Mathematics Education Research Journal
https://doi.org/10.1007/s13394-023-00461-1

ORIGINAL ARTICLE

Children engaging with partitive quotient tasks: elucidating qualitative heterogeneity within the Image Having layer of the Pirie–Kieren model

Children's images—Pirie–Kieren model

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Received: 25 November 2022 / Revised: 3 May 2023 / Accepted: 19 May 2023

Abstract
The paper presents findings from a study that examined, through the lens of the Image Having layer of the Pirie–Kieren model, the qualitative characteristics of the images that different children formed when engaging with eight, novel partitive quotient tasks. The Image Having layer is the first point of abstraction within the Pirie–Kieren model. Therefore, this research is significant in aiming to advance theoretical insight into how the notion of child-created images relates to the development of children's mathematical understanding, in the context of novel for them tasks. This study adopted a qualitative, microgenetic research design and involved nine Year 5 (aged 9–10 years) children. Data based on children's verbal responses and jottings were collected through multiple trials over eight individual sessions with each child. Analysis of 72 interview transcripts showed that children formed and used a range of different images that varied across tasks but also within the same task. This study provides a nuanced description of qualitative distinctions in the nature of child-created images. It thus reveals varied dimensions of a dynamic process of knowledge development and sense-making. This highlights, for educators, the need to be aware of and adaptive to the varied and dynamic dimensions of knowledge that children draw from, when dealing with novel tasks, and which change as children's understanding of new mathematical content develops.

Keywords Pirie–Kieren model · Images · Partitive quotient · Mathematical understanding · Mathematics learning and development · Mathematical representations

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Published online: 04 July 2023

Introduction

The notion of representations in mathematics learning has been captured in well-established theoretical models such as the Pirie–Kieren theory of mathematical understanding (Pirie & Kieren, 1989, 1994) and Duval's theory for the production of representations and relationships between thinking, semiotic system and the main cognitive functions (Duval, 2000, 2006). However, Bobis and Way (2018) emphasise that there is still a need for deeper elaboration of the nature of mathematical representations within these theories. In this regard, they state that 'we are finding that probing the role of representations more deeply in theoretical models, such as the Pirie–Kieren theory reveals the potential to extend and elaborate the theory, particularly regarding children's learning' (p. 69). This paper aims to address this gap in the literature, by examining the notion of representations and 'images' through the lens of the Pirie–Kieren theory.

Martin and Towers (2009) explain that the term 'image', within the Pirie–Kieren theory, refers to any physical or mental ideas or representations that a learner has about a topic, and is not restricted to visual or pictorial ones. For any topic, there are always multiple images that learners create and use (Pirie & Kieren, 1994). Martin and Towers (2009) emphasise that although physical representations of ideas can support the process of making images, within the Pirie–Kieren theory, external representations are not labelled as images. Rather, it is the sense that is made of these representations and the thinking and/or acting around the concept, as well as the personal meaning created by the learner for a concept, which is called an image.

Within mathematics education, the notion of images can therefore be linked to the domain of mathematical representations which can be either internal or external or both. Internal representations refer to an individual's '*mental, cognitive* or brain constructs, concepts, or configurations' (Goldin, 2020, p. 567) that underlie mathematical thinking, reasoning and problem-solving. External representations include manipulatives, diagrams, drawings, written symbols, jottings, verbalisations, facial expressions and gestures that are external to the individual and which are used to signify and communicate mathematical ideas and relationships (Bobis & Way, 2018; Goldin, 2020). Griffiths et al. (2017) note that how children construct and use mental representations in their mathematical activity is still unknown. This is because internal representations are not directly accessible and therefore teachers and researchers often rely on children's externally produced representations to infer the mental ideas that children possibly draw upon, when engaging with mathematical tasks (Goldin, 2020).

The study presented here is aligned with the view that forming and working with mathematical representations involves a dynamic process that underlies children's sense-making in mathematical activity (Cifarelli, 1998). Therefore, analysis of learners' representations requires a detailed analytical approach that values and aims to capture the diverse, individual and ever-evolving nature of representational aspects in mathematical thinking (Finesilver, 2022). On this basis, we have chosen to focus our study on the notion of 'images' that learners create when engaging with mathematical tasks, as depicted in the Pirie–Kieren theoretical model (Pirie & Kieren, 1989), a theory that also proposes that growth of mathematical understanding involves a 'dynamical, leveled, but non-linear, transcendently recursive process of reorganizing one's knowledge structures' (Thom & Pirie, 2006, p. 186).

We present an in-depth examination of the images associated with 9-year-old children's growth of understanding of the partitive quotient meaning of fractions as they engage in tasks involving fair sharing situations. In so doing, the paper aims to add to the empirical and theoretical literature by providing a micro-level, nuanced portrait of children's images related to their engagement with different fair sharing tasks across multiple sessions. The microgenetic design applied to this study enables data collection from dense observation of children's problem-solving behaviour over more than one trial with the same type of task and more than one problem-solving session (Siegler & Crowley, 1991). Thus, this design is appropriate for examining in detail the nature of children's images and any changes that occur as children develop and consolidate their applied images within and across trials.

The research and educational significance of this study is that, despite methodological challenges, examining mathematical representations is important. According to Bobis and Way (2018) this is because mathematics 'essentially consists of ideas that are neither directly visible nor tangible' and therefore, 'to access mathematics and to work with mathematical processes, we must create representations using signs, symbols, and conventions' (pp. 56–57). Representations, examined here through the lens of the notion of 'images' within the Pirie–Kieren model, play a key role for mathematics learning. This is because they provide insights into children's mathematical thinking, experience, processes of coming to understand, as well as their existing and developing knowledge and understanding (MacDonald, 2013).

The paper seeks to address the following research question:

Within the Image Having layer of the Pirie–Kieren model, what are the qualitative characteristics of the images that different children draw from and use when engaging with partitive quotient tasks, over the course of more than one trial with the same type of task?

Theoretical background: the Pirie–Kieren model

The Pirie and Kieren theory (1989, 1994) which characterises growth of mathematical understanding is depicted in a model (Fig. 1) that includes eight nested circles. The nested circles represent eight potential layers-of-action for describing the growth of understanding for an individual, on any specific topic or concept, at any educational level (Martin, 2008). These layers are labelled Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventising.

