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Formulating the r-mode Problem for Slowly Rotating Neutron Stars

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Abstract

We revisit the problem of inertial r-modes in stratified stars, drawing on a more precise description of the composition stratification in a mature neutron star. The results highlight issues with the traditional approach to the problem, leading us to rethink the computational strategy for the r-modes of nonbarotropic neutron stars. We outline two strategies for dealing with the problem. For moderate to slowly rotating neutron stars the only viable alternative may be to approach the problem numerically from the outset, while a meaningful slow-rotation calculation can be carried out for the fastest known spinning stars (which may be close to being driven unstable by the emission of gravitational waves). We demonstrate that the latter approach leads to a problem close, but not identical, to that for barotropic inertial modes. We also suggest that these reformulations of the problem likely resolve the long-standing problem of singular behavior associated with a corotation point in rotating relativistic neutron stars. This issue needs to be resolved in order to guide future gravitational-wave searches.

Unified Astronomy Thesaurus concepts: Neutron stars (1108); Astrophysical fluid dynamics (101); Asteroseismology (73)

1. Motivation

The inertial r-modes of a spinning neutron star have attracted a fair amount of attention. Theorists have explored the precise nature of the r-modes and how they depend on the complex physics of the neutron star's interior, while observers have tried to establish the presence of the predicted r-mode signature in observational data. Much of this interest stems from the discovery (now a quarter of a century ago!) that the r-modes may be driven unstable by the emission of gravitational waves (Andersson 1998; Friedman & Morsink 1998; Lindblom et al. 1998; Andersson et al. 1999a). The r-mode instability may limit the spin-up of accreting neutron stars in low-mass X-ray binaries (Andersson et al. 1999b), providing a natural explanation for the apparent absence of submillisecond radio pulsars. The mechanism may also lead to the emission of detectable gravitational waves from newly born neutron stars (Owen et al. 1998), mainly through the current multipoles associated with the induced fluid motion. This has motivated a sequence of observational gravitational-wave papers (Abadie et al. 2010; Aasi et al. 2015; Fesik & Papa 2020; Abbott et al. 2021a, 2021b, 2022a, 2022b; Guo et al. 2022), so far mainly setting upper limits on the attainable r-mode amplitude. There have also been tantalizing hints of r-mode oscillations in the X-ray emission from two fast-spinning, accreting neutron stars (Strohmayer & Mahmoodifar 2014a, 2014b; see also Andersson et al. 2014; Lee 2014), but these results are far from conclusive.

The need to understand the impact of different aspects of neutron star physics—ranging from the main dissipation channels (shear and bulk viscosity), the state of matter (superfluid mutual friction and the interfaces with the elastic crust), and the role of the star's magnetic field—led to a flurry of activity following the original instability discovery. This work is summarized in early review articles, such as Andersson & Kokkotas (2001) and Andersson (2003), with more recent additions like Lasky (2015) and Haskell & Schwenzer (2021) providing a mature perspective. While many aspects of the problem are fairly well understood, some vexing issues remain. Arguably, the most important of these issues relates to the r-modes in relativity.

In order to make robust predictions for (say) the r-mode frequency in a neutron star we need to involve a realistic matter equation of state. This, in turn, requires a general-relativistic mode calculation. This problem has not—in our view—yet been solved in a satisfactory fashion. Let us explain. The most important contributions to the discussion, from the initial relativistic inertial-mode calculations by Lockitch et al. (2000, 2003) and Ruoff et al. (2003) through to the more recent work for real equations of state by Idrisy et al. (2015), assumes that the matter is barotropic. However, this is not expected to be a realistic assumption. Instead, as established by Reisenegger & Goldreich (1992), the stratification associated with internal composition gradients makes the problem nonbarotropic. This seems to complicate the r-mode calculation. In fact, the problem appears to become singular, leading to a continuous spectrum (Kojima 1998; Beyer & Kokkotas 1999; Kojima & Hosonuma 1999). The implications of this are not well understood, but it seems reasonable to argue that the continuous spectrum arises because of simplifying assumptions introduced in the analysis. Adopting this view, the question becomes how we can regularize the problem. While different strategies have been proposed, such as Lockitch et al. (2004), Yoshida & Lee (2002), Pons et al. (2005), and the recent effort from Kraav et al. (2021, 2022), it is probably fair to suggest that the issue has not yet been resolved—at least not completely. This motivates us to return to the problem.

Our aim is to formulate the r-mode problem for neutron stars stratified by composition gradients from first principles. This forces us to consider how the nonbarotropic aspects arise and how this affects the fluid perturbation equations. The key point is that the matter composition may be considered frozen provided the relevant nuclear reactions are slow compared to the dynamics. The argument was already outlined by

Andersson & Pnigouras (2019), although in that case the main focus was on the composition g-modes. Here we take one step further by framing the discussion in the context of a realistic matter model. This leads to an important—in hindsight probably obvious—insight. While realistic neutron star models are likely to have nonbarotropic high-density cores they will always have barotropic—basically because the matter is composed of single nuclei at lower densities—outer layers. This has two immediate repercussions. First, the standard assumption of a constant adiabatic index (Γ_1) for the perturbations is never appropriate. Second, none of the existing r-mode calculations actually solve the problem we should be considering. We have have to rethink how we approach the problem.

Ultimately, the implications of our new perspective on the problem will depend on the extent to which numerical mode results differ from existing ones. This problem will not be solved here. In this first paper we are mainly interested in the qualitative aspects and so we introduce a number of approximations that sacrifice accuracy for clarity. We want to make the key points as transparent as possible in order to motivate renewed effort in several directions. First and foremost, we need relativistic mode calculations for true equations of state in order to inform future gravitational-wave searches. It may well be that the current models are "good enough" but—as we are arguing for changes to the formulation of the problem—this is by no means guaranteed. Second, we need to revisit the (technically challenging) problem of nonlinear mode coupling and saturation of the r-mode instability (Schenk et al. 2001; Arras et al. 2003; Brink et al. 2004a, 2004b, 2005; Bondarescu et al. 2007, 2009; Bondarescu & Wasserman 2013). Available mode-coupling results relate to barotropic models and hence may change when we add realism to the discussion. Third, the precise nature of the r-modes impact on the gravito-magnetic interaction and the dynamical tides in spinning neutron star binaries (Flanagan & Racine 2007). Again, this is a problem that has so far only been explored for barotropic stellar models (Xu & Lai 2017; Poisson 2020; Poisson & Buisson 2020; Ma et al. 2021).

It appears that there is quite a lot of work to get on with, so let us get started.

2. Low-frequency Oscillations of a Rotating Star

In general, the oscillation modes of a rotating star belong to one of two categories: modes that exist (have a finite eigenfrequency) already in a nonrotating star and modes that owe their existence to the rotation (the Coriolis force). Our main interest here is in the latter class, collectively referred to as inertial modes, but the story will not be complete unless we also touch upon the former. The argument at the center of our discussion also impacts on the gravity g-modes, which are present if the star is stably stratified, either in terms of an entropy gradient or a varying composition. This is a nontrivial issue. Work on main-sequence stars demonstrates that the impact of stratification may vary, with global oscillation modes having different character in different parts of a star (Lee & Saio 1987). An illustrative example concerns slowly pulsating B stars (Lee 2006).

Work on the oscillations of rotating stars is complicated by the fact that the Coriolis force introduces a coupling of the angular harmonics traditionally used to represent the modes. This is particularly significant for modes with frequency comparable to or smaller than the star's rotation frequency. Higher-frequency modes may be approached "perturbatively," adding rotational corrections to the modes of a nonrotating star, but the low-frequency problem is intricate. This issue is not at all new. It was recognized already in the seminal work on rotating ellipoids by Bryan (1889; for a modern version of the calculation, see Lindblom & Ipser 1999). It is also well known from work on waves in shallow ocean/atmospheres, mainly focused on weather and climate studies. The r-modes—the main focus of our attention—are in fact analogous to the Rossby waves from the shallow-water problem (see, e.g., Longuet-Higgins 1968; Zaqarashvili et al. 2021).

For rotating stars, the r-modes were first introduced by Papaloizou & Pringle (1978). Their work was followed by a number of detailed studies in the early 1980s (Provost et al. 1981; Saio 1981, 1982; Smeyers & Martens 1983). The nature of the problem was laid out in detail, relaxing the Cowling approximation (i.e., allowing for perturbations of the gravitational potential), by Smeyers et al. (1981). This body of work establishes the main features of the r-modes. They are represented by retrograde waves in the corotating frame of the star, but appear prograde in the inertial frame. It is precisely this character that makes them unstable to the emission of gravitational waves (Andersson 1998; Friedman & Morsink 1998).

The problem is complicated by the fact that the modes are degenerate in barotropic stars, belonging to the broader class of inertial modes (with frequency proportional to the star's angular frequency; Lockitch & Friedman 1999). This changes when the stellar fluid is stably stratified. Stratification couples the radial layers in the star, breaking the degeneracy of the modes and allowing for a richer spectrum of r-modes (including radial overtones; Provost et al. 1981; Saio 1982). In the context of the r-mode instability, the work by Yoshida & Lee (2000a) is particularly notable, although they consider either models with stable stratification throughout the interior or models that are fully convective. However—and this is important—the arguments they put forward for the stratification relate to entropy gradients, which are only expected to be relevant for newly born neutron stars.

We will set up the problem in such a way that our discussion connects with the inertial-mode analysis of Lockitch & Friedman (1999). This makes sense because one of the questions we want to address involves how the nonbarotropic r-modes morph into barotropic inertial modes as the stratification weakens.

