Piano string vibration modelling using coupled mobilities and a state-space approach

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ABSTRACT

The two transverse motions of the string in planes parallel and perpendicular to the soundboard are believed to contribute to some of the characteristic effects of piano tones. In this work we aim to model the coupling between a single string and soundboard, considering the two transverse motions. A stiff string model is coupled to the bridge of the soundboard. Modelling approaches are presented in both frequency and time domain, using mobility coupling and state-space equations, respectively. For time domain modelling, the soundboard mobilities are approximated by lumped elements, springs and dampers in this case. In the frequency domain model, the mobilities of the elongated string, detached from the bridge, corresponding to the transverse response to a transverse force, are the same in both directions and the coupling between them is neglected, while the soundboard mobilities are represented by a 2×2 matrix. The state-space model has the benefit of allowing inclusion of the non-linear hammer excitation. Although the frequency domain model can represent the linear characteristics of the coupling and give an insight into the importance of the cross-terms of the coupled mobility, it is the time domain model that is able to represent typical features of the interaction.

1. INTRODUCTION

In piano acoustics the vibration of the strings is initiated by the hammer strike and plays an important role in determining the sound radiated from the soundboard. The two ends of the string are connected to the cast-iron frame of the piano by means of connection pins. These are the agraffe, near the action, and the hitch pin, at the opposite end [1]. At a distance, defined as the speaking length, the string is pressed against the bridge of the soundboard. This connection is the main path of vibration transmission from the string to the soundboard and can also be responsible for coupling the two transverse directions of string vibration [2] . Once the hammer hits the string at the striking point, the two remain in contact for a few milliseconds. The hammer force creates pulses that travels along the string and are reflected at the ends. After the hammer loses contact with the string, this is left free to vibrate and decay. Alongside other phenomena, decay rates in piano tones contribute to forming its

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distinguishing sound and have been extensively studied in the past decades. The decay of piano tones and how this is related to the connection with the soundboard has been one of the areas of research in piano acoustics.

Martin [3] recorded decay rates for different pianos, discussing the effect of the soundboard and showing that each piano tone has different decay rates along the piano range. The phenomena was later explained by Weinrich [2], who showed that double decay is caused by the coupling between strings of the same tone at the bridge and by double polarisation. This is the presence of two transverse vibrations on the string, one perpendicular to the soundboard, and the other parallel to it. The author showed experimentally that the vibration in the vertical direction decays faster because of the better impedance match between the string and the soundboard in this direction. According to the double-polarisation explanation, after the initial sound decays the remaining and more sustained section of the envelope is caused by the vibration in the plane parallel to the soundboard.

Cheng et al. [4] have focused on modelling the specific characteristics of string's vibration decay and its fitting to linear and non-linear models, showing that non-linear models may fit better to measured data and that the dynamics of the piano touch does not influence the decay rates. The effects of non-linearities in double polarisation, causing the pairs of modes in both directions to have different natural frequencies, have been addressed by Tan et al. [5], proving that whirling motion of the string occurs when accounting for these differences. More recently, Woodhouse [6] provided a criterion for double decay in different string instruments, showing that for double decay to exist, the loss factor given by the coupling to the bridge/soundboard needs to be higher than the loss factor of the medium in which the string vibrates.

This study presents frequency and time domain models for string vibration including coupling to the soundboard. Two transverse motions of a single string are considered which are normal and parallel to the soundboard. These are coupled to each other at the bridge either by means of a mobility matrix, in the frequency domain, or by an equivalent lumped parameter model for time domain simulations. The string dynamics are represented with a stiff-string model [7] while the dynamical properties of the soundboard at the connection point are obtained by means of Finite Elements (FE).

2. STRING AND SOUNDBOARD

The string-soundboard system is shown in Figure 1 together with the conventions adopted for velocities, v, and forces, F. For the different variables defined in Figure 1, the subscript e indicates the excitation point where the hammer interacts with the string, the subscript s denotes the string at the connection point with the bridge while r is the connection point on the bridge side. The superscripts T and P indicates the two vibration directions transverse to the string: normal to the soundboard (T) and parallel to it (P).

Figure 1: String and soundboard system.

