# A Novel Approach to Predictive Accuracy Testing in Nested Environments

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## Abstract

We introduce a new approach for comparing the predictive accuracy of two nested models that bypasses the difficulties caused by the degeneracy of the asymptotic variance of forecast error loss differentials used in the construction of commonly used predictive comparison statistics. Our approach continues to rely on the out of sample MSE loss differentials between the two competing models, leads to nuisance parameter free Gaussian asymptotics and is shown to remain valid under flexible assumptions that can accommodate heteroskedasticity and the presence of mixed predictors (e.g. stationary and local to unit root). A local power analysis also establishes their ability to detect departures from the null in both stationary and persistent settings. Simulations calibrated to common economic and financial applications indicate that our methods have strong power with good size control across commonly encountered sample sizes.

Keywords: Forecasting, Nested Models, Variance degeneracy, MSE differentials.

**Running Head:** PREDICTIVE ACCURACY IN NESTED ENVIRONMENTS

# 1. Introduction

This paper is concerned with comparing the forecasting performance of two nested models through tests that rely on out of sample mean squared error (MSE) loss differentials. Our proposed approach bypasses the widely documented complications caused by the degenerate asymptotic variances of these differentials that occur in nested environments while also leading to nuisance parameter free standard normal asymptotics. Our approach remains valid under both stationary and persistent predictors thus also greatly expanding its practical relevance in economics and finance.

Since the early work of Diebold and Mariano (1995) and West (1996) a vast body of theoretical research has been concerned with developing new methods for comparing the out of sample predictive ability of competing models. Such tests typically compare the out of sample forecast errors generated from two models under a variety of loss functions and forecasting schemes (e.g. recursive, rolling or fixed updating of model parameter estimates) with the aim of testing the null hypothesis of equal predictive accuracy. Most of the test statistics introduced in this literature are based on estimated out of sample MSE loss differentials associated with the two competing forecast error series and have been shown to be asymptotically normally distributed provided that the models being compared are non-nested and a set of standard regularity conditions hold.

The fundamental difficulties that arise as one moves from a non-nested to a nested environment have also generated a vast and growing literature aiming to operationalise and adapt the above approach to nested models. In a nested modelling context a key complication comes from the fact that under the null of interest the population errors of the two models are identical thus leading to sample MSE loss differentials that are identically zero in the limit with null asymptotic variances. These in turn result in test statistics that are not well defined asymptotically and in the failure of normal approximations for popular test statistics such as the Diebold-Mariano statistic (henceforth referred to as DM).

Alternative normalisations applied to the MSE loss differentials in nested contexts have subsequently been shown to lead to test statistics with well defined but no longer Gaussian limiting null distributions expressed as functionals of stochastic integrals in Brownian Motions (Clark and McCracken (2001, 2005), McCracken (2007), Hansen and Timmermann (2015)). With the exception of restrictive frameworks that rule out heteroskedasticity or allow only a

single additional predictor in the nesting model these distributions typically depend on a variety of model specific parameters that cannot be eliminated via standard HAC type corrections, requiring simulation based approaches for their implementation (see (West, 2006, pp. 126-127), and (Clark and McCracken, 2013, pp. 10-15)). The asymptotics of these test statistics are further influenced by how the in-sample observations are allowed to grow relative to the out of sample observations and the particular choice of the forecasting scheme used to generate forecasts.

Rather than relying on these non-standard and non-Gaussian distributions this same literature has also proposed to bypass the difficulties underlying nested model comparisons by continuing to use normal approximations for adjusted versions of DM type statistics. In Clark and West (2007) for instance the authors introduced an adjustment to the spread of the out of sample MSEs of the two competing models and argued that although asymptotic normality cannot be established per se such an approach results in reasonably accurate inferences with acceptable size distortions. The adjustment essentially corrects for the fact that under the null hypothesis of equal predictive accuracy the MSE of the larger model is contaminated with estimation noise. This adjusted DM type statistic proposed in Clark and West (2007) has become the norm in economic applications involving out of sample forecast comparisons with recent examples found in Molodotsova and Papell (2009), Molodotsova and Papell (2009), Engel and Wu (2021) amongst numerous others.

In this paper we introduce an alternative formulation of the out of sample MSE loss differential between two models that is not subject to the variance degeneracy problem of existing procedures. This subsequently allows us to construct novel test statistics for testing the null hypothesis of equal out of sample population MSEs which are shown to have simple nuisance parameter free normal distributions. The main idea underlying our proposed approach is based on the observation that MSE comparisons across two competing models need not be performed within the same out of sample span of available forecast error observations. These can be performed over partially overlapping segments instead, leading to test statistics that accumulate MSE spreads over all possible such segments. This new setting can trivially accommodate desirable features such as conditional heteroskedasticity and persistent predictors and is also shown to lead to both consistent and locally powerful test statistics. As we discuss further below, our approach can also be adapted to broader contexts where the nestedness of models is an important

consideration for inferences such as model selection testing.

Besides conventional forecasting objectives, nested models are commonly encountered environments when it comes to testing economic hypotheses and validating theories. Notable examples include forecast accuracy comparisons against random walk models in the exchange rate literature spurred by the early work of Meese and Rogoff (1983) and more recently reconsidered in Rossi (2005), Molodotsova and Papell (2009) amongst others, equity premium predictability issues as recently investigated in Ferson, Nallareddy and Biqin (2013), Avdis and Wachter (2017) and numerous others. Our key aim here is to propose a way of addressing and resolving a long standing issue that has generated a vast agenda on the formal comparison of such models via their out-of-sample predictive accuracy. The important auxiliary debate on the advantages or disadvantages of using out-of-sample versus in-sample approaches is not part of our focus. It is also important to emphasise that our interest here is on testing population level predictive ability when forecasts are generated recursively as opposed to finite sample based predictive ability as considered for instance in Giacomini and White (2006). This latter approach is able to avoid the complications induced by the nestedness of models being compared by proceeding via a rolling fixed window based forecasting scheme so that the issue of competing models becoming identical in the limit can be bypassed.

Throughout this paper we also followed the common practice of referring to statistics based on MSE differentials obtained from competing estimated models as Diebold-Mariano type statistics. We must acknowledge however that the specific testing approach initially developed by these authors was not concerned with model specific considerations or specification testing motives as its underlying theory was developed for *given* sequences of forecast errors assumed to satisfy certain regularity conditions (see Diebold (2015)). Nevertheless, the forecasting literature of the past decade has generally amalgamated the notion of forecast evaluation with the evaluation of models on the basis of their forecasting abilities.

The paper is organised as follows. Section 2 introduces the nested forecasting environment and establishes the limiting null distributions of two novel test statistics. Section 3 concentrates on their asymptotic power properties, establishing their consistency and ability to detect local departures from the null. Section 4 introduces a simple adjustment to the same statistics shown to further enhance their power properties without affecting their null distributions. Section 5 provides a comprehensive finite sample evaluation of our methods based on two DGPs calibrated

to commonly encountered applications. Section 6 illustrates the use of our proposed methods via an application to exchange rate models. Section 7 overviews our key results and discusses extensions. Proofs are given in the Appendix. Further simulation results are provided in an online supplement.

## 2. Models and Test Statistics: Theory

We consider the following predictive regressions

$$y_{t+1} = \boldsymbol{x}_{1t}'\boldsymbol{\delta}_1 + v_{t+1} \tag{1}$$

$$y_{t+1} = \boldsymbol{x}_{1t}'\boldsymbol{\beta}_1 + \boldsymbol{x}_{2t}'\boldsymbol{\beta}_2 + u_{t+1} \tag{2}$$

where the $\boldsymbol{x}_{it}$'s are the $(p_i \times 1)$ vectors of predictors, $\boldsymbol{\delta}_1$ and $\boldsymbol{\beta}_i$ the $(p_1 \times 1)$ and $(p_i \times 1)$ parameter vectors and $v_t$ and $u_t$ the random disturbance terms. We let $\boldsymbol{x}_t = (\boldsymbol{x}_{1t}', \boldsymbol{x}_{2t}')'$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$ and set $p = p_1 + p_2$. Here model (1) is nested within the larger model in (2) and under $\boldsymbol{\beta}_2 = 0$ we have $\boldsymbol{\delta}_1 \equiv \boldsymbol{\beta}_1$ and $v_{t+1} \equiv u_{t+1}$. The formulation of the above two nested models is standard and parallels closely the most commonly encountered setting considered in the predictive accuracy testing literature as for instance in Hansen and Timmermann (2015).

One step ahead forecasts of $y_{t+1}$ from (1) and (2) are generated recursively as $\hat{y}_{1,t+1|t} = \boldsymbol{x}_{1t}'\hat{\boldsymbol{\delta}}_{1t}$ and $\hat{y}_{2,t+1|t} = \boldsymbol{x}_t'\hat{\boldsymbol{\beta}}_t$ for $t = k_0, \ldots, T-1$ where $\hat{\boldsymbol{\delta}}_{1t} = (\sum_{j=1}^{t} \boldsymbol{x}_{1j-1}\boldsymbol{x}_{1j-1}')^{-1} \sum_{j=1}^{t} \boldsymbol{x}_{1j-1}'y_j$, $\hat{\boldsymbol{\beta}}_t = (\sum_{j=1}^{t} \boldsymbol{x}_{j-1}\boldsymbol{x}_{j-1}')^{-1} \sum_{j=1}^{t} \boldsymbol{x}_{j-1}'y_j$ and the resulting pseudo out of sample forecast errors are then obtained as $\hat{e}_{1,t+1} = y_{t+1} - \boldsymbol{x}_{1t}'\hat{\boldsymbol{\delta}}_{1t}$ and $\hat{e}_{2,t+1} = y_{t+1} - \boldsymbol{x}_t'\hat{\boldsymbol{\beta}}_t$. Here $k_0$ is the sample location used to initiate the first recursive forecasts that lead to the first out of sample forecast errors $\hat{e}_{1,k_0+1}$ and $\hat{e}_{2,k_0+1}$ and subsequently resulting in $(T - k_0)$ out of sample forecast error observations. Throughout this paper we take $k_0$ to be a given fraction of the sample size, setting $k_0 = [T\pi_0]$ for some $\pi_0 \in (0, 1)$.

Following the early work of Diebold and Mariano (1995), West (1996) and others, a common approach for comparing the predictive accuracy of the two models under MSE loss involves

testing the null hypothesis

$$H_0 \colon E[y_{t+1} - \hat{y}_{1,t+1}(\boldsymbol{\delta}_1)]^2 = E[y_{t+1} - \hat{y}_{2,t+1}(\boldsymbol{\beta})]^2 \tag{3}$$

using a test statistic based on suitably normalised versions of the sample average MSE loss differentials

$$\overline{D}_T = \frac{1}{T - k_0} \left( \sum_{t=k_0}^{T-1} \hat{e}_{1,t+1}^2 - \sum_{t=k_0}^{T-1} \hat{e}_{2,t+1}^2 \right). \tag{4}$$

Within a non-nested setting and a strictly stationary and ergodic environment Diebold and Mariano (1995) and West (1996) established a standard normal limit theory for this class of test statistics (e.g. $\sqrt{T - k_0}\ \overline{D}_T / \hat{\sigma}_{\overline{D}_T}$ with $\hat{\sigma}_{\overline{D}_T}^2$ denoting some suitable long run variance estimator) leading to their systematic use in applied work and a voluminous literature on their refinements. Within a nested context where $u_{t+1} \equiv v_{t+1}$ however it is straightforward to observe that $\sqrt{T - k_0}\ \overline{D}_T \xrightarrow{p} 0$ and $\hat{\sigma}_{\overline{D}_T} \xrightarrow{p} 0$ invalidating the limiting standard normal approximation and the use of these test statistics for inference purposes. This degeneracy problem is not solely confined to the Diebold-Mariano type statistics but universally affects all existing methods that compare forecast errors (or models) in nested settings with recursively generated forecasts.

These observations have led to a vast body of research on out of sample predictive accuracy testing in nested models due to their importance in empirical applications in areas such as asset pricing and the modelling of expected returns in particular. For their validity, inferences in nested contexts such as (1) and (2) must rely on the observation that under $H_0$ it is $(T - k_0)\overline{D}_T$ rather than $\sqrt{T - k_0}\ \overline{D}_T$ that turns out to have a non-degenerate limit which could be used for developing suitable inferences (see Clark and McCracken (2001, 2005), McCracken (2007), Hansen and Timmermann (2015)). This however is also problematic due to the non-standard and non-pivotal nature of the resulting asymptotic distributions. These take the form of functionals of stochastic integrals in Brownian Motions and with the exception of some special cases contain nuisance parameters that are difficult to remove via standard HAC type normalisations. Even under special instances such as conditional homoskedasticity these distributions continue to depend on the number of extra predictors included in the nesting models and the fraction of the sample used to build the first recursive forecasts. Equally importantly these results have been

obtained under stationarity and ergodicity assumptions ruling out the important and frequently encountered case of predictors having roots near unity in their autoregressive representations.

Instead of evaluating the two sequences of squared forecast errors $\{\hat{e}_{1,t+1}^2\}$ and $\{\hat{e}_{2,t+1}^2\}$ over the entire *and* same interval $[k_0 + 1, T]$ as it is done in the formulation of all commonly used test statistics based on $\overline{D}_T$ we here propose to compare the two out of sample MSEs over partially overlapping segments of the $[k_0 + 1, T]$ interval instead. For this purpose we introduce the following generalised MSE spread

$$\widetilde{D}_T(\ell_1, \ell_2) = \frac{\sum\limits_{t=k_0}^{k_0+\ell_1-1} \hat{e}_{1,t+1}^2}{\ell_1} - \frac{\sum\limits_{t=k_0}^{k_0+\ell_2-1} \hat{e}_{2,t+1}^2}{\ell_2} \tag{5}$$

where $\ell_1$ and $\ell_2$ control the range over which the two squared forecast error sequences are evaluated. Note that setting $\ell_1 = \ell_2 = T - k_0$ in (5) reduces it to $\overline{D}_T$ which can be viewed as a special case of $\widetilde{D}_T(\ell_1, \ell_2)$. In line with the analysis based on (4) we take $\ell_1 = [(T - k_0)\lambda_1]$ and $\ell_2 = [(T - k_0)\lambda_2]$ with $\lambda_1$ and $\lambda_2$ referring to the fraction of the $(T - k_0)$ squared forecast errors associated with models (1) and (2) respectively.

From a theoretical standpoint, proceeding with the use of $\widetilde{D}_T(\ell_1, \ell_2)$ instead of $\overline{D}_T$ has no bearing on the null hypothesis being tested in the sense that when $u_{t+1} \equiv v_{t+1}$ the population counterpart of $\widetilde{D}_T(\ell_1, \ell_2)$ also equals zero. A key feature of (5) that distinguishes it from $\overline{D}_T$ however is that the variance of its suitably normalised version will no longer be degenerate provided that $\ell_1 \neq \ell_2$ (equivalently $\lambda_1 \neq \lambda_2$). This normalised version of (5) which forms the basis of our proposed test statistics is given by $Z_T(\ell_1, \ell_2) = \sqrt{T - k_0}\, \widetilde{D}_T(\ell_1, \ell_2)$,

$$Z_T(\ell_1, \ell_2) = \frac{T - k_0}{\ell_1} \left[ \frac{\sum\limits_{t=k_0}^{k_0+\ell_1-1} \hat{e}_{1,t+1}^2}{\sqrt{T - k_0}} - \frac{\ell_1}{\ell_2} \frac{\sum\limits_{t=k_0}^{k_0+\ell_2-1} \hat{e}_{2,t+1}^2}{\sqrt{T - k_0}} \right]. \tag{6}$$

Note that (6) is simply the normalised difference in the means of the two sample MSEs evaluated over the two relevant segments of the effective sample size.

REMARK 1: A key point to observe here is that the variance of (6) is well-defined and no longer collapses to zero in the limit provided that $\lambda_1$ and $\lambda_2$ are bounded away from zero and

bounded away from each other. To illustrate and motivate this point heuristically let us replace both $\hat{e}_{1t+1}^2$ and $\hat{e}_{2t+1}^2$ in (6) with $(u_{t+1}^2 - \sigma_u^2)$ for $\sigma_u^2 \equiv E[u_t^2]$. Taking the $u_t's$ to be IID(0,1) with $E[u_{t+1}^4] < \infty$ it follows that

$$V[Z_T(\ell_1, \ell_2)] \to V[u_{t+1}^2] \frac{|\lambda_2 - \lambda_1|}{\lambda_1 \lambda_2} \tag{7}$$

suggesting that a test statistic based on $\widetilde{D}_T(\ell_1, \ell_2)$ will not have a degenerate distribution as it was the case with the use of $\overline{D}_T$ in nested contexts. We may also wish to point out that having $\lambda_1$ and $\lambda_2$ bounded away from zero is merely a technical requirement in the asymptotics that follow as in practice these parameters will naturally be set at or near their maximum boundary of one.

The quantity in (6) forms the building block of our proposed test statistics for testing the null hypothesis in (3) against one sided right tail based alternatives as it is the norm in this literature. We consider two types of test statistics that operationalise (6). Our choice is guided by the simplicity of the ensuing asymptotics and their intuitive interpretation while recognising that alternative constructions/normalisations of $\widetilde{D}_T(\ell_1, \ell_2)$ may also be considered.

The first test statistic that we consider is denoted $Z_T^0(\lambda_1^0, \lambda_2^0)$ and is based on implementing inferences for given magnitudes $\ell_1^0 = [(T - k_0)\lambda_1^0]$ and $\ell_2^0 = [(T - k_0)\lambda_2^0]$. We write

$$Z_T^0(\lambda_1^0, \lambda_2^0) = \frac{1}{\hat{\sigma}} Z_T([(T - k_0)\lambda_1^0], [(T - k_0)\lambda_2^0]) \tag{8}$$

with $\hat{\sigma}^2$ denoting a consistent estimator of $V[u_{t+1}^2]$.

Our second test statistic is based on averaging (6) across the $\ell_j$'s. The averaging can be implemented over $\ell_1 \in [1, T - k_0]$ for a given $\ell_2^0$ (e.g., $\ell_2^0 = T - k_0$) so that the MSE of the smaller model accumulates progressively as $\ell_1$ increases. More generally, this averaging can be performed over any desired and feasible range of $\ell_1$. To allow such level of generality we introduce the fractional parameter $\tau_0$ and write

$$\overline{Z}_T(\tau_0; \lambda_2^0) = \frac{1}{\hat{\sigma}} \frac{1}{[(T - k_0)(1 - \tau_0)]} \sum_{\ell_1 = [(T - k_0)\tau_0] + 1}^{T - k_0} Z_T(\ell_1, [(T - k_0)\lambda_2^0]) \tag{9}$$

where the choice of $\tau_0$ determines the user-chosen range of $\ell_1$ over which the average of $Z_T(\ell_1, \ell_2^0)$

is taken (given $\ell_2^0 = [(T-k_0)\lambda_2^0]$). Rather than imposing a fixed and given $\lambda_1^0$ as in (8) this average based statistic essentially considers a range of such magnitudes and subsequently aggregates outcomes via averaging. Given the role played by $\ell_1$ and $\ell_2$ in our inferences we can expect that choosing the averaging range in a way that excludes low magnitudes of $\ell_1$ (so that the estimated MSEs associated with model 1 remain sufficiently accurate) will result in more reliable inferences. The issue of how best to select these user-inputs is postponed until Section 3 where we provide precise guidelines informed by a theoretical local power analysis. One motivation behind this average based statistic when compared with (8) is that one can remain partly more agnostic about the specific magnitude to use for one of the two required user-inputs in $Z_T([(T - k_0)\lambda_1^0], [(T - k_0)\lambda_2^0])$ while setting the other one (e.g., $\lambda_2^0$) at or near its maximum boundary of one. Although our context is different, this is also reminiscent of the various approaches used in the structural break testing literature when one does not wish to take a stance on the location of a potential change-point. More importantly, and borrowing from the same literature, we may also conjecture that the averaging process may result in tests with more favorable size-power trade-offs.

At this stage it is also important to point out that there are numerous alternative possibilities for designing test statistics in the spirit of (8) and (9) (e.g., double averaging across $\ell_1$ and $\ell_2$, alternative functional forms etc.). An interesting avenue for future research could be the design of a class of test statistics based on $Z_T(\ell_1, \ell_2)$ and having desirable optimality properties as it has been attempted in the structural break literature.

Although both (8) and (9) allow for a broad range of theoretically feasible magnitudes for $(\lambda_1^0, \lambda_2^0)$ in $Z_T^0(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0)$ in $\overline{Z}_T(\tau_0; \lambda_2^0)$ one naturally expects that choosing $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0)$ to lie in the vicinity of unity would capture the greatest amount of information from the two competing models. As we show further below such a choice does indeed lead to remarkably powerful tests with excellent size control. Given a sequence of forecast errors available to the investigator the practical implementation of either (8) or (9) is also as straightforward as calculating standard DM type test statistics.

To establish the limiting properties of our test statistics under the null hypothesis in (3) we introduce a set of high level assumptions ensuring a flexible environment that encompasses the vast majority of settings considered in the literature while also allowing for a richer temporal structure. As we wish to highlight the generality and usefulness of our methods based on the use

of (5)-(6) we abstain from primitive conditions that may unnecessarily suggest a restrictive scope for their use. More importantly our use of high level assumptions is motivated by the fact that our proposed methods can be immediately seen to be robust to a very rich dynamic structure of predictors including highly persistent processes, strictly stationary and ergodic processes, long memory processes etc.

ASSUMPTION A:

$(i)$ $\displaystyle\sup_{\lambda\in(0,1]}\left|\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}\hat{e}_{j,t+1}^2}{\sqrt{T-k_0}} - \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}u_{t+1}^2}{\sqrt{T-k_0}}\right| \overset{H_0}{=} o_p(1)$ *for* $j=1,2.$

*(ii) The sequence of demeaned squared errors* $\eta_t = u_{t+1}^2 - \sigma_u^2$ *has autocovariances* $\gamma_j^\eta$ *that satisfy* $\sum_{j=0}^\infty |\gamma_j^\eta| < \infty$ *and fulfills a functional central limit theorem, that is* $T^{-\frac{1}{2}}\sum_{t=1}^{[Ts]}(u_{t+1}^2 - \sigma_u^2) \overset{\mathcal{D}}{\to} \sigma W_\eta(s)$ *on* $D_{\mathbb{R}}([0,1])$ *the space of cadlag functions on* $[0,1]$ *with* $W_\eta(.)$ *denoting a standard Brownian Motion and* $\sigma^2 = \gamma_0^\eta + 2\sum_{j=1}^\infty \gamma_j^\eta > 0.$

*(iii) A consistent estimator* $\hat{\sigma}^2$ *of* $\sigma^2$ *exists, that is* $\hat{\sigma}^2 \overset{p}{\to} \sigma^2 \in (0,\infty).$

We note that condition A(i) holds for both $j=1$ and $j=2$ highlighting the fact that we operate within a nested environment with (1) being the true model. A1(i) is trivially satisfied under a very broad range of settings used to obtain the large sample properties of DM type statistics in nested models. An important feature to also highlight here is the fact that A(i) does not restrict the persistence properties of the predictors which could be highly persistent in the sense of following local to unit root processes for instance. This greatly expands and enriches the environment in which predictive accuracy inferences have commonly been introduced. The robustness of A(i) to the persistence properties of the predictors is an important and useful feature of the squared forecast errors as opposed to their level for which a result such as A(i) would not hold. For more primitive conditions illustrating specialised environments under which A(i) holds see Deng and Perron (2008a) and Hansen and Timmermann (2015) for strictly stationary and ergodic/mixing settings and Berenguer-Rico and Nielsen (2020) for environments

where A(i) is shown to hold under both stationary and unit-root or near unit-root regressors.

Assumption A(ii) requires the centered squared errors driving (1)-(2) to satisfy a functional central limit theorem with $\sigma^2$ referring to their long run variance. The absolute summability of the autocovariances of $\eta_t$ ensures that $\sigma^2$ the limit of $V[\sum_{t=k_0}^{T-1} \eta_{t+1}/\sqrt{T-k_0}]$ exists. Examples of processes which satisfy Assumption A(ii) include a broad range of conditionally heteroskedastic ARCH/GARCH processes under suitable existence of moments restrictions. For a detailed set of primitive assumptions ensuring that the stated FCLT holds see (Giraitis, Kokoszka and Leipus, 2000, Theorem 5.1), (Giraitis, Kokoszka and Leipus, 2001, Example 2.2 and Theorem 2.1), Berkes, Hörmann and Horvath (2008), Linder (2009) and references therein.

Assumption A(iii) requires that the long run variance of $\eta_t$ be estimated consistently. Such an estimator could be trivially constructed using least squares residuals from either the null or alternative models. Under conditional homoskedasticity an obvious candidate would be $\hat{\sigma}_{hom}^2 = \sum_{t=k_0}^{T-1} \hat{\eta}_{t+1}^2/(T-k_0)$ while under dependent errors (e.g. if the $u_t's$ follow a GARCH type process) a Newey-West type formulation as in Deng and Perron (2008b) would be suitable and ensure that A(iii) holds.

REMARK 2: As pointed out in Remark 1, our asymptotic theory for $Z_T^0(\lambda_1^0, \lambda_2^0)$ in (8) imposes $\lambda_1^0$ and $\lambda_2^0$ to be bounded away from zero and to be bounded away from each other, say $0 < \underline{\lambda} \leq \lambda_i^0 \leq 1$ for $i = 1, 2$ and $|\lambda_1^0 - \lambda_2^0| \geq \epsilon$ for some positive fraction $\epsilon$. In what follows we refer to such a set from which these two user-inputs can be selected as $\Lambda^0$. The implementation of the average based statistic $\overline{Z}_T(\tau_0; \lambda_2^0)$ in (9) requires setting $\lambda_2^0$ as above and averaging $Z_T(\ell_1, [(T-k_0)\lambda_2^0])$ across $\ell_1 = [(T-k_0)\tau_0] + 1, \ldots, (T-k_0)$ for some $\tau_0$ bounded away from zero and one. We refer to this set as $\overline{\Lambda}^0$.

The following two propositions summarise the large sample behavior of our two test statistics under the null hypothesis stated in (3).

**PROPOSITION 1**. *Under Assumptions A(i)-(iii), the null hypothesis in (3), and for given* $(\lambda_1^0, \lambda_2^0) \in \Lambda^0$, *we have as* $T \to \infty$

$$Z_T^0(\lambda_1^0, \lambda_2^0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, v^0(\lambda_1^0, \lambda_2^0)) \tag{10}$$

*where*

$$v^0(\lambda_1^0, \lambda_2^0) = \frac{|\lambda_1^0 - \lambda_2^0|}{\lambda_1^0 \lambda_2^0}. \tag{11}$$

**PROPOSITION 2**. *Under Assumptions A(i)-(iii), the null hypothesis in (3), and for given* $(\tau_0, \lambda_2^0) \in \overline{\Lambda}^0$, *we have as* $T \to \infty$

$$\overline{Z}_T(\tau_0; \lambda_2^0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \bar{v}(\tau_0; \lambda_2^0)) \tag{12}$$

*where*

$$\bar{v}(\tau_0; \lambda_2{}^0) = \begin{cases} \dfrac{(1-\tau_0)^2 + 2\lambda_2^0(1-\tau_0 + \ln \tau_0)}{\lambda_2^0(1-\tau_0)^2} & \lambda_2^0 \leq \tau_0 \qquad (13) \\[4mm] \dfrac{1 - \tau_0^2 + 2\lambda_2^0((1-\tau_0)\ln \lambda_2^0 + \tau_0 \ln \tau_0)}{\lambda_2^0(1-\tau_0)^2} & \lambda_2^0 > \tau_0. \quad (14) \end{cases}$$

The variance components of the distributional outcomes in (10) and (12) are of course known to the investigator so that both $Z_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{Z}_T(\tau_0; \lambda_2^0)$ can be trivially standardised as

$$\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \equiv \frac{Z_T^0(\lambda_1^0, \lambda_2^0)}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \tag{15}$$

and

$$\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0) \equiv \frac{\overline{Z}_T(\tau_0; \lambda_2^0)}{\sqrt{\bar{v}(\tau_0; \lambda_2^0)}} \tag{16}$$

to proceed with standard normal inferences for testing $H_0$.