This paper focuses on investigating the nature of 'images' which is the terminology used by the Pirie–Kieren theory to capture the notion of representations. Therefore, in the next three sub-sections, we have included concise definitions related to the first three layers of understanding: Primitive Knowing and Image Making, which are the first two layers preceding the layer of our focus, and Image

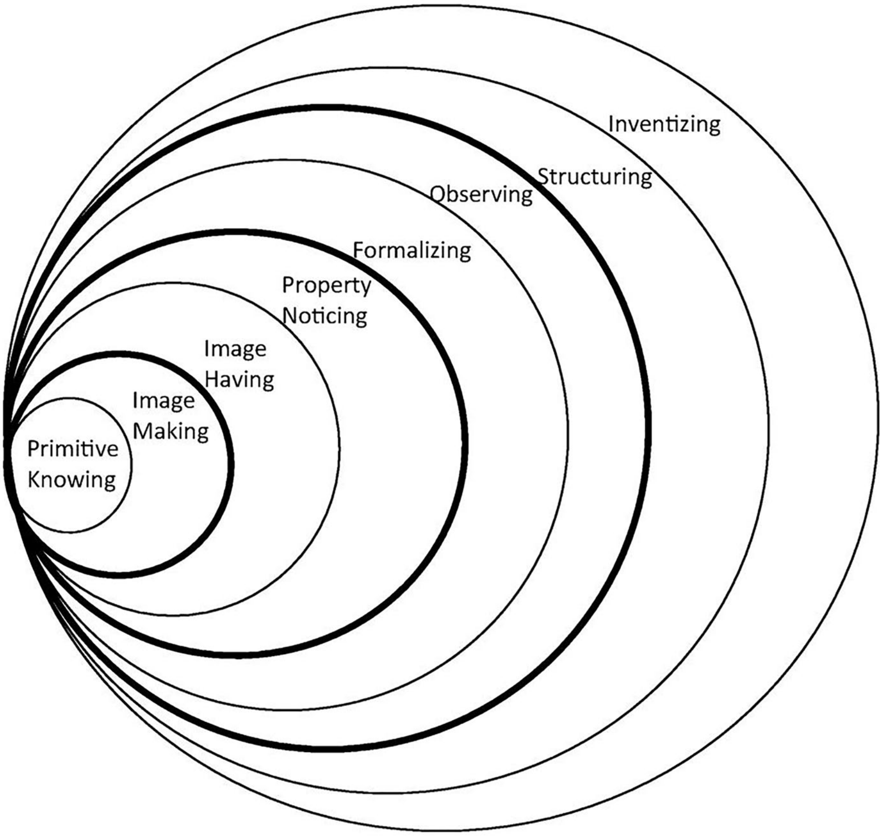


Fig. 1 The Pirie–Kieren model (Yao & Manouchehri, 2022, p. 245, adapted from Pirie & Kieren, 1994, p. 167)`

Having, which is the layer we are zooming into for our analysis. Previous work by Susan Pirie and Thomas Kieren (Kieren & Pirie, 1996; Kieren et al., 1999; Pirie & Kieren, 1991a, 1992, 1994) has provided comprehensive definitions of all layers of understanding actions.

Layer 1: Primitive Knowing (PK)

Primitive Knowing is the starting point in the process of developing understanding of a mathematical topic. It comprises:

> everything that a learner knows (and can do) *except* the knowledge about the particular topic that is being considered. Anything that the learner may already know about that topic is seen, through the lens of the Pirie–Kieren Theory, to be an understanding on one of the other outer layers. (Martin, 2008, p. 65)

An observer, such as a teacher or researcher, can never know exactly what primitive knowing an individual might have at an initial engagement with a mathematical topic. They can, however, make reasonable assumptions about what an individual already knows. This is based on initial engagement with learners, their own schooling experiences and previous experience of teaching that topic at that grade level or engagement with curricular documents.

Layer 2: Image Making (IM)

At the Image Making layer, an individual engages in activities aimed at facilitating the development of particular images or representations for the topic under consideration. They may also make changes to previously held images (Martin, 2008). An example of a child operating within the Image Making layer involves 12-year-old Teresa who is representing the fraction amount of ¾ in different ways using halves, thirds, fourths, sixths, eighths, twelfths and twenty-fourths. Operating within the Image Making layer, using trial and error, she places different combinations of fraction pieces on a ¾ segment of a rectangle, to see which combinations would cover the space completely (Pirie & Kieren, 1994). In this example, Teresa has not yet formulated an image of what combinations would work but is merely mixing and matching pieces in an exploratory manner.

Layer 3: Image Having (IH)

After employing the actions related to operating within the Image Making layer, an individual can exchange these actions for a mental organisation of these or their effects (Kieren et al., 1999). This forms the third layer of the Pirie–Kieren model, Image Having. A learner operating within this layer does not require particular physical actions or examples when engaging with a particular content or problem. However, this does not mean that an individual who is engaging with tasks within the Image Having layer necessarily has an appropriate or complete image for the given task. In this regard, Thom and Pirie (2006) inform that 'to have a mathematical image is to "know" some piece of mathematics as a "matter of fact" (right or wrong!)' (p. 190). Kieren et al. (1999) describe the inner Image Having layer as involving 'less formal, less sophisticated, less abstract and more local ways of acting' (p. 218) which form the basis for operating securely at a higher layer where one has an image *for all* problems of a particular kind. This layer within the model and the notion that it embodies hold great significance indispensable to any comprehensive model of mathematical understanding and children's mathematical content development. This is because it serves as the first point of abstraction in the model at which an individual can operate mentally and symbolically without reference to the meanings of basic concepts or images.

The excerpt provided next, as presented in Thom and Pirie (2006), illustrates Image Having involving a pair of children.

> Sammy: Sam, let's do it. Something about that secret number? What equals that number? Oh! [exclaims excitedly]

> Sam: How 'bout… divided! … will make it equal…
> Sammy: No, times! [i.e., multiplication] We know something that equals that number.
> Sam: Yes [smiles and nods his head].
> Sammy: What is it? [shakes his pencil at Sam and says...] Don't say it, just let me think. A few seconds later, Sammy records onto the paper, "$8 \times 9 = 72$."
> Sam: How do you know? [points to "$8 \times 9 = 72$" on the sheet of paper].
> Sammy: [smiles] I know the times-table! (Thom & Pirie, 2006, p. 190).

Thom and Pirie (2006) explain that:

> Sam and Sammy search for an image that they already "have" in order to describe 72… Sammy states the image he has of, "$8 \times 9 = 72$," … and exclaims that he just knows this to be "true" because of the "times-table!" He did not need to resort to the specific activities that gave rise to his understanding of multiplication as grouping. (Thom & Pirie, 2006, p. 190)

Folding back

Pirie and Kieren (1991b) describe folding back as follows:

> A person functioning at an outer level of understanding when challenged may invoke or fold back to inner, perhaps more specific, local or intuitive understandings. This return to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing one's current understanding on an earlier understanding, and the idea that understanding is somehow 'thicker' when inner levels are revisited. This folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding. (p. 172)

When a learner encounters a problem, question or situation that is not readily solvable, folding back occurs, facilitating the extension of current, inadequate and incomplete understanding. Martin (2008) states that this occurs through a learner's reflection on and subsequent reorganisation of their previously held constructs for the mathematical content, or formation of new images if the current constructs prove inadequate for finding an answer to the problem at hand.