Starting from the velocity perturbations (δv^i), we consider the perturbed Euler equation in the rotating frame of the star. We then have (in a coordinate basis, making due distinction between co- and contra-variant components, and using δ to indicate Eulerian variations)

$$\partial_t \delta v_i + 2\epsilon_{ijk}\Omega^j \delta v^k + \frac{1}{\rho}\nabla_i \delta p - \frac{1}{\rho^2}\delta\rho \nabla_i p + \nabla_i \delta\Phi = 0, \quad (1)$$

where p is the pressure, ρ is the mass density, and Φ the gravitational potential—along with the continuity equation:

$$\partial_t \delta\rho + \nabla_i(\rho \delta v^i) = 0, \quad (2)$$

and the Poisson equation for the perturbed gravitational potential:

$$\nabla^2 \delta\Phi = 4\pi G \delta\rho. \tag{3}$$

In the static limit (where time variations vanish and $\Omega^i \to 0$) the equations decouple into two sets. First, we have

$$\nabla_i \delta\Phi + \frac{1}{\rho}\nabla_i \delta p - \frac{1}{\rho^2}\delta\rho \nabla_i p = 0, \tag{4}$$

along with (3) and, second,

$$\nabla_i(\rho \delta v^i) = 0. \tag{5}$$

At this point, Lockitch & Friedman (1999) note that Equations (3) and (4) represent perturbations that take us to a neighboring equilibrium star. The argument for this is straightforward: a static perturbation of the equation for hydrostatic equilibrium takes us to a new configuration with pressure $\bar{p} = p + \delta p$. In a slowly rotating star, this solution would pick up rotational corrections at order Ω, which suggests a solution such that

$$[\delta\rho, \delta p, \delta\Phi] = \mathcal{O}(1) \quad \text{and} \quad \delta v^i = \mathcal{O}(\Omega). \tag{6}$$

This problem is equivalent to considering the dynamics of the original configuration (albeit for a slight different central density). The dynamical aspects of the problem are contained in the second set of perturbations, represented by Equation (2), for which we would have

$$\delta v^i = \mathcal{O}(1) \quad \text{and} \quad [\delta\rho, \delta p, \delta\Phi] = \mathcal{O}(\Omega). \tag{7}$$

This is the assumption that leads to the inertial modes. Alternatively, as we can always multiply the linearized equations by a constant, we may normalize the Lagrangian displacement associated with the perturbation, in the rotation frame simply given by

$$\delta v^i = \partial_t \xi^i, \tag{8}$$

such that

$$\xi^i = \mathcal{O}(1) \Longrightarrow \delta v^i = \mathcal{O}(\Omega) \quad \text{and} \quad [\delta\rho, \delta p, \delta\Phi] = \mathcal{O}(\Omega^2). \tag{9}$$

This is the convention we assume in the following. It is important to appreciate that, regardless of the chosen normalization, we cannot (completely) determine the density perturbations, etc., without accounting for the change in shape due to the centrifugal force (which also enters at order Ω^2). This obviously complicates the analysis and it would be natural to turn to numerics. However, in order to understand the nature of the problem we wish to proceed analytically (as far as we can). We consider the numerical problem in a companion paper (Gittins & Andersson 2023). In order to make progress analytically, we will study the perturbations to second order in Ω/Ω_0, where

$$\Omega_0^2 = \frac{GM}{R^3}. \tag{10}$$

However, in the interests of clarity, we will keep the background spherical (Lee & Saio 1987). This is a useful simplification as it removes some of the rotational multipole couplings from the problem. It also makes "sense" since the centrifugal force does not introduce additional oscillation modes. In parts of the discussion we will also, again for clarity, make use of the Cowling approximation (neglect the perturbed gravitational potential, $\delta\Phi$). Both assumptions are relaxed in the companion numerical work.

2.1. The Frozen Composition Argument

It is easy to see how the assumption of nonbarotropic perturbations upsets the inertial-mode logic. The usual argument introduces the adiabatic index Γ_1 in such a way that (with Δ representing Lagrangian variations)

$$\Delta p = \frac{p\Gamma_1}{\rho}\Delta\rho \Longrightarrow \delta p = \frac{p\Gamma_1}{\rho}\delta\rho + \frac{p\Gamma_1}{\rho}\xi^i\nabla_i\rho - \xi^i\nabla_i p. \tag{11}$$

This immediately leads to a conflict with the assumed ordering for the rotating-star perturbations (Lockitch & Friedman 1999). Since the background is spherical we must either have

$$\frac{p\Gamma_1}{\rho}\frac{d\rho}{dr} = \frac{dp}{dr} \Longrightarrow \Gamma_1 = \frac{\rho}{p}\frac{dp}{d\rho} = \frac{\rho c_s^2}{p} \equiv \Gamma, \tag{12}$$

(introducing both the adiabatic index Γ and the speed of sound $c_s{}^2$ for the background configuration), which would represent a barotropic model, or the ordering of the solution must change in such a way that $\xi^r = \mathcal{O}(\Omega^2)$ to balance the density and pressure perturbations. This then leads to the different starting assumption for nonbarotropic models suggested by Lockitch & Friedman (1999), and eventually brings us to the vexing issue for relativistic r-modes; see Lockitch et al. (2004). However, for realistic neutron star physics, the argument turns out to be a bit more subtle.

In order to explain the issue we need to explore the physics that give rise to the stratification in a mature neutron star (internal composition gradients) in the first place, leading to $\Gamma_1 \neq \Gamma$. The argument draws heavily on the discussion of g-modes and reactions from Andersson & Pnigouras (2019). Essentially, once we account for out-of equilibrium nuclear reactions,[1] the perturbed proton fraction x_p evolves according to (for small deviations from equilibrium, i.e., in the so-called

[1] Our discussion assumes a star dominated by neutron–proton–electron matter, for which the key reactions are due to the Urca processes. This is the simplest relevant case. The problem will change if (say) hyperons or deconfined quarks are present at high densities. The reaction rates will then be different as may be the outcome for the stratification. Similarly, the state of matter is important. For example, it is known that the composition gradient in a superfluid neutron star core arises not due to an imbalance between neutron and protons but as a result of the presence of muons (Gusakov & Kantor 2013; Passamonti et al. 2016).

subthermal limit; Alford & Harris 2018)

$$(\partial_t + v^j\nabla_j)\Delta x_{\rm p} = \frac{\gamma}{n}\Delta\beta, \tag{13}$$

where n is the baryon number density. Here $\Delta\beta$ represents the deviation from beta equilibrium and γ encodes the (dominant) reaction rate. Considering $\beta = \mu_{\rm n} - \mu_{\rm p} - \mu_{\rm e}$ (with $\mu_{\rm x}$ the chemical potential for particle species "x") a function of ρ and the proton fraction $x_{\rm p}$, assuming that the star is nonrotating (so that $v^i = 0$, which is also true in the rotating frame), and working in the frequency domain (with a harmonic time dependence, $e^{i\omega t}$, for the perturbations), the discussion of Andersson & Pnigouras (2019) leads to

$$\Delta\beta = \frac{\mathcal{B}}{1 + i\mathcal{A}/\omega}\Delta\rho, \tag{14}$$

where the background coefficients

$$\mathcal{A} = \left(\frac{\partial\beta}{\partial x_{\rm p}}\right)_\rho \frac{\gamma}{n}, \quad \mathcal{B} = \left(\frac{\partial\beta}{\partial\rho}\right)_{x_{\rm p}} \tag{15}$$

are time independent. This relation is commonly taken as the starting point for discussions of bulk viscosity; see Schmitt & Shternin (2018) for a recent review of this issue. Here we want to make a slightly different emphasis.

Let us consider the timescales involved. Noting that $\mathcal{A}$ needs to be negative in order for the system to relax toward equilibrium, we introduce a characteristic reaction time as

$$t_R = -\frac{1}{\mathcal{A}}. \tag{16}$$

Then we see that, if the reactions are fast compared to the dynamics (associated with a timescale $\sim 1/\omega$), then $|t_R\omega| \ll 1$ and we have $\Delta\beta \approx 0$. In effect, the fluid elements reach equilibrium before executing an oscillation. The fluid remains in beta equilibrium and hence the perturbations are (effectively) barotropic.

As a ballpark estimate of the relevant timescale, we draw on the discussion by Haensel et al. (2002) and assume

$$t_R \sim 10^{13}\left(\frac{10^8\ {\rm K}}{T}\right)^6 {\rm s} \tag{17}$$

for the modified Urca reactions (ignoring density dependence as we only need a rough idea here). This estimate suggests that —for all modes/rotating rates we may conceivably be interested in—mature neutron star matter will not be in the fast-reaction regime. We have to consider the slow-reaction (stratified) problem.

In the limit of slow reactions we have $|t_R\omega| \gg 1$ and we can Taylor expand (14) to get

$$\Delta\beta \approx \mathcal{B}(1 - i\mathcal{A}/\omega)\Delta\rho \approx \mathcal{B}\Delta\rho, \tag{18}$$

which in turn leads to, for $p = p(\rho, \beta)$,

$$\Delta p = \left[\left(\frac{\partial p}{\partial\rho}\right)_\beta + \left(\frac{\partial p}{\partial\beta}\right)_\rho\left(\frac{\partial\beta}{\partial\rho}\right)_{x_{\rm p}}\right]\Delta\rho. \tag{19}$$

Comparing to (11), we have an expression for Γ_1 in terms of thermodynamical derivatives:

$$\frac{p\Gamma_1}{\rho} = \left(\frac{\partial p}{\partial\rho}\right)_\beta + \left(\frac{\partial p}{\partial\beta}\right)_\rho\left(\frac{\partial\beta}{\partial\rho}\right)_{x_{\rm p}} = \left(\frac{\partial p}{\partial\rho}\right)_{x_{\rm p}}, \tag{20}$$

where the last equality—demonstrated by Andersson & Pnigouras (2019)—holds as this limit represents frozen composition ($\Delta x_{\rm p} = 0$). In this case we need to consider the impact of composition stratification on the fluid dynamics. The relations we have provided link Γ_1 to the thermodynamical derivatives required from the equation of state. It should be fairly evident that taking this parameter to be constant—as is common in the literature—is unlikely to be very realistic.

Moving on, introducing the gravitational acceleration (for a spherical star)

$$g = -\frac{1}{\rho}\frac{dp}{dr} = \frac{d\Phi}{dr}, \tag{21}$$

we see that (11) leads to

$$\delta p = \frac{p\Gamma_1}{\rho}\delta\rho + \rho g\xi^r\left(1 - \frac{\Gamma_1}{\Gamma}\right). \tag{22}$$

It is important to note that the composition of matter impacts on both terms on the right-hand side of this relation.

In the following, when we consider the impact of composition stratification on the oscillations of a slowly rotating star, it is convenient to consider the Brunt–Väisälä frequency, given by

$$\mathcal{N}^2 = \frac{\rho g^2}{p}\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right). \tag{23}$$

This relation illustrates some of the subtleties we need to consider. For example, if Γ and Γ_1 are taken to be constant (as is commonly assumed) then $\mathcal{N}^2$ must diverge as we approach the surface of the star, where $p/\rho \to 0$. The only way to avoid this problem is if the fluid becomes barotropic in the low-density limit, as we then have $\Gamma_1 \to \Gamma$ as $\rho \to 0$. Also, it is important to keep in mind that the assumption that Γ_1 is constant is different from holding $\mathcal{N}^2$ fixed. However, as we will soon see, neither assumption is realistic.

The equation of state relation allows us to remove the density perturbation from the discussion. We then have

$$\delta\rho = \frac{\rho}{p\Gamma_1}\delta p - \frac{\rho g}{c_s^2}\xi^r\left(\frac{\Gamma}{\Gamma_1} - 1\right) = \frac{1}{c_s^2}\delta p - \frac{\mathcal{N}^2}{g^2}(\delta p - \rho g\xi^r) = \frac{1}{c_s^2}\delta p - \frac{\mathcal{N}^2}{g^2}\Delta p. \tag{24}$$

Evidently, and quite intuitively, stratification does not affect incompressible flows, for which $\Delta\rho = 0 \Longrightarrow \Delta p = 0$.