The results in (10)-(14) highlight the simplicity and practicality of our proposed inferences while at the same time offering a solution to an important problem that has not been satisfactorily resolved in this literature. The test statistics in (15) and (16) allow us to generalise the widely used DM style forecast accuracy testing approach to a broad class of empirically relevant models including specifications with highly persistent predictors with or without conditional heteroskedasticity.

We naturally expect the quality of inferences (e.g. power, size vs power trade-offs) to be influenced by the specific choices of $(\lambda_1^0, \lambda_2^0)$ in (15) and $(\tau_0, \lambda_2^0)$ in (16). Although the above null asymptotics hold under a very broad range of parameterisations for those user-inputs a formal analysis of their local asymptotic power allows us to provide precise and tight guidelines ensuring excellent power properties with good size control.

# 3. Asymptotic Power and Test Parameterisations

We here deviate from Assumption A(i) in order to evaluate the large sample behavior of $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ when the DGP is given by (2). Assumption A(i) continues to hold for $j = 2$ but no longer for $j = 1$ since model (1) is misspecified due to the omitted $\boldsymbol{x}_{2,t}$ predictors. As $\hat{e}_{1,t+1}^2$ will now be contaminated by those omitted predictors we expect the stochastic properties of the latter (e.g. the variance of the $\boldsymbol{x}_{2,t}$'s and their correlation with the $\boldsymbol{x}_{1,t}$'s) to influence the power properties of both test statistics. Unlike their null distributions we thus also expect the test statistics to diverge at different rates depending on whether the predictors are stationary or highly persistent. Our analysis of the consistency and local power properties of $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ is guided by these two distinct scenarios which we consider separately.

## 3.1. Consistency and Local Power under Stationarity

We initially concentrate on the case where the predictors driving both (1) and (2) are stationary and ergodic. Specifically, we operate under the following set of high level assumptions that mirror closely the most common environments considered in the predictive accuracy testing literature.

ASSUMPTION B1:

(i) $\displaystyle\sup_{\lambda \in [0,1]} \left\| \frac{\sum_{t=1}^{[T\lambda]} \boldsymbol{x}_t \boldsymbol{x}_t'}{T} - \lambda \, \boldsymbol{Q} \right\| = o_p(1)$ with $\boldsymbol{Q}$ a $p \times p$ nonrandom positive definite matrix,

(ii) $\displaystyle\frac{\sum_{t=1}^{[T\lambda]} \boldsymbol{x}_t u_{t+1}}{\sqrt{T}} \xrightarrow{\mathcal{D}} \boldsymbol{\Omega}^{1/2} \, \boldsymbol{W}(\lambda)$ with $\boldsymbol{W}(.)$ denoting a $p$-dimensional standard Brownian Motion and $\boldsymbol{\Omega} = E[\boldsymbol{x}_t \boldsymbol{x}_t' u_{t+1}^2] > 0$,

*(iii) Assumptions A(ii)-(iii) hold.*

*(iv) The user-inputs in $\mathcal{S}_T(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ are such that $(\lambda_1^0, \lambda_2^0) \in \Lambda^0$ and $(\tau_0, \lambda_2^0) \in \overline{\Lambda}^0$ respectively.*

Assumptions B1(i)-(iii) mirror closely the environment of Hansen and Timmermann (2015) and can be viewed as more primitive conditions ensuring that Assumption A(i) holds. B1(i) requires that the predictors satisfy a uniform law of large numbers and rules out trending or local to unit root predictors while B1(ii) ensures that $\{\boldsymbol{x}_t u_{t+1}\}$ satisfies a multivariate functional central limit theorem. Our main result regarding the asymptotic power properties of the two tests within such a stationary environment is now summarised in Proposition 3 below.

**PROPOSITION 3**. *(i) Suppose model (2) holds with $\boldsymbol{\beta}_2 \neq 0$ and fixed, then under assumption B1 and as $T \to \infty$ we have $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \xrightarrow{p} \infty$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0) \xrightarrow{p} \infty$. (ii) Suppose model (2) holds with $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{1/4}$ for $\boldsymbol{\gamma} \neq 0$. Under assumption B1, $\lim_{||\boldsymbol{\gamma}|| \to \infty} \lim_{T \to \infty} \mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) = \infty$ and $\lim_{||\boldsymbol{\gamma}|| \to \infty} \lim_{T \to \infty} \overline{\mathcal{S}}_T(\tau_0; \lambda_2^0) = \infty$ in probability.*

The above results highlight the consistency of both test statistics as well as their ability to detect local departures from the null hypothesis under stationary settings. It is here also important to point out that the local to the null parameterisation of $\boldsymbol{\beta}_2$ based on $T^{1/4}$ rather than the usual $T^{1/2}$ rate commonly encountered in stationary settings is not in any way due to our specific test statistics or assumptions. The same scenario would also occur in a conventional regression based testing environment and is due to the fact that we are dealing with inferences about the behavior of *squared* errors rather than their level.

To gain further insights into the specific role played by key factors influencing power it is useful to also present the explicit asymptotic local power functions of the two tests for a given size $\alpha \in (0, 1)$. These will in turn be used to provide explicit guidance on selecting suitable parameterisations of our two test statistics (i.e. $(\lambda_1^0, \lambda_2^0)$ in (15) and $(\tau_0; \lambda_2^0)$ in (16)). In what follows it is useful to also recall that $\pi_0$ refers to the given fraction of the sample size used to initiate the recursive computation of forecasts.

**COROLLARY 1**. *Suppose model (2) holds with $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{1/4}$ for $\boldsymbol{\gamma} \neq 0$. Under assumption B1 and letting $q_\alpha$ denote the upper $\alpha$-quantile of the standard normal distribution with CDF $\Phi(.)$, the asymptotic local power functions of the tests based on $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ are given by $1 - \Phi(q_\alpha - \psi^0)$ and $1 - \Phi(q_\alpha - \overline{\psi})$ respectively, where*

$$\psi^0 = \left[ \frac{\sqrt{1-\pi_0}}{\sigma\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \boldsymbol{\gamma}'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\gamma} \right], \tag{17}$$

$$\overline{\psi} = \left[ \frac{\sqrt{1-\pi_0}}{\sigma\sqrt{\overline{v}(\tau_0; \lambda_2^0)}} \boldsymbol{\gamma}'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\gamma} \right], \tag{18}$$

*with $v^0(\lambda_1^0, \lambda_2^0)$ and $\overline{v}(\tau_0; \lambda_2^0)$ as in (11) and (13)-(14) and the $\boldsymbol{Q}_{ij}$'s referring to the components of the population moment matrix $\boldsymbol{Q}$ in assumption B1(i).*

We note that power is monotonic in the sense that both $\psi^0$ and $\overline{\psi}$ are non-decreasing as $||\boldsymbol{\gamma}||$ gets large. For a given significance level, the larger the two non-centrality parameters are the greater the associated probabilities of rejecting the null hypothesis.

The expressions in (17)-(18) are particularly useful for highlighting the factors that influence power by shifting the center of the null asymptotic standard normal distributions away from zero. Viewing the asymptotic local power functions $\Phi(\psi^0 - q_\alpha)$ and $\Phi(\overline{\psi} - q_\alpha)$ in Corollary 1 as providing approximations to the correct decision frequencies of the two test statistics under a sufficiently large $T$ and specific alternatives, we note that for a given size $\alpha$ both test statistics are expected to exhibit a stronger ability to detect departures from the null when the variances of the omitted predictors are large and their correlation with the included predictors small. This feature is particularly important since it hints at the fact that the presence of nearly integrated predictors may help enhance power, a scenario we formally consider further below.

To highlight these points with greater clarity it is useful to focus on the simplified case of two centered predictors $\boldsymbol{x}_t = (x_{1,t}, x_{2,t})$ so that (17)-(18) simplify as

$$\psi^0 = \left[ \frac{\sqrt{1-\pi_0}}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \frac{\gamma^2}{\sigma}(1 - \rho_{12}^2)E[x_{2,t}^2] \right] \tag{19}$$

$$\overline{\psi} = \left[ \frac{\sqrt{1-\pi_0}}{\sqrt{\overline{v}(\tau_0; \lambda_2^0)}} \frac{\gamma^2}{\sigma}(1-\rho_{12}^2)E[x_{2,t}^2] \right] \tag{20}$$

with $\rho_{12} = Corr[x_{1,t}, x_{2,t}]$. All other things being equal, power is expected to deteriorate under a noisy omitted predictor that has low variance (low $E[x_{2,t}^2]$) and/or that is highly correlated with the included predictor (e.g. $|\rho_{12}| \approx 1$). Interestingly, this also suggests that an ideal setting in terms of power implications is one where omitted predictors are highly persistent while included predictors are stationary so that $\rho_{12} \approx 0$ with $E[x_{2,t}^2]$ large. Another important factor affecting power is the variance of the $u_t^2$'s which impacts the magnitudes of $\psi^0$ and $\overline{\psi}$ via $\sigma \equiv \sqrt{V[u_{t+1}^2]}$. Within an NID errors setting for instance we have $V[u_{t+1}^2] = E[u_{t+1}^4] - \sigma_u^4 = 2\sigma_u^4$ so that *all other things being equal*, an environment with high kurtosis will have a detrimental impact on the power properties of both test statistics.

### *Power enhancing choices for* $(\lambda_1^0, \lambda_2^0)$ *and* $(\tau_0; \lambda_2^0)$

The non-centrality parameters in (17)-(18) are also useful for assessing the impact of $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0)$ on both the absolute and relative local powers of the two tests and for providing useful guidance on suitable choices for those user-inputs. From Corollary 1, since the mapping $m \mapsto P[Z > q_\alpha - m]$ is increasing in $m$ on $[0, \infty)$, a test of size $\alpha$ based on $S_T^0(\lambda_1^0, \lambda_2^0)$ will be preferable, in terms of its local power, to a test of the same size based on $S_T^0(\lambda_1^{0\prime}, \lambda_2^{0\prime})$ whenever $\psi^0(\lambda_1^0, \lambda_2^0) > \psi^0(\lambda_1^{0\prime}, \lambda_2^{0\prime})$, holding all other parameters entering $\psi^0$ constant. Given $\psi^0$ in (17) with $v^0(\lambda_1^0, \lambda_2^0)$ defined as in (11) it follows that those two parameters should be set near their boundary of 1 and in close vicinity of one another (e.g. $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ for $\lambda_2^0 \approx 0.9$ as a possibility).

Regarding the average based statistic $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ we note from (13)-(14) that $\overline{\psi}$ in (18) viewed as a function of $\lambda_2^0$ and $\tau_0$ (holding all other parameters constant) reaches its unique maximum for

$$\lambda_2^0 = 0.5\,\tau_0 + 0.5 \tag{21}$$

supporting the use of $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0 = 0.5\tau_0 + 0.5)$ in its practical implementation. If $\tau_0 = 0.5$ for instance, which corresponds to a test statistic that averages across the largest half of the $\ell_1$ magnitudes, this approximate asymptotic power based metric points to an implementation based on $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0 = 0.75)$. Since $\overline{\psi}$ is also a monotonically increasing function of $\tau_0$ however, it also follows that the same average based statistic should be operationalised with a choice of $\tau_0$ that is in the vicinity of 1 (e.g. $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0 = 0.5(0.8) + 0.5)$ or $\overline{\mathcal{S}}_T(\tau_0 = 0.9; \lambda_2^0 = 0.5(0.9) + 0.5)$ as possibilities). A practical side to this power enhancing choice of $\lambda_2^0$ is that the implementation of $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ essentially requires only a single user-input.

Given these preferred parameterisations of the two test statistics it is also useful to evaluate whether either of the two statistics is expected to dominate the other in the sense of $\psi^0$ being greater or smaller than $\overline{\psi}$ over particular regions of the pairs $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0 = 0.5\tau_0 + 0.5)$, holding all other parameters constant. Given the standard normal asymptotics of both test statistics a useful metric for comparing their local powers is Pitman's Asymptotic Relative Efficiency (ARE) which here takes particularly simple forms, following directly from Corollary 1. To avoid confusion between the $\lambda_2^0$ parameter used in $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\lambda_2^0$ used in $\overline{\mathcal{S}}_T(\tau_0, \lambda_2^0)$ we write the two statistics as $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \overline{\lambda}_2^0)$ with $\overline{\lambda}_2^0 = 0.5\tau_0 + 0.5$ as in (21). Their ARE is now given by

$$\mathrm{ARE}(\mathcal{S}^0, \overline{\mathcal{S}}) = \left[ \frac{\overline{v}(\tau_0; 0.5\tau_0 + 0.5)}{v^0(\lambda_1^0; \lambda_2^0)} \right] \tag{22}$$

and more specifically

$$\mathrm{ARE}(\mathcal{S}^0, \overline{\mathcal{S}}) = \frac{\lambda_1^0 \lambda_2^0}{|\lambda_1^0 - \lambda_2^0|} \frac{2(1 - \tau_0)(1 + \ln((1 + \tau_0)/2)) + 2\tau_0 \ln \tau_0}{(1 - \tau_0)^2}. \tag{23}$$

From (23) we can observe a clear trade-off between $\tau_0$ and the magnitudes of $\lambda_1^0$ and $\lambda_2^0$ used in $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$. If we focus on $\lambda_1^0 = 1$ it follows from (23) that $\mathrm{ARE} \geq 1$ for

$$\lambda_2^0 \geq \frac{(1 - \tau_0)^2}{(1 - \tau_0)(3 - \tau_0) + 2[\ln 0.5(1 + \tau_0) - \tau_0 \ln((1 + \tau_0)/2\tau_0)]} \tag{24}$$

which is a monotonically increasing function of $\tau_0$ and highlights the fact that the average based statistic will dominate $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ in terms of its local power (i.e. $\mathrm{ARE} < 1$) unless

impractically large magnitudes of $\lambda_2^0$ are used in its implementation. If the average based statistic is implemented with $\tau_0 = 0.8$ for instance, its power properties will dominate $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ unless $\lambda_2^0 > 0.9798$. If it is implemented with $\tau_0 = 0.9$ the average based statistic will again dominate $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ unless $\lambda_2^0 > 0.9908$. These values suggest that the average based statistic with $\tau_0$ set in the vicinity of unity (e.g. $\overline{\mathcal{S}}(\tau_0 = 0.8; \overline{\lambda}_2^0 = 0.9)$) will dominate $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ in terms of its local power unless impractically large magnitudes of $\lambda_2^0$ are used in $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$.

## 3.2. Consistency and Local Power under Persistence

We now consider an environment where the $p$ predictors $\boldsymbol{x}_t$ entering (1)-(2) are modelled as local to unit root processes specified as

$$\boldsymbol{x}_t = \left(\boldsymbol{I}_p - \frac{\boldsymbol{C}}{T}\right)\boldsymbol{x}_{t-1} + \epsilon_t \tag{25}$$

where $\boldsymbol{C} = diag(c_1, \ldots, c_p)$ for $c_i > 0$, $i = 1, \ldots, p$ and $\epsilon_t$ some stationary and ergodic random disturbance process. The new set of assumptions under which we establish our results are now summarised in Assumption B2 below where $\boldsymbol{J}_C(s) = (\boldsymbol{J}_{1C}(s), \boldsymbol{J}_{2C}(s))'$ denotes a p-dimensional Ornstein-Uhlenbeck process whose two components $\boldsymbol{J}_{1C}(s)$ and $\boldsymbol{J}_{2C}(s)$ are associated with the dynamics of $\boldsymbol{x}_{1,t}$ and $\boldsymbol{x}_{2,t}$ respectively.

ASSUMPTION B2:

(i) $\left(\dfrac{\boldsymbol{x}_{[Ts]}}{\sqrt{T}}, \dfrac{\sum_{t=1}^{[Ts]} u_t}{\sqrt{T}}, \dfrac{\sum_{t=1}^{[Ts]}(u_t^2 - \sigma_u^2)}{\sqrt{T}}\right) \xrightarrow{\mathcal{D}} (\boldsymbol{J}_C(s), \sigma_u W_u(s), \sigma W(s)), \ s \in [0, 1].$

*(ii) Assumption A(iii) holds.*

*(iii) The user-inputs in $\mathcal{S}_T(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ are such that $(\lambda_1^0, \lambda_2^0) \in \Lambda^0$ and $(\tau_0, \lambda_2^0) \in \overline{\Lambda}^0$ respectively.*

The asymptotic power properties of $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2)$ are now summarised in Proposition 4 below.

**PROPOSITION 4**. *(i) Suppose model (2) holds with $\boldsymbol{\beta}_2 \neq 0$ and fixed, then under Assumption*

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*B2 and as $T \to \infty$ we have $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \overset{p}{\to} \infty$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2) \overset{p}{\to} \infty$. (ii) Suppose model (2) holds with $\beta_2 = \gamma/T^{3/4}$ for $\gamma \neq 0$. Under Assumption B2, $\lim_{\|\gamma\| \to \infty} \lim_{T \to \infty} \mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) = \infty$ and $\lim_{\|\gamma\| \to \infty} \lim_{T \to \infty} \overline{\mathcal{S}}_T(\tau_0; \lambda_2) = \infty$ in probability.*

A key message that is conveyed by Proposition 4 when contrasted with Proposition 3 is the important impact of persistence on the power properties the test statistics. The presence of persistent predictors leads to a faster divergence rate for both statistics as reflected in the faster convergence rate towards zero of $\beta_2$ that can be accommodated. With highly persistent predictors, both test statistics diverge at the same $T^{3/2}$ rate compared with a rate of $T^{1/2}$ when predictors were stationary.

   A more explicit formulation of the departure from the null distribution in this local to unit-root context can also be highlighted through the following formulations of the limiting distributions of the two test statistics under the local alternative of interest.

**COROLLARY 2**. *Suppose model (2) holds with $\beta_2 = \gamma/T^{3/4}$ for $\gamma \neq 0$. Under assumption B2 and as $T \to \infty$ we have*

$$\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \overset{\mathcal{D}}{\to} \mathcal{N}(0,1) + \xi^0 \tag{26}$$

$$\overline{\mathcal{S}}_T(\tau_0; \lambda_2) \overset{\mathcal{D}}{\to} \mathcal{N}(0,1) + \overline{\xi} \tag{27}$$

*with*

$$\xi^0 = \frac{\sqrt{1-\pi_0}}{\sigma\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \, \gamma' \left( \frac{1}{(1-\pi_0)\lambda_1^0} \int_{\pi_0}^{\pi_0+(1-\pi_0)\lambda_1^0} \boldsymbol{J}_C^*(s) \boldsymbol{J}_C^*(s)' \right) \gamma, \tag{28}$$

$$\overline{\xi} = \frac{\sqrt{1-\pi_0}}{\sigma\sqrt{\overline{v}(\tau_0; \lambda_2^0)}} \, \gamma' \left( \frac{1}{(1-\tau_0)} \int_{\tau_0}^{1} \frac{1}{(1-\pi_0)\lambda_1} \left( \int_{\pi_0}^{\pi_0+(1-\pi_0)\lambda_1} \boldsymbol{J}_C^*(s) \boldsymbol{J}_C^*(s)' \right) d\lambda_1 \right) \gamma \tag{29}$$

*where $\boldsymbol{J}_C^*(s) = \boldsymbol{J}_{2C}(s) - \boldsymbol{M}(s)\boldsymbol{J}_{1C}(s)$ and $\boldsymbol{M}(s) = (\int_0^s \boldsymbol{J}_{1C}\boldsymbol{J}_{1C}')^{-1}(\int_0^s \boldsymbol{J}_{1C}\boldsymbol{J}_{2C}')$.*

It is here interesting to compare (28)-(29) with the non-centrality parameters (17)-(18) obtained in the stationary context. The two pairs are essentially analogous in the sense that the constant population moments of predictors (i.e. the $\boldsymbol{Q}_{i,j}$'s) are now replaced by stochastic integrals in

Ornstein-Uhlenbeck processes (i.e. $\boldsymbol{J}_{i,C}$). The homogeneity throughout the sample of the limit moment matrix in Assumption B1(i) is of course no longer valid in the context of the stochastic integrals in (28)-(29). The above results also imply that the role played by the pairs $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0)$ in this local to unit-root context will mirror our earlier analysis based on a stationary setting, supporting the same practical implementation of both test statistics in terms of their parameterisations i.e. the power enhancing choices for $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0)$ discussed above continue to hold in the current context.

## 4. Power Enhancements

Here we explore a particular adjustment that can be applied to our two test statistics $\mathcal{S}_T(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ with the purpose of boosting their asymptotic local power properties without affecting their limiting null distributions. The theoretical principle underlying our proposed approach mirrors the idea in Fan, Liao and Yao (2015) where the authors proposed to augment Wald type statistics with a component that vanishes asymptotically under the null while diverging under alternatives of interest. More formally we seek to augment our proposed two test statistics as

$$\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0) \equiv \mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) + h_T^0(\lambda_1^0, \lambda_2^0) \tag{30}$$

$$\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0) \equiv \overline{\mathcal{S}}_T(\tau_0; \lambda_2^0) + \overline{h}_T(\tau_0; \lambda_2^0) \tag{31}$$

for some suitably chosen $h_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{h}_T(\tau_0; \lambda_2^0)$ terms which are such that these adjusted versions of our two test statistics maintain the same limiting null distributions as in Proposition 1 while at the same time displaying more favorable power properties.

In what follows we show that a particular transformation of the forecast errors $\hat{e}_{2,t+1}$ associated with the larger forecasting model can be used to design such augmentation terms in a way that fulfils the requirement that $h_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{h}_T(\tau_0; \lambda_2^0)$ vanish asymptotically under the null while diverging at a desirable rate under the alternative. The augmentation we propose to consider is motivated by the well known Clark and West adjustment to DM type statistics introduced in Clark and West (2007). The original motivation behind Clark and West's adjustment relied on

the intuition that under the null hypothesis estimation noise contaminates the $\hat{e}_{2,t+1}$'s due to the estimation of parameters that are zero in the population. This in turn translates into an inflated $\text{MSE}_2$ resulting in test statistics that are severely undersized. Clark and West proposed to correct for such distortions by suitably adjusting the magnitudes of the forecast errors estimated from the larger model.

To lay down the context and with no loss of generality it is useful to operate within a simplified version of (1)-(2), setting $\boldsymbol{\delta}_1 = 0$ and $\boldsymbol{\beta}_1 = 0$ so that $\hat{e}_{1,t+1} = u_{t+1}$ and $\hat{e}_{2,t+1} = u_{t+1} - \boldsymbol{x}'_{2,t}(\hat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2)$. We can now write $Z_T(\ell_1, \ell_2)$ in (6) as

$$
\frac{\sum_{t=k_0}^{k_0+\ell_1-1} \hat{e}^2_{1,t+1}}{\sqrt{T-k_0}} - \frac{\ell_1}{\ell_2} \frac{\sum_{t=k_0}^{k_0+\ell_2-1} \hat{e}^2_{2,t+1}}{\sqrt{T-k_0}} = \frac{\sum_{t=k_0}^{k_0+\ell_1-1} u^2_{t+1}}{\sqrt{T-k_0}} - \frac{\ell_1}{\ell_2} \frac{\sum_{t=k_0}^{k_0+\ell_2-1} u^2_{t+1}}{\sqrt{T-k_0}}
$$
$$
+ 2\frac{\ell_1}{\ell_2} \frac{\sum_{t=k_0}^{k_0+\ell_2-1} (\hat{\boldsymbol{\beta}}_{2,t} - \boldsymbol{\beta}_2)' \boldsymbol{x}_{2,t} u_{t+1}}{\sqrt{T-k_0}}
$$
$$
- \frac{\ell_1}{\ell_2} \frac{\sum_{t=k_0}^{k_0+\ell_2-1} (\hat{\boldsymbol{\beta}}_{2,t} - \boldsymbol{\beta}_2)' \boldsymbol{x}_{2,t} \boldsymbol{x}'_{2,t} (\hat{\boldsymbol{\beta}}_{2,t} - \boldsymbol{\beta}_2)}{\sqrt{T-k_0}}. \tag{32}
$$

Although it is implicit in our Assumption A(i) that the last two terms in the right hand side of (32) vanish asymptotically under the null hypothesis, in finite samples the rightmost quadratic form is likely to pull down the spread in MSEs causing their null distribution to be mis-centered. Noting that $(\hat{\boldsymbol{\beta}}_{2,t} - \boldsymbol{\beta}_2)' \boldsymbol{x}_{2,t} \boldsymbol{x}'_{2,t} (\hat{\boldsymbol{\beta}}_{2,t} - \boldsymbol{\beta}_2) \equiv (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})^2$, Clark and West (2007)'s proposal was to reformulate the sample MSE spreads between Models 1 and 2 with an adjusted version of $\hat{e}^2_{2,t+1}$, say $\tilde{e}^2_{2,t+1}$, given by

$$
\tilde{e}^2_{2,t+1} = \hat{e}^2_{2,t+1} - (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})^2. \tag{33}
$$

It turns out that implementing the adjustment in (33) within our two test statistics (i.e., using $\tilde{e}^2_{2,t+1}$ instead of $\hat{e}^2_{2,t+1}$ in $\mathcal{S}^0_T(\lambda^0_1, \lambda^0_2)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda^0_2)$) allows us to reformulate them as in (30)-(31) with $h^0_T(\lambda^0_1, \lambda^0_2)$ and $\overline{h}_T(\tau_0; \lambda^2_0)$ fulfilling the desirable requirements in Fan, Liao and Yao (2015) in the sense that the adjustments do not alter the asymptotic null distributions of the test statistics while at the same time leading to an increase in the associated non-centrality parameters under the local alternatives of interest.

The expression in (32) is also useful for highlighting what distinguishes our framework that operates under $\ell_1 \neq \ell_2$ with a standard approach that sets $\ell_1 = \ell_2 = T - k_0$ as for instance in all Diebold-Mariano type statistics. Under $\ell_1 = \ell_2$ we note that the first two terms in the right hand side of (32) cancel out so that the asymptotic behavior of the expression is determined by the two rightmost quadratic forms whose *non-normalised* versions have been shown to be $O_p(1)$ with non-standard limits (see Clark and McCracken (2001, 2005)). Allowing $\ell_1 \neq \ell_2$ essentially forces the asymptotics of the MSE spreads to be driven solely by the first two components in the right hand side of (32).

