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A constrained topology optimization methodology based on budget constrained min-cut

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Abstract

Purpose – The purpose of this paper is to propose a novel methodology based on budget constrained Min-Cut theorem to solve constrained topology optimization (TO).

Design/methodology/approach – This paper establishes a weighted network with budget, which is derived from the sensitivity with respect to the constraint function. The total budget carried by the topology evaluates the extent to which the constraint is satisfied. By finding the Min-Cut under budget constraint in each step, the proposed method is able to solve constrained TO problem.

Findings – The results obtained from a magnetic actuator including a yoke, a coil and an armature have demonstrated that the proposed method is effective to solve constrained TO problem.

Originality/value – A novel methodology based on budget constrained Min-Cut is proposed to solve constrained TO problem.

Keywords Topology optimization, Checkerboard, Min-Cut, Budget constraint optimal design, Design optimization methodology

Paper type Research paper

1. Introduction

Topology optimization (TO) is to find the optimal material distribution of a device under some performance criteria in the initial stage of product manufacturing process. Thanks to the rapid advancement and maturation of TO theory, TO approach is currently being applied to different industrial problems. In practical engineering, TO problem often includes variant constraints. Basically, there are two types of constraint in TO. In the first case, for example, constraint imposed on the maximum stress of the element in the design domain (Holmberg *et al.*, 2013), the number of the constraint is equal to the number of the design variable. The second type of constraint, called global constraint, evaluates the overall performance of the model. This paper focuses on the second type, i.e. the global constraint.

One fundamental constraint handling technique is to use augmented Lagrange or penalty methods (Pereira *et al.*, 2004; Senhora *et al.*, 2020) in which the constraints are added to the objective function multiplied by a penalty function/constant. Following this strategy, one solves a sequence of unconstrained problems, which only requires the solution to one additional adjoint problem. Several issues exist in this type of method. Firstly, the optimized result is sensitive to the step size. Improper tuning of step size can result in unsuccessful convergence and local minima trap. Secondly, checkerboard pattern (Diaz and Sigmund, 1995) is one common by-product in the outcome of the TO, which could cause

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manufacturing difficulty. To hinder checkerboard pattern, the sensitivity value, the derivative with respect to the objective function, is usually filtered so that the topology could be smoothened. When dealing with constraints, the primary objective function is usually combined with augmented Lagrange or penalty function. This modification of the objective function will, to some degree, weaken the suppression of checkerboard pattern.

To overcome the aforementioned issues, a budget constrained min-cut theorem (BCMC) is used to solve constrained TO. A Min-Cut theory based TO method has proved its efficiency to simultaneously enhance the performance parameter and constrain checkerboard pattern (Xia *et al.*, 2021). However, this method deals with constraints in a primitive way. It does not involve the constraints in the process of optimization, while terminates when the constraint is violated and selects the best topology satisfying the constraints as the final solution. For the sake of brevity, this mechanism to tackle with constraint will be referred to as method for nonconstraint TO. Based on previous work, this paper establishes a weighted network with budget, which is derived from the sensitivity with respect to the constraint function. The total budget carried by the topology evaluates the extent to which the constraint is satisfied. By finding the Min-Cut under budget constraint in each step, the proposed method is able to solve constrained TO problem efficiently and effectively without checkerboard pattern.

2. Finding optimal direction using Min-Cut in nonconstraint topology optimization

2.1 An s-t cut and material assignment in topology optimization

For a weighted network, $G(V, E)$, with two distinguished vertices "s" and "t" called the terminals, an *s-t* cut is a set of edges whose removal will disconnect the graph into two disjoint parts, and leaves the terminals "s" and "t" to be in different parts in the partitioned graph $G(C) = (V, E-C)$ (Xia *et al.*, 2021). (Figure 1)

To extend Min-Cut theorem to TO, the mesh-grid in the finite element analysis is firstly transformed to a network. As shown in Figure 2, for a two-dimensional problem, there are two types of elements in the surroundings of the element of interest in the mesh grid. The elements connecting with element p by at least two nodes are called eight-neighborhood elements, and the elements connecting with p by only one node are eight-neighborhood elements. Accordingly, in the transformed network, the element p is expected to connect