2.1 String

In pianos, it is common to have more than one string for the same note, particularly in the mid-high frequency range. In this paper only one string is simulated to emphasise and isolate the effect of the double polarisation. The string used in this study corresponds to a C4 string with parameters obtained from Chaigne and Askenfelt in [8]. These parameters are used in a stiff string model as given in [7]:

$$\mu\frac{\partial^2 y}{\partial t^2} = T_0\frac{\partial^2 y}{\partial x^2} - ESK^2\frac{\partial^4 y}{\partial x^4} \tag{1}$$

where μ is mass per unit length, T_0 is the string tension, E is the Young's modulus, S is the area of the cross-section and K is the radius of gyration. The n-th natural circular frequency of the string is calculated as:

$$\omega_n = n2\pi f_0(1 + Bn^2)^{\frac{1}{2}} \tag{2}$$

where f_0 is the fundamental frequency of the ideal string, B is the inharmonic coefficient given by:

$$B = \pi^2 ESK^2/T_0 L^2 \tag{3}$$

For a pinned string with length L the mode shapes are given as $\sin(n\pi x/L)$. Equation (1) is adopted for the two transverse motions of the string which are considered uncoupled. String damping is added at a later stage in the form of a constant damping ratio $\zeta_{s,n}$. The mode shapes matrices at the positions e and r can therefore be expressed as:

$$\boldsymbol{\Phi}_\mathbf{e} = \begin{bmatrix} \phi_{e,T,1} & 0 \\ 0 & \phi_{e,P,1} \\ \vdots & \vdots \\ \phi_{e,T,n} & 0 \\ 0 & \phi_{e,P,n} \end{bmatrix}, \boldsymbol{\Phi}_\mathbf{r} = \begin{bmatrix} \phi_{r,T,1} & 0 \\ 0 & \phi_{r,P,1} \\ \vdots & \vdots \\ \phi_{r,T,n} & 0 \\ 0 & \phi_{r,P,n} \end{bmatrix} \tag{4}$$

In a frequency domain approach, modal summation can be performed to obtain the mobility at a point j produced by forcing at point k as:

$$Y_{jk} = \sum_{n=1}^{N} i\omega\frac{\phi_{j,n}\phi_{k,n}}{-\omega^2 + \omega_n^2 + 2\zeta_{s,n}\omega\omega_n} \tag{5}$$

where $\phi_{j,n}$ is the n-th mass-normalized mode shape at response point j, in a given direction.

2.2 Soundboard

The soundboard used corresponds to a Brinsmead & Sons baby grand piano from the 19th century which has been made available to the authors for validation measurements. The FE model is developed in COMSOL Multiphysics, the material properties of the soundboard are unknown and were initially set as in [9]. The geometry of the model, the mesh and an example mode shape are shown in Figure 2.

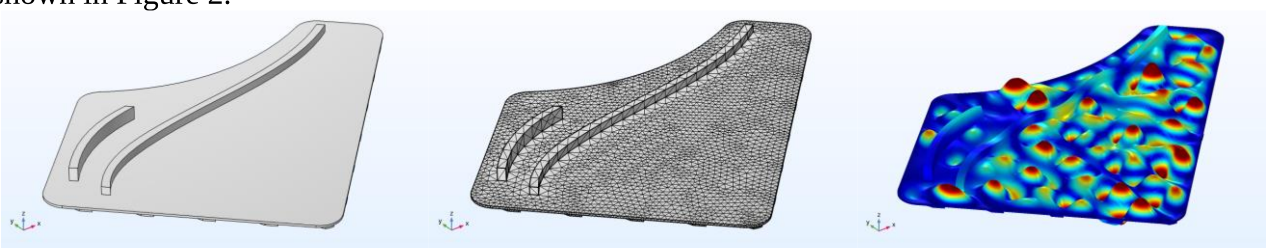

Figure 2: Soundboard FE model and mode at 2 kHz.

The eigenfrequencies are extracted from the FE software and, to obtain mobilities, modal summation is performed as in Equation (5) making use of the modal information of the soundboard. The damping ratio was set to $\zeta_{r,n} = 0.03$ for all the modes. These soundboard mobilities can be directly used in a frequency domain model or can be fitted to obtain an equivalent lumped parameter representation, as explained below, for a time domain approach.