Letting $\mathcal{S}^0_{T,adj}(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ denote the adjusted versions of our two test statistics it immediately follows from (33) and standard algebra that

$$
\mathcal{S}^0_{T,adj}(\lambda_1^0, \lambda_2^0) = \mathcal{S}^0_T(\lambda_1^0, \lambda_2^0) + \frac{1}{\hat{\sigma}} \frac{1}{\lambda_2^0 \sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \frac{\sum_{t=k_0}^{k_0 + [(T-k_0)\lambda_2^0]} (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})^2}{\sqrt{T - k_0}}
$$

$$
\equiv \mathcal{S}^0_T(\lambda_1^0, \lambda_2^0) + h^0_T(\lambda_1^0, \lambda_2^0) \tag{34}
$$

and

$$
\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0) = \overline{\mathcal{S}}_T(\tau_0; \lambda_2^0) + \frac{1}{\hat{\sigma}} \frac{1}{\lambda_2^0 \sqrt{\overline{v}(\tau_0; \lambda_2^0)}} \frac{\sum_{t=k_0}^{k_0 + [(T-k_0)\lambda_2^0]} (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})^2}{\sqrt{T - k_0}}
$$

$$
\equiv \overline{\mathcal{S}}_T(\tau_0; \lambda_2^0) + \overline{h}_T(\tau_0; \lambda_2^0). \tag{35}
$$

The expressions in (34) and (35) highlight the fact that the adjustment to the MSE of the larger model results in test statistics that are augmented versions of $\mathcal{S}_T(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$. As we establish formally below the presence of the additional terms $h^0_T(\lambda_1^0, \lambda_2^0)$ and $\overline{h}_T(\tau_0; \lambda_2^0)$ leaves the limiting null distributions unchanged as both quantities vanish asymptotically. Under the alternative both $h^0_T(\lambda_1^0, \lambda_2^0)$ and $\overline{h}_T(\tau_0; \lambda_2^0)$ diverge to infinity at the same rate as $\mathcal{S}_T(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ implying that $\mathcal{S}_{T,adj}(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ will also share the consistency and local power characteristics of their unadjusted counterparts in the sense of diverging to infinity as $||\boldsymbol{\gamma}|| \to \infty$. More importantly however, the presence of $h^0_T(\lambda_1^0, \lambda_2^0)$ and $\overline{h}_T(\tau_0; \lambda_2^0)$ does result in different (strictly larger) non-centrality parameters that make these adjusted statistics have more favorable power properties. These features are formalised in Proposition 5 and Corollary 3 below.

**PROPOSITION 5**. *The results in Propositions 1-4 continue to hold when $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ are replaced with $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ respectively.*

**COROLLARY 3**. *(i) Under the assumptions of Corollary 1 (stationary predictors), the asymptotic local power functions of the tests based on $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ are given by $1 - \Phi(q_\alpha - 2\psi^0)$ and $1 - \Phi(q_\alpha - 2\overline{\psi})$ with $\psi^0$ and $\overline{\psi}$ as in (17) and (18). (ii) Under the assumptions of Corollary 2 (persistent predictors) we have $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) + \xi_{adj}^0$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) + \overline{\xi}_{adj}$ where*

$$\xi_{adj}^0 = \xi^0 + \frac{\sqrt{1-\pi_0}}{\sigma\sqrt{v^0(\lambda_1^0, \lambda_2^0)}}\,\boldsymbol{\gamma}'\left(\frac{1}{(1-\pi_0)\lambda_2^0}\int_{\pi_0}^{\pi_0+(1-\pi_0)\lambda_2^0}\boldsymbol{J}_C^*(s)\boldsymbol{J}_C^*(s)'\right)\boldsymbol{\gamma}, \tag{36}$$

$$\overline{\xi}_{adj} = \overline{\xi} + \frac{\sqrt{1-\pi_0}}{\sigma\sqrt{\overline{v}(\tau_0; \lambda_2^0)}}\,\boldsymbol{\gamma}'\left(\frac{1}{(1-\pi_0)\lambda_2^0}\int_{\pi_0}^{\pi_0+(1-\pi_0)\lambda_2^0}\boldsymbol{J}_C^*(s)\boldsymbol{J}_C^*(s)'\right)\boldsymbol{\gamma} \tag{37}$$

*with $\xi^0$ and $\overline{\xi}$ as in (28) and (29).*

Proposition 5 essentially implies that all our results regarding the null limiting distributions of $\mathcal{S}_T(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ and their general power properties (consistency and detectability of local departure from the null) continue to hold for their adjusted counterparts while Corollary 3 documents important differences in their specific non-centrality terms.

Indeed, the results in Corollary 3 are particularly interesting and useful for the practical assessment of the power properties of the adjusted versus unadjusted statistics. We have an environment whereby the limiting distributions of the two types of test statistics are the same under the null hypothesis while their non-centrality parameters differ under the alternative, pointing to a more favorable behavior for the adjusted statistics when it comes to detecting local departures from the null.

Letting $\psi_{adj}^0$ and $\overline{\psi}_{adj}$ denote the non-centrality parameters associated with the adjusted statistics, Corollary 3(i) establishes that in a stationary context we have $\psi_{adj}^0 = 2\psi^0$ and $\overline{\psi}_{adj} = 2\overline{\psi}$ so that $\psi_{adj}^0/\psi_0 = 2$ and $\overline{\psi}_{adj}/\overline{\psi} = 2$. In the case of persistent predictors the comparison between $\xi^0$ and $\xi_{adj}^0$ and between $\overline{\xi}$ and $\overline{\xi}_{adj}$ also indicates that the adjusted quantities will stochastically dominate their non-adjusted counterparts in the sense that $P[\xi_{adj}^0 > q] \geq P[\xi^0 > q]$ and

$P[\bar{\xi}_{adj} > q] \geq P[\bar{\xi} > q]$ for some given critical value $q$ and this is again expected to translate into more favorable power outcomes for the adjusted statistics under persistent predictors as well.

# 5. Empirical Size and Power

In this section we investigate the size and power properties of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ together with their adjusted versions across two DGPs calibrated to commonly encountered applications and sample sizes in macroeconomics and finance. The experiments are designed to emphasise the role of the pairs $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0; \lambda_2^0)$ on inferences with the choice of their magnitudes guided by the analysis surrounding our results in Corollaries 1 and 2.

More specifically, the implementation of $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ is restricted to $\lambda_1^0 = 1$ across $\lambda_2^0 \in \{0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95\}$ and similarly for $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$, thus providing a very broad coverage across a range of user-inputs. The average based statistic $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ and its adjusted version $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ are implemented for $\tau_0 \in \{0.5, 0.8\}$ across $\lambda_2^0 \in \{0.50, 0.60, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.00\}$.

All our size and power simulations below set $\pi_0 = 0.25$ (i.e., $k_0 = [T\ 0.25]$) to initiate the recursively generated forecasts.

## 5.1. DGP1

A specification that mimics a frequently encountered setting in the asset pricing literature is one where the null model is the martingale difference sequence, $y_{t+1} = u_{t+1}$, and the larger model the single predictor based predictive regression, $y_{t+1} = \beta x_t + u_{t+1}$ with $x_t = \phi_1 x_{t-1} + v_t$. Letting $\Sigma = \{\{\sigma_u^2, \rho_{uv}\sigma_u\sigma_v\}, \{\rho_{uv}\sigma_u\sigma_v, \sigma_v^2\}\}$ denote the covariance of $(u_t, v_t)$, in line with commonly encountered magnitudes from the equity premium predictability literature we set $\sigma_u^2 = 3$, $\sigma_v^2 = 0.01$, $\rho_{uv} = -0.8$ and experiment with $\phi_1 \in \{0.75, 0.95, 0.98\}$. The conditionally homoskedastic setting takes $(u_t, v_t) \sim NID(0, \Sigma)$ while conditional heteroskedasticity is modelled via an ARCH(1) specification, writing $u_t = \epsilon_t \sqrt{h_t}$ with $h_t = \alpha_0 + \alpha_1 u_{t-1}^2$ and $\epsilon_t \sim NID(0, 1)$. This latter choice naturally influences the magnitudes of $\sigma_u^2$ and $\rho_{uv}$ chosen above and we parameterise $\{\alpha_0, \alpha_1\}$ in a way that maintains the same magnitude for $\sigma_u^2$ as in the conditionally homoskedastic case i.e. $\alpha_0/(1 - \alpha_1) = \sigma_u^2$. For this purpose we set $(\alpha_0, \alpha_1) = (1.8, 0.4)$ throughout.

Size experiments set $\beta = 0$ while for the power properties of the tests we fix the sample size at $T = 500$ and evaluate correct decision frequencies as $\beta$ moves away from the null with $\beta \in \{0, -1.5, -1.75, -2.0, -2.25, -2.5, -3, -3.5\}$. For $\beta = \gamma/T^{1/4}$ this is equivalent to $|\gamma|$ increasing with $\gamma \in \{0, -7.1, -8.3, -9.5, -10.6, -11.8, -14.2, -16.6\}$. Lastly, all of the above experiments are conducted using two alternative estimators for $\sigma$. The first one denoted $\hat{\sigma}^2_{hom}$ is suitable under conditional homoskedasticity while the second one denoted $\hat{\sigma}^2_{nw}$ is its robustified version à la Newey-West. Both estimators are based on the residuals from the model estimated under the alternative.

As our Monte-Carlo simulations encompass a very broad range of scenarios and test statistic parameterisations we provide an extensive selection of outcomes in a supplementary online appendix accompanying this paper. Our focus below is on a selection of key size/power results under conditional homoskedasticity and test statistic parameterisations that mainly rely on our recommendations based on our theoretical local power analysis above.

### *Empirical Size*

Table 1 presents size estimates for $\mathcal{S}^0_T(\lambda^0_1 = 1, \lambda^0_2)$ and $\mathcal{S}^0_{T,adj}(\lambda^0_1 = 1, \lambda^0_2)$ across a broad range of parameterisations in a conditionally homoskedastic setting. For both test statistics we note good to excellent matches of the nominal size of 10% across almost all choices of $\lambda^0_2$ for $T \geq 500$. The adjusted statistic $\mathcal{S}^0_{T,adj}(\lambda^0_1 = 1, \lambda^0_2)$ in particular has empirical sizes that almost perfectly match 10% for virtually all magnitudes of $\lambda^0_2$. Under $\phi_1 = 0.75$ for instance, $\mathcal{S}^0_{T,adj}(\lambda^0_1 = 1, \lambda^0_2 = 0.8)$ has resulted in an empirical size of 10.4% for $T = 1000$ and 11.0% for $T = 500$. The corresponding figures for $\phi_1 = 0.95$ were 10.6% and 11.2% respectively, thus also highlighting the robustness of the test statistics to the degree of persistence of the predictors as expected from our results in Propositions 1-2. Similar outcomes also characterise $\mathcal{S}^0_{T,adj}(\lambda^0_1 = 1, \lambda^0_2 = 0.9)$ suggesting that these test statistics maintain good size control for $T \geq 500$ even when $\lambda^0_2$ is as large as 0.90 or 0.95 and $\lambda^0_1$ is set equal to one.

### Table 1

Regarding the size properties of the unadjusted $\mathcal{S}^0_T(\lambda^0_1, \lambda^0_2)$ statistic we note a mild undersizeness for magnitudes of $\lambda^0_2$ that are in the vicinity of 1, with its empirical sizes clustered around 7-8%.

Overall the outcomes in Table 1 have highlighted remarkably stable size properties for both the unadjusted and adjusted statistics across the different magnitudes of $\lambda_2^0$ including when it is set at 0.90 or 0.95. This is particularly reassuring given our earlier theoretical power analysis which pointed at desirable parameterisations that satisfy $\lambda_1^0 \approx \lambda_2^0$ with both $\lambda_1^0$ and $\lambda_2^0$ set in the vicinity of 1 in the practical implementation of $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$.

Before proceeding further it is also useful to briefly rationalise the size behavior of these two statistics when $\lambda_2^0$ is chosen to lie almost at its boundary as when we set $\lambda_2^0 = 0.95$. In such instances we noted the mild undersizeness of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and mild oversizeness of $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ when operating with small to moderately sized samples. A magnitude of $\lambda_2^0$ that is close to 1 essentially translates into more 'MSE content' from the larger model and hence a greater exposure to estimation noise when the null model holds true. This results in the unadjusted $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ test statistic's distribution being pushed leftward with fewer than expected rejections of the null. On the other hand the adjusted statistic which aims to correct for estimation error via (32) sees its correction factor's contribution increase as $\lambda_2^0 \to 1$, a correction factor that is overly inflated in small samples. At this stage it is also useful to point out that the effective sample size is given by $T - k_0$ so that under $\pi_0 = 0.25$ and $T = 250$ we have only about 188 data points when implementing the tests. A highly persistent predictor combined with such a small sample size can be seen to result in some degree of oversizeness for $S_{T,adj}(\lambda_1^0, \lambda_2^0)$ based inferences when $|\lambda_1^0 - \lambda_2^0|$ is particularly small (e.g., for $(\lambda_1^0, \lambda_2^0) = (1, 0.95)$). Nevertheless, these finite sample distortions quickly fade away as we increase the sample size to $T = 500$.

For comparison purposes the last column of Table 1 also includes the corresponding size estimates for the DM and CW statistics. These conform with the consensus view that the DM statistic is severely undersized under such nested settings while the CW statistics' empirical sizes are clustered around 5% for a nominal size of 10%, in line with the simulation results in Clark and West (2007).

We next consider the finite sample size properties of the average based statistics $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$. Recall that the averaging is performed across a portion of the null model's MSE as captured by $\tau_0$ and for a given fraction of the second model's MSE $\lambda_2^0$. Here we present outcomes obtained under $\tau_0 = 0.8$ which only sum across the larger magnitudes of $\lambda_1$. Such a choice is theoretically justified by our earlier power analysis with further scenarios presented in the supplementary appendix. Results are presented in Table 2 from which we note that the

adjusted statistic $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ displays good to excellent size control (e.g. empirical size estimates near 10% under $\lambda_2^0 = 1$) under moderate to large sample size choices.

## Table 2

An exception to this is when $\lambda_2^0 \approx 0.5 \ \tau_0 + 0.5 (\approx 0.9 \ here)$ under which it shows a tendency to overreject the null hypothesis in smaller samples. This is in complete agreement with our earlier theoretical power analysis where we showed that holding all else constant the power of the test statistic must peak under $\lambda_2^0 = 0.5\tau_0 + 0.5$. Thus the empirical sizes peaking for $\lambda_2^0$ in the vicinity of $0.5(0.8) + 0.5 = 0.90$ highlight the size vs power trade-off that will characterise this average based test statistic.

Regarding the unadjusted statistic $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ we can note a tendency to underreject (e.g. empirical sizes in the vicinity of 7% under $\tau_0 = 0.8$) and this undersizeness deteriorating as $\lambda_2^0 \to 1$ and $\phi_1$ gets closer to 1. This behavior conforms with the intuition that estimation noise caused by the estimation of parameters that are zero in the population pushes the test statistic too much to the left, a feature that was the key motivation behind Clark and West's adjustment to the DM statistic. Note for instance that these distortions are substantially dampened when the test statistic is implemented with smaller magnitudes of $\lambda_2^0$ for which it shows good to excellent size control.

### *Empirical Power*

Table 3 presents empirical power estimates for $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ across $\lambda_2^0 \in \{0.80, 0.85, 0.90, 0.95\}$. The sample size is fixed at $T = 500$ and power is evaluated as the DGP moves further away from the null hypothesis. The choices of $\lambda_2^0$ are dictated by our theoretical results in Corrollaries 1-2 which pointed to magnitudes satisfying $\lambda_1^0 \approx \lambda_2^0 \approx 1$. For both test statistics we note the tendency of their empirical power to converge to 1 as $|\gamma|$ is allowed to increase. We can also clearly observe the particularly favorable impact that the degree of persistence of predictors has on power. As expected from our findings in Propositions 3-4 and their corollaries, power improves as $\lambda_2^0 \to 1$ and as $\phi_1 \to 1$.

## Table 3

Given the good overall size control displayed by $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ Table 3 clearly highlights the benefits of basing inferences on this adjusted version of our first test statistic calibrated to $\lambda_1^0 = 1$

and $\lambda_2^0 \approx 0.9$, also noting that its performance improves considerably as $\phi_1 \to 1$. For $\beta_2 = -2$ for instance it displays power in the vicinity of 70%-80% under $\phi_1 = 0.75$ and 100% under $\phi_1 = 0.95$ or $\phi_1 = 0.98$. It is here important to relate our simulation outcomes with the theoretical results of Corollary 3 where we established an ARE of 2 for the adjusted statistic $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ relative to its unadjusted counterpart. This theoretical power enhancement is clearly supported by the empirical power estimates of Table 3. Compare for instance the empirical power of 51.2% for the unadjusted statistic under $\phi_1 = 0.75$ and $\beta = -2.25$ with 81.2% for its adjusted version, a power gain of 30 percentage points.

At this stage it is also important to recall that our theoretical analysis based on the asymptotic relative efficiency of $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ versus $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ clearly pointed to the potentially superior power performance of the average based statistic, for larger magnitudes of $\tau_0$ in particular. This is clearly corroborated by the comparison between the power outcomes in Table 3 and Table 4 with the latter presenting power outcomes for the $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ statistics. Focusing on the "optimal" choice of $\lambda_2^0 = 0.5\tau_0 + 0.5 = 0.90$ when $\tau_0 = 0.8$ we note from Table 4 that $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 0.9)$ clearly dominates all configurations of $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ in terms of its power properties, typically resulting in relative power gains in excess of 10 percentage points.

**Table 4**

Before proceeding further it is also useful to comment on the power behavior of the DM and CW statistics in comparison to $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0 = 0.90)$. Despite being severely undersized and theoretically unsuitable in the present nested context we note that the DM statistic does show a reasonable ability to detect departures from the null. However we can also observe that it is uniformly dominated by $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 0.90)$ which under $\phi_1 = 0.75$ for instance exceeds its power by about ten percentage points. Comparing the power performance of $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 0.90)$ with that of the CW statistic we note that these two test statistics display very similar power outcomes across most scenarios. Although the CW statistic does not have a well defined limiting distribution due to the nestedness of the competing models it appears to display reasonably good power properties across the DGPs we have considered, despite being far-off the standard normal distribution under the null (as implied by its size properties).

## 5.2. DGP2

The second DGP allows for multiple predictors and is calibrated to mimic US inflation based predictive regressions as considered in Stock and Watson (2010) and Granziera, Hubrich and Moon (2014). We use the same setting as in Granziera, Hubrich and Moon (2014) and consider a DGP given by $y_{t+1} = \mu + \rho y_t + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + u_{t+1}$ with $\mu = 1$ and $\rho = 0.25$. The predictors $\boldsymbol{x}_t = (x_{1,t}, x_{2,t}, x_{3,t})'$ follow the VAR(1) process $\boldsymbol{x}_t = \Phi \, \boldsymbol{x}_{t-1} + \boldsymbol{v}_t$ with $\Phi = \{\{0.6, 0.1, 0\}, \{0.6, 0.25, 0\}, \{0, 0, 0.9\}\}$ thus encompassing both persistent and much noisier processes while also being interdependent. The conditionally homoskedastic scenario takes $(u_t, v_{1,t}, v_{2,t}, v_{3,t})' \sim NID(0, I_4)$ while conditional heteroskedasticity is captured via an ARCH(1) process for $u_t$ as in DGP1 with $\alpha_0 = 0.6$ and $\alpha_1 = 0.4$ so that its unconditional variance matches unity as in the homoskedastic scenario.

For our size experiments we set $\beta_1 = \beta_2 = \beta_3 = 0$ and our power analysis focuses on alternatives to $\beta_1 = \beta_2 = \beta_3 = 0$ by fixing $(\beta_1, \beta_2, \beta_3) = (0.15, 0.15, -0.15)$ and evaluating rejection rates of the null hypothesis for $T = 250, 500, 1000$.

### *Empirical Size*

Tables 5 and 6 present empirical size estimates corresponding to the null DGP under $\beta_1 = \beta_2 = \beta_3 = 0$ for $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0 = 0.80; \lambda_2^0)$ respectively together with their adjusted versions. As DGP2 contains a larger number of predictors than DGP1 we expect the impact of estimation error on the MSE of the second/larger model to be more pronounced under the null. This is indeed corroborated by the size estimates in Table 5 where we note the undersizeness of the unadjusted $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ statistic which is biased downward and thus results in too few rejections of the null (e.g. 6.4% under $T = 1000$ and $\lambda_2^0 = 0.9$ versus a nominal size of 10%). Furthermore, its undersizeness tends to deteriorate for larger magnitudes of $\lambda_2^0$ as this translates into an increased influence of the larger model's MSE.

### Tables 5-6

The adjusted version of the test statistic $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ on the other hand is quite effective in adjusting for estimation noise (e.g. the earlier empirical size of 6.4% is now pushed up to 12%), while also showing a tendency to "over-adjust" in small to moderately size samples, in particular

for larger magnitudes of $\lambda_2^0$. Table 6 presents the corresponding size estimates for the average based statistic $\overline{\mathcal{S}}_{T,adj}(\boldsymbol{\tau}_0 = 0.8; \lambda_2^0)$. We note that this latter statistic maintains good to excellent size control for moderate sample sizes across all magnitudes of $\lambda_2^0$ but requires larger samples when $\lambda_2^0$ is set near one.

*Empirical Power*

For this DGP our power experiments focus on documenting the rejection frequencies of the null hypothesis for a fixed alternative as the sample size is allowed to increase. Results are presented in Tables 7-8 for $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\boldsymbol{\tau}_0 = 0.8; \lambda_2^0)$ and their adjusted versions. Under either $T = 500$ or $T = 1000$ all test statistics have powers at or near 100%.

**Tables 7-8**

Tables 7-8 also clearly corroborate our theoretical power analysis that highlighted peaking powers under $\lambda_2^0 = 0.5\tau_0 + 0.5$. Focusing on $\overline{\mathcal{S}}_T(\boldsymbol{\tau}_0 = 0.8; \lambda_2^0)$ we can clearly observe the empirical powers to be largest under $\lambda_2^0 = 0.9$ for all sample sizes (e.g. 98.2% versus 95.7% when $\lambda_2^0 = 1$ and for $T = 250$).

The main findings from our simulation experiments can be summarised as follows. (i) The adjusted versions of the two test statistics $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\boldsymbol{\tau}_0; \lambda_2^0)$ displayed good to excellent size control across most parameterisations of their respective inputs $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0)$. (ii) Both test statistics are consistent and have non-trivial local asymptotic power while their finite sample power properties are strongly influenced by the respective magnitudes of those same inputs. The guidelines provided by our theoretical power analysis do however lead to highly favorable power outcomes with implementations such as $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0 \approx 0.9)$ and $\overline{\mathcal{S}}_{T,adj}(\boldsymbol{\tau}_0 \approx 0.8; \lambda_2^0 \approx 0.9)$ standing out in terms of their size/power trade-offs, especially for moderately sized samples such as $T \geq 500$. (iii) The proposed methods are valid irrespective of the degree of persistence of the predictors as also corroborated by our finite sample simulations. (iv) Our Monte-Carlo analysis did show that Clark and West's CW statistic which although not grounded on formal standard normal asymptotics also performed particularly well in terms of its power, despite relatively important size distortions. Strictly speaking the CW statistic has

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been introduced for handling nested models estimated via a rolling as opposed to a recursive approach since from a theoretical standpoint it continues to suffer from the variance degeneracy problem characterising DM type constructions.

### 5.3. Summary and tuning-parameter guidelines

The above simulation based outcomes combined with our earlier local power analysis point to precise guidelines for the choice of tuning parameters required in the implementation of our test statistics. For the $\mathcal{S}^0_{T,adj}(\lambda^0_1, \lambda^0_2)$ statistic we argued that $\lambda^0_1$ and $\lambda^0_2$ should be set near their boundary of one and in close vicinity of one another. Our simulations based on $\mathcal{S}^0_{T,adj}(\lambda^0_1 = 1, \lambda^0_2)$ for $\lambda^0_2$ set in the 0.80-0.95 range have indeed resulted in good to excellent size-power tradeoffs and good to excellent size control. The robustness of the empirical size outcomes to a much broader range of the $\lambda^0_2$ magnitudes is also noteworthy as illustrated by the outcomes in Tables 1 and 5.

Regarding the $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda^0_2)$ statistic, our theoretical local power analysis led us to argue for $\tau_0$ to be set in the vicinity of unity and $\lambda^0_2$ as $\lambda^0_2 = 0.5\,\tau_0 + 0.5$. Simulations based on $\tau_0 = 0.8$ did indeed result in good to excellent size and power properties. As the above choice for $\lambda^0_2$ is a maximiser of local power it is perhaps natural to expect some size distortions in smaller samples when $\lambda^0_2$ is set in this way and in particular when this is combined with the presence of highly persistent predictors. This is indeed confirmed by our size experiments implemented under $(\tau_0; \lambda^0_2) = (0.8, 0.9)$ and a local-to-unity type predictor. Nevertheless, these specific finite sample distortions can also be seen to progressively vanish as the sample size is allowed to grow.

## 6. Application

We illustrate the implementation of our proposed methods by revisiting a widely considered puzzle in the international economics literature, namely the random walk like behavior of exchange rates. Our goal is to use our new test statistics in order to evaluate whether past exchange rate levels have any predictive power for subsequent exchange rate changes. Letting $s_t$ denote the log of the spot exchange rate, we compare the out of sample predictive accuracy of

the larger model $\Delta s_{t+1} = \alpha + \beta\, s_t + u_{t+1}$ (model 2) with the random walk with drift specification $\Delta s_{t+1} = \alpha + u_{t+1}$ (model 1).

We consider six major currencies (EUR, YEN, GBP, CHF, AUD and CAD) and implement our tests on daily spot rates spanning the period between 1999/01/04 and 2021/07/16, sourced from the Saint-Louis Fred database. An important advantage of the methods developed in this paper is their robustness to the persistence properties of predictors which is particularly relevant when considering exchange rate series. Indeed, for all three daily series we have considered the first order autocorrelation coefficient from an AR(1) fit is 0.99.

**Table 9**

Predictive accuracy testing outcomes (p-values) based on $\mathcal{S}^0_{T,adj}(\lambda^0_1, \lambda^0_2)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda^0_2)$ are presented in Table 9 where we used $\pi_0 = 0.5$ to initiate the expanding window estimation (i.e., starting from the middle of the sample). For robustness considerations inferences based on $\mathcal{S}_{T,adj}(\lambda^0_1 = 1, \lambda^0_2)$ are implemented across $\lambda^0_2 = \{0.80, 0.85, 0.90, 0.95\}$ while for $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda^0_2 = 1)$ we consider $\tau_0 = 0.80$ and $\tau_0 = 0.90$. Looking at the top and middle panels of Table 9 we note that results unanimously corroborate the fact that the level of exchange rates does not have any meaningful forecasting power for future currency returns over the period considered and across all major currencies. This result based on the use of daily data also corroborates the recent findings in Engel and Wu (2021) based on monthly frequencies. The bottom panel of Table 9 displays the p-values associated with the standard DM and the CW statistics. It is here interesting to point out that inferences based on the standard CW statistic lead to a rejection of the random walk specification for the EURUSD and CADUSD series when implementing the test at a 5% level or above which is in sharp contrast with the large p-values obtained using our two test statistics.

## 7. Concluding Remarks

The main motivation of this paper was to provide a way of bypassing the variance degeneracy problem that arises in the context of out-of-sample nested model comparisons. We did so by developing two new test statistics shown to have nuisance parameter free standard normal asymptotics and good power properties including in the close vicinity of the null hypothesis.

Our proposed inferences can trivially accommodate conditional heteroskedasticity and are also shown to be robust to the presence of highly persistent predictors. Although our power analysis has ruled out the case of deterministic trends via Assumption B1 for instance, these can also be easily accommodated within our framework without any changes to the implementation of the tests provided that the trends are formulated in a scaled form as $(t/T)$ and its powers. Nested comparisons in purely deterministic environments would be particularly relevant in areas such as temperature modelling (e.g., Wu and Zhao (2017), Gades-Riva and Gonzalo (2020)) or the recent literature on modelling pandemic dynamics (e.g., Jiang, Zhao and Shao (2020), Li and Linton (2021)).

Although our proposed test statistics require two user inputs each, both have been shown to display good to excellent size control across a very broad range of such parameterisations in a multitude of empirically relevant settings. Although these user-inputs do have considerable influence on the finite sample power properties of both test statistics their choices can be accurately guided by examining the power functions associated with each test statistic, as demonstrated by our simulations and their theoretical backing.

