a moment-angle manifold whose cohomology contains torsion by choosing $\mathcal{K}$ to be the triangulation of $\mathbb{R}P^2$ from Example 3.3.1. That the resulting manifold has torsion in its cohomology follows from the Hochster decomposition (2.25).

To apply Theorem 3.4.1 and Proposition 3.4.4 we require simplicial operations which alter the skeleta of a simplicial complex without changing its PL-homeomorphism type. Bistellar moves are a modification of stellar subdivision introduced by Pachner [Pac87] defined as follows.

Definition 3.4.6. Let $\mathcal{K}$ be a simplicial complex on $[m]$ of dimension $(n-1)$ and let $\sigma \in \mathcal{K}$ be a $(n-1-k)$-simplex such that $\mathrm{lk}_{\mathcal{K}}(\sigma)$ is the boundary $\partial\tau$ of a k-simplex τ which is not a simplex of $\mathcal{K}$. A *bistellar k-move on $\mathcal{K}$ at σ* is the assignment

$$\chi_\sigma \colon \mathcal{K} \longmapsto (\mathcal{K} - \sigma * \partial\tau) \cup (\partial\sigma * \tau).$$

Since the link of any maximal simplex is empty, a bistellar 0-move is always defined and coincides with stellar subdivision at a maximal simplex. In general, a bistellar move at a simplex σ is not the same as stellar subdivision at σ since the latter will always add a new vertex to $\mathcal{K}$. We apply bistellar moves by identifying minimal missing k-faces τ which are the link in $\mathcal{K}$ of some $(n-1-k)$-simplex σ. As long as $k < n-1-k$, the bistellar move at σ fills in the missing face τ while creating missing faces of strictly bigger dimension. Applying this iteratively allows us to fill in skeleta of a given complex. Moreover, the following means we do not change the PL-homeomorphism type of our complex.

Theorem 3.4.7 ([Pac87, Theorem 1])**.** *Two simplicial manifolds are PL-homeomorphic if and only if one can be obtained from the other by a sequence of bistellar moves.* □

Since the complexes we will encounter are typically very large, another very useful operation is edge contraction, which if applied appropriately, reduces the vertex set of a simplicial complex without altering its PL-homeomorphism type.

Definition 3.4.8. Let $\mathcal{K}$ be a simplicial complex on $[m]$ and let $i, j \in [m]$ be vertices such that the edge $(i,j) \in \mathcal{K}$. Let k be a vertex not in $[m]$ and let $[m'] = ([m] - \{i,j\}) \sqcup \{k\}$. Define a map $f\colon [m] \longrightarrow [m']$ by

$$f(v) = \begin{cases} k & \text{if } v \in \{i,j\} \\ v & \text{if } v \notin \{i,j\}. \end{cases}$$

The *edge contraction* $\mathcal{K}'$ of $\mathcal{K}$ at (i,j) is the simplicial complex

$$\mathcal{K}' = \{(f(v_1), \dots, f(v_k)) \mid (v_1, \dots, v_k) \in \mathcal{K}\}\,.$$

Often, the new vertex k is relabeled to j after performing the edge contraction. We use the notation $i \longmapsto j$ to denote when this is the case.

Edge contractions in simplicial complexes were studied by Attali, Lieutier and Salinas in [ALS11]. The following result of the authors gives a condition which ensures performing an edge contraction to $\mathcal{K}$ does not change its homotopy type.

Theorem 3.4.9 ([ALS11, Theorem 2])**.** *For any simplicial complex K, if an edge (i,j) satisfies*

$$\mathrm{lk}_{\mathcal{K}}(i) \cap \mathrm{lk}_{\mathcal{K}}(j) = \mathrm{lk}_{\mathcal{K}}((i,j)) \tag{3.15}$$

then the edge contraction of (i,j) preserves the homotopy type of $\mathcal{K}$. □

Despite using edge contractions, the simplicial complexes involved in the following examples are very large, and computations are computer-assisted using Sage [S$^+$22].

Example 3.4.10. Theorem 3.4.1 applies even when $\mathcal{K}$ is not a triangulation of a 3-sphere. The Poincaré homology sphere is a 3-manifold whose homology coincides with that of S^3, but is not simply-connected. We construct a triangulation of the Poincaré sphere which has a full 1-skeleton.

There is a 16-vertex triangulation of the Poincaré 3-sphere constructed by Björner and Lutz [BL00, Theorem 5]. This triangulation does not have a full 1-skeleton. The set of minimal missing 1-faces is given by

$$(12,16),(3,8),(5,16),(7,16),(3,16),(3,9),(1,16),(2,16),(3,6),(6,8),(4,16),\\(8,16),(6,16),(6,9)$$

Performing a series of bistellar moves at

$$(9,10,11),(1,7,10),(9,10,15),(9,5,15),(11,13,14),(11,14,16),(9,11,12),(3,11,13),\\(7,12,13),(2,12,15),(1,9,12),(4,9,12),(10,11,12),(1,2,14)$$

in the order given fills in all the missing 1-faces, again in the order given. Since bistellar moves preserve PL-homeomorphism type, the resulting simplicial complex $\mathcal{L}$ is still a triangulation of the Poincaré sphere and has full 1-skeleton. Therefore by Theorem 3.4.1 the manifold $\mathcal{Z}_{\mathcal{L}}$ is a connected sum of sphere products. To demonstrate the size of this example, by computing the Betti numbers of $\mathcal{Z}_{\mathcal{L}}$ we obtain

$$\mathcal{Z}_{\mathcal{L}} \cong (S^5\times S^{15})^{\#340}\#(S^6\times S^{14})^{\#2{,}794}\#(S^7\times S^{13})^{\#11{,}012}\#(S^8\times S^{12})^{\#26{,}961}\\ \#(S^9\times S^{11})^{\#44{,}968}\#(S^{10}\times S^{10})^{\#26{,}565}.$$

This shows that those simplicial complexes which give rise to connected sums of sphere products do not have to be simplicial spheres.

Next, we give a reduction of the manifold of [BM04] to give a moment-angle manifold $\mathcal{Z}_{\mathcal{L}}$ with 2-torsion in its cohomology for which $\mathcal{L}$ is minimally non-Golod.

Example 3.4.11. Let $\mathcal{K}$ be the minimal 6-vertex triangulation of $\mathbb{R}P^2$ from Example 3.3.1, see Figure 3.2. Its minimal missing faces are

$$MF(\mathcal{K}) = \{(1,2,3),(1,2,4),(1,3,5),(1,4,6),(1,5,6),(2,3,6),(2,4,5),(2,5,6),\\(3,4,5),(3,4,6)\}.$$

Let $\mathcal{S} = \partial\Delta^5$ and form the complex $\mathcal{S}_{10}$ as per Construction 3.4.5. We then apply the series of edge contractions $15 \longmapsto 11$, $13 \longmapsto 11$, $12 \longmapsto 7$ and $9 \longmapsto 8$ to reduce the vertex set from 16 to 12 vertices. The resulting simplicial complex $\mathcal{K}$ is a simplicial 4-sphere with 12 vertices. Moreover its minimal missing faces are

$$\{(4,10),(2,8),(1,3,5),(9,5,11),(0,7,10),(6,7,8),(3,5,9),(2,9),(1,4,6),(1,3,7),\\(0,4,5,6),(8,10,11),(3,4,6),(2,5,6),(3,4,5),(1,2,3),(0,7,11),(0,5,6,7),(0,5,9),\\(7,8,9),(0,4,5,11),(0,4,6,8),(0,8,11),(0,9,10),(0,8,9),(9,10,11),(2,3,6),(3,6,7),\\(3,7,9),(1,2,4),(1,11),(2,4,5),(3,10),(1,7,8),(7,9,11),(7,8,10),(1,5,6)\}.$$

As well as not having a full 1-skeleton, we can explicitly check that the complex $\mathcal{K}$ is not minimally non-Golod. For example, the map

$$H^0\left(\mathcal{K}_{\{4,10\}}\right) \otimes H^0\left(\mathcal{K}_{\{2,9\}}\right) \to H^1\left(\mathcal{K}_{\{2,4,9,10\}}\right)$$

is non-trivial, inducing a non-trivial cup product in $\mathcal{Z}_{\mathcal{K}_{\{2,4,9,10\}}}$ by formula (2.24).

We use bistellar moves to fill in the 1-skeleton while keeping $\mathcal{K}$ as a full subcomplex. The bistellar 1-moves at simplices $(5,9,7,6)$, $(0,3,4,7)$, $(0,4,6,11)$, $(0,3,4,9)$ and $(0,5,6,11)$ fill in the five remaining missing 1-simplices $(4,10)$, $(2,8)$, $(2,9)$, $(1,11)$, $(3,10)$, respectively. Denote the new simplicial complex $\mathcal{L}$. Then by Proposition 3.4.4, the simplicial 4-sphere $\mathcal{L}$ is minimally non-Golod. Additionally, $\mathcal{L}$ contains the triangulation $\mathcal{K}$ of $\mathbb{R}P^2$ as a full subcomplex. In particular, $\mathcal{L}$ is a minimally non-Golod complex such that the moment-angle manifold $\mathcal{Z}_{\mathcal{L}}$ contains torsion in its cohomology. Therefore $\mathcal{Z}_{\mathcal{L}}$ is not homotopy equivalent to a connected sum of sphere products.

The complex $\mathcal{Z}_{\mathcal{L}}$ is a minimal example, in the sense that both its dimension and number of vertices are minimal, as the following result justifies.

Lemma 3.4.12. *Let $\mathcal{L}$ be a GHS^{n-1} on $[m]$, so that $\mathcal{Z}_{\mathcal{L}}$ is an $(m+n)$-manifold. Suppose that $H^*(\mathcal{Z}_{\mathcal{L}})$ contains torsion, then $m \geqslant 12$ and $n \geqslant 5$.*

Proof. Since $\mathcal{L}$ is a GHS^{n-1}, the full subcomplexes $\mathcal{L}_J$ of $\mathcal{L}$ satisfy the duality formula (3.12),

$$\widetilde{H}^l(\mathcal{L}_J) \cong \widetilde{H}_{n-l-2}(\mathcal{L}_{[m]-J}).$$

It is known that the 6-vertex triangulation $\mathcal{K}$ of $\mathbb{R}P^2$ is the smallest simplicial complex containing torsion, both in the sense of number of vertices and homological dimension of the torsion. The above isomorphism therefore implies that if $\mathcal{L}_J$ contains torsion, then both $|J| \geqslant 6$ and $|[m]-J| \geqslant 6$, and so $m \geqslant 12$. Additionally, since $n-l-2 \geqslant 1$ in this case, the bound $l \geqslant 2$ implies that $n \geqslant 5$, as stated. □

This example demonstrates that moment-angle manifolds associated to a minimally non-Golod complex need not be connected sums of sphere products. Moreover, this example shows the torsion-free assumption in Theorem 3.4.1 is necessary.

Remark 3.4.13. Comparing the statements of Theorem 3.4.1 and Proposition 3.4.4, Example 3.4.11 for $n = 5$ is the lowest n for which the two statements differ, the former statement requiring a torsion-free assumption on cohomology. The smallest n for which there is dimensional difference in the conditions on the skeleta occurs at $n = 8$. Concretely, any generalised homology 7-sphere with full 2-skeleton is minimally non-Golod, but we require a full 3-skeleton to conclude the corresponding moment-angle manifold is a connected sum of sphere products. We can only speculate that this is a strict gap, and a construction similar to Example 3.4.11 using the 9-vertex triangulation of $\mathbb{C}P^2$ in place of $\mathbb{R}P^2$ would be a candidate. So far, attempts to construct such a complex have been too computationally intensive.

Our final example constructs a moment-angle manifold whose cohomology is that of a connected sum of sphere products, but this correspondence is not due to a homotopy equivalence.

Example 3.4.14. Let $\mathcal{K}$ be the simplicial complex from Example 3.3.5. Then

$$MF(\mathcal{K}) = \{(1,2,3),(4,5,6),(7,8,9),(1,4,7),(1,2,7,8),(2,3,5,6),(4,5,8,9),(2,3,5,8,9)\}.$$

Let $\mathcal{S} = \partial\Delta^8$ and form the complex $\mathcal{S}_8$ as per Construction 3.4.5. Then $\mathcal{S}_8$ is a simplicial complex with 17 vertices with $\mathcal{K}$ as a full subcomplex.

We perform edge contractions $14 \longmapsto 0$, $10 \longmapsto 16$, $15 \longmapsto 0$ and $13 \longmapsto 12$ so $\mathcal{S}_8$. The resulting simplicial $\mathcal{L}$ complex has 13 vertices, so that $\mathcal{Z}_\mathcal{L}$ is a 21-manifold, and moreover $\mathcal{L}$ has $\mathcal{K}$ as a full subcomplex by construction. Its minimal missing faces are

$$\begin{aligned}&\{(2,3,5,6),(2,3,5,8,9),(0,1,6,11,16),(8,9,12),(8,9,4,5),(0,16,11,12),(4,5,6),\\&(8,1,2,7),(0,1,7,11,16),(1,4,7),(0,4,6,11,16),(0,4,7,11,16),(3,12,5,6),(2,3,12),\\&(1,2,3),(9,12,5,6),(8,9,7)\}.\end{aligned}$$

We eliminate the missing 2-faces not in $MF(\mathcal{K})$ by performing bistellar moves at $(0,3,4,6,11,1)$, filling in $(8,9,12)$, and $(0,4,5,7,8,11)$, filling in $(2,3,12)$. Let $\mathcal{L}$ be the resulting simplicial complex. Then $\mathcal{Z}_\mathcal{L}$ is a 21-manifold.

Moreover, $\mathcal{Z}_\mathcal{L}$ has the cohomology of a connected sum of sphere products. To see this, by construction of $\mathcal{K}$ in Example 3.3.6, all cup products of the classes supported by $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$ or $\{1,4,7\}$ are trivial. Therefore using the proof of Proposition 3.4.11, it is sufficient to check that there are no non-trivial maps

$$H^k(\mathcal{L}_I) \otimes H^l(\mathcal{L}_J) \longrightarrow H^{k+l+1}(\mathcal{L}_{I\sqcup J}) = \widetilde{H}_{5-k-l}(\mathcal{L}_{[m]-(I\sqcup J)}) \tag{3.16}$$

for $k = 1$ and $l = 2,3$, or $k = l = 2$. We take cases to show that all such maps are trivial. First, let $\mathcal{M}_k$ be the set of all missing k-faces of $\mathcal{L}$. Then by construction of $\mathcal{L}$,

$$\mathcal{M}_2 = \{(1,2,3),(4,5,6),(7,8,9),(1,4,7)\}$$

and we further compute $\mathcal{M}_3$ and $\mathcal{M}_4$ as

$$\begin{aligned}\mathcal{M}_3 = \{&(8,9,12,5),(2,3,5,6),(8,9,12,16),(2,3,12,6),(8,9,4,5),(0,16,11,12),\\&(8,1,2,7),(9,2,3,12),(16,2,3,12),(3,12,5,6),(8,9,2,12),(9,12,5,6)\},\\\mathcal{M}_4 = \{&(2,3,4,8,9),(0,1,6,11,16),(0,1,7,11,16),(0,4,6,11,16),(0,4,7,11,16)\}.\end{aligned}$$

First consider $k = 1$ and $l = 3$. Then since $5 - k - l = 1$, it follows from (3.16) that I and $[m] - (I \sqcup J)$ must be distinct elements of $\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$. Therefore $[m] - J$ must be contained in one of $\{\{1,2,3,4,5,6\},\{1,2,3,7,8,9\},\{4,5,6,7,8,9\}\}$. From the table in Example 3.3.6 we conclude that

$$H^l(\mathcal{L}_J) = H^3(\mathcal{L}_J) \cong H_3\left(\mathcal{L}_{[m]-J}\right) = 0.$$

for any choice of J.

Now we turn to the cases $k = 1$, $l = 2$ and $k = l = 2$. We begin with the following claim. Suppose that J is a subset of the vertex set of $\mathcal{L}$. Then if $|J| = 5$ and J contains either 0 or 11, or if $|J| = 6$ and J contains both 0 and 11, then $H^2(\mathcal{L}_J) = 0$.

To see this, suppose that $|J| = 5$ and contains the vertex 0, and suppose that $\mathcal{L}_J$ has some minimal missing face, so that it is not contractible. Then by inspection the missing faces of J are either one of the last four elements in $\mathcal{M}_4$, in which case $H_2(\mathcal{L}_J) = 0$, or exactly one of the elements in $\mathcal{M}_2$ or $\mathcal{M}_3$, in which case $\mathcal{L}_J$ has a cone vertex and is contractible. The case that J contains 11 instead is identical.

If instead $|J| = 6$ and contains both 0 and 11, then the minimal missing faces of $\mathcal{L}_J$ are either

- J itself;
- $\{0, 11, 12, 16\}$ along with exactly one of the last four elements in $\mathcal{M}_4$;
- exactly two of the last four elements in $\mathcal{M}_4$;
- $\{0, 11, 12, 16\}$ and either $\{8, 9, 12, 16\}$ or $\{2, 3, 12, 16\}$;
- $\{1, 4, 7\}$, $\{0, 4, 7, 11, 16\}$ and $\{0, 1, 7, 11, 16\}$;
- $\{4, 5, 6\}$ and $\{0, 4, 6, 11, 16\}$;
- exactly one of those listed in $\mathcal{M}_2, \mathcal{M}_3$ or $\mathcal{M}_4$.

In all cases, a direct check gives that $H_2(\mathcal{L}) = 0$. This establishes the claim.

Returning to the main claim, first suppose that $k = 1$ and $l = 2$, so that $5 - k - l = 2$. Then since I is one of $\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\}$, by the claim we must have that either $J = \{0, 11, 12, 16\}$ or $[m] - (I \cup J) = \{0, 11, 12, 16\}$. In the first case, $[m] - (I \cup J)$ is one of $\{4,5,6,7,8,9\},\{1,2,3,7,8,9\},\{1,2,3,4,5,6\}$ or $\{2,3,5,6,8,9\}$, in which case $H_2(\mathcal{L}_{[m]-(I\cup J)}) = 0$ by the table in Example 3.3.6. In the second case, we similarly obtain $H^2(\mathcal{L}_J) = 0$. Therefore all maps (3.16) are trivial.

The case $k = l = 2$ is dealt similarly, interchanging the roles of I and $[m] - (I \cup J)$.

Therefore since $\mathcal{Z}_{\mathcal{L}}$ is a manifold and all cohomology classes are torsion-free, the cohomology ring of $\mathcal{Z}_{\mathcal{L}}$ is that of a connected sum of sphere products. Specifically $H^*(\mathcal{Z}_{\mathcal{L}})$ is isomorphic to

$$H^*\left((S^5 \times S^{16})^{\#4}\#(S^7 \times S^{14})^{\#12}\#(S^8 \times S^{13})^{\#30}\#(S^9 \times S^{12})^{\#20}\#(S^{10} \times S^{11})^{\#19}\right).$$

On the other hand, $\mathcal{Z}_{\mathcal{L}}$ is not a connected sum of sphere products since the inclusion $\mathcal{K} \longrightarrow \mathcal{L}$ implies the existence of a non-trivial triple Massey product in $H^*(\mathcal{Z}_{\mathcal{L}})$. This is the first known example of such a manifold in which the cohomology isomorphism is not due to a homotopy equivalence. A previous claim for such an example was made in [ALL07], but this was shown to be incorrect in [GdM13]. We conclude that the cohomology ring is not a strong enough invariant to identify moment-angle manifolds as connected sums of sphere products.

Chapter 4

Relations among higher Whitehead maps in polyhedral products

4.1 Introduction

The graded commutator

$$[a, b] = ab - (-1)^{\deg a \deg b} ba$$

equips a graded algebra A with the structure of a graded quasi-Lie algebra, that is, the commutator is a bilinear map satisfying graded skew symmetry

$$[a, b] = -(-1)^{\deg a \deg b}[b, a] \tag{4.1}$$

and the graded Jacobi identity

$$(-1)^{\deg a \deg c}[a, [b, c]] + (-1)^{\deg a \deg b}[b, [c, a]] + (-1)^{\deg b \deg c}[c, [a, b]] = 0 \tag{4.2}$$

for all $a, b, c \in A$. A quasi-Lie algebra differs from a Lie algebra by demanding skew symmetry (4.1) in place of the alternativity assumption $[a, a] = 0$. Therefore in a quasi-Lie algebra the element $[a, a]$ is 2-torsion, if it does not vanish.

The question of whether an analogous structure exist in homotopy theory has existed since the early 20th century. Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ and $\gamma \in \pi_r(X)$ for $p, q, r \geqslant 1$. The Whitehead product of α and β is an element of $\pi_{p+q-1}(X)$ defined by J. H. C. Whitehead [Whi41], see Section 2.1.4. Whitehead showed that the operation

$$[\cdot, \cdot] \colon \pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)$$

on homotopy groups is a bilinear operation which satisfies graded symmetry, that is, $[\alpha, \beta] = (-1)^{pq}[\beta, \alpha]$. The question of whether the Whitehead product satisfies the Jacobi identity is non-trivial.

Let $\theta\colon \pi_*(X) \longrightarrow H_{*-1}(\Omega X)$ be the composite of the adjunction isomorphism $\pi_*(X) \longrightarrow \pi_{*-1}(\Omega X)$ with the Hurewicz map, and let a, b, c be the images of α, β, γ under θ, respectively. Using graded symmetry of the Whitehead product and the identity $\theta([\alpha, \beta]) = (-1)^{p-1}[a, b]$ of Samelson [Sam53], we compute that

$$(-1)^{p+q+r+1}\theta((-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{pq}[[\beta,\gamma],\alpha] + (-1)^{qr}[[\gamma,\alpha],\beta]) = 0$$

in $H_*(\Omega X)$. For $p, q, r \geqslant 2$, proofs that this relation occurs on the level of homotopy groups, that is that

$$(-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{pq}[[\beta,\gamma],\alpha] + (-1)^{qr}[[\gamma,\alpha],\beta] = 0 \tag{4.3}$$

were initially given independently by Nakaoka–Toda [NT54] and Massey–Uehara [UM57], almost 15 years after the definition of the Whitehead product.

It follows that the Whitehead product endows the direct sum $\bigoplus_{n\geqslant 1} \pi_n(X)$ with the structure of a graded quasi-Lie algebra, when the grading is given by setting the degree of $\pi_n(X)$ to be $n-1$. Such structure is used, for example, by Hilton [Hil55] in giving a decomposition of the loops on a wedge of spheres, and more generally by Milnor [Mil72] of the loops on a wedge of suspension spaces, see also [Sel97, Theorem 7.9.4].

The study of Whitehead products in polyhedral product spaces was initiated by Grbić and Theriault [GT16] to study the map $\mathcal{Z}_\mathcal{K} \longrightarrow DJ_\mathcal{K}$ in the homotopy fibration

$$\mathcal{Z}_\mathcal{K} \longrightarrow DJ_\mathcal{K} \longrightarrow \textstyle\prod_{i=1}^m \mathbb{C}P^\infty.$$

Abramyan and Panov [AP19] give a systematic combinatorial analysis of these maps as follows. Let $\mathcal{K}$ be a simplicial complex on $[m]$ for $m \geqslant 2$ and recall that the Davis–Januszkiewicz space $DJ_\mathcal{K}$ is the polyhedral product $(\mathbb{C}P^\infty, *)^\mathcal{K}$. For $i = 1, \ldots, m$, let $\mu_i\colon S^2_i \longrightarrow \mathbb{C}P^\infty_i \longrightarrow DJ_\mathcal{K}$ be the inclusion of the bottom cell followed by the inclusion of the ith coordinate of $DJ_\mathcal{K}$. The Whitehead product of μ_i and μ_j, is the map

$$[\mu_i, \mu_j]\colon S^3 = S^1 * S^1 \longrightarrow S^2_i \vee S^2_j \longrightarrow \mathbb{C}P^\infty_i \vee \mathbb{C}P^\infty_j \longrightarrow DJ_\mathcal{K} \vee DJ_\mathcal{K} \xrightarrow{\nabla} DJ_\mathcal{K}.$$

The structure of the polyhedral product allows for a generalisation of the Whitehead product given by [AP19] as follows. Suppose that $\mathcal{K}$ contains $\partial\Delta^{k-1}$ on vertex set $\{1, \ldots, k\} \subseteq [m]$. The higher Whitehead map

$$h_w(\mu_1, \ldots, \mu_k)\colon S^{2k-1} \longrightarrow DJ_\mathcal{K}$$

is the homotopy class of the composite

$$S^{2k-1} = \mathop{\ast}_{i=1}^{k} S^1 \longrightarrow FW(S^2_1, \ldots, S^2_k) \longrightarrow FW(\mathbb{C}P^\infty_1, \ldots, \mathbb{C}P^\infty_k) \longrightarrow DJ_\mathcal{K}$$

where the first map is the restriction of the quotient map

$$D^2 \times \cdots \times D^2 \longrightarrow S^2 \times \cdots \times S^2$$

to $\mathop{\ast}\limits_{i=1}^{k} S^1 = \bigcup_{i=1}^{k} D_1^2 \times \cdots \times S_i^1 \times \cdots \times D_k^2$, and the last map is the map of polyhedral products induced by the inclusion $\partial\Delta^{k-1} \longrightarrow \mathcal{K}$, see Proposition 2.5.3(ii). The authors of [AP19] use the term higher Whitehead product for the above. We use the term higher Whitehead map to avoid any confusion with the higher Whitehead product in the sense of Porter [Por65], which is a higher topological operation with indeterminacy. In general, the higher Whitehead map is an element of a higher Whitehead product, which can contain multiple other non-homotopic elements.

Abramyan and Panov [AP19] also gave combinatorial conditions for the triviality of higher Whitehead maps. By [AP19, Proposition 3.3] the map $h_w(\mu_1, \ldots, \mu_k) \in \pi_{2k-1}(DJ_{\mathcal{K}})$ is trivial if and only if $\mathcal{K}$ contains the simplex Δ^{k-1}.

The definition of the higher Whitehead map is readily iterated to give higher Whitehead maps whose factors are higher Whitehead maps using the method of simplicial substitution. Moreover the triviality of substituted higher Whitehead maps can again be expressed combinatorially [AP19, Theorem 5.2]. As a particular case we have the following. The Jacobi identity (4.3) gives a relation

$$h_w(h_w(\mu_1, \mu_2), \mu_3) + h_w(h_w(\mu_2, \mu_3), \mu_1) + h_w(h_w(\mu_3, \mu_1), \mu_2) = 0$$

in which all terms are non-trivial if $\mathcal{K}$ does not contain any of the simplices $(1,2)$, $(2,3)$ or $(1,3)$, or equivalently if $\mathcal{K}$ contains three disjoint points on vertices $\{1,2,3\}$ as a full subcomplex. Similarly, a result of Hardie [Har61, Theorem 2.3] gives a relation

$$\sum_{i=1}^{k} h_w(h_w(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_k), \mu_i) = 0 \tag{4.4}$$

in $\pi_{2m-2}(DJ_{\mathcal{K}})$. We recognise that if $\mathcal{K}$ contains $\operatorname{sk}^{k-3}\Delta^{k-1}$ as a full subcomplex, then every term is non-trivial.

We give a generalisation of the above on two fronts. First, we consider non-spherical maps and define the higher Whitehead map $h_w(f_1, \ldots, f_m)$ for any maps $f_i\colon \Sigma X_i \longrightarrow Y_i$ as a map

$$h_w(f_1, \ldots, f_m)\colon \mathop{\ast}\limits_{i=1}^{m} X_i \longrightarrow FW(Y_1, \ldots, Y_m).$$

In Corollary 4.2.11, for certain sets of maps $\{f_1, \ldots, f_m\}$ satisfying a modest homological compatibility condition, we give a triviality result extending [AP19, Proposition 3.3]. We expand these results to consider higher Whitehead maps where each map f_i is itself a higher Whitehead map, and in Theorem 4.2.20 give a full characterisation of triviality extending [AP19, Theorem 5.2], again for maps satisfying a certain compatibility condition.

Second, we consider non-trivial relations between higher Whitehead maps. A k-partition of the vertex set $[m]$ is a collection of disjoint subsets $\Pi = \{I_1, \dots, I_k\}$ whose union is $[m]$. Given such a partition for $k \geqslant 3$, we define the identity complex $\mathcal{K}_\Pi$ on $[m]$ to have minimal missing faces $[m] - I_j$ for $j = 1, \dots, k$. We show that identity complexes can also be obtained from $\operatorname{sk}^{k-3} \Delta^{k-1}$, which governed the form of relation (4.4), via the method of simplicial composition. As a consequence, we show that the substituted higher Whitehead maps

$$h_w \left(h_w \left(f_{j_1}, \dots, f_{j_{r_i}}\right), f_{i_1}, \dots, f_{i_{n_i}}\right) : \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$$

are defined, where $I_i = \{i_1, \dots, i_{n_i}\}$ and $[m] - I_i = \{j_1, \dots, j_{r_i}\}$. In Theorem 4.3.7 we give our main result generalising (4.4), which is that if each X_i is a suspension then there is a relation

$$\sum_{i=1}^{k} h_w \left(h_w \left(f_{j_1}, \dots, f_{j_{r_i}}\right), f_{i_1}, \dots, f_{i_{n_i}}\right) \circ \sigma_i = 0 \tag{4.5}$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}\right]$, where

$$\sigma_i \colon \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m \longrightarrow \Sigma^{r_i - 2}(X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}) * X_{i_1} * \cdots * X_{i_{n_i}}$$

is induced by a permutation of coordinates. Relation (4.5) also generalises the work of Cohen [Coh57], who defined the Whitehead product in the case that the f_i are not assumed to be spherical and gave an appropriate Jacobi identity.

We extend our results by considering a novel approach to derive relations between Whitehead products with some maps repeated. We define a folded higher Whitehead map by composing the higher Whitehead map with a map from $(\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$ into another polyhedral product $(\underline{Y}, \underline{*})^{\overline{\mathcal{K}}}$ induced by an associative H-space structure on some of the Y_i, which we call a fold map. We establish that such fold maps are induced on polyhedral products by simplicial maps $\mathcal{K}_\Pi \longrightarrow \overline{\mathcal{K}}$. Therefore composing the relations of Theorem 4.3.7 with fold maps provides relations among folded higher Whitehead maps whose form is again governed purely combinatorially. In particular, we give three families for which the complex $\overline{\mathcal{K}}$ is identified explicitly, and give the associated relations between folded higher Whitehead maps.

To prove relation (4.5) we examine the original proofs of the Jacobi identity (4.3) of Massey–Uehara [UM57] and Nakaoka–Toda [NT54]. The two proofs are very different in character. Massey–Uehara analyse the homotopy groups of a wedge sum to deduce that the three iterated Whitehead products in (4.3) are linearly dependent. Deciding the coefficients is considerably more involved, requiring the use of the triple Massey product.

Indeed, this problem was one of the motivating examples for the definition of the triple Massey product. In general, higher Massey products can only be used to distinguish iterated 2-fold Whitehead products. Therefore this method does not readily generalise for our purposes.

Nakaoka–Toda approach the problem geometrically using techniques of relative homotopy theory and the relative Whitehead product of Blakers–Massey [BM53]. This approach is better suited to our geometric definition of the higher Whitehead map, and we adapt this method, along with a similar one of Hardie [Har64], to cover the more general form of higher Whitehead maps that we consider. In particular, we define the relative higher Whitehead product and examine it through a long exact sequence of relative homotopy sets. The main difficulty of the generalisation is establishing the result for non-spherical maps f_i from an arbitrary suspension space, which we accomplish by constructing decompositions of the join $X_1 * \cdots * X_m$.

For spherical maps $f_i\colon S^{r_i} \longrightarrow Y_i$, relation (4.5) implies that the higher Whitehead map endows the group $\bigoplus_{n\geqslant 1} \pi_n\left((\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}\right)$ with the structure of an L_∞-algebra, also known as a homotopy Lie algebra, extending the graded quasi-Lie algebra structure given by the Whitehead product.

Acknowledgement and declaration

The material in this chapter was jointly produced with Matthew Staniforth, under the supervision of Jelena Grbić. The development of the necessary tools to prove the main Theorem 4.3.7, as well as its corollaries to folded maps, Theorems 4.4.5, 4.4.10 and 4.4.13 is my own work. The techniques used to define and analyse the triviality of higher Whitehead maps and their folded versions were developed by Matthew Staniforth. The presentation and contextualisation of all results given is my own.

