Modular flavour symmetry and orbifolds

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Abstract

We develop a bottom-up approach to flavour models which combine modular symmetry with orbifold constructions. We first consider a 6d orbifold $\mathbb{T}^2/\mathbb{Z}_N$, with a single torus defined by one complex coordinate z and a single modulus field τ, playing the role of a flavon transforming under a finite modular symmetry. We then consider 10d orbifolds with three factorizable tori, each defined by one complex coordinate z_i and involving the three moduli fields τ_1, τ_2, τ_3 transforming under three finite modular groups. Assuming supersymmetry, consistent with the holomorphicity requirement, we consider all 10d orbifolds of the form $(\mathbb{T}^2)^3/(\mathbb{Z}_N \times \mathbb{Z}_M)$, and list those which have fixed values of the moduli fields (up to an integer). The key advantage of such 10d orbifold models over 4d models is that the values of the moduli are not completely free but are constrained by geometry and symmetry. To illustrate the approach we discuss a 10d modular seesaw model with S_4^3 modular symmetry based on $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ where $\tau_1 = i,\ \tau_2 = i + 2$ are constrained by the orbifold, while $\tau_3 = \omega$ is determined by imposing a further remnant S_4 flavour symmetry, leading to a highly predictive example in the class CSD(n) with $n = 1 - \sqrt{6}$.

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1 Introduction

The Standard Model (SM), despite its many successes, does not account for the origin of neutrino mass nor the quark and lepton family replication, and gives no insight into the fermion masses and mixing parameters. One approach is to introduce a family symmetry which may be a finite discrete or continuous, gauged or global, Abelian or non-Abelian. Large lepton mixing has motivated studies of non-Abelian finite discrete groups such as A_4, S_4, A_5 (for reviews see e.g. [1,2]). However such family symmetries must eventually be spontaneously broken by new Higgs fields called flavons, and it turns out that the vacuum alignment of such flavon fields plays a crucial role determining the physical predictions of such models.

Another interesting class of symmetries arise from the modular group $SL(2,\mathbb{Z})$, which is the group of 2×2 matrices with integer elements, the kind you first learn about in high school with positive or negative elements, but with unit determinant. Geometrically, such a group is the symmetry of a torus, which essentially has a flat geometry in two dimensions (when it is cut open) and the symmetry corresponds to the discrete coordinate transformations which leave the torus invariant, in other words the different choices of two dimensional lattice vectors describing the same torus. The two dimensional space may be conveniently associated with the real and imaginary directions of the complex plane, with the lattice vectors becoming complex vectors in the Argand plane. The modular symmetry in the upper half of the complex plane, $PSL(2,\mathbb{Z})$, has particularly nice features which rely on holomorphicity, the lack of complex conjugation symmetry, reminiscent of supersymmetry.

At first sight, modular symmetry does not look like a promising starting point for a family symmetry, for one thing it is an infinite group, since there are an infinite number of 2×2 matrices with integer elements and unit determinant. Secondly, it is not immediately obvious what a torus has got to do with particle physics. With the advent of superstring theory and extra dimensions, this second question may at least find an answer, since orbifold compactifications of two extra dimensions are often done on a torus [3,4], and in superstring theory, the single lattice vector which describes the torus (in the convention that the other lattice vector has unit length and lies along the real axis) is promoted to the status of a field, called the modulus field τ, where its vacuum expectation value (VEV) fixes the geometry of the torus. Moreover, it is possible to obtain a finite discrete group from the infinite modular group as discussed below.

The infinite modular group has a series of infinite normal subgroups called the principle congruence subgroups $\Gamma(N)$ of level N, whose elements are equal to the 2×2 unit matrix mod N (where typically N is an integer called the level of the group). For a given choice of level $N > 2$, the quotient group $\Gamma_N = PSL(2,\mathbb{Z})/\Gamma(N)$ is finite and may be identified with the groups $\Gamma_N = A_4, S_4, A_5$ for levels $N = 3, 4, 5$, which may be subsequently be used as a family symmetry [5]. Indeed the only flavon present in such theories is the single modulus field τ, whose VEV fixes the value of Yukawa couplings which form representations of Γ_N and are modular forms, leading to very predictive theories independent of flavons [5].

Following the above observations [5], there has been considerable activity in applying modular symmetry to flavour models, and also in extending the framework to more general settings, following the bottom-up approach (see [6] for more details and extensive references). For example the modular S_4 group was studied in [7–9]. To enhance the predictivity of such models, rather than considering the VEV of τ to be a free complex parameter, it is interesting to consider fixed points or stabilizers which are special values for the modulus field τ such as $\tau = i, \omega, i\infty$ where part of the modular transformations are preserved. However such an approach with one modulus ‡ is rather too restrictive and generally calls for additional moduli fields which can be introduced in a straightforward way by considering additional modular groups, with one modulus per modular group, as suggested in [11–15]. A recent example of a model of this kind was based on three finite modular groups S_4^3 broken to its diagonal subgroup S_4, with three moduli fields in the low energy theory located at three different fixed points, for example $\tau_1 = i,\ \tau_2 = i{+}2,\ \tau_3 = \omega$, leading to a very predictive and successful phenomenological description of the neutrino and charged lepton masses and lepton mixing based on a version of the littlest seesaw [16].

While there has been considerable effort devoted to studying modular symmetry arising from orbifolds in top-down heterotic string constructions [17], § there has been little work on bottom-up approaches which combine orbifolds together with modular symmetry. In the bottom-up approach to modular symmetry as applied to flavour models, orbifolds are usually not considered at all. Instead the formalism of modular symmetry and modular forms is adopted and flavour models then constructed, without any reference to the underlying orbifold [5]. However there have been some bottom-up attempts to relate modular symmetry to orbifold GUTs, such as the model based on supersymmetric $SU(5)$ in 6d, where the two extra dimensions are compactified on a $\mathbb{T}^2/\mathbb{Z}_2$, leading to a remnant A_4 with single modulus field located at the fixed point $\tau = \omega$ of the orbifold [23]. In this model, there was also an A_4 flavour symmetry commuting with the A_4 modular symmetry, which was a pre-curser to the eclectic flavour symmetry approach [23].

In this paper we develop a bottom-up approach to flavour models which combines modular symmetry with orbifold constructions. We shall consider orbifolds in 10d which can provide three modular groups and three moduli fields in the low energy theory (below the compactification scales). We assume that the 6 extra dimensions are factorisable into 3 tori, each defined by one complex coordinate z_i. Assuming supersymmetry, consistent with the holomorphicity requirement, we consider all the orbifolds of the form $(\mathbb{T}^2)^3/(\mathbb{Z}_N \times$

‡Recently it has been claimed that a single modulus at $\tau = i$ can provide a good phenomenological description of leptons, but this requires that the neutrino mass matrix is infinite at the fixed point [10].

§Top-down approaches suggest that the finite modular symmetry will typically be accompanied by a flavour symmetry leading to so called eclectic symmetry [18–22].