For later convenience, it is worth noting that

$$\frac{d\rho}{dr} = -\frac{\rho g}{c_s^2} \tag{25}$$

can be used to remove one of the three background quantities from the discussion.

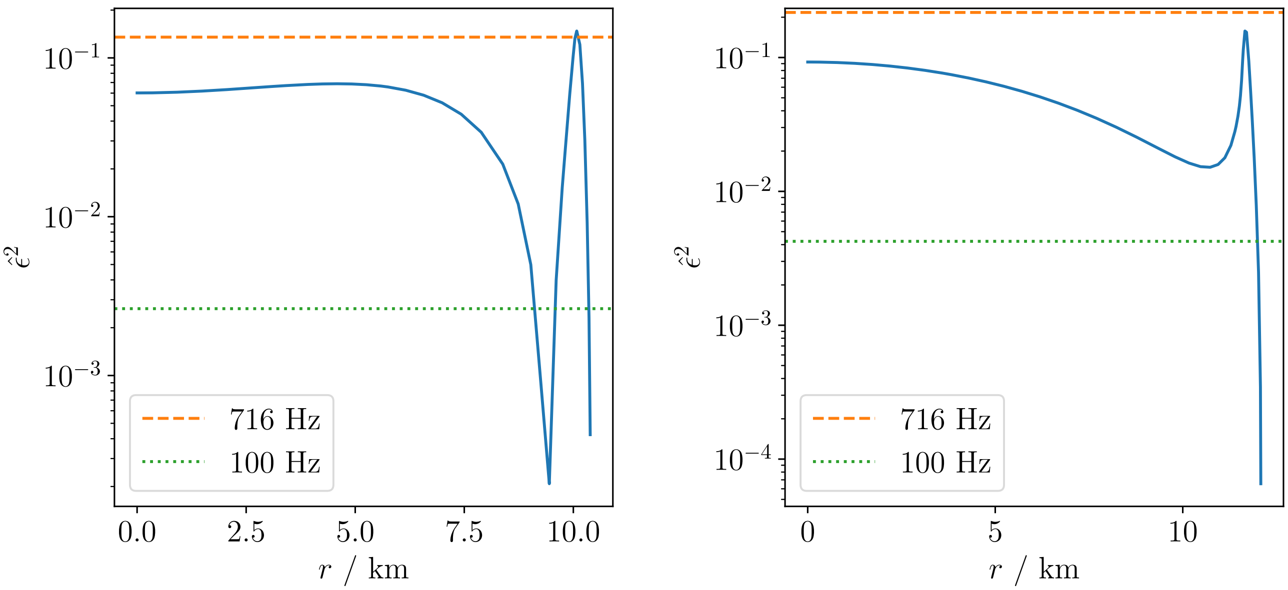

Figure 1. The dimensionless quantity $\hat{\epsilon}^2$ inside two $M = 1.4\,M_\odot$ neutron stars described by the BSk19 (left panel) and BSk21 (right panel) equations of state from Fantina et al. (2013). The two models are suitably indicative because the dependence of the proton fraction with density is very different in the two cases (see Figure 4 from Potekhin et al. 2013). The stellar models are obtained by integrating the relativistic equations of stellar structure with appropriate generalizations for the Brunt–Väisälä frequency $\mathcal{N}$ and gravitational acceleration g. The horizontal lines indicate the value of $\epsilon^2 = \Omega^2/\Omega_0^2$ for the two spin frequencies 100 and 716 Hz, the latter representing the fastest known spinning neutron star (Hessels et al. 2006).

2.2. The Impact of Stratification

As already stated, we will work with the displacement rather than the velocity perturbations in the following. We may then write the perturbed continuity equation

$$\delta\rho + \nabla_i(\rho\xi^i) = 0, \tag{26}$$

as (ignoring the rotational deformation of the background star, as advertised, and making use of the equation of state relation (24))

$$\begin{aligned}\nabla_i\xi^i &= -\frac{1}{\rho c_s^2}\left(1 - \frac{\mathcal{N}^2 c_s^2}{g^2}\right)\delta p + \frac{g}{c_s^2}\left(1 - \frac{\mathcal{N}^2 c_s^2}{g^2}\right)\xi^r \\ &= -\frac{1}{\rho c_s^2}\left(1 - \frac{\mathcal{N}^2 c_s^2}{g^2}\right)\Delta p.\end{aligned} \tag{27}$$

Clearly, it makes sense to consider the dimensionless quantity

$$\hat{\epsilon}^2 = \frac{\mathcal{N}^2 c_s^2}{g^2}. \tag{28}$$

An example of this quantity, for the BSk19 and BSk21 equations of state (Fantina et al. 2013; Potekhin et al. 2013; two models with very different proton fraction profiles), is provided in Figure 1. The results show that $\hat{\epsilon}^2$ typically varies by about an order of magnitude throughout the star's core, reaches a peak in the low-density region and then drops sharply toward the surface. This behavior will guide the discussion in the following.

Next consider the Euler equations (in a rotating frame), which take the form

$$-\omega^2\xi_i + 2i\omega\epsilon_{ijk}\Omega^j\xi^k + \frac{1}{\rho}\nabla_i\delta p - \frac{1}{\rho^2}\delta\rho\nabla_i p + \nabla_i\delta\Phi = 0. \tag{29}$$

For an axisymmetric rotating star, the φ component becomes (noting that all perturbations behave as $e^{im\varphi}$)

$$im\left(\frac{\delta p}{\rho} + \delta\Phi\right) = \omega^2\xi_\varphi - 2i\omega\epsilon_{\varphi jk}\Omega^j\xi^k. \tag{30}$$

In essence, given that the inertial modes we are interested in have frequency $\omega \sim \Omega$, we must have $\delta p \sim \mathcal{O}(\Omega^2)$ in order to have $\xi^i \sim \mathcal{O}(1)$. The radial component of the Euler equation then tells us that we must also have $\delta\rho \sim \mathcal{O}(\Omega^2)$ and the equation of state relation (24) then leads to

$$\frac{\mathcal{N}^2}{g}\xi^r \lesssim \mathcal{O}(\Omega^2). \tag{31}$$

For a given stratification, expressed in terms of $\mathcal{N}^2$, this constrains the slow-rotation ordering of the radial displacement. This accords with the point we made earlier (Lockitch & Friedman 1999).

It is, however, useful to make this argument more precise. For inertial modes of uniformly rotating stars it is natural to use the slow-rotation expansion with $\epsilon = \Omega/\Omega_0$ as the small parameter, noting that the Kepler break-up frequency corresponds to roughly

$$\Omega_K \approx \frac{2}{3}\sqrt{\pi G\rho_0} \Longrightarrow \Omega_K^2 \approx \frac{1}{3}\Omega_0^2 \Longrightarrow \epsilon_K \approx 0.6. \quad (32)$$

Given the previous discussion we know that we should have $\delta p = \epsilon^2\delta\tilde{p}$, using the tilde to indicate an $\mathcal{O}(1)$ quantity (a notation we will adopt in the following), and $\delta\rho = \epsilon^2\delta\tilde{\rho}$, as well. If, in addition, we focus on inertial modes we know that $\omega \sim \Omega$, and it is evident from (29) that we do not have to consider terms of order ϵ; we can take ϵ^2 as the slow-rotation parameter.[2] Hence, we expand the mode frequency as

$$\omega_n = \Omega(\omega_0 + \epsilon^2\tilde{\omega}_2). \quad (33)$$

The equation of state relation (24) then becomes

$$\epsilon^2 c_s^2\delta\tilde{\rho} = \epsilon^2\delta\tilde{p} - \hat{\epsilon}^2(\epsilon^2\delta\tilde{p} - \rho g\xi^r). \quad (34)$$

Let us see what we learn from this.

Noting that the Kepler frequency corresponds to $\epsilon \lesssim 1$ while the results from Figure 1 show that $\hat{\epsilon}^2 < 1$, as well, we may consider a double expansion in ϵ and $\hat{\epsilon}$. Of course, since $\hat{\epsilon}$ varies throughout the star this would have to be a local argument. The first few terms of such an expansion would be

$$\xi^i \approx \xi_0^r + \hat{\epsilon}^2\hat{\xi}_2^r + \epsilon^2\xi_2^r, \quad (35)$$

and it is easy to see that we need to consider several different cases.

First, suppose $\hat{\epsilon} \gg \epsilon$, as would be the case if we combine the results in Figure 1 with a fairly slowly spinning star. Effectively, this situation corresponds to taking $\hat{\epsilon} \sim \mathcal{O}(\epsilon^0)$ and—formally letting $\xi_0^r + \hat{\epsilon}^2\hat{\xi}_2^r \to \xi_0^r$ —it follows that

$$\xi_0^r = 0, \quad \text{at order } \epsilon^0, \quad (36)$$

$$c_s^2\delta\tilde{\rho} = \delta\tilde{p} - \hat{\epsilon}^2(\delta\tilde{p} - \rho g\xi_2^r), \quad \text{at order } \epsilon^2. \quad (37)$$

This is the usual stratified r-mode ordering (see, e.g., Provost et al. 1981), but it is not clear that this is the case we should consider. The argument would only apply to slow and perhaps moderately fast-spinning neutron stars (say, the 100 Hz case illustrated in Figure 1), but not the fastest observed systems.

Instead, the results in Figure 1 suggest that, for stars spinning with $\epsilon \sim 0.3$ (about half the Kepler rate), roughly corresponding to the fastest known pulsars with spin frequency close to 700 Hz, we should instead consider $\hat{\epsilon} \sim \mathcal{O}(\epsilon)$, in which case we may ignore terms of order $\epsilon^2\hat{\epsilon}^2$ in the expansion. This then leads to

$$c_s^2\delta\tilde{\rho} = \delta\tilde{p} + \hat{\epsilon}^2\rho g\xi_0^r, \quad \text{at order } \epsilon^2. \quad (38)$$

This is different from what we usually assume, yet seems a case we ought to consider. Perhaps significantly, the relation suggests that we may have $\xi_0^r \neq 0$, which changes the mode structure. The model is also interesting as it limits to the barotropic case in regions where $\hat{\epsilon}^2 \to 0$.

Finally, from Figure 1 it is clear that there will always be a low-density region where $\hat{\epsilon} \lesssim \mathcal{O}(\epsilon^2)$. If the $\hat{\epsilon}^2$ are smaller than (say) ϵ^4 then we may ignore the stratification and consider the barotropic result:

$$c_s^2\delta\tilde{\rho} = \delta\tilde{p}, \quad \text{at order } \epsilon^2. \quad (39)$$

If these assumptions hold throughout the star, then we must end up with the inertial modes from Lockitch & Friedman (1999). Globally, this is unlikely to be the relevant case, but Figure 1 suggests that we always have to consider the region close to the surface as barotropic. This impacts on the surface boundary condition and may, in turn, also affect the modes.