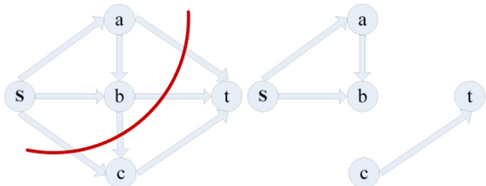


Figure 1.
An example of an s-t cut

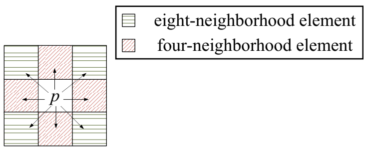


Figure 2.
Eight- and four-neighborhood element of the element p (a) and (b)

with both its eight-neighborhood elements and four-neighborhood elements (rule I), or solely four-neighborhood elements (rule II). Figure 3(b) and 3(c) shows the transformation of a mesh grid to a network based on rule I and rule II, respectively. Compared to rule I, the adoption of rule II can suppress the checkerboard patterned in the optimized topology to a larger extent. Then, for a TO problem with material α and β to be reassigned, two terminals α and β are added in the primary network [as shown in Figure 4(a)]. An *s-t* cut in TO problem is a set of edges whose removal will disconnect the network, and each element is solely connected with either α or β in the partitioned network.
Furthermore, the correspondence between the cut C and the assignment of materials for each element p is:

$$f_p{}^C = \begin{cases} \alpha \text{ if } t_p^{\alpha} \in C \\ \beta \text{ if } t_p^{\beta} \in C \end{cases} \tag{1}$$

where $f_p{}^C$ is the new material attribute of element p determined by cut C.

According to the definition of an *s-t* cut, either of the two edges, t_p^{α} or t_p^{β}, has to be included in the cut C. In other words, each element will be assigned to one material once a cut C has been established. An arbitrary material distribution could be uniquely represented by a cut. Figure 3 demonstrates a cut and its corresponding material assignment.

2.2 The best optimal direction

The key to a TO problem is to find the best way to relocate the material distribution in each iteration. Accuracy and piecewise smoothness are the two critical indicators to evaluate the quality of the optimized results. The accuracy indicates whether the elements carrying the same level sensitivity value are assigned to the same material, while the piecewise smoothness is an evaluation of the checkerboard pattern. In this paper, the two indicators are measured simultaneously by using the following energy function containing both a smooth term $S(\bullet)$ and a data term $D(\bullet)$ (Boykov *et al.*, 2001):

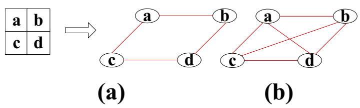
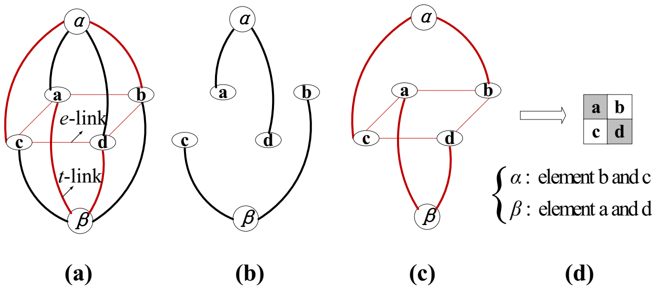


Figure 3.
One way to transform a mesh-grid (a) to a network (b)



Notes: (a) transformed network; (b) partitioned network; (c) cut set; (d) updated material distribution

Figure 4.
A cut in the network for TO

$$\min : E(f^C) = \sum_{\{p,q\}\in N} S_{p,q}(f_p, f_q) + \sum_{p\in P} D_p(f_p)$$
$$= K \times \sum_{\{p,q\}\in N} Connection_{(p,q)} + (-\sum_{p\in P} SE_{f_p}(p))) \quad (2)$$

where P is the set of elements, N is the set of the interacting pairs of elements, K is the weight, f_p is the updated material state determined by cut C, and *Connection* is the degree of connections given by:

$$Connection_{(p,q)} = \begin{cases} 1 \; f_p \neq f_q \\ 0 \; f_p \neq f_q \end{cases} \quad (3)$$

Obviously, the minimization of the energy function can hinder the checkerboard pattern.