In this section, we have delineated the key theoretical notions that frame our analysis. In the section devoted to data analysis, we will provide further detail on how the model is operationalised and applied in the context of this study. In the following section, we present a review of existing research related to young children's understanding of the partitive quotient fraction sub-construct, which is the core concept underpinning the tasks that we used in our investigation.

Review of related literature on partitive quotient tasks

Partitive division stems from sharing situations where one or multiple items are shared equally into a specified number of parts (Adu-Gyamfi et al., 2019; Hiebert & Behr, 1988). In many countries, children first encounter division through this context of sharing (Anghileri & Beishuizen, 1998; Ching & Wu, 2021; Streefland, 1991), and empirical research has found that this model of division is intuitive for children (Correa et al., 1998; Fischbein et al., 1985; Roche & Clarke, 2013). For continuous items, such as cakes or pizzas, the partitive quotient represents the amount that each person receives when one shares the items equally. Within the context of the partitive quotient which is one of five sub-constructs (part-whole, quotient, operator, measure and ratio) that a fraction can assume (Kieren, 1980), the fraction ¾ may represent the amount of pizza that each person gets when four people share three pizzas fairly. In these situations, there is sharing between two separate measure spaces (Mitchell, 2012) where the denominator and the numerator represent the number of recipients and the number of items shared, respectively.

Partitive quotient tasks afford rich environments for children to develop several aspects of fraction knowledge such as partitioning, unit fractions, equivalences and fraction addition (Adu-Gyamfi et al., 2019; Empson et al., 2006). This paper focuses on children's partitioning. Kieren (1995) describes partitioning as the drawing and/or folding actions of children as they try to create parts of the same size when engaging with partitive quotient tasks. Empson et al. (2006) include in their definition of partitioning 'mental as well as materially expressed actions' (p. 3) which are consistent with internal and external representations, respectively. Post et al. (1982) point out that partitioning is the main cognitive configuration that forms a part of and is produced by the partitive quotient sub-construct. Developing learners' understanding of fractions through activities and experiences that involve partitioning is therefore important and continues to be emphasised in research and practice (Empson et al., 2006; Lamon, 2020; Mulligan & Mitchelmore, 2018; Wilkins & Norton, 2018).

Partitioning within the context of the partitive quotient has been explored in previous research involving primary school-aged children. Pothier and Sawada (1983) involved 43 children (aged 4 years to 9 years) from kindergarten to grade 3 in their study. They found that children's strategies for engaging with the partitive quotient tasks included various forms of halving which children were likely exposed to within social settings, well before formal schooling. Empson et al. (2006) suggested that children use halves or quarters because these partitions are easy to make. Consistent with other research (e.g. Charles & Nason, 2000; Lamon, 2012; Subramanian & Verma, 2009), they noted that for this partitioning approach, the children generally partition the items into halves and quarters, distribute the parts to the sharers individually and then deal 'with left-over parts by creating new partitions of these parts into smaller parts' (p. 11).

Streefland (1991) explored different aspects of fraction development with sixteen fourth grade students (ages 9–10) and found that when the number of items was less than the number of sharers, students regularly partitioned *each* of the items into the number of people involved in the sharing situation. When the number of sharers was greater than the number of items being shared, Streefland (1991) reported that the children distributed wholes to the sharers and then divided the remaining items into the number of people engaged in the sharing situation. For example, in sharing five eggs among four people, each person got a whole and then ¼ from the last egg. Charles and Nason (2000) reported that at least one of the 12 Year 3 children (aged 7.9–8.3) from Eastern Australia partitioned the items into a multiple of the number of sharers. For example, Ben shared one pizza between two people by partitioning the pizza into eight which is a multiple of two (number of people sharing).

Partitioning within the context of fractions is an important mathematical content area for young learners, and as such, there has been much related research (e.g. Charles & Nason, 2000; Lamon, 2020; Pothier & Sawada, 1990; Siemon et al., 2012). However, the research reported in the current paper differs from earlier studies in that it involves, similar to Zhang et al. (2015), students who have only been taught the part-whole sub-construct of fractions in formal schooling. To date, internationally, the part-whole fraction sub-construct is still the first and most widely taught fraction sub-construct in elementary school (Boyce & Norton, 2016; Wilkins & Norton, 2018), and therefore, the focus on this group of learners in empirical research is current and important for educational practice. Finally, although previous research related to the Pirie–Kieren theory has explored fractional content (e.g. Duzenli-Gokalp & Sharma, 2010; Gokalp & Bulut, 2018; Martin et al., 1994; Pirie & Kieren, 1992), the partitive quotient, which is the focus of this study, has not been previously used to explore aspects of the theory.

Research design and methods

The present study employed a qualitative, microgenetic research design. Microgenetic designs involve intensive collection of data over a specific period of time with the aim 'to generate a very rich picture of moment-to-moment learning processes' (Chinn, 2006, p. 439). Research participants typically engage with a similar type of tasks and/or measures repeatedly, across sessions that are scheduled to take place close in time. Increased-in-density opportunities to engage with a particular type of task and discover more advanced strategies and concepts (Luwel et al., 2008) allow for systematic comparisons across and within individuals.

Within the field of mathematics education and psychology, microgenetic designs have been applied to explore areas such as single-digit addition (Siegler, 2007; Siegler & Crowley, 1991; Voutsina, 2012), multiplication and division (Dubé & Robinson, 2018), numerical–spatial relations in a number board (Laski & Siegler, 2014) and low attaining students' multiplicative thinking when working with 3D array tasks (Finesilver, 2017).

Kuhn (1995) states that microgenetic studies provide researchers with the opportunity to observe both the developing knowledge base for a particular topic or concept and the strategies through which this knowledge is developed. Microgenetic designs are highly appropriate for research such as the present study that aims at tracing children's movement across different layers of the adopted theoretical model as they engage with a particular type of task, with the view of closely examining the nature of particular instances, namely the nature of the images that they create and use as they develop their solving approach.