$\mathbb{Z}_M$), and list all the available orbifolds, which have fixed values of the moduli fields (up to an integer). The key advantage of such 10d orbifold models over 4d models is that the values of the moduli are not completely free but are constrained geometry and symmetry.

To illustrate the approach, we focus on the orbifold example $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$, and discuss in detail the fixed points, with the choices $\tau_1 = i,\ \tau_2 = i+2$ being constrained by the orbifold, while τ_3 is unconstrained but may be fixed by specifying a remnant symmetry. Motivated by model building considerations we consider $\tau_3 = \omega$ determined by imposing a remnant S_4 flavour symmetry. We assume an S_4^3 modular symmetry, associated with each of the three moduli. We show that such a model can reproduce a minimal 4d modular seesaw model of leptons based on three finite modular groups S_4^3 broken to the diagonal modular subgroup S_4. In the 4d models the three moduli fields were simply assumed to lie at the fixed points, $\tau_1 = i,\ \tau_2 = i+2,\ \tau_3 = \omega$ [16], but in the 10d model, these values are constrained by geometry and symmetry. The resulting model is in the class CSD(n) with $n = 1 - \sqrt{6}$, where the atmospheric angle θ_{23} is restricted to the second octant.

The bottom-up approach to modular symmetry from orbifolds followed here can readily be extended to Grand Unified Theories (GUTs), with up to three moduli groups and moduli fields, including a remnant flavour symmetry, leading to a bottom-up version of the ecletic flavour symmetry in orbifold GUTs as anticipated in [23].

The layout of the remainder of the paper is as follows: In Sec. 2, the general SUSY preserving orbifolding is presented and shown how it fixes the modulus. This is shown for the case of 6 and 10 spacetime dimensions, as well as an specific detailed example. In Sec. 3 we describe the basics of the modular symmetry S_4, its corresponding modular forms at the fixed points as well as how it can arise as a remnant symmetry in orbifolding. In Sec. 4 we present a viable and predictive lepton model which uses S_4^3 modular symmetry in a $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ orbifold. Finally in Sec. 5 we present our conclusions.

2 Orbifolding

Modular symmetries have proved themselves very useful in model building. They may provide predictive flavor structure specially for the lepton sector without requiring the addition of extra fields nor complicated symmetry breaking mechanisms. A model with modular symmetry requires to be built in 6 dimensions (at least) and start with $\mathcal{N} = 1$ SUSY, as the modular transformations are essentially transformations of the extra dimensional part of the enhanced Poincaré symmetry coupled with a SUSY transformation on the fields.

Most models assume a 6 dimensional spacetime with SUSY where the extra dimensions are compactified as a torus (with twist angle τ) and build a model using the assumed modular symmetries. However assuming the extra dimensions to be a torus can't lead to a viable theory as the resulting model after compactification would have no chirality and $\mathcal{N} = 2$ SUSY. The standard solution is to compactify the extra dimensions as an orbifold, which we now present its basics.

2.1 The orbifold $\mathbb{T}^2/\mathbb{Z}_N$

The two extra dimensional coordinates can be treated as a single complex coordinate $z = x_5 + ix_6$. The torus compactification is done by identifying

$$z \sim z + 1, \quad z \sim z + \tau, \tag{1}$$

where τ is called the twist angle and, for now, it is an arbitrary complex number. This identification restricts the range of the complex coordinate. The $\{1, \tau\}$ are called the basis vectors which generate the lattice of the extra dimensional plane and define the torus.

The torus by itself leads to a non chiral theory after compactification. The solution is to assume orbifolding, which is equivalent to assume that the extra dimensional part of the Poincaré group is not a full symmetry. This is done by modding out a discrete subgroup of the extra dimensional Lorentz group, which is called orbifolding. In 6 dimensions, the extra dimensional part of the Lorentz group is

$$SO(1,5)/SO(1,3) \simeq SO(2) \simeq U(1), \tag{2}$$

which correspond to rotation in the 2 extra dimensions. One can mod out by any discrete subgroup $F \in U(1)$, which can only be $F = \mathbb{Z}_N$, with N an arbitrary integer (for now). It has to be a discrete group to avoid reducing the dimensionality. The $\mathbb{Z}_N$ orbifolding is achieved by the identification

$$z \sim e^{2i\pi/N} z, \tag{3}$$

which further restricts the range of the extra dimensional coordinates. The orbifold has fixed points which allow boundary conditions that generate chirality, may break the gauge symmetry and reduce the enhanced SUSY. Therefore they may lead to a consistent model after compactification.

To avoid dimensional reduction and therefore for the orbifold to be consistent, the orbifold action in Eq.3 must be equivalent to an integer number of lattice transformations as in Eq. 3. In other words, there must exist integer numbers $a, b \in \mathbb{Z}$ such that a solution exists for

$$e^{2i\pi/N} z = z + a + b\tau. \tag{4}$$

It is enough to find a solution for each of the basis vectors $\{1, \tau\}$,

$$e^{2i\pi/N} = a + b\tau, \quad e^{2i\pi/N}\tau = c + d\tau, \tag{5}$$

where there must exist $a, b, c, d \in \mathbb{Z}$ that solve these equations. It is clear that there is no solution for arbitrary N and τ. This restricts the N and τ to be one of

$$\begin{aligned} &N = 2, \quad \tau = z \in \mathbb{C}, \\ &N = 3, \quad \tau = \omega, \\ &N = 4, \quad \tau = i, \\ &N = 6, \quad \tau = \{\omega, \rho/\sqrt{3}\}, \end{aligned} \tag{6}$$

where $\omega = e^{2i\pi/3}$ and $\rho = e^{i\pi/6}$ and all the solutions are valid up to an integer.

Therefore working with an orbifold may fix τ geometrically, adding predictivity, and solves the chirality problem therefore allowing a viable model.

2.2 The orbifold $(\mathbb{T}^2)^3/(\mathbb{Z}_N \times \mathbb{Z}_M)$

Many models may require various independent modular symmetries or different τ values to achieve a better fit. One such model is presented in Sec. 4. As it needs 3 independent modular symmetries, we focus on 10 dimensional spaces with $\mathcal{N} = 1$ SUSY before and after compactification.

In the 10 dimensional case, one can orbifold by a discrete subgroup of the extra dimensional part of the Lorentz group

$$SO(1, 9)/SO(1, 3) \simeq SO(6) \simeq SU(4), \tag{7}$$

which corresponds to rotations in the extra 6 dimensions. The former $SU(4)$ can be identified with the $SU(4)_{\mathcal{R}}$ of the enhanced $\mathcal{N} = 4$ SUSY. As we want to preserve simple SUSY after compactification, the discrete orbifolding group must be $F \subset SU(3)$. As it is rank 2, a general 10d SUSY preserving abelian factorisable orbifolding is

$$(\mathbb{T}^2)^3/(\mathbb{Z}_N \times \mathbb{Z}_M) \tag{8}$$

which can be compactified by the basis vectors

$$z_i \sim z_i + 1, \quad z_i \sim z_i + \tau_i, \tag{9}$$

and the orbifolding defined by

$$\begin{aligned} \theta_N : \ (x, z_1, z_2, z_3) \sim (x, \alpha_N z_1, \beta_N z_2, \gamma_N z_3), \\ \theta_M : \ (x, z_1, z_2, z_3) \sim (x, \alpha_M z_1, \beta_M z_2, \gamma_M z_3), \end{aligned} \tag{10}$$

where $\alpha_{N,M}, \beta_{N,M}, \gamma_{N,M}$ are Nth, Mth roots of unity.