4.2 The higher Whitehead map

In this section we define the higher Whitehead map and give some elementary properties. We then define the substituted higher Whitehead map, a combinatorial generalisation of the higher Whitehead map, for which each component map is itself a higher Whitehead map. We build up to results giving classes of maps for which the triviality of the higher and substituted higher Whitehead maps are determined purely combinatorially. Finally, we define folded higher Whitehead maps, which are higher Whitehead maps composed with a fold map induced by H-space structure in the codomain.

4.2.1 Preliminaries

We begin by giving the definition of exterior Whitehead products due to Hardie [Har61], on which our generalisation is based.

Let $f_i\colon S^{r_i} \longrightarrow Y_i$ be maps with $r_i \geqslant 1$ for $i = 1, \ldots, k$. Hardie [Har61] defined the exterior Whitehead product, hereon called the higher Whitehead map, as follows. Let $r = r_1 + \cdots + r_k$ and define the map

$$\rho\colon S^{r-1} = \mathop{\circledast}_{i=1}^{k} S^{r_i-1} = \bigcup_{i=1}^{m} D^{r_1} \times \cdots \times S^{r_i-1} \times \cdots \times D^{r_k} \longrightarrow FW(S^{r_1}, \ldots, S^{r_k})$$

to be the restriction to the join of the product of quotient maps $D^{r_i} \longrightarrow S^{r_i}$ identifying $\partial D^{r_i} = S^{r_i-1}$ to the basepoint. The *higher Whitehead map* of $f_1, \ldots, f_k$, denoted $h_w(f_1, \ldots, f_k)$, is the homotopy class of the composite

$$FW(f_1, \ldots, f_k) \circ \rho\colon \mathop{\circledast}_{i=1}^{k} S^{r_i-1} \longrightarrow FW(Y_1, \ldots, Y_k). \tag{4.6}$$

By [Har61], the map (4.6) depends only on the homotopy classes of the maps f_i. Therefore $h_w(f_1, \ldots, f_k)$ is a well-defined element of $\pi_{r-1}(FW(Y_1, \ldots, Y_k))$.

The higher Whitehead map $h_w(f_1, \ldots, f_k)$ is an element of a set of homotopy classes of maps called the higher Whitehead product, a higher operation of homotopy groups introduced by Porter in [Por65] given as follows.

Consider spaces $X_1, \ldots, X_k, Y$. Given maps $f_i\colon \Sigma X_i \longrightarrow Y$ for $i = 1, \ldots, k$, denote by

$$\omega(f_1, \ldots, f_k) = \{\phi\colon FW(\Sigma X_1, \ldots, \Sigma X_k) \longrightarrow Y \mid \phi|_{\Sigma X_i} \simeq f_i \text{ for } i = 1, \ldots, k\}$$

the set of extensions up to homotopy of $\bigvee_{i=1}^{k} f_i\colon \bigvee_{i=1}^{k} \Sigma X_i \longrightarrow Y$ to $FW(\Sigma X_1, \ldots, \Sigma X_k)$. When $k = 2$ an extension is uniquely given by the composite of $f_1 \vee f_2$ with the fold map $\nabla\colon Y \vee Y \longrightarrow Y$. When $k \geqslant 3$ the cardinality of the set of extensions can be 0, 1, finite, or even infinite.

The *k-fold higher Whitehead product* of the maps f_i is the set of homotopy classes

$$[f_1, ..., f_k] = \{\phi \circ \rho \mid \phi \in \omega(f_1, \ldots, f_k)\} \subseteq [X_1 * \cdots * X_k, Y] \tag{4.7}$$

where

$$\rho\colon X_1 * \cdots * X_m = \bigcup_{i=1}^{k} CX_1 \times \cdots \times X_i \times \cdots \times CX_k \to FW(\Sigma X_1, \ldots, \Sigma X_k) \tag{4.8}$$

is the restriction of the product of quotient maps $CX_i \to \Sigma X_i$.

Returning to the map of Hardie (4.6), let $Y = FW(Y_1, \ldots, Y_k)$ and for $f_i \colon S^{r_i} \longrightarrow Y_i$ denote by $\widehat{f_i}$ the composite of f_i with the inclusion $Y_i \longrightarrow FW(Y_1, \ldots, Y_k)$. Then $FW(f_1, \ldots, f_k) \in \omega(\hat{f}_1, \ldots, \hat{f}_k)$ and therefore

$$h_w(f_1, \ldots, f_k) \in [\widehat{f}_1, \ldots, \widehat{f}_k].$$

4.2.2 The higher Whitehead map

The higher Whitehead product (4.7) allows for the maps $f_i \colon \Sigma X_i \longrightarrow Y_i$ to be non-spherical, that is, the spaces X_i need not be spheres. This extends the generalised Whitehead product of [Coh57, Ark62], which we studied in Section 2.1.4. In a similar way, the higher Whitehead map of Hardie (4.6) also admits a generalisation to non-spherical maps, as follows.

Definition 4.2.1. Let $m \geqslant 2$. For $i = 1, \ldots, m$, let $f_i \colon \Sigma X_i \longrightarrow Y_i$ be maps and let

$$\rho \colon \mathop{\ast}_{i=1}^{m} X_i \longrightarrow FW(\Sigma X_1, \ldots, \Sigma X_m)$$

be the map (4.8). The *higher Whitehead map* is the composite

$$h_w(f_1, \ldots, f_m) \colon \mathop{\ast}_{i=1}^{m} X_i \xrightarrow{\rho} FW(\Sigma X_1, \ldots, \Sigma X_m) \longrightarrow FW(Y_1, \ldots, Y_m) \tag{4.9}$$

where the second map is induced by the maps $f_i \colon \Sigma X_i \longrightarrow Y_i$. The higher Whitehead map of Hardie (4.6) is the special case that X_i is a sphere for all i.

The polyhedral product is a natural setting to study the higher Whitehead map. Let $(\underline{CX}, \underline{X})$, and $(\underline{Y}, \underline{*})$ denote the m-tuples of pairs $\{(CX_i, X_i)\}_{i=1}^m$ and $\{Y_i, *\}_{i=1}^m$, respectively. We first recall that

$$(\underline{CX}, \underline{X})^{\partial\Delta^{m-1}} = \bigcup_{i=1}^{m} CX_1 \times \cdots \times CX_{i-1} \times X_i \times CX_{i+1} \times \cdots \times CX_m = \mathop{\ast}_{i=1}^{m} X_i$$

and

$$(\underline{Y}, \underline{*})^{\partial\Delta^{m-1}} = \bigcup_{i=1}^{m} (Y_1 \times \cdots \times Y_{i-1} \times * \times Y_{i+1} \times \cdots \times Y_m) = FW(Y_1, \ldots, Y_m).$$

We freely interchange between the polyhedral product notation, on the left, and the notation on the right, where which we use will depend on context. It is immediate from the definition and Proposition 2.5.3(i) that $h_w(f_1, \ldots, f_k)$ is the map of polyhedral products

$$h_w(f_1, \ldots, f_m) \colon (\underline{CX}, \underline{X})^{\partial\Delta^{m-1}} \longrightarrow (\underline{Y}, \underline{*})^{\partial\Delta^{m-1}} \tag{4.10}$$

induced by maps of pairs $(CX_i, X_i) \longrightarrow (Y_i, *)$ representing $f_i \in [\Sigma X_i, Y_i]$.

By homotopy invariance of the polyhedral product, the homotopy class of the second map in (4.9), and thus the homotopy class of the higher Whitehead map, depends only on the homotopy classes of the maps $f_1, \ldots, f_m$.

The higher Whitehead map satisfies familiar properties of the Whitehead product. The following three results establish that it is a multilinear map, which satisfies a generalisation of graded symmetry and naturality properties.

Proposition 4.2.2. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \ldots, m$ and for some $j \in \{1, \ldots, m\}$ let $f_j'\colon \Sigma X_j \longrightarrow Y_j$. Then if X_j is a suspension,*

$$h_w(f_1, \ldots, f_j + f_j', \ldots, f_m) = h_w(f_1, \ldots, f_j, \ldots, f_m) + h_w(f_1, \ldots, f_j', \ldots, f_m). \quad (4.11)$$

Proof. For $i = 1, \ldots, m$ write f_i as a map $f_i\colon (CX_i, X_i) \longrightarrow (Y_i, *)$ and f_j' as a map $f_j'\colon (CX_j, X_j) \longrightarrow (Y_j, *)$. Since X_j is a suspension we write

$$X_j = \Sigma\widetilde{X}_j = C_+\widetilde{X}_j \cup C_-\widetilde{X}_j.$$

Consider the composite

$$f\colon (CX_j, X_j) \longrightarrow (CX_j \cup_{C+\widetilde{X}_j} CX_j, C_-\widetilde{X}_j \cup_{\widetilde{X}_j} C_-\widetilde{X}_j) \longrightarrow (Y_j, *)$$

where the first map is the aforementioned homotopy equivalence of pairs, and the second is the map from the colimit Z determined by f_j and f_j'. Then f is homotopic to the sum $f_j + f_j'$.

Therefore $h_w(f_1, \ldots, f_j + f_j', \ldots, f_m)$ is the map of polyhedral products

$$(\underline{CX}, \underline{X})^{\partial\Delta} \simeq (\underline{CX}', \underline{X}')^{\partial\Delta} \longrightarrow (\underline{Y}, \underline{*})^{\partial\Delta}$$

induced by f_i for $i \neq j$ and f, where $(CX_i', X_i') = (CX_i, X_i)$ for $i \neq j$ and $(CX_j', X_j') = (CX_j \cup_{C_+\widetilde{X}_j} CX_j, C_-\widetilde{X}_j \cup_{\widetilde{X}_j} C_-\widetilde{X}_j)$.

On the other hand $X_1 * \cdots X_{j-1} * X_j' * X_{j+1} * \cdots * X_m$ decomposes as the union of two copies of $X_1 * \cdots * X_m$ which intersect in $Z' = CX_1 \times \cdots \times C_+\widetilde{X}_j \times \cdots \times CX_m$. Applying the contraction $Z' \longrightarrow *$ produces a homotopy from g to a map g' such that $g'(Z') = *$, which represents the sum

$$h_w(f_1, \ldots, f_j, \ldots, f_m) + h_w(f_1, \ldots, f_j', \ldots, f_m)$$

and the result follows. □

Proposition 4.2.3. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ for $i = 1, \ldots, m$, and let $\rho\colon (i_1, \ldots, i_m) \longmapsto (1, \ldots, m)$ be a permutation. Let Z be a space with maps $\iota\colon FW(Y_1, \ldots, Y_m) \longrightarrow Z$ and*

$\iota'\colon FW(Y_{i_1},\dots,Y_{i_m}) \longrightarrow Z$ such that $\iota'(Y_{i_j}) = \iota(Y_{\rho(i_j)})$ for $j = 1,\dots,m$. Define

$$h_w^Z(f_1,\dots,f_m) = \iota \circ h_w(f_1,\dots,f_m), \qquad h_w^Z(f_{i_1},\dots,f_{i_m}) = \iota' \circ h_w(f_{i_1},\dots,f_{i_m})$$

and define a map $\sigma\colon X_{i_1} * \cdots * X_{i_m} \longrightarrow X_1 * \cdots * X_m$ permuting coordinates as the restriction of the product of maps $CX_{i_j} \longrightarrow CX_{\rho(i_j)}$ to the join. Then

$$h_w^Z(f_{i_1},\dots,f_{i_m}) = h_w^Z(f_1,\dots,f_m) \circ \sigma \tag{4.12}$$

and furthermore if $X_i = S^{p_i-1}$ for $i = 1,\dots,m$ then

$$h_w^Z(f_{i_1},\dots,f_{i_m}) = \epsilon(\rho) h_w^Z(f_1,\dots,f_m) \tag{4.13}$$

where $\epsilon(\rho)$ is the Koszul sign of ρ, that is, $\epsilon(\rho)$ has a factor $(-1)^{p_ip_j}$ for every transposition (i,j) of ρ.

Proof. Relation (4.12) follows from the definition of the higher Whitehead map. Relation (4.13) then follows since σ is then a map $S^{p_1+\cdots+p_m-1} \longrightarrow S^{p_1+\cdots+p_m-1}$ of degree $\epsilon(\rho)$. □

Example 4.2.4. If $X_1 = S^{p_1-1}$ and $X_2 = S^{p_2-1}$ are spheres and ρ is the permutation $(2,1) \longrightarrow (1,2)$ then we recover the graded symmetry of the Whitehead product [Whi41], namely

$$h_w^Z(f_2,f_1) = (-1)^{p_1p_2} h_w^Z(f_1,f_2).$$

Proposition 4.2.5. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$, $g_i\colon Y_i \longrightarrow Z_i$, and $h_i\colon W_i \longrightarrow X_i$ for all $i = 1,\dots,m$. Denote by $g\colon (\underline{Y},\underline{*})^{\partial\Delta^{m-1}} \longrightarrow (\underline{Z},\underline{*})^{\partial\Delta^{m-1}}$ the induced map of polyhedral products. Then*

$$h_w(g_1 \circ f_1,\dots,g_m \circ f_m) = g \circ h_w(f_1,\dots,f_m) \tag{4.14}$$

and

$$h_w(f_1 \circ \Sigma h_1,\dots,f_m \circ \Sigma h_m) = h_w(f_1,\dots,f_m) \circ \mathop{\Huge *}_{i=1}^{m} h_i. \tag{4.15}$$

Proof. Both statements follow immediately from functoriality of the polyhedral product by viewing $h_w(f_1,\dots,f_m)$ as the map of polyhedral products (4.10). □

We now consider when the map $h_w(f_1,\dots,f_m)$ is trivial. There is a homotopy cofibration

$$X_1 * \cdots * X_m \xrightarrow{\rho} FW(\Sigma X_1,\dots,\Sigma X_m) \longrightarrow \Sigma X_1 \times \cdots \times \Sigma X_m$$

which is derived analogously to Proposition 2.1.6, see [Por65, Theorem 2.3]. Therefore $h_w(f_1,\dots,f_m) = FW(f_1,\dots,f_m) \circ \rho$ is trivial if and only if there is a map $\phi\colon \Sigma X_1 \times$

$\cdots \times \Sigma X_m \longrightarrow FW(Y_1, \ldots, Y_m)$ extending $FW(f_1, \ldots, f_m)$. In particular, if f_i is trivial for some $i = 1, \ldots, m$, then ϕ can be chosen to be

$$f_1 \times \cdots \times f_m \simeq f_1 \times \cdots \times f_{i-1} \times * \times f_{i+1} \times \cdots \times f_m$$

and so $h_w(f_1, \ldots, f_m)$ is trivial.

On the other hand, the higher Whitehead map $h_w(f_1, \ldots, f_m)$ can be trivial even when the maps f_i are not. For example, let $f_1 \colon \Sigma M(\mathbb{Z}_2, 1) \longrightarrow Y_1$ and $f_2 \colon \Sigma M(\mathbb{Z}_3, 1) \longrightarrow Y_2$ be non-trivial, where $M(G, n)$ is the Moore space with reduced homology G concentrated in degree n. Then $M(\mathbb{Z}_2, 1) \wedge M(\mathbb{Z}_3, 1)$ is simply-connected and moreover, by the Künneth formula (2.3), has trivial homology in all positive degrees. Therefore it is contractible by Whitehead's theorem. In particular, the domain of the higher Whitehead map

$$h_w(f_1, f_2) \colon \Sigma M(\mathbb{Z}_2, 1) \wedge M(\mathbb{Z}_3, 1) \longrightarrow Y_1 \vee Y_2$$

is contractible, so $h_w(f_1, f_2)$ is trivial.

We give conditions on the maps $f_i \colon \Sigma X_i \longrightarrow Y_i$ which ensure that $h_w(f_1, \ldots, f_m)$ is non-trivial. Let $\hat{f}_i \colon X_i \longrightarrow \Omega Y_i$ be the adjoint to f_i for $i = 1, \ldots, m$.

Lemma 4.2.6. *The higher Whitehead map $h_w(f_1, \ldots, f_m)$ is trivial if and only if the map*

$$\hat{f}_1 * \cdots * \hat{f}_m \colon X_1 * \cdots * X_m \longrightarrow \Omega Y_1 * \cdots * \Omega Y_m$$

is trivial.

Proof. Consider the following diagram

$$\begin{tikzcd}
\prod_{i=1}^m \Omega Y_i \arrow[r, "*"] & \Omega Y_1 * \cdots * \Omega Y_m \arrow[r] & FW(Y_1, \ldots, Y_m) \arrow[r, "\iota"] & \prod_{i=1}^m Y_i \\
& & X_1 * \cdots * X_m \arrow[ul, dashed, "\hat{f}_1 * \cdots * \hat{f}_m"] \arrow[u, "h_w(f_1, \ldots, f_m)"'] \arrow[r] & \prod_{i=1}^m CX_i \arrow[u, "\theta"']
\end{tikzcd} \tag{4.16}$$

where the top row is homotopy fibration sequence (2.15) for $\mathcal{K} = \partial\Delta^{m-1}$. By definition, the map $h_w(f_1, \ldots, f_m)$ factors through the restriction of the map

$$\overline{\rho} \colon CX_1 \times \cdots \times CX_m \longrightarrow \Sigma X_1 \times \cdots \times \Sigma X_m$$

to the join $X_1 * \cdots * X_m$. Therefore the square in the diagram commutes, where $\theta = f_1 \times \cdots \times f_m \circ \overline{\rho}$, so the composite $\iota \circ h_w(f_1, \ldots, f_m)$ is trivial. Hence there is the dashed lift to the homotopy fibre of ι given by $\hat{f}_1 * \cdots * \hat{f}_m$. Therefore if $\hat{f}_1 * \cdots * \hat{f}_m$ is trivial, so is $h_w(f_1, \ldots, f_m)$.

On the other hand, if $h_w(f_1, \ldots, f_m)$ is trivial there is a further lift $X_1 * \cdots * X_m \longrightarrow \Omega Y_1 \times \cdots \times \Omega Y_m$. Then $\hat{f}_1 * \cdots * \hat{f}_m$ factors through the trivial map $\Omega Y_1 \times \cdots \times \Omega Y_m \longrightarrow \Omega Y_1 * \cdots * \Omega Y_m$, so is itself trivial. □

A homological condition which gives $\hat{f}_1 * \cdots * \hat{f}_m$ non-trivial is the following.

Definition 4.2.7. Let $f_i \colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \ldots, m$. Let $\hat{f}_i \colon X_i \longrightarrow \Omega Y_i$ be the adjoint to f_i for $i = 1, \ldots, m$. We say that the maps f_i are *compatible* if for each $i = 1, \ldots, m$ the induced map

$$(\hat{f}_i)_* \colon H_{n_i}(X_i) \longrightarrow H_{n_i}(\Omega Y_i)$$

is non-trivial for some $n_i > 0$, and moreover the tensor products $H_{n_1}(X_1) \otimes \cdots \otimes H_{n_m}(X_m)$ and $H_{n_1}(\Omega Y_1) \otimes \cdots \otimes H_{n_m}(\Omega Y_m)$ are non-trivial.

The conditions on the tensor products are automatically satisfied if, for example, the groups $H_{n_i}(X_i)$ and $H_{n_i}(\Omega Y_i)$ contain a $\mathbb{Z}$ summand for each $i = 1, \ldots, m$.

Example 4.2.8.

(i) The maps $f_1 \colon \Sigma M(\mathbb{Z}_2, 1) \longrightarrow Y_1$ and $f_2 \colon \Sigma M(\mathbb{Z}_3, 1) \longrightarrow Y_2$ are not compatible since $H_1(M(\mathbb{Z}_2, 1)) = \mathbb{Z}_2$, while $H_1(M(\mathbb{Z}_3, 1)) = \mathbb{Z}_3$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_3 = 0$.

(ii) The maps $\mu_i \colon S^2 \longrightarrow \mathbb{C}P^\infty$ are compatible since $\hat{\mu}_i \colon S^1 \longrightarrow \Omega\mathbb{C}P^\infty = S^1$ induces a non-trivial map on H_1 between $\mathbb{Z}$ summands.

(iii) For $n_i \geqslant 2$, the identity maps $f_i \colon M(\mathbb{Z}_{p_i}, n_i) \longrightarrow M(\mathbb{Z}_{p_i}, n_i)$ are compatible if $\gcd(p_1, \ldots, p_m) > 1$.

Lemma 4.2.9. *Suppose that $f_1, \ldots, f_m$ are compatible. Then the map*

$$\hat{f}_1 * \cdots * \hat{f}_m \colon X_1 * \cdots * X_m \longrightarrow \Omega Y_1 * \cdots * \Omega Y_m$$

is non-trivial.

Proof. Let $N = n_1 + \cdots + n_m$. Then by the Künneth Theorem (2.1) and the conditions on $H_{n_i}(X_i)$, the group $H_N(X_1 \wedge \cdots \wedge X_m)$ contains

$$H_{n_1}(X_1) \otimes \cdots \otimes H_{n_m}(X_m)$$

as a non-trivial summand, and similarly for $H_N(\Omega Y_1 \wedge \cdots \wedge \Omega Y_m)$. Moreover the map

$$(\hat{f}_1)_* \otimes \cdots \otimes (\hat{f}_m)_* \colon H_{n_1}(X_1) \otimes \cdots \otimes H_{n_m}(X_m) \longrightarrow H_{n_1}(\Omega Y_1) \otimes \cdots \otimes H_{n_m}(\Omega Y_m)$$

is non-trivial by assumption. Therefore by naturality of the sequence (2.1), the map

$$H_N(X_1 \wedge \cdots \wedge X_m) \longrightarrow H_N(\Omega Y_1 \wedge \cdots \wedge \Omega Y_m)$$

is non-trivial. Therefore the map

$$\begin{aligned} H_{N+m-1}(X_1 * \cdots * X_m) &\xrightarrow{\cong} H_{N+m-1}(\Sigma^{m-1} X_1 \wedge \cdots \wedge X_m) \\ &\xrightarrow{\cong} H_N(X_1 \wedge \cdots \wedge X_m) \\ &\longrightarrow H_N(\Omega Y_1 \wedge \cdots \wedge \Omega Y_m) \\ &\xrightarrow{\cong} H_{N+m-1}(\Sigma^{m-1} \Omega Y_1 \wedge \cdots \wedge \Omega Y_m) \\ &\xrightarrow{\cong} H_{N+m-1}(\Omega Y_1 * \cdots * \Omega Y_m) \end{aligned}$$

is also non-trivial. The claim follows. □

Putting the above together we have established the following.

Theorem 4.2.10. *Let $f_i \colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \ldots, m$. Then the higher Whitehead map*

$$h_w(f_1, \ldots, f_m) \colon \mathop{\Huge *}_{i=1}^{m} X_i \longrightarrow FW(Y_1, \ldots, Y_m)$$

is trivial if f_i is trivial for some $i = 1, \ldots, m$, and non-trivial if the maps $f_1, \ldots, f_m$ are compatible. □

We rephrase the previous result in the following combinatorial way, which will be useful when considering generalisations of the higher Whitehead map.

Corollary 4.2.11. *Let $\mathcal{K}$ be a simplicial complex which contains $\partial\Delta^{m-1}$. Let $h_w^{\mathcal{K}}(f_1, \ldots, f_m)$ denote the composite of $h_w(f_1, \ldots, f_m)$ with the map $\iota \colon (\underline{Y}, \underline{*})^{\partial\Delta} \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}}$ induced by the inclusion $\partial\Delta^{m-1} \longrightarrow \mathcal{K}$. Then if $f_1, \ldots, f_m$ are compatible, the map $h_w^{\mathcal{K}}(f_1, \ldots, f_m)$ is trivial if and only if $\mathcal{K}$ contains Δ^{m-1}.*

Proof. Suppose that $\mathcal{K}$ contains Δ^{m-1}. Then $h_w^{\mathcal{K}}(f_1, \ldots, f_m) = \iota \circ h_w(f_1, \ldots, f_m)$ is trivial by the right-hand square of diagram (4.16) since $(\underline{Y}, \underline{*})^{\mathcal{K}}$ contains $Y_1 \times \cdots \times Y_m$. Conversely, if $\mathcal{K}$ does not contain Δ^{m-1}, then the full subcomplex of $\mathcal{K}$ on $[m]$ is $\partial\Delta^{m-1}$, and the result follows by Theorem 4.2.10. □

Before exemplifying our results, we relate the higher Whitehead map to the higher Whitehead product of Porter [Por65].

Proposition 4.2.12. *Denote by $\iota_i \colon Y_i \longrightarrow FW(Y_1, \ldots, Y_m)$ the inclusion of Y_i into the ith coordinate. Then the higher Whitehead map is an element of the higher Whitehead product*

$$h_w(f_1, \ldots, f_m) \in [\iota_1 \circ f_1, \ldots, \iota_m \circ f_m].$$

Proof. This is immediate from the definitions of the higher Whitehead map and higher Whitehead product since the map $FW(f_1, \ldots, f_k)$ is an extension of the wedge of maps $f_1, \ldots, f_k$. □

Example 4.2.13. For $m \geqslant 2$, let $\mu_i \colon S^2 \longrightarrow \mathbb{C}P^\infty$ be cellular inclusions for $i = 1, \ldots, m$. Then since the μ_i are compatible, the map

$$h_w(\mu_1, \ldots, \mu_m) \colon S^{2m-1} \longrightarrow FW(\mathbb{C}P^\infty, \ldots, \mathbb{C}P^\infty)$$

is non-trivial by Theorem 4.2.10. Furthermore, the map $h_w(\mu_1, \ldots, \mu_m)$ is an element of the higher Whitehead product $[\iota_1 \circ \mu_1, \ldots, \iota_m \circ \mu_m]$ by Proposition 4.2.12.

In this case, this higher Whitehead product contains only $h_w(\mu_1, \ldots, \mu_m)$. To see this, by a result of Porter [Por65, Theorem 2.7], the higher Whitehead product $[\iota_1 \circ \mu_1, \ldots, \iota_m \circ \mu_m]$ is non-empty if and only if all subproducts $[\iota_{i_1} \circ \mu_{i_1}, \ldots, \iota_{i_{m-1}} \circ \mu_{i_{m-1}}]$, for $\{i_1, \ldots, i_{m-1}\}$ an ordered subset of $\{1, \ldots, m\}$, contain the trivial map; and moreover any element of $[\iota_1 \circ \mu_1, \ldots, \iota_m \circ \mu_m]$ is determined by choices of null-homotopy of the trivial maps in the length $(m-1)$ subproducts. Working inductively, suppose each length $(m-1)$ subproduct $[\iota_{i_1} \circ \mu_{i_1}, \ldots, \iota_{i_{m-1}} \circ \mu_{i_{m-1}}]$ contains only $h_w(\mu_{i_1}, \ldots, \mu_{i_{m-1}})$, which is trivial by Corollary 4.2.11 since $\mathcal{K} = \partial\Delta^{m-1}$ contains the simplex $(i_1, \ldots, i_{m-1})$. Then any choice of null-homotopy of $h_w(\mu_{i_1}, \ldots, \mu_{i_{m-1}}) \in \pi_{2m-1}(FW(\mathbb{C}P^\infty_{i_1}, \ldots, \mathbb{C}P^\infty_{i_{m-1}}))$ is an element of the relative homotopy group

$$\pi_{2m}\left(\mathbb{C}P^\infty_{i_1} \times \cdots \times \mathbb{C}P^\infty_{i_{m-1}}, FW\left(\mathbb{C}P^\infty_{i_1}, \ldots, \mathbb{C}P^\infty_{i_{m-1}}\right)\right).$$

Then since $\mathbb{C}P^\infty_{i_1} \times \cdots \times \mathbb{C}P^\infty_{i_{m-1}}$ has trivial homotopy groups above dimension 2, the long exact sequence of Proposition 2.1.3(i) implies that the above relative homotopy group is isomorphic to $\pi_{2m-1}(FW(\mathbb{C}P^\infty_{i_1}, \ldots, \mathbb{C}P^\infty_{i_{m-1}}))$. Therefore there is a unique choice of null-homotopy for the trivial map $h_w(\mu_{i_1}, \ldots, \mu_{i_{m-1}})$ and the claim follows.

In Example 4.2.23, we use the higher Whitehead map to construct higher Whitehead products with non-trivial indeterminacy, that is, higher Whitehead products which contain more than one homotopy class of maps.

4.2.3 The higher Whitehead map with substitution

Viewing the higher Whitehead map as the map of polyhedral products (4.10) readily leads to a much wider class of higher Whitehead maps whose triviality can be controlled combinatorially. We begin with a motivating example.

Example 4.2.14. Let Y_1 and Y_2 be spaces and suppose that $Y_3 = Y_{3_1} \vee Y_{3_2}$ for spaces Y_{3_1} and Y_{3_2}. Consider maps $f_i \colon \Sigma X_i \longrightarrow Y_i$ for $i \in \{1, 2, 3_1, 3_2\}$ and define $f_3 = h_w(f_{3_1}, f_{3_2}) \colon \Sigma X_3 \longrightarrow Y_3$ where $X_3 = X_{3_1} \wedge X_{3_2}$.

Consider the higher Whitehead map

$$h_w(f_1, f_2, f_3) \colon X_1 * X_2 * X_3 \longrightarrow FW(Y_1, Y_2, Y_3).$$

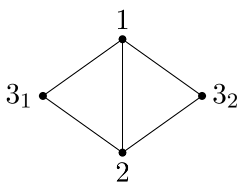
We rewrite the fat wedge $FW(Y_1, Y_2, Y_3)$ as a different polyhedral product which explicitly sees the codomain $Y_{3_1} \vee Y_{3_2}$ of f_3 as follows. Using the definition of the fat wedge we decompose

$$\begin{aligned} FW(Y_1, Y_2, Y_3) &= FW(Y_1, Y_2, Y_{3_1} \vee Y_{3_2}) \\ &= (Y_1 \times Y_2 \times *) \cup (* \times Y_2 \times (Y_{3_1} \vee Y_{3_2})) \cup (Y_1 \times * \times (Y_{3_1} \vee Y_{3_2})). \end{aligned}$$

Writing $Y_{3_1} \vee Y_{3_2} = (Y_{3_1} \times *) \cup (* \times Y_{3_2})$ and viewing its basepoint as the product of basepoints of Y_{3_1} and Y_{3_2} we obtain

$$\begin{aligned} &(Y_1 \times Y_2 \times * \times *) \cup (* \times Y_2 \times Y_{3_1} \times *) \cup (* \times Y_2 \times * \times Y_{3_2}) \\ &\qquad\qquad \cup (Y_1 \times * \times Y_{3_1} \times *) \cup (Y_1 \times * \times * \times Y_{3_2}) \end{aligned}$$

which is the polyhedral product $(\underline{Y}, \underline{*})^{\mathcal{K}}$ where $(\underline{Y}, \underline{*}) = \{(Y_i, *_i)\}_i$ for $i \in \{1, 2, 3_1, 3_2\}$ and $\mathcal{K}$ is the following simplicial complex



The complex $\mathcal{K}$ is an example of a substitution complex, which are studied extensively in [AP19]. It carries not only the structure of the boundary simplex $\partial\Delta[1, 2, 3]$ required to define $h_w(f_1, f_2, f_3)$, as seen by the missing faces $(1, 2, 3_1)$ and $(1, 2, 3_2)$, but also carries as a full subcomplex the boundary $\partial\Delta[3_1, 3_2]$ defining $f_3 = h_w(f_{3_1}, f_{3_2})$. Our subsequent analysis will show how the combinatorial conditions of Theorem 4.2.10 controlling the triviality of $h_w(f_1, f_2, f_3)$ and $h_w(f_{3_1}, f_{3_2})$ amalgamate to give combinatorial conditions controlling the triviality of the nested higher Whitehead map $h_w(f_1, f_2, h_w(f_{3_1}, f_{3_2}))$.

We first define substitution complexes as a special case of the more general polyhedral join product. A simplicial pair $(\mathcal{S}, \mathcal{T})$ consists of simplicial complexes $\mathcal{S}$ and $\mathcal{T}$ both with vertex set $[l]$ such that $\mathcal{T}$ is a subcomplex of $\mathcal{S}$.