It is important to recognise that our proposed test statistics do involve the discarding of some information albeit very limited and with its amount under the control of the user. As a result some power loss is of course unavoidable but the absence of any alternative approach that uses more information while achieving the same purpose in an environment that can accommodate both stationary and persistent predictors as well as conditional heteroskedasticity makes such power losses only notional. Our simulation results have indeed shown that very little needs to be discarded for our methods to work well and to provide reliable inferences. In this sense they are not subject to the disadvantages of sample splitting based techniques used for instance in the goodness of fit literature (e.g. half-sample methods).

The principles underlying our proposed inferences based on (6) should also be portable beyond out of sample forecasting considerations to areas involving model selection testing à la Vuong (1989) where nestedness versus non-nestedness or the overlapping nature of models being compared influences test procedures due to variance degeneracy problems (see also Shi (2015)). In Schennah and Wilhelm (2017) for instance, the authors developed a model selection test for choosing between two parametric likelihoods based on sample splitting principles which although different from our approach based on MSE comparisons on overlapping intervals was

driven by similar concerns. Adapting the analysis of this paper to such model selection testing contexts is a promising avenue currently being explored.

Table 1: **DGP1** Empirical size of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity and 10% nominal size

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $\phi = 0.75$ | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.080 | 0.086 | 0.086 | 0.084 | 0.086 | 0.084 | 0.077 | 0.080 | 0.080 | 0.072 | 0.006 |
| | T=500 | 0.088 | 0.086 | 0.089 | 0.087 | 0.092 | 0.090 | 0.093 | 0.093 | 0.088 | 0.083 | 0.006 |
| | T=1000 | 0.098 | 0.098 | 0.099 | 0.089 | 0.088 | 0.089 | 0.090 | 0.093 | 0.096 | 0.086 | 0.007 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.099 | 0.106 | 0.106 | 0.107 | 0.110 | 0.110 | 0.109 | 0.116 | 0.120 | 0.126 | 0.055 |
| | T=500 | 0.102 | 0.100 | 0.104 | 0.104 | 0.108 | 0.108 | 0.110 | 0.114 | 0.117 | 0.124 | 0.055 |
| | T=1000 | 0.106 | 0.107 | 0.108 | 0.102 | 0.101 | 0.103 | 0.104 | 0.109 | 0.117 | 0.111 | 0.055 |
| | | | | | | $\phi = 0.95$ | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.076 | 0.077 | 0.077 | 0.074 | 0.074 | 0.072 | 0.069 | 0.069 | 0.069 | 0.062 | 0.006 |
| | T=500 | 0.083 | 0.084 | 0.084 | 0.083 | 0.086 | 0.085 | 0.081 | 0.078 | 0.075 | 0.075 | 0.008 |
| | T=1000 | 0.088 | 0.089 | 0.085 | 0.088 | 0.089 | 0.086 | 0.090 | 0.087 | 0.089 | 0.084 | 0.007 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.102 | 0.107 | 0.107 | 0.109 | 0.108 | 0.115 | 0.112 | 0.118 | 0.124 | 0.140 | 0.064 |
| | T=500 | 0.098 | 0.102 | 0.103 | 0.104 | 0.108 | 0.108 | 0.108 | 0.108 | 0.110 | 0.124 | 0.059 |
| | T=1000 | 0.098 | 0.099 | 0.096 | 0.098 | 0.101 | 0.101 | 0.106 | 0.105 | 0.112 | 0.114 | 0.056 |
| | | | | | | $\phi = 0.98$ | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.078 | 0.079 | 0.078 | 0.077 | 0.075 | 0.073 | 0.073 | 0.071 | 0.074 | 0.068 | 0.011 |
| | T=500 | 0.082 | 0.081 | 0.081 | 0.081 | 0.080 | 0.078 | 0.080 | 0.075 | 0.072 | 0.062 | 0.010 |
| | T=1000 | 0.085 | 0.086 | 0.090 | 0.088 | 0.087 | 0.090 | 0.088 | 0.086 | 0.084 | 0.073 | 0.007 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.115 | 0.117 | 0.119 | 0.122 | 0.121 | 0.128 | 0.128 | 0.133 | 0.150 | 0.170 | 0.084 |
| | T=500 | 0.104 | 0.103 | 0.106 | 0.106 | 0.107 | 0.105 | 0.113 | 0.119 | 0.122 | 0.129 | 0.072 |
| | T=1000 | 0.098 | 0.100 | 0.104 | 0.103 | 0.102 | 0.104 | 0.108 | 0.110 | 0.113 | 0.109 | 0.060 |

Table 2: **DGP1** Empirical size of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity and 10% nominal size

| $\lambda_2^0$ | | 0.500 | 0.600 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $\phi = 0.75$ | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.085 | 0.088 | 0.086 | 0.082 | 0.075 | 0.076 | 0.061 | 0.042 | 0.047 | 0.006 |
| | T=500 | 0.090 | 0.092 | 0.088 | 0.084 | 0.085 | 0.080 | 0.066 | 0.056 | 0.065 | 0.006 |
| | T=1000 | 0.097 | 0.097 | 0.089 | 0.089 | 0.086 | 0.083 | 0.071 | 0.073 | 0.073 | 0.007 |
| | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.104 | 0.111 | 0.117 | 0.120 | 0.124 | 0.151 | 0.154 | 0.121 | 0.110 | 0.055 |
| | T=500 | 0.103 | 0.110 | 0.108 | 0.107 | 0.119 | 0.130 | 0.139 | 0.116 | 0.106 | 0.055 |
| | T=1000 | 0.108 | 0.108 | 0.103 | 0.106 | 0.108 | 0.118 | 0.121 | 0.113 | 0.099 | 0.055 |
| | | | | | | $\phi = 0.95$ | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.078 | 0.077 | 0.078 | 0.072 | 0.068 | 0.067 | 0.053 | 0.038 | 0.043 | 0.006 |
| | T=500 | 0.083 | 0.085 | 0.087 | 0.085 | 0.079 | 0.072 | 0.055 | 0.046 | 0.054 | 0.008 |
| | T=1000 | 0.086 | 0.086 | 0.088 | 0.081 | 0.084 | 0.077 | 0.067 | 0.058 | 0.069 | 0.007 |
| | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.108 | 0.112 | 0.121 | 0.126 | 0.138 | 0.161 | 0.169 | 0.140 | 0.115 | 0.064 |
| | T=500 | 0.102 | 0.106 | 0.114 | 0.118 | 0.122 | 0.134 | 0.139 | 0.118 | 0.104 | 0.059 |
| | T=1000 | 0.097 | 0.097 | 0.104 | 0.102 | 0.108 | 0.116 | 0.126 | 0.104 | 0.098 | 0.056 |
| | | | | | | $\phi = 0.98$ | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.079 | 0.082 | 0.074 | 0.072 | 0.070 | 0.067 | 0.057 | 0.043 | 0.039 | 0.011 |
| | T=500 | 0.080 | 0.079 | 0.078 | 0.075 | 0.077 | 0.065 | 0.053 | 0.043 | 0.053 | 0.010 |
| | T=1000 | 0.083 | 0.088 | 0.088 | 0.082 | 0.084 | 0.074 | 0.061 | 0.059 | 0.067 | 0.007 |
| | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.119 | 0.127 | 0.131 | 0.144 | 0.157 | 0.189 | 0.206 | 0.174 | 0.143 | 0.084 |
| | T=500 | 0.104 | 0.109 | 0.110 | 0.116 | 0.134 | 0.148 | 0.159 | 0.129 | 0.119 | 0.072 |
| | T=1000 | 0.097 | 0.102 | 0.107 | 0.104 | 0.115 | 0.127 | 0.130 | 0.108 | 0.101 | 0.060 |

Table 3: **DGP1** Empirical Power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity

| $\beta$ | -1.500 | -1.750 | -2.000 | -2.250 | -2.500 | -3.000 | -3.500 |
|---|---|---|---|---|---|---|---|
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | | | $\phi_1 = 0.75$ | | | |
| $\lambda_2^0 = 0.80$ | 0.194 | 0.241 | 0.308 | 0.371 | 0.442 | 0.615 | 0.770 |
| $\lambda_2^0 = 0.85$ | 0.213 | 0.272 | 0.349 | 0.427 | 0.519 | 0.685 | 0.831 |
| $\lambda_2^0 = 0.90$ | 0.245 | 0.328 | 0.412 | 0.512 | 0.605 | 0.779 | 0.895 |
| $\lambda_2^0 = 0.95$ | 0.317 | 0.425 | 0.542 | 0.653 | 0.741 | 0.877 | 0.956 |
| DM | 0.312 | 0.440 | 0.566 | 0.666 | 0.752 | 0.877 | 0.946 |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.350 | 0.447 | 0.565 | 0.671 | 0.758 | 0.895 | 0.966 |
| $\lambda_2^0 = 0.85$ | 0.399 | 0.511 | 0.635 | 0.732 | 0.815 | 0.933 | 0.983 |
| $\lambda_2^0 = 0.90$ | 0.468 | 0.590 | 0.712 | 0.812 | 0.879 | 0.960 | 0.992 |
| $\lambda_2^0 = 0.95$ | 0.579 | 0.708 | 0.819 | 0.897 | 0.941 | 0.986 | 0.998 |
| CW | 0.740 | 0.853 | 0.924 | 0.965 | 0.984 | 0.997 | 1.000 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | | | $\phi_1 = 0.95$ | | | |
| $\lambda_2^0 = 0.80$ | 0.615 | 0.747 | 0.843 | 0.917 | 0.952 | 0.990 | 0.997 |
| $\lambda_2^0 = 0.85$ | 0.681 | 0.801 | 0.885 | 0.945 | 0.968 | 0.994 | 0.999 |
| $\lambda_2^0 = 0.90$ | 0.753 | 0.860 | 0.924 | 0.967 | 0.984 | 0.998 | 0.999 |
| $\lambda_2^0 = 0.95$ | 0.845 | 0.924 | 0.962 | 0.984 | 0.993 | 0.999 | 1.000 |
| DM | 0.857 | 0.925 | 0.959 | 0.979 | 0.989 | 0.998 | 0.999 |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.850 | 0.925 | 0.965 | 0.988 | 0.995 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.885 | 0.949 | 0.980 | 0.992 | 0.997 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.919 | 0.966 | 0.986 | 0.996 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.955 | 0.986 | 0.993 | 0.999 | 1.000 | 1.000 | 1.000 |
| CW | 0.985 | 0.996 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | | | $\phi_1 = 0.98$ | | | |
| $\lambda_2^0 = 0.80$ | 0.853 | 0.927 | 0.964 | 0.984 | 0.992 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.886 | 0.949 | 0.977 | 0.990 | 0.996 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.921 | 0.966 | 0.986 | 0.994 | 0.998 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.956 | 0.981 | 0.994 | 0.997 | 0.999 | 1.000 | 1.000 |
| DM | 0.956 | 0.982 | 0.992 | 0.997 | 0.998 | 1.000 | 1.000 |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.957 | 0.982 | 0.995 | 0.998 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.968 | 0.988 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.981 | 0.992 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.991 | 0.996 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| CW | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 4: **DGP1** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity

| $\beta$ | -1.500 | -1.750 | -2.000 | -2.250 | -2.500 | -3.000 | -3.500 |
|---|---|---|---|---|---|---|---|
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | | | $\phi_1 = 0.75$ | | | |
| $\lambda_2^0 = 0.80$ | 0.272 | 0.345 | 0.450 | 0.548 | 0.635 | 0.800 | 0.911 |
| $\lambda_2^0 = 0.85$ | 0.363 | 0.475 | 0.600 | 0.695 | 0.785 | 0.904 | 0.965 |
| $\lambda_2^0 = 0.90$ | 0.449 | 0.567 | 0.685 | 0.782 | 0.853 | 0.942 | 0.980 |
| $\lambda_2^0 = 0.95$ | 0.383 | 0.495 | 0.615 | 0.723 | 0.803 | 0.911 | 0.967 |
| $\lambda_2^0 = 1$ | 0.305 | 0.397 | 0.498 | 0.602 | 0.695 | 0.840 | 0.926 |
| DM | 0.312 | 0.440 | 0.566 | 0.666 | 0.752 | 0.877 | 0.946 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.501 | 0.613 | 0.735 | 0.824 | 0.889 | 0.969 | 0.992 |
| $\lambda_2^0 = 0.85$ | 0.632 | 0.747 | 0.849 | 0.914 | 0.955 | 0.990 | 0.998 |
| $\lambda_2^0 = 0.90$ | 0.704 | 0.815 | 0.892 | 0.945 | 0.973 | 0.994 | 0.999 |
| $\lambda_2^0 = 0.95$ | 0.642 | 0.763 | 0.853 | 0.920 | 0.955 | 0.990 | 0.998 |
| $\lambda_2^0 = 1$ | 0.541 | 0.659 | 0.768 | 0.853 | 0.915 | 0.974 | 0.993 |
| CW | 0.740 | 0.853 | 0.924 | 0.965 | 0.984 | 0.997 | 1.000 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | | | $\phi_1 = 0.95$ | | | |
| $\lambda_2^0 = 0.80$ | 0.771 | 0.870 | 0.925 | 0.967 | 0.982 | 0.997 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.865 | 0.934 | 0.969 | 0.987 | 0.994 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.904 | 0.956 | 0.976 | 0.992 | 0.996 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.876 | 0.936 | 0.966 | 0.986 | 0.993 | 0.999 | 1.000 |
| $\lambda_2^0 = 1$ | 0.802 | 0.889 | 0.939 | 0.970 | 0.985 | 0.997 | 0.999 |
| DM | 0.857 | 0.925 | 0.959 | 0.979 | 0.989 | 0.998 | 0.999 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.922 | 0.970 | 0.989 | 0.996 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.960 | 0.987 | 0.995 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.974 | 0.990 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.964 | 0.989 | 0.994 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 1$ | 0.941 | 0.975 | 0.989 | 0.997 | 0.999 | 1.000 | 1.000 |
| CW | 0.985 | 0.996 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | | | $\phi_1 = 0.98$ | | | |
| $\lambda_2^0 = 0.80$ | 0.925 | 0.966 | 0.985 | 0.994 | 0.997 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.960 | 0.983 | 0.994 | 0.998 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.972 | 0.990 | 0.996 | 0.999 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.960 | 0.982 | 0.994 | 0.998 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 1$ | 0.933 | 0.970 | 0.986 | 0.995 | 0.998 | 1.000 | 1.000 |
| DM | 0.956 | 0.982 | 0.992 | 0.997 | 0.998 | 1.000 | 1.000 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.978 | 0.992 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.990 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.994 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.991 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 1$ | 0.984 | 0.995 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 |
| CW | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 5: **DGP2** Empirical Size of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.055 | 0.055 | 0.055 | 0.054 | 0.054 | 0.050 | 0.047 | 0.046 | 0.043 | 0.037 | 0.002 |
| | T=500 | 0.066 | 0.065 | 0.065 | 0.063 | 0.065 | 0.061 | 0.060 | 0.055 | 0.051 | 0.044 | 0.002 |
| | T=1000 | 0.078 | 0.079 | 0.078 | 0.073 | 0.073 | 0.071 | 0.073 | 0.070 | 0.064 | 0.053 | 0.001 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.110 | 0.110 | 0.114 | 0.117 | 0.119 | 0.120 | 0.124 | 0.134 | 0.151 | 0.172 | 0.074 |
| | T=500 | 0.099 | 0.099 | 0.103 | 0.105 | 0.109 | 0.110 | 0.114 | 0.118 | 0.123 | 0.139 | 0.065 |
| | T=1000 | 0.105 | 0.104 | 0.104 | 0.102 | 0.103 | 0.102 | 0.110 | 0.115 | 0.120 | 0.127 | 0.066 |

Table 6: **DGP2** Empirical Size of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.054 | 0.054 | 0.054 | 0.052 | 0.049 | 0.046 | 0.039 | 0.034 | 0.028 | 0.022 | 0.019 | 0.002 |
| | T=500 | 0.066 | 0.066 | 0.066 | 0.061 | 0.064 | 0.058 | 0.052 | 0.039 | 0.028 | 0.023 | 0.027 | 0.002 |
| | T=1000 | 0.074 | 0.074 | 0.070 | 0.070 | 0.069 | 0.064 | 0.061 | 0.049 | 0.037 | 0.034 | 0.043 | 0.001 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.112 | 0.115 | 0.123 | 0.129 | 0.132 | 0.144 | 0.165 | 0.208 | 0.233 | 0.198 | 0.157 | 0.074 |
| | T=500 | 0.104 | 0.107 | 0.112 | 0.110 | 0.118 | 0.121 | 0.133 | 0.158 | 0.180 | 0.154 | 0.124 | 0.065 |
| | T=1000 | 0.100 | 0.104 | 0.101 | 0.105 | 0.109 | 0.111 | 0.117 | 0.137 | 0.151 | 0.129 | 0.111 | 0.066 |

Table 7: **DGP2** Empirical Power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.520 | 0.572 | 0.620 | 0.668 | 0.724 | 0.778 | 0.835 | 0.885 | 0.929 | 0.971 | 0.884 |
| | T=500 | 0.784 | 0.836 | 0.877 | 0.914 | 0.945 | 0.967 | 0.986 | 0.994 | 0.998 | 1.000 | 0.998 |
| | T=1000 | 0.960 | 0.976 | 0.987 | 0.995 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.912 | 0.936 | 0.956 | 0.969 | 0.979 | 0.987 | 0.993 | 0.997 | 0.999 | 1.000 | 1.000 |
| | T=500 | 0.991 | 0.996 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | T=1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 8: **DGP2** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.557 | 0.619 | 0.677 | 0.739 | 0.805 | 0.871 | 0.933 | 0.970 | 0.982 | 0.975 | 0.957 | 0.884 |
| | T=500 | 0.821 | 0.874 | 0.918 | 0.951 | 0.974 | 0.991 | 0.998 | 1.000 | 1.000 | 1.000 | 0.999 | 0.998 |
| | T=1000 | 0.974 | 0.988 | 0.994 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.930 | 0.953 | 0.970 | 0.982 | 0.991 | 0.995 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | T=500 | 0.995 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | T=1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 9: Exchange Rate Predictability

| | EURUSD | YENUSD | GBPUSD | CHFUSD | AUDUSD | CADUSD |
|---|---|---|---|---|---|---|
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | | | | | | |
| $\lambda_2^0 = 0.80$ | 1.000 | 1.000 | 0.360 | 0.924 | 0.764 | 0.997 |
| $\lambda_2^0 = 0.85$ | 1.000 | 0.998 | 0.219 | 0.874 | 0.130 | 0.862 |
| $\lambda_2^0 = 0.90$ | 1.000 | 1.000 | 0.731 | 0.838 | 0.846 | 0.989 |
| $\lambda_2^0 = 0.95$ | 0.997 | 0.997 | 0.814 | 0.755 | 0.711 | 0.945 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)_{nw}$ | | | | | | |
| $\tau_0 = 0.80$ | 0.640 | 0.928 | 0.957 | 0.508 | 0.993 | 0.954 |
| $\tau_0 = 0.90$ | 0.471 | 0.607 | 0.695 | 0.500 | 0.216 | 0.374 |
| $DM_{nw}$ | 0.091 | 0.438 | 0.530 | 0.643 | 0.276 | 0.096 |
| $CW_{nw}$ | 0.048 | 0.248 | 0.299 | 0.532 | 0.155 | 0.044 |

## APPENDIX B: PROOFS

**PROOF OF PROPOSITION 1**. We consider the asymptotic behavior of $Z_T(\ell_1, \ell_2)$ in (6). Rescaling the time axis we write $Z_T(\lambda_1, \lambda_2) \equiv Z_T([(T - k_0)\lambda_1], [(T - k_0)\lambda_2])$ and focus on $Z_T(\lambda_1, \lambda_2)$. Using $\hat{e}_{j,t+1}^2 = u_{t+1}^2 + (\hat{e}_{j,t+1}^2 - u_{t+1}^2)$ $(j = 1, 2)$ in (6) yields

$$
\begin{aligned}
Z_T(\lambda_1, \lambda_2) = {} & \frac{T - k_0}{[(T - k_0)\lambda_1]} \left( \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_1]} u_{t+1}^2}{\sqrt{T - k_0}} - \frac{[(T - k_0)\lambda_1]}{[(T - k_0)\lambda_2]} \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_2]} u_{t+1}^2}{\sqrt{T - k_0}} \right) \\
& + \frac{T - k_0}{[(T - k_0)\lambda_1]} \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_1]} (\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T - k_0}} \\
& - \frac{T - k_0}{[(T - k_0)\lambda_2]} \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_2]} (\hat{e}_{2,t+1}^2 - u_{t+1}^2)}{\sqrt{T - k_0}} \\
\equiv {} & \frac{T - k_0}{[(T - k_0)\lambda_1]} \mathcal{N}_{1T}(\lambda_1, \lambda_2) + \frac{T - k_0}{[(T - k_0)\lambda_1]} \mathcal{N}_{2T}(\lambda_1) - \frac{T - k_0}{[(T - k_0)\lambda_2]} \mathcal{N}_{3T}(\lambda_2). \quad (38)
\end{aligned}
$$

From Assumption A(i) we have $\sup_{\lambda_1} |\mathcal{N}_{2T}(\lambda_1)| = o_p(1)$ and $\sup_{\lambda_2} |\mathcal{N}_{3T}(\lambda_2)| = o_p(1)$. Combining with

$$
\sup_{\lambda_1, \lambda_2} \left| \frac{[(T - k_0)\lambda_1]}{[(T - k_0)\lambda_2]} - \frac{\lambda_1}{\lambda_2} \right| = O(1/(T - k_0)) \quad (39)
$$

and

$$
\sup_{\lambda_j} \left| \frac{(T - k_0)}{[(T - k_0)\lambda_j]} - \frac{1}{\lambda_j} \right| = O(1/(T - k_0)) \quad j = 1, 2 \quad (40)
$$

gives

$$
Z_T(\lambda_1, \lambda_2) = \frac{1}{\lambda_1} \left( \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_1]} u_{t+1}^2}{\sqrt{T - k_0}} - \frac{\lambda_1}{\lambda_2} \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_2]} u_{t+1}^2}{\sqrt{T - k_0}} \right) + o_p(1). \quad (41)
$$

It is now convenient to reformulate (41) as

$$
\begin{aligned}
Z_T(\lambda_1, \lambda_2) = {} & \frac{1}{\lambda_1} \left( \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_1]} (u_{t+1}^2 - \sigma_u^2)}{\sqrt{T - k_0}} - \frac{\lambda_1}{\lambda_2} \frac{\sum_{t=k_0}^{k_0 - 1 + [(T-k_0)\lambda_2]} (u_{t+1}^2 - \sigma_u^2)}{\sqrt{T - k_0}} \right) \\
& + \sigma_u^2 \sqrt{T - k_0} \left( \frac{[(T - k_0)\lambda_1]}{(T - k_0)\lambda_1} - \frac{[(T - k_0)\lambda_2]}{(T - k_0)\lambda_2} \right) + o_p(1), \quad (42)
\end{aligned}
$$

and note that the second component in the right hand side of (42) is $O(1/\sqrt{T - k_0})$. We now recall that our setting operates under fixed and given magnitudes of $(\lambda_1, \lambda_2)$, say $(\lambda_1^0, \lambda_2^0)$ chosen such that

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$(\lambda_1^0, \lambda_2^0) \in \Lambda^0$. We have

$$Z_T(\lambda_1^0, \lambda_2^0) = \frac{1}{\lambda_1^0} \left( \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_1^0]}(u_{t+1}^2 - \sigma_u^2)}{\sqrt{T-k_0}} - \frac{\lambda_1^0}{\lambda_2^0} \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_2^0]}(u_{t+1}^2 - \sigma_u^2)}{\sqrt{T-k_0}} \right) + o_p(1). \quad (43)$$

It follows from Assumptions A(ii)-A(iii), the continuous mapping theorem and Slutsky's theorem that

$$Z_T^0(\lambda_1^0, \lambda_2^0) \equiv \frac{Z_T(\lambda_1^0, \lambda_2^0)}{\hat{\sigma}} \xrightarrow{\mathcal{D}} \frac{1}{\lambda_1^0} \left( W_\eta(\lambda_1^0) - \frac{\lambda_1^0}{\lambda_2^0} W_\eta(\lambda_2^0) \right). \quad (44)$$

The right hand side of (44) is a centered Gaussian random variable with variance $v^0(\lambda_1^0, \lambda_2^0) = |\lambda_1^0 - \lambda_2^0|/\lambda_1^0\lambda_2^0$ as stated in (11). Specifically, the statement in (44) is equivalent to $Z_T^0(\lambda_1^0, \lambda_2^0) \xrightarrow{\mathcal{D}} N(0, |\lambda_1^0 - \lambda_2^0|/\lambda_1^0\lambda_2^0)$. This also establishes that $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \equiv Z_T^0(\lambda_1^0, \lambda_2^0)/\sqrt{v^0(\lambda_1^0, \lambda_2^0)} \xrightarrow{\mathcal{D}} N(0, 1)$. ■

**PROOF OF PROPOSITION 2**. We view $Z_T(\lambda_1, \lambda_2)$ in (42) as a functional of $\lambda_1$ whose range is determined by the choice of $\tau_0$, and for a given $\lambda_2 = \lambda_2^0$, satisfying $(\tau_0, \lambda_2^0) \in \overline{\Lambda}^0$. Assumption A(ii) combined with standard continuous mapping arguments applied to (42) yields

$$Z_T(\lambda_1; \lambda_2^0) \xrightarrow{\mathcal{D}} \sigma \frac{1}{\lambda_1} \left( W_\eta(\lambda_1) - \frac{\lambda_1}{\lambda_2^0} W_\eta(\lambda_2^0) \right). \quad (45)$$

The asymptotic behavior of $\overline{Z}_T(\tau_0; \lambda_2^0)$ in (9) now follows by appealing to Assumptions A(i)-(iii), the continuity of the average operation and (45). Specifically,

$$\overline{Z}_T(\tau_0; \lambda_2^0) \xrightarrow{\mathcal{D}} \frac{1}{1-\tau_0} \int_{\tau_0}^1 \left[ \frac{W_\eta(\lambda_1)}{\lambda_1} - \frac{W_\eta(\lambda_2^0)}{\lambda_2^0} \right] d\lambda_1. \quad (46)$$

Note that

$$E \left| \int_{\tau_0}^1 \frac{W_\eta(\lambda_1)}{\lambda_1} d\lambda_1 \right| \leq \int_{\tau_0}^1 E \left| \frac{W_\eta(\lambda_1)}{\lambda_1} \right| d\lambda_1 = \int_{\tau_0}^1 \frac{E|W_\eta(1)|}{\sqrt{\lambda_1}} d\lambda_1 < \infty \quad (47)$$

so that (46) is well defined almost surely and by construction centered Gaussian. It now suffices to

obtain its variance. We have

$$\frac{1}{(1-\tau_0)^2} Cov\left[\int_{\tau_0}^1 \left(\frac{W_\eta(s_1)}{s_1} - \frac{W_\eta(\lambda_2^0)}{\lambda_2^0}\right) ds_1, \int_{\tau_0}^1 \left(\frac{W_\eta(s_2)}{s_2} - \frac{W_\eta(\lambda_2^0)}{\lambda_2^0}\right) ds_2\right] =$$

$$\frac{1}{(1-\tau_0)^2} \int_{\tau_0}^1 \left[\int_{\tau_0}^1 Cov\left[\left(\frac{W_\eta(s_1)}{s_1} - \frac{W_\eta(\lambda_2^0)}{\lambda_2^0}\right), \left(\frac{W_\eta(s_2)}{s_2} - \frac{W_\eta(\lambda_2^0)}{\lambda_2^0}\right)\right] ds_2\right] ds_1 =$$

$$\frac{1}{(1-\tau_0)^2} \int_{\tau_0}^1 \int_{\tau_0}^1 \left[\frac{s_1 \wedge s_2}{s_1 s_2} - \frac{s_1 \wedge \lambda_2^0}{s_1 \lambda_2^0} - \frac{\lambda_2^0 \wedge s_2}{s_2 \lambda_2^0} + \frac{1}{\lambda_2^0}\right] ds_1 \ ds_2, \qquad (48)$$

where we appealed to Fubini's Theorem for interchanging expectations with integration in the second row of (48). Standard integral calculus now leads to (13)-(14). ■

The following lemma collects some key results used in the proofs of Proposition 3 and Corollary 1 on the power properties of the proposed tests under stationarity.

