

Algebraic criteria for stable diffeomorphism of spin 4-manifolds

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ABSTRACT. We study closed, connected, spin 4-manifolds up to stabilisation by connected sums with copies of $S^2 \times S^2$. For a fixed fundamental group, there are primary, secondary and tertiary obstructions, which together with the signature lead to a complete stable classification. The primary obstruction exactly detects $\mathbb{C}\mathbb{P}^2$ -stable diffeomorphism and was previously related to algebraic invariants by Kreck and the authors.

In this article we formulate conjectural relationships of the secondary and tertiary obstructions with algebraic invariants: the secondary obstruction should be determined by the (stable) equivariant intersection form and the tertiary obstruction via a τ -invariant recording intersection data between 2-spheres, with trivial algebraic self-intersection, and their Whitney discs.

We prove our conjectures for the following classes of fundamental groups: groups of cohomological dimension at most 3, right-angled Artin groups, abelian groups, and finite groups with quaternion or abelian 2-Sylow subgroups.

We apply our theory to give a complete algebraic stable classification of spin 4-manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}/2$.

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Introduction

Two closed, connected, smooth 4-manifolds are *stably diffeomorphic* if they become diffeomorphic after taking connected sums with finitely many copies of $S^2 \times S^2$, where we allow different numbers of copies for the two manifolds. An analogous notion of *stable homeomorphism* applies in the topological category. Kreck [Kre99, Theorem C] reduced these classification problems to computations of bordism groups, which we review in Section 1.2 in the spin case. He showed that two spin 4-manifolds N_1 and N_2 with fundamental group π are stably diffeomorphic if and only there are maps $N_i \rightarrow B\pi$ and choices of spin structures such that the N_i represent equivalent elements of the bordism group $\Omega_4^{\text{Spin}}(B\pi)$. While this is extremely useful, explicitly calculating invariants of spin bordism groups is highly geometric, and usually intractable.

In this article we introduce new *algebraic* invariants for spin 4-manifolds. The invariants are independent of category (smooth or topological) as one might expect in the stable setting. We shall focus on the smooth category, and explain why our classification also applies in the topological category in Section 2.2.

Theorem 1.1. *Fix a group π with finite 3-dimensional classifying space, or an abelian group, a right angled Artin group, or a finite group π with abelian or quaternion 2-Sylow subgroup.*

One can decide whether two closed, connected, spin, smooth 4-manifolds N_1 and N_2 are stably diffeomorphic over their fundamental group π by computing, for all possible spin structures α_i on N_i , the following algebraic invariants on a 1-skeleton sum $M := (N_1, \alpha_1) \#_1 (N_2, -\alpha_2)$:

- (0) *the signature $\sigma(M) = \sigma(N_1) - \sigma(N_2)$ of M , an integer;*
- (1) *the extension class of the stable $\mathbb{Z}\pi$ -module $\pi_2(M)$, a ‘linear’ invariant;*
- (2) *the stable intersection form λ_M on $\pi_2(M)$, a $\mathbb{Z}\pi$ -valued ‘quadratic’ form;*
- (3) *the Kervaire-Milnor invariant τ_M , a $\mathbb{Z}/2$ -valued ‘cubical’ refinement of λ_M .*

There are handle decompositions of N_i with diffeomorphic 1-skeleta (the union of the 0- and 1-handles). To obtain a 1-skeleton sum $(N_1, \alpha_1) \#_1 (N_2, -\alpha_2)$, remove the 1-skeleta, and identify the boundaries by a spin structure preserving diffeomorphism. The stable diffeomorphism class turns out not to depend on the choices.

The invariants in Theorem 1.1 precisely detect the steps in an obstruction theory to decide whether M is trivial in the spin bordism group $\Omega_4^{\text{Spin}}(B\pi)$. While the signature is an invariant of each N_i , it remains an open problem to formulate the other invariants solely in terms of the N_i , in such a way that they detect stable diffeomorphism. Our solution is to use the 1-skeleton sum described above. We shall give precise definitions of the invariants in points (1), (2), and (3) in Sections 1.3.1 to 1.3.3.

The linear invariant in (1) exactly detects $\mathbb{C}\mathbb{P}^2$ -stable diffeomorphism of closed, connected 4-manifolds. This was the focus of [KPT18], so in this article we shall focus on the other invariants.