The choice of the phases of the orbifolding are restricted by the preservation of $\mathcal{N} = 1$ SUSY. The τ_i must be fixed so that the lattice is unchanged by the orbifold transformation. The τ_i are fixed, as they must such that the orbifolding identification does not change the lattice and therefore the torus remains unchanged. Therefore there must exist integers a, b, c, d such that

$$(\delta, \delta\tau) = (a + b\tau, c + d\tau), \tag{11}$$

for each corresponding $\delta = \alpha_{N,M}, \beta_{N,M}, \gamma_{N,M}$

These restrictions limit the available (SUSY preserving [24]) orbifolds to be as in Table 1, which displays all the available orbifolds with some of the τ_i fixed as shown (up to an integer), while the non-fixed values are indicated by the complex number z.

(N, M)	$(\alpha_N, \beta_N, \gamma_N)$	$(\alpha_M, \beta_M, \gamma_M)$	(τ_1, τ_2, τ_3)
$(3,1)$	(ω, ω, ω)	$(1,1,1)$	(ω, ω, ω)
$(4,1)$	$(i, i, -1)$	$(1,1,1)$	(i, i, z)
$(6,1)_I$	$(-\omega^2, -\omega^2, \omega^2)$	$(1,1,1)$	$(\{\omega, \rho/\sqrt{3}\}, \{\omega, \rho/\sqrt{3}\}, \omega)$
$(6,1)_{II}$	$(-\omega^2, \omega, -1)$	$(1,1,1)$	$(\{\omega, \rho/\sqrt{3}\}, \omega, z)$
$(2,2)$	$(1,-1,-1)$	$(-1,1,-1)$	(z, z, z)
$(4,2)$	$(i, -i, 1)$	$(1,-1,-1)$	(i, i, z)
$(6,2)_I$	$(-\omega^2, 1, -\omega)$	$(1,-1,-1)$	$(\{\omega, \rho/\sqrt{3}\}, z, \{\omega, \rho/\sqrt{3}\})$
$(6,2)_{II}$	$(\omega^2, -\omega^2, -\omega^2)$	$(1,-1,-1)$	$(\omega, \{\omega, \rho/\sqrt{3}\}, \{\omega, \rho/\sqrt{3}\})$
$(3,3)$	$(1, \omega, \omega^2)$	$(\omega, 1, \omega^2)$	(ω, ω, ω)
$(6,3)$	$(-\omega^2, 1, -\omega)$	$(1, \omega, \omega^2)$	$(\{\omega, \rho/\sqrt{3}\}, \{\omega, \rho/\sqrt{3}\}, \omega)$
$(4,4)$	$(1, i, -i)$	$(i, 1, -i)$	(i, i, i)
$(6,6)$	$(1, -\omega^2, -\omega)$	$(-\omega^2, 1, -\omega)$	$(\{\omega, \rho/\sqrt{3}\}, \{\omega, \rho/\sqrt{3}\}, \{\omega, \rho/\sqrt{3}\})$

Table 1: Comprehensive list of 6d abelian factorisable and SUSY preserving orbifolds $(\mathbb{T}^2)^3/(\mathbb{Z}_N \times \mathbb{Z}_M)$, where $\omega = e^{2i\pi/3}$ and $\rho = e^{i\pi/6}$, and the fixed points of τ_i are specified only up to an integer. For example, $\tau_2 = i, i+1, i+2$ and so on are all equivalent. The values of the complex numbers z are not restricted by the orbifold, but particular values of z may be fixed by a remnant global symmetry.

2.3 The orbifold $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$

In this subsection we discuss an example of an orbifold chosen from Table 1 corresponding to $(N, M) = (4, 2)$ which leads to an interesting model [¶]. The full model based on the resulting orbifold $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ will be presented in Sec. 4. The model we have in mind is an extra dimensional version of a four dimensional model based on three finite modular groups S_4^3 broken to a diagonal subgroup S_4, with the three moduli fields in the low energy theory located at three different fixed points, namely $\tau_1 = i,\ \tau_2 = i + 2,\ \tau_3 = \omega$. In a 4d framework, this was shown to lead to a very predictive and successful phenomenological

[¶]This example is not unique, there are other choices which also lead to viable models.

description of the neutrino and charged lepton masses and lepton mixing based on a type of littlest seesaw [16].

In the 10d framework considered here, the desired moduli fields τ_i for such model are in principle consistent with the orbifold divisors $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$. However $\mathbb{Z}_2 \times \mathbb{Z}_2$ does not fix any of the τ_i, so is not so restrictive. The $\mathbb{Z}_4$ orbifold divisor fixes the τ_i as needed by the model, but does not have the necessary fixed branes to build consistent interactions. We are then left with the only viable and predictive choice being the orbifold divisor $\mathbb{Z}_4 \times \mathbb{Z}_2$, which can lead to the desired fixed points, as we discuss below.

We assume, then, a 10d spacetime where the 6 extra dimensions are factorisable into 3 torii, each defined by one complex coordinate z_i with $i = 1, 2, 3$, and compactified as in Eq. 9

$$z_i \sim z_i + 1, \quad z_i \sim z_i + \tau_i, \tag{12}$$

The orbifold $(\mathbb{T}^2)^3/\mathbb{Z}_4 \times \mathbb{Z}_2$ as defined by the orbifolding actions in Eq. 10, using Table 1 with $(N, M) = (4, 2)$ then implies,

$$\begin{aligned} \theta_4 &: (x, z_1, z_2, z_3) \sim (x, iz_1, -iz_2, z_3), \\ \theta_2 &: (x, z_1, z_2, z_3) \sim (x, z_1, -z_2, -z_3). \end{aligned} \tag{13}$$

In the orbifold approach, $(1, \tau_i)$ define the twist and the basis vectors of each torus. For the orbifold to be consistent, the orbifolding actions $\theta_{2,4}$ must not change the lattice, i.e. its action over the lattice basis vectors $(1, \tau_i)$ must be a linear combination of the original lattice vectors, with integer coefficients. Therefore there must exist integers $a_{1,2,3}, b_{1,2,3}, c_{1,2,3}, d_{1,2,3} \in \mathbb{Z}$ such that, as in Eq. 11

$$\begin{aligned} (i, i\tau_{1,2}) &= (a_{1,2} + b_{1,2}\tau_{1,2}, c_{1,2} + d\tau_{1,2}), \\ (-1, -\tau_3) &= (a_3 + b_3\tau_3, c_3 + d\tau_3), \end{aligned} \tag{14}$$

In the present example, solving Eq. 14 gives,

$$\begin{aligned} &\tau_{1,2} = i + n_{1,2}, \quad | \quad n_{1,2} \in \mathbb{Z}, \\ &\tau_3 \in \mathbb{C}. \end{aligned} \tag{15}$$

which corresponds to the result given in Table 1 with $(N, M) = (4, 2)$. We emphasise that the twists τ_i are fixed geometrically by the orbifold actions. Therefore in the orbifold approach to modular symmetries, the moduli fields are not a completely free choice, but are constrained as in Table 1.