Clearly, the profile for $\hat{\epsilon}$ is fixed for any given stellar model, while ϵ can be varied (up to the Kepler limit, which means that we should have $\epsilon \lesssim 0.6$). From Figure 1 it is easy to see that all the suggested orderings may apply locally in a neutron star core, making the formulation of a consistent model tricky. The main conclusion is that we have to consider all results obtained with the standard "constant-Γ_1" prescription as unrealistic. In fact, if we take the results in Figure 1 at face value—and there is no reason why we should not—then we have to reconsider our strategy for the fastest spinning stratified neutron stars. For the core of these stars, relation (38) should apply, in which case the perturbation problem is closer to—but not exactly the same as—that for inertial modes (Lockitch & Friedman 1999). To what extent this affects the mode frequencies remains to be established.

3. Formulating the Mode Problem

A mode solution to the perturbation problem must satisfy the perturbation equations (obviously) and relevant boundary conditions (typically, regularity at the center of the star and the vanishing of the Lagrangian variation of the pressure at the star's surface). It is well established that the equations allow for oscillation modes of different character. Moreover, the problem gets richer as more detailed neutron star physics is considered. Somewhat simplistically, each aspect of physics added to the model—matter composition, rotation, superfluidity, electromagnetism, elasticity, and so on—brings new families of modes into play. This makes the general problem complex.

In order to build useful intuition, it is natural to focus on particular aspects. If we are mainly interested in the oscillations of a rotating star, the natural starting point would be to work out how the rotation impacts on modes that exist already in a nonrotating star. For a mode with frequency ω_n in the nonrotating star, the strategy is—at least in principle—fairly straightforward, although the mode calculation may get quite involved as the rotation couples different multipole contributions. Still, for slow-to-moderate rotation rates, this problem can be dealt with perturbatively as long as $\Omega \lesssim \omega_n$. This should be the case for the fundamental mode of the star, which has frequency of order the Kepler break-up frequency, and the (even higher frequency) pressure p-modes. We will not consider those problems here. We will also not consider the rotational corrections to the gravity g-modes, a slightly more subtle issue given that the high-overtone g-modes are expected to have very low frequencies in a nonrotating star. Hence, for these modes the Coriolis force may dominate over the

[2] Note that this is not true for modes that have a finite frequency in the nonrotating limit.

buoyancy already at fairly low rotation rates and they would then become part of the problem we are considering. An example of this behavior can be found in the study by Yoshida & Lee (2000b), where it is shown how inertial modes are strongly modified when the buoyancy force becomes comparable to, or stronger than, the Coriolis force.

In this exploratory analysis our main focus is on the qualitative nature of the low-frequency modes of a rotating star. In this spirit, we rely on simplifying assumptions. In particular, following Lee & Saio (1997), we ignore the change in shape of the background star associated with the centrifugal force. This assumption is not expected to affect the qualitative nature of the problem.

As we have already seen, the discussion necessarily gets somewhat involved and some of the issues we need to consider are subtle. In general, we need to consider both the impact of rotation on oscillation modes that exist already in a nonrotating star and modes that are brought into existence when we consider the impact of the Coriolis force. Given this, it makes sense to work out the general perturbation equations to first slow-rotation order. This involves making choices already at the outset.

There are three common strategies for investigating the oscillations of rotating stars (Unno et al. 1989). The first, and formally most elegant, approach expresses the rotational corrections to a given mode as a sum over all the modes of the corresponding nonrotating star (which form a suitable complete basis as long as we ignore dissipation). The second option builds on an explicit expansion in angular harmonics, while the third involves time evolving the perturbation equations. The last strategy has the advantage that one can readily deal with fast-spinning stars, for which the algebra of the other approaches becomes daunting, but it also has the drawback that one loses track of the fine details of the problem (you get what you get from the simulation, depending on the chosen initial data). Examples of work in this direction can be found in Jones et al. (2002), Passamonti et al. (2009) for Newtonian models, and Gaertig & Kokkotas (2009, 2011), Gaertig et al. (2011), and Kruger et al. (2021) for efforts in relativity. In the following we will carry out an expansion in harmonics. This approach has the advantage that it highlights the nature of the fluid motion.

Assuming that the oscillation modes—with label n and frequency ω_n —are associated with a displacement vector

$$\xi_n^i(t, r, \theta, \varphi) = \hat{\xi}^i(r, \theta, \varphi)e^{i\omega_n t}, \tag{40}$$

(where the hat indicates a quantity that is independent of t and we adopt the Lockitch & Friedman 1999 sign convention) we have (in a coordinate basis)

$$\hat{\xi}^i = \sum_l \left[\frac{1}{r}W_l Y_l^m \delta_r^i + \left(\frac{1}{r^2}V_l \partial_\theta Y_l^m + \frac{m}{r^2\sin\theta}U_l Y_l^m\right)\delta_\theta^i + \frac{i}{r^2\sin^2\theta}(mV_l Y_l^m + U_l \sin\theta\partial_\theta Y_l^m)\delta_\varphi^i\right], \tag{41}$$

where we refer to W_l and V_l as polar perturbations, while U_l is axial (and noting that the m multipoles decouple for an axisymmetric system, like a rotating star). For a given multipole l, these perturbations have different parity. This follows since the equilibrium state of a rotating star is invariant under the parity transformation (defined by a reflection through the origin, $\theta \to \pi - \theta$ and $\varphi \to \varphi + \pi$), the linear perturbations have definite parity for this transformation. Alternatively, the different modes are sometimes described as even and odd; see, for example, Lee & Saio (1987).

Along with the decomposition of the displacement, all scalar perturbations are expanded in spherical harmonics. That is, we have

$$\delta\rho_n = \delta\hat{\rho}(r, \theta, \varphi)e^{i\omega_n t}, \tag{42}$$

with (dropping the hats on the individual l-multipole components to keep the equations that follow as tidy as possible)

$$\delta\hat{\rho} = \sum_l \delta\rho_l Y_l^m, \tag{43}$$

and similar for all other scalar quantities. We also know that the rotating equilibrium remains spherical to linear order in Ω so all background quantities depend only on r as long as we consider the first-order slow-rotation corrections. Working to this order of approximation, let us summarize the equations we need.

3.1. The Perturbation Equations

First, it follows from (41) that

$$\nabla_i\hat{\xi}^i = \sum_l \frac{1}{r^2}[\partial_r(rW_l) - l(l+1)V_l]Y_l^m. \tag{44}$$

This result is important because it shows that—up to order Ω—only the polar contributions to a given mode $[W_l, V_l]$ contribute to the density perturbation $\delta\rho_l$. This is brought out by the continuity equation, which leads to (changing $l \to j$ for consistency with the recurrence relations to be derived in the following)

$$\frac{1}{r^2}\partial_r(r\rho W_j) - j(j+1)\frac{\rho}{r^2}V_j = -\delta\rho_j. \tag{45}$$

Notably—as long as we ignore the rotational deformation—this equation does not involve the coupling of different multipoles.

Turning to the perturbed Euler equations, in the rotating frame we have

$$-\omega_n^2\hat{\xi}_i + 2i\omega_n\epsilon_{ijk}\Omega^j\hat{\xi}^k + \frac{1}{\rho}\nabla_i\delta\hat{p} - \frac{1}{\rho^2}\delta\hat{\rho}\nabla_i p + \nabla_i\delta\tilde{\Phi} = 0, \tag{46}$$

leading to the radial component:

$$\mathcal{E}_r = \frac{1}{r}\sum_l\left[\left(-\omega_n^2 W_l + 2m\omega_n\Omega V_l + r\partial_r\delta\Phi_l + \frac{1}{\rho}r\partial_r\delta p_l - \frac{1}{\rho^2}\delta\rho_l r\partial_r p\right)Y_l^m + 2\omega_n\Omega U_l\sin\theta\partial_\theta Y_l^m\right] = 0, \tag{47}$$

the θ component:

$$\mathcal{E}_\theta = \frac{1}{\sin\theta}\sum_l [-\omega_n^2(V_l \sin\theta\partial_\theta Y_l^m + mU_l Y_l^m) + 2\omega_n\Omega\cos\theta(mV_l Y_l^m + U_l\sin\theta\partial_\theta Y_l^m) + \left(\delta\Phi_l + \frac{\delta p_l}{\rho}\right)\sin\theta\partial_\theta Y_l^m\Bigg] = 0 \tag{48}$$

and the φ component:

$$\mathcal{E}_\varphi = i\sum_l \{-\omega_n^2(mV_l Y_l^m + U_l\sin\theta\partial_\theta Y_l^m) + 2\omega_n\Omega\sin\theta\Bigg[\sin\theta W_l Y_l^m + \cos\theta\Bigg(V_l\partial_\theta Y_l^m + \frac{m}{\sin\theta}U_l Y_l^m\Bigg)\Bigg] + m\left(\delta\Phi_l + \frac{\delta p_l}{\rho}\right)Y_l^m\Bigg\} = 0. \tag{49}$$

One possible strategy would be to work with these equations as they are, deal with the fact that different multipoles couple head-on and solve the problem numerically (see, e.g., Lee 2006). However, this may not be the most "transparent" option as it obscures the nature of the different mode solutions. As we will see later, it follows from the angular components (Equations (48) and (49)) that we have to consider coupling between the l components and the ones for $l \pm 2$. This leads to a larger set of equations to solve, so it makes sense to ask if this coupling can be avoided. It turns out that it cannot, but we can find a somewhat more "intuitive" set of equations to solve.

From the Euler equations it is easy to see that it makes sense to introduce

$$\delta\mathcal{U}_l = \frac{\delta p_l}{\rho} + \delta\Phi_l, \tag{50}$$

(not to be confused with the axial amplitude U_l). This variable is used both in the classic (dimensionless) formulation from Unno et al. (1989) and the two-potential formalism used by, for example, Lindblom & Ipser (1999).