**LEMMA A1**. Suppose model (2) holds with $\beta_2 = \gamma/T^{1/4}$. Under Assumption B1 and as $T \to \infty$ we have

(i)
$$\sup_\lambda \left|\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} - \lambda\sqrt{1-\pi_0}\ \gamma'(Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})\gamma\right| = o_p(1), \qquad (49)$$

(ii)
$$\sup_\lambda \left|\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(\hat{e}_{2,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}}\right| = o_p(1). \qquad (50)$$

**PROOF OF LEMMA A1**. (i) As we operate under model (2) with $\beta_2 = \gamma/T^{1/4}$ we have $\hat{e}_{1,t+1} - u_{t+1} = x'_{2,t}\beta_2 - x'_{1,t}(\hat{\delta}_{1,t} - \beta_1)$ so that the following identity holds

$$\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} = A_{1T}(\lambda) + A_{2T}(\lambda) - 2A_{3T}(\lambda) + 2A_{4T}(\lambda) - 2A_{5T}(\lambda) \qquad (51)$$

where

$$A_{1T}(\lambda) = T^{-\frac{1}{2}}(T-k_0)^{-\frac{1}{2}}\gamma' \sum_t x_{2,t}x'_{2,t}\gamma,$$

$$A_{2T}(\lambda) = (T-k_0)^{-\frac{1}{2}} \sum_t (\hat{\delta}_{1,t} - \beta_1)'x_{1,t}x'_{1,t}(\hat{\delta}_{1,t} - \beta_1)$$

$$A_{3T}(\lambda) = T^{-\frac{1}{4}}(T-k_0)^{-\frac{1}{2}} \sum_t (\hat{\delta}_{1,t} - \beta_1)'x_{1,t}x'_{2,t}\gamma$$

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$$A_{4T}(\lambda) = T^{-\frac{1}{4}}(T - k_0)^{-\frac{1}{2}} \boldsymbol{\gamma}' \sum_t \boldsymbol{x}_{2,t} u_{t+1}$$

$$A_{5T}(\lambda) = (T - k_0)^{-\frac{1}{2}} \sum_t (\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1)' \boldsymbol{x}_{1,t} u_{t+1}$$

with $t = k_0, \ldots, k_0 - 1 + [(T - k_0)\lambda]$ in all of the above summations and below, unless otherwise indicated.

For $A_{1T}(\lambda)$, we write

$$A_{1T}(\lambda) = \sqrt{\frac{T - k_0}{T}} \boldsymbol{\gamma}' \left( \frac{\sum_t \boldsymbol{x}_{2,t} \boldsymbol{x}'_{2,t}}{T - k_0} - \lambda \boldsymbol{Q}_{22} \right) \boldsymbol{\gamma} + \sqrt{\frac{T - k_0}{T}} \lambda \boldsymbol{\gamma}' \boldsymbol{Q}_{22} \boldsymbol{\gamma} \tag{52}$$

and as $|\sqrt{(T - [T\pi_0])/T} - \sqrt{1 - \pi_0}| = o(1)$ we have

$$\left| A_{1T}(\lambda) - \lambda \sqrt{1 - \pi_0} \, \boldsymbol{\gamma}' \boldsymbol{Q}_{22} \boldsymbol{\gamma} \right| \leq \sqrt{1 - \pi_0} \, \|\boldsymbol{\gamma}\|^2 \left\| \frac{\sum_t \boldsymbol{x}_{2,t} \boldsymbol{x}'_{2,t}}{T - k_0} - \lambda \, \boldsymbol{Q}_{22} \right\| \tag{53}$$

so that Assumption B1(i) directly implies

$$\sup_\lambda \left| A_{1T}(\lambda) - \lambda \sqrt{1 - \pi_0} \, \boldsymbol{\gamma}' \boldsymbol{Q}_{22} \boldsymbol{\gamma} \right| = o_p(1). \tag{54}$$

Before focusing on the remainder quantities we consider the limiting behavior of $(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1)$. Setting $t = [Ts]$ we write

$$T^{1/4}(\hat{\boldsymbol{\delta}}_{1,[Ts]} - \boldsymbol{\beta}_1) = \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{1,j-1}}{T} \right)^{-1} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{2,j-1}}{T} \right) \boldsymbol{\gamma}$$

$$+ T^{-1/4} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{1,j-1}}{T} \right)^{-1} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} u_j}{\sqrt{T}} \right). \tag{55}$$

For the second term in the right hand side of (55) we have

$$\sup_s \frac{1}{T^{1/4}} \left\| \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{1,j-1}}{T} \right)^{-1} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} u_j}{\sqrt{T}} \right) \right\| = o_p(1) \tag{56}$$

due to Assumptions B1(i)-(ii). For the first term in the right hand side of (55) we can write

$$\left\| \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{1,j-1}}{T} \right)^{-1} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{2,j-1}}{T} \right) \boldsymbol{\gamma} - \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma} \right\|$$

$$\leq \left\| \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{1,j-1}}{T} \right)^{-1} - \boldsymbol{Q}_{11}^{-1} \right\| \left( \left\| \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{2,j-1}}{T} \boldsymbol{\gamma} - \boldsymbol{Q}_{12} \boldsymbol{\gamma} \right\| + \| \boldsymbol{Q}_{12} \boldsymbol{\gamma} \| \right)$$

$$+ \left\| \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{1,j-1}}{T} \right)^{-1} \right\| \left\| \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{2,j-1}}{T} \boldsymbol{\gamma} - \boldsymbol{Q}_{12} \boldsymbol{\gamma} \right\| \qquad (57)$$

so that Assumptions B1(i)-(ii) also ensure that

$$\sup_s \left\| \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{1,j-1}}{T} \right)^{-1} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}'_{2,j-1}}{T} \right) \boldsymbol{\gamma} - \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma} \right\| = o_p(1). \qquad (58)$$

Combining (56) and (58) and using the triangle inequality in (55) yields

$$\sup_s \left\| T^{1/4} (\hat{\boldsymbol{\delta}}_{1,[Ts]} - \boldsymbol{\beta}_1) - \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma} \right\| = o_p(1). \qquad (59)$$

We now focus on $A_{2T}(\lambda)$. Using suitable normalisations and appealing to (59), we can express $A_{2T}(\lambda)$ as

$$A_{2T}(\lambda) = \sqrt{\frac{T-k_0}{T}} (T-k_0)^{-1} \sum_t T^{1/4} (\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1)' \boldsymbol{x}_{1,t} \boldsymbol{x}'_{1,t} T^{1/4} (\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1)$$

$$= \sqrt{\frac{T-k_0}{T}} (T-k_0)^{-1} \sum_t (T^{1/4} (\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1) - \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma})' \boldsymbol{x}_{1,t} \boldsymbol{x}'_{1,t} T^{1/4} (\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1)$$

$$+ \sqrt{\frac{T-k_0}{T}} (T-k_0)^{-1} \boldsymbol{\gamma}' \boldsymbol{Q}_{21} \boldsymbol{Q}_{11}^{-1} \sum_t \boldsymbol{x}_{1,t} \boldsymbol{x}'_{1,t} (T^{1/4} (\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1) - \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma})$$

$$+ \sqrt{\frac{T-k_0}{T}} \boldsymbol{\gamma}' \boldsymbol{Q}_{21} \boldsymbol{Q}_{11}^{-1} \left( \frac{\sum_t \boldsymbol{x}_{1,t} \boldsymbol{x}'_{1,t}}{T-k_0} - \lambda \boldsymbol{Q}_{11} \right) \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma}$$

$$+ \lambda \sqrt{\frac{T-k_0}{T}} \boldsymbol{\gamma}' \boldsymbol{Q}_{21} \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma}. \qquad (60)$$

Assumption B1(i) combined with the result in (59) give

$$\sup_\lambda \left| A_{2T}(\lambda) - \lambda \sqrt{1-\pi_0} \boldsymbol{\gamma}' \boldsymbol{Q}_{21} \boldsymbol{Q}_{11}^{-1} \boldsymbol{Q}_{12} \boldsymbol{\gamma} \right| = o_p(1). \qquad (61)$$

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For $A_{3T}(\lambda)$ we write

$$A_{3T}(\lambda) = \sqrt{\frac{T-k_0}{T}}(T-k_0)^{-1}\sum(T^{1/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1) - \boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12}\boldsymbol{\gamma})'\boldsymbol{x}_{1,t}\boldsymbol{x}_{2,t}'\boldsymbol{\gamma}$$

$$+ \sqrt{\frac{T-k_0}{T}}\boldsymbol{\gamma}'\boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\left(\frac{\sum \boldsymbol{x}_{1,t}\boldsymbol{x}_{2,t}'}{T-k_0} - \lambda\boldsymbol{Q}_{12}\right)\boldsymbol{\gamma} + \sqrt{\frac{T-k_0}{T}}\lambda\boldsymbol{\gamma}'\boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12}\boldsymbol{\gamma} \qquad (62)$$

so that using (59), Assumption B1(i) and the triangle inequality yields

$$\sup_\lambda \left|A_{3T}(\lambda) - \lambda\sqrt{1-\pi_0}\boldsymbol{\gamma}'\boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12}\boldsymbol{\gamma}\right| = o_p(1). \qquad (63)$$

Next, as an immediate consequence of Assumption B1(ii) we have

$$\sup_\lambda |A_{4T}(\lambda)| = o_p(1). \qquad (64)$$

Finally, using (59) together with Assumption B1(ii) yields

$$\sup_\lambda |A_{5T}(\lambda)| = o_p(1). \qquad (65)$$

Combining (54), (61) and (63)-(65) with successive uses of the triangle inequality yields the stated result in Lemma A1(i). The statement in (50) follows an identical line of argument as above and details are therefore omitted from the exposition here.

**PROOF OF PROPOSITION 3**. (i) We initially consider the case of a fixed and non-zero $\boldsymbol{\beta}_2$ and establish that $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \overset{p}{\to} \infty$. Using (38) and appealing to Lemma A1(ii) we have

$$\frac{Z_T(\lambda_1^0, \lambda_2^0)}{\sqrt{T-k_0}} = \frac{1}{\lambda_1^0}\frac{\mathcal{N}_{1T}(\lambda_1^0, \lambda_2^0)}{\sqrt{T-k_0}} + \frac{1}{\lambda_1^0}\frac{\mathcal{N}_{2T}(\lambda_1^0)}{\sqrt{T-k_0}} + o_p(1). \qquad (66)$$

We can now note from (44)-(45) that the first term in the right hand side of (66) is $O_p(T^{-1/2})$ so that

$$\frac{Z_T(\lambda_1^0, \lambda_2^0)}{\sqrt{T-k_0}} = \frac{1}{\lambda_1^0}\left[\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_1^0]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{T-k_0}\right] + o_p(1). \qquad (67)$$

It is now straightforward to adapt the result in Lemma A1(i) to a fixed $\boldsymbol{\beta}_2$ setting and infer that

$$\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_1^0]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{T-k_0} \overset{p}{\to} \lambda_1^0\,\boldsymbol{\beta}_2'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\beta}_2, \qquad (68)$$

yielding (for fixed $\boldsymbol{\beta}_2$)

$$\frac{Z_T^0(\lambda_1^0, \lambda_2^0)}{\sqrt{T - k_0}} \overset{p}{\to} \frac{1}{\sigma} \boldsymbol{\beta}_2'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\beta}_2 \tag{69}$$

where we also made use of Assumption B1(iii) ensuring that $\hat{\sigma} \overset{p}{\to} \sigma \in (0, \infty)$. It now follows that

$$\frac{\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)}{\sqrt{T - k_0}} \equiv \frac{1}{\sqrt{T - k_0}} \frac{Z_T^0(\lambda_1^0, \lambda_2^0)}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \overset{p}{\to} \frac{1}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \frac{1}{\sigma} \boldsymbol{\beta}_2'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\beta}_2 \tag{70}$$

with $v^0(\lambda_1^0, \lambda_2^0)$ given by (11), thus leading to $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \overset{p}{\to} \infty$ as stated. Proceeding similarly for $\overline{Z}_T(\tau_0; \lambda_2^0)$ we have

$$\frac{\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)}{\sqrt{T - k_0}} \overset{p}{\to} \frac{1}{\sqrt{\overline{v}(\tau_0; \lambda_2^0)}} \frac{1}{\sigma} \boldsymbol{\beta}_2'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\beta}_2 \tag{71}$$

with $\overline{v}(\tau_0; \lambda_2^0)$ as in (13)-(14), thus also establishing that $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0) \overset{p}{\to} \infty$.

(ii) We next focus on the local asymptotic behavior of the two test statistics with $\boldsymbol{\beta}_2$ parameterised as $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{1/4}$. Using (38) in conjunction with Lemma A1(i)-(ii) we have

$$\frac{Z_T^0(\lambda_1^0, \lambda_2^0)}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} = \frac{1}{\hat{\sigma}} \frac{1}{\lambda_1^0 \sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \left[ \frac{\sum_{t=k_0}^{k_0 - 1 + [(T - k_0)\lambda_1^0]} (\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T - k_0}} \right] +$$

$$\frac{1}{\hat{\sigma}} \frac{1}{\lambda_1^0 \sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \left[ \frac{\sum_{t=k_0}^{k_0 - 1 + [(T - k_0)\lambda_1^0]} u_{t+1}^2}{\sqrt{T - k_0}} - \frac{\lambda_1^0}{\lambda_2^0} \frac{\sum_{t=k_0}^{k_0 - 1 + [(T - k_0)\lambda_2^0]} u_{t+1}^2}{\sqrt{T - k_0}} \right]$$

$$+ o_p(1). \tag{72}$$

It now follows directly from (49) in Lemma A1, Assumption B1(iii) and Slutsky's theorem that

$$\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0) \equiv \frac{Z_T^0(\lambda_1^0, \lambda_2^0)}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \overset{\mathcal{D}}{\to} \frac{\sqrt{1 - \pi_0}}{\sigma \sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \boldsymbol{\gamma}'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\gamma} + N(0, 1) \tag{73}$$

as required. The result for $\overline{Z}_T(\tau_0; \lambda_2^0)$ follows identical arguments and is therefore omitted. ∎

**PROOF OF COROLLARY 1**. Follows directly from (71)-(73) in the proof of Proposition 3. ∎

**LEMMA A2**.

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(i) Under Assumption B2 and as $T \to \infty$ we have for $\lambda \in [0, 1]$

$$\frac{1}{(T-k_0)^2} \sum_{t=k_0}^{k_0+[(T-k_0)\lambda]} \boldsymbol{x}_t \boldsymbol{x}_t' \overset{\mathcal{D}}{\to} \frac{1}{(1-\pi_0)^2} \int_{\pi_0}^{\pi_0+(1-\pi_0)\lambda} \boldsymbol{J}_C \boldsymbol{J}_C' dr \tag{74}$$

(ii) Under Assumption B2 and as $T \to \infty$ we have for $\lambda \in [0, 1]$

$$\sup_\lambda \left\| \frac{1}{T-k_0} \sum_{t=k_0}^{k_0+[(T-k_0)\lambda]} \boldsymbol{x}_t u_{t+1} \right\| = O_p(1) \tag{75}$$

(iii) Suppose model (2) holds with $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{3/4}$. Under Assumption B2 and as $T \to \infty$ we have

$$T^{3/4}(\boldsymbol{\delta}_{1,[Ts]} - \boldsymbol{\beta}_1) \overset{\mathcal{D}}{\to} \left( \int_0^s \boldsymbol{J}_{1C} \boldsymbol{J}_{1C}' dr \right)^{-1} \left( \int_0^s \boldsymbol{J}_{1C} \boldsymbol{J}_{2C}' dr \right) \boldsymbol{\gamma} \equiv \boldsymbol{M}(s)\, \boldsymbol{\gamma}. \tag{76}$$

**PROOF OF LEMMA A2**. For (74) we have

$$\frac{1}{(T-k_0)^2} \sum_{t=k_0}^{k_0+[(T-k_0)\lambda]} \boldsymbol{x}_t \boldsymbol{x}_t' = \left( \frac{T}{T-k_0} \right)^2 \frac{1}{T^2} \sum_{t=k_0}^{k_0+[(T-k_0)\lambda]} \boldsymbol{x}_t \boldsymbol{x}_t'$$

$$= \left( \frac{T}{T-k_0} \right)^2 \sum_{t=k_0}^{k_0+[(T-k_0)\lambda]} \int_{\frac{t-1}{T}}^{\frac{t}{T}} \left( \frac{\boldsymbol{x}_{[Tr]}}{\sqrt{T}} \right) \left( \frac{\boldsymbol{x}_{[Tr]}}{\sqrt{T}} \right)' dr$$

$$\overset{\mathcal{D}}{\to} \frac{1}{(1-\pi_0)^2} \int_{\pi_0}^{\pi_0+(1-\pi_0)\lambda} \boldsymbol{J}_C \boldsymbol{J}_C' dr \tag{77}$$

due to Assumption B2(i). For (75) we have

$$\sup_\lambda \left\| \frac{1}{T-k_0} \sum_{t=k_0}^{k_0+[(T-k_0)\lambda]} \boldsymbol{x}_t u_{t+1} \right\| \leq \sup_\lambda \left| \frac{\sum u_{t+1}}{\sqrt{T-k_0}} \right| \sup_\lambda \left\| \frac{\boldsymbol{x}_{[(T-k_0)\lambda]}}{\sqrt{T-k_0}} \right\|$$

$$= O_p(1) \tag{78}$$

which also follows from Assumption B2(i). For (76) we write

$$T^{3/4}(\boldsymbol{\delta}_{1,[Ts]} - \boldsymbol{\beta}_1) = \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}_{1,j-1}'}{T^2} \right)^{-1} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}_{2,j-1}'}{T^2} \right) \boldsymbol{\gamma}$$

$$+ \frac{1}{T^{1/4}} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} \boldsymbol{x}_{1,j-1}'}{T^2} \right)^{-1} \left( \frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1} u_j}{T} \right). \tag{79}$$

From (75) it also follows that

$$T^{3/4}(\boldsymbol{\delta}_{1,[Ts]} - \boldsymbol{\beta}_1) = \left(\frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1}\boldsymbol{x}_{1,j-1}'}{T^2}\right)^{-1} \left(\frac{\sum_{j=1}^{[Ts]} \boldsymbol{x}_{1,j-1}\boldsymbol{x}_{2,j-1}'}{T^2}\right) \boldsymbol{\gamma} + o_p(1) \tag{80}$$

and the statement in (76) follows directly using (74) in (80) and appealing to the continuous mapping theorem.

**LEMMA A3**.

(i) Suppose model (2) holds with $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{3/4}$. Under Assumptions B2 and as $T \to \infty$ we have

$$\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} \xrightarrow{\mathcal{D}} \frac{1}{\sqrt{1-\pi_0}} \boldsymbol{\gamma}' \left(\int_{\pi_0}^{\pi_0+(1-\pi_0)\lambda} \boldsymbol{J}_C^*(s)\boldsymbol{J}_C^*(s)'ds\right) \boldsymbol{\gamma} \tag{81}$$

where $\boldsymbol{J}_C^*(s) = \boldsymbol{J}_{2C}(s) - \boldsymbol{M}(s)\boldsymbol{J}_{1C}(s)$ for $\boldsymbol{M}(s) = (\int_0^s \boldsymbol{J}_{1C}\boldsymbol{J}_{1C}'dr)^{-1}(\int_0^s \boldsymbol{J}_{1C}\boldsymbol{J}_{2C}'dr)$.

(ii) Suppose model (2) holds with $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{3/4}$. Under Assumptions B2 and as $T \to \infty$ we have

$$\sup_{\lambda \in (0,1]} \left| \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(\hat{e}_{2,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} \right| \xrightarrow{p} 0. \tag{82}$$

**PROOF OF LEMMA A3**. We consider (81) first. We operate under $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{3/4}$. Recalling that

$\hat{e}_{1,t+1} - u_{t+1} = \boldsymbol{x}_{2,t}'\boldsymbol{\beta}_2 - \boldsymbol{x}_{1,t}'(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1)$ and using $\lim_{T\to\infty}((T - k_0)/T)^j \to (1 - \pi_0)^j$, we write

$$\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T - k_0}}$$

$$= \frac{(1 - \pi_0)^{3/2}}{(T - k_0)^2}\boldsymbol{\gamma}'\left(\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} \boldsymbol{x}_{2,t}\boldsymbol{x}_{2,t}'\right)\boldsymbol{\gamma}$$

$$+ \frac{(1 - \pi_0)^{3/2}}{(T - k_0)^2}\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(T^{3/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1))'\boldsymbol{x}_{1,t}\boldsymbol{x}_{1,t}'(T^{3/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1))$$

$$- 2\frac{(1 - \pi_0)^{3/2}}{(T - k_0)^2}\left(\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(T^{3/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1))'\boldsymbol{x}_{1,t}\boldsymbol{x}_{2,t}'\right)\boldsymbol{\gamma}$$

$$+ 2\frac{(1 - \pi_0)^{1/2}}{T^{1/4}}\left(\frac{1}{T - k_0}\boldsymbol{\gamma}'\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} \boldsymbol{x}_{2,t}u_{t+1}\right)$$

$$- 2\frac{(1 - \pi_0)^{1/2}}{T^{1/4}}\left(\frac{1}{T - k_0}\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(T^{3/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1))'\boldsymbol{x}_{1,t}u_{t+1}\right) + o(1) \qquad (83)$$

It next follows from (75)-(76) that the last two terms in the right hand side of (83) are $O_p(T^{-1/4})$ so that we also have

$$\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T - k_0}} =$$

$$\frac{(1 - \pi_0)^{3/2}}{(T - k_0)^2}\boldsymbol{\gamma}'\left(\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} \boldsymbol{x}_{2,t}\boldsymbol{x}_{2,t}'\right)\boldsymbol{\gamma}$$

$$+ \frac{(1 - \pi_0)^{3/2}}{(T - k_0)^2}\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(T^{3/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1))'\boldsymbol{x}_{1,t}\boldsymbol{x}_{1,t}'(T^{3/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1))$$

$$- 2\frac{(1 - \pi_0)^{3/2}}{(T - k_0)^2}\left(\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]}(T^{3/4}(\hat{\boldsymbol{\delta}}_{1,t} - \boldsymbol{\beta}_1))'\boldsymbol{x}_{1,t}\boldsymbol{x}_{2,t}'\right)\boldsymbol{\gamma} + o_p(1). \qquad (84)$$

Using (74) and (76) from Lemma A2 together with the continuous mapping theorem, (84) leads to the required result in (81). The result in (82) is established following similar arguments and details are omitted. ∎

**PROOF OF PROPOSITION 4 and COROLLARY 2**. We focus on Part (ii) of the Proposition as the test consistency property stated in Part (i) follows as its direct consequence. From (38) and

Lemma A2, under the local alternative $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{3/4}$ we have

$$\frac{Z_T^0(\lambda_1^0, \lambda_2^0)}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}}$$

$$= \frac{1}{\hat{\sigma}} \frac{1}{\lambda_1^0 \sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \left[ \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_1^0]} u_{t+1}^2}{\sqrt{T-k_0}} - \frac{\lambda_1}{\lambda_2^0} \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_2^0]} u_{t+1}^2}{\sqrt{T-k_0}} \right] +$$

$$\frac{1}{\hat{\sigma}} \frac{1}{\lambda_1^0 \sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \left[ \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_1^0]} (\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} \right] + o_p(1)$$

$$\xrightarrow{\mathcal{D}} N(0,1)$$

$$+ \frac{1}{\sigma} \frac{1}{\lambda_1^0 \sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \frac{1}{\sqrt{1-\pi_0}} \boldsymbol{\gamma}' \left( \int_{\pi_0}^{\pi_0 + (1-\pi_0)\lambda_1^0} \boldsymbol{J}_{C}^{*}(s) \boldsymbol{J}_{C}^{*}(s)' ds \right) \boldsymbol{\gamma} \tag{85}$$

using (81), Slutsky and the continuous mapping theorems (note that the standard normality of the first component in the right hand side of (85) has been established in Proposition 1). It now follows directly from (85) that $\lim_{\|\gamma\|\to\infty} \lim_{T\to\infty} \mathcal{S}_T(\lambda_1^0, \lambda_2^0)$ as required. The result for $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ follows identical lines and its details are omitted. ∎

**PROOF OF PROPOSITION 5 and COROLLARY 3.** We have $\hat{e}_{1,t+1}^2 - \tilde{e}_{2,t+1}^2 = (\hat{e}_{1,t+1}^2 - \hat{e}_{2,t+1}^2) + (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})^2$ which leads to the formulations of $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ in (34) and (35) respectively. Under the null hypothesis and for both test statistics the result follows by verifying that

$$\sup_{\lambda} \left| \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})^2}{\sqrt{T-k_0}} \right| \xrightarrow{p} 0. \tag{86}$$

Noting that

$$\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})^2}{\sqrt{T-k_0}} = \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} (\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} +$$

$$\frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} (\hat{e}_{2,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}}$$

$$- 2 \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda]} (\hat{e}_{1,t+1}\hat{e}_{2,t+1} - u_{t+1}^2)}{\sqrt{T-k_0}}, \tag{87}$$

the statement in (86) follows directly from Assumption A(i) since we operate under the null hypothesis noting also that $(\hat{e}_{1,t+1}\hat{e}_{2,t+1} - u_{t+1}^2) = (\hat{e}_{1,t+1} - \hat{e}_{2,t+1})\hat{e}_{2,t+1} + (\hat{e}_{2,t+1}^2 - u_{t+1}^2)$ from which we infer the

$o_p(1)$'ness of the third component in the right hand side of (87). It now follows that Propositions 1 and 2 continue to hold for the two adjusted statistics.

For the behavior of the adjusted statistics under the alternative we initially consider the case of stationary predictors and operate under $\boldsymbol{\beta}_2 = \boldsymbol{\gamma}/T^{1/4}$ as in the setting of Corollary 1. Using (87) with Lemma A1 we can write

$$h_T^0(\lambda_1^0, \lambda_2^0) = \frac{1}{\hat{\sigma}} \frac{1}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}\lambda_2^0} \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_2^0]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} + o_p(1)$$
$$\xrightarrow{p} \frac{1}{\sigma} \frac{1}{\sqrt{v^0(\lambda_1^0, \lambda_2^0)}} \sqrt{1-\pi_0}\boldsymbol{\gamma}'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\gamma} \equiv \psi^0 \qquad (88)$$

and

$$\bar{h}_T(\tau_0; \lambda_2^0) = \frac{1}{\hat{\sigma}} \frac{1}{\sqrt{\bar{v}(\tau_0; \lambda_2^0)}\lambda_2^0} \frac{\sum_{t=k_0}^{k_0-1+[(T-k_0)\lambda_2^0]}(\hat{e}_{1,t+1}^2 - u_{t+1}^2)}{\sqrt{T-k_0}} + o_p(1)$$
$$\xrightarrow{p} \frac{1}{\sigma} \frac{1}{\sqrt{\bar{v}(\tau_0; \lambda_2^0)}} \sqrt{1-\pi_0}\boldsymbol{\gamma}'(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12})\boldsymbol{\gamma} \equiv \bar{\psi}. \qquad (89)$$

Using (88)-(89) it follows that $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0) \xrightarrow{\mathcal{D}} N(2\psi^0, 1)$ and similarly for $\bar{\mathcal{S}}_{T,adj}^0(\lambda_1^0, \lambda_2^0) \xrightarrow{\mathcal{D}} N(2\bar{\psi}, 1)$ which establishes the fact that Proposition 3 continues to hold for the two adjusted statistics in addition to part (i) of Corollary 3. The result for the case of persistent predictors follows identical lines, making use of Lemma A3 and Proposition 3 which in turn establishes part (ii) of Corollary 3 and that Proposition 4 also holds for the two adjusted statistics. ∎

# Online Supplementary Material

Pitarakis, Jean-Yves (2023): Supplement to "A novel approach to predictive Accuracy Testing in Nested Environments", Econometric Theory Supplementary Material. To view, please visit:

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# Supplementary Appendix:
# A Novel Approach to Predictive Accuracy Testing in Nested Environments

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## 1. Introduction

This supplementary appendix provides further simulation results illustrating the size and power properties of the two test statistics $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ and their adjusted versions $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ introduced in the paper. Emphasis is placed on robustness considerations as well as additional DGPs designed to capture commonly encountered stylised facts in applied work (e.g., presence of unit-roots and cointegration amongst predictors).