Each orbifold action in Eq. 13, leaves some invariant subspaces which are called fixed branes

$$\begin{aligned}
\theta_4 &: \left(x, \left\{0, \frac{i+1}{2}\right\}, \left\{0, \frac{i+1}{2}\right\}, z_3\right), \\
\theta_4^2 &: \left(x, \left\{0, \frac{1}{2}, \frac{i}{2}, \frac{i+1}{2}\right\}, \left\{0, \frac{1}{2}, \frac{i}{2}, \frac{i+1}{2}\right\}, z_3\right), \\
\theta_2 &: \left(x, z_1, \left\{0, \frac{1}{2}, \frac{i}{2}, \frac{i+1}{2}\right\}, \left\{0, \frac{1}{2}, \frac{\tau_3}{2}, \frac{\tau_3+1}{2}\right\}\right), \\
\theta_2\theta_4 &: \left(x, \left\{0, \frac{i+1}{2}\right\}, \left\{0, \frac{i+1}{2}\right\}, \left\{0, \frac{1}{2}, \frac{\tau_3}{2}, \frac{\tau_3+1}{2}\right\}\right), \\
\theta_2\theta_4^2 &: \left(x, \left\{0, \frac{1}{2}, \frac{i}{2}, \frac{i+1}{2}\right\}, z_2, \left\{0, \frac{1}{2}, \frac{\tau_3}{2}, \frac{\tau_3+1}{2}\right\}\right).
\end{aligned} \tag{16}$$

When building a model, fields can be chosen to be located in any of the previous branes or in the bulk.

We want a minimal model where all fields can behave as modular forms (with different τ_i depending on their location) but can interact with each other, we will only use the 6d branes

$$\begin{aligned}
\mathbb{T}^2_A &= (x, z_1, 0, 0), \\
\mathbb{T}^2_B &= (x, 0, z_2, 0), \\
\mathbb{T}^2_C &= (x, 0, 0, z_3),
\end{aligned} \tag{17}$$

where all of them touch at the origin brane, where all interactions happen.

From Eq. 13, we note that the z_1 only feels the θ_4 action, therefore the $\mathbb{T}^2_A$ is a $\mathbb{Z}_4$ orbifold. As the action of θ_2 on z_2 is also contained in θ_4, the $\mathbb{T}^2_B$ is also a $\mathbb{Z}_4$ orbifold. Finally the z_3 only feels the θ_2 action, therefore the $\mathbb{T}^2_C$ is a $\mathbb{Z}_2$ orbifold.

3 Modular S_4 symmetries in the orbifold approach

So far we have considered possible orbifolds in which the VEVs of the moduli fields τ_i are fixed at least partially by the geometry. We now turn to the modular symmetries of the fields τ_i which are broken by the VEVs of the moduli fields τ_i. In general such modular symmetries are infinite but have a series of infinite normal subgroups called the principle congruence subgroups $\Gamma(N)$ of level N, whose elements are equal to the 2×2 unit matrix mod N (where typically N is an integer called the level of the group). For a given choice of level $N > 2$, the quotient group $\Gamma_N = PSL(2, \mathbb{Z})/\Gamma(N)$ is finite and may be identified with the groups $\Gamma_N = A_4, S_4, A_5$ for levels $N = 3, 4, 5$, which may be subsequently be used as a family symmetry [5]. In this section we consider the case $N = 4$ which corresponds to modular S_4 symmetries.

With two extra dimensions the single complex modulus τ has an infinite modular symmetry $\overline{\Gamma} = SL(2, \mathbb{Z})$ as follows. The modular group $\overline{\Gamma}$ is the group of linear fraction transformations which acts on the complex modulus τ in the upper half complex plane as follow,

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \text{with} \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \qquad \mathrm{Im}\,\tau > 0\,. \tag{18}$$

The modular group $\overline{\Gamma}$ can be generated by two generators S and T

$$S : \tau \mapsto -\frac{1}{\tau}, \qquad T : \tau \mapsto \tau + 1. \tag{19}$$

From the infinite modular group the finite subgroup $\Gamma_N = PSL(2, \mathbb{Z})/\Gamma(N)$ may be obtained. A crucial element of the modular invariance approach is the modular form $f(\tau)$ of weight k and level N. The modular form $f(\tau)$ is a holomorphic function of the complex modulus τ and it is required to transform under the action of $\overline{\Gamma}(N)$ as follows,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \tag{20}$$

The modular forms of level $N = 4$ have been constructed in [7, 25] .

The associated finite modular group Γ_4 has two generators S and T which fulfill the following rations

$$S^2 = (ST)^3 = (TS)^3 = T^4 = 1\,. \tag{21}$$

The finite modular group Γ_4 is isomorphic to the permutation group S_4 of four objects. In order to see the correlation between S_4 and tri-bimaximal mixing and the connection to S_3, A_4 groups more easily, it is convenient to generate the S_4 group in terms of three generators $\hat{S}$, $\hat{T}$ and $\hat{U}$ with the multiplication rules [26, 27],

$$\hat{S}^2 = \hat{T}^3 = \hat{U}^2 = (\hat{S}\hat{T})^3 = (\hat{S}\hat{U})^2 = (\hat{T}\hat{U})^2 = (\hat{S}\hat{T}\hat{U})^4 = 1\,, \tag{22}$$

where $\hat{S}$ and $\hat{T}$ alone generate the group A_4, while $\hat{T}$ and $\hat{U}$ alone generate the group S_3. The generators S, T can be expressed in terms of $\hat{S}$, $\hat{T}$ and $\hat{U}$

$$S = \hat{S}\hat{U},\, T = \hat{S}\hat{T}^2\hat{U},\, ST = \hat{T} \tag{23}$$

or vice versa

$$\hat{S} = (ST^2)^2, \quad \hat{T} = ST, \quad \hat{U} = T^2 S T^2\,, \tag{24}$$

with the explicit matrices being

$$\hat{S} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \hat{U} = \mp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{25}$$

where the minus sign in $\hat{U}$ applies for the **3** representation while the plus sign is for the **3′** representation.

	S	T
$\mathbf{1}$, $\mathbf{1'}$	± 1	± 1
$\mathbf{2}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$
$\mathbf{3}$, $\mathbf{3'}$	$\pm\frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$	$\pm\frac{1}{3}\begin{pmatrix} 1 & -2\omega^2 & -2\omega \\ -2 & -2\omega^2 & \omega \\ -2 & \omega^2 & -2\omega \end{pmatrix}$

Table 2: The representation matrices of the generators S and T in the five irreducible representations of S_4, where $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ is a cubic root of unity.