The algebraic obstructions that we will discuss are generalisations of the ones studied in [KLPT17] by the present authors together with Land. There the fundamental group was that of a closed, oriented, aspherical 3-manifold, and we obtained our results by computing the obstructions on explicit models for the stable diffeomorphism classes. The current investigation proceeds more abstractly, as suggested for the secondary obstruction in the third author's thesis [Tei92].

1.1. The stable classification for fundamental group $\mathbb{Z} \times \mathbb{Z}/2$

As a concrete application, we provide an intrinsic stable classification for spin 4-manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}/2$. We will recall in Lemma 7.2 that $\Omega_4^{\text{Spin}}(B(\mathbb{Z} \times \mathbb{Z}/2))$ is isomorphic to $16\mathbb{Z} \oplus \Omega_2^{\text{Pin}^-} \cong 16\mathbb{Z} \oplus \mathbb{Z}/8$, where the isomorphism is given by the signature and the Brown invariant [Bro72] of a $\mathbb{Z}/4$ -valued quadratic enhancement of the $\mathbb{Z}/2$ -valued intersection form of a surface, determined by the Pin^- structure. By Lemma 7.3, changing the spin structure acts on the $\mathbb{Z}/8$ summand by -1 . Thus by Kreck's theorem, smooth, closed, connected, spin 4-manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}/2$ are stably classified by their signature and the Brown invariant in $(\mathbb{Z}/8)/\pm 1$. In particular, there are 5 stable diffeomorphism classes with zero signature that have Brown invariant $0, \pm 1, \pm 2, \pm 3$ and 4. We apply our classification results to obtain an intrinsic description of the Brown invariant in this case. Let $\pi := \mathbb{Z} \times \mathbb{Z}/2 = \langle T, t \mid [T, t], T^2 \rangle$.

Theorem 1.2. *For a smooth, closed, connected, spin 4-manifold with fundamental group π , the Brown invariant $\beta(M) \in \mathbb{Z}/8$ is detected as follows:*

- (1) $\beta(M) = \pm 1$ if and only if $\pi_2(M)$ is stably free and λ_M is stably isometric to a form induced up to $\mathbb{Z}\pi$ from a form over \mathbb{Z} ;
- (2) $\beta(M) = \pm 3$ if and only if $\pi_2(M)$ is stably free but λ_M is not stably induced from a form over \mathbb{Z} ;
- (3) $\beta(M) = \pm 2$ if and only if $\pi_2(M)$ is not stably free and λ_M is odd;
- (4) $\beta(M) \in \{0, 4\}$ if and only if $\pi_2(M)$ is not stably free and λ_M is even.

In the last case, $\pi_2(M)$ is stably isomorphic to $I \oplus I^*$, where $I := I\pi$ is the augmentation ideal, and there exists such a stable isomorphism for which the restriction of λ_M to I^* is trivial. Let $x \in \pi_2(M)$ be the image of $\varphi \in I^*$ under such an isomorphism, where $\varphi \in I^*$ is determined by $\varphi(1-t) = 1+T$ and $\varphi(1-T) = 0$. Then

- (5) $\beta(M) = 0$ if and only if $\tau_M(x) = 0$, where τ_M is the Kervaire–Milnor invariant.

Remark 1.3. If $\pi_2(M)$ is stably free, we can determine whether the first case holds by computing the signature $\sigma(M)$, then asking whether λ_M becomes stably hyperbolic after subtracting $\sigma(M)$ copies of $\mathbb{Z}\pi \otimes_{\mathbb{Z}} E_8$. In particular, if $\sigma(M) = 0$, then the first case is equivalent to λ_M being stably hyperbolic.

The Kervaire–Milnor invariant is also detected by the stable equivariant intersection form for 2-dimensional [HKT09, KPT21] and 3-dimensional groups [KLPT17, HH19]. This leads us to the following question.

Question 1.4. *Is the difference between $\beta(M) = 0$ and $\beta(M) = 4$ also detected by the stable intersection form?*