The structure of this supplementary appendix is as follows:

Section 2 considers the same DGPs and simulations as in the main text and repeats all size and power experiments allowing for conditional heteroskedasticity and the use of Newey-West

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type adjustments in the variance normalisers of the test statistics. We also use this set of additional experiments to document the impact of using alternative magnitudes for $\tau_0$ in the implementation of the average based statistics $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$.

Section 3 revisits the local power experiments of the main text (i.e., Tables 3-4) considering smaller departures from the null hypothesis. This is motivated by the fact that under highly persistent predictors (e.g., $\phi_1 = 0.95, 0.98$) our empirical power estimates quickly reached magnitudes such as 99%-100% making it important to also gauge power for magnitudes that are more local to zero.

Section 4 focuses on the sensitivity of outcomes to the sample size via an alternative choice for $k_0 = [T\pi_0]$. We may recall that $k_0$ is the starting point of the recursive estimation procedure so that the effective size for the sample of forecast errors is $T - k_0$. All our experiments in the main text were implemented using $\pi_0 = 0.25$. Here we repeat the same experiments using $\pi_0 = 0.5$ instead so that the effective sample size under $T = 250$ for instance is now $(T - k_0) = 125$.

Finally, Section 5 introduces an additional DGP designed to capture additional features commonly encountered in applied work (e.g., exact unit-roots and cointegration among predictors).

## 2. Size and Power under Conditional Heteroskedasticity

Our analysis continues to be based on the two DGPs (DGP1 and DGP2) whose features are as described in the main text. The emphasis here is on documenting performance under conditional heteroskedasticity when using Newey-West type adjustments for the variance normalisers of the statistics. Recall from (8) and (9) and Assumption A that $\hat{\sigma}^2$ refers to a consistent estimator of the variance of $\eta_t = u_{t+1}^2 - \sigma_u^2$. Our analysis under conditional homoskedasticity used $\hat{\sigma}_{hom}^2 = \sum_t \hat{\eta}_{t+1}^2/(T - k_0)$ while in what follows we use a Newey-West type formulation as $\hat{\sigma}_{nw}^2 = \hat{\gamma}_0 + 2\sum_\ell k(\ell/M_T)\hat{\gamma}_\ell$ where $\hat{\gamma}_\ell = \sum_t \hat{\eta}_{t+1}\hat{\eta}_{t+1-\ell}/(T - k_0)$. Here $k(.)$ denotes a kernel function (set at Parzen's kernel) and $M$ the bandwidth satisfying $M \to \infty$ and $M/T \to 0$. Besides addressing the issue of robustness to conditional heteroskedasticity our additional simulations also consider an extended set of parameterisations for the user-inputs required when

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implementing our proposed inferences. All results are based on a 10% nominal size and rely on 10000 replications.

### *Size Properties: DGP1 and DGP2 under conditional heteroskedasticity*

Table S1 presents empirical size outcomes for $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$. We note that virtually all outcomes fall in the vicinity of the 10% nominal size suggesting that the NW type adjustment applied to these test statistics is able to accurately neutralise the underlying heteroskedasticity (albeit with some mild overrejections in small samples for $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ which are quickly resorbed as $T$ exceeds 500).

Tables S2 and S3 repeat the same exercise for the average based statistics $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ across $\tau_0 \in \{0.5, 0.8\}$. Outcomes continue to indicate good size control across alternative parameterisations. The adjusted versions of the statistics show a tendency to overreject under $T = 250$ and $\phi$ close to 1. These overrejections are again quickly resorbed as $T$ is allowed to grow. Finally, we note little difference in size outcomes between the implementation of $\overline{\mathcal{S}}_T(\tau_0; \lambda_2^0)$ or $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ using either $\tau_0 = 0.5$ and $\tau_0 = 0.8$.

At this stage it is also interesting to contrast the outcomes in Tables S2 and S3 which differ in the magnitude of $\tau_0$ used in the implementation of the average based statistics. We may recall that $\tau_0$ determines the range of $\lambda_1$'s used in the averaging of the $Z_T(\lambda_1, \lambda_2^0)$ statistic (see equation (9) in the main text) with larger magnitudes of $\tau_0$ indicating an averaging across the larger magnitudes of $\lambda_1$. We may also recall that our local power analysis implied more favorable power outcomes for $\tau_0$ near one combined with $\lambda_2^0 = 0.5\tau_0 + 0.5$. Comparing the size estimates of $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ in Table S2 based on $\tau_0 = 0.5$ with Table S3 based on $\tau_0 = 0.8$ we note greater size distortions that occur for the $\tau_0 = 0.8$ implementation, albeit solely for $T = 250$ and $\lambda_2^0 = 0.85$ and/or $\lambda_2^0 = 0.90$. This is clearly not surprising given our earlier point about a power maximising configuration of $(\tau_0; \lambda_2^0 = 0.5\tau_0 + 0.5)$. Nevertheless we can also note that the documented oversizeness nearly disappears under $T = 1000$ in addition to being mainly confined to DGPs with highly persistent predictors (e.g., $\phi = 0.98$).

Next, in Tables S4-S6 we repeat the size experiments associated with DGP2 under conditional

Table S1: **DGP1** Empirical size of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $\phi = 0.75$ | | | | | | $DM_{nw}$ |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.101 | 0.106 | 0.111 | 0.110 | 0.112 | 0.114 | 0.109 | 0.107 | 0.102 | 0.080 | 0.008 |
| | T=500 | 0.102 | 0.105 | 0.113 | 0.117 | 0.122 | 0.120 | 0.118 | 0.119 | 0.108 | 0.091 | 0.007 |
| | T=1000 | 0.108 | 0.111 | 0.113 | 0.113 | 0.114 | 0.116 | 0.118 | 0.116 | 0.114 | 0.097 | 0.007 |
| | | | | | | | | | | | | $CW_{nw}$ |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.120 | 0.123 | 0.130 | 0.129 | 0.133 | 0.135 | 0.129 | 0.132 | 0.134 | 0.118 | 0.060 |
| | T=500 | 0.112 | 0.119 | 0.124 | 0.132 | 0.134 | 0.135 | 0.135 | 0.133 | 0.129 | 0.116 | 0.059 |
| | T=1000 | 0.115 | 0.117 | 0.121 | 0.120 | 0.123 | 0.126 | 0.128 | 0.126 | 0.127 | 0.113 | 0.058 |
| | | | | | | $\phi = 0.95$ | | | | | | $DM_{nw}$ |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.090 | 0.097 | 0.098 | 0.101 | 0.105 | 0.104 | 0.102 | 0.099 | 0.093 | 0.075 | 0.008 |
| | T=500 | 0.099 | 0.103 | 0.107 | 0.110 | 0.116 | 0.113 | 0.110 | 0.106 | 0.099 | 0.086 | 0.008 |
| | T=1000 | 0.097 | 0.101 | 0.106 | 0.107 | 0.115 | 0.114 | 0.115 | 0.112 | 0.107 | 0.096 | 0.007 |
| | | | | | | | | | | | | $CW_{nw}$ |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.111 | 0.121 | 0.123 | 0.126 | 0.134 | 0.135 | 0.133 | 0.131 | 0.129 | 0.120 | 0.071 |
| | T=500 | 0.113 | 0.116 | 0.121 | 0.125 | 0.130 | 0.130 | 0.127 | 0.125 | 0.119 | 0.111 | 0.064 |
| | T=1000 | 0.103 | 0.108 | 0.112 | 0.113 | 0.123 | 0.122 | 0.124 | 0.123 | 0.120 | 0.110 | 0.059 |
| | | | | | | $\phi = 0.98$ | | | | | | $DM_{nw}$ |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.095 | 0.101 | 0.104 | 0.106 | 0.108 | 0.105 | 0.107 | 0.105 | 0.097 | 0.079 | 0.011 |
| | T=500 | 0.095 | 0.100 | 0.106 | 0.106 | 0.108 | 0.109 | 0.113 | 0.107 | 0.099 | 0.080 | 0.009 |
| | T=1000 | 0.103 | 0.104 | 0.104 | 0.109 | 0.107 | 0.110 | 0.112 | 0.109 | 0.107 | 0.091 | 0.008 |
| | | | | | | | | | | | | $CW_{nw}$ |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.126 | 0.132 | 0.137 | 0.140 | 0.142 | 0.144 | 0.146 | 0.145 | 0.142 | 0.142 | 0.091 |
| | T=500 | 0.112 | 0.119 | 0.122 | 0.125 | 0.128 | 0.128 | 0.133 | 0.133 | 0.123 | 0.112 | 0.076 |
| | T=1000 | 0.111 | 0.113 | 0.114 | 0.120 | 0.117 | 0.119 | 0.125 | 0.121 | 0.120 | 0.109 | 0.060 |

Table S2: **DGP1** Empirical size of $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\phi = 0.75$ | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.102 | 0.104 | 0.103 | 0.094 | 0.079 | 0.068 | 0.049 | 0.052 | 0.057 | 0.065 | 0.070 | 0.008 |
| | T=500 | 0.105 | 0.104 | 0.112 | 0.105 | 0.090 | 0.074 | 0.065 | 0.069 | 0.075 | 0.076 | 0.083 | 0.007 |
| | T=1000 | 0.108 | 0.113 | 0.112 | 0.102 | 0.091 | 0.078 | 0.069 | 0.069 | 0.074 | 0.079 | 0.083 | 0.007 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.126 | 0.135 | 0.138 | 0.133 | 0.127 | 0.119 | 0.106 | 0.102 | 0.101 | 0.103 | 0.105 | 0.060 |
| | T=500 | 0.120 | 0.124 | 0.135 | 0.131 | 0.120 | 0.106 | 0.099 | 0.101 | 0.104 | 0.098 | 0.102 | 0.059 |
| | T=1000 | 0.117 | 0.124 | 0.125 | 0.120 | 0.110 | 0.101 | 0.091 | 0.088 | 0.091 | 0.093 | 0.095 | 0.058 |
| | | | | | | | $\phi = 0.95$ | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.094 | 0.095 | 0.095 | 0.088 | 0.074 | 0.057 | 0.044 | 0.045 | 0.047 | 0.058 | 0.063 | 0.008 |
| | T=500 | 0.101 | 0.099 | 0.100 | 0.096 | 0.091 | 0.081 | 0.062 | 0.060 | 0.066 | 0.072 | 0.076 | 0.008 |
| | T=1000 | 0.099 | 0.102 | 0.100 | 0.100 | 0.094 | 0.078 | 0.075 | 0.077 | 0.084 | 0.084 | 0.092 | 0.007 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.124 | 0.131 | 0.140 | 0.143 | 0.135 | 0.120 | 0.106 | 0.105 | 0.097 | 0.106 | 0.104 | 0.071 |
| | T=500 | 0.120 | 0.120 | 0.125 | 0.125 | 0.125 | 0.120 | 0.097 | 0.099 | 0.098 | 0.101 | 0.100 | 0.064 |
| | T=1000 | 0.109 | 0.112 | 0.112 | 0.117 | 0.110 | 0.097 | 0.097 | 0.097 | 0.102 | 0.099 | 0.104 | 0.059 |
| | | | | | | | $\phi = 0.98$ | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.095 | 0.100 | 0.097 | 0.085 | 0.076 | 0.057 | 0.047 | 0.041 | 0.049 | 0.056 | 0.057 | 0.011 |
| | T=500 | 0.105 | 0.098 | 0.099 | 0.093 | 0.081 | 0.064 | 0.058 | 0.058 | 0.067 | 0.071 | 0.080 | 0.009 |
| | T=1000 | 0.099 | 0.102 | 0.103 | 0.099 | 0.088 | 0.078 | 0.070 | 0.072 | 0.080 | 0.087 | 0.088 | 0.008 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.137 | 0.146 | 0.153 | 0.149 | 0.151 | 0.141 | 0.125 | 0.119 | 0.112 | 0.113 | 0.108 | 0.091 |
| | T=500 | 0.126 | 0.127 | 0.132 | 0.128 | 0.122 | 0.110 | 0.105 | 0.102 | 0.102 | 0.100 | 0.106 | 0.076 |
| | T=1000 | 0.110 | 0.115 | 0.120 | 0.118 | 0.110 | 0.104 | 0.096 | 0.096 | 0.101 | 0.102 | 0.102 | 0.060 |

Table S3: **DGP1** Empirical size of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $\phi = 0.75$ | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.103 | 0.113 | 0.113 | 0.111 | 0.109 | 0.109 | 0.096 | 0.091 | 0.066 | 0.028 | 0.030 | 0.008 |
| | T=500 | 0.105 | 0.110 | 0.118 | 0.116 | 0.113 | 0.110 | 0.106 | 0.097 | 0.070 | 0.036 | 0.044 | 0.007 |
| | T=1000 | 0.110 | 0.112 | 0.114 | 0.112 | 0.108 | 0.110 | 0.101 | 0.095 | 0.077 | 0.049 | 0.058 | 0.007 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.122 | 0.132 | 0.135 | 0.134 | 0.135 | 0.136 | 0.131 | 0.143 | 0.141 | 0.097 | 0.084 | 0.060 |
| | T=500 | 0.116 | 0.121 | 0.133 | 0.131 | 0.126 | 0.128 | 0.128 | 0.130 | 0.126 | 0.084 | 0.082 | 0.059 |
| | T=1000 | 0.116 | 0.120 | 0.124 | 0.122 | 0.119 | 0.120 | 0.116 | 0.117 | 0.107 | 0.081 | 0.079 | 0.058 |
| | | | | | | $\phi = 0.95$ | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.096 | 0.100 | 0.103 | 0.104 | 0.105 | 0.099 | 0.089 | 0.083 | 0.057 | 0.023 | 0.030 | 0.008 |
| | T=500 | 0.104 | 0.108 | 0.111 | 0.109 | 0.114 | 0.115 | 0.103 | 0.089 | 0.059 | 0.030 | 0.038 | 0.008 |
| | T=1000 | 0.100 | 0.105 | 0.105 | 0.110 | 0.113 | 0.102 | 0.104 | 0.097 | 0.071 | 0.045 | 0.056 | 0.007 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.118 | 0.126 | 0.134 | 0.135 | 0.139 | 0.135 | 0.135 | 0.148 | 0.147 | 0.104 | 0.090 | 0.071 |
| | T=500 | 0.118 | 0.119 | 0.125 | 0.124 | 0.132 | 0.133 | 0.129 | 0.128 | 0.114 | 0.081 | 0.074 | 0.064 |
| | T=1000 | 0.107 | 0.112 | 0.113 | 0.121 | 0.121 | 0.113 | 0.117 | 0.117 | 0.107 | 0.077 | 0.077 | 0.059 |
| | | | | | | $\phi = 0.98$ | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.100 | 0.104 | 0.111 | 0.104 | 0.104 | 0.101 | 0.093 | 0.086 | 0.060 | 0.026 | 0.025 | 0.011 |
| | T=500 | 0.101 | 0.099 | 0.105 | 0.102 | 0.103 | 0.100 | 0.101 | 0.090 | 0.062 | 0.026 | 0.040 | 0.009 |
| | T=1000 | 0.103 | 0.106 | 0.106 | 0.109 | 0.107 | 0.102 | 0.103 | 0.096 | 0.069 | 0.044 | 0.056 | 0.008 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.132 | 0.137 | 0.145 | 0.139 | 0.146 | 0.147 | 0.154 | 0.168 | 0.175 | 0.129 | 0.103 | 0.091 |
| | T=500 | 0.119 | 0.117 | 0.127 | 0.124 | 0.125 | 0.123 | 0.134 | 0.140 | 0.133 | 0.090 | 0.085 | 0.076 |
| | T=1000 | 0.111 | 0.115 | 0.115 | 0.119 | 0.118 | 0.116 | 0.120 | 0.122 | 0.104 | 0.079 | 0.080 | 0.060 |

heteroskedasticity. Outcomes continue to corroborate the fact that all of these test statistics show good to excellent size control regardless of the chosen user-inputs for $(\lambda_1^0, \lambda_2^0)$ and $(\tau_0, \lambda_2^0)$. An exception to this occurs for the unadjusted statistics $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ under smaller sample sizes and $\lambda_2^0$ in the vicinity of 0.8-0.9 as it was the case in our homoskedastic experiments. Nevertheless, these distortions are nearly eliminated as we increase the sample size.

Table S4: **DGP2** Empirical size of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | $DM_{nw}$ |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.075 | 0.079 | 0.080 | 0.083 | 0.084 | 0.083 | 0.079 | 0.080 | 0.078 | 0.057 | 0.003 |
| | T=500 | 0.084 | 0.087 | 0.091 | 0.094 | 0.098 | 0.095 | 0.096 | 0.092 | 0.086 | 0.068 | 0.002 |
| | T=1000 | 0.095 | 0.097 | 0.098 | 0.099 | 0.104 | 0.102 | 0.107 | 0.100 | 0.097 | 0.082 | 0.002 |
| | | | | | | | | | | | | $CW_{nw}$ |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.112 | 0.120 | 0.128 | 0.133 | 0.135 | 0.136 | 0.134 | 0.137 | 0.143 | 0.140 | 0.075 |
| | T=500 | 0.108 | 0.114 | 0.120 | 0.123 | 0.129 | 0.128 | 0.128 | 0.129 | 0.122 | 0.113 | 0.064 |
| | T=1000 | 0.110 | 0.112 | 0.115 | 0.118 | 0.122 | 0.124 | 0.128 | 0.124 | 0.125 | 0.114 | 0.067 |

***Power Properties: DGP1 and DGP2 under conditional heteroskedasticity***

Table S5: **DGP2** Empirical size of $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.072 | 0.071 | 0.072 | 0.061 | 0.047 | 0.035 | 0.023 | 0.024 | 0.028 | 0.037 | 0.042 | 0.003 |
| | T=500 | 0.086 | 0.087 | 0.083 | 0.072 | 0.067 | 0.049 | 0.040 | 0.041 | 0.050 | 0.052 | 0.060 | 0.002 |
| | T=1000 | 0.092 | 0.090 | 0.088 | 0.080 | 0.072 | 0.059 | 0.052 | 0.057 | 0.061 | 0.069 | 0.073 | 0.002 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.125 | 0.137 | 0.152 | 0.153 | 0.150 | 0.145 | 0.136 | 0.126 | 0.120 | 0.120 | 0.117 | 0.075 |
| | T=500 | 0.121 | 0.129 | 0.132 | 0.129 | 0.127 | 0.118 | 0.109 | 0.107 | 0.108 | 0.107 | 0.103 | 0.064 |
| | T=1000 | 0.113 | 0.114 | 0.117 | 0.118 | 0.116 | 0.106 | 0.099 | 0.095 | 0.098 | 0.100 | 0.100 | 0.067 |

Table S6: **DGP2** Empirical size of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.076 | 0.083 | 0.083 | 0.083 | 0.080 | 0.075 | 0.063 | 0.055 | 0.036 | 0.015 | 0.014 | 0.003 |
| | T=500 | 0.091 | 0.092 | 0.097 | 0.095 | 0.092 | 0.089 | 0.079 | 0.063 | 0.037 | 0.016 | 0.022 | 0.003 |
| | T=1000 | 0.094 | 0.094 | 0.096 | 0.097 | 0.096 | 0.091 | 0.087 | 0.074 | 0.052 | 0.026 | 0.038 | 0.003 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.118 | 0.131 | 0.136 | 0.137 | 0.139 | 0.140 | 0.147 | 0.173 | 0.188 | 0.147 | 0.114 | 0.075 |
| | T=500 | 0.115 | 0.121 | 0.128 | 0.126 | 0.127 | 0.127 | 0.129 | 0.137 | 0.136 | 0.106 | 0.087 | 0.064 |
| | T=1000 | 0.111 | 0.113 | 0.117 | 0.118 | 0.119 | 0.116 | 0.119 | 0.123 | 0.119 | 0.088 | 0.080 | 0.067 |

Table S7 presents empirical power estimates for the Newey-West corrected versions of $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0, \lambda_2^0)$ under DGP1 while Tables S8-S9 repeat the same exercise for $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ and $\tau_0 \in \{0.5, 0.8\}$.

In Table S7 we continue to observe the tendency of power to converge towards 1 as $\beta$ moves further away from the null. Relative to the homoskedastic setting however (see Table 3 in the main text) we can also note an overall deterioration of power under $\phi_1 = 0.75$ in particular. This of course is a well known phenomenon caused by the use of the Newey-West correction which affects the rate of divergence of all test statistics under the alternative.

Tables S8 and S9 consider the power properties of the average based test statistics under $\tau_0 = 0.5$ and $\tau_0 = 0.8$ respectively. As discussed above we continue to observe a drop in empirical power for these heteroskedasticity adjusted statistics. Nevertheless all empirical powers of $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ lie above 90% for $\phi_1 = 0.95$ and $\phi_1 = 0.98$.

Tables S10-S12 present the power properties related to DGP2 when conditional heteroskedasticity is present. Focusing first on Table S10 we note that the power outcomes are in close agreement with our homoskedasticity based experiments in the main text. Power quickly reaches

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Table S7: **DGP1** Empirical Power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional heteroskedasticity

| $\beta$ | -1.500 | -1.750 | -2.000 | -2.250 | -2.500 | -3.000 | -3.500 |
|---|---|---|---|---|---|---|---|
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.75$ | | | |
| $\lambda_2^0 = 0.80$ | 0.175 | 0.200 | 0.235 | 0.263 | 0.308 | 0.414 | 0.545 |
| $\lambda_2^0 = 0.85$ | 0.178 | 0.214 | 0.249 | 0.295 | 0.342 | 0.466 | 0.605 |
| $\lambda_2^0 = 0.90$ | 0.180 | 0.229 | 0.271 | 0.334 | 0.397 | 0.548 | 0.690 |
| $\lambda_2^0 = 0.95$ | 0.205 | 0.275 | 0.352 | 0.439 | 0.515 | 0.692 | 0.823 |
| $DM_{nw}$ | 0.226 | 0.327 | 0.443 | 0.547 | 0.638 | 0.791 | 0.888 |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.267 | 0.323 | 0.397 | 0.474 | 0.545 | 0.703 | 0.831 |
| $\lambda_2^0 = 0.85$ | 0.285 | 0.358 | 0.441 | 0.525 | 0.608 | 0.766 | 0.876 |
| $\lambda_2^0 = 0.90$ | 0.321 | 0.412 | 0.506 | 0.605 | 0.690 | 0.839 | 0.926 |
| $\lambda_2^0 = 0.95$ | 0.403 | 0.516 | 0.628 | 0.737 | 0.810 | 0.920 | 0.971 |
| $CW_{nw}$ | 0.640 | 0.766 | 0.859 | 0.922 | 0.958 | 0.988 | 0.997 |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.95$ | | | |
| $\lambda_2^0 = 0.80$ | 0.438 | 0.552 | 0.658 | 0.765 | 0.833 | 0.929 | 0.969 |
| $\lambda_2^0 = 0.85$ | 0.493 | 0.611 | 0.716 | 0.816 | 0.877 | 0.955 | 0.980 |
| $\lambda_2^0 = 0.90$ | 0.565 | 0.694 | 0.787 | 0.875 | 0.922 | 0.972 | 0.991 |
| $\lambda_2^0 = 0.95$ | 0.697 | 0.809 | 0.879 | 0.935 | 0.965 | 0.990 | 0.997 |
| $DM_{nw}$ | 0.828 | 0.908 | 0.942 | 0.973 | 0.984 | 0.996 | 0.999 |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.692 | 0.799 | 0.871 | 0.931 | 0.960 | 0.990 | 0.997 |
| $\lambda_2^0 = 0.85$ | 0.743 | 0.844 | 0.903 | 0.953 | 0.976 | 0.995 | 0.998 |
| $\lambda_2^0 = 0.90$ | 0.808 | 0.897 | 0.943 | 0.972 | 0.987 | 0.997 | 0.999 |
| $\lambda_2^0 = 0.95$ | 0.885 | 0.948 | 0.972 | 0.989 | 0.996 | 0.999 | 1.000 |
| $CW_{nw}$ | 0.979 | 0.993 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.98$ | | | |
| $\lambda_2^0 = 0.80$ | 0.709 | 0.815 | 0.883 | 0.931 | 0.960 | 0.987 | 0.995 |
| $\lambda_2^0 = 0.85$ | 0.762 | 0.857 | 0.915 | 0.952 | 0.972 | 0.993 | 0.997 |
| $\lambda_2^0 = 0.90$ | 0.822 | 0.899 | 0.946 | 0.971 | 0.985 | 0.996 | 0.999 |
| $\lambda_2^0 = 0.95$ | 0.892 | 0.948 | 0.976 | 0.987 | 0.995 | 0.999 | 1.000 |
| $DM_{nw}$ | 0.950 | 0.978 | 0.990 | 0.995 | 0.997 | 0.999 | 1.000 |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.883 | 0.939 | 0.971 | 0.985 | 0.993 | 0.998 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.913 | 0.959 | 0.980 | 0.991 | 0.996 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.941 | 0.974 | 0.989 | 0.995 | 0.998 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.969 | 0.987 | 0.995 | 0.999 | 0.999 | 1.000 | 1.000 |
| $CW_{nw}$ | 0.995 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table S8: **DGP1** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ under conditional heteroskedasticity