We assume $\mathcal{N} = 1$ SUSY in 10d and this abelian orbifold preserves $\mathcal{N} = 1$ SUSY in 4d after compactification [24]. Therefore we can assume 3 independent modular symmetry groups, each associated with a different tori [11–14]. We assume three discrete modular symmetries $S_4^{A,B,C}$ associated to each complex coordinate $z_{1,2,3}$ correspondingly.

With the assumed S_4 modular symmetries, the corresponding moduli from Eq.15, which have an arbitrary integer, now can only be

$$n = 0, 1, 2, 3, \tag{26}$$

where it is now limited to a choice of one in four.

3.1 Fixed points and S_4 modular forms

In most models using modular symmetries, the τ is a free parameter that is minimized by a potential and treated as a VEV. A standard strategy to increase the predictivity of the model is to restrict to fixed points which are geometrically preferred. These point $\bar{\tau}$ are defined as the points that are invariant under some element of the modular group $\gamma \in S_4$ called the stabilizer.

In an orbifold, the τ is not a free parameter and it is fixed by the geometry of the orbifold itself. However, there are a finite number of choices, which allow specific modular forms which are listed in Table 3 [28]. All the presented S_4 modular forms are defined in the basis from Table 2.

In the $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ orbifold, it will be assumed that

$$\tau_1 = i, \quad \tau_2 = i + 2, \tag{27}$$

which are particular cases of Eq. 15 which are phenomenologicaly preferred, as described in the Sec. 4. However the choice of τ_3 is undetermined by the $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ orbifold, and instead shall be fixed by assuming a remnant S_4 symmetry, as discussed in the next subsection.

τ	$Y_{\mathbf{3}}^{(2)}(\tau)$, $Y_{\mathbf{3,I}}^{(6)}(\tau)$	$Y_{\mathbf{3}}^{(4)}(\tau)$, $Y_{\mathbf{3'}}^{(6)}(\tau)$	$Y_{\mathbf{3'}}^{(4)}(\tau)$, $Y_{\mathbf{3,II}}^{(6)}(\tau)$
i	$(1, 1+\sqrt{6}, 1-\sqrt{6})$	$(1, -\frac{1}{2}, -\frac{1}{2})$	$(1, 1-\sqrt{\frac{3}{2}}, 1+\sqrt{\frac{3}{2}})$
$i+1$	$(1, -\frac{\omega}{3}(1+i\sqrt{2}), -\frac{\omega^2}{3}(1+i\sqrt{2}))$	$(0, 1, -\omega)$	$(1, \frac{i\omega}{\sqrt{2}}, \frac{i\omega^2}{\sqrt{2}})$
$i+2$	$(1, \frac{1}{3}(-1+i\sqrt{2}), \frac{1}{3}(-1+i\sqrt{2}))$	$(0, 1, -1)$	$(1, -\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}})$
$i+3$	$(1, \omega(1+\sqrt{6}), \omega(1-\sqrt{6}))$	$(1, -\frac{\omega}{2}, -\frac{\omega^2}{2})$	$(1, \omega(1-\sqrt{\frac{3}{2}}), \omega^2(1+\sqrt{\frac{3}{2}}))$

τ	$Y_{\mathbf{3}}^{(2)}(\tau)$	$Y_{\mathbf{3}}^{(4)}(\tau)$,$Y_{\mathbf{3'}}^{(4)}(\tau)$	$Y_{\mathbf{3,II}}^{(6)}(\tau)$,$Y_{\mathbf{3'}}^{(6)}(\tau)$	$Y_{\mathbf{3,I}}^{(6)}(\tau)$
ω	$(0, 1, 0)$	$(0, 0, 1)$	$(1, 0, 0)$	
$\omega+1$	$(1, 1, -\frac{1}{2})$	$(1, -\frac{1}{2}, 1)$	$(1, -2, -2)$	
$\omega+2$	$(1, -\frac{\omega^2}{2}, \omega)$	$(1, \omega^2, -\frac{\omega}{2})$	$(1, -2\omega^2, -2\omega)$	
$\omega+3$	$(1, \omega, -\frac{\omega^2}{2})$	$(1, -\frac{\omega}{2}, \omega^2)$	$(1, -2\omega, -2\omega^2)$	$(0, 0, 0)$
$\rho/\sqrt{3}$	$(1, -\frac{\omega}{2}, \omega^2)$	$(1, \omega, -\frac{\omega^2}{2})$	$(1, -2\omega, -2\omega^2)$	
$\rho/\sqrt{3}+1$	$(0, 0, 1)$	$(0, 1, 0)$	$(1, 0, 0)$	
$\rho/\sqrt{3}+2$	$(1, -\frac{1}{2}, 1)$	$(1, 1, -\frac{1}{2})$	$(1, -2, -2)$	
$\rho/\sqrt{3}+3$	$(1, \omega^2, -\frac{\omega}{2})$	$(1, -\frac{\omega^2}{2}, \omega)$	$(1, -2\omega^2, -2\omega)$	

Table 3: The alignments of triplet modular forms $Y_{\mathbf{3},\mathbf{3'}}(\tau)$ of level 4 up to weight 6 with the available fixed moduli in orbifolds. We have ignored the overall constant appearing in each alignment.

3.2 S_4 Remnant Symmetry

The orbifold $\mathbb{Z}_2$, associated with the third torus $\mathbb{T}^2_C$, does not fix τ. However, supposing that the twist angle is $\tau = \omega = e^{2i\pi/3}$ would leave a remnant S_4 symmetry (which is a subgroup of the extra dimensional Poincaré group) after compactification [29, 30]. We shall assume that there is a remnant S_4 after compactification, therefore fixing uniquely

$$\tau_3 = \omega. \tag{28}$$

We focus on the branes of the fixus torus $\mathbb{T}_C$ [30–32],

$$\bar{z} = \{0,\ 1/2,\ \omega/2,\ \omega^2/2\}, \tag{29}$$

which are naturally invariant under the orbifold transformations

$$T_1 :\ \bar{z} \to \bar{z} + 1, \quad T_2 :\ \bar{z} \to \bar{z} + \omega, \quad Z : \bar{z} \to -\bar{z}. \tag{30}$$

The set of branes is invariant under the permutation set of them. However not all permutations are Poincaré transformations.

These fixed branes and are permuted by the Poincaré transformations

$$S_1 : \bar{z} \to \bar{z} + 1/2, \quad S_2 : \bar{z} + \omega/2, \quad R : \bar{z} \to \omega\bar{z}, \quad P : \bar{z} \to \bar{z}^*, \quad P' : \bar{z} \to -\bar{z}^*, \tag{31}$$

which, after orbifolding, generate the remnant symmetry. We can write these operations explicitly $S_1[(12)(34)]$, $S_2[(13)(24)]$, $R[(243)(1)]$, $P[(34)(1)(2)]$, $P'[(34)(1)(2)]$. There are only 3 independent transformations since $S_2 = R^2 \cdot S_1 \cdot R$, $\ P = P'$.