Definition 4.2.15. Let $\mathcal{K}$ be a simplicial complex on $[m]$, and let $(\mathcal{S}_i, \mathcal{T}_i)$ be a simplicial pair on $[l_i]$ for $i = 1, \ldots, m$. Let $(\underline{\mathcal{S}}, \underline{\mathcal{T}}) = \{(\mathcal{S}_i, \mathcal{T}_i)\}_{i=1}^m$ be an m-tuple of simplicial pairs. The *polyhedral join product* is the simplicial complex on vertex set $[l_1] \sqcup \cdots \sqcup [l_m]$ defined by

$$(\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}} (\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\sigma} \subseteq \mathop{\Large *}_{i=1}^{m} \mathcal{S}_i, \quad \text{where } (\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\sigma} = \mathop{\Large *}_{i=1}^{m} \mathcal{Y}_i, \quad \mathcal{Y}_i = \begin{cases} \mathcal{S}_i & \text{for } i \in \sigma \\ \mathcal{T}_i & \text{for } i \notin \sigma. \end{cases}$$

If $\mathcal{T}_i = \{\varnothing\}$ for all i, then the polyhedral join $(\underline{\mathcal{S}}, \underline{\{\varnothing\}})^{*\mathcal{K}}$ is called the substitution of $\mathcal{S}_1, ..., \mathcal{S}_m$ into $\mathcal{K}$, and denoted by $\mathcal{K}\langle \mathcal{S}_1, ..., \mathcal{S}_m \rangle$, see [AP19]. If $\mathcal{S}_i = \Delta^{l_i - 1}$ for all i, then $(\underline{\Delta}^{l_i-1}, \underline{\mathcal{T}})^{*\mathcal{K}}$ is called the composition of $\mathcal{K}$ with $\mathcal{T}_1, ..., \mathcal{T}_m$, and denoted by $\mathcal{K}(\mathcal{T}_1, ..., \mathcal{T}_m)$, see [Ayz13]. In the case that $\mathcal{K} = \Delta^{m-1}$ or $\mathcal{K} = \partial\Delta^{m-1}$, we suppress notation by denoting the corresponding substitution complexes by $\Delta\langle \mathcal{S}_1, \ldots, \mathcal{S}_m \rangle$ and $\partial\Delta\langle \mathcal{S}_1, \ldots, \mathcal{S}_m \rangle$, respectively, and adopt the similar abbreviation for composition complexes. We observe that for any simplicial complex $\mathcal{K}$, $\mathcal{K}\langle \bullet, \ldots, \bullet \rangle = \mathcal{K} = \mathcal{K}(\circ, \ldots, \circ)$, where $\circ$ denotes the empty complex on a single vertex.

The polyhedral join product is functorial with respect to simplicial inclusions $\mathcal{K} \longrightarrow \mathcal{L}$, up to how $\mathcal{K}$ sits inside $\mathcal{L}$, and maps of pairs of simplicial complexes.

It was proven by Vidaurre [Vid17, Theorem 2.9] that the polyhedral join product and the polyhedral product are related in the following way

$$(\underline{X}, \underline{A})^{(\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}} = \left(\underline{(\underline{X}, \underline{A})^{\mathcal{S}_i}}, \underline{(\underline{X}, \underline{A})^{\mathcal{T}_i}}\right)^{\mathcal{K}}. \tag{4.17}$$

where on the right hand side of the equality, we abbreviate notation by using $\underline{X}$ to denote the tuple corresponding to the vertex set of $\mathcal{K}_i$ and $\mathcal{L}_i$, for each i. This formalises the calculation of Example 4.2.14.

We will return to composition complexes when we consider relations among higher Whitehead maps. For now, we continue to build towards the definition of substituted Whitehead maps.

Example 4.2.16. Continuing Example 4.2.14, the simplicial complex $\mathcal{K}$ is the substitution complex

$$\mathcal{K} = \partial\Delta\langle \bullet_1, \bullet_2, \partial\Delta\langle \bullet_{3_1}, \bullet_{3_2} \rangle\rangle.$$

Therefore the higher Whitehead map $h_w(f_1, f_2, f_3) \colon X_1 * X_2 * X_3 \longrightarrow FW(Y_1, Y_2, Y_3)$ is the same as the higher Whitehead map

$$h_w(\mu_1, \mu_2, h_w(\mu_{3_1}, \mu_{3_2})) \colon X_1 * X_2 * (X_{3_1} \wedge X_{3_2}) \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}}.$$

We are therefore led to define substituted higher Whitehead maps as follows.

Definition 4.2.17. Let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be simplicial complexes on $[l_1], \ldots, [l_m]$, respectively, and let $f_i \colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i}$ be maps for $i = 1, \ldots, m$. Then the *substituted higher Whitehead map* $h_w(f_1, \ldots, f_m)$ is given by the composite

$$h_w(f_1, \ldots, f_m) \colon (\underline{CX}, \underline{X})^{\partial\Delta} \longrightarrow (\underline{(\underline{Y}, \underline{*})^{\mathcal{K}_i}}, \underline{*})^{\partial\Delta} = (\underline{Y}, \underline{*})^{\partial\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_m \rangle}.$$

If $\mathcal{K}$ is a simplicial complex such that $\partial\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle \subseteq \mathcal{K}$, we define $h_w^{\mathcal{K}}(f_1,\dots,f_m)$ to be the composite

$$h_w^{\mathcal{K}}(f_1,\dots,f_m)\colon (\underline{CX},\underline{X})^{\partial\Delta^{m-1}} \longrightarrow (\underline{Y},\underline{*})^{\partial\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}} \tag{4.18}$$

where the first map is $h_w(f_1,\dots,f_m)$ and the second map is induced by the inclusion $\partial\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle \longrightarrow \mathcal{K}$.

Example 4.2.18. A special case of substituted higher Whitehead maps that we consider are nested higher Whitehead map, defined as follows.

A nested higher Whitehead map of depth 0 is any map $f\colon \Sigma X \longrightarrow Y$. A nested higher Whitehead map of depth $n \geqslant 1$ is a higher Whitehead map of maps $f_1,\dots,f_m$, where each f_i is a nested higher Whitehead map of depth at most $n-1$, with the depth of some f_i equal to $n-1$.

Moreover, to a nested higher Whitehead map we associate a *defining complex* $\mathcal{K}$ as follows. For a higher Whitehead map of depth 0 we set $\mathcal{K} = \bullet$. Then to a nested higher Whitehead map $h_w(f_1,\dots,f_m)$ of depth $n \geqslant 1$ we set $\mathcal{K} = \partial\Delta\langle\mathcal{L}_1,\dots,\mathcal{L}_m\rangle$, where $\mathcal{L}_i$ is the defining complex of f_i for each $i = 1,\dots,m$.

We consider when substituted higher Whitehead maps are trivial, before specialising to nested higher Whitehead maps. It follows from (4.17) that if there exists $i = 1,\dots,m$ such that $\mathcal{K}_i$ has at least two vertices, there exist complexes $\mathcal{K}$ such that $\partial\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle \subsetneq \mathcal{K} \subsetneq \Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle$. We characterise when the composite

$$h_w^{\mathcal{K}}(f_1,\dots,f_m)\colon \mathop{\ast}_{i=1}^{m} X_i \xrightarrow{h_w} (\underline{Y},\underline{*})^{\partial\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}} \tag{4.19}$$

is trivial in terms of the combinatorics of $\mathcal{K}$.

We introduce the following terminology. Given a map $f\colon \Sigma X \longrightarrow (\underline{Y},*)^{\mathcal{K}}$ which is non-trivial, then for a simplicial complex $\mathcal{K}'$ on $[m]$ such that $\mathcal{K} \subseteq \mathcal{K}'$, if the composite $f'\colon \Sigma X \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}'}$ is trivial, we say that $\mathcal{K}'$ is a trivialising complex for f.

Proposition 4.2.19. *Let $\mathcal{K}$ be a complex on $[l_1]\sqcup\cdots\sqcup[l_m]$ such that $\partial\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle \subseteq \mathcal{K}$. For all $i = 1,\dots,m$, suppose that $f_i\colon \Sigma X_i \longrightarrow (\underline{Y}_i,\underline{*})^{\mathcal{K}_i}$. Then $h_w^{\mathcal{K}}(f_1,\dots,f_m)$ is trivial if at least one of the following is satisfied:*

(i) the map $f_i\colon \Sigma X_i \longrightarrow (\underline{Y}_i,\underline{*})^{\mathcal{K}_i}$ is trivial for some $i = 1,\dots,m$;

(ii) $\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle \subseteq \mathcal{K}$;

(iii) $\mathcal{K}$ contains $\partial\Delta\langle\mathcal{K}_1,\dots,\mathcal{K}_i',\dots,\mathcal{K}_m\rangle$, where $\mathcal{K}_i'$ is a trivialising complex for f_i for some $i = 1,\dots,m$.

Moreover, if (i) and (iii) are not satisfied and the f_i are compatible, the map $h_w^{\mathcal{K}}(f_1, \dots, f_m)$ is trivial if and only if $\mathcal{K}$ contains $\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle$.

Proof. First we show that if any of (i) – (iii) hold, then $h_w^{\mathcal{K}}(f_1, \dots, f_m)$ is trivial. If either (i) or (ii) hold, then this is immediate from Theorem 4.2.10.

Alternatively, if (iii) holds, then there is a map $f_i \colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i}$ such that the composite $f_i' \colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i} \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i'}$ is trivial. Then by naturality of the polyhedral product we have

$$h_w^{\mathcal{K}}(f_1, \dots, f_m) = h_w^{\mathcal{K}}(f_1, \dots, f_i', \dots, f_m)$$

where the latter is trivial by Theorem 4.2.10.

For the second claim, if neither (i) nor (iii) hold and the f_i are compatible, then by Corollary 4.2.11 we conclude that since

$$h_w(f_1, \dots, f_m) \colon \mathop{\circledast}_{i=1}^{m} X_i \longrightarrow ((\underline{Y}, \underline{*})^{\mathcal{K}_i}, \underline{*})^{\partial\Delta} = (\underline{Y}, \underline{*})^{\partial\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle}$$

is non-trivial, then $\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle \subseteq \mathcal{K}$, as claimed. $\square$

We now specialise to the case that the maps f_i are nested higher Whitehead maps. In this case the triviality of $h_w(f_1, \dots, f_m)$ can be determined combinatorially provided that all the maps used to build $f_1, \dots, f_m$ are compatible.

Theorem 4.2.20. *Suppose that $g_{i_j} \colon \Sigma X_i \longrightarrow Y_i$ are compatible maps for $i = 1, \dots, m$ and $j = 1, \dots, k_i$. Let $f_i = h_w(g_{i_1}, \dots, g_{i_{k_i}})$ and let $\mathcal{K}_i$ be the defining complex for f_i. Then if $\mathcal{K}$ is a simplicial complex on $[l_1] \sqcup \cdots \sqcup [l_m]$ containing $\partial\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle$ which does not contain $\partial\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_i', \dots, \mathcal{K}_m\rangle$ for any trivialising complex $\mathcal{K}_i'$ of f_i, then*

$$h_w^{\mathcal{K}}(f_1, \dots, f_m) \colon \mathop{\circledast}_{i=1}^{m} (\Sigma^{k_i-2} X_{i_1} \wedge \cdots \wedge X_{i_{k_i}}) \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}}$$

is trivial if and only if $\mathcal{K}$ contains $\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle$.

Proof. This will follow immediately from Proposition 4.2.19 once we show that the maps f_i are compatible. More generally, let $g_i \colon \Sigma X_i \longrightarrow Y_i$ for $i = 1, \dots, m$ be compatible maps, where $\hat{g}_i \colon X_i \longrightarrow \Omega Y_i$ induces a non-trivial map $H_{n_i}(X_i) \longrightarrow H_{n_i}(\Omega Y_i)$ for $n_i > 0$. Then there is a commutative diagram

$$\begin{array}{ccc}
H_{N+m-2}(\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m) & \longrightarrow & H_{N+m-2}(\Omega(\Omega Y_1 * \cdots * \Omega Y_m)) \\
\big\downarrow & & \big\downarrow \\
H_{N+m-1}(\Sigma^{m-1} X_1 \wedge \cdots \wedge X_m) & \longrightarrow & H_{N+m-1}(\Omega Y_1 * \cdots * \Omega Y_m)
\end{array}$$

where $N = n_1 + \cdots + n_m$, the left map is an isomorphism, the bottom map is induced by $\hat{g}_1 * \cdots * \hat{g}_m$, the top map induced by its adjoint and the right map is the homology suspension, see [Whi55]. By the proof of Lemma 4.2.9, since the g_i are compatible, the composite around the bottom of the square is non-trivial. Therefore the composite around the top of the square is non-trivial, and in particular the map

$$H_{N+m-2}(\Sigma^{m-2}X_1 \wedge \cdots \wedge X_m) \longrightarrow H_{N+m-2}(\Omega(\Omega Y_1 * \cdots * \Omega Y_m)$$

is non-trivial.

Therefore using decomposition (2.16)

$$\Omega FW(Y_1, \ldots, Y_m) \simeq \Omega(\Omega Y_1 * \cdots * \Omega Y_m) \times \prod_{i=1}^{m} \Omega Y_i$$

we conclude that

$$\begin{aligned}\hat{h}_*\colon H_{N+m-2}(\Sigma^{m-2}X_1 \wedge \cdots \wedge X_m) &\longrightarrow H_{N+m-2}(\Omega(\Omega Y_1 * \cdots * \Omega Y_m) \\ &\longrightarrow H_{N+m-2}(\Omega FW(Y_1, \ldots, Y_m))\end{aligned}$$

is non-trivial. Therefore the adjoint of $h_w(g_1, \ldots, g_m)$ induces a non-trivial map in homology. Moreover, the Künneth Theorem (2.1) and the conditions on $H_{n_i}(X_i)$ and $H_{n_i}(\Omega Y_i)$ ensure that the domain and codomain of $\hat{h}_*$ also satisfy the non-vanishing conditions of compatibility. □

Before giving some examples of Theorem 4.2.20, we observe that the higher Whitehead map $h_w^{\mathcal{K}}(f_1, \ldots, f_m)$ is an element of a certain higher Whitehead product. We exploit this fact firstly in constructing non-trivial elements of certain higher Whitehead products, and secondly in constructing an infinite family of trivial higher Whitehead products with non-trivial indeterminacy.

Proposition 4.2.21. *Let $\mathcal{K}$ be a complex on $[l_1] \sqcup \cdots \sqcup [l_m]$ such that $\partial\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_m \rangle \subseteq \mathcal{K}$. Then*

$$h_w^{\mathcal{K}}(f_1, \ldots, f_m) \in [f_1^{\mathcal{K}}, \ldots, f_m^{\mathcal{K}}]$$

where $f_i^{\mathcal{K}}\colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}}$ denotes the composite of f_i with the map of polyhedral products $(\underline{Y}, \underline{*})^{\mathcal{K}_i} \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}}$ induced by the simplicial inclusion $\mathcal{K}_i \longrightarrow \mathcal{K}$.

Proof. The following diagram commutes

$$\begin{tikzcd} \bigvee_{i=1}^m \Sigma X_i \arrow[r, equal] \arrow[d, hook] & (\underline{\Sigma X}, \underline{*})^{\bullet^{m-1}} \arrow[r] \arrow[d, hook] & (\underline{Y}, \underline{*})^{\bullet^{m-1}} \arrow[r] \arrow[d, hook] & (\underline{Y}, \underline{*})^{\mathcal{K}} \arrow[d, equal] \\ FW(\Sigma X_1, \ldots, \Sigma X_m) \arrow[r, equal] & (\underline{\Sigma X}, \underline{*})^{\partial\Delta^{m-1}} \arrow[r] & (\underline{Y}, \underline{*})^{\partial\Delta^{m-1}} \arrow[r] & (\underline{Y}, \underline{*})^{\mathcal{K}} \end{tikzcd}$$

by naturality of the polyhedral product with respect to simplicial inclusions and continuous maps of pairs. The map $h_w^{\mathcal{K}}(f_1,\ldots,f_m)$ is the composite of $\rho\colon X_1 * \cdots * X_m \longrightarrow FW(\Sigma X_1,\ldots,\Sigma X_m)$ with the bottom row of the diagram, while the composite along the top row is the map $\bigvee_{i=1}^m f_i^{\mathcal{K}}$. Therefore $h_w^{\mathcal{K}}(f_1,\ldots,f_m)$ is the composite of ρ with an extension of the map $\bigvee_{i=1}^m f_i^{\mathcal{K}}$ to $FW(\Sigma X_1,\ldots,\Sigma X_m)$. $\square$

Example 4.2.22. Let $\mathcal{K}_1 = \mathcal{K}_2 = \bullet$ and let $\mathcal{K}_3 = \bullet_{[3_1,3_2]}$. Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be compatible maps for $i \in \{1,2,3_1,3_2\}$. Let $Y_3 = Y_{3_1} \vee Y_{3_2}$ and define $f_3 = h_w(f_{3_1},f_{3_2})\colon X_{3_1} * X_{3_2} \longrightarrow Y_3$. Let $\mathcal{K} = \partial\Delta\langle\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\rangle$ and consider the complexes $\mathcal{L}_1 = \mathcal{K} \cup \Delta[3_1,3_2]$, $\mathcal{L}_2 = \mathcal{K} \cup \Delta[1,2,3_1] \cup \Delta[1,2,3_2]$ and $\mathcal{L}_3 = \mathcal{K} \cup \Delta[1,3_1,3_2] \cup \Delta[2,3_1,3_2]$.

Consider the map

$$h_w^{\mathcal{L}_i}(f_1,f_2,h_w(f_{3_1},f_{3_2}))\colon X_1 * X_2 * X_3 \longrightarrow (\underline{Y},\underline{*})^{\partial\Delta\langle\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\rangle} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{L}_i}$$

for $i = 1,2,3$. By Theorem 4.2.20, this map is trivial for $i = 2$ since $\mathcal{L}_2 = \Delta\langle\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\rangle$, and also trivial for $i = 3$ since $\Delta[3_1,3_2]$ is a trivialising complex for $f_3 = h_w(3_1,3_2)$ and $\mathcal{L}_3 = \partial\Delta\langle\mathcal{K}_1,\mathcal{K}_2,\Delta[3_1,3_2]\rangle$. For $i = 1$, on the other hand, notice that $\Delta[3_1,3_2]$ is the only trivialising complex for $h_w(f_{3_1},f_{3_2})$ on the vertex set $\{3_1,3_2\}$ by Corollary 4.2.11. Therefore since the substitution complex $\partial\Delta\langle\mathcal{K}_1,\mathcal{K}_2,\Delta[3_1,3_2]\rangle$ is not contained in $\mathcal{L}_1$, condition (iii) of Theorem 4.2.20 is not satisfied. Therefore since the f_i are compatible and $\mathcal{L}_1$ does not contain $\Delta\langle\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\rangle$, then $h_w^{\mathcal{L}_1}(f_1,f_2,h_w(f_{3_1},f_{3_2}))$ is non-trivial.

Moreover, using Proposition 4.2.21 we have that $h_w^{\mathcal{L}_i}(f_1,f_2,h_w(f_{3_1},f_{3_2}))$ is a non-trivial element of the higher Whitehead product

$$\left[f_1^{\mathcal{L}_i}, f_2^{\mathcal{L}_i}, h_w^{\mathcal{L}_i}(f_{3_1},f_{3_2})\right]$$

for $i = 1$, while it is a trivial element for $i = 2,3$.

In general, the higher Whitehead product $[f_1^{\mathcal{K}},\ldots,f_m^{\mathcal{K}}]$ can contain many more elements than just $h_w^{\mathcal{K}}(f_1,\ldots,f_m)$. If this is the case, the higher Whitehead product is said to have non-trivial indeterminacy. In the following example, we construct an infinite family of higher Whitehead products with non-trivial indeterminacy.

Example 4.2.23. Let $p \geqslant 2$ and $q \geqslant 3$, $\mathcal{K}_1 = \partial\Delta^{p-1}$ and for all $2 \leqslant i \leqslant q$, let $\mathcal{K}_i = \bullet$. Define a complex $\mathcal{K}$ on $[p+q-1]$ by

$$\begin{aligned}\mathcal{K} &= \partial\Delta^{q-1}\langle\mathcal{K}_1,\ldots,\mathcal{K}_q\rangle \cup \Delta[1,\ldots,p]\\ &= \partial\Delta^{q-1}\langle\partial\Delta[1,\ldots,p], p+1,\ldots,p+q-1\rangle \cup \Delta[1,\ldots,p].\end{aligned}$$

For $i = 1,\ldots,p$, let $f_{1_i}\colon \Sigma X_{1_i} \longrightarrow (\underline{Y}^{1_i},*)^{\mathcal{K}_{1_i}}$ be non-trivial, and let $f_1 = h_w^{\partial\Delta}(f_{1_1},\ldots,f_{1_p})$. For $i = 2,\ldots,q$, let $f_i\colon \Sigma X_i \longrightarrow (\underline{Y}^i,*)^{\mathcal{K}_i}$ be non-trivial. By Proposition 4.2.21, the map $h_w^{\mathcal{K}}(f_1,\ldots,f_q)$ is an element of the higher Whitehead product $[f_1^{\mathcal{K}},\ldots,f_q^{\mathcal{K}}]$.

Since $\mathcal{K}$ contains $\Delta[1,\dots,p]$, the map $f_1^{\mathcal{K}}$ is trivial. Therefore $[f_1^{\mathcal{K}},\dots,f_q^{\mathcal{K}}]$ contains a trivial element by [Por65, Theorem 2.4]. On the other hand, if all the f_i are compatible, then by Theorem 4.2.20, the map $h_w^{\mathcal{K}}(f_1,\dots,f_q)$ is non-trivial since $\mathcal{K}$ contains neither $\Delta^{q-1}\langle\partial\Delta[1,\dots,p],p+1,\dots,p+q-1\rangle$ nor $\partial\Delta^{q-1}\langle\Delta[1,\dots,p],p+1,\dots,p+q-1\rangle$. We therefore obtain that the higher Whitehead product $[f_1^{\mathcal{K}},\dots,f_q^{\mathcal{K}}]$ has non-trivial indeterminacy.

The case where $p=q=3$ was considered in [AP19] in the case that $f_i=\mu_i\colon S^2\longrightarrow\mathbb{C}P^\infty$ for each i. In particular it was observed that there is a map $S^{10}\longrightarrow(\mathbb{C}P^\infty,*)^{\mathcal{K}}=DJ_{\mathcal{K}}$ which is not an element of a non-trivial higher Whitehead product. Our results show that this composite is the higher Whitehead map $h_w^{\mathcal{K}}(f_1,\dots,f_q)$, which is a non-trivial element of the trivial higher Whitehead product $[f_1^{\mathcal{K}},\dots,f_q^{\mathcal{K}}]$.

4.2.4 The folded higher Whitehead map

Let $f_i\colon\Sigma X_i\longrightarrow Y$ for $i=1,2$. The composite of the higher Whitehead map

$$h_w(f_1,f_2)\colon X_1*X_2\longrightarrow\Sigma X_1\vee\Sigma X_2\longrightarrow Y\vee Y.$$

with the fold map $\nabla\colon Y\vee Y\longrightarrow Y$ is the Whitehead product $[f_1,f_2]\in[X_1*X_2,Y]$. Determining whether such a Whitehead product is trivial or not is a classical problem, whose solution depends upon internal properties of the spaces in question, see [Mah65], for example.

In the case that Y is an H-space, the Whitehead product $[f_1,f_2]$ factors through the map

$$\Sigma X_1\times\Sigma X_2\xrightarrow{f_1\times f_2}Y\times Y\xrightarrow{\mu}Y$$

where $\mu\colon Y\times Y\longrightarrow Y$ is the H-multiplication map on Y. Therefore $[f_1,f_2]$ is trivial, since it factors through $\Sigma X_1\times\Sigma X_2$, which is the homotopy cofibre of the map $X_1*X_2\longrightarrow\Sigma X_1\vee\Sigma X_2$ by Proposition 2.1.6.

Let $\mathcal{K}$ be a simplicial complex on the vertex set $[l]=[l_1]\sqcup\cdots\sqcup[l_m]$ and consider a substituted higher Whitehead map of the form

$$h_w^{\mathcal{K}}(f_1,\dots,f_m)\colon\mathop{\Huge *}_{i=1}^{m}X_i\longrightarrow(\underline{Y},\underline{*})^{\mathcal{K}}$$

as defined in (4.19). Suppose that there is an associative H-space Y and $I\subseteq[l]$ such that $Y=Y_i$ for all $i\in I$. Then we use the H-space structure to construct a map of polyhedral products

$$\nabla\colon(\underline{Y},\underline{*})^{\mathcal{K}}\longrightarrow(\underline{Y},\underline{*})^{\overline{\mathcal{K}}}\tag{4.20}$$

which generalises the H-multiplication map $Y\times Y\longrightarrow Y$ above, where $\overline{\mathcal{K}}$ is a simplicial complex on $[l]-I$. We call such a map a fold map of polyhedral products. In this setup, the Whitehead product $[f_1, f_2]$ is then the composite $\nabla \circ h_w(f_1, f_2)$, with $\mathcal{K}$ a 1-simplex, $\overline{\mathcal{K}}$ a single vertex, and $Y_1 = Y_2 = Y$ an H-space.

The main result of this section is specifying combinatorial conditions on $\overline{\mathcal{K}}$ for the triviality of $\nabla \circ h_w^{\mathcal{K}}(f_1, \ldots, f_m)$

We begin by defining a fold of a simplicial complex.

Definition 4.2.24. A *fold* of a vertex set $[m]$ consists of disjoint subsets $I, J \subseteq [m]$ and a surjective map $\psi\colon I \longrightarrow J$. Given a fold $\psi\colon I \longrightarrow J$ of $[m]$ we denote $I_j = \psi^{-1}(j)$ for each $j \in J$, so that $I = \bigsqcup_{j\in J} I_j$.

Let $\mathcal{K}$ be a simplicial complex on $[m]$. A fold $\psi\colon I \longrightarrow J$ of $[m]$ extends to a map on $\mathcal{K}$ by sending a simplex $(i_1, \ldots, i_k) \in \mathcal{K}$ to $(\overline{\psi}(i_1), \ldots, \overline{\psi}(i_k))$, where $\overline{\psi}(i) = \psi(i)$ if $i \in I$, and $\overline{\psi}(i) = i$, otherwise. We interchange freely between referring to $\psi\colon I \longrightarrow J$ as a fold of $\mathcal{K}$ and of its vertex set $[m]$.

We define the fold of $\mathcal{K}$ under $\psi\colon I \longrightarrow J$ to be the image $\overline{\psi}(\mathcal{K})$, denoted by $\mathcal{K}_{\nabla(I,J)}$.

It follows from the definition that

$$\mathcal{K}_{\nabla(I,J)} = \{\sigma \subseteq [m] - I \mid \sigma \in \mathcal{K} \text{ or } (\sigma - j) \sqcup i \in \mathcal{K} \text{ for some } j \in J, i \in I_j\}. \tag{4.21}$$

When $J = \{j\}$ consists of one element we abbreviate $\mathcal{K}_{\nabla(I,\{j\})}$ to $\mathcal{K}_{\nabla(I,j)}$. If further $I = \{i\}$, then we abbreviate $\mathcal{K}_{\nabla(\{i\},j)} = \mathcal{K}_{\nabla(i,j)}$.

We observe the following properties which follow immediately from Definition 4.2.24.

Proposition 4.2.25. *Let $\mathcal{K}$ be a simplicial complex on $[m]$. Then:*

(i) if $i, j \in [m]$ with $i \neq j$, then $\mathcal{K}_{\nabla(i,j)} \cong \mathcal{K}_{\nabla(j,i)}$;

(ii) if $I = \{i_1, \ldots, i_n\} \subseteq [m]$, and $j \in [m]$ with $j \notin I$, then

$$\mathcal{K}_{\nabla(I,j)} = \mathcal{K}_{\nabla(i_1,j)_{\nabla(i_2,j)\ldots\nabla(i_n,j)}};$$

(iii) if $I_1, I_2 \subseteq [m]$ are such that $I_1 \cap I_2 = \varnothing$, and $j \notin I_1 \sqcup I_2$, then

$$\mathcal{K}_{\nabla(I_1,j)_{\nabla(I_2,j)}} = \mathcal{K}_{\nabla(I_2,j)_{\nabla(I_1,j)}}.$$

$\square$

We now describe the map of polyhedral products induced by a fold of simplicial complexes. Let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be simplicial complexes on $[l_1], \ldots, [l_m]$, respectively. Let $f_i\colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i}$ be maps for $i = 1, \ldots, m$ and let $\mathcal{K}$ be a simplicial complex on $[l] = [l_1] \sqcup \cdots \sqcup [l_m]$ containing $\partial\Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_m\rangle$.

Let $\psi\colon I \longrightarrow J$ be a fold of $\mathcal{K}$. Suppose that for each $j \in J$ that Y_j is an associative H-space, and that $Y_i = Y_j$ for each $i \in I_j$. For $j \in J$, the H-multiplication map $\mu_j\colon Y_j \times Y_j \longrightarrow Y_j$ extends up to homotopy to a map

$$\prod_{i\in I_j} Y_i \times Y_j \longrightarrow Y_j.$$

Therefore the fold $\psi\colon I \longrightarrow J$ induces a map $\prod_{i\in[m]} Y_i \longrightarrow \prod_{i\in([m]-I)} Y_i$ given by

$$\prod_{j\in J}\prod_{i\in I_j}(Y_i \times Y_j) \times \prod_{i\in([m]-(I\sqcup J))} Y_i \longrightarrow \prod_{j\in J} Y_j \times \prod_{i\in([m]-(I\sqcup J))} Y_i$$

which restricts to a map

$$\nabla_{(I,J)}\colon (\underline{Y},\underline{*})^{\mathcal{K}} = \bigcup_{\sigma\in\mathcal{K}}\prod_{i\in\sigma} Y_i \longrightarrow \bigcup_{\sigma\in\mathcal{K}}\prod_{i\in\overline{\psi}(\sigma)} Y_i = (\underline{Y},\underline{*})^{\mathcal{K}_{\nabla(I,J)}}. \tag{4.22}$$

We call $(\underline{Y},\underline{*})^{\mathcal{K}_{\nabla(I,J)}}$ the fold of $(\underline{Y},\underline{*})^{\mathcal{K}}$ under the map $\psi\colon I \longrightarrow J$. We call the map $\nabla(I,J)$ the fold map on polyhedral products induced by the map $\psi\colon I \longrightarrow J$.

Let

$$h_w^{\mathcal{K}}(f_1,\ldots,f_m)\colon \mathop{\Large\ast}_{i=1}^{m} X_i \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}}$$

be the substituted higher Whitehead map (4.19). The *folded higher Whitehead map* is the composite

$$\nabla_{(I,J)} \circ h_w^{\mathcal{K}}(f_1,\ldots,f_m)\colon \mathop{\Large\ast}_{i=1}^{m} X_i \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}_{\nabla(I,J)}}.$$

We often drop the composition symbol $\circ$ for folded higher Whitehead maps from hereon. In analogy with the substituted higher Whitehead map, the folded higher Whitehead map is an element of a particular higher Whitehead product.

Proposition 4.2.26. *The folded higher Whitehead map is an element of the higher Whitehead product*

$$\nabla_{(I,J)}h_w^{\mathcal{K}}(f_1,\ldots,f_m) \in [\nabla_{(I,J)} \circ f_1^{\mathcal{K}},\ldots,\nabla_{(I,J)} \circ f_m^{\mathcal{K}}] \subseteq \left[X_1 * \cdots * X_m, (\underline{Y},\underline{*})^{\mathcal{K}_{\nabla(I,J)}}\right]$$

Proof. This follows from Proposition 4.2.21 together with the observation that the map $\nabla_{(I,J)}\colon (\underline{Y},\underline{*})^{\mathcal{K}} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}_{\nabla(I,J)}}$ is by definition an extension of the partial fold $\bigvee_{i\in[m]} Y_i \longrightarrow \bigvee_{i\in([m]-I)} Y_i$. $\square$

In general, determining necessary and sufficient conditions for the triviality of a folded higher Whitehead map is hard. Using homology to detect non-triviality as for the substituted higher Whitehead map is no longer adequate. Nevertheless, the main result of this section gives sufficient combinatorial conditions for the triviality of folded higher Whitehead maps. We begin with some preparatory results.