The soundboard FE model has been verified against lab measurements in terms of mobility in the T direction at some positions along the bridge. The results are not reported here for brevity but show an acceptable level of agreement.

3. FREQUENCY DOMAIN MODEL

The frequency domain model is based on classical procedures developed to obtain the driving point mobility matrix of a coupled system [10]. Although it is known that the hammer string interaction is not well suited to be represented with a linear frequency domain approach, it is still useful to preliminary evaluate the driving point mobility at the hammer excitation point in the assumption of a linear excitation. This can give useful information about the amount of coupling in the different directions introduced by the connection with the bridge.

Considering the conventions in Figure 1, the velocities at the different points e, s and r can be expressed as:

$$\begin{aligned} \mathbf{v_e} &= \mathbf{Y_{ee}F_e} + \mathbf{Y_{es}F_s} \\ \mathbf{v_s} &= \mathbf{Y_{ss}F_s} + \mathbf{Y_{se}F_e} \\ \mathbf{v_r} &= \mathbf{Y_{rr}F_r} \end{aligned} \qquad (6)$$

The matrices and vectors in Equation (6) can be described as follows. $\mathbf{Y_{ee}}$ and $\mathbf{Y_{ss}}$ are the (2×2) direct mobility matrices of the string at the excitation and connection point. $\mathbf{Y_{es}}$ and $\mathbf{Y_{se}}$ are the transfer mobility matrices of the string between excitation and connection point. $\mathbf{Y_{rr}}$ is the direct mobility matrix of the soundboard at the position where the bridge is connected to the string. The force vector $\mathbf{F_e}$ represents the interaction force between the hammer and the string and will only have a component in the T direction, i.e. $F_e^P = 0$ by assumption. The force vectors $\mathbf{F_s}$ and $\mathbf{F_r}$ represent the force exchanged between the string and the bridge; these are equal to ensure continuity. In the first instance the mobility matrices of the string can be assumed to be diagonal. The direct mobility matrix for the soundboard is on the contrary full due to the more complex nature of this component.

The equations in (6) can be combined to obtain the mobility matrix of the coupled system as seen from the excitation point:

$$\mathbf{Y_c} = \mathbf{Y_{ee}} - \mathbf{Y_{se}^T}[\mathbf{Y_{rr}} + \mathbf{Y_{ss}}]^{-1}\mathbf{Y_{se}} \qquad (7)$$

Due to $\mathbf{Y_{rr}}$, the combined matrix $\mathbf{Y_c}$ is full and the system will exhibit coupling between the two transverse directions.

4. TIME DOMAIN MODEL

A state-space modal model is developed to obtain a time domain solution of the string-soundboard system. The connection with the soundboard is simplified and represented by a pair of spring-damper elements set at an angle with respect to the horizontal axis, as shown in Figure 3. The forces exerted by these elements are given by:

$$\begin{bmatrix} F_T \\ F_P \end{bmatrix} = \mathbf{K_r d} + \mathbf{C_r v} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} d_T \\ d_P \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} v_T \\ v_P \end{bmatrix} \qquad (8)$$

The different elements of the stiffness matrix can be expressed as [11]:

$$\begin{aligned} k_{11} &= k_1 \cos^2 \alpha_1 + k_2 \cos^2 \alpha_2 \\ k_{12} &= k_1 \sin \alpha_1 \cos \alpha_1 + k_2 \sin \alpha_2 \cos \alpha_2 \\ k_{22} &= k_1 \sin^2 \alpha_1 + k_2 \sin^2 \alpha_2 \end{aligned} \qquad (9)$$

with the damping matrix taking an equivalent form.

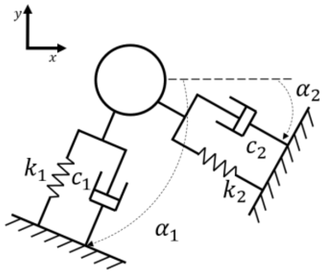

Figure 3: Representation of the soundboard by spring-damper elements. The values of k_1, k_2, c_1, c_2 are tuned such that the pairs of springs and dampers approximate locally the behaviour of soundboard at the connection point.