Participants

Nine children (three girls and six boys, aged 9–10 years) in Year 5 from the Commonwealth of Dominica were purposively selected to participate in this study by the first author, in consultation with the children's class teacher. Children who were likely to verbalise their thoughts as they worked on tasks and with regular attendance records were selected to participate to maximise the likelihood of collecting a rich data set from children across the full sequence of sessions. While more boys than girls participated in the research, consideration and analysis of any gender-related differences associated with children's engagement with the tasks were not relevant to the research questions, and therefore, the gender imbalance is not a problematic element for this study. The study was conducted in term 2 (February and March) of the academic year. The research followed the ethical guidelines of the British Educational Research Association (BERA) (2018) as well as the research ethics procedures of the authors' academic institution in relation to recruitment, data collection and data storage. For all research participants, written informed parental consent and child assent were obtained.

Methods

Data were collected by the first author using individual, video-recorded task-based interviews. The task-based interview is a form of clinical interview in which an individual or group of individuals talk while working on a mathematical task or set of tasks (Maher & Sigley, 2020). The 'talk' that learners typically engage in may include thinking aloud as they complete the tasks or responding to probes by the researcher. Goldin (1997) adds that task-based interviews are used for '(a) observing the mathematical behavior of children, usually in an exploratory problem solving context, and (b) drawing inferences from the observations to allow something to be said about the problem solver's possible meanings, knowledge structures, cognitive processes, affect, or changes in these in the course of the interview' (p. 40). These purposes align with the focus of the current study, and therefore, the task-based interview was considered an appropriate data collection method for this research. Task-based interviews have also been used in previous research that has investigated similar interests to this research, such as learners' understanding of fractions (Clarke et al., 2006), and students' ways of reasoning and connections made to existing ideas as learners extend their understanding (Houssart & Evens, 2011).

All of the research participants engaged in solving a sequence of eight, similar, novel for them tasks that involved fair sharing situations over a 6-week period. The novelty of the tasks was confirmed by a close examination of the curriculum content covered up to that grade level and discussion with the children's class teachers. Tasks were adapted from previous empirical work (Charles & Nason, 2000; Lamon, 2012; Streefland, 1991). The main modification made was to present the tasks in contexts familiar to the children such as sharing pizzas and cakes among friends. The children in this study engaged in solving problems of the type: 'Share *two* cakes/pizzas fairly among *three* children.' After the children offered one solution, the interviewer asked 'How else can you share the cakes/pizzas among the three children?' to elicit additional ways of partitioning the diagrams and solving the tasks.

As noted in the review of related literature, partitive quotient tasks provide multiple opportunities for children to demonstrate and develop several aspects of fraction knowledge. The focus of the analysis of this paper is on the partitioning aspect of the children's engagement with the partitive quotient tasks. While the children provided a fraction amount that represented the amount of cake/pizza that each person would receive in each of the iterations of solving a given problem, the analysis presented here does not focus on this aspect of partitive quotient engagement. The number of items in the sharing situations for each of the tasks ranged from 2 to 4, whereas the number of people included 3, 5, 6, 7 and 8. The children were provided with rectangular region models to represent cakes and pizzas in the tasks since previous research found that children find it easier to work with these models rather than with circular ones (e.g. Charles et al., 1999). Generally, two tasks were solved each week by each child, depending on the children's availability due to school commitments. The maximum duration of interviews was 30 min.

A semi-structured interview protocol served as a guide for conducting the task-based interview(s). The semi-structured nature of the protocol allowed for flexibility in questioning based on the research participants' responses. This is 'essential to allow for the enormous differences that occur in individual problem-solving behaviors… and to avoid "leading" the child in a predetermined direction' (Goldin, 1993, p. 305). If the child appeared to struggle to provide another way of partitioning when the researcher asked: 'How else can you share the cakes/pizzas among the children?", the session would end.

The collected data consisted of 72 interview transcripts that included children's verbal reports and gestures as well as problem-solving activity (i.e. written activity and diagrams) as the nine children engaged with the same type of task over eight individual sessions. In the following sections, when referring to one of the eight tasks across the eight sessions, we use numbering (T01–T08) and the associated, short description of the task. When presenting interview extracts, 'R' refers to the researcher/interviewer.

Data analysis: using the Pirie–Kieren model as an analytic tool

The data were analysed using the Pirie–Kieren model as the analytical framework to trace children's understanding actions within and across the layers. Since this paper focuses on exploring the nature of children's images within the Image Having layer, in this section, we present the image(s) for partitioning used for the present research. We analysed the video data and interview transcriptions for the different solutions that the children offered and the number of partitions the children shared each item into, as part of a given task and across multiple tasks.

Subsequently, using the transcripts, along with associated video recordings, for each of the eight task-based interviews, we coded the actions, overt behaviour and verbalisations of the research participants. This allowed us to infer their operation within each of the two layers of the Pirie–Kieren model: Image Making and Image Having. For both Image Making and Image Having, children exhibited one or more of the following actions specific to the tasks:

- Partitioning of diagrams with lines drawn on diagrams
- Partitioning motion with fingers/pencil/head
- Counting, recounting partitions
- Inspecting partitioned diagrams
- Distributing pieces of cake/pizza by finger motions from the diagram to the picture of the people, drawing lines from partitions to people sharing, shading partitions, numbering/labelling partitions.

The actions characteristic of the children operating within the Image Making and Image Having layers therefore appeared to be similar. However, the key distinction in how the children engaged with the tasks within the Image Making and Image Having layers related to their general overt behaviour and verbalisations presented in Table 1.

After this coding was completed, the research data were again reviewed for the purpose of explicitly studying children's operating within the Image Having layer. This examination of data sometimes included a focus on children's operating within the Image Making layer since, in some cases, children folded back from working within the Image Having layer.

Partitioning images in this study

A close examination of the data revealed that children in this study appeared to use five different images to find the number of partitions to share the items, depicted as rectangles. These images are presented in Table 2, with associated illustrative excerpts of enactment of each image.

Our analysis zoomed into the different types of images that children created when engaging with the same type of task over multiple trials. This was done to examine, in detail, the qualitative nature and application approach of the images that children produced, and how this might differ across children, as well as across and within tasks. The following section presents the outcomes of this analysis.