Lemma 4.2.27. *Let $\mathcal{K}_1, \dots, \mathcal{K}_m$ be simplicial complexes on $[l_1], \dots, [l_m]$, respectively. Let $f_i\colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i}$ be maps for $i = 1, \dots, m$.*

Let $\mathcal{K}, \mathcal{K}'$ be simplicial complexes on $[l] = [l_1] \sqcup \cdots \sqcup [l_m]$ containing $\partial\Delta\langle\mathcal{K}_1, \dots, \mathcal{K}_m\rangle$ and let $\psi\colon I \longrightarrow J$ be a fold of both $\mathcal{K}$ and $\mathcal{K}'$ such that $\mathcal{K}_{\nabla(I,J)} = \mathcal{K}'_{\nabla(I,J)}$. If for each $j \in J$ we have that Y_j is an associative H-space and that $Y_i = Y_j$ for each $i \in I_j$, then

$$\nabla_{(I,J)} h_w^{\mathcal{K}}(f_1, \dots, f_m) = \nabla_{(I,J)} h_w^{\mathcal{K}'}(f_1, \dots, f_m).$$

Proof. Since $\mathcal{K}_{\nabla(I,J)} = \mathcal{K}'_{\nabla(I,J)}$, there is a commutative diagram

$$\begin{tikzcd}
\partial\Delta\langle\mathcal{K}_1, \dots, \mathcal{K}_m\rangle \arrow[r] \arrow[d] & \mathcal{K} \arrow[d, "\overline{\psi}"] \\
\mathcal{K}' \arrow[r, "\overline{\psi}"] & \mathcal{K}_{\nabla(I,J)}.
\end{tikzcd}$$

Therefore by functoriality of the polyhedral product with respect to simplicial inclusions, the following diagram commutes

$$\begin{tikzcd}
X_1 * \cdots * X_m \arrow[r] & (\underline{Y}, \underline{*})^{\partial\Delta^{m-1}\langle\mathcal{K}_1, \dots, \mathcal{K}_m\rangle} \arrow[r] \arrow[d] & (\underline{Y}, \underline{*})^{\mathcal{K}} \arrow[d] \\
 & (\underline{Y}, \underline{*})^{\mathcal{K}'} \arrow[r] & (\underline{Y}, \underline{*})^{\mathcal{K}_{\nabla(I,J)}}
\end{tikzcd}$$

where the composite around the top of the square is $\nabla_{(I,J)} h_w^{\mathcal{K}}(f_1, \dots, f_m)$ and the composite around the bottom of the square is $\nabla_{(I,J)} h_w^{\mathcal{K}'}(f_1, \dots, f_m)$. The result follows. $\square$

Definition 4.2.28. Let $\mathcal{K}$ be a simplicial complex on $[l]$ and let $\psi\colon I \longrightarrow J$ be a fold of $\mathcal{K}$. Let $\mathcal{L}_\psi$ be the substitution complex on $[l]$ obtained from $\mathcal{K}_{\nabla(I,J)}$ by substituting the simplex $\Delta[I_j]$ at vertex j for each $j \in J$.

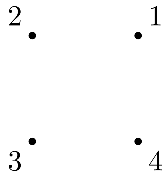
The complex $\mathcal{L}_\psi$ is the largest simplicial complex which folds to $\mathcal{K}_{\nabla(I,J)}$ under $\psi\colon I \longrightarrow J$, in the sense of the following.

Lemma 4.2.29. *Suppose that $\mathcal{L}'$ is such that $\mathcal{L}'_{\nabla(I,J)} \subseteq \mathcal{K}_{\nabla(I,J)}$. Then $\mathcal{L}' \subseteq \mathcal{L}_\psi$.*

Proof. Let $J = \{j_1, \dots, j_r\}$. Then for any $\sigma \in \mathcal{L}'$ write $\sigma = \sigma_{j_1} \sqcup \cdots \sqcup \sigma_{j_r} \sqcup \sigma'$, where $\sigma_{j_k} \subseteq I_{j_k}$ and $\sigma' \in [m] - I$. Since $\mathcal{L}'_{\nabla(I,J)} \subseteq \mathcal{K}_{\nabla(I,J)}$, then $\overline{\psi}(\sigma) = (j_1 \sqcup \cdots \sqcup j_r) \cup \sigma' \in$

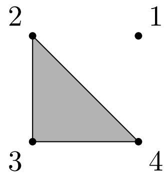
$\mathcal{K}_{\nabla(I,J)}$. Then by construction $(I_{j_1} \sqcup \cdots \sqcup I_{j_r}) \cup (j_1 \sqcup \cdots \sqcup j_r) \cup \sigma' \in \mathcal{L}_\psi$. Then since $\sigma_{j_k} \subseteq I_{j_k}$, it follows that $\sigma \in \mathcal{L}_\psi$. □

Example 4.2.30. Let $\mathcal{K}$ consist of 4 disjoint points $\{\{1\},\{2\},\{3\},\{4\}\}$ and suppose that $Y_2 = Y_3 = Y_4$ is an associative H-space. Define a fold of $\mathcal{K}$ by $\psi\colon \{3,4\} \longrightarrow \{2\}$. Then $\mathcal{K}_{\nabla(I,J)}$ consists of 2 disjoint points, and $\mathcal{L}_\psi$ has maximal simplices $\{\{1\},\{2,3,4\}\}$. This example is shown in Figure 4.1.



(a) $\mathcal{K}$

(b) $\mathcal{K}_{\nabla(I,J)}$



(c) $\mathcal{L}_\psi$

Figure 4.1: The construction of the complex $\mathcal{L}_\psi$.

We now give our main result.

Proposition 4.2.31. *Let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be simplicial complexes on $[l_1], \ldots, [l_m]$, respectively. Let $f_i\colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i}$ be maps for $i = 1, \ldots, m$. Let $\mathcal{K}$ be a simplicial complex on $[l_1] \sqcup \cdots \sqcup [l_m]$ containing $\partial\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_m\rangle$ and let $\psi\colon I \longrightarrow J$ be a fold of $\mathcal{K}$.*

Then the folded higher Whitehead map

$$\nabla_{(I,J)} h_w^{\mathcal{K}}(f_1, \ldots, f_m)\colon X_1 * \cdots * X_m \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_{\nabla(I,J)}}$$

is trivial if either

(i) $\mathcal{K}_{\nabla(I,J)}$ contains $\mathcal{K}'_{\nabla(I,J)}$ where either $\mathcal{K}' = \partial\Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}'_i, \ldots, \mathcal{K}_m\rangle$, where $\mathcal{K}'_i$ is a trivialising complex for f_i, or $\mathcal{K}' = \Delta\langle \mathcal{K}_1, \ldots, \mathcal{K}_m\rangle$;

(ii) the map

$$h_w^{\mathcal{L}_\psi}(f_1, \ldots, f_m)\colon X_1 * \cdots * X_m \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{L}_\psi}$$

is trivial, where $\mathcal{L}_\psi$ is the complex defined in Definition 4.2.28.

Proof. Suppose that (i) holds. Then $h_w^{\mathcal{K}'}(f_1, \ldots, f_m)$ is trivial by Theorem 4.2.20. Then by Lemma 4.2.27 we have $\nabla_{(I,J)} h_w^{\mathcal{K}}(f_1, \ldots, f_m) = \nabla_{(I,J)} h_w^{\mathcal{K}'}(f_1, \ldots, f_m)$, and so the map $\nabla_{(I,J)} h_w^{\mathcal{K}}(f_1, \ldots, f_m)$ is trivial. The same argument words if instead (ii) holds, since $\mathcal{L}_\psi$ folds to $\mathcal{K}_{\nabla(I,J)}$ by Lemma 4.2.29. □

Example 4.2.32. Let $\mathcal{K}$ and $\psi\colon \{3,4\} \longrightarrow \{2\}$ be the complex and the fold map from Example 4.2.30. Then the folded higher Whitehead map $\nabla_{(\{3,4\},2)} h_w(h_w(h_w(f_2, f_3), f_4), f_1)$ is trivial since $h_w^{\mathcal{L}_\psi}(h_w(h_w(f_2, f_3), f_4), f_1)$ is trivial as $\Delta[2,3,4]$ is a trivialising complex for $h_w(h_w(h_w(f_2, f_3), f_4)$.

4.3 Relations among higher Whitehead maps

In this section we study relations among higher Whitehead maps which generalise the Jacobi identity (4.3). The study of relations among higher Whitehead maps was established by Hardie [Har61, Har64]. Let $f_i\colon S^{r_i} \longrightarrow Y_i$ for $i = 1, \dots, m$. Define

$$Z = \{(y_1, \dots, y_m) \in Y_1 \times \cdots \times Y_m \mid \text{ there is } i \neq j \text{ such that } y_i = * = y_j\} \qquad (4.23)$$

and for $i = 1, \dots, m$ let

$$h_w^Z(f_i, h_w(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m))\colon S^{r_1+\cdots+r_m-2} \longrightarrow Z$$

denote the composite of $h_w(f_i, h_w(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m))$ with the inclusion $Y_i \vee FW(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_m) \longrightarrow Z$.

Theorem 4.3.1 ([Har61, Theorem 2.3])**.** *Suppose $m \geqslant 3$ and that $r_i \geqslant 2$ for $i = 1, \dots, m$. Then*

$$\sum_{i=1}^{m} (-1)^{\eta(i)} h_w^Z\left(f_i, h_w(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m)\right) = 0 \qquad (4.24)$$

where $\eta(i) = r_i(r_1 + \cdots + r_i) + 1$. □

In the case that $k = 3$ identity (4.24) recovers the graded Jacobi identity (4.3).

For maps $f_i\colon \Sigma X_i \longrightarrow Y_i$ where X_i is not necessarily a sphere, there is a Jacobi identity among generalised Whitehead products of the form

$$h_w(f_1, h_w(f_2, f_3)) + h_w(f_2, h_w(f_3, f_1)) \circ \sigma + h_w(f_3, h_w(f_1, f_2)) \circ \tau \qquad (4.25)$$

where σ and τ are respectively the maps $\Sigma X_2 \wedge X_3 \wedge X_1 \longrightarrow \Sigma X_1 \wedge X_2 \wedge X_3$ and $\Sigma X_3 \wedge X_1 \wedge X_2 \longrightarrow \Sigma X_1 \wedge X_2 \wedge X_3$ obtained by permuting coordinates, see [Coh57, Ark62]. This relation is obtained by a method of G. Whitehead [Whi54], deriving an equivalent relation between the adjoints of the maps in (4.25) and using the group-like structure of loop spaces.

We use the polyhedral product to give a combinatorial method of constructing relations between higher Whitehead maps which generalises (4.24) to relations among more general forms of higher Whitehead maps, while also generalising (4.25) by showing such relations hold for non-spherical maps. Moreover, we use the underlying combinatorics together with the results of the previous section to determine spaces and maps for which our relations contain non-trivial summands.

4.3.1 Identity complexes

The space Z (4.23) is the polyhedral product $(\underline{Y}, \underline{*})^{\mathcal{K}}$, where $\mathcal{K} = \mathrm{sk}^{m-3}\,\Delta^{m-1}$. The minimal missing faces of $\mathcal{K}$ are $\{1, \ldots, i-1, i+1, \ldots, m \mid i = 1, \ldots, m\}$, and each missing face $\{1, \ldots, i-1, i+1, \ldots, m\}$ corresponds to the higher Whitehead map

$$h_w^{\mathcal{K}}(f_i, h_w(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m))$$

being defined in $(\underline{Y}, \underline{*})^{\mathcal{K}}$. We therefore view relation (4.24) as being governed by the missing faces of the complex $\mathrm{sk}^{m-3}\,\Delta^{m-1}$.

We construct a more general family of relations among higher Whitehead maps by using the simplicial composition operation to propagate the missing face structure of $\mathrm{sk}^{m-3}\,\Delta^{m-1}$ as follows. Recall that given a simplicial $\mathcal{K}$ on $[m]$ and simplicial complexes $\mathcal{K}_1, \ldots, \mathcal{K}_m$ on $[l_1], \ldots, [l_m]$, respectively, the composition complex $\mathcal{K}(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ is the polyhedral join

$$\mathcal{K}(\mathcal{K}_1, \ldots, \mathcal{K}_m) = (\underline{\Delta}^{l_i-1}, \underline{\mathcal{K}_i})^{*\mathcal{K}}.$$

Given a vertex set $[m]$, a k-partition Π of $[m]$ is a collection of pairwise disjoint subsets $\{I_1, \ldots, I_k\}$ of $[m]$ such that $\bigcup_{i=1}^{k} I_i = [m]$.

Definition 4.3.2. Let $\Pi = \{I_1, \ldots, I_k\}$ be a k-partition of of $[m]$. For $i = 1, \ldots, k$ write $I_i = \{i_1, \ldots, i_{n_i}\}$ and let $\partial\Delta[I_i]$ be the complex $\partial\Delta^{n_i-1}$ on vertex set I_i. We define the simplicial complex $\mathcal{K}_\Pi$ to be

$$\mathcal{K}_\Pi = \mathrm{sk}^{k-3}\,\Delta^{k-1}(\partial\Delta[I_1], \ldots, \partial\Delta[I_k]).$$

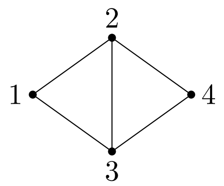
We call $\mathcal{K}_\Pi$ the *identity complex* associated to the partition Π of $[m]$.

Example 4.3.3. (i) Let $\Pi = \{\{1\}, \ldots, \{m\}\}$ be the partition of $[m]$ into singletons. Then $n_j = 1$ for $j = 1, \ldots, m$ and $\mathcal{K}_\Pi = \mathrm{sk}^{m-3}\,\Delta^{m-1}$.

(ii) Let $\Pi = \{\{1\}, \{2,3\}, \{4\}\}$ be a 3-partition of $[4]$. Then

$$\mathcal{K}_\Pi = \bullet_1 * \partial\Delta[2,3] * \circ_4 \cup \circ_1 * \Delta[2,3] * \circ_4 \cup \circ_1 * \partial\Delta[2,3] * \bullet_4$$

which is the simplicial complex shown below.



The minimal missing faces of $\mathcal{K}_\Pi$ are $\{(1,2,3), (2,3,4), (1,4)\}$. We observe that these have the form $[m] - I_j$, where $I_1 = \{1\}$, $I_2 = \{2,3\}$ and $I_3 = \{4\}$ are the elements of the partition Π. This leads to the following alternate description of identity complexes.

Proposition 4.3.4. *Let $\Pi = \{I_1, \ldots, I_k\}$ be a k-partition of $[m]$. Then*

$$MF(\mathcal{K}_\Pi) = \{[m] - I_j \mid j = 1, \ldots, k\}.$$

To prove Proposition 4.3.4, we begin with the following result describing the minimal missing faces of the polyhedral join product, which is of independent interest.

Proposition 4.3.5. *Let $\mathcal{K}$ be a simplicial complex on $[m]$, and let $(\mathcal{S}_1, \mathcal{T}_1), \ldots, (\mathcal{S}_m, \mathcal{T}_m)$ be simplicial pairs on vertex sets $[l_1], \ldots, [l_m]$ respectively. Then,*

$$MF\left((\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}\right) = \{J \in MF(\mathcal{S}_i) \mid i \in \mathcal{K}\} \sqcup \left\{ \bigsqcup_{i \in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{T}_i), J_i \in \mathcal{S}_i \right\}.$$

Proof. We first show that

$$\{J \in MF(\mathcal{S}_i) \mid i \in \mathcal{K}\} \sqcup \left\{ \bigsqcup_{i \in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{T}_i), J_i \in \mathcal{S}_i \right\} \subseteq MF\left((\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}\right).$$

For any $\{i\} \in \mathcal{K}$, $J \in MF(\mathcal{S}_i)$ implies that $J \in MF\left((\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}\right)$. Now consider $\bigsqcup_{i \in L} J_i$, where $L \in MF(\mathcal{K})$, $J_i \in MF(\mathcal{T}_i)$ and $J_i \in \mathcal{S}_i$ for all $i \in L$. This is a missing face of $(\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}$ by definition of the polyhedral join. Moreover, it is minimal since for any $i \in L$ and $s \in J_i$,

$$\bigsqcup_{i \neq k \in L} J_k \sqcup (J_i - \{s\}) = \bigsqcup_{k \in \tau} J_k \sqcup \sigma_i \in (\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\tau}$$

where $\tau \in \mathcal{K}$, since L is a minimal missing face; and $\sigma_i \in \mathcal{T}_i$, since J_i is a minimal missing face.

Now we show that

$$MF\left((\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}\right) \subseteq \{J \in MF(\mathcal{S}_i) \mid i \in \mathcal{K}\} \sqcup \left\{ \bigsqcup_{i \in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{T}_i), J_i \in \mathcal{S}_i \right\}.$$

Let $F \in MF((\underline{\mathcal{K}}, \underline{\mathcal{L}})^{*\mathcal{K}})$. We show that either $F \in \{J \in MF(\mathcal{K}_i) \mid i \in \mathcal{K}\}$ or $F \in \{\bigsqcup_{i \in L} F_i \mid L \in MF(\mathcal{K}), F_i \in MF(\mathcal{L}_i), F_i \in \mathcal{K}_i\}$.

For any $i = 1, \ldots, m$ define the restriction $F_i = F|_{[l_i]}$. Then if $F_i \in MF(\mathcal{S}_i)$ then $F = F_i \in MF(\mathcal{K})$. Otherwise F would not be minimal, since $F_i \subseteq F$ is a missing face.

On the other hand, suppose that $F_i \in \mathcal{S}_i$ for all i. Denote by $\sigma = \{i \in [m] \mid F|_i \neq \varnothing\}$. Firstly, $\sigma \notin \mathcal{K}$, as otherwise $F \in \mathcal{K}$, since $F = \bigsqcup_{i\in\sigma} F_i$ where $F_i \in \mathcal{S}_i$ for all i. For all $i \in \sigma$, F_i is a non-face of $\mathcal{T}_i$. Otherwise, by removing vertices from F_i we obtain a smaller non-face of $(\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}$. For such i, $F_i \in MF(\mathcal{T}_i)$. It follows that $\sigma \in MF(\mathcal{K})$, as otherwise we restrict to a minimal missing face $\tau \in MF(\mathcal{K})$ with $\tau \subseteq \sigma$, and $\bigsqcup_{i\in\tau} F_i$ is a missing face of $(\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}$. Finally, for all i, F_i is a minimal missing face of $\mathcal{T}_i$, since otherwise, for $\widehat{F}_i \subsetneq F_i$ with $\widehat{F}_i \in MF(\mathcal{S}_i)$, $(F - F_i) \sqcup \widehat{F}_i \in (\underline{\mathcal{S}}, \underline{\mathcal{T}})^{*\mathcal{K}}$. □

Proof of Proposition 4.3.4. By Proposition 4.3.5, the minimal missing faces of the composition complex $\mathcal{K}(\mathcal{T}_1, \ldots, \mathcal{T}_m)$ are

$$MF(\mathcal{K}(\mathcal{T}_1, \ldots, \mathcal{T}_m)) = \left\{ \bigsqcup_{i\in L} J_i \mid L \in MF(\mathcal{K}), J_i \in MF(\mathcal{T}_i) \right\}$$

so that

$$MF(\mathrm{sk}^{k-3}\,\Delta^{k-1}(\partial\Delta[I_1], \ldots, \partial\Delta[I_k])) = \{\{[m] - \{i_1, \ldots, i_{n_i}\}\} \mid i = 1, \ldots, k\}$$

and the claim follows. □

Since a simplicial complex is determined by its set of minimal missing faces, an equivalent definition of $\mathcal{K}_\Pi$ is to specify its minimal missing faces to be $\{[m] - I_j \mid j = 1, \ldots, k\}$.

4.3.2 Relations among higher Whitehead maps

Let Π be a k-partition of $[m]$. To identify higher Whitehead maps which are defined in $(\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$, we decompose $\mathcal{K}_\Pi$ as a union of substitution complexes which each define a certain higher Whitehead map.

Proposition 4.3.6. *Let* $\Pi = \{I_1, \ldots, I_k\}$ *be a* k*-partition of* $[m]$ *and write* $I_i = \{i_1, \ldots, i_{n_i}\}$ *for* $i = 1, \ldots, k$*. Then*

$$\mathcal{K}_\Pi = \bigcup_{i=1}^{k} \partial\Delta\langle \partial\Delta[j_1, \ldots, j_{r_i}], i_1, \ldots, i_{n_i} \rangle$$

where $\{j_1, \ldots, j_{r_i}\} = [m] - \{i_1, \ldots, i_{n_i}\}$.

Proof. The Alexander dual $\hat{\mathcal{K}}$ of a simplicial complex $\mathcal{K}$ on $[m]$ is the simplicial complex with simplices $\{\sigma \mid [m] - \sigma \notin \mathcal{K}\}$. In particular, the maximal faces of $\hat{\mathcal{K}}$ are the complements in $[m]$ of the minimal missing faces of $\mathcal{K}$. Moreover, two finite simplicial complexes are equal if and only if their Alexander duals are equal.

By Proposition 4.3.4, the Alexander dual of $\mathcal{K}_\Pi$ has maximal faces $\bigsqcup_{i=1}^{k}\Delta[i_1,\dots,i_{n_i}]$. On the other hand, let $\mathcal{K}^i = \partial\Delta\langle\partial\Delta[j_1,\dots,j_{r_i}], i_1,\dots,i_{n_i}\rangle$. Then by Proposition 4.3.5 the maximal faces $\hat{\mathcal{K}}^i_{max}$ of $\hat{\mathcal{K}^i}$ are given by

$$\{[m]-\{j_1,\dots,j_{r_i}\}\sqcup\{[m]-\{j,i_1,\dots,i_{n_i}\}\mid j\in J_i\} = \{i_1,\dots,i_{n_i}\}\sqcup\{J_i - j\mid j\in J_i\}.$$

Then

$$\widehat{\bigcup_{i=1}^{k}\mathcal{K}^i} = \bigcap_{i=1}^{k}\widehat{\mathcal{K}^i} = \bigcap_{i=1}^{k}\left(\widehat{\mathcal{K}^i}_{max}\right) = \bigsqcup_{i=1}^{k}\Delta[i_1,\dots,i_{n_i}]$$

and the result follows. □

Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i=1,\dots,m$. It follows from Definition 4.2.1 and Proposition 4.3.6 that the space $(\underline{Y},\underline{*})^{\mathcal{K}_\Pi}$ contains the codomains of the higher Whitehead maps

$$\begin{aligned} &h_w\left(h_w\left(f_{j_1},\dots,f_{j_{r_i}}\right), f_{i_1},\dots,f_{i_{n_i}}\right):\\ &\qquad \Sigma^{n_i}\left(\Sigma^{r_i-2}X_{j_1}\wedge\cdots\wedge X_{j_{r_i}}\right)\wedge X_{i_1}\wedge\cdots\wedge X_{i_{n_i}} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}^i} \end{aligned} \tag{4.26}$$

for $i=1,\dots,k$. Our main result is that there is a relation between the maps (4.26) for $i=1,\dots,k$.

Theorem 4.3.7. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i=1,\dots,m$. Let $\Pi=\{I_1,\dots,I_k\}$ be a k-partition of $[m]$ for $k\geqslant 3$ and denote $I_i=\{i_1,\dots,i_{l_i}\}$ and $J_i=[m]-I_i=\{j_1,\dots,j_{r_i}\}$ for $i=1,\dots,k$. Then if X_i is a suspension for each $i=1,\dots,m$,*

$$\sum_{i=1}^{k} h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_{j_1},\dots,f_{j_{r_i}}\right), f_{i_1},\dots,f_{i_{n_i}}\right)\circ\sigma_i = 0 \tag{4.27}$$

in $\left[\Sigma^{m-2}X_1\wedge\cdots\wedge X_m, (\underline{Y},\underline{*})^{\mathcal{K}_\Pi}\right]$, where

$$\sigma_i\colon \Sigma^{m-2}X_1\wedge\cdots\wedge X_m \longrightarrow \Sigma^{n_i}(\Sigma^{r_i-2}X_{j_1}\wedge\cdots\wedge X_{j_{r_i}})\wedge X_{i_1}\wedge\cdots\wedge X_{i_{n_i}}$$

is the restriction of the coordinate permutation

$$CX_1\times\cdots\times CX_m \longrightarrow CX_{j_1}\times\cdots\times CX_{j_{r_i}}\times CX_{i_1}\times\cdots\times CX_{i_{n_i}}.$$

The proof of Theorem 4.3.7 is delayed to Section 4.5. We first discuss the triviality of the terms in relation (4.27), before giving examples and applications.

Proposition 4.3.8. *Suppose that $f_1,\dots,f_m$ are compatible. Then for each $i=1,\dots,k$, the map*

$$h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_{j_1},\dots,f_{j_{r_i}}\right), f_{i_1},\dots,f_{i_{l_i}}\right)$$

is non-trivial.

Proof. Observe that by Corollary 4.2.11, the only trivialising complex of $h_w\left(f_{j_1},\ldots,f_{j_{r_i}}\right)$ is $\Delta[j_1,\ldots,j_{r_i}]$. By Proposition 4.3.4 the complex $\partial\Delta\langle\Delta[j_1,\ldots,j_{r_i}],i_1,\ldots,i_{n_i}\rangle$ is not a subcomplex $\mathcal{K}_\Pi$ since $(j_1,\ldots,j_{r_i})$ is a minimal missing face of $\mathcal{K}_\Pi$. Therefore by Theorem 4.2.20, the map $h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_{j_1},\ldots,f_{j_{r_i}}\right),f_{i_1},\ldots,f_{i_{n_i}}\right)$ is trivial if and only if $\Delta\langle\partial\Delta[j_1,\ldots,j_{r_i}],i_1,\ldots,i_{n_i}\rangle\subseteq\mathcal{K}_\Pi$. But by definition of the partition Π there exists $I_k\in J_i$ such that $I_k\cap I_i=\varnothing$. Then $J_k=[m]-I_k$ is a missing face of $\mathcal{K}_\Pi$ containing the vertices $\{i_1,\ldots,i_{n_i}\}$. Therefore $\Delta\langle\partial\Delta[j_1,\ldots,j_{r_i}],i_1,\ldots,i_{n_i}\rangle$ is not a subcomplex of $\mathcal{K}_\Pi$. □

In particular, by Proposition 4.3.8, each term in (4.27) is non-trivial if all the f_i are compatible. We now give examples of Theorem 4.3.7.

Example 4.3.9. Let $f_i\colon\Sigma X_i\longrightarrow Y_i$ be compatible maps such that X_i is a suspension for $i=1,\ldots,m$.

(i) Let $\Pi=\{\{1\},\{2\},\{3\}\}$. Then $J_1=\{2,3\}$, $J_2=\{1,3\}$ and $J_3=\{1,2\}$, and $\mathcal{K}_\Pi=\bullet_{[3]}$. Then

$$h_w^{\mathcal{K}_\Pi}(h_w(f_2,f_3),f_1)\circ\sigma_1+h_w^{\mathcal{K}_\Pi}(h_w(f_1,f_3),f_2)\circ\sigma_2+h_w^{\mathcal{K}_\Pi}(h_w(f_1,f_2),f_3)\circ\sigma_3=0$$

in the group $[\Sigma X_1\wedge X_2\wedge X_3,Y_1\vee Y_2\vee Y_3]$. Moreover, each term is non-trivial. This recovers the generalised Jacobi identity (4.25).

(ii) Generalising the previous example, let $\Pi=\{\{1\},\ldots,\{m\}\}$. Then $J_i=\{1,\ldots,\hat{i},\ldots,m\}$ for $i=1,\ldots,m$ and $\mathcal{K}_\Pi=\operatorname{sk}^{m-3}\Delta^{m-1}$. Then

$$h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_2,\ldots,f_m\right),f_1\right)\circ\sigma_1+\cdots+h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_1,\ldots,f_{m-1}\right),f_m\right)\circ\sigma_m=0$$

in the group $\left[\Sigma^{m-2}X_1\wedge\cdots\wedge X_m,(\underline{Y},\underline{*})^{\operatorname{sk}^{m-3}\Delta^{m-1}}\right]$. Moreover, each term is non-trivial. This establishes Hardie's identity, Theorem 4.3.1, for non-spherical maps.

(iii) Define a 3-partition $\Pi=\{\{1\},\{2,3\},\{4\}\}$ of $[4]$. Then $J_1=\{2,3,4\}$, $J_2=\{1,4\}$ and $J_3=\{1,2,3\}$ and $\mathcal{K}_\Pi$ is the simplicial complex shown in Example 4.3.3(ii). Then

$$\begin{aligned}h_w^{\mathcal{K}_\Pi}(h_w(f_2,f_3,f_4),f_1)\circ\sigma_1+h_w^{\mathcal{K}_\Pi}(h_w(f_1,f_4),f_2,f_3)\circ\sigma_2&\\+h_w^{\mathcal{K}_\Pi}(h_w(f_1,f_2,f_3),f_4)\circ\sigma_3&=0\end{aligned}$$

in the group $\left[\Sigma^2X_1\wedge X_2\wedge X_3\wedge X_4,(\underline{Y},\underline{*})^{\mathcal{K}_\Pi}\right]$, and moreover each term is non-trivial. Observe that this is a relation among elements of not only 2-fold Whitehead products but also a higher Whitehead product. That is, by Lemma 4.2.21, the summand $h_w^{\mathcal{K}_\Pi}(h_w(f_1,f_4),f_2,f_3)$ is an element of the higher Whitehead product $[h_w^{\mathcal{K}_\Pi}(f_1,f_4),f_2^{\mathcal{K}_\Pi},f_3^{\mathcal{K}_\Pi}]$.

In the special case that the maps f_i are spherical, then we obtain the following, which is a generalisation of Theorem 4.3.1. Recall the definiton of the Koszul sign $\epsilon(\rho)$ of a permutation ρ, given in Proposition 4.2.3.

Corollary 4.3.10. *Let $f_i \in \pi_{p_i}(Y_i)$ for $i = 1, \ldots, m$. Let $\Pi = \{I_1, \ldots, I_k\}$ be a k-partition of $[m]$ for $k \geqslant 3$ and denote $I_i = \{i_1, \ldots, i_{n_i}\}$ and $J_i = [m] - I_i = \{j_1, \ldots, j_{r_i}\}$. Then if $p_i \geqslant 2$ for each $i = 1, \ldots, m$ there is a relation*

$$\sum_{i=1}^{k} \epsilon(\rho) h_w^{\mathcal{K}_\Pi} \left(h_w\left(f_{j_1}, \ldots, f_{j_{r_i}}\right), f_{i_1}, \ldots, f_{i_{n_i}}\right) = 0$$

in $\pi_{p_1+\cdots+p_m-2}\left((\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}\right)$ where $\epsilon(\rho)$ is the Koszul sign of the permutation $\rho\colon (1, \ldots, m) \longmapsto (j_1, \ldots, j_{r_i}, i_1, \ldots, i_{n_i})$.

Proof. The map $\sigma_i\colon S^{p_1+\cdots+p_m-2} \longrightarrow S^{p_1+\cdots+p_m-2}$ from the statement of Theorem 4.3.7 has degree $\epsilon(\rho)$ and the result follows. □

Example 4.3.11. In some cases, $\epsilon(\rho)$ can be calculated explicitly. We revisit the relations from Example 4.3.9 in the case that $f_i\colon S^{p_i} \longrightarrow Y_i$, with $p_i \geqslant 2$ for $i = 1, \ldots, m$.