Suppose α and β are two material to be relocated (α has higher material property), then $SE_{fp}(p)$ is the calibrated sensitivity of the original set:

$$SE^p_\alpha = \begin{cases} |se^p|, se < 0 \\ -|se^p|, se > 0 \end{cases}, SE^p_\beta = \begin{cases} -|se^p|, se < 0 \\ |se^p|, se < 0 \end{cases} \quad (4)$$

where se is the original sensitivity with respect to the objective function [usually calculated by an adjoint variable method (Hong *et al.*, 2015)], while SE_α is the likelihood that the element is assigned to material α. For an element of material property α, a negative sensitivity se suggests an increase of the permeability of the element, and thus has a positive SE_α and a negative SE_β, and vice versa. As the energy function is minimized, the data term D (•) is defined in such way that the elements with high tendency of changing to α or β is accordingly updated to α or β as much as possible.

Therefore, the minimization of this energy function will redistribute the material distribution accurately and simultaneously reduce the variance of the material assignment between two interacting elements with a high connectivity degree. Moreover, the weight K has an impact on the optimized results. Obviously, the checkerboard pattern is barely constrained if K is small. Nevertheless, a wide range of the values of K has been proved efficient in paper (Xia *et al.*, 2021).

2.3 Min-Cut and energy minimization

The edge weights on the transformed network for TO are defined in Table 1. It has been proved that the cost of a cut $|C|$ in the weighted network, the sum of the edge weights of it, equals to the energy function in (2) (Boykov *et al.*, 2001). Furthermore, it has been explained previously that a cut C in the transformed network for the TO corresponds to a change in the existing topology. Thus, finding the best way of the material variation based on the

Table 1. Edge weights of t-Links and e-Links for the network in TO

Edge	Weight	For
t^α_p	$SE_a(p)$	$p \in P$
t^β_p	$SE_\beta(p)$	$p \in P$
$e\{p, q\}$	$K \times Connection(p, q)$	$\{p, q\} \in N$

present topology, evaluated by the energy function in (4), can be achieved by finding the Min-Cut in the transformed network. A number of polynomial-time algorithms have been proposed to solve the Min-Cut problem. Therefore, finding the minimum cut in the transformed network will not bear any additional computational cost.

3. Budget constrained Min-Cut and constrained topology optimization

In the network for unconstrained TO, the weights on t-links are derived from sensitivity value of objective function. Thus, finding the Min-Cut in each iteration ensures the enhancement of the objective function. To further satisfy the constraints in constrained TO, the t-links are additionally assigned with budget value, which is derived from the sensitivity with respect to constraint function. Considering inequality constraint $g(x) \leq 0$, the budgets on the links t_p^α and t_p^β, respectively, are calculated from:

$$BE_\alpha^p = \begin{cases} |be^p|, be < 0 \\ -|be^p|, be > 0 \end{cases}, BE_\beta^p = \begin{cases} -|be^p|, be < 0 \\ |be^p|, be > 0 \end{cases} \tag{5}$$

where be is the sensitivity with respect to constraint function. Similar to equation (4), BE_α and BE_β, respectively, evaluate the reasonability that element p should change to α and β.