Table 1 Examples of overt behaviour and verbalisations from the present research, characteristic of the Image Making and Image Having layers of the Pirie–Kieren model

Layer	Overt behaviour
Image Making (IM)	• Spending an extended period staring ahead or looking at the task sheet • Speaking extremely slowly and haltingly, with hesitation in verbalisations and actions (e.g. counting, partitioning) • Not providing an explanation or a coherent explanation for the choice of number of partitions or method of solving • Providing an explicit statement such as ‘I am exploring’, ‘I am just trying this to see’, ‘I am not sure if this will work’ and ‘I do not have a way to share the cakes’
Image Having (IH)	• Beginning to solve the task immediately or with little hesitation • Reporting solving approaches without hesitation, for example, for the problem of sharing three cakes among five children: ‘I am sharing each of my cakes into five pieces’ • Giving a clear and coherent explanation for the choice of number of partitions or solving approach, for example, when sharing four cakes among three children, after verbalising that each cake would be shared into three, they state ‘because there are three children here’ • Providing explicit statements such as ‘I know the answer already’ or ‘I know how to share the cake’

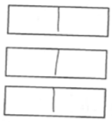
Findings

In this section, we present examples from five children (David, Jack, Kenny, Harry, Rebecca) operating within the Image Having layer, to illustrate the qualitative characteristics of the images that different children developed and used. Each example represents a description of the nature of a child’s image(s) when operating within the Image Having layer. In our analysis, we identified five qualitative distinctions in the images that children produced when engaging with partitive quotient tasks, over the course of multiple problem-solving sessions: application of an appropriate image across tasks, application of an appropriate

Table 2 Partitioning images observed in this study

1. Number of partitions in each diagram = Number of people sharing (*Excerpt 1)* Kenny shares two cakes fairly among three children. **R:** Here is our task for today. Share two cakes among three children so that each child gets the same amount of cake, and no cake is left over. **Kenny:** There are two cakes. You have to share them among three children, so you cut the cake into three [partitions the first rectangle into three]. So that means one child would get, one child would get this piece, another child would get this, and the third child would get this one [points on each of the three partitions of the first rectangle in turn]. And then, you do the same [partitions the second rectangle into three], to the second cake so that they each get two pieces. So, the first child would get this piece, the second would get this piece, the third child would get this piece. So that means there... each child takes two cakes, gets two cakes. Two cakes. Two pieces of cakes.
2. Number of partitions in each diagram = Multiple of the number of people sharing (*Excerpt 2*) Samuel shares two cakes among seven people. **Samuel:** If I divide each cake into twenty-one pieces, … in the first cake, child one or A will get three pieces because seven can go into twenty-one three times.
3. Number of partitions in each diagram = Number associated with the half family (*Excerpt 3*) **4. Number of partitions in multiple diagrams = Number of people sharing (*Excerpt 3*).** Harry shares three pizzas fairly among six people. **Harry:** Miss is three pizzas and six child – six people. Then you can cut, share them in halves. [Partitions each diagram in two.]

Table 2 (continued)



5. The child partitions each of the diagrams into a number of pieces (e.g., ten) and distributes evenly. If there is a remainder (one or more partitions), cuts this into number of people (fifths). (*Excerpt 4*)

Harry shares three cakes among five children.

Harry: Well, if we give everyone, each person one [referring to one whole cake] it would still remain two more people to get, and it will have no more. So, we can share them in halves. [Partitions each of the three diagrams in half.] Then each child would get one-half. And it would still remain one, one more [refers to half of a cake/diagram]. So, you can share them in fifths. [Partitions one-half of a diagram into five.]

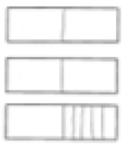


image for the first solution only, application of an inappropriate image, application of multiple appropriate images across tasks and application of an appropriate but overgeneralised image across tasks.

Application of an appropriate image across tasks

Table 3 presents excerpts from the first solution that David provided for T01 and T02, where he shared two cakes among three children and four cakes among three children, respectively.

After operating in the Image Making layer for part of the first solution for T01, David crossed from Image Making to the Image Having layer for the first solution. He remained within the Image Having layer with this appropriate image for the partitive quotient in T02. David operated in a similar way for the

Table 3 Application of appropriate image across tasks—example from David

T01: Share two cakes among three children
Image Making
David: So, what I am thinking now is that I should separate them into quarters and stuff like that, to give each person [quickly moves pencil back and forth from person to diagram three times] and if it doesn't work out, I will go on the next page.
R: Can you show me?
David: [Looks at paper with the written task and diagrams and places pencil on the diagram of one child.]
Image Having
David: Wait! Actually, since there are three children and two cakes, I'm going to separate the cakes into thirds, cause there are chi–, three children. [Looks at paper with the written task and diagrams while speaking.] The (Into?) thirds. [Partitions the first diagram into three using vertical lines]. And so then… make the thirds.
T02: Share four cakes among three children
Image Having
David: Okay. Like the last session we had, am, like yesterday, uuum… we have three children but this time we have four cakes. So right, like yesterday I would… am separate them into thirds? Separate into thirds.

solutions related to the other tasks. He also used partitioning of the form 'Number of partitions in each diagram = Multiple of the number of people sharing'.

Application of an appropriate image for the first solution only

Table 4 includes excerpts from the first solution that Jack provided for T01 and T02. In these tasks, he shared two cakes among three children and four cakes among three children, respectively.

After operating in the Image Making layer for the first solution for T01, Jack crosses to Image Having in T02 for the first solution. Jack's operating is unique in that he was unable to provide a solution for partitioning beyond the first one for each of the tasks. Although he sometimes struggled with partitioning, he had an image as to the number of partitions he wanted to make. This is evident in T02 when he immediately explained that 'because there are three children here' when asked why he chose to share the cakes in three.

Application of an inappropriate image

In T02, Kenny engaged in sharing four cakes among three children (Table 5). In this excerpt, for the first solution, he folds back from operating within the Image Having layer to working within the Image Making layer. He subsequently returns to the outer Image Having layer within the same task. Subsequent to this task, his images for partitioning were 'Number of partitions in each diagram = Number of people sharing' and 'Number of partitions in each diagram = Multiple of the number of people sharing'. This way of operating mirrored that of David.

Table 4 Application of an appropriate image for first solution only - example from Jack

T01: SHARE TWO CAKES AMONG THREE CHILDREN

IMAGE MAKING

Jack: Am.... [Looks at the diagrams for an extended period.]

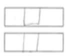
R: Tell me what you are thinking.

Jack: I am thinking of how I have to set it down. I have to share among the three children. I have to share among the three children. [Long pause.]

Miss... Each child gets one-third of a cake, Miss.

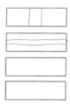
R: Could you show me on the diagram?

Jack: Miss, there are three children... to share it equally you have to, like… He would get one-third [Uses the end of his pencil and traces a vertical partitioning line about a third of the first diagram], then there would be another one-third [uses the end of his pencil and traces a vertical partitioning line of about a third of the second diagram], so it would have no cake left over.