The state space equations of the system are:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned} \tag{10}$$

The matrix **A** includes the effect of the spring-damper systems as:

$$\mathbf{A} = \begin{bmatrix} -\mathbf{C_s} - \boldsymbol{\phi}_\mathbf{r}\mathbf{C_r}\boldsymbol{\phi}_\mathbf{r}^\mathbf{T} & -\mathbf{K_s} - \boldsymbol{\phi}_\mathbf{r}\mathbf{K_r}\boldsymbol{\phi}_\mathbf{r}^\mathbf{T} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{11}$$

with $\mathbf{C_s}$ and $\mathbf{K_s}$ being the $2\text{n} \times 2\text{n}$ diagonal matrices containing the modal damping and stiffness of the string. The state space vector **x** includes the modal velocities and displacements. The hammer force is included in **Bu** as:

$$\mathbf{Bu} = \begin{bmatrix} \boldsymbol{\phi}_\mathbf{e} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} F_{h,T} \\ 0 \end{bmatrix} \tag{12}$$

where $F_{h,T}$ is the hammer force.

The matrix **C** allows transformation from modal to physical velocities and displacements at the excitation or at the connection point as:

$$\mathbf{C} = \begin{bmatrix} \boldsymbol{\phi}_\mathbf{r}^\mathbf{T} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi}_\mathbf{r}^\mathbf{T} \end{bmatrix} \tag{13}$$

The force applied by the hammer to the string is represented using a power law [12-15] as:

$$\begin{aligned} F_{h,T} &= K_h \left| y_h - x_{e,T} \right|^\gamma \\ m_h \ddot{y}_{h,T} + F_{h,T} &= 0 \end{aligned} \tag{14}$$

where the parameters K_h, γ and m_h correspond to stiffness, nonlinear coefficient and hammer mass obtained in [8]. In this study, the ones from a hammer striking a C4 string are used with $K_h = 4.5 \text{ x } 10^9$ N/m, $\gamma = 2.5$ and $m_h = 2.97$ g. The term $x_{e,T}$ corresponds to the displacement of the string at the excitation point in the T direction.

5. RESULTS

5.1 Frequency domain

The mobilities of the full system at the driving point, calculated according to Equation (5), are presented in Figure 4. Due to the connection with the soundboard the fundamental frequency corresponds to the speaking length of the string (note C4, 262 Hz). The small effect visible at 1098 Hz is the so-called duplex scaling which is related to the vibration of the short segment of string between the bridge and the end of the string. The magnitude of the cross terms at the resonance are comparable to the direct ones, suggesting the importance of coupling.

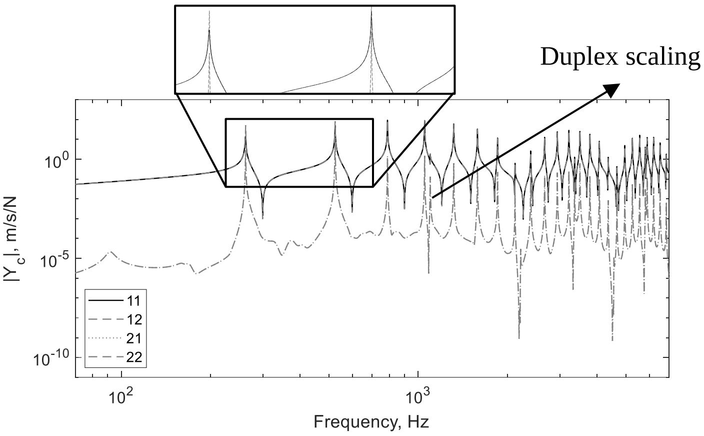

Figure 4: Coupled driving point mobility.

5.2 Equivalent lumped parameter model of the soundboard

The driving point mobilities of the FE model are obtained at the location of C4 string. These have been approximated by fitting a pair of spring-damper elements. The results are presented in Figure 5, and the parameters used are listed in Table 1 following the representation given in Figure 3. Both TT and TP magnitudes tend to behave like a damper, while PP is similar to a stiffness, only behaving as a damper at higher frequencies. The trend of the phase may suggest a more complicated behaviour, but this is not explored further in this paper. At lower frequencies the three mobilities show a clear stiffness-like behaviour. According to this simplified approach, soundboard resonances are not accounted for.