1.2. Review of stable classification

As mentioned above, Kreck [Kre99, Theorem C] showed that two closed, connected, spin 4-manifolds with fundamental group π are stably diffeomorphic if and only if there are choices of spin structures and identifications of the fundamental groups with π , giving rise to equal elements in the bordism group $\Omega_4^{\text{Spin}}(\pi) := \Omega_4^{\text{Spin}}(B\pi)$, as we shall explain in Section 2.1. To understand this group of bordism classes of pairs (M, c) , where M is a closed 4-manifold with spin structure and $c: M \rightarrow B\pi$ classifies the universal cover, we consider the Atiyah-Hirzebruch spectral sequence (AHSS) computing $\Omega_4^{\text{Spin}}(B\pi)$ in terms of $E_{p,q}^2 = H_p(B\pi; \Omega_q^{\text{Spin}})$. The AHSS gives rise to a filtration whose iterated graded quotients are

$$\mathbb{Z} \cong \Omega_4^{\text{Spin}} \underbrace{\subseteq}_{E_{2,2}} F_2 \underbrace{\subseteq}_{E_{3,1}} F_3 \underbrace{\subseteq}_{E_{4,0}} \Omega_4^{\text{Spin}}(\pi).$$

The first isomorphism is determined by the signature. More precisely, it is given by the signature divided by 16 in the smooth case and the signature divided by 8 in the topological case; the closed topological E_8 -manifold cannot be smoothed. This divisibility is the only difference between the stable classification of smooth and topological spin 4-manifolds and therefore we can ignore it in the sequel. The signature extends to the entire group $\Omega_4^{\text{Spin}}(\pi)$ and so we reduce our study to $\tilde{\Omega}_4^{\text{Spin}}(\pi)$, the kernel of the signature map, which is independent of category. The AHSS then reduces to a shorter filtration

$$E_{2,2} \underbrace{\subseteq}_{E_{3,1}} F \underbrace{\subseteq}_{E_{4,0}} \tilde{\Omega}_4^{\text{Spin}}(\pi),$$

where the subgroup F consists of bordism classes represented by signature zero 4-manifolds M with spin structure such that $c: M \rightarrow (B\pi)^{(3)}$ lands in the 3-skeleton of the classifying space $B\pi$. Similarly, the smallest filtration term $E_{2,2}$ is represented by elements (M, c) with $c: M \rightarrow (B\pi)^{(2)}$. Since the $E_{p,q}^2$ term of the spectral sequence is $H_p(\pi; \Omega_q^{\text{Spin}})$, the $E_{p,q}^\infty$ -terms are as follows:

- $E_{2,2}^\infty := E_{2,2}^\infty = H_2(\pi; \mathbb{Z}/2) / \text{im}(d_2, d_3)$;
- $E_{3,1}^\infty := E_{3,1}^\infty = H_3(\pi; \mathbb{Z}/2) / \text{im}(d_2)$;
- $E_{4,0}^\infty := E_{4,0}^\infty = \ker(d_2: H_4(\pi; \mathbb{Z}) \rightarrow H_2(\pi; \mathbb{Z}/2))$.

Following [Tei92], we obtain the primary invariant $\text{pri}(M) = c_*[M] \in E_{4,0}$, the secondary invariant $\text{sec}(M) \in E_{3,1}$ and the tertiary invariant $\text{ter}(M) \in E_{2,2}$. The challenge is to recast these obstructions in terms of algebraic topological data of the 4-manifold. As a preliminary observation, the equivariant intersection form λ_M on $\pi_2(M)$ changes by orthogonal sum with a hyperbolic form (on a free $\mathbb{Z}\pi$ -module of rank 2) if we add $S^2 \times S^2$ to M . So the signature $\sigma(M) \in \mathbb{Z}$ is a stable invariant, as is the isomorphism class of $\pi_2(M)$ up to stabilisation by free $\mathbb{Z}\pi$ -modules, and the isometry class of λ_M up to stabilisation with hyperbolic forms on free modules.

1.3. Translating the bordism invariants into algebra

Let us discuss the three obstructions in order of appearance.