T02: SHARE FOUR CAKES AMONG THREE CHILDREN

IMAGE HAVING



Jack: We are cutting one piece of cake [referring to the first diagram as one piece of cake]. [Partitions the first diagram into three.] So that is one piece of cake for one child [points to the second partition in the first diagram], another piece for another child [points to the first partition in the first diagram], and another piece for another child [points to the third partition in the first diagram].

R: Why did you choose to share the cakes in three?

Jack: Because there are three children here.

At the start of T02, Kenny appears to have an image for the number of partitions to share each diagram. This image for the number of partitions is the number of items (four) equals to the number of partitions (four). This is an example of

Table 5 Application of an inappropriate image—example from Kenny

T02: SHARE FOUR CAKES AMONG THREE CHILDREN

IMAGE HAVING

Kenny: So, I can share the cake into four pieces. Four cakes. [Partitions the first diagram into four.]

R: Please tell me why you chose four?

FOLDS BACK TO IMAGE MAKING

Kenny: Because... [Looks at diagrams, touching the pencil to different parts of each diagram.]

Kenny: Not four.

R: Not four? Why not four?

Kenny: Because when I, if I cut all the cakes into four pieces and I share them among the three people, one slice would be left over. And then, I would have to share, share one and – we must share all the cake so that each person gets the same amount and that no cake can be left over.

IMAGE HAVING

Kenny: Soooo... So, I can share into six pieces.

R: Any reason in particular why you chose six?

Kenny: Because six is a multiple of three and that you can – if you divide it equally, six times four is twenty-four. And if you divide like twenty-four by three you would get six.

operating within the Image Having layer but applying an inappropriate image to the solving of a particular mathematical task.

After the researcher asks ‘Tell me why you chose four’, Kenny seems to fold back to the Image Making layer, where he engages in activities such as inspecting the diagrams and his previous partitioning approach. These actions appear to lead Kenny to reject the current image that he utilises for the number of partitions in each diagram

('Not four. Because when I, if I cut all the cakes into four pieces and I share them among the three people, one slice would be left over'). Kenny's folding back appears to be triggered by the researcher's question.

As part of his Image Making activities, Kenny also touches with his pencil the different parts of the diagrams. Following this, he appears to formulate a new image as to how to share the diagrams. His verbalisation 'So I can share into six pieces. Because six is a multiple of three and that you can – if you divide it equally, six times four is twenty-four. And if you divide like twenty-four by three you would get six' describes his new image. At this juncture, Kenny seemed to have returned to operating within the Image Having layer.

Application of multiple, appropriate images across tasks

Across the eight tasks, Harry utilised four different images for the number of partitions in the diagrams. These included:

Image 1: Halving
Image 2: Number of partitions in each diagram or multiple diagrams = Number of people sharing
Image 3: Partitions each of the diagrams into a number of pieces (e.g. ten) and distributes evenly. If there is a remainder (one or more partitions), he partitions this into the number of people (fifths)
Image 4: Number of partitions in multiple diagrams = Number of people sharing

The excerpts in Table 6 illustrate Harry's engagement with tasks T02–T08. After operating in the Image Making layer for T02, Harry crosses to operate within the Image Having layer in T03.

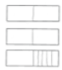
In contrast to David and Kenny, Harry used different images for partitioning across the tasks. By the end of the eight tasks, he did not derive a general solution for all partitive quotient tasks such as 'Number of partitions in each diagram = Multiple of the number of people sharing'. He appeared to have a suite of partitioning possibilities that he used as he engaged with the tasks. In this regard, his way of working was unique among the children.

Application of an appropriate but overgeneralised image across tasks

For the first solution of T03 (Table 7), Rebecca operated in the Image Having layer.

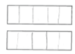
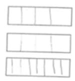
Rebecca is applying the image 'Number of partitions = Multiple of number of people' sharing (T03 excerpt, Table 7). In addition to using this image, Rebecca is utilising halving (and distributing one piece in each half to each person in the sharing situation). This is further confirmed in T05 based on her explanations. The excerpt related to T03 shows that this aspect of the image results in her rejection of the appropriate number of partitions, fifteen. T07 (sharing two pizzas among five people) illustrates

Table 6 Application of multiple appropriate images—example from Harry

T02: SHARE FOUR CAKES AMONG THREE CHILDREN  **Image Making** **Harry:** Miss everyone got, am,... [Looks at diagrams for an extended period and moves pencil in the air over the diagrams.] A whole cake, because is four, and it would still remain one [looks at the diagrams]. And you can just share it in thirds.
T03: SHARE THREE CAKES AMONG FIVE PEOPLE **Image Having** **Harry:** Okay Miss. There is three cakes and five children. Well, if we give everyone, each person one [referring to one whole cake] it would still remain two more people to get, and it will have no more. So, we can share them in halves.  [Partitions each of the three diagrams in half.] Then each child would get one-half. And it would still remain one, one more [refers to half of a cake/diagram]. So, you can share them in fifths. [Partitions one-half of a diagram into five.]
T04: SHARE FOUR CAKES AMONG SIX PEOPLE  **Harry:** There is four cakes and six children. You can share three of them in halves. [Partitions three diagrams in two.] And one in sixths. [Partitions one diagram into six.]

how this image of halving and distributing one piece in each half to each person in the sharing situation constrained the number of partitions that Rebecca considered to be appropriate, in a similar way that it did in T03. In this excerpt (T07), Rebecca first applies the halving approach to each of the two pizzas, which results in the creation of four sections. She then mentally partitions each half into five pieces, which is the number of people in the sharing situation. Rebecca's verbalisation 'Well if you share it in ten pieces, then half of ten is five. Five for one, five for another, then five for another and five for another, one child would stay without a piece' indicates this. Rebecca rejected sharing each of the two diagrams into ten partitions as an appropriate number

Table 6 (continued)

T05: SHARE TWO CAKES AMONG SEVEN PEOPLE  **Harry:** I would share one cake in sevenths. [Partitions the first diagram into seven.] **R:** Could you tell me why sevenths? **Harry:** Because there are seven children and then each child would get one piece. But it still remaining one more cake and you can still share it in seven. [Partitions the second diagram into seven.] Seven.
T06: SHARE THREE PIZZAS AMONG SIX PEOPLE  **Harry:** Yes, Miss. Miss is three pizzas and six child – and six people. Then you can cut, share them in halves. [Partitions each diagram in two.]
T07: SHARE TWO PIZZAS AMONG FIVE PEOPLE  **Harry:** Miss, you can share it in fifths. **R:** Please go ahead. Let me see what you mean. **Harry:** [Partitions the two diagrams in five.]
T08: SHARE THREE PIZZAS AMONG EIGHT PEOPLE  **Harry:** Miss, if you share, this one in fourths [taps on the first diagram]; this one in fourths [taps on the second diagram]; and this one in eighths [taps on the third diagram]. [Partitions two diagrams into four and one diagram into eight.]

of partitions in each diagram, although it was appropriate. When each diagram is partitioned into ten, each person would receive four pieces each instead of five, the number of people involved in the sharing situation. The application of the halving approach and distributing one piece in each half to the person in the sharing situation prevented Rebecca from making this observation. It is for this reason that in addition to classifying Rebecca's image as appropriate, the authors also categorised it as overgeneralised.