| $\beta$ | -1.500 | -1.750 | -2.000 | -2.250 | -2.500 | -3.000 | -3.500 |
|---|---|---|---|---|---|---|---|
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.75$ | | | |
| $\lambda_2^0 = 0.80$ | 0.212 | 0.267 | 0.338 | 0.417 | 0.487 | 0.627 | 0.759 |
| $\lambda_2^0 = 0.85$ | 0.193 | 0.245 | 0.310 | 0.385 | 0.458 | 0.602 | 0.720 |
| $\lambda_2^0 = 0.90$ | 0.180 | 0.231 | 0.280 | 0.352 | 0.412 | 0.550 | 0.674 |
| $\lambda_2^0 = 0.95$ | 0.171 | 0.209 | 0.263 | 0.318 | 0.382 | 0.502 | 0.630 |
| $\lambda_2^0 = 1$ | 0.166 | 0.199 | 0.245 | 0.291 | 0.355 | 0.467 | 0.590 |
| $DM_{nw}$ | 0.226 | 0.327 | 0.443 | 0.547 | 0.638 | 0.791 | 0.888 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.488 | 0.568 | 0.663 | 0.749 | 0.812 | 0.903 | 0.958 |
| $\lambda_2^0 = 0.85$ | 0.463 | 0.554 | 0.633 | 0.719 | 0.784 | 0.892 | 0.947 |
| $\lambda_2^0 = 0.90$ | 0.438 | 0.515 | 0.596 | 0.686 | 0.751 | 0.863 | 0.931 |
| $\lambda_2^0 = 0.95$ | 0.405 | 0.484 | 0.561 | 0.647 | 0.714 | 0.837 | 0.914 |
| $\lambda_2^0 = 1$ | 0.391 | 0.448 | 0.528 | 0.618 | 0.677 | 0.812 | 0.890 |
| $CW_{nw}$ | 0.640 | 0.766 | 0.859 | 0.922 | 0.958 | 0.988 | 0.997 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.95$ | | | |
| $\lambda_2^0 = 0.80$ | 0.623 | 0.734 | 0.822 | 0.890 | 0.926 | 0.974 | 0.990 |
| $\lambda_2^0 = 0.85$ | 0.597 | 0.700 | 0.798 | 0.868 | 0.914 | 0.966 | 0.987 |
| $\lambda_2^0 = 0.90$ | 0.554 | 0.663 | 0.758 | 0.833 | 0.891 | 0.955 | 0.982 |
| $\lambda_2^0 = 0.95$ | 0.516 | 0.623 | 0.724 | 0.801 | 0.864 | 0.941 | 0.975 |
| $\lambda_2^0 = 1$ | 0.479 | 0.587 | 0.687 | 0.771 | 0.838 | 0.925 | 0.968 |
| $DM_{nw}$ | 0.828 | 0.908 | 0.942 | 0.973 | 0.984 | 0.996 | 0.999 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.835 | 0.904 | 0.947 | 0.971 | 0.984 | 0.997 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.811 | 0.885 | 0.936 | 0.965 | 0.981 | 0.996 | 0.998 |
| $\lambda_2^0 = 0.90$ | 0.787 | 0.866 | 0.918 | 0.957 | 0.976 | 0.994 | 0.998 |
| $\lambda_2^0 = 0.95$ | 0.753 | 0.843 | 0.902 | 0.946 | 0.970 | 0.991 | 0.997 |
| $\lambda_2^0 = 1$ | 0.723 | 0.818 | 0.881 | 0.935 | 0.962 | 0.989 | 0.996 |
| $CW_{nw}$ | 0.979 | 0.993 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.98$ | | | |
| $\lambda_2^0 = 0.80$ | 0.833 | 0.903 | 0.944 | 0.968 | 0.981 | 0.995 | 0.998 |
| $\lambda_2^0 = 0.85$ | 0.812 | 0.888 | 0.934 | 0.961 | 0.976 | 0.994 | 0.997 |
| $\lambda_2^0 = 0.90$ | 0.780 | 0.864 | 0.918 | 0.949 | 0.971 | 0.990 | 0.997 |
| $\lambda_2^0 = 0.95$ | 0.746 | 0.838 | 0.899 | 0.938 | 0.963 | 0.987 | 0.996 |
| $\lambda_2^0 = 1$ | 0.715 | 0.814 | 0.877 | 0.925 | 0.954 | 0.984 | 0.994 |
| $DM_{nw}$ | 0.950 | 0.978 | 0.990 | 0.995 | 0.997 | 0.999 | 1.000 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.937 | 0.970 | 0.985 | 0.993 | 0.996 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.932 | 0.965 | 0.983 | 0.993 | 0.996 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.918 | 0.957 | 0.980 | 0.991 | 0.995 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.904 | 0.950 | 0.976 | 0.989 | 0.994 | 0.998 | 1.000 |
| $\lambda_2^0 = 1$ | 0.892 | 0.942 | 0.969 | 0.986 | 0.993 | 0.998 | 1.000 |
| $CW_{nw}$ | 0.995 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table S9: **DGP1** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ under conditional heteroskedasticity

| $\beta$ | -1.500 | -1.750 | -2.000 | -2.250 | -2.500 | -3.000 | -3.500 |
|---|---|---|---|---|---|---|---|
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.75$ | | | |
| $\lambda_2^0 = 0.80$ | 0.209 | 0.247 | 0.308 | 0.371 | 0.439 | 0.584 | 0.732 |
| $\lambda_2^0 = 0.85$ | 0.258 | 0.339 | 0.416 | 0.507 | 0.593 | 0.749 | 0.860 |
| $\lambda_2^0 = 0.90$ | 0.315 | 0.413 | 0.510 | 0.615 | 0.702 | 0.834 | 0.915 |
| $\lambda_2^0 = 0.95$ | 0.263 | 0.348 | 0.454 | 0.560 | 0.642 | 0.778 | 0.884 |
| $\lambda_2^0 = 1$ | 0.211 | 0.279 | 0.355 | 0.444 | 0.522 | 0.674 | 0.796 |
| $DM_{nw}$ | 0.226 | 0.327 | 0.443 | 0.547 | 0.638 | 0.791 | 0.888 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.435 | 0.519 | 0.624 | 0.716 | 0.785 | 0.898 | 0.959 |
| $\lambda_2^0 = 0.85$ | 0.546 | 0.649 | 0.747 | 0.830 | 0.885 | 0.961 | 0.986 |
| $\lambda_2^0 = 0.90$ | 0.634 | 0.735 | 0.825 | 0.889 | 0.932 | 0.977 | 0.992 |
| $\lambda_2^0 = 0.95$ | 0.584 | 0.688 | 0.783 | 0.858 | 0.905 | 0.961 | 0.984 |
| $\lambda_2^0 = 1$ | 0.515 | 0.603 | 0.698 | 0.781 | 0.843 | 0.928 | 0.967 |
| $CW_{nw}$ | 0.640 | 0.766 | 0.859 | 0.922 | 0.958 | 0.988 | 0.997 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.95$ | | | |
| $\lambda_2^0 = 0.80$ | 0.597 | 0.716 | 0.805 | 0.882 | 0.925 | 0.976 | 0.991 |
| $\lambda_2^0 = 0.85$ | 0.737 | 0.840 | 0.902 | 0.949 | 0.972 | 0.994 | 0.997 |
| $\lambda_2^0 = 0.90$ | 0.820 | 0.898 | 0.938 | 0.969 | 0.983 | 0.995 | 0.998 |
| $\lambda_2^0 = 0.95$ | 0.770 | 0.860 | 0.913 | 0.952 | 0.973 | 0.991 | 0.997 |
| $\lambda_2^0 = 1$ | 0.672 | 0.776 | 0.850 | 0.908 | 0.943 | 0.979 | 0.993 |
| $DM_{nw}$ | 0.828 | 0.908 | 0.942 | 0.973 | 0.984 | 0.996 | 0.999 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.824 | 0.903 | 0.945 | 0.974 | 0.985 | 0.998 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.904 | 0.955 | 0.979 | 0.990 | 0.996 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.937 | 0.972 | 0.987 | 0.994 | 0.998 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.916 | 0.961 | 0.978 | 0.991 | 0.996 | 0.999 | 1.000 |
| $\lambda_2^0 = 1$ | 0.865 | 0.928 | 0.958 | 0.982 | 0.989 | 0.998 | 0.999 |
| $CW_{nw}$ | 0.979 | 0.993 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | | | | $\phi_1 = 0.98$ | | | |
| $\lambda_2^0 = 0.80$ | 0.827 | 0.907 | 0.943 | 0.971 | 0.986 | 0.995 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.909 | 0.954 | 0.979 | 0.990 | 0.995 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.942 | 0.971 | 0.988 | 0.994 | 0.997 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.920 | 0.958 | 0.978 | 0.991 | 0.995 | 0.999 | 1.000 |
| $\lambda_2^0 = 1$ | 0.864 | 0.923 | 0.957 | 0.980 | 0.988 | 0.997 | 0.999 |
| $DM_{nw}$ | 0.950 | 0.978 | 0.990 | 0.995 | 0.997 | 0.999 | 1.000 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.938 | 0.972 | 0.989 | 0.996 | 0.997 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.971 | 0.988 | 0.996 | 0.999 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.985 | 0.993 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.977 | 0.990 | 0.995 | 0.998 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 1$ | 0.955 | 0.981 | 0.991 | 0.997 | 0.999 | 1.000 | 1.000 |
| $CW_{nw}$ | 0.995 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table S10: **DGP2** Empirical power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | $DM_{nw}$ |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.521 | 0.571 | 0.620 | 0.669 | 0.722 | 0.776 | 0.833 | 0.883 | 0.930 | 0.971 | 0.889 |
| | T=500 | 0.783 | 0.835 | 0.877 | 0.913 | 0.944 | 0.967 | 0.985 | 0.994 | 0.998 | 1.000 | 0.998 |
| | T=1000 | 0.959 | 0.976 | 0.987 | 0.994 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | | | $CW_{nw}$ |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$ | T=250 | 0.910 | 0.936 | 0.955 | 0.969 | 0.979 | 0.987 | 0.993 | 0.997 | 0.999 | 1.000 | 1.000 |
| | T=500 | 0.992 | 0.996 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | T=1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

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Table S11: **DGP2** Empirical power of $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.528 | 0.612 | 0.692 | 0.768 | 0.818 | 0.840 | 0.837 | 0.816 | 0.785 | 0.751 | 0.714 | 0.854 |
| | T=500 | 0.717 | 0.807 | 0.878 | 0.927 | 0.948 | 0.956 | 0.951 | 0.937 | 0.918 | 0.893 | 0.861 | 0.988 |
| | T=1000 | 0.879 | 0.932 | 0.965 | 0.982 | 0.990 | 0.990 | 0.990 | 0.985 | 0.980 | 0.973 | 0.958 | 1.000 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)_{nw}$ | T=250 | 0.864 | 0.916 | 0.946 | 0.970 | 0.977 | 0.980 | 0.980 | 0.976 | 0.967 | 0.959 | 0.946 | 0.997 |
| | T=500 | 0.954 | 0.977 | 0.988 | 0.993 | 0.996 | 0.997 | 0.996 | 0.994 | 0.993 | 0.989 | 0.985 | 1.000 |
| | T=1000 | 0.989 | 0.994 | 0.998 | 0.999 | 0.999 | 1.000 | 0.999 | 0.999 | 0.999 | 0.998 | 0.997 | 1.000 |

magnitudes near 1 for inferences based on $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)_{nw}$. For the average based statistics in Tables S11-S12 the empirical results also clearly corroborate our theoretical analysis that allowed us to establish that power peaks for $\lambda_2^0 = 0.5\tau_0 + 0.5$ (i.e., $\lambda_2^0 = 0.75$ under $\tau_0 = 0.5$ and $\lambda_2^0 = 0.95$ under $\tau_0 = 0.8$).

Table S12: **DGP2** Empirical power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional heteroskedasticity

| | $\lambda_2^0$ | 0.500 | 0.550 | 0.600 | 0.650 | 0.700 | 0.750 | 0.800 | 0.850 | 0.900 | 0.950 | 1.000 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | $DM_{nw}$ |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.429 | 0.472 | 0.517 | 0.565 | 0.627 | 0.711 | 0.820 | 0.912 | 0.945 | 0.929 | 0.881 | 0.854 |
| | T=500 | 0.589 | 0.639 | 0.696 | 0.754 | 0.820 | 0.895 | 0.951 | 0.986 | 0.992 | 0.984 | 0.963 | 0.854 |
| | T=1000 | 0.759 | 0.810 | 0.858 | 0.903 | 0.944 | 0.973 | 0.992 | 0.998 | 0.999 | 0.998 | 0.992 | 1.000 |
| | | | | | | | | | | | | | $CW_{nw}$ |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)_{nw}$ | T=250 | 0.778 | 0.822 | 0.858 | 0.897 | 0.928 | 0.957 | 0.980 | 0.994 | 0.997 | 0.994 | 0.986 | 0.997 |
| | T=500 | 0.899 | 0.931 | 0.951 | 0.969 | 0.981 | 0.991 | 0.997 | 0.999 | 1.000 | 0.999 | 0.997 | 1.000 |
| | T=1000 | 0.968 | 0.980 | 0.988 | 0.993 | 0.997 | 0.999 | 0.999 | 1.000 | 1.000 | 1.000 | 0.999 | 1.000 |

Recalling that these power experiments were implemented using a fixed $T = 500$ it is also useful to point out that for such sample sizes the size properties of the tests were also not far off their nominal counterparts. Overall, these power outcomes provide an encouraging assessment of the finite sample behavior of the proposed test statistics.

## 3. Power under additional parameterisations

Here we reconsider the power experiments associated with DGP1 in the main text (Tables 3-4) by considering parameterisations that bring the DGP closer to the null (i.e., smaller magnitudes of $\beta$). The experiments in Tables 3-4 of the main text were implemented for

$\beta \in \{-1.50, -1.75, -2.00, -2.25, -2.50, -3.00, -3.50\}$. Here we repeat the same experiments using $\beta \in \{-0.50, -0.75, -1.00, -1.25\}$ with outcomes presented in Tables S13-S15. The sample size for these local power experiments is kept at $T = 500$ as in the main text.

Table S13: **DGP1** Empirical Power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity

| $\beta$ | -0.50 | -0.75 | -1.00 | -1.25 |
|---|---|---|---|---|
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | $\phi_1 = 0.75$ | | |
| $\lambda_2^0 = 0.80$ | 0.098 | 0.109 | 0.128 | 0.155 |
| $\lambda_2^0 = 0.85$ | 0.100 | 0.112 | 0.136 | 0.173 |
| $\lambda_2^0 = 0.90$ | 0.104 | 0.113 | 0.146 | 0.188 |
| $\lambda_2^0 = 0.95$ | 0.099 | 0.132 | 0.179 | 0.241 |
| DM | 0.025 | 0.062 | 0.118 | 0.204 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)_{adj}$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.128 | 0.159 | 0.202 | 0.268 |
| $\lambda_2^0 = 0.85$ | 0.136 | 0.165 | 0.229 | 0.304 |
| $\lambda_2^0 = 0.90$ | 0.147 | 0.188 | 0.261 | 0.360 |
| $\lambda_2^0 = 0.95$ | 0.167 | 0.243 | 0.336 | 0.453 |
| CW | 0.147 | 0.273 | 0.440 | 0.597 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | $\phi_1 = 0.95$ | | |
| $\lambda_2^0 = 0.80$ | 0.124 | 0.194 | 0.323 | 0.464 |
| $\lambda_2^0 = 0.85$ | 0.126 | 0.218 | 0.363 | 0.523 |
| $\lambda_2^0 = 0.90$ | 0.138 | 0.255 | 0.419 | 0.594 |
| $\lambda_2^0 = 0.95$ | 0.170 | 0.317 | 0.526 | 0.711 |
| DM | 0.096 | 0.289 | 0.531 | 0.719 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)_{adj}$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.197 | 0.344 | 0.543 | 0.713 |
| $\lambda_2^0 = 0.85$ | 0.217 | 0.388 | 0.597 | 0.763 |
| $\lambda_2^0 = 0.90$ | 0.251 | 0.446 | 0.666 | 0.820 |
| $\lambda_2^0 = 0.95$ | 0.321 | 0.542 | 0.755 | 0.886 |
| CW | 0.379 | 0.655 | 0.860 | 0.943 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | $\phi_1 = 0.98$ | | |
| $\lambda_2^0 = 0.80$ | 0.179 | 0.353 | 0.558 | 0.735 |
| $\lambda_2^0 = 0.85$ | 0.201 | 0.398 | 0.609 | 0.782 |
| $\lambda_2^0 = 0.90$ | 0.231 | 0.458 | 0.674 | 0.833 |
| $\lambda_2^0 = 0.95$ | 0.288 | 0.557 | 0.763 | 0.896 |
| DM | 0.229 | 0.551 | 0.773 | 0.905 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)_{adj}$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.322 | 0.565 | 0.769 | 0.892 |
| $\lambda_2^0 = 0.85$ | 0.358 | 0.618 | 0.805 | 0.918 |
| $\lambda_2^0 = 0.90$ | 0.408 | 0.679 | 0.849 | 0.942 |
| $\lambda_2^0 = 0.95$ | 0.483 | 0.758 | 0.901 | 0.969 |
| CW | 0.562 | 0.830 | 0.949 | 0.986 |

It is important to point out that Tables S13-S15 consider signal to noise ratios that are particularly low when compared with Tables 3-4 in the main text. Nevertheless we can still observe empirical powers that reach magnitudes as high as 80-99% for a selection of outcomes. For $\beta = -0.75$ for instance and focusing on $\phi_1 = 0.98$ we note empirical powers in the vicinity of 80% for $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 0.90)$. The same frequency declines towards the 60-65% range

Table S14: **DGP1** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)$ under conditional homoskedasticity

| $\beta$ | -0.50 | -0.75 | -1.00 | -1.25 |
|---|---|---|---|---|
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ | | $\phi_1 = 0.75$ | | |
| $\lambda_2^0 = 0.80$ | 0.087 | 0.118 | 0.155 | 0.215 |
| $\lambda_2^0 = 0.85$ | 0.087 | 0.113 | 0.149 | 0.204 |
| $\lambda_2^0 = 0.90$ | 0.087 | 0.113 | 0.144 | 0.191 |
| $\lambda_2^0 = 0.95$ | 0.088 | 0.113 | 0.134 | 0.175 |
| $\lambda_2^0 = 1$ | 0.091 | 0.113 | 0.127 | 0.169 |
| DM | 0.025 | 0.062 | 0.118 | 0.204 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.150 | 0.210 | 0.294 | 0.393 |
| $\lambda_2^0 = 0.85$ | 0.142 | 0.196 | 0.273 | 0.378 |
| $\lambda_2^0 = 0.90$ | 0.139 | 0.184 | 0.249 | 0.344 |
| $\lambda_2^0 = 0.95$ | 0.125 | 0.176 | 0.227 | 0.315 |
| $\lambda_2^0 = 1$ | 0.126 | 0.164 | 0.213 | 0.296 |
| CW | 0.147 | 0.273 | 0.440 | 0.597 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ | | $\phi_1 = 0.95$ | | |
| $\lambda_2^0 = 0.80$ | 0.148 | 0.271 | 0.451 | 0.617 |
| $\lambda_2^0 = 0.85$ | 0.141 | 0.261 | 0.420 | 0.584 |
| $\lambda_2^0 = 0.90$ | 0.131 | 0.244 | 0.384 | 0.548 |
| $\lambda_2^0 = 0.95$ | 0.135 | 0.230 | 0.354 | 0.505 |
| $\lambda_2^0 = 1$ | 0.131 | 0.221 | 0.332 | 0.473 |
| DM | 0.096 | 0.289 | 0.531 | 0.719 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.282 | 0.468 | 0.677 | 0.822 |
| $\lambda_2^0 = 0.85$ | 0.261 | 0.447 | 0.645 | 0.805 |
| $\lambda_2^0 = 0.90$ | 0.246 | 0.421 | 0.616 | 0.775 |
| $\lambda_2^0 = 0.95$ | 0.228 | 0.398 | 0.580 | 0.743 |
| $\lambda_2^0 = 1$ | 0.215 | 0.373 | 0.551 | 0.719 |
| CW | 0.379 | 0.655 | 0.860 | 0.943 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.5; \lambda_2^0)$ | | $\phi_1 = 0.98$ | | |
| $\lambda_2^0 = 0.80$ | 0.235 | 0.466 | 0.672 | 0.832 |
| $\lambda_2^0 = 0.85$ | 0.232 | 0.448 | 0.644 | 0.811 |
| $\lambda_2^0 = 0.90$ | 0.212 | 0.421 | 0.618 | 0.782 |
| $\lambda_2^0 = 0.95$ | 0.198 | 0.390 | 0.584 | 0.757 |
| $\lambda_2^0 = 1$ | 0.189 | 0.368 | 0.551 | 0.727 |
| DM | 0.229 | 0.551 | 0.773 | 0.905 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.5; \lambda_2^0)$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.419 | 0.672 | 0.839 | 0.937 |
| $\lambda_2^0 = 0.85$ | 0.403 | 0.657 | 0.821 | 0.929 |
| $\lambda_2^0 = 0.90$ | 0.376 | 0.632 | 0.805 | 0.919 |
| $\lambda_2^0 = 0.95$ | 0.355 | 0.606 | 0.786 | 0.907 |
| $\lambda_2^0 = 1$ | 0.338 | 0.583 | 0.768 | 0.897 |
| CW | 0.562 | 0.830 | 0.949 | 0.986 |

Table S15: **DGP1** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity

| $\beta$ | -0.50 | -0.75 | -1.00 | -1.25 |
|---|---|---|---|---|
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | $\phi_1 = 0.75$ | | |
| $\lambda_2^0 = 0.80$ | 0.101 | 0.122 | 0.155 | 0.203 |
| $\lambda_2^0 = 0.85$ | 0.102 | 0.137 | 0.194 | 0.274 |
| $\lambda_2^0 = 0.90$ | 0.104 | 0.147 | 0.230 | 0.336 |
| $\lambda_2^0 = 0.95$ | 0.084 | 0.132 | 0.192 | 0.282 |
| $\lambda_2^0 = 1$ | 0.086 | 0.117 | 0.159 | 0.233 |
| DM | 0.025 | 0.062 | 0.118 | 0.204 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.152 | 0.206 | 0.277 | 0.375 |
| $\lambda_2^0 = 0.85$ | 0.185 | 0.257 | 0.370 | 0.499 |
| $\lambda_2^0 = 0.90$ | 0.215 | 0.303 | 0.437 | 0.579 |
| $\lambda_2^0 = 0.95$ | 0.176 | 0.265 | 0.389 | 0.524 |
| $\lambda_2^0 = 1$ | 0.151 | 0.218 | 0.312 | 0.431 |
| CW | 0.147 | 0.273 | 0.440 | 0.597 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | $\phi_1 = 0.95$ | | |
| $\lambda_2^0 = 0.80$ | 0.144 | 0.266 | 0.445 | 0.608 |
| $\lambda_2^0 = 0.85$ | 0.177 | 0.350 | 0.565 | 0.739 |
| $\lambda_2^0 = 0.90$ | 0.209 | 0.414 | 0.637 | 0.799 |
| $\lambda_2^0 = 0.95$ | 0.182 | 0.375 | 0.575 | 0.750 |
| $\lambda_2^0 = 1$ | 0.157 | 0.297 | 0.486 | 0.659 |
| DM | 0.096 | 0.289 | 0.531 | 0.719 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.270 | 0.459 | 0.680 | 0.821 |
| $\lambda_2^0 = 0.85$ | 0.336 | 0.574 | 0.778 | 0.895 |
| $\lambda_2^0 = 0.90$ | 0.402 | 0.634 | 0.828 | 0.927 |
| $\lambda_2^0 = 0.95$ | 0.354 | 0.593 | 0.788 | 0.903 |
| $\lambda_2^0 = 1$ | 0.297 | 0.511 | 0.712 | 0.851 |
| CW | 0.379 | 0.655 | 0.860 | 0.943 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | $\phi_1 = 0.98$ | | |
| $\lambda_2^0 = 0.80$ | 0.240 | 0.462 | 0.685 | 0.840 |
| $\lambda_2^0 = 0.85$ | 0.310 | 0.582 | 0.783 | 0.907 |
| $\lambda_2^0 = 0.90$ | 0.360 | 0.651 | 0.830 | 0.934 |
| $\lambda_2^0 = 0.95$ | 0.317 | 0.603 | 0.793 | 0.913 |
| $\lambda_2^0 = 1$ | 0.265 | 0.519 | 0.720 | 0.865 |
| DM | 0.229 | 0.551 | 0.773 | 0.905 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.421 | 0.676 | 0.850 | 0.943 |
| $\lambda_2^0 = 0.85$ | 0.505 | 0.768 | 0.909 | 0.970 |
| $\lambda_2^0 = 0.90$ | 0.567 | 0.811 | 0.927 | 0.981 |
| $\lambda_2^0 = 0.95$ | 0.521 | 0.783 | 0.913 | 0.975 |
| $\lambda_2^0 = 1$ | 0.457 | 0.718 | 0.872 | 0.957 |
| CW | 0.562 | 0.830 | 0.949 | 0.986 |

for less persistent predictors. Finally, it is also noteworthy to point out that inferences based on $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 0.9)$ (as suggested by our local power analysis) lead to empirical powers similar to those displayed by the CW statistic, despite the latter being severely undersized. If we focus on Table S15 under $\phi_1 = 0.75$ for instance, inferences based on $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 0.9)$ led to power estimates of $(21.5\%, 30.3\%, 43.7\%, 57.9\%)$ compared with $(14.7\%, 27.3\%, 44.0\%, 59.7\%)$ for CW.

# 4. Size and Power under $\pi_0 = 0.50$

Here we repeat our size and power based experiments presented in the main text using an alternative starting point that initiates the recursive forecasts. Recalling that estimation starts at $k_0 = [T\pi_0]$ so that the effective sample size is given by $T - k_0$, our simulations in the main text were implemented using $\pi_0 = 0.25$ throughout. In the Tables that follow we consider the case of $\pi_0 = 0.50$ instead. This is also meant to document the behavior of our test statistics under smaller samples sizes. Under $T = 250$ for instance, the effective number of forecast errors is about 125.

Tables S16-S17 focus on size under DGP1 and can be compared with Tables 1-2 in the main text while Tables S18-S19 correspond to DGP2 and can be compared with Tables 5-6 in the main text. On the basis of these comparisons we can clearly argue that the use of $\pi_0 = 0.50$ instead of $\pi_0 = 0.25$ does not cause any significant differences in the empirical size estimates of both test statistics and across all sample size configurations. Focusing on $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 1)$ under $\phi_1 = 0.75$ for instance we had a size estimate of 11.0% under $(T, \pi_0) = (250, 0.25)$ compared with 10.2% under $(T, \pi_0) = (250, 0.50)$ in Table S17. These results indicate that finite sample approximations typically remain accurate across most parameterisations.