Making use of the standard recurrence relation

$$\sin\theta\partial_\theta Y_l^m = l\mathcal{Q}_{l+1}Y_{l+1}^m - (l+1)\mathcal{Q}_l Y_{l-1}^m, \tag{51}$$

where

$$\mathcal{Q}_l = \left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}, \tag{52}$$

the radial components of the Euler equations leads to

$$\sum_l \Bigg\{\Bigg[-\omega_n^2 W_l + 2m\omega_n\Omega V_l + r\partial_r\delta\mathcal{U}_l + \frac{r}{\rho^2}(\delta p_l\partial_r\rho - \delta\rho_l r\partial_r p)\Bigg]Y_l^m + 2\omega_n\Omega U_l(l\mathcal{Q}_{l+1}Y_{l+1}^m - (l+1)\mathcal{Q}_l Y_{l-1}^m)\Bigg\} = 0. \tag{53}$$

Multiplying this by Y_j^{m} , integrating over the angles and executing the sum over l, we have the recurrence relation:

$$-\omega_n^2 W_j + 2m\omega_n\Omega V_j + r\partial_r\delta\mathcal{U}_j + \frac{r}{\rho^2}(\delta p_j\partial_r\rho - \delta\rho_j r\partial_r p) + 2\omega_n\Omega[(j-1)\mathcal{Q}_j U_{j-1} - (j+2)\mathcal{Q}_{j+1}U_{j+1}] = 0. \tag{54}$$

Next, we consider the combination (the radial component of the vorticity equation)

$$\partial_\varphi\mathcal{E}_\theta - \partial_\theta\mathcal{E}_\varphi = -i\omega_n\sin\theta\sum_l \{[l(l+1)\omega_n - 2m\Omega]U_l Y_l^m - 2\Omega[l(l+1)\cos\theta Y_l^m + \sin\theta\partial_\theta Y_l^m]V_l + 2\Omega[2\cos\theta Y_l^m + \sin\theta\partial_\theta Y_l^m]W_l\} = 0, \tag{55}$$

where we have made use of Legendre's equation,

$$\partial_\theta(\sin\theta\partial_\theta Y_l^m) = \left[\frac{m^2}{\sin\theta} - l(l+1)\sin\theta\right]Y_l^m, \tag{56}$$

to simplify the result.

Using the previous recurrence relation (51), along with

$$\cos\theta Y_l^m = \mathcal{Q}_{l+1}Y_{l+1}^m + \mathcal{Q}_l Y_{l-1}^m, \tag{57}$$

we arrive at the recurrence relation:

$$[j(j+1)\omega_n - 2m\Omega]U_j + 2\Omega(j+1) \times [W_{j-1} - (j-1)V_{j-1}]\mathcal{Q}_j - 2\Omega j[W_{j+1} + (j+2)V_{j+1}]\mathcal{Q}_{j+1} = 0. \tag{58}$$

Keeping Legendre's equation in mind, it may also be useful to consider the combination (Lee & Strohmayer 1996; Glampedakis & Andersson 2006)

$$\partial_\theta(\sin\theta\mathcal{E}_\theta) + \frac{1}{\sin\theta}\partial_\varphi\mathcal{E}_\varphi = \sum_l \Bigg\{\omega_n[l(l+1)\omega_n - 2m\Omega]V_l - 2m\omega_n\Omega W_l - l(l+1)\left(\delta\Phi_l + \frac{\delta p_l}{\rho}\right)\Bigg\}\sin\theta Y_l^m - \sum_l 2\omega_n\Omega\sin\theta[l(l+1)\cos\theta Y_l^m + \sin\theta\partial_\theta Y_l^m]U_l = 0, \tag{59}$$

which leads to

$$\omega_n[j(j+1)\omega_n - 2m\Omega]V_j - 2m\omega_n\Omega W_j - j(j+1)\delta\mathcal{U}_j - 2\omega_n\Omega[(j-1)(j+1)\mathcal{Q}_j U_{j-1} + j(j+2)\mathcal{Q}_{j+1}U_{j+1}] = 0. \tag{60}$$

For inertial modes, it is notable that the radial vorticity Equation (58) links only variables that have a leading-order contribution. Another such relation follows from the θ

component of the vorticity equation:

$$\begin{aligned}\partial_r\mathcal{E}_\varphi - \partial_\varphi\mathcal{E}_r &= \omega_n[2\Omega(1-\mathcal{Q}_j^2-\mathcal{Q}_{j+1}^2)r\partial_r W_j + m\omega_n W_j]\\ &-\omega_n[\{m\omega_n + 2\Omega[(j+1)\mathcal{Q}_j^2 - j\mathcal{Q}_{j+1}^2]\}r\partial_r V_j + 2m^2\Omega V_j]\\ &-\omega_n[(\omega_n(j-1)-2m\Omega)r\partial_r U_{j-1} + 2m(j-1)\Omega U_{j-1}]\mathcal{Q}_j\\ &+\omega_n\{[\omega_n(j+2)+2m\Omega]r\partial_r U_{j+1} + 2m(j+2)\Omega U_{j+1}\}\mathcal{Q}_{j+1}\\ &+2\omega_n\Omega[(j-2)r\partial_r V_{j-2} - r\partial_r W_{j-2}]\mathcal{Q}_{j-1}\mathcal{Q}_j\\ &-2\omega_n\Omega[(j+3)r\partial_r V_{j+2} + r\partial_r W_{j+2}]\mathcal{Q}_{j+2}\mathcal{Q}_{j+1}\\ &=-\frac{mr}{\rho^2}(\delta\rho_j\partial_r p - \delta p_j\partial_r\rho).\end{aligned} \tag{61}$$

This equation is identical to Equation (39) from Lockitch and Friedman, apart from the right-hand side, which only vanishes for barotropic stars. In general, the equation is a bit messy as it couples radial derivatives of all displacement components and it also involves the $l \pm 2$ multipoles. However, we will find the equation useful for nonbarotropic stars satisfying the traditional r-mode slow-rotation ordering as it simplifies considerably in that case. Finally, we also need the perturbed Poisson equation for the gravitational potential. However, we will make the Cowling approximation in our explicit examples (set $\delta\Phi_j = 0$) so will not give the equation here.

4. The "Traditional" r-modes

As already asserted, we will focus on low-frequency modes such that $\Omega \gtrsim \omega_0$. This includes high-order gravity g-modes and inertial modes. We have already seen that the perturbed Euler equations then imply that we must have $[\delta\rho_l, \delta p_l, \delta\Phi_l] \sim \mathcal{O}(\Omega^2)$. Moreover, the continuity equation requires the polar components W_l and V_l to be of the same order, and we already know that if we consider a strongly stratified star (with $\hat{\epsilon} \gg \epsilon$) then we must have $[W_l, V_l] \sim \mathcal{O}(\Omega^2)$. We are then left to consider if it is possible to combine these assumptions with $U_l \sim \mathcal{O}(1)$. The answer to this question is affirmative, but it follows from Equation (58) that we must then have

$$\sum_l [l(l+1)\omega_n - 2m\Omega]U_l Y_l^m = 0. \tag{62}$$

That is, for given values of $l = l'$ (say) and m, we may have $U_{l'} \neq 0$ as long as the leading-order mode frequency is given by

$$\omega_0 = \frac{2m}{l'(l'+1)}. \tag{63}$$

These are the r-modes (Papaloizou & Pringle 1978; Provost et al. 1981; Saio 1982). In addition to the leading-order displacement they will have polar components as well as other axial multipoles $U_{l\neq l'}$, but these enter at $\mathcal{O}(\Omega^2)$. We will deliberate on these contributions in the following.

With our conventions, the pattern speed of a mode is $-\omega/m$. Therefore, all r-modes travel in the same direction across the star (retrograde with respect to spin, in the rotating frame). We also note that there are no axisymmetric r-modes; we must have $m \neq 0$.

Focusing on the r-mode problem, we consider the perturbations for mode solutions such that $\omega_n \sim \mathcal{O}(\Omega)$ and

$$\begin{aligned}&U_{l=l'} \sim \mathcal{O}(1), \quad [W_l, V_l, \delta\rho_l, \delta p_l] \sim \mathcal{O}(\Omega^2),\\ &U_l \sim \mathcal{O}(\Omega^2) \text{ for } l \neq l'.\end{aligned} \tag{64}$$

As before, we use tildes to identify terms that enter at order Ω^2, i.e.,

$$\delta p_l = \epsilon^2\delta\tilde{p}_l = \left(\frac{\Omega}{\Omega_0}\right)^2\delta\tilde{p}_l, \tag{65}$$

with $\delta\tilde{p}_l \sim \mathcal{O}(1)$ by definition. In addition, we need to keep track of the rotational correction to the frequency, so recall Equation (33), from which it is worth noting that assumed slow-rotation ordering is only valid as long as

$$\frac{\tilde{\omega}_2}{\omega_0}\epsilon^2 \ll 1. \tag{66}$$

With these assumptions, the continuity Equation (45) relates order Ω^2 quantities, and we have

$$\begin{aligned}\frac{1}{r^2}\partial_r(r\rho\tilde{W}_j) - \frac{j(j+1)\rho}{r^2}\tilde{V}_j = -\frac{1}{c_s^2}\delta\tilde{p}_j\\ + \frac{\mathcal{N}^2}{g^2}\left(\delta\tilde{p}_j - \frac{\rho g}{r}\tilde{W}_j\right).\end{aligned} \tag{67}$$

So far, the different relations only involve single-multipole polar components. This changes when we turn to the perturbed Euler equations.

Let us first consider the radial component of the vorticity Equation (58). We know already that at leading order we may have a single axial contribution $U_{l'} \neq 0$ as long as the frequency is given by Equation (63). However, at this point we cannot determine the axial eigenfunction. Essentially, the leading-order r-mode solutions are degenerate. To break this degeneracy, we need to go to higher orders, keeping in mind that the polar contributions enter at order Ω^2.