Table 7 Application of an appropriate, overgeneralised image—example from Rebecca

T03: SHARE THREE CAKES AMONG FIVE PEOPLE

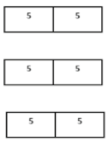
IMAGE HAVING

Rebecca: [Looks at diagrams ~five seconds]. Ten, [points at the first diagram], ten [points at the second diagram], ten [points at the third diagram]. [Looks ahead]. Hmmm.

Five for one child, [points at the first half of the first diagram] five for another child [points at the second half of the first diagram].

Five [points at the first half of the second diagram] for one child, five [points at the second half of the second diagram] for another child. [Looks ahead].

And five [points at the first half of the third diagram] for another child and then 1, 2, 3, 4, 5 [Gestures Back and forth from the fourth diagram to the children while counting softly aloud].



Note: Rebecca did not partition the diagrams provided. This diagram is an illustation of the authors' interpretation of Rebecca's verbalisations. '5' represents the number of partitions in the section.

R: How else can you share the three cakes among five children?

Rebecca: Fifteen, fifteen, fifteen [points at each of the three diagrams in turn.]

IMAGE MAKING

[Stares ahead intently and mouth moves.]

Table 7 (continued)

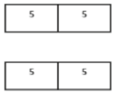
Twenty, twenty, twenty [points at each of the three partitions in turn]. You have ten [points to the first half of the first diagram], ten [points to the second half of the first diagram], ten [points to the first half of the second diagram], ten [points to the second half of the second diagram], ten [points to the first half of the third diagram] and (?). **R:** I heard you talked about fifteen at first. **Rebecca:** Yes. **R:** Then you changed it to twenty. Can you tell me why you thought fifteen would work? **Rebecca:** Well because if I – I choose ten, now I choose fifteen, the times table. I thought fifteen would work, if I put fifteen, then fifteen, then fifteen [points to each of the three diagrams in turn]. Then, let's see [stares ahead]. **Fifteen divided by two... [shakes head negatively]. Fifteen divi– you cannot get fifteen divided by two**. So if you put fifteen, fifteen, fifteen [Points at each of the three diagrams in turn] then each child would have to get– **R:** Could you explain why you are dividing by two? **Rebecca:** Because this cake, two child, two children would get, wait, if you cut this cake in equal length, two children would get from this cake and two from this cake and one from this cake, and then you would just give the rest of the children the other half of the cake.
T05: SHARE TWO CAKES AMONG SEVEN PEOPLE **IMAGE HAVING** **Rebecca:** Fourteen. You have to share the cake in fourteen pieces [looks at the diagrams]. **R:** Why did you choose fourteen?

Table 7 (continued)

Rebecca: 'Cause when you said two cakes among seven children, I was thinking in my mind two sevens a fourteen, so I choose fourteen. So, you share the cake in fourteen pieces. You cut the cake in half [motions a half line in the centre of the 2 diagrams]. Then one [points in the first half of the first diagram then to the picture of the children] for each child and the other half, one for each child [points in the second half of the first diagram then to the picture of the children]. And then in this cake [points on the second diagram] you cut it in half. One for each child [points in the first half of the second diagram then to the picture of the children] and one for each child [points in the second half of the second diagram then to the picture of the children].

T07: SHARE TWO PIZZAS AMONG FIVE PEOPLE

IMAGE MAKING



Rebecca: Okay. Fifteen. If you cut the cake in fifteen pieces, then... you... you cut it in fifteen pieces.

R: How did you choose fifteen?

Rebecca: Fifteen?

R: Aha.

Rebecca: I looked at five. So, I was checking ten, let's see if is ten. So, I looked at ten then I say to myself, I count in my head, and I say ten cannot work so I go on to fifteen and then I would get fifteen.

R: Why do you think ten could not work?

Rebecca: Well, if you share it in ten pieces then half of ten is five. Five for one, five for another, then five for another and five for another, one child would stay without a piece.

Note: Rebecca did not partition the diagrams provided. This diagram is an illustation of the authors' interpretation of Rebecca's verbalisations. '5' represents the number of partitions in the section.

Discussion

The findings show that when operating within the Image Having layer of the Pirie–Kieren model, children formed, held and used a range of different images. We identified five qualitative distinctions: the same appropriate image applied across trials and tasks, an appropriate image applied for a single trial (first solution) before reverting to an inappropriate image, application of an inappropriate image across trials, multiple appropriate images across tasks, appropriate but overgeneralised images across tasks. These images depict varied facets of children's developing understanding relating to partitioning within the context of novel partitive quotient tasks. Below, we outline key insights emerging from our findings, as they relate to theory, children's engagement with partitive quotient tasks and education. We conclude with methodological reflections and suggestions for future research.

Theoretical insights

This paper elucidates and elaborates the notion of images and Image Having in that it provides a portrait for these notions that goes beyond dichotomic characterisations of images as appropriate/inappropriate or complete/incomplete, bringing to the fore that, for individual children, an 'appropriate' image may be characteristically diverse as could be a partial one.

The Image Having layer is a critical component of the Pirie–Kieren model in that it is the first point of abstraction which is essential to learners' mathematical development (Pirie & Kieren, 1994; Sarama & Clements, 2016). This study offers significant insights into what occurs as children create personal meaning and make sense of novel mathematical content (Steffe & Olive, 2010). Bobis and Way (2018) point out that child-created representations are a key constituent of moving a child along the continuum of mathematical understanding and development. The findings provide concrete examples of this as the children, operating within the Image Having layer for the given task, spoke, gestured and partitioned diagrams (external representations), forming images that were gradually clarified, bolstered and consolidated. We do not suggest that the types of images presented here are representative of how children in general engage with the partitioning aspect of partitive quotient tasks or any topic in mathematics, as other qualitative characteristics of child-created images could arise from a different sample and for different mathematical contents. Further research could examine this. Rather, this study aims to advance theoretical understanding by zooming into a specific layer of the Pirie–Kieren model and elucidating its varied and dynamic nature, thus supporting the view of knowledge development and sense-making as a dynamic process that underpins children's mathematical activity (Cifarelli, 1998; MacDonald, 2013).