Table 1: Parameters used for fitting of spring and damper systems

k_1 (MN/m)	k_2 (MN/m)	c_1 (Ns/m)	c_2 (Ns/m)	$\alpha_{1,k}$ (rad)	$\alpha_{2,k}$ (rad)	$\alpha_{1,c}$ (rad)	$\alpha_{2,c}$ (rad)
9	1.5	900	1000	$-\pi/2$	$-\pi/4$	$3\pi/4$	$\pi/4$

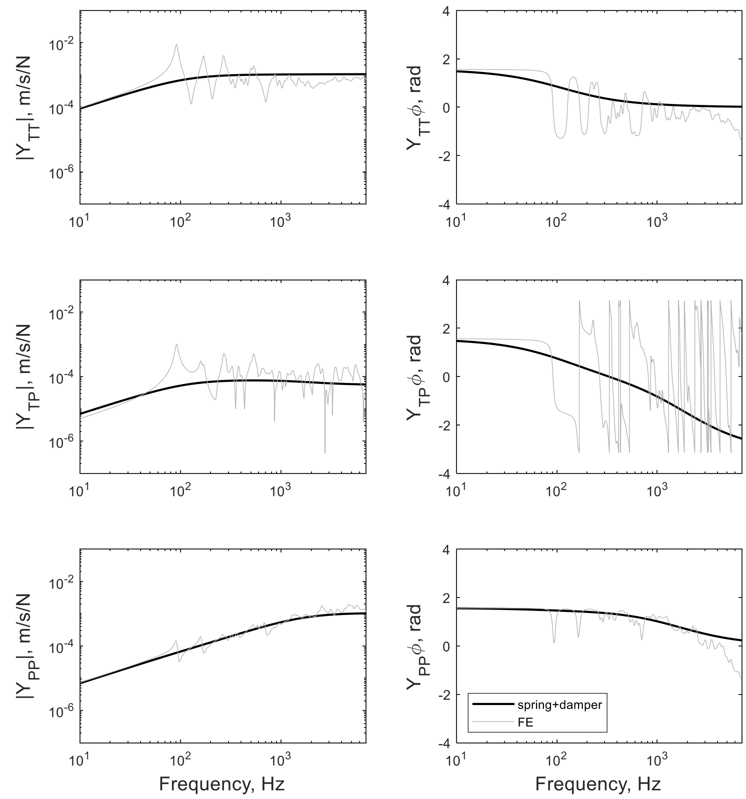

Figure 5: Fitted mobilities. Left: Magnitudes. Right: Phases.

5.3 Time domain results

In the time domain simulations, 40 modes of the string, in both transverse directions, are considered. The hammer excitation corresponds to a C4 hammer and the impact velocity is selected to be 2.5 m/s, similar to the 2 m/s of a *mezzo forte* piano touch according to [16].

The contact force between hammer and string is presented in Figure 6. The different peaks in the profile correspond to reflections in the string from the agraffe. The contact duration is 2.23 ms, which is in line with other literature results, e.g. [17].

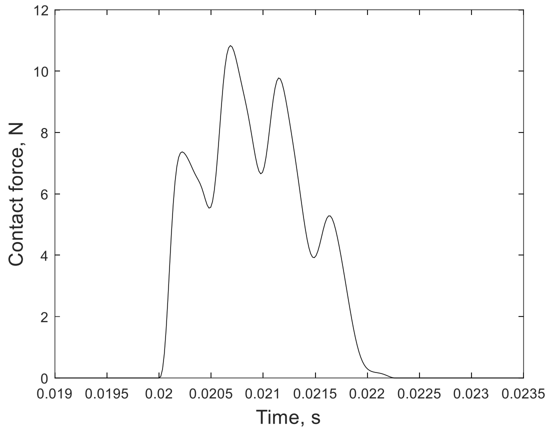

Figure 6: Contact force of hammer-string interaction.