These symmetry transformations relate to the S_4 generators with $\hat{S} = S_1$, $\hat{T} = R$, $\hat{U} = P$ satisfying Eq. 22 which is the presentation rules for the S_4 symmetry [1].

The $\mathbb{T}_{A,B}$ have only 2 branes from Eq. 16. Therefore its remnant symmetry can only be $\mathbb{Z}_2$.

With the assumption of an S_4 remnant symmetry, the τ_3 is fixed geometrically to be equal to ω [23]. .

4 A realistic orbifold model

We now turn to a concrete 10d bottom-up orbifold model with three factorizable tori built from the fundamental space depicted geometrically in Fig. 1. The 10d model is compactified on an orbifold $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ and we assume three finite modular symmetries $S_4^{A,B,C}$. Furthermore there is a remnant S_4 symmetry whose only role is to fix $\tau_3 = \omega$ ‖. This uniquely fixes the moduli geometrically to be $\tau_1 = i,\ \tau_2 = i + 2,\ \tau_3 = \omega$, (up to a choice in four).

The field content which defines the model is given in Table 4.

Field	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$	Loc
L	**1**	**1**	**3**	0	0	0	$\mathbb{T}^2_C$
e^c	**1**	**1**	**1**	0	0	-6	$\mathbb{T}^2_C$
μ^c	**1**	**1**	**1**	0	0	-4	$\mathbb{T}^2_C$
τ^c	**1**	**1**	**1**	0	0	-2	$\mathbb{T}^2_C$
N_a^c	**1**	**1**	**1**	0	-4	0	$\mathbb{T}^2_B$
N_s^c	**1**	**1**	**1**	-2	0	0	$\mathbb{T}^2_A$
Φ_{BC}	**1**	**3**	**3**	0	0	0	Bulk
Φ_{AC}	**3**	**1**	**3**	0	0	0	Bulk

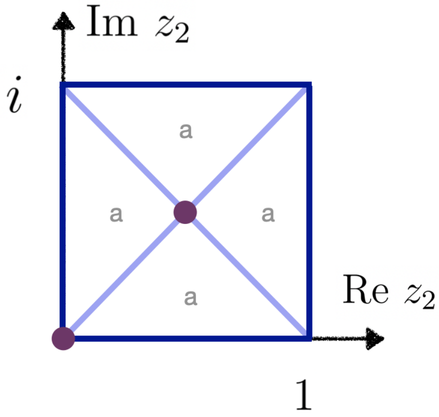
Yuk/Mass	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
$Y_e(\tau_3)$	**1**	**1**	**3**	0	0	6
$Y_\mu(\tau_3)$	**1**	**1**	**3**	0	0	4
$Y_\tau(\tau_3)$	**1**	**1**	**3**	0	0	2
$Y_a(\tau_2)$	**1**	**3**	**1**	0	4	0
$Y_s(\tau_1)$	**3**	**1**	**1**	2	0	0
$M_a(\tau_2)$	**1**	**1**	**1**	0	8	0
$M_s(\tau_1)$	**1**	**1**	**1**	4	0	0

Table 4: Transformation properties of fields and modular forms (Yuk/Mass) under the modular symmetries $S_4^{A,B,C}$ with modular weights $k_{A,B,C}$. The Higgs fields $H_{u,d}$ (not displayed) transform trivially under all the modular S_4 symmetries. The leptons $L \sim (2, -1/2)$, and $e^c, \mu^c, \tau^c \sim (1, 1)$ have the usual SM $SU(2)_L \times U(1)_Y$ quantum numbers and the right-handed neutrinos $N_{a,s}^c$ are SM singlets. The Higgs Φ which break the three modular symmetries to their diagonal subgroup, live in the 10d bulk, while the leptons live in the 2d subspaces as shown.

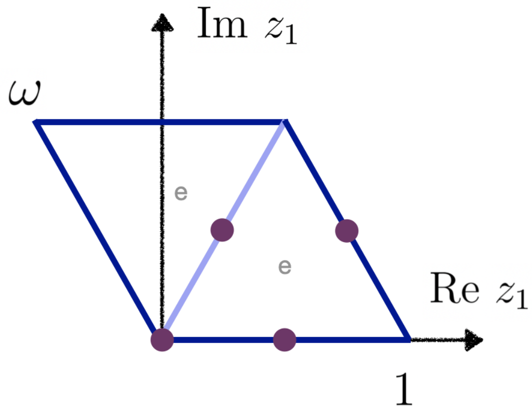
The resulting 4d Lagrangian is [16], ignoring the dimensionless coupling coefficients,

$$\begin{aligned} w_\ell \;=\; & \frac{1}{\Lambda}\left[L\Phi_{BC}Y_a N_a^c + L\Phi_{AC}Y_s N_s^c\right] H_u \\ & + \left[LY_e e^c + LY_\mu \mu^c + LY_\tau \tau^c\right] H_d \\ & + \frac{1}{2} M_a N_a^c N_a^c + \frac{1}{2} M_s N_s^c N_s^c \,. \end{aligned} \tag{32}$$

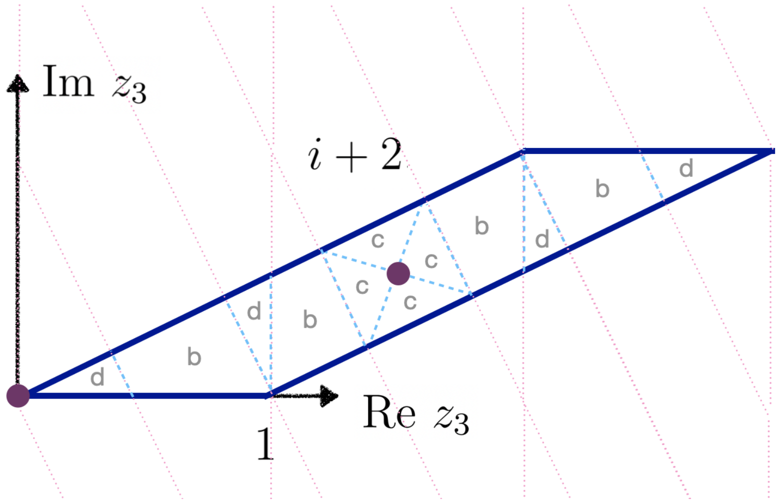
‖As discussed later, remnant S_4 symmetry may be further employed to control the Kähler potential.



(a) The extra dimensional space for $\mathbb{T}^2_A$. The $\mathbb{Z}_4$ orbifolding identifies the four isosceles triangles labeled as a.



(c) The extra dimensional space for $\mathbb{T}^2_C$. I The $\mathbb{Z}_2$ orbifolding identifies the two equilateral triangles labeled as e.