When the current solution dissatisfies the constraint $g(x) \leq 0$, the elements carrying negative be should be changed to α, and the elements carrying positive be should be changed to β as much as possible to decrease $g(x)$. That is to say, the cut set C should consist of t_p^α links carrying positive BE_α^p, and t_p^β links carrying positive BE_β^p as much as possible. Define *Satisfylevel* to evaluate the extent to which the updated material distribution determined by cut C contributes to the satisfaction of constraint:

$$Satisfylevel = -\left(\sum_{t_p^\alpha \in C} BE_\alpha^p + \sum_{t_p^\beta \in C} BE_\alpha^p\right) \tag{6}$$

The smaller *Satisfylevel* is, the higher the contribution is. Obviously, minimization of *Satisfylevel* contradicts the minimization of energy function in (2). When all the t_p^α links carrying positive BE_α^p belong to cut C, *Satisfylevel* is minimized, and is defined as *Satisfylevel_Min*. To balance the minimization of (2) and minimization of (6), define the following equation:

$$-\left(\sum_{t_p^\alpha \in C} CE_\alpha^p + \sum_{t_p^\beta \in C} CE_\alpha^p\right) < relax{*}Satisfylevel\_Min \tag{7}$$

where *relax* is a tuning parameter between [0,1]. Generally, the value of $g(x)$ will increase gradually in the process of optimization. The closer the $g(x)$ is getting to zero, the higher tendency that the constraint is violated, then the larger *relax* should be to ensure the satisfaction of the constraint. The influence of *relax* on the optimization results and the tuning mechanism of it will be elaborated in section 4.

Therefore, by finding the Min-Cut under the budget constraint (7) in each step, the proposed method will find the optimized topology satisfying inequality constraint $g(x) \leq 0$ finally.

The aforementioned network could be interpreted as a weighted network with dependent fixed budget fee. Finding the Min-Cut under budget constraint has been proved a strongly

NP-hard problem (Holzhauser *et al.*, 2016). Contrary to Min-Cut problem, for which a number of polynomial-time algorithms have been proposed, BCMC problem can only be solved by approximation algorithm. However, most of the approximation algorithms are not applicable to engineering application because of large approximation ratio. Pablo A. Maya Duque proposes a strongly polynomial-time algorithm that uses Megiddo's parametric search technique, while the technique only works for simple network, such as series parallel network (Maya Duque *et al.*, 2013).

To solve BCMC problem in each iteration effectively and efficiently, this paper uses particle swarm optimization (PSO) method (Wang *et al.*, 2018). The chromosome can be represented as a binary string of n bits, $X = x_1 x_2 \cdots x_n$, where n is the total number of edges in the weighted network. x_i is defined as

$$x_i = \begin{cases} 1\,, & \text{edge } i \text{ belongs to the cut set} \\ 0\,, & \text{edge } i \text{ does not belong to the cut set} \end{cases} \tag{8}$$

The efficiency of employing PSO to solve BCMC problem will be demonstrated in Section 4.

The pseudo code of the algorithm for constrained TO based on BCMC is given below.

The pseudo code of the algorithm

Algorithm BCMC

```
Input: relax, k, stopCriterion
Output: Material distribution f* satisfying constraint
1. While (! stopCriterion)
    2. Analyze sensitivity of objective function se
    3. Analyze sensitivity of constraint function be
    4. Construct weighted network G with budget
    5. Calculate Satisfylevel_Min, and relax
    6. Find Min-Cut in G under budget constraint (7), and update the
topology to f*
7. End
```

4. Numerical results

To testify the proposed methodology, a magnetic actuator including a yoke, a coil and an armature (Xia *et al.*, 2020) is topologically optimized to maximize the magnetic force in a specific direction. In the finite element analysis, the design domain is discretized into 30×9 quadrilateral elements, and the input current of 1 A is applied to the coils with 400 turns. Meanwhile, to maintain the structural stability, the percentage of air volume *Volume_air* in the design domain is constrained to be smaller than a user-defined value *V_set*. The constraint function is thus: (Figure 5)

$$g(x) = Volume\_air - V\_set < 0 \tag{9}$$

Firstly, to explore the impact of parameter *relax* on the satisfaction of constraint, *relax* ranging from 0.2 to 1 are set to find the budget constrained Min-Cut in the weighted network established based on the initial topology. *V_set* is 0.2. Figure 6 shows the corresponding material distribution of the optimized constrained Min-Cut after the first iteration under different *relax*. Clearly, the higher the *relax* is, the better the updated material distribution satisfies the constraint. In this paper, the adjustment of *relax* is ruled by

$$relax = \begin{cases} \tan^{-1}(g(x) + Advance)\hat{0}.2 & g(x) > -Advance \\ 0.2 & g(x) < -Advance \end{cases} \quad (10)$$

where *Advance* is a predefined positive value to alert the violation of constraint. In this numerical example, *Advance* is set as half of the *V_set*.