Naturally, the picture is mildly different when it comes to power. Contrasting Table S20 with Table 3 in the main text we note a drop in power of about ten percentage points overall. Given a drop of 25% in the effective sample size this comes across as a very reasonably sized loss. Similar orders of magnitude also characterise $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ when comparing Table S21 with Table 4 in the main text. For DGP2 the spreads in power are even smaller when contrasting Table

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Table S16: **DGP1** Empirical size of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity and 10% nominal size $(\pi_0 = 0.50)$

| $\lambda_2^0$ | | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\phi = 0.75$ | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.095 | 0.096 | 0.097 | 0.096 | 0.095 | 0.096 | 0.095 | 0.097 | 0.092 | 0.090 | 0.019 |
| | T=500 | 0.100 | 0.099 | 0.102 | 0.099 | 0.100 | 0.103 | 0.103 | 0.097 | 0.101 | 0.089 | 0.017 |
| | T=1000 | 0.094 | 0.095 | 0.094 | 0.093 | 0.094 | 0.097 | 0.101 | 0.097 | 0.095 | 0.092 | 0.018 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.107 | 0.107 | 0.107 | 0.108 | 0.110 | 0.110 | 0.112 | 0.116 | 0.114 | 0.126 | 0.061 |
| | T=500 | 0.106 | 0.106 | 0.110 | 0.107 | 0.111 | 0.112 | 0.113 | 0.112 | 0.116 | 0.114 | 0.060 |
| | T=1000 | 0.098 | 0.098 | 0.100 | 0.098 | 0.101 | 0.105 | 0.108 | 0.107 | 0.108 | 0.110 | 0.063 |
| | | | | | | | $\phi = 0.95$ | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.094 | 0.092 | 0.090 | 0.085 | 0.089 | 0.087 | 0.088 | 0.089 | 0.085 | 0.086 | 0.022 |
| | T=500 | 0.094 | 0.099 | 0.098 | 0.100 | 0.094 | 0.094 | 0.092 | 0.091 | 0.092 | 0.085 | 0.021 |
| | T=1000 | 0.091 | 0.091 | 0.091 | 0.095 | 0.098 | 0.098 | 0.095 | 0.100 | 0.099 | 0.095 | 0.020 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.106 | 0.105 | 0.105 | 0.101 | 0.106 | 0.109 | 0.108 | 0.114 | 0.114 | 0.125 | 0.070 |
| | T=500 | 0.103 | 0.109 | 0.109 | 0.112 | 0.105 | 0.107 | 0.106 | 0.108 | 0.113 | 0.114 | 0.068 |
| | T=1000 | 0.095 | 0.097 | 0.097 | 0.102 | 0.105 | 0.106 | 0.105 | 0.111 | 0.110 | 0.111 | 0.065 |
| | | | | | | | $\phi = 0.98$ | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.090 | 0.095 | 0.091 | 0.088 | 0.090 | 0.090 | 0.092 | 0.092 | 0.087 | 0.092 | 0.034 |
| | T=500 | 0.093 | 0.094 | 0.093 | 0.095 | 0.094 | 0.093 | 0.090 | 0.093 | 0.088 | 0.083 | 0.026 |
| | T=1000 | 0.094 | 0.097 | 0.096 | 0.097 | 0.099 | 0.095 | 0.096 | 0.100 | 0.092 | 0.092 | 0.022 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.108 | 0.115 | 0.114 | 0.110 | 0.115 | 0.118 | 0.123 | 0.131 | 0.127 | 0.151 | 0.094 |
| | T=500 | 0.103 | 0.106 | 0.107 | 0.109 | 0.109 | 0.110 | 0.110 | 0.115 | 0.115 | 0.120 | 0.078 |
| | T=1000 | 0.100 | 0.104 | 0.102 | 0.103 | 0.107 | 0.105 | 0.105 | 0.112 | 0.108 | 0.113 | 0.071 |

Table S17: **DGP1** Empirical size of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity and 10% nominal size; $(\pi_0 = 0.50)$

| $\lambda_2^0$ | | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 1.00 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\phi = 0.75$ | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.093 | 0.098 | 0.097 | 0.097 | 0.097 | 0.097 | 0.097 | 0.104 | 0.084 | 0.058 | 0.063 | 0.019 |
| | T=500 | 0.098 | 0.098 | 0.104 | 0.098 | 0.100 | 0.098 | 0.100 | 0.097 | 0.083 | 0.074 | 0.072 | 0.017 |
| | T=1000 | 0.095 | 0.094 | 0.094 | 0.095 | 0.094 | 0.094 | 0.097 | 0.098 | 0.081 | 0.071 | 0.080 | 0.018 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.104 | 0.108 | 0.110 | 0.109 | 0.116 | 0.117 | 0.127 | 0.150 | 0.143 | 0.114 | 0.102 | 0.061 |
| | T=500 | 0.104 | 0.107 | 0.111 | 0.108 | 0.112 | 0.112 | 0.122 | 0.132 | 0.126 | 0.110 | 0.096 | 0.060 |
| | T=1000 | 0.100 | 0.097 | 0.100 | 0.102 | 0.101 | 0.103 | 0.112 | 0.118 | 0.109 | 0.095 | 0.099 | 0.063 |
| | | | | | | | $\phi = 0.95$ | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.094 | 0.093 | 0.093 | 0.087 | 0.091 | 0.093 | 0.091 | 0.097 | 0.081 | 0.060 | 0.060 | 0.022 |
| | T=500 | 0.097 | 0.098 | 0.101 | 0.104 | 0.100 | 0.092 | 0.097 | 0.089 | 0.081 | 0.063 | 0.068 | 0.021 |
| | T=1000 | 0.092 | 0.092 | 0.089 | 0.094 | 0.098 | 0.098 | 0.096 | 0.098 | 0.084 | 0.070 | 0.075 | 0.020 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.109 | 0.111 | 0.110 | 0.107 | 0.113 | 0.124 | 0.128 | 0.152 | 0.154 | 0.121 | 0.103 | 0.070 |
| | T=500 | 0.106 | 0.108 | 0.113 | 0.115 | 0.115 | 0.110 | 0.118 | 0.124 | 0.127 | 0.106 | 0.098 | 0.068 |
| | T=1000 | 0.096 | 0.097 | 0.095 | 0.100 | 0.106 | 0.111 | 0.107 | 0.124 | 0.118 | 0.095 | 0.093 | 0.065 |
| | | | | | | | $\phi = 0.98$ | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.090 | 0.092 | 0.093 | 0.091 | 0.094 | 0.092 | 0.098 | 0.107 | 0.093 | 0.066 | 0.059 | 0.034 |
| | T=500 | 0.092 | 0.094 | 0.094 | 0.093 | 0.095 | 0.097 | 0.095 | 0.095 | 0.077 | 0.064 | 0.071 | 0.026 |
| | T=1000 | 0.094 | 0.096 | 0.095 | 0.096 | 0.100 | 0.099 | 0.096 | 0.098 | 0.078 | 0.070 | 0.072 | 0.022 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.108 | 0.113 | 0.119 | 0.117 | 0.123 | 0.129 | 0.146 | 0.179 | 0.183 | 0.150 | 0.121 | 0.094 |
| | T=500 | 0.103 | 0.107 | 0.108 | 0.109 | 0.117 | 0.118 | 0.124 | 0.141 | 0.140 | 0.121 | 0.110 | 0.078 |
| | T=1000 | 0.100 | 0.103 | 0.102 | 0.105 | 0.110 | 0.113 | 0.114 | 0.125 | 0.121 | 0.101 | 0.094 | 0.071 |

Table S18: **DGP2** Empirical Size of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity; $\pi_0 = 0.50$

| | $\lambda_2^0$ | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.080 | 0.080 | 0.082 | 0.073 | 0.076 | 0.075 | 0.074 | 0.073 | 0.069 | 0.069 | 0.009 |
| | T=500 | 0.086 | 0.087 | 0.087 | 0.086 | 0.085 | 0.081 | 0.081 | 0.078 | 0.073 | 0.066 | 0.009 |
| | T=1000 | 0.084 | 0.089 | 0.090 | 0.091 | 0.091 | 0.087 | 0.092 | 0.085 | 0.082 | 0.076 | 0.011 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.109 | 0.118 | 0.117 | 0.113 | 0.115 | 0.120 | 0.121 | 0.131 | 0.139 | 0.162 | 0.079 |
| | T=500 | 0.102 | 0.108 | 0.108 | 0.109 | 0.111 | 0.109 | 0.112 | 0.113 | 0.114 | 0.129 | 0.070 |
| | T=1000 | 0.099 | 0.102 | 0.105 | 0.108 | 0.108 | 0.110 | 0.116 | 0.115 | 0.115 | 0.122 | 0.077 |

Table S19: **DGP2** Empirical Size of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity; $\pi_0 = 0.50$

| | $\lambda_2^0$ | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 1.00 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.079 | 0.080 | 0.081 | 0.075 | 0.075 | 0.072 | 0.071 | 0.074 | 0.069 | 0.048 | 0.039 | 0.009 |
| | T=500 | 0.087 | 0.089 | 0.087 | 0.085 | 0.083 | 0.078 | 0.077 | 0.069 | 0.062 | 0.051 | 0.047 | 0.009 |
| | T=1000 | 0.086 | 0.090 | 0.088 | 0.089 | 0.087 | 0.085 | 0.084 | 0.076 | 0.065 | 0.051 | 0.061 | 0.011 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.113 | 0.119 | 0.122 | 0.119 | 0.124 | 0.132 | 0.150 | 0.190 | 0.210 | 0.168 | 0.132 | 0.079 |
| | T=500 | 0.106 | 0.108 | 0.112 | 0.112 | 0.116 | 0.117 | 0.132 | 0.146 | 0.163 | 0.142 | 0.113 | 0.070 |
| | T=1000 | 0.100 | 0.105 | 0.106 | 0.110 | 0.108 | 0.112 | 0.122 | 0.136 | 0.136 | 0.117 | 0.107 | 0.077 |

S23 with Table 8 in the main text. If we consider the case of $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0 = 0.7)$ under $T = 250$ as an example, we have a power estimate of 99.1% under $\pi_0 = 0.25$ (Table 8 in main text) compared with 97.8% in Table S23.

Table S20: **DGP1** Empirical Power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity; $\pi_0 = 0.50$

| $\beta$ | 0.00 | -1.50 | -1.75 | -2.00 | -2.25 | -2.50 | -3.00 | -3.50 |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | | | $\phi_1 = 0.75$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.095 | 0.187 | 0.229 | 0.282 | 0.333 | 0.396 | 0.533 | 0.671 |
| $\lambda_2^0 = 0.85$ | 0.095 | 0.202 | 0.257 | 0.315 | 0.377 | 0.450 | 0.604 | 0.738 |
| $\lambda_2^0 = 0.90$ | 0.100 | 0.232 | 0.301 | 0.377 | 0.451 | 0.536 | 0.687 | 0.810 |
| $\lambda_2^0 = 0.95$ | 0.092 | 0.290 | 0.387 | 0.494 | 0.576 | 0.662 | 0.804 | 0.895 |
| DM | 0.018 | 0.298 | 0.390 | 0.480 | 0.547 | 0.625 | 0.747 | 0.841 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)_{adj}$ | | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.108 | 0.305 | 0.392 | 0.497 | 0.590 | 0.683 | 0.838 | 0.934 |
| $\lambda_2^0 = 0.85$ | 0.108 | 0.342 | 0.447 | 0.555 | 0.658 | 0.750 | 0.887 | 0.958 |
| $\lambda_2^0 = 0.90$ | 0.117 | 0.406 | 0.520 | 0.639 | 0.739 | 0.827 | 0.930 | 0.979 |
| $\lambda_2^0 = 0.95$ | 0.116 | 0.513 | 0.648 | 0.764 | 0.847 | 0.906 | 0.971 | 0.994 |
| CW | 0.061 | 0.655 | 0.778 | 0.859 | 0.913 | 0.950 | 0.982 | 0.996 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | | | $\phi_1 = 0.95$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.095 | 0.548 | 0.659 | 0.759 | 0.843 | 0.897 | 0.960 | 0.983 |
| $\lambda_2^0 = 0.85$ | 0.094 | 0.602 | 0.713 | 0.804 | 0.880 | 0.922 | 0.974 | 0.989 |
| $\lambda_2^0 = 0.90$ | 0.093 | 0.667 | 0.779 | 0.858 | 0.914 | 0.945 | 0.982 | 0.992 |
| $\lambda_2^0 = 0.95$ | 0.085 | 0.761 | 0.856 | 0.916 | 0.951 | 0.970 | 0.991 | 0.996 |
| DM | 0.022 | 0.728 | 0.813 | 0.876 | 0.917 | 0.941 | 0.977 | 0.988 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)_{adj}$ | | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.110 | 0.790 | 0.882 | 0.937 | 0.971 | 0.986 | 0.998 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.109 | 0.829 | 0.913 | 0.954 | 0.981 | 0.993 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.115 | 0.876 | 0.943 | 0.972 | 0.989 | 0.997 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.115 | 0.931 | 0.972 | 0.986 | 0.996 | 0.999 | 1.000 | 1.000 |
| CW | 0.068 | 0.957 | 0.982 | 0.991 | 0.996 | 0.999 | 1.000 | 1.000 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | | | $\phi_1 = 0.98$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.093 | 0.789 | 0.869 | 0.923 | 0.951 | 0.971 | 0.991 | 0.997 |
| $\lambda_2^0 = 0.85$ | 0.094 | 0.824 | 0.897 | 0.938 | 0.963 | 0.980 | 0.993 | 0.998 |
| $\lambda_2^0 = 0.90$ | 0.092 | 0.862 | 0.921 | 0.956 | 0.973 | 0.987 | 0.995 | 0.998 |
| $\lambda_2^0 = 0.95$ | 0.083 | 0.910 | 0.951 | 0.975 | 0.986 | 0.993 | 0.998 | 1.000 |
| DM | 0.027 | 0.880 | 0.927 | 0.954 | 0.971 | 0.982 | 0.994 | 0.999 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)_{adj}$ | | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.113 | 0.925 | 0.966 | 0.987 | 0.994 | 0.997 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.112 | 0.944 | 0.975 | 0.991 | 0.996 | 0.998 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.116 | 0.960 | 0.984 | 0.995 | 0.997 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.118 | 0.978 | 0.992 | 0.997 | 0.999 | 0.999 | 1.000 | 1.000 |
| CW | 0.082 | 0.988 | 0.994 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 |

Table S21: **DGP1** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity; $\pi_0 = 0.50$

| $\beta$ | 0.00 | -1.50 | -1.75 | -2.00 | -2.25 | -2.50 | -3.00 | -3.50 |
|---|---|---|---|---|---|---|---|---|
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | | | $\phi_1 = 0.75$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.091 | 0.259 | 0.323 | 0.412 | 0.474 | 0.563 | 0.713 | 0.831 |
| $\lambda_2^0 = 0.85$ | 0.094 | 0.346 | 0.442 | 0.542 | 0.625 | 0.711 | 0.833 | 0.909 |
| $\lambda_2^0 = 0.90$ | 0.084 | 0.415 | 0.524 | 0.627 | 0.699 | 0.783 | 0.876 | 0.939 |
| $\lambda_2^0 = 0.95$ | 0.073 | 0.361 | 0.460 | 0.562 | 0.644 | 0.723 | 0.840 | 0.917 |
| $\lambda_2^0 = 1$ | 0.072 | 0.282 | 0.362 | 0.449 | 0.530 | 0.619 | 0.749 | 0.857 |
| DM | 0.018 | 0.298 | 0.390 | 0.480 | 0.547 | 0.625 | 0.747 | 0.841 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.112 | 0.439 | 0.550 | 0.673 | 0.760 | 0.835 | 0.941 | 0.982 |
| $\lambda_2^0 = 0.85$ | 0.122 | 0.575 | 0.693 | 0.798 | 0.872 | 0.923 | 0.975 | 0.994 |
| $\lambda_2^0 = 0.90$ | 0.130 | 0.650 | 0.770 | 0.852 | 0.914 | 0.952 | 0.988 | 0.997 |
| $\lambda_2^0 = 0.95$ | 0.109 | 0.587 | 0.711 | 0.811 | 0.877 | 0.927 | 0.979 | 0.994 |
| $\lambda_2^0 = 1$ | 0.096 | 0.485 | 0.606 | 0.706 | 0.799 | 0.867 | 0.951 | 0.986 |
| CW | 0.061 | 0.655 | 0.778 | 0.859 | 0.913 | 0.950 | 0.982 | 0.996 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | | | $\phi_1 = 0.95$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.099 | 0.690 | 0.790 | 0.861 | 0.917 | 0.946 | 0.983 | 0.991 |
| $\lambda_2^0 = 0.85$ | 0.095 | 0.788 | 0.869 | 0.920 | 0.955 | 0.971 | 0.991 | 0.996 |
| $\lambda_2^0 = 0.90$ | 0.079 | 0.840 | 0.905 | 0.945 | 0.967 | 0.979 | 0.992 | 0.997 |
| $\lambda_2^0 = 0.95$ | 0.072 | 0.800 | 0.874 | 0.923 | 0.956 | 0.972 | 0.990 | 0.996 |
| $\lambda_2^0 = 1$ | 0.067 | 0.724 | 0.813 | 0.879 | 0.925 | 0.953 | 0.983 | 0.994 |
| DM | 0.022 | 0.728 | 0.813 | 0.876 | 0.917 | 0.941 | 0.977 | 0.988 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.120 | 0.882 | 0.944 | 0.973 | 0.988 | 0.996 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.131 | 0.942 | 0.975 | 0.988 | 0.996 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.127 | 0.959 | 0.985 | 0.992 | 0.997 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.113 | 0.944 | 0.976 | 0.988 | 0.995 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 1$ | 0.097 | 0.908 | 0.952 | 0.979 | 0.991 | 0.996 | 1.000 | 1.000 |
| CW | 0.068 | 0.957 | 0.982 | 0.991 | 0.996 | 0.999 | 1.000 | 1.000 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | | | $\phi_1 = 0.98$ | | | | |
| $\lambda_2^0 = 0.80$ | 0.092 | 0.867 | 0.923 | 0.957 | 0.973 | 0.984 | 0.996 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.090 | 0.915 | 0.952 | 0.975 | 0.986 | 0.991 | 0.998 | 0.999 |
| $\lambda_2^0 = 0.90$ | 0.077 | 0.936 | 0.965 | 0.983 | 0.989 | 0.994 | 0.998 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.068 | 0.918 | 0.953 | 0.974 | 0.985 | 0.992 | 0.998 | 1.000 |
| $\lambda_2^0 = 1$ | 0.068 | 0.882 | 0.928 | 0.960 | 0.977 | 0.986 | 0.995 | 0.999 |
| DM | 0.027 | 0.880 | 0.927 | 0.954 | 0.971 | 0.982 | 0.994 | 0.999 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.124 | 0.961 | 0.985 | 0.996 | 0.998 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.135 | 0.981 | 0.993 | 0.998 | 0.999 | 0.999 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.138 | 0.988 | 0.995 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.123 | 0.981 | 0.992 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 |
| $\lambda_2^0 = 1$ | 0.105 | 0.968 | 0.988 | 0.995 | 0.998 | 0.999 | 1.000 | 1.000 |
| CW | 0.082 | 0.988 | 0.994 | 0.998 | 0.999 | 1.000 | 1.000 | 1.000 |

Table S22: **DGP2** Empirical Power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity; $\pi_0 = 0.50$

| | $\lambda_2^0$ | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.488 | 0.525 | 0.573 | 0.618 | 0.671 | 0.716 | 0.773 | 0.830 | 0.883 | 0.937 | 0.796 |
| | T=500 | 0.701 | 0.748 | 0.799 | 0.842 | 0.882 | 0.920 | 0.948 | 0.972 | 0.988 | 0.996 | 0.978 |
| | T=1000 | 0.895 | 0.931 | 0.954 | 0.973 | 0.985 | 0.993 | 0.997 | 0.999 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ | T=250 | 0.859 | 0.891 | 0.914 | 0.940 | 0.960 | 0.973 | 0.985 | 0.991 | 0.996 | 0.999 | 0.998 |
| | T=500 | 0.975 | 0.985 | 0.992 | 0.996 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | T=1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table S23: **DGP2** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity; $\pi_0 = 0.50$

| | $\lambda_2^0$ | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 1.00 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.501 | 0.548 | 0.600 | 0.671 | 0.732 | 0.799 | 0.876 | 0.932 | 0.951 | 0.941 | 0.907 | 0.774 |
| | T=500 | 0.710 | 0.770 | 0.825 | 0.880 | 0.925 | 0.961 | 0.985 | 0.995 | 0.998 | 0.996 | 0.988 | 0.974 |
| | T=1000 | 0.900 | 0.938 | 0.966 | 0.982 | 0.994 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | T=250 | 0.869 | 0.904 | 0.934 | 0.958 | 0.978 | 0.990 | 0.996 | 0.999 | 0.999 | 0.999 | 0.997 | 0.998 |
| | T=500 | 0.979 | 0.988 | 0.995 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | T=1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

## 5. Unit-Roots and Cointegration among predictors

**DGP3** consists of a predictive regression with predictors that form a cointegrated system. The larger model contains four predictors driven by a VECM while the restricted specification contains two predictors. Specifically, $y_{t+1} = \beta_0 + \sum_{j=1}^{4} \beta_j x_{jt} + u_{t+1}$ and the predictors are generated from $\Delta X_t = \Pi X_{t-1} + v_t$ with $\Pi = \alpha\beta'$, $v_t \sim N(0, I_4)$ and $u_t \sim N(0,1)$. We take $\alpha_{4\times1} = (0.0, 0.3, 0.0, 0.2)'$ and $\beta_{1\times4} = (1.0 \ -0.5 \ 0.0 \ 0.0)$. We note that the system of predictors has cointegrating rank equal to two and two common trends.

Our size experiments set $\beta_3 = \beta_4 = 0$ with $\{\beta_0, \beta_1, \beta_2\} = \{0.15, 0.30, -0.15\}$ (see Tables S24-S25), while for power, we fix $T = 500$ and consider a selection of departures from the null for $\beta_3$ and $\beta_4$ (see Tables S26-S27). It is again clear that both $\mathcal{S}_{T,adj}(\lambda_1^0, \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0; \lambda_2^0)$ show good size control across nearly all parameterisations. The average based statistic continues to display a certain degree of oversizeness when $\lambda_2^0$ is set at 0.90 (e.g., 15.1% for $T = 1000$). This is again intuitively explained by the fact that $(\tau_0 = 0.8; \lambda_2^0 = 0.90)$ is the configuration that

maximises local power.

The main motivation behind this DGP3 was to assess whether our earlier results on the size and power properties of the proposed tests continued to hold when the system of predictors is allowed to also contain exact unit roots. Tables S24-S25) make it clear that very little appears to change under such settings, both qualitatively and quantitatively.

Table S24: **DGP3** Empirical Size of $\mathcal{S}_T^0(\lambda_1^0=1,\lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0=1,\lambda_2^0)$ under conditional homoskedasticity

| $\lambda_2^0$ | | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | DM |
| $\mathcal{S}_T^0(\lambda_1^0=1,\lambda_2^0)$ | T=250 | 0.045 | 0.045 | 0.045 | 0.042 | 0.043 | 0.040 | 0.038 | 0.036 | 0.033 | 0.027 | 0.001 |
| | T=500 | 0.059 | 0.057 | 0.060 | 0.060 | 0.057 | 0.056 | 0.056 | 0.051 | 0.046 | 0.040 | 0.001 |
| | T=1000 | 0.063 | 0.064 | 0.063 | 0.064 | 0.066 | 0.067 | 0.061 | 0.061 | 0.057 | 0.047 | 0.001 |
| | | | | | | | | | | | | CW |
| $\mathcal{S}_{T,adj}^0(\lambda_1^0=1,\lambda_2^0)$ | T=250 | 0.097 | 0.094 | 0.097 | 0.103 | 0.108 | 0.112 | 0.117 | 0.123 | 0.141 | 0.161 | 0.061 |
| | T=500 | 0.095 | 0.096 | 0.099 | 0.100 | 0.101 | 0.108 | 0.114 | 0.113 | 0.124 | 0.143 | 0.064 |
| | T=1000 | 0.089 | 0.089 | 0.092 | 0.094 | 0.098 | 0.101 | 0.103 | 0.106 | 0.109 | 0.129 | 0.063 |

Table S25: **DGP3** Empirical Size of $\overline{\mathcal{S}}_T(\tau_0=0.8;\lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0=0.8;\lambda_2^0)$ under conditional homoskedasticity

| $\lambda_2^0$ | | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 1.00 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | DM |
| $\overline{\mathcal{S}}_T(\tau_0=0.8;\lambda_2^0)$ | T=250 | 0.043 | 0.041 | 0.042 | 0.039 | 0.039 | 0.040 | 0.030 | 0.024 | 0.019 | 0.012 | 0.012 | 0.000 |
| | T=500 | 0.058 | 0.057 | 0.058 | 0.057 | 0.050 | 0.048 | 0.043 | 0.034 | 0.022 | 0.017 | 0.027 | 0.001 |
| | T=1000 | 0.072 | 0.072 | 0.071 | 0.069 | 0.068 | 0.056 | 0.054 | 0.038 | 0.029 | 0.033 | 0.038 | 0.001 |
| | | | | | | | | | | | | | CW |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0=0.8;\lambda_2^0)$ | T=250 | 0.099 | 0.098 | 0.105 | 0.109 | 0.121 | 0.134 | 0.149 | 0.190 | 0.217 | 0.186 | 0.154 | 0.066 |
| | T=500 | 0.095 | 0.098 | 0.102 | 0.107 | 0.110 | 0.118 | 0.139 | 0.160 | 0.180 | 0.159 | 0.135 | 0.063 |
| | T=1000 | 0.099 | 0.103 | 0.101 | 0.104 | 0.106 | 0.105 | 0.114 | 0.134 | 0.151 | 0.133 | 0.113 | 0.062 |

Tables S26-S27) document empirical powers associated with DGP3. We continue to note that for both test statistics power converges towards one as we increase the magnitudes of $(\beta_3,\beta_4)$. As expected from our local power analysis, $\overline{\mathcal{S}}_{T,adj}(\tau_0=0.8;\lambda_2^0\approx 0.90)$ displays the most favorable power outcomes and for magnitudes of $(\beta_3,\beta_4)$ near zero dominates $\mathcal{S}_{T,adj}(\lambda_1^0,\lambda_2^0)$ based inferences.

Table S26: **DGP3** Empirical Power of $\mathcal{S}_T^0(\lambda_1^0 = 1, \lambda_2^0)$ and $\mathcal{S}_{T,adj}^0(\lambda_1^0 = 1, \lambda_2^0)$ under conditional homoskedasticity

| $\beta_3$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.10 |
|---|---|---|---|---|---|---|---|
| $\beta_4$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.10 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.073 | 0.189 | 0.394 | 0.625 | 0.805 | 0.913 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.074 | 0.212 | 0.449 | 0.686 | 0.851 | 0.939 | 0.999 |
| $\lambda_2^0 = 0.90$ | 0.086 | 0.248 | 0.523 | 0.760 | 0.896 | 0.963 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.094 | 0.316 | 0.624 | 0.839 | 0.944 | 0.983 | 1.000 |
| DM | 0.020 | 0.204 | 0.532 | 0.797 | 0.922 | 0.973 | 1.000 |
| $\mathcal{S}_T^0(\lambda_1^0, \lambda_2^0)_{adj}$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.187 | 0.446 | 0.710 | 0.886 | 0.961 | 0.989 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.212 | 0.499 | 0.762 | 0.916 | 0.975 | 0.994 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.248 | 0.568 | 0.821 | 0.947 | 0.987 | 0.997 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.317 | 0.666 | 0.884 | 0.975 | 0.995 | 0.999 | 1.000 |
| CW | 0.283 | 0.720 | 0.924 | 0.989 | 0.998 | 1.000 | 1.000 |

Table S27: **DGP3** Empirical Power of $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ and $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ under conditional homoskedasticity

| $\beta_3$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.10 |
|---|---|---|---|---|---|---|---|
| $\beta_4$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.10 |
| $\overline{\mathcal{S}}_T(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.078 | 0.255 | 0.516 | 0.751 | 0.894 | 0.964 | 0.999 |
| $\lambda_2^0 = 0.85$ | 0.084 | 0.334 | 0.631 | 0.850 | 0.948 | 0.984 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.093 | 0.382 | 0.706 | 0.895 | 0.966 | 0.992 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.075 | 0.350 | 0.666 | 0.864 | 0.955 | 0.988 | 1.000 |
| $\lambda_2^0 = 1$ | 0.076 | 0.290 | 0.581 | 0.808 | 0.922 | 0.974 | 1.000 |
| DM | 0.016 | 0.205 | 0.533 | 0.798 | 0.920 | 0.972 | 1.000 |
| $\overline{\mathcal{S}}_{T,adj}(\tau_0 = 0.8; \lambda_2^0)$ | | | | | | | |
| $\lambda_2^0 = 0.80$ | 0.260 | 0.573 | 0.821 | 0.948 | 0.986 | 0.997 | 1.000 |
| $\lambda_2^0 = 0.85$ | 0.339 | 0.682 | 0.897 | 0.975 | 0.995 | 0.999 | 1.000 |
| $\lambda_2^0 = 0.90$ | 0.393 | 0.745 | 0.926 | 0.985 | 0.998 | 1.000 | 1.000 |
| $\lambda_2^0 = 0.95$ | 0.358 | 0.711 | 0.909 | 0.979 | 0.997 | 0.999 | 1.000 |
| $\lambda_2^0 = 1$ | 0.297 | 0.639 | 0.871 | 0.965 | 0.992 | 0.998 | 1.000 |
| CW | 0.281 | 0.712 | 0.929 | 0.987 | 0.999 | 1.000 | 1.000 |