(b) The extra dimensional space for $\mathbb{T}^2_B$. The $\mathbb{Z}_4$ orbifolding is done by rotating the space by $\pi/2$ and creating drawing the lattice (dotted pink). One identifies the overlaps, which is this case are four quadrilaterlas labeled as b, four isosceles triangles labeled as c and four right angle triangles labeled as d.

Figure 1: Visualization of the extra dimensional space for each of the fundamental tori $\mathbb{T}^2_{A,B,C}$. Identifying together opposite sides we obtain $\mathbb{T}^2$. The orbifolding is described in each subfigure. The dots represent the fixed points.

and the modular Yukawa forms are fixed by the moduli $\tau_1 = i, \tau_2 = i+2, \tau_3 = \omega$ resulting in the alignments, using Tables 3 and 4, ignoring the overall constants,

$$\begin{aligned} Y_a &= (0, 1, -1)^T, \\ Y_s &= (1, 1+\sqrt{6}, 1-\sqrt{6})^T, \\ Y_\tau &= (0, 1, 0)^T, \\ Y_\mu &= (0, 0, 1)^T, \\ Y_e &= (1, 0, 0)^T. \end{aligned} \tag{33}$$

The Φ fields are assumed to obtain a diagonal VEV that breaks two modular symmetries into the diagonal one [16].

Hence, the charged-lepton mass matrix is simply given by

$$M_l = v_d \begin{pmatrix} (Y_e)_1 & (Y_\mu)_1 & (Y_\tau)_1 \\ (Y_e)_3 & (Y_\mu)_3 & (Y_\tau)_3 \\ (Y_e)_2 & (Y_\mu)_2 & (Y_\tau)_2 \end{pmatrix}, \tag{34}$$

where v_d stands for $\langle H_d \rangle$, and we ignore the dimensionless coupling coefficients.

Plugging in the specific shapes of the modular forms given in Eq. 33 we arrive at a diagonal charged-lepton mass matrix for $\tau_C = \omega$, including the dimensionless coupling coefficients:

$$M_l = v_d \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}. \tag{35}$$

The Dirac neutrino mass matrix is then given by:

$$M_D = v_u \begin{pmatrix} (Y_a)_1 & (Y_s)_1 \\ (Y_a)_3 & (Y_s)_3 \\ (Y_a)_2 & (Y_s)_2 \end{pmatrix}, \tag{36}$$

where, as usual, v_u denotes the H_u VEV, and the 2×3 structure comes from the CSD with just two RH neutrinos. We have ignored the dimensionless coupling coefficients. Choosing specific stabilisers for the two remaining moduli fields, we can achieve a CSD(3.45) structure with $n = 1 - \sqrt{6}$:

$$M_D = v_u \begin{pmatrix} 0 & b \\ a & b\left(1+\sqrt{6}\right) \\ -a & b\left(1-\sqrt{6}\right) \end{pmatrix}. \tag{37}$$

The type-I seesaw mechanism will lead to an effective mass matrix for the light neutrinos:

$$m_\nu = M_D \cdot M_R^{-1} \cdot M_D^T = v_u^2 \begin{pmatrix} \frac{b^2}{M_s} & \frac{b^2 n}{M_s} & \frac{b^2(2-n)}{M_s} \\ . & \frac{a^2}{M_a} + \frac{b^2 n^2}{M_s} & -\frac{a^2}{M_a} + \frac{b^2 n(2-n)}{M_s} \\ . & . & \frac{a^2}{M_a} + \frac{b^2(2-n)^2}{M_s} \end{pmatrix}, \tag{38}$$

where $n = 1 - \sqrt{6} \approx -1.45$. This can be rewritten in terms of 3 independent physical parameters

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 1 & n & 2-n \\ n & n^2 & n(2-n) \\ 2-n & n(2-n) & (2-n)^2 \end{pmatrix}, \tag{39}$$

where

$$m_a = \left|\frac{v_u^2 a^2}{M_a}\right|, \quad m_b = \left|\frac{v_u^2 b^2}{M_s}\right|, \quad \rho = \text{Arg}\left(\frac{a^2}{b^2}\right). \tag{40}$$

Therefore the model has only these 3 parameters for the whole neutrino sector.

4.1 Eclectic symmetry with the remnant S_4

As discussed in Sec. 3.2, the remnant symmetry S_4 acts on the branes and in the previous model, it has been identified with the modular symmetry S_4^C. Therefore we have built a model whose flavour structure is completely defined by the modular symmetries $S_4^A \times S_4^B \times S_4^C$. However, it is known that having purely modular symmetries to define the flavour structure complicates the Kähler [33].

In our setup, the only modular multiplet is the lepton doublet L. In general, the minimal Kähler potential for a superfield L would be a single term $\mathcal{K} = L\bar{L}$. However as L is modular form, the Kähler potential is enhanced to include terms

$$\mathcal{K} = L\bar{L} + \sum_k a_k (Y_k\overline{Y_k})_{\mathbf{1}}(L\overline{L})_{\mathbf{1}} + \sum_k b_k (Y_k L)_{\mathbf{1}}(\overline{Y_k L})_{\mathbf{1}}, \tag{41}$$

where the sum is over all available modular forms and the fields inside a parenthesis $()_{\mathbf{1}}$ are contracted into a modular symmetry singlet and the a_k, b_k are arbitrary dimensionless constants. The b_k terms appear only the case where L is something larger than a singlet, like in our model, which is a triplet. In that case the sum is done over the three different modular forms from Table 3, depending on the chosen τ.

The a_k terms can be absorbed as an overall normalization of the field L, therefore they are not relevant. The b_k terms are absorbed by the normalization of each component of the field L and therefore introducing parameters that change the flavour structure given from the superpotential.

In our model, the only nontrivial modular form is the lepton doublet L which will have 3 b_k Kähler terms. These parameters will affect the charged lepton mass matrix and can be reabsorbed in the definition of $y_{e,\mu,\tau}$, therefore preserving the same flavour structure there. However these parameters also affect the normalization of each left handed neutrino independently, introducing these 3 extra free parameters to the neutrino mass matrix and therefore reducing the predictiveness of the model. In general, for all modular symmetry models, it is assumed that these parameters are negligible, although they not necessarily need be.

One could avoid the presence of the unwanted terms by enhancing the modular symmetry by adding a standard flavour symmetry, such that the undesired Kähler terms are forbidden by the standard flavour symmetry [34]. Relating flavor symmetries and modular symmetries are called eclectic symmetries [18, 19, 22].

As an alternative model to the one presented in the previous section, we could use the remnant symmetry as a standard flavour symmetry, as described in Sec 3.2. This way the

S_4^C becomes standard flavour symmetry while $S_4^{A,B}$ remain as modular symmetries, thus having a trivial (where they all commute) eclectic symmetry. With this assumption the lepton doublet is no longer a modular form and there are no b_k Kähler terms. The model would require an extra $\mathbb{Z}_3$ shaping symmetry which would differentiate the three charged lepton singlets. Furthermore the modular forms $Y_{e,\mu,\tau}$ would not be available and they would have to be replaced by 3 flavon S_4 triplets $\phi_{e,\mu,\tau}$ whose VEV has the same desired alignments. This could be easily achieved through the orbifold boundary conditions and a very simple alignment superpotential [29]. With these changes, the flavour structure and all the phenomenological implications would be exactly the same as the model described in the previous subsection. Thus the same flavour structure CSD$(1-\sqrt{6})$ can be achieved easily through modular or eclectic S_4^3 symmetry.