The displacements at the excitation and connection point, in both transverse directions, and the FFT of the transverse displacement at the connection point are presented in Figure 7. The responses at the connection point are three orders of magnitude lower than at the excitation point. The response in the P direction is one order of magnitude lower than in the T direction. The FFT shows the inharmonic partials of the C4 string. The duplex scaling is visible at 1098 Hz together with higher order partials.

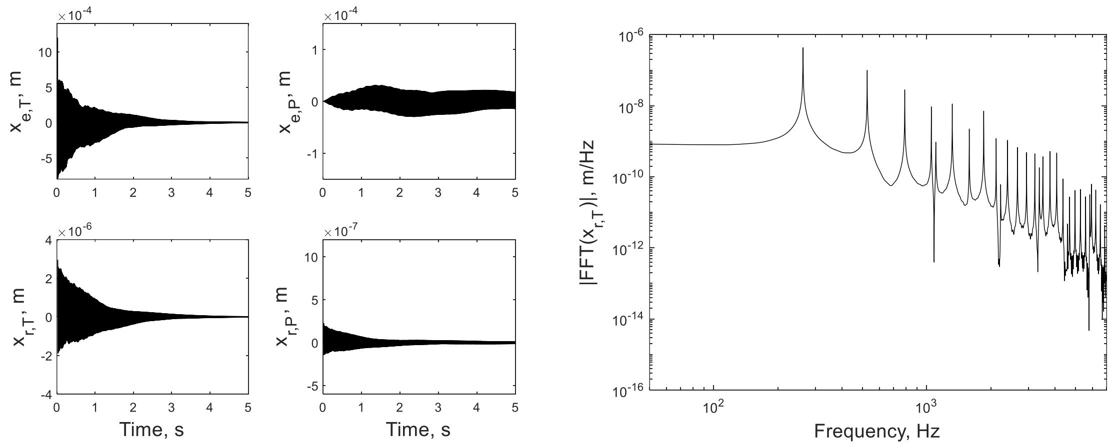

Figure 7: Left: Response at excitation and connection point, in T and P directions. Right: FFT of displacement at connection point in T direction.

The force transmitted to the soundboard is shown in Figure 8. While the transverse component normal to the soundboard F_T is initially dominant and decays through time, the component parallel to the soundboard F_P initially increases and becomes dominant after 3 seconds. These results correspond broadly with what was shown by Weinrich [2]; the final part of the sound envelope will be caused by the transverse motion parallel to the soundboard after the decay of the normal motion has taken place. The trend of the transmitted force can differ from this if multiple strings are considered.

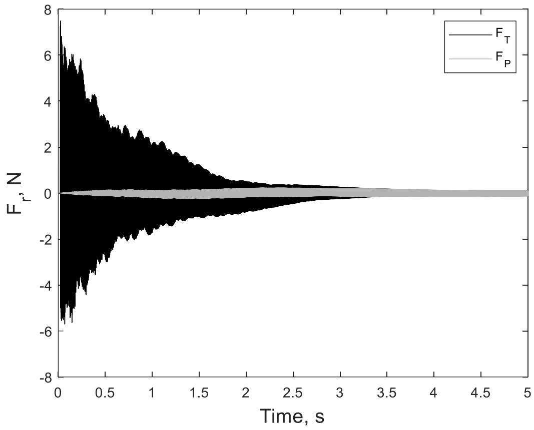

Figure 8: Forces at the connection point.

6. CONCLUSIONS

A frequency domain model considering the two transverse motions of a piano string connected to a soundboard has been developed. While this linear approach cannot be used to simulate sound generation in pianos it shows that the connection with the bridge has an important contribution in defining the double polarisation in the string vibration. A state space time domain model considering a simplified soundboard representation in the two transverse directions can represent some of the decay characteristics of the string. It shows that the force transmitted to the soundboard in the direction parallel to the soundboard is initially negligible in comparison to the first but becomes predominant after a few seconds. The approach used in this study is sufficient to represent the characteristics of piano strings and their coupling with the soundboard but cannot consider the effects of non-linearities produced by changes in string tension, resulting for example in phantom partials [18]. On the other hand, it can be extended to account for multiple strings in the same note and to assess the coupling between the transverse and longitudinal directions due to the connection with the bridge.

7. ACKNOWLEDGEMENTS

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