Insights into children's partitioning images within partitive quotient tasks and implications for education

Both David and Kenny appeared to produce appropriate partitioning images. They created meaning by linking the constituent elements of the task (number of people sharing) to the number of partitions made. They then extended this meaning, in some cases, to note that the number of partitions could also be a multiple of the number of people in the sharing situation. In contrast to David's image that was formed and used consistently across the eight tasks, Kenny's initially formed, appropriate image was temporarily replaced with an inappropriate image before he 'recovered' and returned to his appropriate image for the number of partitions. This is consistent with Siegler (2006) who points out that 'cognitive change involves regression as well as progression, odd transitional states that are present only briefly but that are crucial for the changes to occur… and many other surprising features' (p. 468). Bobis and Way (2018) further assert that 'investigating student-created representations may yield insights into their mathematical thinking… and may more closely reflect their level of conceptual understanding than their responses to imposed conventional representations' (p. 58). We add, in alignment with Pirie and Kieren (1994), that the sense that children make and express as they reflect on external representations, such as their drawings, as well as their verbalisations and gestures, can further deepen our understanding of children's mathematical thinking. The observation related to Kenny's temporary regression also highlights the power of the microgenetic research design to capture knowledge development as it is occurring. A different methodological design would likely not have captured this in-the-moment occurrence. This is significant for education too, as in teaching situations, after a child provides an appropriate answer, there is potentially little need to probe further as it may be perceived that knowledge construction is complete (Empson & Jacobs, 2008).

Harry used four different images for partitioning. Researchers (e.g. Ashlock, 2002; Kazak et al., 2015) have pointed out that a critical component of abstraction, which allows an individual to gain an image for all problems of a particular type, is finding commonalities among the ideas/images that relate to a concept. We posit that if the images held by an individual in relation to a task are too distinct and diverse, key properties may go unobserved in a sequence of problem-solving sessions and may ultimately inhibit an individual from finding a general solution, for all similar problems.

Ashlock (2002) also notes that incomplete or inappropriate images of mathematical concepts/topics occur when learners overgeneralise and draw conclusions before there is adequate data at hand, after having engaged with only a few problems of a particular type. Alternatively, he opines learners may overspecialise as they develop their understanding of a concept, and therefore, the resulting image in such cases restricts its applicability to problems of a particular type only. Both these explanations appear to be applicable to Rebecca's case (Table 5). In T03, Rebecca began applying the halving approach to each item being shared. While this image was appropriate for that task and several subsequent tasks, in T07, the non-applicability of this approach to the current problem became evident to Rebecca. This resulted in folding back and amendments being made to the images held.

Overgeneralising and overspecialising have been observed previously (Ashlock, 2015), including where children often use their natural number knowledge to engage with fraction computation (Siegler & Lortie-Forgues, 2015; Van Hoof et al., 2017). The overgeneralisation explanation, however, has not been previously applied to the partitioning aspect of partitive quotient problems. Gabriel et al. (2013) point out that children are frequently exposed to the notion of a half very early in life and in formal schooling, and 'this limited vision of fractions seems to generate difficulties when it comes to generalization' (p. 9). This might explain why Rebecca applied this knowledge to the solving of partitive quotient problems.

Consistent with Martin (2008), we found that when an individual develops incomplete and/or inappropriate images and reflects on these, restructuring of these constructs can occur or new images can be formed to replace the existing ones through the process of folding back. This finding also aligns with Goldin and Shteingold (2001) who state that 'the apparent limitations in some children's understanding are not intrinsic. Rather they are the result of internal representation systems that are only partially developed' (p. 3).

The findings of this study have particular implications for education which increasingly embraces a blueprint for instruction that prioritises and responds to children's mathematical ideas, strategies and thinking (Jacobs & Empson, 2016). MacDonald (2013) encourages educators to 'embrace children's multifaceted ways of knowing' (p. 65) as they draw on their varied experiences and understandings to make meaning of the mathematical content they encounter. In line with this, our findings suggest that teachers may need to continually adjust 'in response to children's content-specific thinking' (Jacobs & Empson, 2016, p. 185). This is done by probing students, using questioning and varied tasks that are tailored and suited to the different images that individual learners develop in the context of partitive quotient tasks. This may be challenging in a classroom setting, and therefore, future research could explore the components of practice or instructional approaches that would facilitate this.

Concluding remarks

We note, similar to Fritz et al. (2019), that processes of learning and growth of mathematical understanding are complex for both learners and teachers of mathematics in any setting. The research design adopted in this study was powerful in revealing nuanced differences and changes over multiple problem-solving sessions with each child. The small number of research participants may be considered a limitation. However, a small number of participants is a usual characteristic of qualitative microgenetic research that includes dense observation of qualitative changes in individual children's behaviour (Kuhn, 1995). Therefore, we deemed it to be appropriate for the present analysis, where the aim was to explore and elaborate a particular theoretical notion. Future research could explore nuances in the nature and range of images that children create when solving partitive quotient tasks, with a larger sample. This may reveal variations that have not been captured in the current data. We acknowledge that the behaviours observed here and interpretation of findings are situated within the particular interview interactions and a specific type of tasks. Further research could explore how the qualitative variations revealed in this study compare or contrast with images identified within different mathematical topics and for children of different ages or educational levels.

The examples and deep insights into the child-created images that this study revealed in relation to the partitive quotient fraction sub-construct are significant for pedagogy, as there are distinct implications for selecting appropriate pedagogical approaches that can support and extend individual children's distinctive and developing understandings, when they engage with multiple tasks associated with a specific concept (Bobis & Way, 2018). By zooming into the broad notion of the Image

Having layer in the Pirie–Kieren model, and analysing children's images across multiple task trials, the findings have theoretical significance too. This is because they bring to the fore nuanced qualitative variations in the nature of child-created images and thus elucidate the heterogeneity and complexity of the Image Having layer of the Pirie–Kieren theory.

Acknowledgements We are grateful to all children who participated in the study.

Funding The research was supported by a grant from the Commonwealth Scholarship Commission in the UK.

Data Availability Raw data not available due to ethical restrictions. The participants of this study did not give written consent for their data to be shared publicly.

Declarations

Ethical approval The research reported in this paper was conducted while the first author was affiliated with the University of Southampton, UK, before moving to her current institution, the University of the West Indies, Mona Campus, Jamaica. Ethics approval was obtained from the Faculty of Social Sciences Ethics Committee, University of Southampton, UK (reference number: 13550).

Conflict of interest The authors declare no competing interests.

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