Equation (58) provides a recurrence relation involving these multipole contributions. At order Ω^3 we have (with $j = l'$)

$$\begin{aligned}&l'('l+1)\tilde{\omega}_2 U_{l'} + 2(l'+1)[\tilde{W}_{l'-1} - (l'-1)\tilde{V}_{l'-1}]\mathcal{Q}_{l'}\\ &- 2l'[\tilde{W}_{l'+1} + (l'+2)\tilde{V}_{l'+1}]\mathcal{Q}_{l'+1} = 0.\end{aligned} \tag{68}$$

This is the only relation we get that involves the leading-order eigenfunction and the frequency correction $\tilde{\omega}_2$. However, for $j = l' + 2$ we get from Equation (58):

$$\begin{aligned}&[(l'+2)(l'+3)\omega_0 - 2m]\tilde{U}_{l'+2}\\ &+ 2(l'+3)[\tilde{W}_{l'+1} - (l'+1)\tilde{V}_{l'+1}]\mathcal{Q}_{l'+2}\\ &- 2(l'+2)[\tilde{W}_{l'+3} + (l'+4)\tilde{V}_{l'+3}]\mathcal{Q}_{l'+3} = 0,\end{aligned} \tag{69}$$

while $j = l' - 2$ leads to

$$[(l'-2)(l'-1)\omega_0 - 2m]\tilde{U}_{l'-2} + 2(l'-1)[\tilde{W}_{l'-3} - (l'-3)\tilde{V}_{l'-3}]\mathcal{Q}_{l'-2} - 2(l'-2)[\tilde{W}_{l'-1} + l'\tilde{V}_{l'-1}]\mathcal{Q}_{l'-1} = 0. \quad (70)$$

From these relations, the pattern is clear. At order Ω^2, the r-mode solution involves a number of multipoles. The question then becomes, does this sequence truncate? To answer this question, first note that $\mathcal{Q}_m = 0$, which helps establish the lowest-order term in the series. There are three different options for the lowest-order multipole contribution, associated with different classes of mode solutions. First, we may have $l' = m$ in Equation (68). In this case, the (leading order) $U_{l'=m}$ term corresponds to the lowest-order multipole in the solution. In the language of Lockitch & Friedman (1999), the mode is axial led. This case corresponds to the traditional $l' = m$ r-mode (Papaloizou & Pringle 1978). Another option would be to have $l' = m + 2$ in Equation (70). This would also lead to an axial-led mode, but now the lowest multipole is given by $\tilde{U}_{l'-2} \sim \mathcal{O}(\Omega^2)$. A third option follows by setting $l' = m + 1$ in Equation (70), which then decouples and from Equation (68) we arrive at a polar-led mode with the lowest multipole contributions given by $[\tilde{W}_{l'-1}, \tilde{V}_{l'-1}]$ (again at order Ω^2). The main lesson here is that the nature of the r-modes is quite similar to that of the general inertial modes discussed by Lockitch & Friedman (1999): each mode has several multipole contributions, but the $U_{l'}$ term is elevated above the other contributions in the slow-rotation expansion. The close relation between the two problems may not been very clearly explained in the existing literature. It is, however, important for what follows.

In order to complete the formulation of the problem, we will use the other vorticity Equation (61). With the ordering we have, at order Ω^2, this relation reduces to

$$-\omega_0\{[\omega_0(j-1) - 2m]r\partial_r U_{j-1} + 2m(j-1)U_{j-1}\}\mathcal{Q}_j + \omega_0\{[\omega_0(j+2) + 2m]r\partial_r U_{j+1} + 2m(j+2))U_{j+1}\}\mathcal{Q}_{j+1} + \frac{mr}{\rho g}\frac{\mathcal{N}^2}{\Omega_0^2}\left(\delta\tilde{p}_j - \frac{\rho g}{r}\tilde{W}_j\right) = 0. \quad (71)$$

From this relation, we infer two relations involving the leading-order $U_{l'}$ term. First, with $j = l' + 1$ we have

$$\left(\frac{2m}{l'+1}\right)^2 \mathcal{Q}_{l'+1}[r\partial_r U_{l'} - (l'+1)U_{l'}] + \frac{mr}{\rho g}\frac{\mathcal{N}^2}{\Omega_0^2}\left(\delta\tilde{p}_{l'+1} - \frac{\rho g}{r}\tilde{W}_{l'+1}\right) = 0. \quad (72)$$

Second, with $j = l' - 1$ we get

$$\left(\frac{2m}{l'}\right)^2 \mathcal{Q}_{l'}[r\partial_r U_{l'} + l' U_{l'}] + \frac{mr}{\rho g}\frac{\mathcal{N}^2}{\Omega_0^2}\left(\delta\tilde{p}_{l'-1} - \frac{\rho g}{r}\tilde{W}_{l'-1}\right) = 0. \quad (73)$$

Finally, we have the divergence Equation (60), which leads to

$$-j(j+1)\delta\tilde{\mathcal{U}}_j - 2\omega_0\Omega_0^2[(j-1)(j+1)\mathcal{Q}_j U_{j-1} + j(j+2)\mathcal{Q}_{j+1}U_{j+1}] = 0. \quad (74)$$

For $j = l' + 1$ we have

$$\delta\tilde{\mathcal{U}}_{l'+1} = -\frac{2\omega_0 l'}{(l'+1)}\Omega_0^2\mathcal{Q}_{l'+1}U_{l'} = -\frac{4m\Omega_0^2}{(l'+1)^2}\mathcal{Q}_{l'+1}U_{l'}, \quad (75)$$

while $j = l' - 1$ leads to

$$\delta\tilde{\mathcal{U}}_{l'-1} = -\frac{2\omega_0(l'+1)}{(l')}\Omega_0^2\mathcal{Q}_{l'}U_{l'} = -\frac{4m\Omega_0^2}{(l')^2}\mathcal{Q}_{l'}U_{l'}. \quad (76)$$

In essence, if we want to determine the leading-order eigenfunction and the frequency correction $\tilde{\omega}_2$, we need to solve a coupled system for $U_{l'}$ and $\tilde{W}_{l'\pm1}$. The other contributions to the mode solution (like $\tilde{U}_{l'\pm2}$) can be calculated as a second step.

Finally, the mode solution must satisfy the condition that the Lagrangian perturbation in the pressure vanishes at the surface. That is, we require

$$\Delta\tilde{p}_l = \delta\tilde{p}_l - \frac{\rho g}{r}\tilde{W}_l = 0 \quad \text{at} \quad r = R. \quad (77)$$

At this point, we may return to the question of whether the multipole sum truncates for the r-modes. First, the relation (71) also tells us, for $j = l' + 3$ and $j = l' - 3$, respectively, that we must have (as long as $\mathcal{N}^2 \neq 0$)

$$\delta\tilde{p}_{l'\pm3} - \frac{\rho g}{r}\tilde{W}_{l'\pm3} = \Delta\tilde{p}_{l'\pm3} = 0. \quad (78)$$

Second, in Equations (75) and (76), we can use $j = l' \pm 2$ to show that we must have

$$\delta\tilde{\mathcal{U}}_{l'\pm3} = \delta\tilde{\Phi}_{l'\pm3} + \frac{\delta\tilde{p}_{l'\pm3}}{\rho} = 0. \quad (79)$$

Third, with the assumed r-mode ordering, the radial component of the Euler Equations (54) leads to

$$r\partial_r\delta\tilde{\mathcal{U}}_j + 2\omega_0\Omega_0^2[(j-1)\mathcal{Q}_j U_{j-1} - (j+2)\mathcal{Q}_{j+1}U_{j+1}] = -\frac{r}{\rho^2}(\delta\tilde{p}_j\partial_r\rho - \delta\tilde{\rho}_j r\partial_r p) = \frac{\mathcal{N}^2 r}{g}\left(\frac{1}{\rho}\delta\tilde{p}_j - \frac{g}{r}\tilde{W}_j\right) = \frac{\mathcal{N}^2 r}{g}\left(\delta\tilde{\mathcal{U}}_j - \delta\tilde{\Phi}_j - \frac{g}{r}\tilde{W}_j\right), \quad (80)$$

so, for $j = l' + 1$, and making use of Equation (75), we have

$$r\partial_r\delta\tilde{\mathcal{U}}_{l'+1} - \left[(l'+1) + \frac{\mathcal{N}^2 r}{g}\right]\delta\tilde{\mathcal{U}}_{l'+1} = -\frac{\mathcal{N}^2 r}{g}\left(\delta\tilde{\Phi}_{l'+1} + \frac{g}{r}\tilde{W}_{l'+1}\right). \quad (81)$$

It also follows that

$$\delta\tilde{\Phi}_{l'+3} + \frac{g}{r}\tilde{W}_{l'+3} = 0. \quad (82)$$

Similarly, with $j = l' - 1$ we get

$$r\partial_r \delta\tilde{\mathcal{U}}_{l'-1} + \left[l' - \frac{\mathcal{N}^2 r}{g}\right]\delta\tilde{\mathcal{U}}_{l'-1} = -\frac{\mathcal{N}^2 r}{g}\left(\delta\tilde{\Phi}_{l'-1} + \frac{g}{r}\tilde{W}_{l'-1}\right) \quad (83)$$

and we also have

$$\delta\tilde{\Phi}_{l'-3} + \frac{g}{r}\tilde{W}_{l'-3} = 0. \quad (84)$$

Combining the results, we see that we must have $\tilde{W}_{l'\pm3} = \delta\tilde{p}_{l'\pm3} = \delta\tilde{\Phi}_{l'\pm3} = 0$. Finally, the continuity equation leads to $\tilde{V}_{l'\pm3} = 0$, while $\delta\tilde{\rho}_{l'\pm3} = 0$ follows from the equation of state relation. Is essence, all polar $l' \pm 3$ multipole contributions will vanish. In turn, this means that a general r-mode must truncate with the $\tilde{U}_{l'\pm2}$ terms obtained from Equations (69) and (70). This accords with the discussion in Smeyers et al. (1981) and Smeyers & Martens (1983).

4.1. The $l' = m$ Modes

Having written down the equations we need to solve to determine the frequency correction $\tilde{\omega}_2$ and the multipole structure of an r-mode to order Ω^2—notably without any simplifying assumptions other than neglecting the rotational change in shape of the star—we have a decision to make. Do we want to consider a model that is as "realistic" as possible—which will require a numerical solution—or are we more focused on the formal structure of the mode solution? The initial answer is quite simple. As we are not including the rotational shape change it is natural to focus on the qualitative nature of the solution. This leads us to the question of which further simplifying assumptions we may consider.

As already advertised, we are now going to make the Cowling approximation. That is, we assume that $\delta\tilde{\Phi}_l = 0$. For the problem at hand, this is pragmatic (as we are focusing on qualitative aspects) and reasonable (as we do not have to solve the Poisson equation for the perturbed gravitational potential). We want to keep the problem tractable enough that we may proceed to solve it by analytic means. We also know from available numerical results that the r-modes are determined with reasonable precision within this approximation (at least in the context of Newtonian gravity). In the Cowling approximation, we have

$$\delta\tilde{\mathcal{U}}_{l\pm1} = \frac{\delta\tilde{p}_{l\pm1}}{\rho}. \quad (85)$$

In the $l' = m$ case, we have $\mathcal{Q}_{l'=m} = 0$, which means that we only need to consider the coupling between the leading-order $U_{l'}$ and the polar $l' + 1$ contributions. (The axial second order contribution $\tilde{U}_{l'+2}$ can be calculated at a second stage.) The set of equations to consider now are as follows.