4.2 Numerical Fit

	without SK atmospheric data		with SK atmospheric data	
	NuFit $\pm 1\sigma$	Model	NuFit $\pm 1\sigma$	Model
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	34.34	$33.41^{+0.75}_{-0.72}$	34.30
$\theta_{23}/^\circ$	$49.1^{+1.0}_{-1.3}$	48.31	$42.2^{+1.1}_{-0.9}$	46.98
$\theta_{13}/^\circ$	$8.54^{+0.11}_{-0.12}$	8.54	$8.58^{+0.11}_{-0.11}$	8.75
$\delta/^\circ$	197^{+42}_{-25}	284	232^{+36}_{-26}	278
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.41^{+0.21}_{-0.20}$	7.42	$7.41^{+0.21}_{-0.20}$	7.13
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.511^{+0.028}_{-0.021}$	2.510	$+2.507^{+0.026}_{-0.027}$	2.520
$\frac{m_a}{10^{-3}\ \mathrm{eV}}$		31.47		30.50
$\frac{m_b}{10^{-3}\ \mathrm{eV}}$		2.28		2.32
η/π		1.24		1.26
χ^2		6.3		26.61

Table 5: Normal Ordering NuFit 5.2 values [35, 36] for the neutrino observables, and the best fit point from the model. The best fit is for NuFit data without SK atmospheric data where the atmospheric angle θ_{23} is in the second octant, as preferred by the model.

With the CSD$(1-\sqrt{6})$ structure, we can achieve the fits shown in Table 5. Note that in both best fits, there is a unique physical phase $\eta \approx 5\pi/4$ which could point to a geometrical origin.

To quantify how good the fit is we use

$$\chi^2 = \sum_i \left(\frac{x_i^{\mathrm{exp}} - x_i^{\mathrm{model}}}{\sigma_i^{\mathrm{exp}}} \right)^2 , \tag{42}$$

where it is summed over all 6 experimental neutrino values $(\theta_{12}, \theta_{13}, \theta_{23}, \delta, \Delta m_{21}^2, \Delta m_{3\ell}^2)$. The model fits these 6 observables plus the lightest neutrino mass (which is zero), the Majorana phases (where one is unphysical) which determine neutrinoless beta decay parameter which is just equal to the $(1,1)$ element of the neutrino mass matrix, $m_{ee} = m_b$ [16]. Note that the 6 experimentally constrained observables are being fit with only 3 real parameters (m_a, m_b, η), which is a non-trivial achievement. Overall these 3 parameters are predicting 9 neutrino observables, which shows that the model is highly predictive. In particular the model requires a normal neutrino mass squared ordering with the lightest neutrino being massless, and predicts the atmospheric angle to be in the second octant, $\theta_{23} \approx 48^\circ$, with close to maximal leptonic CP violation, $\delta \approx 280^\circ$.

5 Conclusions

In recent years modular symmetries have been applied to flavour models in bottom-up approaches, where finite modular groups Γ_N may result from the quotient group of the modular symmetry by its principal congruence subgroup of level N, where for example $N = 4$ corresponds to S_4. In such approaches the role of the flavon field is played by a complex modulus field τ in orbifold models with two extra dimensions.

In this paper we have discussed modular symmetry models arising from bottom-up orbifold constructions. The simplest example in 6d involves the orbifold $\mathbb{T}^2/\mathbb{Z}_N$ with a single torus defined by one complex coordinate z and a single modulus field τ, playing the role of a flavon transforming under a finite modular symmetry. More generally we have considered bottom-up orbifolds in 10d, where the 6 extra dimensions are factorisable into 3 tori, each defined by one complex coordinate z_i and involving the three moduli fields τ_1, τ_2, τ_3 transforming under three independent finite modular groups. Assuming supersymmetry, consistent with the holomorphicity requirement, we consider all the orbifolds of the form $(\mathbb{T}^2)^3/(\mathbb{Z}_N \times \mathbb{Z}_M)$, and list all the available orbifolds, which have fixed values of the moduli fields (up to an integer). The key advantage of such 10d orbifold models over 4d models is that the values of the moduli are not completely free but are constrained by geometry and symmetry.

To illustrate the approach we have shown how a recently proposed littlest modular seesaw model with S_4^3 modular symmetry could result from such an orbifold construction. We have shown how this model may arise from an $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$ orbifold with $\tau_1 = i$, $\tau_2 = i+2$ being fixed by the geometry of the $\mathbb{Z}_4$ orbifold, while $\tau_3 = \omega$ is determined by imposing a further remnant S_4 flavour symmetry, commuting with the three S_4 modular symmetries. The $\tau_1 = i$ leads to an alignment $(1, 1+\sqrt{6}, 1-\sqrt{6})$ and CSD(n) with $n = 1 - \sqrt{6}$, where the atmospheric angle θ_{23} is restricted to lie in the second octant. The χ^2 fit shows that this is a highly predictive and successful description of all neutrino phenomenology, with two real parameters describing all neutrino mixing and mass ratios.

An alternative case $n = 1 + \sqrt{6}$, which prefers the atmospheric angle θ_{23} to be in the first octant, which was possible in the 4d model [16], is not allowed here since it corresponds to an alignment $(1, 1-\sqrt{6}, 1+\sqrt{6})$ and a stabilizer $\tau_1 = (-8+i)/13$ which is not achievable in the 10d orbifold model considered here. Whereas in the 4d model [16], the fixed points were selected in an *ad hoc* way, in the 10d orbifold model considered here the values of $\tau_{1,2}$ are fixed by the geometry to be equal to i (up to an integer).

In the modular symmetry model considered here, the remnant S_4 plays no role apart from fixing $\tau_3 = \omega$. However in an alternative orbifold model, the combination of flavour symmetry and modular symmetry could be used to control the corrections to the Kähler potential, as in top-down eclectic flavour symmetry. This would reintroduce flavons, and lead to a more complicated model which is beyond the scope of the main discussion here, although we have briefly sketched the consequences.

Finally we note that the bottom-up approach to modular symmetry from orbifolds followed here can readily be extended to GUTs, with up to three moduli groups and moduli fields. One could similarly include a remnant flavour symmetry, leading to a bottom-up version of the ecletic flavour symmetry in orbifold GUTs.

Acknowledgements

SFK acknowledges the STFC Consolidated Grant ST/L000296/1 and the European Union's Horizon 2020 Research and Innovation programme under Marie Sklodowska-Curie grant agreement HIDDeN European ITN project (H2020-MSCA-ITN-2019//860881-HIDDeN). SFK would also like to thank CERN for its hospitality.

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