(i) Let $\Pi = \{\{1\}, \{2\}, \{3\}\}$. There is a relation

$$(-1)^{p_1(p_2+p_3)} h_w^{\mathcal{K}_{P_i}}(h_w(f_2, f_3), f_1) + (-1)^{p_2 p_3} h_w^{\mathcal{K}_{P_i}}(h_w(f_1, f_3), f_2) + h_w^{\mathcal{K}_{P_i}}(h_w(f_1, f_2), f_3) = 0$$

in $\pi_{p_1+p_2+p_3-2}(Y_1 \vee Y_2 \vee Y_3)$. Let ι_j denote the inclusion $Y_j \longrightarrow Y_1 \vee Y_2 \vee Y_3$ and let $g_j = \iota_j \circ f_j$ for $j = 1, 2, 3$. Then multiplying by $(-1)^{p_1 p_3}$ and applying (4.13) and Proposition 4.2.21, we recover the graded Jacobi identity for Whitehead products

$$(-1)^{p_1 p_2}[[g_2, g_3], g_1] + (-1)^{p_2 p_3}[[g_1, g_3], g_2] + (-1)^{p_1 p_3}[[g_1, g_2], g_3] = 0.$$

(ii) Let $\Pi = \{\{1\}, \ldots, \{m\}\}$. Similar to above, if ι_i denotes the inclusion $Y_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$ and κ_i the inclusion $FW(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_m) \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}$, we recover the relation of Hardie

$$\sum_{i=1}^{m} (-1)^{p_i(p_{i+1}+\cdots+p_m)}[\kappa_i \circ h_w(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m), \iota_i \circ f_i] = 0$$

in $\pi_{p_1+\cdots+p_m-2}\left((\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}\right)$, see Theorem 4.3.1.

(iii) Let $\Pi = \{\{1\}, \{2, 3\}, \{4\}\}$. Then there is a relation

$$\begin{aligned}(-1)^{p_1(p_2+p_3)} h_w^{\mathcal{K}_\Pi}(h_w(f_2, f_3, f_4), f_1) + (-1)^{(p_2+p_3)p_4} h_w^{\mathcal{K}_\Pi}(h_w(f_4, f_1), f_2, f_3)& \\ + (-1)^{p_1 p_4} h_w^{\mathcal{K}_\Pi}(h_w(f_1, f_2, f_3), f_4) &= 0\end{aligned}$$

in $\pi_{p_1+\cdots+p_4-2}\left((\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}\right)$.

4.3.3 Relations in general complexes

Let Π be a k-partition of $[m]$ and suppose that $\mathcal{L}$ is a simplicial complex containing $\mathcal{K}_\Pi$. Then composing each summand of (4.27) with the inclusion $(\underline{Y}, \underline{*})^{\mathcal{K}_\Pi} \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{L}}$ we obtain the following Corollary of Theorem 4.3.7.

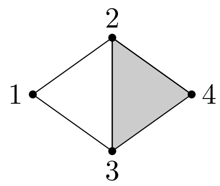
Corollary 4.3.12. *Let $f_i \colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \ldots, m$. Let $\Pi = \{I_1, \ldots, I_k\}$ be a k-partition of $[m]$ for $k \geqslant 3$ and denote $I_i = \{i_1, \ldots, i_{l_i}\}$ and $J_i = [m] - I_i = \{j_1, \ldots, j_{r_j}\}$. Then if X_i is a suspension for each $i = 1, \ldots, m$ there is a relation*

$$\sum_{i=1}^{k} h_w^{\mathcal{L}}(h_w(f_{j_1}, \ldots, f_{j_{r_i}}), f_{i_1}, \ldots, f_{i_{n_i}}) \circ \sigma_i = 0 \tag{4.28}$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{L}}\right]$. □

Contrary to Theorem 4.3.7, it is no longer in general true that every term in the relation (4.28) is non-trivial, as the following example shows.

Example 4.3.13. Let $\mathcal{L}$ be the following simplicial complex.



The identity complex $\mathcal{K}_\Pi$ for $\Pi = \{\{1\}, \{2, 3\}, \{4\}\}$ considered in Example 4.3.9(iii) is a subcomplex of $\mathcal{L}$. Therefore by Corollary 4.3.12 there is a relation

$$\begin{aligned} h_w^{\mathcal{L}}(h_w(f_2, f_3, f_4), f_1) \circ \sigma_1 + h_w^{\mathcal{L}}(h_w(f_1, f_4), f_2, f_3) \circ \sigma_2 \\ + h_w^{\mathcal{L}}(h_w(f_1, f_2, f_3), f_4) \circ \sigma_3 = 0 \end{aligned}$$

in the group $\left[\Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4, (\underline{Y}, \underline{*})^{\mathcal{L}}\right]$. On the other hand by Theorem 4.2.20, the higher Whitehead map $h_w^{\mathcal{L}}(h_w(f_2, f_3, f_4), f_1)$ is trivial since $\mathcal{L}$ contains the simplex $(2, 3, 4)$, while the remaining terms in the relation are non-trivial if the f_i are compatible. Therefore the relation reduces to

$$h_w^{\mathcal{L}}(h_w(f_1, f_4), f_2, f_3) \circ \sigma_2 + h_w^{\mathcal{L}}(h_w(f_1, f_2, f_3), f_4) \circ \sigma_3 = 0$$

where each term is non-trivial.

By combining different identity complexes, Corollary 4.3.12 produces more exotic relations.

Example 4.3.14. Define two 3-partitions of $[5]$ as

$$\Pi_1 = \{\{1\},\{2,3\},\{4,5\}\}$$
$$\Pi_2 = \{\{1\},\{2,3,4\},\{5\}\}$$

and let $\mathcal{K}_{\Pi_1}$ and $\mathcal{K}_{\Pi_2}$ be the corresponding identity complexes. Define $\mathcal{L} = \mathcal{K}_{\Pi_1} \cup \mathcal{K}_{\Pi_2}$, where the union is taken over all common faces. By Corollary 4.3.12, from the inclusion $\mathcal{K}_{\Pi_1} \longrightarrow \mathcal{L}$ we deduce the relation

$$\begin{aligned} h_w^{\mathcal{L}}\left(h_w\left(f_2,f_3,f_4,f_5\right),f_1\right)\circ\sigma_1 + h_w^{\mathcal{L}}\left(h_w\left(f_1,f_4,f_5\right),f_2,f_3\right)\circ\sigma_2 \\ + h_w^{\mathcal{L}}\left(h_w\left(f_1,f_2,f_3\right),f_4,f_5\right) = 0 \end{aligned}$$

and similarly from the inclusion $\mathcal{K}_{\Pi_2} \longrightarrow \mathcal{L}$ we get the relation

$$\begin{aligned} h_w^{\mathcal{L}}\left(h_w\left(f_2,f_3,f_4,f_5\right),f_1\right)\circ\sigma_1 + h_w^{\mathcal{L}}\left(h_w\left(f_1,f_5\right),f_2,f_3,f_4\right)\circ\sigma_2 \\ + h_w^{\mathcal{L}}\left(h_w\left(f_1,f_2,f_3,f_4\right),f_5\right) = 0 \end{aligned}$$

where σ_j and τ_j are the appropriate permutation maps. Then since

$$MF(\mathcal{L}) = \{\{1,2,3,4\},\{1,2,3,5\},\{2,3,4,5\},\{1,4,5\}\}$$

we deduce from Theorem 4.2.20 that if the f_i are compatible then each summand in both relations is non-trivial. Moreover, since the term $h_w^{\mathcal{L}}\left(h_w\left(f_2,f_3,f_4,f_5\right),f_1\right)$ is shared and $\sigma_1 = \tau_1$, it follows that

$$\begin{aligned} & h_w^{\mathcal{L}}\left(h_w\left(f_1,f_4,f_5\right),f_2,f_3\right)\circ\sigma_2 + h_w^{\mathcal{L}}\left(h_w\left(f_1,f_2,f_3\right),f_4,f_5\right) \\ & \quad = h_w^{\mathcal{L}}\left(h_w\left(f_1,f_5\right),f_2,f_3,f_4\right)\circ\tau_2 + h_w^{\mathcal{L}}\left(h_w\left(f_1,f_2,f_3,f_4\right),f_5\right) \end{aligned}$$

in $\left[\Sigma^3 X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5, (\underline{Y},\underline{*})^{\mathcal{L}}\right]$.

4.3.4 Propagation of relations

The relations of Theorem 4.3.7 are between higher Whitehead maps

$$h_w(h_w(f_{j_1},\ldots,f_{j_{r_i}}),f_{i_1},\ldots,f_{i_{n_i}})$$

such that the sets $I_i = \{i_1,\ldots,i_{n_i}\}$ are pairwise disjoint for $i = 1,\ldots,k$. We present a method of constructing a space in which there are relations where the sets I_i are allowed to intersect non-trivially.

Proposition 4.3.15. *Let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1,\ldots,m$. Let $l \leqslant m-3$ and let $\Pi = \{I_1,\ldots,I_k\}$ be a k-partition of $[n] = [m]-[l]$ for $k \geqslant 3$. Let $\mathcal{K}_\Pi$ be the* *corresponding identity complex. Let* $\mathcal{L} = \partial\Delta^l(\mathcal{K}_\Pi, \circ_1, \dots, \circ_l)$. *Then there are relations*

$$\sum_{i=1}^{k} h_w^{\mathcal{L}}(h_w(f_{j_1}, \dots, f_{j_{r_i}}, f_1, \dots, f_l), f_{i_1}, \dots, f_{i_{n_i}}) \circ \sigma_i = 0$$

and

$$\sum_{i=1}^{k} h_w^{\mathcal{L}}(h_w(f_{j_1}, \dots, f_{j_{r_i}}), f_{i_1}, \dots, f_{i_{n_i}}, f_1, \dots, f_l) \circ \tau_i = 0$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{L}}\right]$, *where* σ_i *and* τ_i *are appropriate permutation maps.*

Proof. Let $\Pi' = \{I_1, \dots, I_k, \{1, \dots, l\}\}$ be a $(k+1)$-partition of $[m]$ and let $\mathcal{K}_{\Pi'}$ be the corresponding identity complex. Then there is a relation

$$\sum_{i=1}^{k} h_w^{\mathcal{L}}(h_w(f_{j_1}, \dots, f_{j_{r_i}}, f_1, \dots, f_l), f_{i_1}, \dots, f_{i_{n_i}}) \circ \sigma_i + h_w(h_w(f_{l+1}, \dots, f_m), f_1, \dots, f_l) \circ \sigma = 0$$

in $[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_{\Pi'}}]$. We claim that $\mathcal{K}_{\Pi'} \subseteq \mathcal{L}$. Since

$$\mathcal{K}_{\Pi'} = \mathrm{sk}^{k-2}\, \Delta^k(\partial\Delta[1, \dots, l], \partial\Delta[I_1], \dots, \partial\Delta[I_k])$$

we have

$$\begin{aligned}
\mathcal{K}_{\Pi'} &= \partial\Delta[1, \dots, l] * \left(\bigcup_{j=1}^{k} \Delta[[n] - I_j]\right) \cup \Delta[1, \dots, l] * \left(\bigcup_{1 \leqslant i < j \leqslant k} \Delta[[n] - (I_i \cup I_j)]\right) \\
&= \partial\Delta[1, \dots, l] * (\Delta[[m] - I_1] \cup \cdots \cup \Delta[[m] - I_k]) \cup \Delta[1, \dots, l] * \mathcal{K}_\Pi
\end{aligned}$$

while the complex $\mathcal{L}$ decomposes as

$$\mathcal{L} = \partial\Delta^l(\mathcal{K}_\Pi, \circ_1, \dots, \circ_l) = \bigcup_{k=1}^{l} \Delta[1, \dots, k-1, k+1, \dots, m] \cup \mathcal{K}_\Pi * \Delta[1, \dots, l].$$

Therefore $\mathcal{K}_{\Pi'} \subseteq \mathcal{L}$. From this decomposition of $\mathcal{L}$ we further see that it contains the subcomplex

$$\partial\Delta\langle\Delta[l+1, \dots, m], \bullet_1, \dots, \bullet_l\rangle$$

and therefore $h_w^{\mathcal{L}}(h_w(f_{l+1}, \dots, f_m), f_1, \dots, f_l)$ is trivial in $[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{L}}]$ while the remaining terms are non-trivial and the first relation follows.

For the second relation, the same method as above shows that $\mathcal{L}$ also contains the identity complex corresponding to the partition $\{\{1, \dots, l\} \sqcup I_i, I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_k\}$ of $[m]$ for each $i = 1, \dots, k$. Therefore the term

$$h_w^{\mathcal{L}}(h_w(f_{j_1}, \dots, f_{j_{r_i}}), f_{i_1}, \dots, f_{i_{n_i}}, f_1, \dots, f_l) \circ \tau_i$$

can be rewritten as

$$-\sum_{i'\neq i} h_w^{\mathcal{L}}(h_w(f_{j'_1},\ldots,f_{j'_{r_{i'}}},f_1,\ldots,f_l),f_{i'_1},\ldots,f_{i'_{n_{i'}}})\circ\tau_{i'}$$

and therefore

$$\begin{aligned}
&\sum_{i=1}^{k} h_w^{\mathcal{L}}(h_w(f_{j_1},\ldots,f_{j_{r_i}}),f_{i_1},\ldots,f_{i_{n_i}},f_1,\ldots,f_l)\circ\tau_i\\
&=-\sum_{i=1}^{k}\sum_{i'\neq i} h_w^{\mathcal{L}}(h_w(f_{j'_1},\ldots,f_{j'_{r_{i'}}},f_1,\ldots,f_l),f_{i'_1},\ldots,f_{i'_{n_{i'}}})\circ\tau_{i'}\\
&=-k\sum_{i=1}^{k} h_w^{\mathcal{L}}(h_w(f_{j_1},\ldots,f_{j_{r_i}},f_1,\ldots,f_l),f_{i_1},\ldots,f_{i_{n_i}})\circ\sigma_i\\
&=0
\end{aligned}$$

with the final equality following from the first relation. □

4.4 Relations among folded higher Whitehead maps

We combine the relations among higher Whitehead maps given in Theorem 4.3.7 with the fold maps of polyhedral products introduced in Section 4.2.4 to produce relations among folded higher Whitehead maps. By analysing certain fold maps, we produce several families of relations for which trivial summands can be identified.

Let $\mathcal{K}_1,\ldots,\mathcal{K}_m$ be simplicial complexes on $[l_1],\ldots,[l_m]$, respectively. Suppose that $f_{i_j}\colon \Sigma X_i \longrightarrow Y_i$ are maps for $i=1,\ldots,m$ and $j=1,\ldots,k_i$ and let

$$g_i = h_w^{\mathcal{K}_i}(f_{i_1},\ldots,f_{i_{k_i}})\colon \Sigma X_i \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}_i} \tag{4.29}$$

be nested higher Whitehead maps for $i=1,\ldots,m$, where $X_i=\Sigma^{k_i-2}X_{i_1}\wedge\cdots\wedge X_{i_{k_i}}$. Let $\Pi=\{P_1,\ldots,P_k\}$ be a k-partition of $[m]$ and write $P_i=\{i_1,\ldots,i_{n_i}\}$ and $[m]-P_i=\{j_1,\ldots,j_{r_i}\}$. Let $\mathcal{K}=\mathcal{K}_\Pi\langle\mathcal{K}_1,\ldots,\mathcal{K}_m\rangle$, where $\mathcal{K}_\Pi$ is the identity complex corresponding to Π. Recall that by Theorem 4.3.7,

$$\sum_{i=1}^{k} h_w^{\mathcal{K}}\left(h_w\left(g_{j_1},\ldots,g_{j_{r_i}}\right),g_{i_1},\ldots,g_{i_{n_i}}\right)\circ\sigma_i=0 \tag{4.30}$$

in $\left[\Sigma^{m-2}X_1\wedge\cdots\wedge X_m,(\underline{Y},\underline{*})^{\mathcal{K}}\right]$. Let $\psi\colon I\longrightarrow J$ be a fold of $\mathcal{K}$ and suppose that for each $j\in J$ that Y_j is an associative H-space, and that $Y_i=Y_j$ for every $i\in I_j=\psi^{-1}(j)$.

By composing each term in (4.30) with the fold map

$$\nabla_{(I,J)}\colon (\underline{Y},\underline{*})^{\mathcal{K}} \longrightarrow (\underline{Y},\underline{*})^{\mathcal{K}_{\nabla(I,J)}}$$

we obtain a relation among folded higher Whitehead maps

$$\sum_{i=1}^{k} \nabla_{(I,J)} h_w^{\mathcal{K}} \left(h_w \left(g_{j_1}, \ldots, g_{j_{r_i}}\right), g_{i_1}, \ldots, g_{i_{n_i}}\right) \circ \sigma_i = 0 \tag{4.31}$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_{\nabla(I,J)}}\right]$. We are interested in cases where the complex $\mathcal{K}_{\nabla(I,J)}$ can be determined, and therefore trivial summands in (4.31) can be identified by applying Proposition 4.2.31.

4.4.1 Folds of identity complexes

The first family comes from considering the case that $\mathcal{K}_i = \bullet$ for each $i = 1, \ldots, m$. We give a full characterisation of fold maps $\psi\colon I \longrightarrow J$ which produce relations among folded higher Whitehead maps.

We first determine the dependence of $\mathcal{K}_{\nabla(I,J)}$ on the partition Π and the fold $\psi\colon I \longrightarrow J$ of $\mathcal{K} = \mathcal{K}_\Pi$. For $A \subseteq [m]$, let $\Delta[A]$ denote the $(|A| - 1)$-simplex on vertex set A.

Lemma 4.4.1. *We have*

(i) if $I \cup J \subseteq P_l$ for some $l = 1, \ldots, k$ then

$$\mathcal{K}_{\nabla(I,J)} = \partial\Delta[[m] - P_l] * \Delta[P_l - I];$$

(ii) if $|I| = |J| = 1$ and $I \subseteq P_i$ and $J \subseteq P_j$ for $i \neq j$, then

$$\mathcal{K}_{\nabla(I,J)} = \partial\Delta[[m] - I];$$

(iii) otherwise, $\mathcal{K}_{\nabla(I,J)} = \Delta[[m] - I]$.

Proof. By definition $\mathcal{K} = \mathcal{K}_\Pi$ decomposes as the polyhedral join

$$\begin{aligned}\mathcal{K} &= ((\Delta[P_1], \partial\Delta[P_1]), \ldots, (\Delta[P_k], \partial\Delta[P_k]))^{*sk_{k-3}\Delta[1,\ldots,k]} \\ &= (\Delta[P_i] * \Delta[P_j] * \cdots) \cup ((\Delta[P_i] * \partial\Delta[P_j] \cup \partial\Delta[P_i] * \Delta[P_j]) * \cdots) \cup (\partial\Delta[P_i] * \partial\Delta[P_j] * \cdots)\end{aligned}$$

for $i \neq j$. We first consider the case that $I = \{u\}$ and $J = \{v\}$. Suppose that $u \in P_i$ and $v \in P_j$. Then,

$$\begin{aligned}\nabla_{(u,v)}(\Delta[P_i] * \Delta[P_j]) &= \Delta[P_i \sqcup P_j - \{u\}] \\ \nabla_{(u,v)}(\Delta[P_i] * \partial\Delta[P_j]) &= \Delta[P_i \sqcup P_j - \{u\}] \\ \nabla_{(u,v)}(\partial\Delta[P_i] * \Delta[P_j]) &= \Delta[P_i \sqcup P_j - \{u\}] \\ \nabla_{(u,v)}(\partial\Delta[P_i] * \partial\Delta[P_j]) &= \partial\Delta[P_i \sqcup P_j - \{u\}].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{K}_{\nabla(I,J)} &= (\Delta[P_i \sqcup P_j - \{u\}] * \cdots) \cup (\partial\Delta[P_i \sqcup P_j - \{u\}] * \cdots) \\ &= ((\Delta[P_1], \partial\Delta[P_1]), \ldots, (\Delta[P_i \sqcup P_j - \{u\}], \partial\Delta[P_i \sqcup P_j - \{u\}]), \ldots \\ &\qquad \ldots, (\Delta[P_k], \partial\Delta[P_k]))^{*\partial\Delta[[k]-i]} \\ &= \partial\Delta[[m] - \{u\}]\end{aligned}$$

proving (ii).

If instead $u, v \in P_l$, the folding $\nabla_{(I,J)}$ sends $\Delta[P_l] \longmapsto \Delta[P_l - \{u\}]$ and $\partial\Delta[P_l] \longmapsto \Delta[P_l - \{u\}]$. Therefore

$$\begin{aligned}\mathcal{K}_{\nabla(I,J)} &= \Delta[P_l - \{u\}] * ((\Delta[P_1], \partial\Delta[P_1]), \ldots, (\Delta[P_k], \partial\Delta[P_k]))^{*\partial\Delta[[k]-\{l\}]} \\ &= \Delta[P_l - \{u\}] * \partial\Delta[[m] - P_l].\end{aligned}$$

Claims (i) and (iii) then follow from the above and claim (ii), using Proposition 4.2.25. □

Example 4.4.2. Let $\Pi = \{\{1,2,3\},\{4\},\{5\}\}$ and let $\mathcal{K} = \mathcal{K}_\Pi$. Then:

(i) if $I = \{1,2\}$ and $J = \{3\}$ then $\mathcal{K}_{\nabla(I,J)} = \partial\Delta[4,5] * \{3\}$;

(ii) if $I = \{1,2\}$ and $J = \{4\}$ then $\mathcal{K}_{\nabla(I,J)} = \Delta[3,4,5]$;

(iii) if $I = \{1\}$ and $J = \{2\}$ then $\mathcal{K}_{\nabla(I,J)} = \partial\Delta[4,5] * \Delta[2,3]$;

(iv) if $I = \{1\}$ and $J = \{4\}$ then $\mathcal{K}_{\nabla(I,J)} = \partial\Delta[2,3,4,5]$.

We use Proposition 4.2.31 to determine which terms of relation (4.31) are trivial. Specifically we use the characterisation that $\nabla_{(I,J)} h_w^{\mathcal{K}} \left(h_w\left(g_{j_1}, \ldots, g_{j_{r_i}}\right), g_{i_1}, \ldots, g_{i_{n_i}}\right)$ is trivial if $h_w^{\mathcal{L}} \left(h_w\left(g_{j_1}, \ldots, g_{j_{r_i}}\right), g_{i_1}, \ldots, g_{i_{n_i}}\right)$ is trivial, where $\mathcal{L} = \mathcal{L}_\psi$ is the simplicial complex obtained from $\mathcal{K}_{\nabla(I,J)}$ by substituting the simplex $\Delta^{|I_j|-1}$ at vertex j for each $j \in J$, see Definition 4.2.28.

Lemma 4.4.3. *Let $\mathcal{L} = \mathcal{L}_\psi$ be as above. Then*

(i) if $I \cup J \subseteq P_l$ for some $l = 1, \ldots, k$, then

$$\mathcal{L} = \partial\Delta[[m] - P_l] * \Delta[P_l];$$

(ii) if $|I| = |J| = 1$ and $I \subseteq P_i$ and $J \subseteq P_j$ for $i \neq j$ then $\mathcal{L}$ has minimal missing faces $\Delta[[m] - \{u\}]$ and $\Delta[[m] - \{v\}]$;

(iii) otherwise, $\mathcal{L} = \Delta[[m]]$.

Proof. By Proposition 4.3.5 we have

$$MF(\mathcal{L}) = \left\{ \bigsqcup_{i \in L} J_i \mid L \in MF\left(\mathcal{K}_{\nabla(I,J)}\right), J_i \in \Delta^{|I_j|-1} \right\}.$$

Statement (i) then follows from Lemma 4.4.1 since the missing face of $\partial\Delta[[m] - P_l] * \Delta[P_l - I]$ is not supported by any vertex in J. Statements (ii) and (iii) also follow similarly from Lemma 4.4.1. □

Proposition 4.4.4. *Suppose that I and J are not single elements, one from $\{j_1, \ldots, j_{r_i}\}$ and the other from $\{i_1, \ldots, i_{n_i}\}$. Then*

$$H_i = \nabla_{(I,J)} h_w^{\mathcal{K}} \left(h_w \left(g_{j_1}, \ldots, g_{j_{r_i}}\right), g_{i_1}, \ldots, g_{i_{n_i}}\right)$$

is trivial.

Proof. First, suppose that $I \sqcup J$ is not contained in some P_r for $r = 1, \ldots, k$, and that at least one of I and J contains more than one element. Then by Lemma 4.4.3(iii), $\mathcal{L} = \Delta[[m]]$, and so by Proposition 4.2.31, the map H_i is trivial for each $i = 1, \ldots, k$.

Next, suppose that $I \sqcup J \subseteq P_r$ for some $r = 1, \ldots, k$. Then by Lemma 4.4.3(i), $\mathcal{L} = \partial\Delta[[m] - P_r] * \Delta[P_r]$. Then certainly H_r is trivial, and furthermore for $i \neq r$, observe that $\partial\Delta\langle\Delta[j_1, \ldots, j_{r_i}], i_1, \ldots, i_{n_i}\rangle \subseteq \mathcal{L}$, and therefore H_i is also trivial for $i \neq r$.

Finally, suppose that $|I| = |J| = 1$ and that $I = \{u\} \in P_s$ and $J = \{v\} \in P_t$ for $s \neq t$. By Lemma 4.4.3(ii), $\mathcal{L}$ has minimal missing faces $\Delta[[m] - \{u\}]$ and $\Delta[[m] - \{v\}]$. Therefore if $u = j_l$ and $v = j_k$, then $\partial\Delta\langle\Delta[j_1, \ldots, j_{r_i}], i_1, \ldots, i_{n_i}\rangle \subseteq \mathcal{L}$, and so H_i is trivial. Observe that since $s \neq t$, it is not possible for $u = i_l$ and $v = i_k$ in this case. □

We therefore consider folds $\psi\colon I \longrightarrow J$ of $\mathcal{K}_\Pi$ for which I and J are both singletons not contained in the same P_l. Our main result is then the following.

Theorem 4.4.5. *Let $\Pi = \{P_1, \ldots, P_k\}$ be a k-partition of $[m]$ and let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \ldots, m$. Let $i, j \in [m]$ be such that $i \in P_u$ and $j \in P_v$ for $u \neq v$ and let $\psi\colon \{i\} \longrightarrow \{j\}$ be a fold of $\mathcal{K} = \mathcal{K}_\Pi$. Then if each X_i is a suspension and $Y_i = Y_j$ is an H-space,*

$$\begin{aligned}\nabla_{(i,j)} h_w^{\mathcal{K}}(h_w(f_{i'_1}, \ldots, f_{i'_{r_i}}), f_{i_1}, \ldots, f_{i_{n_i}}) \circ \sigma_i \\ + \nabla_{(i,j)} h_w^{\mathcal{K}}(h_w(f_{j'_1}, \ldots, f_{j'_{r_j}}), f_{j_1}, \ldots, f_{j_{n_j}}) \circ \sigma_j = 0\end{aligned}$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, FW(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_m)\right]$, where $P_u = \{i_1, \ldots, i_{n_i}\}$, $[m] - P_u = \{i'_1, \ldots, i'_{r_i}\}$, $P_v = \{j_1, \ldots, j_{n_j}\}$ and $[m] - P_v = \{j'_1, \ldots, j'_{r_j}\}$.

Moreover, if $i, j \in [m]$ with $i \neq j$, let $\{i_1, \ldots, i_k\} \subseteq [m] - \{i, j\}$ be any non-empty proper subset. Then if each X_i is a suspension and $Y_i = Y_j$ is an H-space,

$$\nabla_{(i,j)} h_w^{\mathcal{K}}(h_w(f_{j_1}, \ldots, f_{j_r}, f_i), f_{i_1}, \ldots, f_{i_k}, f_j) \circ \sigma \\ + \nabla_{(i,j)} h_w^{\mathcal{K}}(h_w(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m), f_i) \circ \tau = 0$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, FW(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_m)\right]$, where σ and τ are appropriate permutation maps.

Proof. The first part follows immediately by applying Proposition 4.4.4 to relation (4.31). The second part then follows by applying the first part to the partition

$$\Pi = \{\{i\}, \{j, i_1, \ldots, i_k\}, \{j_1, \ldots, j_r\}\}$$

of $[m]$. □

Example 4.4.6. Certain folded higher Whitehead maps coincide with 2-fold Whitehead products. For example, let $m = 5$. Then combining Theorem 4.4.5 with Proposition 4.2.26, the folded higher Whitehead maps $\nabla_{(5,1)} h_w(h_w(f_1, f_2, f_3), f_4, f_5)$ and $\nabla_{(5,1)} h_w(h_w(f_1, f_2), f_3, f_4, f_5)$ are both identified, up to sign, with the Whitehead product

$$[h_w(f_1, f_2, f_3, f_4), \iota \circ f_1]$$

for the inclusion map $\iota \colon Y_1 \longrightarrow FW(Y_1, Y_2, Y_3, Y_4)$.

Of particular interest, Theorem 4.4.5 allows us to detect new instances of 2-torsion in homotopy groups.

Example 4.4.7. Let $\Pi = \{\{1\}, \{2, \ldots, m-1\}, \{m\}\}$ and consider the higher Whitehead maps $h_w^{\mathcal{K}_\Pi}(h_w(f_1, \ldots, f_{m-1}), f_m)$ and $h_w^{\mathcal{K}_\Pi}(h_w(f_2, \ldots, f_m), f_1)$. Applying the fold $\psi \colon \{m\} \longrightarrow \{1\}$ of $\mathcal{K}_\Pi$, we obtain that

$$\nabla_{(m,1)} h_w^{\mathcal{K}_\Pi}(h_w(f_1, \ldots, f_{m-1}), f_m) = \nabla_{(m,1)} h_w^{\mathcal{K}_\Pi}(h_w(f_2, \ldots, f_m), f_1) \circ \sigma.$$

Suppose that $X_i = S^{p_i - 1}$ with $p_i \geqslant 2$ for $i = 1, \ldots, m$. Then applying Theorem 4.4.5 for any maps $f_i \colon S^{p_i} \longrightarrow Y_i$ with $p_1 p_m$ even and $Y_i = Y_j$ an H-space, we obtain that

$$2\nabla_{(m,1)} h_w^{\mathcal{K}_\Pi}(h_w(f_1, \ldots, f_{m-1}), f_m) = 0$$

in $\pi_{p_1 + \cdots + p_m - 2}((\underline{Y}, \underline{*})^{\partial\Delta})$. Therefore our methods allow us to study, via the higher Whitehead map, elements of homotopy groups which cannot be seen with rational methods.

In general, the triviality of the map $\nabla_{(m,1)} h_w^{\mathcal{K}_\Pi}(h_w(f_1, \ldots, f_{m-1}), f_m)$ depends on the internal properties of the space Y_i. To demonstrate this, let $p_i = 2$ for $i = 1, \ldots, m$, so that $f_i \colon S^2 \longrightarrow Y_i$. First, suppose that $Y_i = S^2$ for $i = 1, \ldots, m$ and that each f_i is an identity map. In this case, the fold $\nabla_{(I,J)}$ can be defined without requiring Y_i to be an H-space, since there is no edge between $\{1\}$ and $\{m\}$ in $\mathcal{K}_\Pi$. In this case, relation (4.31) has the form

$$2\nabla_{(m,1)}h_w^{\mathcal{K}_\Pi}(h_w(f_1,\dots,f_{m-1}),f_m)+\nabla_{(m,1)}h_w^{\mathcal{K}_\Pi}(h_w(f_1,f_m),f_2,\dots,f_{m-1})=0$$

in $\pi_{2m-2}((S^2,*)^{\partial\Delta})$. By naturality of the higher Whitehead map and Proposition 4.2.26, we have

$$\nabla_{(m,1)}h_w(h_w(f_1,f_m),f_2,\dots,f_{m-1})=h_w([f_1,f_1],f_2,\dots,f_{m-1}).$$

Since $[f_1,f_1]=2\eta$, where $\eta\in\pi_3(S^2)$ is the Hopf map, we therefore obtain from linearity of the higher Whitehead map that

$$\nabla_{(m,1)}h_w(h_w(f_1,\dots,f_{m-1}),f_m)=h_w(\eta,f_2,\dots,f_{m-1})$$

which is moreover non-trivial since $\eta,f_2,\dots,f_{m-1}$ are compatible.