(i) The continuity Equation (67):

$$\frac{1}{r^2}\partial_r(r\rho\tilde{W}_{l'+1}) - \frac{(l'+1)(l'+2)\rho}{r^2}\tilde{V}_{l'+1} = -\frac{1}{c_s^2}\delta\tilde{p}_{l'+1} + \frac{\mathcal{N}^2}{g^2}\left(\delta\tilde{p}_{l'+1} - \frac{\rho g}{r}\tilde{W}_{l'+1}\right). \quad (86)$$

(ii) The differential Equation (72):

$$r\partial_r U_{l'} - (l'+1)U_{l'} = -\frac{(l'+1)^2}{4m\mathcal{Q}_{l'+1}}\frac{r}{\rho g}\frac{\mathcal{N}^2}{\Omega_0^2} \times \left(\delta\tilde{p}_{l'+1} - \frac{\rho g}{r}\tilde{W}_{l'+1}\right). \quad (87)$$

Along with (iii) the algebraic relation (68):

$$\frac{(l'+2)\rho}{r}\tilde{V}_{l'+1} = \frac{(l'+1)}{2\mathcal{Q}_{l'+1}}\frac{\rho}{r}\tilde{\omega}_2 U_{l'} - \frac{\rho}{r}\tilde{W}_{l'+1}. \quad (88)$$

(iv) The relation (75):

$$\frac{\delta\tilde{p}_{l'+1}}{\rho} = -\frac{4m\Omega_0^2}{(l'+1)^2}\mathcal{Q}_{l'+1}U_{l'}. \quad (89)$$

And the surface boundary condition (77) (obviously). It is easy to see that we end up with two coupled first-order equations for $U_{l'}$ and $\tilde{W}_{l'+1}$.

It is instructive to introduce $U_{l'} = r^{l'+1}\bar{U}_{l'}$ and rewrite Equation (87) as

$$r\partial_r\bar{U}_{l'} = \frac{r}{\rho g}\mathcal{N}^2\bar{U}_{l'} + \frac{(l'+1)^2}{4m\mathcal{Q}_{l'+1}}\frac{\mathcal{N}^2}{\Omega_0^2}\tilde{W}_{l'+1}. \quad (90)$$

It follows immediately that, in the barotropic limit (when $\mathcal{N}^2 \to 0$ for a fixed Ω), we must have

$$\bar{U}_{l'} = \text{constant} \Longrightarrow U_{l'} = r^{l'+1}. \quad (91)$$

The only alternative would be for

$$\frac{\mathcal{N}^2}{\Omega_0^2}\tilde{W}_{l'+1} = \frac{\mathcal{N}^2}{\Omega^2}W_{l'+1} \quad (92)$$

to remain finite in the barotropic limit. However, this would violate the assumed slow-rotation ordering for the r-mode solution. That this happens should not be a surprise given the general discussion in Section 2. The result is simply an illustration of the fact that we need to make different assumptions in barotropic regions of the star.

In general, we need to solve Equation (90) along with the continuity Equation (86), which becomes

$$\frac{1}{r^2}\partial_r(r\rho\tilde{W}_{l'+1}) + \left[\frac{(l'+1)}{r^2} + \frac{\mathcal{N}^2}{rg}\right]\rho\tilde{W}_{l'+1} = \left[\frac{(l'+1)^2}{2\mathcal{Q}_{l'+1}^2}\frac{\tilde{\omega}_2}{r^2} - \frac{4m\Omega_0^2}{(l'+1)^2}\frac{1}{c_s^2}\left(1 - \frac{\mathcal{N}^2 c_s^2}{g^2}\right)\right]\rho\mathcal{Q}_{l'+1}r^{l'+1}\bar{U}_{l'}. \quad (93)$$

The two equations (plus the boundary conditions) constitute a Sturm–Liouville problem (Provost et al. 1981) so we expect to have an infinite set of eigenvalues (see Saio 1982 and Gittins & Andersson 2023 for indicative results for the r-mode overtones). However, as we have seen, the problem changes in barotropic limit. The overtones disappear and we are left with a single r-mode, represented by Equation (91).

As a simple example of the single remaining r-mode, we may consider an incompressible barotropic star, for which $\rho =$ constant, so $c_s^2 \to \infty$, and we are left with

$$\partial_r(r^{l'+2}\tilde{W}_{l'+1}) = \frac{(l'+1)^2}{2\mathcal{Q}_{l'+1}}\tilde{\omega}_2 r^{2(l'+1)}\bar{U}_{l'}, \tag{94}$$

with $\bar{U}_{l'} = A =$ constant. This leads to

$$\tilde{W}_{l'+1} = \frac{(l'+1)^2}{2(2l'+3)\mathcal{Q}_{l'+1}}\tilde{\omega}_2 A r^{l'+1}, \tag{95}$$

and we also have

$$\delta\tilde{p}_{l'+1} = -\frac{4m\Omega_0^2}{(l'+1)^2}\mathcal{Q}_{l'+1}\rho r^{l'+1}A. \tag{96}$$

Finally, the surface boundary condition becomes

$$-\frac{4m}{(l'+1)^2}\mathcal{Q}_{l'+1} - \frac{(l'+1)^2}{2(2l'+3)\mathcal{Q}_{l'+1}}\tilde{\omega}_2 = 0, \tag{97}$$

so we arrive at

$$\tilde{\omega}_2 = -\frac{8m}{(l'+1)^4}, \tag{98}$$

since

$$Q_m^2 = \frac{1}{2m+3} \tag{99}$$

when $l' = m$. We briefly compare this result to available results from the literature in the Appendix.

In summary, our arguments clearly illustrate the known fact that the nature of the $l' = m$ r-mode problem changes as stratification weakens. The evidence is clear. We have to approach the $\mathcal{N}^2 \to 0$ limit with care. In fact, for neutron stars the problem is particularly intricate. Taking the results in Figure 1 at face value, we always have to assume the region close to the surface of the star to be barotropic, while the high-density region may not be. This, in turn, means that the assumed slow-rotation ordering associated with the nonbarotropic r-mode must break (as $\hat{\epsilon} \lesssim \epsilon$ close to the star's surface). In effect, the formulation of the problem—as we have presented it—is not consistent. This presents a technical challenge as the solution needs to smoothly join the stratified region where the stratified assumptions hold with a barotropic region where the equation becomes those associated with the general inertial modes. As far as we are aware, this problem has not been considered (at least not for neutron stars), although the required strategy—basically abandoning the slow-rotation ordering for the perturbation—has been developed and employed to good effect for main-sequence stars (Lee 2006).

4.2. The $l' \neq m$ Modes

Let us now turn to the $l' \neq m$ r-modes. In general, we then need to consider the continuity equations for the polar $l' \pm 1$ contributions. In particular, we need to solve the differential Equations (72) and (73):

$$r\partial_r U_{l'} - (l'+1)U_{l'} = -\frac{(l'+1)^2}{4m\mathcal{Q}_{l'+1}}\frac{r}{\rho g}\frac{\mathcal{N}^2}{\Omega_0^2}\left(\delta\tilde{p}_{l'+1} - \frac{\rho g}{r}\tilde{W}_{l'+1}\right), \tag{100}$$

and

$$r\partial_r U_{l'} + l'U_{l'} = -\frac{(l')^2}{4m\mathcal{Q}_{l'}}\frac{r}{\rho g}\frac{\mathcal{N}^2}{\Omega_0^2}\left(\delta\tilde{p}_{l'-1} - \frac{\rho g}{r}\tilde{W}_{l'-1}\right). \tag{101}$$

It is easy to see that we run into trouble in the barotropic limit. Combining Equations (100) and (101) we have an algebraic relation:

$$(2l'+1)U_{l'} = -\frac{1}{4m}\frac{\mathcal{N}^2}{\Omega_0^2}\left[\frac{(l'+1)^2}{\mathcal{Q}_{l'+1}}\tilde{W}_{l'+1} - \frac{(l')^2}{\mathcal{Q}_{l'}}\tilde{W}_{l'-1}\right], \tag{102}$$

or, alternatively,

$$\tilde{W}_{l'-1} = \frac{\mathcal{Q}_{l'}}{(l')^2}\left[(2l'+1)4m\frac{\Omega_0^2}{\mathcal{N}^2}U_{l'} + \frac{(l'+1)^2}{\mathcal{Q}_{l'+1}}\tilde{W}_{l'+1}\right]. \tag{103}$$

The first relation shows that, if $\tilde{W}_{l'\pm1}$ remain finite then we must have $U_{l'} \to 0$ in the barotropic limit. This would be incompatible with Equation (100). We get a hint of the resolution to the problem from the alternative version, which suggests that if we insist that $U_{l'}$ remains $\mathcal{O}(1)$ when $\mathcal{N}^2 \to 0$, then $\tilde{W}_{l'-1}$ must diverge.

The unavoidable conclusion is that the $l' \neq m$ r-modes cannot exist in the barotropic limit—as expected from the arguments by Lockitch & Friedman (1999). If $\mathcal{N}^2 \to 0$ for a fixed rotation rate Ω, then the assumed r-mode ordering must break. In fact, if $\mathcal{N}^2 = 0$ at some point in the star we have a problem. Given the available equations there does not seem to be a way to avoid dividing by $\mathcal{N}^2$ so the problem will be singular.

In summary, while the stratified problem can be solved for $l' \neq m$ r-modes (Saio 1982), the solution does not apply for realistic neutron star models; see Gittins & Andersson (2023) for related numerical results. If we want to consider the actual problem, then we have to rethink our strategy. This again suggests that we may need to abandon the slow-rotation ordering and tackle the general problem numerically (as in the body of work by Lee and collaborators; Lee & Saio 1987, 1997; Lee & Baraffe 1995; Yoshida & Lee 2000a, 2000b; Lee 2006).

5. A Physically Motivated Alternative

Based on the stratification results from Figure 1, it makes sense—for the fastest spinning stars—to consider the stratification to be second order in the slow-rotation expansion. We then have Equation (38), which leads to (at order ϵ^2)

$$c_s^2 \delta\tilde{\rho}_j = \delta\tilde{p}_j + \frac{\mathcal{N}^2 c_s^2 \rho}{rg} W_j, \tag{104}$$

where $W_j \sim \mathcal{O}(1)$. This suggests that we change the assumed ordering in such a way that $W_j \to W_j + \epsilon^2 \tilde{W}_j$ and similar for V_j. The axial displacement remains as before. With the polar displacement components present already at leading order, the problem is close to that for a general inertial mode.