Next, the proposed method is used to maximize the magnetic force under three constraint: *V_set* = 0.17, 0.2 and 0.23. The three constraints with different *V_set* are all well-satisfied. Figure 7(a) and 7(b) records the iterative process of $g(x)$ and magnetic force obtained from BCMC under three values of *V_set*, respectively. Table 2 shows the comparison of $g(x)$ and magnetic force obtained from BCMC and the method for nonconstrained TO (old method), respectively. It is clear that BCMC has better ability to enhance the objective function and satisfy the constraint properly. On the contrary, when using the nonconstrained method, the process of optimization indeed terminates inside of the feasible region, but too distant to the boundary, and results in insufficient search of design domain, lower force, in other words. Figure 8 illustrates the corresponding optimized topology, where no obvious checkerboard pattern is presented.

5. Conclusion

This paper proposes a novel methodology BCMC for constrained TO. Based on previously proposed method for nonconstrained TO based on Min-Cut theorem, this paper establishes a weighted network with budget, which is derived from the sensitivity with respect to the constraint function. The total budget carried by the topology evaluates the extent to which the constraint is satisfied. By finding the Min-Cut under budget constraint in each step, the proposed method is able to solve constrained TO problem efficiently and effectively without checkerboard pattern.

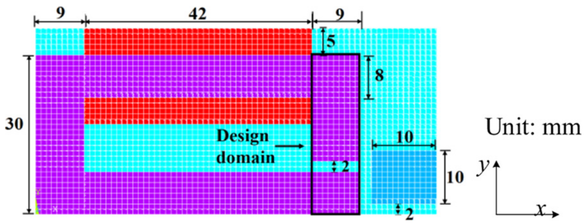


Figure 5. Schematic diagram of the actuator

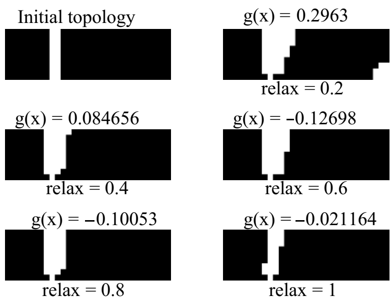
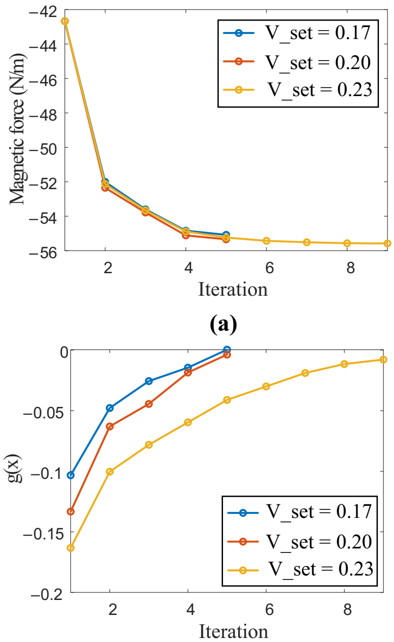


Figure 6. One-step material distribution updation using different values of relax



(b)

Figure 7.
Iterative process of (a) magnetic force and (b) g(x)

Table 2.
Comparison of results obtained from the proposed method and the old method

		V_set = 0.17	*V_set* = 0.20	*V_set* = 0.23
g(x)	Proposed	0.00	−0.004	−0.01
	Old	0.152	0.018	0.029
Force(N/m)	Proposed	55.08	55.34	55.58
	Old	54.92	55.20	55.39

(a) (b) (c)

Notes: (a) *V*_set = 0.17; (b) *V*_set = 0.20; (c) *V*_set = 0.23

Figure 8.
Optimized topology under three constraint functions

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