Applying the map $(S^2,*)^{\partial\Delta}\longrightarrow(\mathbb{C}P^\infty,*)^{\partial\Delta}$ trivialises the map $h_w(\eta,f_2,\dots,f_{m-1})$, since $\eta=0$ in $\pi_3(\mathbb{C}P^\infty)$. Therefore by naturality of the higher Whitehead map, the folded map

$$\nabla_{(m,1)}h_w(h_w(g_1,\dots,g_{m-1}),g_m)$$

is trivial in $\pi_{2m-2}((\mathbb{C}P^\infty,*)^{\partial\Delta})$, where $g_i\colon S^2\longrightarrow\mathbb{C}P^\infty$ is the inclusion of the bottom cell.

Alternatively, applying the map $(S^2,*)^{\partial\Delta}\longrightarrow(\Omega S^3,*)^{\partial\Delta}$ induced by the suspension $S^2\longrightarrow\Omega\Sigma S^2$ does not trivialise the map $h_w(\eta,f_2,\dots,f_{m-1})$, since the composite $S^3\xrightarrow{\eta} S^2\longrightarrow\Omega\Sigma S^2$ is adjoint to $\Sigma\eta$, which generates $\pi_4(S^3)$. Therefore the folded map

$$\nabla_{(m,1)}h_w(h_w(g_1,\dots,g_{m-1}),g_m)$$

is non-trivial in $\pi_{2m-2}((\Omega S^3,*)^{\partial\Delta})$, where $g_i\colon S^2\longrightarrow\Omega S^3$ is the suspension homomorphism. Moreover, since ΩS^3 is an H-space, the map $h_w([g_1,g_1],g_2,\dots,g_{m-1})$ is trivial. It follows that $\nabla_{(m,1)}h_w(h_w(g_1,\dots,g_{m-1}),g_m)$ is a 2-torsion element in $\pi_{2m-2}((\Omega S^3,*)^{\partial\Delta})$.

4.4.2 Folding and substitution

We return to the case that $\mathcal{K}_1,\dots,\mathcal{K}_m$ are general simplicial complexes. Let $\mathcal{K}=\mathcal{K}_\Pi\langle\mathcal{K}_1,\dots,\mathcal{K}_m\rangle$ with vertex set $[l]=[l_1]\sqcup\cdots\sqcup[l_m]$. Unlike the case with $\mathcal{K}_i=\bullet$ for each i, for an arbitrary fold $\psi\colon I\longrightarrow J$ of $\mathcal{K}$ the folded complex $\mathcal{K}_{\nabla(I,J)}$ cannot be determined in general. The main difficulty is that the simplicial folding operation does not commute with substitution.

Example 4.4.8. Let $\Pi = \{\{1\}, \{2, 3\}, \{4\}\}$ from Example 4.3.9(iii) and let $\mathcal{K}_\Pi$ be the corresponding identity complex. Let $\mathcal{K} = \mathcal{K}_\Pi\langle\partial\Delta[1_1, 1_2], 2, 3, 4\rangle$. Let ψ be the map sending $I = \{2\}$ to $J = \{1_1\}$. Then the complex $\mathcal{K}_{\nabla(I,J)}$ is shown in Figure 4.2.

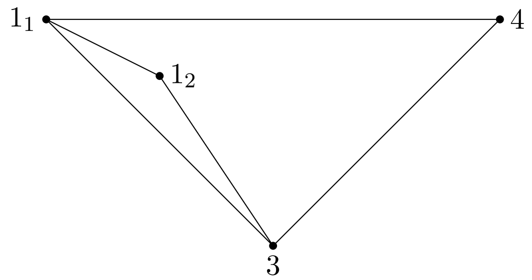


Figure 4.2: A folded substitution complex.

In particular, it is not a substitution complex.

We analyse folds $\psi\colon I \longrightarrow J$ of $\mathcal{K}$ for which the folded complex $\mathcal{K}_{\nabla(I,J)}$ is a substitution complex. A simple scenario is the case that the fold takes the vertex set $[l_i]$ of $\mathcal{K}_i$ to itself for $i = 1, \ldots, m$. Concretely, suppose that $\psi\colon I \longrightarrow J$ satisfies $\psi([l_i] \cap I) \subseteq [l_i]$ for each $i = 1, \ldots, m$.

Lemma 4.4.9. *Let $\mathcal{L}$ be a simplicial complex on $[m]$ and let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be simplicial complexes on $[l_1], \ldots, [l_m]$, respectively. Let $[l] = [l_1] \sqcup \cdots \sqcup [l_m]$ and suppose that $\psi\colon I \longrightarrow J$ is a fold of $\mathcal{L}\langle\mathcal{K}_1, \ldots, \mathcal{K}_m\rangle$ such that $\psi([l_i] \cap I) \subseteq [l_i]$ for each $i = 1, \ldots, m$. Then*

$$\mathcal{L}\langle\mathcal{K}_1, \ldots, \mathcal{K}_m\rangle_{\nabla(I,J)} = \mathcal{L}\langle(\mathcal{K}_1)_{\nabla(I,J)}, \ldots, (\mathcal{K}_m)_{\nabla(I,J)}\rangle$$

where if $[l_i] \cap I = \varnothing$ we set $(\mathcal{K}_i)_{\nabla(I,J)} = \mathcal{K}_i$.

Proof. First, define a map $\overline{\psi}\colon [l] \longrightarrow [l]$ by $\overline{\psi}(i) = \psi(i)$ if $i \in I$ and $\overline{\psi}(i) = i$ otherwise. Then ψ and $\overline{\psi}$ induce the same fold of simplicial complexes. By definition,

$$\mathcal{L}\langle\mathcal{K}_1, \ldots, \mathcal{K}_m\rangle = \left\{\bigsqcup_{j\in\tau} \sigma_j \mid \sigma_j \in \mathcal{K}_j, \tau \in \mathcal{L}\right\}$$

and therefore

$$\mathcal{L}\langle\mathcal{K}_1, \ldots, \mathcal{K}_m\rangle_{\nabla(I,J)} = \left\{\bigsqcup_{j\in\tau} \overline{\psi}(\sigma_j) \mid \sigma_j \in \mathcal{K}_j, \tau \in \mathcal{L}\right\}.$$

Now any simplex of $(\mathcal{K}_i)_{\nabla(I,J)}$ is either a simplex of $\mathcal{K}_i$, or is of the form $\psi(\sigma_i)$ for some $\sigma_i \in \mathcal{K}_i$. Either way, every simplex $(\mathcal{K}_i)_{\nabla(I,J)}$ can be written $\overline{\psi}(\sigma_i)$ for some $\sigma_i \in \mathcal{K}_i$, and so the right-hand side above defines $\mathcal{L}\langle(\mathcal{K}_1)_{\nabla(I,J)}, \ldots, (\mathcal{K}_m)_{\nabla(I,J)}\rangle$. $\square$

Theorem 4.4.10. *Let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be simplicial complexes on $[l_1], \ldots, [l_m]$, respectively and let $[l] = [l_1] \sqcup \cdots \sqcup [l_m]$. Let $\Pi = \{P_1, \ldots, P_k\}$ be a k-partition of $[m]$ and denote*

$I_i = \{i_1, \dots, i_{n_i}\}$ and $[m] - P_i = \{j_1, \dots, j_{r_i}\}$. Let $\mathcal{K} = \mathcal{K}_\Pi\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle$. Let $\psi\colon I \longrightarrow J$ be a fold of $\mathcal{K}$ such that $\psi([l_i] \cap I) \subseteq [l_i]$ for each $i = 1, \dots, m$.

Let $g_i\colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i}$ be nested higher Whitehead maps (4.29) for $i = 1, \dots, m$. Then if each X_i is a suspension,

$$\sum_{i=1}^{k} h_w^{\mathcal{K}}\left(h_w\left(\nabla_{(I,J)} \circ g_{j_1}, \dots \nabla_{(I,J)} \circ g_{j_{r_j}}\right), \nabla_{(I,J)} \circ g_{i_1}, \dots, \nabla_{(I,J)} \circ g_{i_{n_i}}\right) = 0 \tag{4.32}$$

in $\left[\Sigma^{m-2} X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_{\nabla(I,J)}}\right]$. Furthermore, all summands are trivial if

$$\nabla_{(I,J)} \circ g_j\colon \Sigma X_j \longrightarrow (\underline{Y}, \underline{*})^{(\mathcal{K}_j)_{\nabla(I,J)}}$$

is trivial for some $j = 1, \dots, m$.

Proof. By Lemma 4.4.9, $\mathcal{K}_{\nabla(I,J)} = \mathcal{K}_\Pi\langle (\mathcal{K}_1)_{\nabla(I,J)}, \dots, (\mathcal{K}_m)_{\nabla(I,J)}\rangle$. Therefore by naturality of the higher Whitehead map

$$\begin{aligned}\nabla_{(I,J)} h_w^{\mathcal{K}}&\left(h_w\left(g_{j_1}, \dots, g_{j_{r_i}}\right), g_{i_1}, \dots, g_{i_{n_i}}\right)\\ &= h_w^{\mathcal{K}}\left(h_w\left(\nabla_{(I,J)} \circ g_{j_1}, \dots \nabla_{(I,J)} \circ g_{j_{r_j}}\right), \nabla_{(I,J)} \circ g_{i_1}, \dots, \nabla_{(I,J)} \circ g_{i_{n_i}}\right)\end{aligned}$$

establishing the claimed relation. Moreover, if $\nabla_{(I,J)} \circ g_j = 0$ for some $j = 1, \dots, m$, every term in relation (4.32) is trivial by Theorem 4.2.20. □

For the second family, we consider the case that $\Pi = \{\{1\}, \{2, \dots, m-1\}, \{m\}\}$ and that $\mathcal{K}_m$ is isomorphic to a full subcomplex of $\mathcal{K}_1$. Let $I = [l_m]$ and J be the vertex set of the isomorphic copy of $\mathcal{K}_m$ inside $\mathcal{K}_1$, with the isomorphism given by $\psi\colon I \longrightarrow J$.

Lemma 4.4.11. *We have that*

$$\mathcal{K}_{\nabla(I,J)} = \partial\Delta\langle \mathcal{K}_1, \dots, \mathcal{K}_{m-1}\rangle.$$

Proof. Since $\psi(\mathcal{K}_m) = (\mathcal{K}_1)_J$, then $(\mathcal{K}_1 \sqcup \mathcal{K}_m)_{\nabla(I,J)} = \mathcal{K}_1$. Then since

$$\mathcal{K} = \mathcal{K}_\Pi\langle \mathcal{K}_1, \dots, \mathcal{K}_m\rangle = \partial\Delta\langle \mathcal{K}_1 \sqcup \mathcal{K}_m, \mathcal{K}_2, \dots, \mathcal{K}_{m-1}\rangle$$

the result follows by Lemma 4.4.9. □

If each X_i is a suspension then there is a relation

$$\begin{aligned}\nabla_{(I,J)} h_w^{\mathcal{K}}\left(h_w(g_2, \dots, g_m), g_1\right) \circ \sigma_1 + \nabla_{(I,J)} h_w^{\mathcal{K}}\left(h_w(g_1, g_m), g_2, \dots, g_{m-1}\right) \circ \sigma_2\\ + \nabla_{(I,J)} h_w^{\mathcal{K}}\left(h_w(g_1, \dots, g_{m-1}), g_m\right) = 0\end{aligned} \tag{4.33}$$

in $\left[\Sigma^{m-2}X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_{\nabla(I,J)}}\right]$. We analyse the triviality of each term using Proposition 4.2.31. Of particular interest is the second term, which is trivial if

$$(\partial\Delta\langle\mathcal{K}_1 * \mathcal{K}_m, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1}\rangle)_{\nabla(I,J)} = \partial\Delta\langle(\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)}, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1}\rangle$$

is a subcomplex of $\mathcal{K}_{\nabla(I,J)} = \partial\Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}\rangle$. While $\mathcal{K}_1 \subseteq (\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)}$, it is not true that the converse containment holds, even though $(\mathcal{K}_m)_{\nabla(I,J)} \subseteq \mathcal{K}_1$. For example, if $\mathcal{K}_1$ consists of two disjoint points and $\mathcal{K}_m$ is a single point, then $(\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)}$ is a 1-simplex. The following gives an easy combinatorial condition for when the converse containment holds.

Lemma 4.4.12. *The containment $\mathcal{K}_1 \subseteq (\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)}$ is strict if and only if there is $j \in J$ and $L \in MF(\mathcal{K}_1)$ such that $j \in L$.*

Proof. Suppose there exists $j \in J$ and $L \in MF(\mathcal{K}_1)$ with $j \in L$. Write $L = \{l_1, \ldots, l_k, j\}$ and let $i \in I$ be such that $\psi(i) = j$. Then since $\Delta[l_1, \ldots, l_k] \in \mathcal{K}_1$ and $i \in \mathcal{K}_m$, the simplex $\Delta[l_1, \ldots, l_k, i]$ is in $\mathcal{K}_1 * \mathcal{K}_m$. Therefore $\Delta[l_1, \ldots, l_k, j]$ is a simplex of $(\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)}$, but not of $\mathcal{K}_1$.

Conversely suppose that the claimed containment is strict. Then there is $\sigma \in (\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)}$ such that $\sigma \notin \mathcal{K}_1$. Furthermore, σ can always be chosen to be a minimal missing face of $\mathcal{K}_1$. Since $\sigma \notin \mathcal{K}_1$, then $\sigma = \psi(\tau)$ for some $\tau \in \mathcal{K}_1 * \mathcal{K}_m$. In particular, $\sigma \cap J$ is non-empty. Therefore any $j \in \sigma \cap J$ gives a $j \in J$ such that $j \in \sigma$. □

Our main result is that the first and third terms in relation (4.33) are not trivialised by Proposition 4.2.31, while the triviality of the second term can be expressed in terms of the minimal missing faces of $\mathcal{K}_1$.

Theorem 4.4.13. *Let $g_i \colon \Sigma X_i \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_i}$ be nested higher Whitehead maps for $i = 1, \ldots, m$.*

Suppose that $\mathcal{K}_m$ is isomorphic to a full subcomplex $(\mathcal{K}_1)_J$, where the isomorphism is given by $\psi \colon I \longrightarrow J$. Then if each X_i is a suspension, there is a relation

$$\begin{aligned}\nabla_{(I,J)} h_w^{\mathcal{K}}\left(h_w(g_2, \ldots, g_m), g_1\right) \circ \sigma_1 + \nabla_{(I,J)} h_w^{\mathcal{K}}\left(h_w(g_1, g_m), g_2, \ldots, g_{m-1}\right) \circ \sigma_2 \\ + \nabla_{(I,J)} h_w^{\mathcal{K}}\left(h_w(g_1, \ldots, g_{m-1}), g_m\right) = 0\end{aligned}$$

in $\left[\Sigma^{m-2}X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_{\nabla(I,J)}}\right]$, where the second term is trivial if there is no $j \in J$ and $L \in MF(\mathcal{K}_1)$ such that $j \in L$.

Proof. We prove a slightly stronger statement, showing that the first and third terms of (4.33) are not immediately trivialised by Proposition 4.2.31, and that the second term is not trivialised if there is $j \in J$ and $L \in MF(\mathcal{K}_1)$ such that $j \in L$. Let $\mathcal{L} = \mathcal{K}_{\nabla(I,J)} = \partial\Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}\rangle$ and define simplicial complexes

$$\begin{aligned}
\mathcal{L}_1 &= (\Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}\rangle)_{\nabla(I,J)} & \mathcal{L}_2 &= (\partial\Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}\rangle * \mathcal{K}_m)_{\nabla(I,J)} \\
\mathcal{L}_3 &= (\Delta\langle\mathcal{K}_2, \ldots, \mathcal{K}_m\rangle)_{\nabla(I,J)} & \mathcal{L}_4 &= (\partial\Delta\langle\mathcal{K}_2, \ldots, \mathcal{K}_m\rangle * \mathcal{K}_1)_{\nabla(I,J)} \\
\mathcal{L}_5 &= (\Delta\langle\mathcal{K}_1 \sqcup \mathcal{K}_m, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1}\rangle)_{\nabla(I,J)} & \mathcal{L}_6 &= (\partial\Delta\langle\mathcal{K}_1 * \mathcal{K}_m, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1})_{\nabla(I,J)}.
\end{aligned}$$

By Proposition 4.2.31, the map $\nabla_{(I,J)}h_w(h_w(f_1, \ldots, f_{m-1}), f_m)$ trivial if either $\mathcal{L}_1$ or $\mathcal{L}_2$ is contained in $\mathcal{L}$; the map $\nabla_{(I,J)}h_w\,(h_w(g_1, \ldots, g_{m-1}), g_m)$ is trivial if either $\mathcal{L}_3$ or $\mathcal{L}_4$ is contained in $\mathcal{L}$; and the map $\nabla_{(I,J)}h_w\,(h_w(g_1, g_m), g_2, \ldots, g_{m-1})$ is trivial if either $\mathcal{L}_5$ or $\mathcal{L}_6$ is contained in $\mathcal{L}$.

First observe that $\mathcal{L}_1 = \mathcal{L}_5 = \Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}\rangle$, so neither $\mathcal{L}_1$ nor $\mathcal{L}_5$ is contained in $\mathcal{L}$. For $\mathcal{L}_3$, take a simplex $\Delta[k_2, \ldots, k_{m-1}, k_m] \in \Delta\langle\mathcal{K}_2, \ldots, \mathcal{K}_m\rangle$, where $k_j \in \mathcal{K}_j$. Then $\Delta[k_2, \ldots, k_{m-1}, \psi(k_m)]$ is a simplex of $\mathcal{L}_3$, but not a simplex of $\mathcal{L}$ since $\psi(k_m) \in \mathcal{K}_1$. Therefore $\mathcal{L}_3$ is also not contained in $\mathcal{L}$.

Next, observe that

$$\mathcal{L}_6 = (\partial\Delta\langle\mathcal{K}_1 * \mathcal{K}_m, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1}\rangle)_{\nabla(I,J)} = \partial\Delta\langle(\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)}, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1}\rangle$$

is contained in $\mathcal{L}$ if and only if $(\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)} = \mathcal{K}_1$. Therefore if there is no $j \in J$ and $L \in MF(\mathcal{K}_1)$ with $j \in L$, then by Lemma 4.4.12 $(\mathcal{K}_1 * \mathcal{K}_m)_{\nabla(I,J)} = \mathcal{K}_1$, and so $\nabla_{(I,J)}h_w\,(h_w(g_1, g_m), g_2, \ldots, g_{m-1})$ is trivial.

Furthermore, observe that since $\partial\Delta\langle\mathcal{K}_1 * \mathcal{K}_m, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1}\rangle \subseteq \partial\Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}\rangle * \mathcal{K}_m$ and $\partial\Delta\langle\mathcal{K}_1 * \mathcal{K}_m, \mathcal{K}_2, \ldots, \mathcal{K}_{m-1}\rangle \subseteq \partial\Delta\langle\mathcal{K}_2, \ldots, \mathcal{K}_m\rangle * \mathcal{K}_1$, then $\mathcal{L}_6 \subseteq \mathcal{L}_2$ and $\mathcal{L}_6 \subseteq \mathcal{L}_4$. Therefore it remains to show that the remaining two terms are not trivialised in this case, or equivalently that the containments $\mathcal{L}_6 \subseteq \mathcal{L}_2$ and $\mathcal{L}_6 \subseteq \mathcal{L}_4$ are strict. Take a simplex $\Delta[k_2, \ldots, k_{m-1}, k_m] \in \partial\Delta\langle\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}\rangle * \mathcal{K}_m$, where $k_j \in \mathcal{K}_j$. Then $\Delta[k_2, \ldots, k_{m-1}, \psi(k_m)]$ is a simplex of $\mathcal{L}_2$, but not of $\mathcal{L}_6$ since $\psi(k_m) \in \mathcal{K}_1$. Therefore the containment $\mathcal{L}_6 \subseteq \mathcal{L}_2$ is strict. A similar argument shows that $\mathcal{L}_6 \subseteq \mathcal{L}_4$ is strict, and the result follows. □

Example 4.4.14. Let $\Pi = \{\{1\}, \{2, 3\}, \{4\}\}$ and let $\mathcal{K}_1 = \partial\Delta[1_1, 1_2, 1_3]$. Let

$$g_1 = h_w(g_{1_1}, g_{1_2}, g_{1_3})\colon \Sigma^2 X_{1_1} \wedge X_{1_2} \wedge X_{1_3} \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}_1}$$

and let $g_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 2, 3, 4$. Let $I = \{4\}$, $J = \{1_1, 1_2, 1_3\}$ and $\psi\colon I \longrightarrow J$ be defined by $\psi(4) = 1_1$. Suppose that $Y_4 = Y_{1_1}$ is an H-space and X_i is a suspension for $i = 2, 3, 4$. Since $1_1 \in J$ is contained in the missing face $\{1_1, 1_2, 1_3\}$ of $\mathcal{K}_1$, by Theorem 4.4.13 there is a relation

$$\begin{aligned}\nabla_{(4,1_1)} h_w(h_w(g_2, g_3, g_4), h_w(g_{1_1}, g_{1_2}, g_{1_3}) \circ \sigma_1 \\ + \nabla_{(4,1_1)} h_w(h_w(h_w(g_{1_1}, g_{1_2}, g_{1_3}), g_4), g_2, g_3) \circ \sigma_2 \\ + \nabla_{(4,1_1)} h_w(h_w(h_w(g_{1_1}, g_{1_2}, g_{1_3}), g_2, g_3), g_4) = 0\end{aligned} \tag{4.34}$$

in $\left[\Sigma^2 X_1 \wedge X_2 \wedge X_3 \wedge X_4, (\underline{Y}, \underline{*})^{\mathcal{L}}\right]$, where $\mathcal{L} = \partial\Delta\langle\partial\Delta[1_1, 1_2, 1_3], 2, 3\rangle$.

This integrally recovers relations detected in the rational homotopy groups of the Davis–Jaunszkiewicz space $DJ_{\mathcal{L}}$ by Zhuravleva [Zhu21]. Let each g_j be the cellular inclusion $S^2 \longrightarrow \mathbb{C}P^\infty$ and let μ_j be the composite of g_j with the inclusion $\mathbb{C}P^\infty \longrightarrow DJ_{\mathcal{L}}$. Then the methods of Zhuravleva show that

$$[[\mu_2, \mu_3, \mu_{1_1}]_{\mathbb{Q}}, [\mu_{1_1}, \mu_{1_2}, \mu_{1_3}]_{\mathbb{Q}}] + [[[\mu_{1_1}, \mu_{1_2}, \mu_{1_3}]_{\mathbb{Q}}, \mu_2, \mu_3]_{\mathbb{Q}}, \mu_{1_1}] = 0 \tag{4.35}$$

where $[\cdot, \cdot, \cdot]_{\mathbb{Q}}$ is the rational triple Whitehead bracket obtained by rationalising the higher Whitehead map.

Our results show that integrally the Whitehead products $[h_w^{\mathcal{L}}(g_2, g_3, g_{1_1}), h_w^{\mathcal{L}}(g_{1_1}, g_{1_2}, g_{1_3})]$ and $[h_w^{\mathcal{L}}(h_w(g_{1_1}, g_{1_2}, g_{1_3}), g_2, g_3), \mu_{1_1}]$ differ up to sign by

$$\nabla_{(4,1_1)} h_w(h_w(h_w(g_{1_1}, g_{1_2}, g_{1_3}), g_4), g_2, g_3) \in [[h_w^{\mathcal{L}}(g_{1_1}, g_{1_2}, g_{1_3}), \mu_{1_1}], \mu_2, \mu_3] \tag{4.36}$$

which is null-homotopic by Example 4.4.7, since the folded map $\nabla_{(4,1_1)} h_w(h_w(g_{1_1}, g_{1_2}, g_{1_3}), g_4)$ is null-homotopic.

More generally, however, each term in relation (4.34) can be non-trivial. In this case, the term (4.36) a 2-torsion element since $\nabla_{(4,1_1)} h_w(h_w(g_{1_1}, g_{1_2}, g_{1_3}), g_4)$ is itself 2-torsion by Example 4.4.7. For example, this occurs if $Y_i = \Omega S^3$ and $g_i \colon S^2 \longrightarrow \Omega\Sigma S^2$ is the adjoint to the suspension map on S^2 for $i = 1, \ldots, m$. We therefore recover further integral relations which cannot be detected rationally.

4.5 Proof of main theorem

In this section we prove Theorem 4.3.7, which we state again.

Theorem 4.3.7. *Let $f_i \colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \ldots, m$. Let $\Pi = \{I_1, \ldots, I_k\}$ be a k-partition of $[m]$ for $k \geqslant 3$ and denote $I_i = \{i_1, \ldots, i_{l_i}\}$ and $J_i = [m] - I_i = \{j_1, \ldots, j_{r_j}\}$. Then if X_i is a suspension for each $i = 1, \ldots, m$,*

$$\sum_{i=1}^{k} h_w^{\mathcal{K}_\Pi}\left(h_w\left(f_{j_1}, \ldots, f_{j_{r_i}}\right), f_{i_1}, \ldots, f_{i_{n_i}}\right) \circ \sigma_i = 0$$

in $\left[\Sigma^{m-2}X_1 \wedge \cdots \wedge X_m, (\underline{Y}, \underline{*})^{\mathcal{K}_\Pi}\right]$, *where*

$$\sigma_i\colon \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m \longrightarrow \Sigma^{n_i}(\Sigma^{r_i-2}X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}) \wedge X_{i_1} \wedge \cdots \wedge X_{i_{n_i}}$$

is the restriction of the coordinate permutation

$$CX_1 \times \cdots \times CX_m \longrightarrow CX_{j_1} \times \cdots \times CX_{j_{r_i}} \times CX_{i_1} \times \cdots \times CX_{i_{n_i}}.$$

We adapt the methods used by Nakaoka–Toda [NT54] and Hardie [Har64] in proving the Jacobi identity and Theorem 4.3.1, respectively. We extend these methods in two main ways, first by using the combinatorial structure of $\mathcal{K}_\Pi$ to detect the form of the nested higher Whitehead maps appearing in Theorem 4.3.7, and second deriving the claimed relations for maps $f_i\colon \Sigma X_i \longrightarrow Y_i$ where X_i is not necessarily a sphere. The central object of the proof is a long exact sequence constructed by generalising the method given by Arkowitz [Ark11, pp. 138–139].

Let X be a space and A a subspace. Denote by X_A be the homotopy fibre of the inclusion $A \longrightarrow X$. Then by Proposition 2.1.3(i) for any space W there is long exact sequence

$$\cdots \longrightarrow [\Sigma W, X] \xrightarrow{j} [W, X_A] \xrightarrow{\partial} [W, A] \xrightarrow{i} [W, X] \longrightarrow \cdots. \tag{4.37}$$

If (X, A) and (Y, B) are pairs, we denote by $[(Y, B), (X, A)]$ the set of homotopy classes of maps of pairs $(Y, B) \longrightarrow (X, A)$. There is a bijection between $[W, X_A]$ and $[(CW, W), (X, A)]$ constructed as follows. We model the homotopy fibre X_A as the space

$$\{(\gamma, a) \in PX \times A \mid \gamma(1) = a\}.$$

Let $f\colon W \longrightarrow X_A$ be a map. We define a map $CW \longrightarrow X$ by sending $(w, t) \longmapsto \gamma_w(t)$ where $f(w) = (\gamma_w, a_w)$. Moreover, this is a map of pairs $(CW, W) \longrightarrow (X, A)$ since $\gamma_w(1) = a_w \in A$. This constructs a map $[W, X_A] \longrightarrow [(CW, W), (X, A)]$. Conversely any map $g \in (CW, W) \longrightarrow (X, A)$ defines a map $W \longrightarrow X_A$ by sending $w \longmapsto (\gamma, a)$ where $\gamma(t) = g(w, t)$ and $a = g(w, 1) \in A$. This gives a map $[(CW, W), (X, A)] \longrightarrow [W, X_A]$ and the claimed bijection follows. Combining this with long exact sequence (4.37), we obtain a long exact sequence

$$\cdots \longrightarrow [\Sigma W, X] \xrightarrow{j} [(CW, W), (X, A)] \xrightarrow{\partial} [W, A] \longrightarrow \cdots \tag{4.38}$$

where moreover the maps j and ∂ can be specified explicitly. In particular, the map j is the composite of the isomorphism $[\Sigma W, X] \longrightarrow [(CW, W), (X, *)]$ with the map induced by the inclusion $(X, *) \longrightarrow (X, A)$, and ∂ is the restriction sending $f \in [(CW, W), (X, A)]$ to $f|_W \in [W, A]$.

Let $\mathcal{K} = \mathcal{K}_\Pi$ and fix $W = \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m$, $X = FW(Y_1, \ldots, Y_m)$ and $A = (\underline{Y}, \underline{*})^{\mathcal{K}}$. Since W is a suspension, the terms appearing in (4.38) are groups. The proof proceeds in two main stages.

In Section 4.5.1 we construct maps $\varphi_i \colon (CW, W) \longrightarrow (X, A)$ called relative higher Whitehead maps with the property that

$$\partial \varphi_i = h_w^{\mathcal{K}} \left(h_w \left(f_{j_1}, \ldots, f_{j_{r_i}} \right), f_{i_1}, \ldots, f_{i_{n_i}} \right). \tag{4.39}$$

These maps generalise the construction of relative Whitehead products given by Blakers–Massey [BM53].

The remainder of the proof is an analysis of the map $j \colon [\Sigma W, X] \longrightarrow [(CW, W), (X, A)]$. In particular we show that

$$j \circ h_w(f_1, \ldots, f_m) \simeq \sum_{i=1}^{k} \varphi_i \circ \sigma_i \tag{4.40}$$

from which Theorem 4.3.7 follows by applying ∂ to both sides, using relation (4.39), and exactness of sequence (4.38).

4.5.1 The relative higher Whitehead map

The relative Whitehead product was introduced by Blakers and Massey [BM53] as an operation on relative homotopy groups defined as follows. For a sphere S^r, let D^r_+ and D^r_- denote the upper and lower hemispheres, respectively. Let $\alpha \in \pi_p(X, A)$ and $\beta \in \pi_q(A)$ be represented by $f \colon (D^p, S^{p-1}, D^{p-1}_+) \longrightarrow (X, A, *)$ and $g \colon (D^q, S^{q-1}) \longrightarrow (A, *)$, respectively. The relative Whitehead product $[\alpha, \beta] \in \pi_{p+q-1}(X, A)$ is the homotopy class of the map of pairs

$$\begin{aligned}(D^{p+q-1}, S^{p+q-2}) &= (D^p \times S^{q-1} \cup D^{p-1}_+ \times D^q, D^{p-1}_- \times S^{q-1} \cup S^{p-2} \cup D^q) \\ &\longrightarrow (X \vee A, A \vee A) \longrightarrow (X, A)\end{aligned}$$

where the first map is induced by the product $f \times g$ and the second map is induced by the fold $\nabla \colon X \vee X \longrightarrow X$.

The relative Whitehead product shares many of the same properties as the Whitehead product, including naturality and bilinearity, see [BM53] for details. Furthermore it satisfies the following. Let $\partial_n \colon \pi_n(X, A) \longrightarrow \pi_{n-1}(A)$ be the boundary operator from long exact sequence

$$\cdots \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \xrightarrow{\partial_n} \pi_{n-1}(A) \longrightarrow \cdots.$$

Then

$$\partial_{p+q-1}[\alpha, \beta] = -[\partial_p \alpha, \beta]. \tag{4.41}$$

This is a key property used by Nakaoka–Toda [NT54] in deriving the Jacobi identity (4.2). To analyse relations between higher Whitehead maps, we introduce a generalisation of the relative Whitehead product analogous to the way the higher Whitehead map generalises the Whitehead product.

Let $f\colon (C\Sigma X, \Sigma X) \longrightarrow (Z, B)$ be a map of pairs and let $f_i\colon \Sigma X_i \longrightarrow Y_i$ be maps for $i = 1, \dots, m$.