With these assumptions, the leading-order continuity equation requires

$$\frac{1}{r^2}\partial_r(r\rho W_j) - j(j+1)\frac{\rho}{r^2}V_j = 0. \tag{105}$$

The radial vorticity equation leads to, at order ϵ:

$$[j(j+1)\omega_0 - 2m]U_j + 2(j+1)[W_{j-1} - (j-1)V_{j-1}]\mathcal{Q}_j - 2j[W_{j+1} + (j+2)V_{j+1}]\mathcal{Q}_{j+1} = 0. \tag{106}$$

It is also convenient to use the algebraic relation from the divergence equation, which at order ϵ^2 provides

$$\omega_0[j(j+1)\omega_0 - 2m]V_j - 2m\omega_0 W_j - j(j+1)\delta\tilde{\mathcal{U}}_j - 2\omega_0[(j-1)(j+1)\mathcal{Q}_j U_{j-1} + j(j+2)\mathcal{Q}_{j+1}U_{j+1}] = 0. \tag{107}$$

Finally, in this case it seems natural (given that the horizontal vorticity equation involves derivatives of all three displacement components) to use the radial Euler equation, which leads to, at order ϵ^2:

$$\begin{aligned} &-\omega_0^2 W_j + 2m\omega_0 V_j + r\partial_r \delta\tilde{\mathcal{U}}_j \\ &\quad +2\omega_0[(j-1)\mathcal{Q}_j U_{j-1} - (j+2)\mathcal{Q}_{j+1}U_{j+1}] \\ &= -\frac{r}{\rho^2\Omega_0^2}(\delta\tilde{\rho}_j \partial_r \rho - \delta\tilde{\rho}_j \partial_r p) \\ &= \frac{r}{\rho^2\Omega_0^2}\frac{d\rho}{dr}(c_s^2\delta\tilde{\rho}_j - \delta\tilde{p}_j) = \frac{\mathcal{N}^2}{\Omega_0^2}W_j. \end{aligned} \tag{108}$$

Finally, in the Cowling approximation we have Equation (85) and we also need Equation (61), which at order ϵ^2 leads to

$$\begin{aligned} &\omega_0[2(1 - \mathcal{Q}_j^2 - \mathcal{Q}_{j+1}^2)r\partial_r W_j + m\omega_0 W_j] \\ &- \omega_0[\{m\omega_0 + 2[(j+1)\mathcal{Q}_j^2 - j\mathcal{Q}_{j+1}^2]\}r\partial_r V_j + 2m^2 V_j] \\ &- \omega_0[(\omega_0(j-1) - 2m)r\partial_r U_{j-1} + 2m(j-1)U_{j-1}]\mathcal{Q}_j \\ &+ \omega_0\{[\omega_0(j+2) + 2m]r\partial_r U_{j+1} + 2m(j+2)U_{j+1}\}\mathcal{Q}_{j+1} \\ &+ 2\omega_0[(j-2)r\partial_r V_{j-2} - r\partial_r W_{j-2}]\mathcal{Q}_{j-1}\mathcal{Q}_j \\ &- 2\omega_0[(j+3)r\partial_r V_{j+2} + r\partial_r W_{j+2}]\mathcal{Q}_{j+2}\mathcal{Q}_{j+1} \\ &= -\frac{mr}{\rho^2\Omega_0^2}(\delta\tilde{\rho}_j\partial_r p - \delta\tilde{p}_j\partial_r\rho) = -\frac{m\mathcal{N}^2}{\Omega_0^2}W_j. \end{aligned} \tag{109}$$

Let us focus on the problem we would have to solve in order to identify a solution "close to" the traditional r-mode. That is, we are looking for modes such that $U_{l'} \sim \mathcal{O}(1)$ with $l' = m$ and with a frequency given by Equations (33) and (63). With the usual ordering for stratified stars, this would include the r-mode overtones. The only difference here is that we are no longer (necessarily) assuming that the polar displacement contributions enter at higher slow-rotation order. The equations that involve $U_{l'}$ are then, first of all, the leading-order relation:

$$[l'(l'+1)\omega_0 - 2m]U_{l'} - 2l'[W_{l'+1} + (l'+2)V_{l'+1}]\mathcal{Q}_{l'+1} = 0, \tag{110}$$

which, for a mode with the usual leading-order r-mode frequency (Equation (63)), reduces to

$$-2l'[W_{l'+1} + (l'+2)V_{l'+1}]\mathcal{Q}_{l'+1} = 0. \tag{111}$$

This can be combined with the leading-order continuity equation to give

$$\partial_r(\rho W_{l'+1}) + (l'+2)\rho W_{l'+1} = 0, \tag{112}$$

which leads to, with A constant,

$$r^{l'+2}\rho W_{l'+1} = A \Longrightarrow W_{l'+1} = \frac{A}{\rho r^{l'+2}}. \tag{113}$$

This is clearly problematic as the solution diverges at the center of the star (and at the surface as well, if $\rho \to 0$ as $r \to R$). The only way to avoid trouble is to have the trivial solution, $A = 0$, and move on to the equations for the higher-order terms. If we do this then we immediately see that the problem is *identical* to the one we already solved for barotropic stars. There will be a single r-mode for each $l' = m$. This tells us that the frequencies of the r-mode overtones can no longer be given by Equation (33). As expected—and in accordance with the results from Figure 4 of Yoshida & Lee (2000a)—they have to change character.

Next, for $l' \neq m$ it is easy to see that the equations we are now considering still lead to a singular problem *unless* the polar components W_j and V_j are $\mathcal{O}(1)$. This is as expected: We need to consider solutions close to the barotropic inertial modes.

Finally, for the modes of the fastest spinning neutron stars we see that the problem (to leading order) is very close to the inertial-mode problem as formulated by Lockitch & Friedman (1999). The only difference is the right-hand side of Equation (109). In essence, the problem we need to consider if we want to establish the astrophysical role of the gravitational-wave driven r-mode instability is close, but not identical, to the inertial-mode problem. As far as we are aware, this problem has not been stated despite the numerous discussions of the r-mode instability in the literature. This problem clearly needs further attention and our intention is to approach it numerically (also accounting for the rotational shape corrections, following the strategy outlined in Gittins & Andersson 2023) in the near future.

6. Conclusions and Outlook

We have revisited the problem of inertial r-modes in stratified neutron stars. Our motivation for this was twofold. First, we wanted to add realism to the discussion by introducing a more precise description of the composition stratification in a mature neutron star. Our analysis of the problem highlights issues with the traditional approach to the problem. In order to account for the expected variation of the internal composition stratification with density, we need to rethink the computational strategy for determining the r-modes. There appears to be two strategies for dealing with this problem. The first would simply involve introducing the standard slow-rotation expansion for the perturbation and approach the problem numerically from the outset. This approach has been championed in a series of papers (albeit not for realistic neutron star stratification) by Lee and colleagues (Lee & Saio 1987, 1997; Lee & Baraffe 1995; Yoshida & Lee 2000a, 2000b; Lee 2006). Our discussion suggests this may be the only viable alternative for moderate-to-slow rotating neutron stars. An alternative approach would be to focus on the fastest (known) spinning stars. For these, the stratification is expected to be relatively weak and the slow-rotation expansion is (again) viable. We have shown that this leads to a problem close to that for inertial modes, as formulated by Lockitch & Friedman (1999).

This brings us to our second—somewhat deeper—motivation. We wanted to shed light on the (still unresolved) problem of r-modes in stratified relativistic stars (Kojima 1998; Beyer & Kokkotas 1999; Kojima & Hosonuma 1999; Yoshida & Lee 2002; Lockitch et al. 2004; Pons et al. 2005; Kraav et al. 2021, 2022). In this context, our analysis also suggests issues with the standard formulation of the problem. We expect that the long-standing issue of a singularity associated with internal corotation points will be resolved once the r-mode problem is reformulated as a generalized inertial-mode problem (in the spirit of the discussion in Section 5 above). This is likely to lead to mode solutions fairly close to the inertial modes and hence results similar to those of Lockitch et al. (2000, 2003), Ruoff et al. (2003), and Idrisy et al. (2015). The latter may be particularly important as the modes are determined for realistic (barotropic) equations of state. If it turns out to be the case that the mode solutions shift only slightly once we account for the stratification then the result we need for (say) gravitational-wave searches may already be at hand. Of course, at this point this is speculation. There are calculations to be done in order to verify the assertion. In addition, the implications of our discussion for the range of problems where the r-modes are thought to play a role, from limiting the spin of neutron stars to the dynamical tide in a neutron star binary, also remain to be explored.

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Appendix
Comparing to the Literature

As a slight addendum to our discussion, we worked out the order Ω^2 frequency correction for the single r-mode that remains (for $l' = m$) in barotropic stars. It is interesting to compare the result to similar results in the literature. From Equation (98) we have (with $l' = m$)

$$\tilde{\omega}_2 = -\frac{8m}{(m+1)^4}. \tag{A1}$$

As our calculation assumed constant density, it is natural to first compare to the results from Kokkotas & Stergioulas (1999). Working with the equations from Saio (1982), hence including the rotational change in shape, they arrive at (with our conventions)

$$\tilde{\omega}_2 = \frac{5m}{(m+1)^2}. \tag{A2}$$

Evidently, the result is different from ours. The first clue to the origin of the difference comes once we note that the Kokkotas & Stergioulas (1999) frequency correction vanishes if we ignore the rotational change in shape. This suggests that we are not comparing like-for-like. This becomes apparent when we turn to the results from the Appendix of Provost et al. (1981). They have

$$\tilde{\omega}_2 = -\frac{8}{(m+1)^4} + \frac{5m}{(m+1)^2}. \tag{A3}$$

Here we recognize the second term as the result from Kokkotas & Stergioulas (1999). In essence, this is the rotationally induced frequency correction. This seems quite intuitive. Of course, the first term from the Provost et al. (1981) result still differs from ours (there is a missing factor of m). While we have not been able to pinpoint the origin of the discrepancy, we have reworked the calculation from Provost et al. (1981) and the result we get accords with Equation (A1).

For constant-density stars it turns out to be straightforward to account for the rotational change in shape. Following the strategy from Saio (1982) we note that that shape correction only impacts on the surface boundary condition. Working this out, we arrive at (still in the Cowling approximation)

$$\tilde{\omega}_2 = -\frac{8m}{(m+1)^4} + \frac{5m}{(m+1)^2}. \tag{A4}$$

This agree with the identification of Equation (A2) as the rotational correction to the r-mode.

It is also relativity easy to relax the Cowling approximation. Again, for constant-density stars this only affects the surface boundary condition. Working this out, we find that the inclusion of the perturbed gravitational potential adds a multiplicative factor to the frequency correction. Instead of Equation (A1), we get

$$\tilde{\omega}_2 = -\frac{8m}{(m+1)^4}\left(\frac{2m+3}{2m}\right). \tag{A5}$$

Now, Provost et al. (1981) state that their result follows if one expands the result from Bryan (1889) in spherical harmonics. This assertion is difficult to confirm, but it is supported by the result from Lindblom & Ipser (1999). Their result leads to

$$\tilde{\omega}_2 = -\frac{4(2m+3)}{m(m+1)^4} + \frac{5m}{(m+1)^2}. \tag{A6}$$

Again, we recognize the shape correction. Assuming that the factor due to the Cowling approximation is the one we determined, the result agrees with Equation (A4). This still leaves us with the missing factor of m compared to Equation (A1).

In essence, the results in the available literature are not quite consistent. Having said that, as we have checked our calculation leading to Equation (A1) several times, we stand by this as the correct result.

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