Definition 4.5.1. Let $W = X * X_1 * \cdots * X_m$. The relative higher Whitehead map of $f, f_1, \dots, f_m$ is the homotopy class of the composite

$$\begin{aligned} h_w(f, f_1, \dots, f_m)\colon CW \simeq CX * X_1 * \cdots * X_m & \\ &\xrightarrow{\rho} FW(\Sigma CX, \Sigma X_1, \dots, \Sigma X_m) \\ &\xrightarrow{\simeq} FW(C\Sigma X, \Sigma X_1, \dots, \Sigma X_m) \\ &\longrightarrow FW(Z, Y_1, \dots, Y_m). \end{aligned}$$

The restriction of $h_w(f, f_1, \dots, f_m)$ to W is the composite

$$\begin{aligned} W = X * X_1 * \cdots * X_m & \\ &\xrightarrow{\rho} FW(\Sigma X, \Sigma X_1, \dots, \Sigma X_m) \\ &\longrightarrow FW(B, Y_1, \dots, Y_m) \end{aligned}$$

which is the higher Whitehead map $h_w(f|_{\Sigma X}, f_1, \dots, f_m)$. It follows that the relative higher Whitehead map $h_w(f, f_1, \dots, f_m)$ is an element of the relative homotopy group

$$[(CW, W), (FW(Z, Y_1, \dots, Y_m), FW(B, Y_1, \dots, Y_m))].$$

The higher and relative higher Whitehead maps satisfy a relation analogous to (4.41) as follows.

Proposition 4.5.2. *The map $h_w(f, f_1, \dots, f_m)$ satisfies*

$$\partial h_w(f, f_1, \dots, f_m) = h_w(f|_{\Sigma X}, f_1, \dots, f_m)$$

where

$$\partial\colon [(CW, W), (FW(Z, Y_1, \dots, Y_m), FW(B, Y_1, \dots, Y_m))] \longrightarrow [W, FW(B, Y_1, \dots, Y_m)]$$

is the map from exact sequence (4.37).

Proof. Since

$$\partial h_w(f, f_1, \ldots, f_m) = h_w(f, f_1, \ldots, f_m)|_W$$

the result follows immediately from the definition of $h_w(f, f_1, \ldots, f_m)$. □

It should be noted that in the spherical case this only recovers (4.41) up to sign, due to the choices of orientation made in [BM53].

4.5.2 The inclusion map j

We now turn to proving formula (4.40). For the partition $\Pi = \{I_1, \ldots, I_k\}$ of $[m]$ with $I_i = \{i_1, \ldots, i_{n_i}\}$ and $[m] - I_i = \{j_1, \ldots, j_{r_i}\}$, let $\psi = f_1 \times \cdots \times f_m$ and $\psi_i = f_{j_1} \times \cdots \times f_{j_{r_i}}$ for $i = 1, \ldots, k$. Then ψ is a map whose restriction to $X_1 * \cdots * X_m$ is $h_w(f_1, \ldots, f_m)$ and ψ_i is a map whose restriction to $X_{j_1} * \cdots * X_{j_{r_i}}$ is $h_w\left(f_{j_1}, \ldots, f_{j_{r_i}}\right)$.

Let $W = \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m$. We construct subspaces Z_i and F_i of ΣW such that the restriction $\psi|_{F_i}$ is homotopic to the relative higher Whitehead map $h_w(\psi_i, f_{i_1}, \ldots, f_{i_{n_i}}) \circ \sigma_i$ and whose further restriction to $Z_i \subseteq F_i$ is a representative of

$$h_w^{\mathcal{K}}\left(h_w\left(f_{j_1}, \ldots, f_{j_{r_i}}\right), f_{i_1}, \ldots, f_{i_{n_i}}\right) \circ \sigma_i.$$

We construct the subspaces F_i and Z_i to have the property that $F_i \cap F_{i'} \subseteq Z_i \cap Z_{i'}$ for each distinct pair $i, i' \in \{1, \ldots, k\}$. Then formula (4.40) is derived from a property of the map j given in Lemma 4.5.11.

In the case that the X_i are spheres, the subspaces F_i can be constructed using the methods of Nakaoka–Toda [NT54] and Hardie [Har64]. For general X_i, more care must be taken, and we carefully build these spaces starting with decompositions of the spaces CX_i. We first use the suspension structure of the X_i to construct the maps f_i on subspaces of $CX_i = C\Sigma\widetilde{X}_i$, which in turn are used to construct the maps ψ_i on subspaces of $CX_{j_1} \times \cdots \times CX_{j_{n_i}}$. These are finally used to construct the subspaces F_i on which the relative higher Whitehead maps are defined.

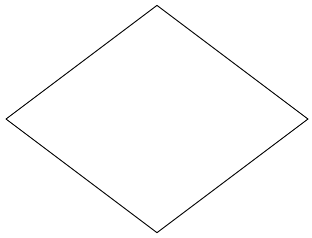
4.5.2.1 Decompositions of CX_i

Fix $i = 1, \ldots, m$. Since X_i is a suspension, write $X_i = \Sigma\widetilde{X}_i$. We consider $C\Sigma\widetilde{X}_i$ given by

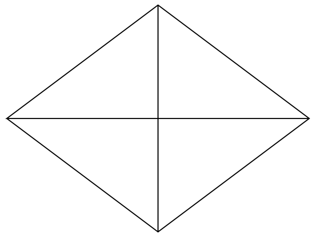
$$C\Sigma\widetilde{X}_i = \{(s, t, x) \in I \times I \times \widetilde{X}_i\}/(1, t, x) \sim (s, 0, x) \sim (s, 1, x) \sim (s, t, *)$$

and realise $X_i = \Sigma\widetilde{X}_i$ as the subspace at $s = 0$. We also define $C_-\widetilde{X}_i$ to be the subspace corresponding to $s = 0$, $t \leqslant \frac{1}{2}$ and $C_+\widetilde{X}_i$ the subspace for $s = 0$, $t \geqslant \frac{1}{2}$. In Figure 4.3 we show the situation for $\widetilde{X}_i = S^0$, which gives a useful reference for our next constructions.

(a) $\widetilde{X}_i$



(b) $X_i = \Sigma\widetilde{X}_i = C_-\widetilde{X}_i \cup C_+\widetilde{X}_i$



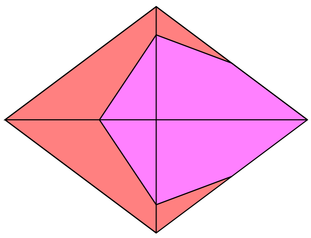
(c) $CX_i = C\Sigma\widetilde{X}_i$

Figure 4.3: The spaces $\widetilde{X}_i$, X_i and CX_i for $\widetilde{X}_i = S^0$.

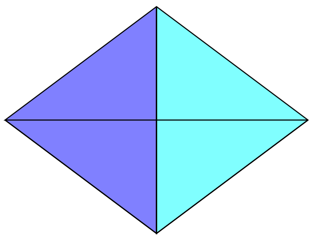
We define two decompositions of $CX_i = C\Sigma\widetilde{X}_i$ by setting

$$D_i^+ = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_i \mid t \geqslant \frac{1}{2}\right\} \qquad D_i^- = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_i \mid t \leqslant \frac{1}{2}\right\}$$
$$D_i^1 = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_i \mid s \geqslant \frac{3}{4} - t\right\} \qquad D_i^2 = \left\{(s,(t,x)) \in C\Sigma\widetilde{X}_i \mid s \leqslant \frac{3}{4} - t\right\}$$

for which $C\Sigma\widetilde{X}_i = D_i^- \cup D_i^+ = D_i^1 \cup D_i^2$. The decompositions for $\widetilde{X}_i = S^0$ are shown in Figure 4.4. On the left, the darker-shaded red area is D_i^2 and the lighter-shaded area is D_i^1. On the right, the darker-shaded blue area is D_i^- and the lighter-shaded area is D_i^+.



(a) $C\Sigma\widetilde{X}_i = D_i^1 \cup D_i^2$



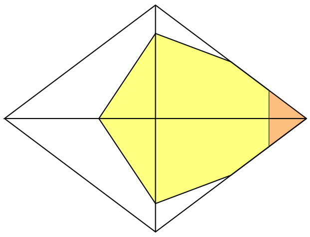
(b) $C\Sigma\widetilde{X}_i = D_i^+ \cup D_i^-$

Figure 4.4: Two decompositions of $C\Sigma\widetilde{X}_i$ for $\widetilde{X}_i = S^0$.

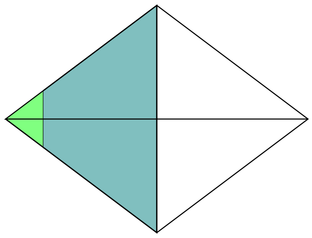
Next, we define the subspaces of D_i^- and D_i^1 that we will require throughout the rest of this proof. We begin with further decompositions of D_i^- and D_i^1. Let

$$E_i^- = \left\{(s,(t,x)) \in D_i^- \mid t \leqslant \frac{1}{8}\right\} \qquad E_i^1 = \left\{(s,(t,x)) \in D_i^1 \mid t \geqslant \frac{7}{8}\right\}$$
$$C_i^- = \left\{(s,(t,x)) \in D_i^- \mid t \geqslant \frac{1}{8}\right\} \qquad C_i^1 = \left\{(s,(t,x)) \in D_i^1 \mid t \leqslant \frac{7}{8}\right\}$$

so that by construction we have $D_i^1 = E_i^1 \cup C_i^1$ and $D_i^- = E_i^- \cup C_i^-$. These decompositions for $\widetilde{X}_i = S^0$ are shown in Figure 4.5. On the left, the darker-shaded yellow area is E_i^1 and the lighter-shaded area is C_i^1. On the right, the darker-shaded green area is C_i^- and the lighter-shaded area is E_i^-. We observe that by construction E_i^1 is disjoint from D_i^-, and E_i^- is disjoint from D_i^1, which we will require later.



(a) $D_i^1 = E_i^1 \cup C_i^1$



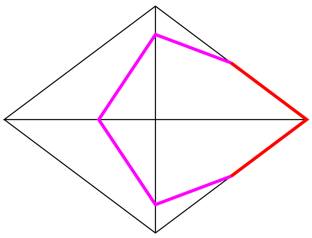
(b) $D_i^- = E_i^- \cup C_i^-$

Figure 4.5: Decompositions of D_i^1 and D_i^- for $\widetilde{X}_i = S^0$.

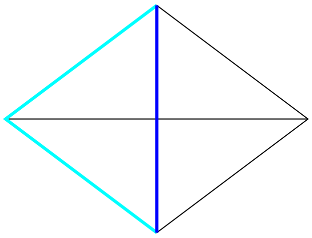
Finally we define further subspaces of D_i^- and D_i^1 as

$$\begin{aligned}
(B_2)_i^- &= \left\{(s,(t,x)) \in D_i^- \mid t \leqslant \frac{1}{2}, s = 0\right\} & (B_2)_i^1 &= \left\{(s,(t,x)) \in D_i^1 \mid s = \frac{3}{4} - t\right\} \\
(B_1)_i^- &= \left\{(s,(t,x)) \in D_i^- \mid t = \frac{1}{2}\right\} & (B_1)_i^1 &= \left\{(s,(t,x)) \in D_i^1 \mid t \geqslant \frac{3}{4}, s = 0\right\} \\
R_i^- &= \left\{(s,(t,x)) \in D_i^- \mid t = \frac{1}{8}, s = 0\right\} & R_i^1 &= \left\{(s,(t,x)) \in D_i^1 \mid t = \frac{7}{8}, s = 0\right\} \\
S_i^- &= \left\{(s,(t,x)) \in D_i^- \mid t = \frac{1}{8}\right\} & S_i^1 &= \left\{(s,(t,x)) \in D_i^1 \mid t = \frac{7}{8}\right\} \\
T_i^- &= \left\{(s,(t,x)) \in D_i^- \mid t \leqslant \frac{1}{8}, s = 0\right\} & T_i^1 &= \left\{(s,(t,x)) \in D_i^1 \mid t \geqslant \frac{7}{8}, s = 0\right\}.
\end{aligned}$$

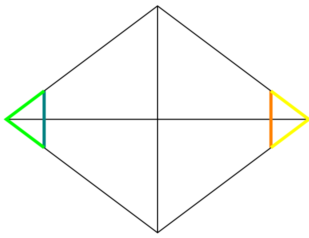
We also denote by $B_i^- = (B_1)_i^- \cup (B_2)_i^-$ and $B_i^1 = (B_1)_i^1 \cup (B_2)_i^1$. These subspaces are shown in Figure 4.6 for $\widetilde{X}_i = S^0$. In (a), the lighter-shaded red area is $(B_2)_i^1$ and the darker-shaded area is $(B_1)_i^1$. In (b), the lighter-shaded blue area is $(B_2)_i^-$ and the darker-shaded area is $(B_1)_i^-$. In (c), the lighter-shaded green area is T_i^-, the darker-shaded green area is S_i^-, the lighter-shaded yellow area is T_i^1, and the darker-shaded yellow area is S_i^1.



(a) $B_i^1 = (B_1)_i^1 \cup (B_2)_i^1$



(b) $B_i^- = (B_1)_i^- \cup (B_2)_i^-$



(c) S_i^-, T_i^-, S_i^1 and T_i^1

Figure 4.6: Other subspaces of D_i^1 and D_i^- for $\widetilde{X}_i = S^0$.

When $X_i = S^{r_i-1}$, the spaces D_i^- and D_i^1 are discs D^{r_i} and their topological boundaries B_i^- and B_i^1, respectively, are also spheres S^{r_i-1}. For general X_i, the spaces B_i^- and B_i^1 are no longer the topological boundaries of D_i^- and D_i^1, respectively. Nevertheless, we construct B_i^- and B_i^+ to have the homotopy type of X_i. This is summarised by the following result, with a similar observation for the spaces S_i^-, S_i^1, T_i^- and T_i^1.

Lemma 4.5.3. *Let $\delta \in \{1, -\}$. There is a homotopy equivalence $C\Sigma\widetilde{X}_i \longrightarrow D_i^\delta \longrightarrow E_i^\delta$, which restricts to a homotopy commutative diagram*

$$\begin{tikzcd}
\widetilde{X}_i \arrow[rr] \arrow[dr] \arrow[dd] & & C_+\widetilde{X}_i \arrow[dr] \arrow[dd] & \\
& C_-\widetilde{X}_i \arrow[rr] \arrow[dd] & & \Sigma\widetilde{X}_i \arrow[dd] \\
(B_1)_i^\delta \cap (B_2)_i^\delta \arrow[rr] \arrow[dr] \arrow[dd] & & (B_1)_i^\delta \arrow[dr] \arrow[dd] & \\
& (B_2)_i^\delta \arrow[rr] \arrow[dd] & & B_i^\delta \arrow[dd] \\
R_i^\delta \arrow[rr] \arrow[dr] & & T_i^\delta \arrow[dr] & \\
& S_i^\delta \arrow[rr] & & T_i^\delta \cup S_i^\delta
\end{tikzcd}$$

in which the horizontal squares are pushouts and all vertical arrows are homotopy equivalences.

Proof. $C\Sigma\widetilde{X}_i$ is contracted to D_i^1 first by sending $C_+\widetilde{X}_i$ to $(B_1)_i^1$ and then sending D_i^2 to $(B_2)_i^1$. This then takes $C_-\widetilde{X}_i$ to $(B_2)_i^1$, giving the first layer of the diagram. In a similar way, D_i^1 can be further contracted to E_i^1 in a way consistent with the second layer of the diagram. The case for $\delta = -$ is dealt with similarly. □

4.5.2.2 Subspaces of $CX_{j_1} \times \cdots \times CX_{j_{r_i}}$

Let

$$\begin{aligned}
V &= \prod_{i=1}^{m} CX_i \\
V^* &= \bigcup_{i=1}^{m} CX_1 \times \cdots \times X_i \times \cdots \times CX_m.
\end{aligned}$$

We recall that for the partition $\Pi = \{I_1, \ldots, I_k\}$ of $[m]$ we write $I_i = \{i_1, \ldots, i_{n_i}\}$ and $[m] - I_i = \{j_1, \ldots, j_{r_i}\}$. For $i = 1, \ldots, k$ we define subspaces of V^* as

$$\begin{aligned}
V_i &= CX_{j_1} \times \cdots \times CX_{j_{r_i}} \\
V_i^* &= \bigcup_{l=1}^{r_i} CX_{j_1} \times \cdots \times X_{j_l} \times \cdots \times CX_{j_{r_i}}.
\end{aligned}$$

For each $i = 1, \ldots, k$, we define a pair (G_i, G_i^*) homotopy equivalent to $(V_i, V_i{}^*)$. We use matrix notation to encode the factors in the product. If $k \geqslant 4$ is even, define the $k \times k$ matrix $H' = [\eta_k'(i,j)]$ by

$$\eta_k'(i,j) = \begin{cases} * & \text{if } i = j \\ 1 & \text{if } i+j < k+1, \text{ or } i+j = k+1 \text{ and } i \geqslant \frac{k}{2}+1 \\ - & \text{if } i+j > k+1, \text{ or } i+j = k+1 \text{ and } i \leqslant \frac{k}{2} \end{cases}$$

and define the matrix $H = [\eta_k(i,j)]$ from H' by swapping the entries $(i,j) = (\frac{k}{2}+1, \frac{k}{2}-1)$ and $(i,j) = (\frac{k}{2}+1, \frac{k}{2})$. If $k \geqslant 3$ is odd, define the $k \times k$ matrix H by

$$\eta_k(i,j) = \begin{cases} * & \text{if } i = j \\ 1 & \text{if } i+j < k+1, \text{ or } i+j = k+1 \text{ and } i > \frac{k+1}{2} \\ - & \text{if } i+j > k+1, \text{ or } i+j = k+1 \text{ and } i < \frac{k+1}{2}. \end{cases}$$

Figure 4.7 shows the matrix H, on the left for $k = 8$, and on the right for $k = 7$.

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & * & 1 & 1 & 1 & 1 & 1 & - \\ 1 & 1 & * & 1 & 1 & 1 & - & - \\ 1 & 1 & 1 & * & 1 & - & - & - \\ 1 & 1 & - & 1 & * & - & - & - \\ 1 & 1 & - & - & - & * & - & - \\ 1 & - & - & - & - & - & * & - \\ - & - & - & - & - & - & - & * \end{bmatrix}$$

(a) $k = 8$

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & * & 1 & 1 & 1 & 1 & - \\ 1 & 1 & * & 1 & 1 & - & - \\ 1 & 1 & 1 & * & - & - & - \\ 1 & 1 & - & - & * & - & - \\ 1 & - & - & - & - & * & - \\ - & - & - & - & - & - & * \end{bmatrix}$$

(b) $k = 7$

Figure 4.7: The matrix H for different values of k.

We define

$$G_i = \prod_{n=1}^{r_i} D_{j_n}^{\eta_k(i,\pi(j_n))} = D_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times D_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}$$
$$G_i^* = \bigcup_{n=1}^{r_i} D_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times B_{j_n}^{\eta_k(i,\pi(j_n))} \times \cdots \times D_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}$$

where if $j_i \in P_l$ then we set $\pi(j_i) = l$.

Example 4.5.4. Consider the partition $\Pi = \{\{1\}, \{2,3\}, \{4\}\}$ where we set $P_1 = \{1\}$, $P_2 = \{2,3\}$ and $P_3 = \{4\}$ so that $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$ and $\pi(4) = 3$. Since Π is a

3-partition we choose the 3×3 matrix H as follows.

$$\begin{bmatrix} * & 1 & 1 \\ 1 & * & - \\ - & - & * \end{bmatrix}$$

This then defines

$$G_1 = D_2^{\eta_3(1,\pi(2))} \times D_3^{\eta_3(1,\pi(3))} \times D_4^{\eta_3(1,\pi(4))} = D_2^{\eta_3(1,2))} \times D_3^{\eta_3(1,2)} \times D_4^{\eta_3(1,3)} = D_2^1 \times D_3^1 \times D_4^1$$

and similarly we obtain

$$\begin{aligned} G_2 &= D_1^{\eta_3(2,1)} \times D_4^{\eta_3(2,3)} = D_1^1 \times D_4^- \\ G_3 &= D_1^{\eta_3(3,1)} \times D_2^{\eta_3(3,2)} \times D_3^{\eta_3(3,2)} = D_1^- \times D_2^- \times D_3^-. \end{aligned}$$

Moreover we have

$$\begin{aligned} G_1^* &= (B_2^1 \times D_3^1 \times D_4^1) \cup (D_2^1 \times B_3^1 \times D_4^1) \cup (D_2^1 \times D_3^1 \times B_4^1) \\ G_2^* &= (B_1^1 \times D_4^-) \cup (D_1^1 \times B_4^-) \\ G_3^* &= (B_1^- \times D_2^- \times D_3^-) \cup (D_1^- \times B_2^- \times D_3^-) \cup (D_1^- \times D_2^- \times B_3^-). \end{aligned}$$

By construction, we have the following.

Lemma 4.5.5. *For all $i = 1, \dots, k$, there is a homotopy equivalence of pairs $(V_i, V_i^*) \simeq (G_i, G_i^*)$.*

Proof. This follows from Lemma 4.5.3, since for each i there are homotopy equivalences of pairs $(D_i^-, B_i^-) \simeq (C\Sigma\widetilde{X}_i, \Sigma\widetilde{X}_i)$ and $(D_i^1, B_i^1) \simeq (C\Sigma\widetilde{X}_i, \Sigma\widetilde{X}_i)$. □

When each X_i is a sphere, it suffices to define G_i^* as the topological boundary of G_i. In general, however, the topological boundary of G_i will not have the homotopy type of V_i^*.

The following is an important consequence of our construction which will be used later.

Lemma 4.5.6. *Suppose that $1 \leqslant i < j \leqslant k$. Then there exists $i \neq r \neq j$ such that G_i contains a factor D_r^1 and G_j contains a factor D_r^-.*

Proof. Observe that given any two rows i and j of a matrix H, there exists $i \neq r \neq j$ such that $\eta_k(i, r) = 1$ and $\eta_k(j, r) = -$. Then by construction G_i has a factor D_r^1 and G_j has a factor D_r^-. □

Example 4.5.7. To demonstrate Lemma 4.5.6, consider G_1, G_2 and G_3 from Example 4.5.4. Then G_1 has a factor D_4^1, while G_2 as a factor D_4^-. Similarly, G_1 has a factor D_2^1 while G_3 has a factor D_2^-; and G_2 has a factor D_1^1 while G_3 has a factor D_1^-.

Next, we give a decomposition of $G_i^* \simeq \Sigma^{r_i-1}X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}$ into two contractible spaces τ_i and κ_i with $\tau_i \simeq C\Sigma^{r_i-2}X_{j_1} \wedge \cdots \wedge X_{j_{r_i}} \simeq \kappa_i$ which intersect in $\sigma_i \simeq \Sigma^{r_i-2}X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}$. When G_i^* is a sphere, it is sufficient to specify τ_i as any open ball. In general, however, the complement of a contractible subspace in a suspension need not be contractible.

Instead, we build τ_i and κ_i from decompositions of D_i^δ as follows. Since $D_i^- = E_i^- \cup C_i^-$ and $D_i^1 = E_i^1 \cup C_i^1$, we have

$$G_i = \prod_{n=1}^{r_i} D_{j_n}^{\eta_k(i,\pi(j_n))} = \prod_{n=1}^{r_i} E_{j_n}^{\eta_k(i,\pi(j_n))} \cup C_{j_n}^{\eta_k(i,\pi(j_n))} = \bigcup_{\substack{\{t_1,\ldots,t_{r_i}\} \\ \in\{0,1\}^{r_i}}} \left(\prod_{n=1}^{r_i} (Y_{t_n})_{j_n}^{\eta_k(i,\pi(j_n))} \right)$$

where $(Y_0)_{j_n}^\delta = E_{j_n}^\delta$ and $(Y_1)_{j_n}^\delta = C_{j_n}^\delta$.

We define

$$\begin{aligned}
\tau_i &= G_i^* \cap \prod_{n=1}^{r_i} E_{j_n}^{\eta_k(i,\pi(j_n))} \\
\kappa_i &= G_i^* \cap \left(\bigcup_{\substack{\{t_1,\ldots,t_{r_i}\}\in\{0,1\}^{r_i} \\ \{t_1,\ldots,t_{r_i}\}\neq\{0,\ldots,0\}}} \prod_{n=1}^{r_i} (Y_{t_n})_{j_n}^{\eta_k(i,\pi(j_n))} \right) \\
\sigma_i &= \tau_i \cap \kappa_i
\end{aligned}$$

so that by construction $G_i^* = \tau_i \cup \kappa_i$.

The construction of these spaces is demonstrated in Figure 4.8, which shows the space $CX_1 \times CX_2$. In Figure 4.8(a) the area shaded green is $E_1 \times E_2$, the area shaded grey is $E_1 \times C_2 \cup C_1 \times E_2 \cup C_1 \times C_2$, and their union is G_i. In Figure 4.8(b) The bold light green line is τ_i and the dark green line is κ_i. Their union is G_i^* and their intersection is σ_i.

(a) The decomposition of G_i (b) The spaces τ_i and κ_i inside G_i

Figure 4.8: The construction of the subspaces of $CX_{j_1} \times \cdots \times CX_{j_{r_i}}$.

We show that τ_i, κ_i and σ_i give the desired decomposition of G_i^*.

Lemma 4.5.8. *Let $W = \Sigma^{r_i-2}X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}$. There is a homotopy commutative diagram*

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[dd] & & C_+W \arrow[dr] \arrow[dd] & \\
& C_-W \arrow[rr] \arrow[dd] & & \Sigma W \arrow[dd] \\
\sigma_i \arrow[rr] \arrow[dr] & & \tau_i \arrow[dr] & \\
& \kappa_i \arrow[rr] & & G_i^*
\end{tikzcd}$$

where the top and bottom squares are pushouts and the vertical maps are homotopy equivalences.

Proof. The top and bottom faces are pushouts since all maps are inclusions. By Lemma 4.5.5, the homotopy equivalence $V_i \longrightarrow G_i$ restricts to a homotopy equivalence

$$\Sigma W = V_i^* \longrightarrow G_i^*.$$

We show that this further restricts to homotopy equivalences $W \longrightarrow \sigma_i$ and $C_+W \longrightarrow \tau_j$. By commutativity it then follows that $C_-W \longrightarrow \kappa_i$ is also homotopy equivalence.

We first analyse σ_i. Since $E_i^\delta \cap C_i^\delta = S_i^\delta$,

$$\begin{aligned}
\sigma_i &= \tau_i \cap \kappa_i \\
&= G_i^* \cap \left(\prod_{n=1}^{r_i} E_{j_n}^{\eta_k(i,\pi(j_n))} \cap \left(\bigcup_{\substack{\{t_1,\dots,t_{r_i}\}\in\{0,1\}^{r_i} \\ \{t_1,\dots,t_{r_i}\}\neq\{0,\dots,0\}}} \prod_{n=1}^{r_i} (Y_{t_n})_{j_n}^{\eta_k(i,\pi(j_n))} \right)\right) \\
&= G_i^* \cap \left(\bigcup_{\substack{\{t_1,\dots,t_{r_i}\}\in\{0,1\}^{r_i} \\ \{t_1,\dots,t_{r_i}\}\neq\{0,\dots,0\}}} \prod_{n=1}^{r_i} (Z_{t_n})_{j_n}^{\eta_k(i,\pi(j_n))} \right)
\end{aligned}$$

where $(Z_0)_{j_n}^\delta = E_{j_n}^\delta$ and $(Z_1)_{j_n}^\delta = S_{j_n}^\delta$. Since $S_i^\delta \subseteq E_i^\delta$, the above union simplifies to give

$$\sigma_i = G_i^* \cap \left(\bigcup_{n=1}^{r_i} E_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times S_{j_n}^{\eta_k(i,\pi(j_n))} \times \cdots \times E_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))} \right).$$

Next, since

$$G_i^* = \bigcup_{n=1}^{r_i} D_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times B_{j_n}^{\eta_k(i,\pi(j_n))} \times \cdots \times D_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}$$

and since $B_i^\delta \cap E_i^\delta = T_i^\delta$ and $B_i^\delta \cap S_i^\delta = R_i^\delta$, we have

$$\begin{aligned}
\sigma_i = &\left(B_{j_1}^{\eta_k(i,\pi(j_1))} \times B_{j_2}^{\eta_k(i,\pi(j_2))} \times \cdots \times D_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}\right) \\
&\qquad \cap \left(\bigcup_{n=1}^{r_i} E_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times S_{j_n}^{\eta_k(i,\pi(j_n))} \times \cdots \times E_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}\right) \\
&\cup \cdots \cup \left(D_{j_1}^{\eta_k(i,\pi(j_1))} \times B_{j_2}^{\eta_k(i,\pi(j_2))} \times \cdots \times B_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}\right) \\
&\qquad \cap \left(\bigcup_{n=1}^{r_i} E_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times S_{j_n}^{\eta_k(i,\pi(j_n))} \times \cdots \times E_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}\right) \\
= &\, R_{j_1}^{\eta_k(i,\pi(j_1))} \times E_{j_2}^{\eta_k(i,\pi(j_2))} \times \cdots \times E_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))} \\
&\cup T_{j_1}^{\eta_k(i,\pi(j_1))} \times \left(S_{j_2}^{\eta_k(i,\pi(j_2))} \times \cdots \times E_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))} \cup E_{j_2}^{\eta_k(i,\pi(j_2))} \times \cdots \times S_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}\right) \\
&\cup \cdots \cup E_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times E_{j_{r_i-1}}^{\eta_k(i,\pi(j_{r_i-1}))} \times R_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))} \\
&\cup \left(S_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times E_{j_{r_i-1}}^{\eta_k(i,\pi(j_{r_i-1}))} \cup E_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times S_{j_{r_i-1}}^{\eta_k(i,\pi(j_{r_i-1}))}\right) \times T_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}
\end{aligned}$$

and so applying Lemma 4.5.3 we obtain

$$\begin{aligned}
\sigma_i \simeq &\, \widetilde{X}_{j_1} \times C\Sigma\widetilde{X}_{j_2} \times \cdots \times C\Sigma\widetilde{X}_{j_{r_i}} \\
&\cup C_-\widetilde{X}_{j_1} \times \left(C_+\widetilde{X}_{j_2} \times \cdots \times C\Sigma\widetilde{X}_{j_{r_i}} \cup \cdots \cup C\Sigma\widetilde{X}_{j_2} \times \cdots \times C_+\widetilde{X}_{j_{r_i}}\right) \\
&\cup \cdots \cup C\Sigma\widetilde{X}_{j_1} \times \cdots \times C\Sigma\widetilde{X}_{j_{r_i-1}} \times \widetilde{X}_{j_{r_i}} \\
&\left(C_+\widetilde{X}_{j_1} \times \cdots \times C\Sigma\widetilde{X}_{j_{r_i-1}} \cup \cdots \cup C\Sigma\widetilde{X}_{j_1} \times \cdots \times C_+\widetilde{X}_{j_{r_i-1}}\right) \times C_-\widetilde{X}_{j_{r_i}} \\
\simeq &\, \widetilde{X}_{j_1} * C_+\widetilde{X}_{j_2} * \cdots * C_+\widetilde{X}_{j_{r_i}} \cup \cdots \cup C_+\widetilde{X}_{j_1} * C_+\widetilde{X}_{j_2} * \cdots * \widetilde{X}_{j_{r_i}}.
\end{aligned}$$

To conclude that $\sigma_i \simeq W$, we use the following observation. Suppose that B_1 and B_2 are contractible spaces containing A_1 and A_2, respectively. Since

$$\begin{aligned}
(A_1 * B_2) \cap (B_1 * A_2) &= (CA_1 \times B_2 \cup A_1 \times CB_2) \cap (CB_1 \times A_2 \cup B_1 \times CA_2) \\
&= CA_1 \times A_2 \cup A_1 \times CA_2 \\
&= A_1 * A_2
\end{aligned}$$

then $(A_1 * B_2) \cup (B_1 * A_2) \simeq \Sigma A_1 * A_2$. Now suppose for $i = 1, \ldots, k$ that B_i is a contractible space containing A_i and let

$$D = (A_1 * B_2 * \cdots * B_k) \cup (B_1 * A_2 * \cdots * B_k) \cup \cdots \cup (B_1 * B_2 * \cdots * A_k).$$

Applying the above observation inductively we obtain that

$$\begin{aligned}D &= (A_1 * B_2 * \cdots * B_k) \cup (B_1 * A_2 * \cdots * B_k) \cup \cdots \cup (B_1 * B_2 * \cdots * A_k)\\ &= A_1 * (B_2 * \cdots * B_k) \cup B_1 * ((A_2 * \cdots * B_k) \cup \cdots \cup (B_2 * \cdots * A_k))\\ &\simeq \Sigma(A_1 * \Sigma^{k-2}(A_2 * \cdots * A_k))\\ &\simeq \Sigma^{k-1} A_1 * \cdots * A_k.\end{aligned}$$

Applying this to the above expression for σ_i we obtain

$$\begin{aligned}\sigma_i &\simeq \widetilde{X}_{j_1} * C_+\widetilde{X}_{j_2} * \cdots * C_+\widetilde{X}_{j_{r_i}} \cup \cdots \cup C_+\widetilde{X}_{j_1} * C_+\widetilde{X}_{j_2} * \cdots * \widetilde{X}_{j_{r_i}}\\ &= \Sigma^{r_i-1}\widetilde{X}_{j_1} * \cdots * \widetilde{X}_{j_{r_i}}\\ &= \Sigma^{2r_i-2}\widetilde{X}_{j_1} \wedge \cdots \wedge \widetilde{X}_{j_{r_i}}\\ &= \Sigma^{r_i-2}X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}\end{aligned}$$

which is equal to W, as claimed.

It remains to show that τ_i is contractible. Since $B_i^\delta \cap E_i^\delta = T_i^\delta$,

$$\begin{aligned}\tau_i &= G_i^* \cap \prod_{n=1}^{r_i} E_{j_n}^{\eta_k(i,\pi(j_n))}\\ &= \left(\bigcup_{n=1}^{r_i} D_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times B_{j_n}^{\eta_k(i,\pi(j_n))} \times \cdots \times D_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}\right) \cap \prod_{n=1}^{r_i} E_{j_n}^{\eta_k(i,\pi(j_n))}\\ &= \bigcup_{n=1}^{r_i} E_{j_1}^{\eta_k(i,\pi(j_1))} \times \cdots \times T_{j_n}^{\eta_k(i,\pi(j_n))} \times \cdots \times E_{j_{r_i}}^{\eta_k(i,\pi(j_{r_i}))}\end{aligned}$$

so applying Lemma 4.5.3 again,

$$\begin{aligned}\tau_i &\simeq \bigcup_{n=1}^{r_i} C\Sigma\widetilde{X}_{j_1} \times \cdots \times C_-\widetilde{X}_{j_n} \times \cdots \times C\Sigma\widetilde{X}_{j_{r_i}}\\ &\simeq C_-\widetilde{X}_{j_1} * \cdots * C_-\widetilde{X}_{j_{r_i}}\end{aligned}$$

from which it follows that $\tau_i \simeq CW$, as claimed. □

4.5.2.3 Subspaces of V^*

Let ρ_i be the permutation map defined by

$$\rho_i\left(\left(x_{j_1}, \ldots, x_{j_{r_i}}\right), \left(x_{i_1}, \ldots, x_{i_{n_i}}\right)\right) = (x_1, \ldots, x_m)$$

and define

$$F_i = \rho_i \left(G_i \times \left(\bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}} \right) \cup \tau_i \times \left(CX_{i_1} \times \cdots \times CX_{i_{n_i}} \right) \right)$$
$$Z_i = \rho_i \left(\kappa_i \times \left(\bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}} \right) \cup \sigma_i \times \left(CX_{i_1} \times \cdots \times CX_{i_{n_i}} \right) \right).$$

To demonstrate the construction, Figure 4.9 shows the arrangement of F_1, F_2 and F_3 in V^* for the partition $\Pi = \{\{1\},\{2\},\{3\}\}$. The green segment of the top face is the space shown in Figure 4.8.

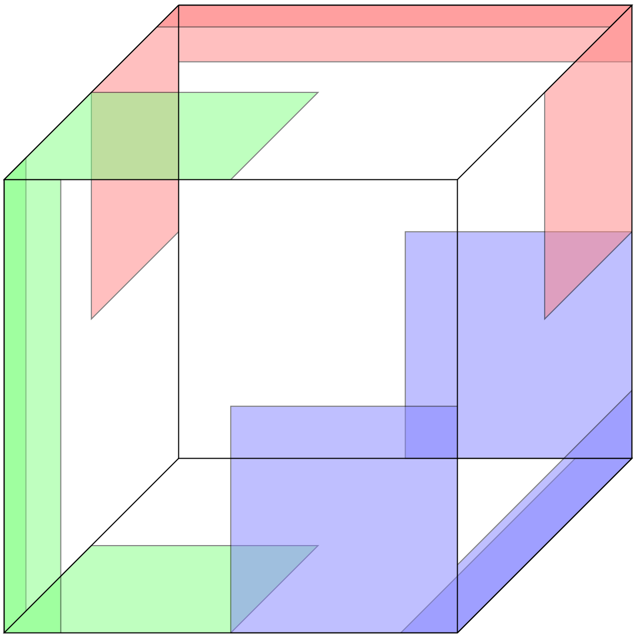


Figure 4.9: The arrangements of the spaces F_1, F_2 and F_3 in V^* for the partition $\Pi = \{\{1\},\{2\},\{3\}\}$. Observe that the subspaces pairwise intersect only at their boundaries.

The pairs (F_i, Z_i) are constructed to have the following properties.

Lemma 4.5.9. *Let* $W = \Sigma^{m-2} X_1 \wedge \cdots \wedge X_m$.

(i) There is a homotopy equivalence of pairs

$$(CW, W) \longrightarrow (F_i, Z_i)$$

for each $i = 1, \ldots, k$.

(ii) For any $i \neq i'$, *we have* $F_i \cap F_{i'} \subseteq Z_i \cap Z_{i'}$.

Proof. We first prove (i). Let $W_i = \Sigma^{r_i - 2} X_{j_1} \wedge \cdots \wedge X_{j_{r_i}}$. Then by Lemma 4.5.8, there are homotopy equivalences of pairs $(V_i, C_+W_i) \longrightarrow (G_i, \tau_i)$ and $(C_-W_i, W_i) \longrightarrow (\kappa_i, \sigma_i)$.

Therefore

$$
\begin{aligned}
F_i &\simeq \rho_i\left(C\Sigma W_i \times \left(\bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}}\right) \cup C_+W_i \times \left(CX_{i_1} \times \cdots \times CX_{i_{n_i}}\right)\right)\\
&\simeq \rho_i\left(C_+W_i * X_{i_1} * \cdots * X_{i_{n_i}}\right)\\
&\simeq \rho_i\left(C(X_{j_1} * \cdots * X_{j_{r_i}}) * X_{i_1} * \cdots * X_{i_{n_i}}\right)\\
&\simeq CW
\end{aligned}
$$

and

$$
\begin{aligned}
Z_i &\simeq \rho_i\left(C_-W \times \left(\bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}}\right) \cup W_i \times \left(CX_{i_1} \times \cdots \times CX_{i_{n_i}}\right)\right)\\
&\simeq \rho_i\left(W_i * X_{i_1} * \cdots * X_{i_{n_i}}\right)\\
&\simeq W
\end{aligned}
$$

which establishes the claimed homotopy equivalence of pairs.

Now we prove (ii). Specifically, for $i < i'$ we show that $F_i \cap F_{i'} \subseteq Z_i \cap Z_{i'}$. Define

$$
\begin{aligned}
H = \rho_i&\left(G_i \times \bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}}\right)\\
&\cap \rho_{i'}\left(G_{i'} \times \bigcup_{l=1}^{n_{i'}} CX_{i'_1} \times \cdots \times X_{i'_l} \times \cdots \times CX_{i'_{n_{i'}}}\right).
\end{aligned}
$$

Comparing coordinates, since $D_k^\delta \cap X_k \subseteq B_k^\delta$, we have

$$
\begin{aligned}
H \subseteq \rho_i&\left(G_i^* \times \bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}}\right)\\
&\cap \rho_{i'}\left(G_{i'}^* \times \bigcup_{l=1}^{n_{i'}} CX_{i'_1} \times \cdots \times X_{i'_l} \times \cdots \times CX_{i'_{n_{i'}}}\right).
\end{aligned}
$$

By Lemma 4.5.6, there is $i \neq r \neq i'$ such that G_i has a factor D_r^1 and $G_{i'}$ has a factor D_r^-. Then since $\pi_r(\tau_i) \cap D_r^- = \pi_r(\tau_{i'}) \cap D_r^1 = \varnothing$,

$$
H \cap \rho_i\left(\tau_i \times \bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}}\right) = \varnothing
$$

and similarly switching i' for i. Therefore since $G_i^* - \tau_i = \kappa_i$

$$H \subseteq \rho_i \left(\kappa_i \times \bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}} \right)$$
$$\cap\, \rho_{i'} \left(\kappa_{i'} \times \bigcup_{l=1}^{n_{i'}} CX_{i'_1} \times \cdots \times X_{i'_l} \times \cdots \times CX_{i'_{n_{i'}}} \right) \subseteq Z_i \cap Z_{i'}.$$

Finally, since $\pi_r(G_i) = D_r^1$ and $\pi_r(\tau_{i'}) \cap D_r^1 = \varnothing$,

$$\rho_i \left(G_i \times \bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}} \right)$$
$$\cap\, \rho_{i'} \left(\tau_{i'} \times CX_{i'_1} \times \cdots \times CX_{i'_{n_{i'}}} \right) = \varnothing$$

and vice-versa switching i and i'. Therefore $F_i \cap F_{i'} \subseteq Z_i \cap Z_{i'}$. □

4.5.3 Final proof

We state some final preparatory results before we prove Theorem 4.3.7. The following were given by [NT54] and [Har64] for spherical maps but are equally true for our purposes. For completeness we give their proofs.

Lemma 4.5.10. *Let $[f] \in [(X,A),(Y,B)]$ be represented by $f\colon (X,A) \longrightarrow (Y,B)$ and let $A' \subseteq X'$ be subspaces of X such that (X',A') is a deformation retract of (X,A). Suppose that $f((X-A) \cup A') \subseteq B$, then $f|_{X'}$ also represents $[f]$.*

Proof. Since (X',A') is a deformation retract of (X,A) there is a map $H_t\colon I \times X \longrightarrow X$ such that $H_0((X,A)) = (X,A)$ and $H_1((X,A)) = (X',A')$. Then we define a homotopy $f_t\colon (X,A) \longrightarrow (Y,B)$ by $f_t = f \circ H_t$. Then $f_0 = f$, $f_1 = f|_{X'}$ and $f_t(A) = f \circ H_t(A) \subseteq f((X-A) \cup A') \subseteq B$. □

Lemma 4.5.11. *Let W be a suspension space. For $i = 1,\dots,k$, consider pairs (F_i, Z_i) with $F_i \subseteq \Sigma W$ satisfying the following conditions. For each $i = 1,\dots,k$, Z_i is closed in F_i, pairwise intersections are such that $F_i \cap F_j \subseteq Z_i \cap Z_j$ for all $j \neq i$, and there exists a homotopy equivalence $e_i\colon \Sigma W \longrightarrow \Sigma W$ such that $e_i(F_i) = C_+W$, $e_i(Z_i) = W$, and $e_i((\Sigma W \backslash F_i) \cup Z_i) = C_-W$.*

Let (X,A) be a pair and suppose that $f\colon \Sigma W \longrightarrow X$ satisfies $f((\Sigma W \backslash \bigcup_i F_i) \cup (\bigcup_i Z_i)) \subseteq A$. Then

$$j(f) = \sum_{i=1}^{k} f_i$$

where $f_i = f|_{F_i} \in [(F_i, Z_i),(X,A)] \cong [(CW,W),(X,A)]$ for each $i = 1,\dots,k$ and $j\colon [\Sigma W, X] \longrightarrow [(CW,W),(X,A)]$ is the map in the long exact sequence (4.38).

To prove Lemma 4.5.11, we first establish the following lemma.

Lemma 4.5.12. *Let $Z = C_1\widetilde{Z} \cup C_2\widetilde{Z}$ and write $\Sigma Z = C_+Z \cup C_-Z$. Let (X, A) be a CW-pair. Let $g\colon \Sigma Z \longrightarrow X$ be a map such that $g(Z) \subseteq A$. Then $j(g) = g|_{C_+Z} + g|_{C_-Z}$.*

Proof. The map j is the composite of the isomorphism $\psi\colon [\Sigma Z, X] \longrightarrow [(CZ, Z), (X, *)]$ with the map j' induced by the inclusion $(X, *) \longrightarrow (X, A)$. We describe a choice of map of pairs $\phi\colon (CZ, Z) \longrightarrow (\Sigma Z, *)$ which induces the isomorphism ψ.

Consider $CZ = C\Sigma\widetilde{Z}$ and define

$$G_1 = \left\{(s,t,z) \in C\Sigma\widetilde{Z} \mid 0 \leqslant t \leqslant \frac{1}{2}\right\} \qquad G_2 = \left\{(s,t,z) \in C\Sigma\widetilde{Z} \mid \frac{1}{2} \leqslant t \leqslant 1\right\}$$
$$G_3 = \left\{(s,t,z) \in C\Sigma\widetilde{Z} \mid \frac{1}{2} \leqslant s \leqslant 1\right\} \qquad G_4 = \left\{(s,t,z) \in C\Sigma\widetilde{Z} \mid 0 \leqslant s \leqslant \frac{1}{2}\right\}.$$

Define the map $\phi\colon (CZ, Z) \longrightarrow (\Sigma Z, *)$ by $\phi(G_1) = C_+Z$ and $\phi(G_2) = C_-Z$, which moreover sends $\phi(G_3) = \Sigma C_1\widetilde{Z}$ and $\phi(G_4) = \Sigma C_2\widetilde{Z}$. Then $\psi(g) = g\phi$.

In particular, ϕ sends $G_1 \cap G_2$ to $Z \subseteq \Sigma Z$, $G_1 \cap G_3$ to $C_+C_1\widetilde{Z}$ and $G_1 \cap G_4$ to $C_+C_2\widetilde{Z}$. Since $g(Z) \subseteq A$, then $\psi(g) = g\phi$ is a map such that $G_1 \cap G_2 \subseteq A$. Since $G_1 \cap G_2$ is contractible, there is a homotopy from $g\phi$ to a map sending $G_1 \cap G_2$ to the basepoint. Therefore $j(g) = j'(\psi g)$ is homotopic to the sum $g\phi|_{G_1} + g\phi|_{G_2} = g|_{C_+Z} + g|_{C_-Z}$. □

We now prove Lemma 4.5.11.

Proof of Lemma 4.5.11. Write $\Sigma W = F_k \cup_{Z_k} ((\Sigma W \backslash F_k) \cup Z_k)$. Pre-composing with the homotopy equivalence e_k, the map f is homotopic to a map $f' = f \circ e_k\colon \Sigma W \longrightarrow X$, where $f'(W) \subseteq A$ since $f(Z_k) \subseteq A$. Then, by Lemma 4.5.12, $j(f) = j(f') = f'|_{C_-W} + f'|_{C_+W} = f|_{(\Sigma W\backslash F_k)\cup Z_k} + f|_{F_k}$. Since $f((\Sigma W \backslash \bigcup_i F_i) \cup (\bigcup_i Z_i)) \subseteq A$, then collapsing $(\Sigma W \backslash \bigcup_i F_i) \cup (\bigcup_i Z_i)$ gives a homotopy of pairs between the map $f|_{(\Sigma W\backslash F_k)\cup Z_k}\colon (CW, W) \longrightarrow (X, A)$ and $\sum_{i=1}^{k-1} f|_{F_i}$. □

We now have everything we need to complete the proof of Theorem 4.3.7.

Proof of Theorem 4.3.7. Since $X_j \subseteq D_j^+ \cup D_j^2$, we represent $f_j\colon \Sigma X_j \longrightarrow Y_j$ by a map of pairs $f_j\colon (C\Sigma\widetilde{X}_j, \Sigma\widetilde{X}_j) \longrightarrow (Y_j, *)$ such that $f_j(D_j^+ \cup D_j^2) = *$.

Let $\mathcal{L} = \partial\Delta[1, \ldots, m]$ and for $i = 1, \ldots, k$ let $\mathcal{L}_i = \partial\Delta[j_1, \ldots, j_{r_i}]$. Define the map $\psi\colon (V, V^*) \longrightarrow (\prod_{l=1}^m Y_l, (\underline{Y}, \underline{*})^{\mathcal{L}})$ by

$$\psi(x_1, \ldots, x_m) = (f_1(x_1), \ldots, f_m(x_m))$$

and for $i = 1, \ldots, k$ define maps $\psi_i\colon (V_i, V_i^*) \longrightarrow (\prod_{l=1}^{r_i} Y_{j_l}, (\underline{Y}, \underline{*})^{\mathcal{L}_i})$ by $\psi_i = \psi|_{V_i}$.

Since $f_j(D_j^+ \cup D_j^2) = *$, then ψ_i maps $x_i \in C\Sigma\widetilde{X}_i$ away from the basepoint only if $x_i \in D_i^1 \cap D_i^-$. Therefore ψ_i maps all of $\left(x_{j_1}, \ldots, x_{j_{r_i}}\right)$ away from the basepoint only if $\left(x_{j_1}, \ldots, x_{j_{r_i}}\right) \in \prod_{l=1}^{r_i} D_l^1 \cap D_l^- \subseteq G_i$. Equivalently, $\psi_i((V_i - G_i) \cup G_i^*) \subseteq (\underline{Y}, *)^{\mathcal{L}_i}$. Therefore by Lemma 4.5.10, $\psi_i|_{G_i}$ is homotopic to ψ_i.

Now consider

$$\begin{aligned}
\psi\rho_i|_{\rho_i^{-1}F_i} \colon G_i \times &\left(\bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}}\right) \cup \tau_i \times \left(CX_{i_1} \times \cdots \times CX_{i_{n_i}}\right) \\
&\longrightarrow \psi_i(G_i) \times \left(\bigcup_{l=1}^{n_i} \Sigma X_{i_1} \times \cdots \times * \times \cdots \times \Sigma X_{i_{n_i}}\right) \cup * \times \left(\Sigma X_{i_1} \times \cdots \times \Sigma_{i_{n_i}}\right) \\
&\longrightarrow \prod_{l=1}^{r_i} Y_{j_l} \times (\underline{Y}, \underline{*})^{\partial\Delta[i_1,\ldots,i_{n_i}]} \cup * \times \prod_{l=1}^{n_i} Y_{i_l} \\
&= FW\left(\prod_{l=1}^{r_i} Y_{j_l}, Y_{i_1}, \ldots, Y_{i_{n_i}}\right) \\
&= (\underline{Y}, \underline{*})^{\partial\Delta\langle\Delta[j_1,\ldots,j_{r_i}],i_1,\ldots,i_{n_i}\rangle} \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{L}}
\end{aligned}$$

and the restriction

$$\begin{aligned}
\psi\rho_i|_{\rho_i^{-1}Z_i} \colon \kappa_i \times &\left(\bigcup_{l=1}^{n_i} CX_{i_1} \times \cdots \times X_{i_l} \times \cdots \times CX_{i_{n_i}}\right) \cup \sigma_i \times \left(CX_{i_1} \times \cdots \times CX_{i_{n_i}}\right) \\
&\longrightarrow (\underline{Y}, \underline{*})^{\mathcal{L}_i} \times (\underline{Y}, \underline{*})^{\partial\Delta[i_1,\ldots,i_{n_i}]} \cup * \times \prod_{l=1}^{n_i} Y_{i_l} \\
&= FW\left((\underline{Y}, \underline{*})^{\mathcal{L}_i}, Y_{i_1}, \ldots, Y_{i_{n_i}}\right) \\
&= (\underline{Y}, \underline{*})^{\partial\Delta\langle\partial\Delta[j_1,\ldots,j_{r_i}],i_1,\ldots,i_{n_i}\rangle} \longrightarrow (\underline{Y}, \underline{*})^{\mathcal{K}}.
\end{aligned}$$

It follows that $\psi\rho_i|_{\rho_i^{-1}F_i} \colon (\rho_i^{-1}F_i, \rho_i^{-1}Z_i) \longrightarrow ((\underline{Y}, \underline{*})^{\mathcal{L}}, (\underline{Y}, \underline{*})^{\mathcal{K}})$ is the relative higher Whitehead map $h_w(\psi_i|_{G_i}, f_{i_1}, \ldots, f_{i_{n_i}})$. Therefore $\psi|_{F_i} \colon (F_i, Z_i) \longrightarrow ((\underline{Y}, \underline{*})^{\mathcal{L}}, (\underline{Y}, \underline{*})^{\mathcal{K}})$ is the composite $h_w(\psi_i|_{G_i}, f_{i_1}, \ldots, f_{i_{n_i}}) \circ \rho_i^{-1}$. Finally, since $\psi_i|_{G_i}$ is homotopic to ψ_i and $\rho_i^{-1} = \sigma_i$, then $\psi|_{F_i}$ is homotopic to $h_w(\psi_i, f_{i_1}, \ldots, f_{i_{n_i}}) \circ \sigma_i$.

Let $W = \Sigma^{m-2}X_1 \wedge \cdots \wedge X_m$, $X = (\underline{Y}, \underline{*})^{\mathcal{L}}$ and $A = (\underline{Y}, \underline{*})^{\mathcal{K}}$. Consider long exact sequence (4.38)

$$\cdots \longrightarrow [\Sigma W, X] \xrightarrow{f} [(CW, W), (X, A)] \xrightarrow{\partial} [W, A] \longrightarrow \cdots .$$

By Proposition 4.5.2 and Lemma 4.5.11,

$$\begin{aligned}
\partial j(\psi|_{V*}) &\simeq \partial\left(\sum_{i=1}^{k} \psi|_{F_i}\right) \\
&\simeq \sum_{i=1}^{k} \partial h_w(\psi_i, f_{i_1}, \ldots, f_{i_{n_i}}) \circ \sigma_i \\
&= \sum_{i=1}^{k} h_w(\psi_i|_{V_i^*}, f_{i_1}, \ldots, f_{i_{n_i}}) \circ \sigma_i \\
&= \sum_{i=1}^{k} h_w(h_w(f_{j_1}, \ldots, f_{j_{r_i}}), f_{i_1}, \ldots, f_{i_{n_i}}) \circ \sigma_i.
\end{aligned}$$

On the other hand, $\partial j \simeq 0$, completing the proof. $\square$

Bibliography

[Ada56] J. F. Adams, *On the cobar construction*, Proceedings of the National Academy of Sciences of the United States of America **42** (1956), no. 7, 409–412.

[AH56] J. F. Adams and P. J. Hilton, *On the chain algebra of a loop space*, Commentarii Mathematici Helvetici **30** (1956), 305–330.

[ALL07] D. Allen and J. La Luz, *A counterexample to a conjecture of Bosio and Meersseman*, arXiv:0712.3801 (2007).

[ALS11] D. Attali, A. Lieutier, and D. Salinas, *Efficient data structure for representing and simplifying simplicial complexes in high dimensions*, Proceedings of the Twenty-Seventh Annual Symposium on Computational Geometry (New York, NY, USA), Association for Computing Machinery, 2011, pp. 501—-509.

[Ame20] S. Amelotte, *Connected sums of sphere products and minimally non-Golod complexes*, arXiv:2006.16320 (2020).

[AP19] S. A. Abramyan and T. E. Panov, *Higher Whitehead products for moment-angle complexes and substitutions of simplicial complexes*, Tr. Mat. Inst. Steklova **305** (2019), 7–28.

[Ark62] M. Arkowitz, *The generalized Whitehead product*, Pacific Journal of Mathematics **12** (1962), no. 1, 7–23.

[Ark11] ______, *Introduction to homotopy theory*, Universitext, Springer New York, 2011.

[Ayz13] A. Ayzenberg, *Composition of simplicial complexes, polytopes and multigraded Betti numbers*, arXiv:1301.4459 (2013).

[Bas02] I. V. Baskakov, *Cohomology of k-powers of spaces and the combinatorics of simplicial divisions*, Russian Mathematical Surveys **57** (2002), no. 5, 989–990.

[BBCG10] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: A method of decomposition for moment-angle complexes, arrangements and related spaces*, Advances in Mathematics **225** (2010), no. 3, 1634–1668.

[BJ07] A. Berglund and M. Jöllenbeck, *On the Golod property of Stanley–Reisner rings*, Journal of Algebra **315** (2007), no. 1, 249–273.

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Springer, 1972.

[BL00] A. Björner and F. H. Lutz, *Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere*, Experimental Mathematics **9** (2000), no. 2, 275–289.

[BM53] A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Annals of Mathematics **58** (1953), no. 2, 295–324.

[BM04] F. Bosio and L. Meersseman, *Real quadrics in $\mathbb{C}^n$, complex manifolds and convex polytopes*, Acta Mathematica **197** (2004), 53–127.

[BP00] V. M. Buchstaber and T. E. Panov, *Torus actions, combinatorial topology, and homological algebra*, Russian Mathematical Surveys **55** (2000), no. 5, 825–921.

[BP15] ———, *Toric topology*, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, 2015.

[BS53] R. F. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, Commentarii Mathematici Helvetici **27** (1953), 320–337.

[BW15] P. Beben and J. Wu, *The homotopy type of a Poincaré duality complex after looping*, Proceedings of the Edinburgh Mathematical Society **58** (2015), no. 3, 581—-616.

[Cai17] L. Cai, *On products in a real moment-angle manifold*, Journal of the Mathematical Society of Japan **69** (2017), no. 2, 503–528.

[Coh57] D. E. Cohen, *Products and carrier theory*, Proceedings of the London Mathematical Society **s3-7** (1957), no. 1, 219–248.

[DHKS04] W. G. Dwyer, P. S. Hirchhorn, D. M. Kan, and J. H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical surveys and monographs, vol. 113, American Mathematical Society, 2004.

[DJ91] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Mathematical Journal **62** (1991), no. 2, 417–451.

[DS07] G. Denham and A. I. Suciu, *Moment-angle complexes, monomial ideals and Massey products*, Pure and Applied Mathematics Quarterly **3** (2007), no. 1, 25—60.

[DV73] E. Dyer and A. T. Vasquez, *Some small aspherical spaces*, Journal of the Australian Mathematical Society **16** (1973), no. 3, 332—352.

[FW15] F. Fan and X. Wang, *On the cohomology of moment-angle complexes associated to Gorenstein* complexes*, arXiv:1508.00159 (2015).

[Gan67] T. Ganea, *Induced fibrations and cofibrations*, Transactions of the American Mathematical Society **127** (1967), no. 3, 442–459.

[GdM13] S. Gitler and S. López de Medrano, *Intersections of quadrics, moment-angle manifolds and connected sums*, Geometry & Topology **17** (2013), no. 3, 1497–1534.

[GIPS22] J. Grbić, M. Ilyasova, T. E. Panov, and G. Simmons, *One-relator groups and algebras related to polyhedral products*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **152** (2022), no. 1, 128—147.

[GL21] J. Grbić and A. Linton, *Non-trivial higher Massey products in moment-angle complexes*, Advances in Mathematics **387** (2021), 107837.

[GPTW12] J. Grbić, T. E. Panov, S. Theriault, and J. Wu, *Homotopy types of moment-angle complexes for flag complexes*, Transactions of the American Mathematical Society **368** (2012), 6663–6682.

[GT07] J. Grbić and S. Theriault, *The homotopy type of the complement of a coordinate subspace arrangement*, Topology **46** (2007), no. 4, 357–396.

[GT16] ———, *Homotopy theory in toric topology*, Russian Mathematical Surveys **71** (2016), no. 2, 185–251.

[Har61] K. A. Hardie, *Higher Whitehead products*, The Quarterly Journal of Mathematics **12** (1961), no. 1, 241–249.

[Har64] ———, *A proof of the Nakaoka–Toda formula*, Pacific Journal of Mathematics **14** (1964), no. 4, 1249–1254.

[Hat02] A. Hatcher, *Algebraic topology*, Algebraic Topology, Cambridge University Press, 2002.

[Hil55] P. J. Hilton, *On the homotopy groups of the union of spheres*, Journal of the London Mathematical Society **s1-30** (1955), no. 2, 154–172.

[Hoc77] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Lecture Notes in Pure and Appl. Math. (Univ. Oklahoma), vol. 26, Dekker, 1977, pp. 171–223.

[IK19] K. Iriye and D. Kishimoto, *Fat-wedge filtration and decomposition of polyhedral products*, Kyoto Journal of Mathematics **59** (2019), no. 1, 1–51.

[Jam55] I. M. James, *Reduced product spaces*, Annals of Mathematics **62** (1955), no. 1, 170–197.

[Kat17] L. Katthän, *A non-Golod ring with a trivial product on its Koszul homology*, Journal of Algebra **479** (2017), 244–262.

[Lim15] I. Y. Limonchenko, *Families of minimally non-Golod complexes and their polyhedral products*, Dal'nevost. Mat. Zh. **15** (2015), 222–237.

[LP22] I. Y. Limonchenko and T. E. Panov, *Minimally non-Golod face rings and Massey products*, Uspekhi Mat. Nauk **77** (2022), 203–204.

[LW19] X. Li and G. Wang, *A moment-angle manifold whose cohomology has torsion*, Homology, Homotopy and Applications **21** (2019), no. 2, 199–212.

[Lyn50] R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Annals of Mathematics **52** (1950), no. 3, 650–665.

[Mah65] M. Mahowald, *Some Whitehead products in S^n*, Topology **4** (1965), no. 1, 17–26.

[May69] J. P. May, *Matric Massey products*, Journal of Algebra **12** (1969), 533–568.

[McG79] D. McGavran, *Adjacent connected sums and torus actions*, Transactions of the American Mathematical Society **251** (1979), 235–254.

[Mil72] J. W. Milnor, *On the construction FK*, London Mathematical Society Lecture Note Series, pp. 118—-136, Cambridge University Press, 1972.

[NT54] M. Nakaoka and H. Toda, *On Jacobi identity for Whitehead products*, Journal of the Institute of Polytechnics, Osaka City University Series A: Mathematics **5** (1954), no. 1, 1–13.

[Pac87] U. Pachner, *Konstruktionsmethoden und das kombinatorische homöomorphieproblem für triangulationen kompakter semilinearer mannigfaltigkeiten*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **57** (1987), 69–86, (in German).

[Por65] G. J. Porter, *Higher order Whitehead products*, Topology **3** (1965), 123–135.

[PV16] T. E. Panov and Y. A. Veryovkin, *Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups*, Sbornik: Mathematics **207** (2016), 1582–1600.

[S+22] W. A. Stein et al., *Sage Mathematics Software (Version 9.3.1)*, The Sage Development Team, 2022, `http://www.sagemath.org`.

[Sam53] H. Samelson, *A connection between the Whitehead and the Pontryagin product*, American Journal of Mathematics **75** (1953), no. 4, 744–752.

[Sel97] P. Selick, *Introduction to homotopy theory*, Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs, American Mathematical Society, 1997.

[Suz54] H. Suzuki, *A product in homotopy theory*, Tohoku Mathematical Journal **6** (1954), no. 1, 78–88.

[Tay66] D. Taylor, *Ideals generated by monomials in an R-sequence*, Ph.D. thesis, University of Chicago, 1966.

[UM57] H. Uehara and W. S. Massey, *The Jacobi identity for Whitehead products*, Algebraic Geometry and Topology: A Symposium in Honor of Solomon Lefschetz (Princeton) (Ralph Hartzler Fox, ed.), Princeton University Press, 1957, pp. 361–377.

[Ver16] Y. A. Veryovkin, *Pontryagin algebras of some moment-angle complexes*, Dal'nevost. Mat. Zh. **16** (2016), 9–23.

[Vid17] E. Vidaurre, *On polyhedral product spaces over polyhedral joins*, Homology, Homotopy and Applications **20** (2017), 259–280.

[Vog73] R. M. Vogt, *Homotopy limits and colimits.*, Mathematische Zeitschrift **134** (1973), 11–52.

[Whi41] J. H. C. Whitehead, *On adding relations to homotopy groups*, Annals of Mathematics **42** (1941), 409–428.

[Whi54] G. W. Whitehead, *On mappings into group-like spaces*, Commentarii mathematici Helvetici **28** (1954), 320–328.

[Whi55] ———, *On the homology suspension*, Annals of Mathematics **62** (1955), no. 2, 254–268.

[Zhu21] E. Zhuravleva, *Adams-Hilton models and higher Whitehead brackets for polyhedral products*, arXiv:2112.10013 (2021).