- (1) The *Primary Obstruction Theorem 1.6* [KPT18] reinterprets the invariant $\mathbf{pri}(M) = c_*[M]$ as the extension class of a short exact sequence of $\mathbb{Z}\pi$ -modules whose central module is stably isomorphic to $\pi_2(M)$. In loc. cit. we gave examples of various fundamental groups for which $\mathbf{pri}(M)$ is and is not determined by the stable isomorphism class of the $\mathbb{Z}\pi$ -module $\pi_2(M)$.
- (2) We say that a group π has the *Secondary Property* if $\mathbf{sec}(M)$ is detected by $\lambda_{M,s}$ for all spin 4-manifolds M with fundamental group π . Here $\lambda_{M,s}$ is the restriction of λ_M to a certain summand of $\pi_2(M)$. This summand arises as the image of a splitting s of the extension in (1), which exists if and only if $\mathbf{pri}(M) = 0$. We shall show in Section 1.3.2 that if $\mathbf{sec}(M)$ vanishes then there are choices of a splitting for which λ_M vanishes on the image.
- (3) We say that a group π has the *Tertiary Property* if $\mathbf{ter}(M)$ is detected by the Kervaire-Milnor invariant $\tau_{M,s}$ for all spin 4-manifolds M with fundamental group π . Here $\tau_{M,s}$ records intersection data between 2-spheres (with trivial algebraic self-intersection) and their Whitney discs. It is the Kervaire-Milnor invariant τ_M from [FQ90], as corrected by [Sto94] and [ST01], restricted to the image of a splitting in (1) on which λ_M vanishes. This cubical intersection invariant is defined if and only if $\mathbf{pri}(M) = 0 = \mathbf{sec}(M)$, and it does not depend on the choice of splitting, as we will discuss in Section 1.3.3.

We will state these properties of a group in detail as Secondary Property 1.11 and Tertiary Property 1.15. Then the following statement will be made precise.

Theorem 1.5. *Consider a group π with finite 3-dimensional classifying space $B\pi$, or an abelian group, a right angled Artin group, or a finite group π with abelian or quaternion 2-Sylow subgroup. Then π has the Secondary and Tertiary properties. As a consequence, Theorem 1.1 holds.*

Next we give more detailed descriptions of the obstructions and the statements of our theorems.

1.3.1. The algebraic primary obstruction. Let M be a closed, connected, oriented 4-manifold with fundamental group π . There is a map $c: M \rightarrow B\pi$ inducing the identification on fundamental groups, that is well defined up to based homotopy. The primary obstruction $\mathbf{pri}(M)$ corresponds to the edge homomorphism in the AHSS and is given by the image $c_*[M]$ of the fundamental class $[M] \in H_4(M; \mathbb{Z})$ in $H_4(B\pi; \mathbb{Z}) = H_4(\pi; \mathbb{Z})$. We will build on the following result.

Theorem 1.6 (Primary obstruction theorem [KPT18, Theorem 1.8]). *Let K be a finite connected 2-complex with fundamental group π . There is an isomorphism*

$$\mathrm{Ext}_{\mathbb{Z}\pi}^1(H^2(K; \mathbb{Z}\pi), \pi_2(K)) \cong H_4(\pi; \mathbb{Z})$$

mapping $\mathbf{pri}(M) = c_*[M]$ to an extension

$$0 \longrightarrow \pi_2(K) \longrightarrow (\mathbb{Z}\pi)^r \oplus \pi_2(M) \longrightarrow H^2(K; \mathbb{Z}\pi) \longrightarrow 0.$$

Recall that a finite presentation of π gives a 2-complex K as above by using a single 0-cell, one 1-cell for each generator and one 2-cell for each relation. Using Tietze transformations, we showed that the choice of presentation disappears when

considering the extension group above. To make this precise, we need the following notion and the next lemma.

Definition 1.7. We say that two R -modules P and Q are *stably isomorphic*, and write $P \cong_s Q$, if there exist non-negative integers p and q such that $P \oplus R^p \cong Q \oplus R^q$.

Lemma 1.8. [HAM93, (40)] *Let K_i be finite connected 2-complexes with fundamental group π . Then there exist $k_i \in \mathbb{N}_0$ such that $K_1 \vee \bigvee_{r=1}^{k_1} S^2 \simeq K_2 \vee \bigvee_{r=1}^{k_2} S^2$. In particular, $\pi_2(K_1) \cong_s \pi_2(K_2)$ and $H^2(K_1; \mathbb{Z}\pi) \cong_s H^2(K_2; \mathbb{Z}\pi)$.*

In [KPT18, Lemma 5.11] we checked that wedge sum with S^2 does not change the extension group in 1.6 and that a homotopy self-equivalence of K inducing the identity on $\pi_1(K)$ determines the identity map on this extension group [KPT18, Lemma 5.12]. We concluded that the image of $\mathbf{pri}(M)$ in this extension group is a well-defined algebraic invariant of M in a group that depends only on the group π .

1.3.2. The algebraic secondary obstruction. Consider the equivariant intersection form

$$\lambda_M: \pi_2(M) \times \pi_2(M) \rightarrow \mathbb{Z}\pi$$

of a closed, connected, oriented 4-manifold M with fundamental group π . It is given by either counting geometric intersections (with signs and group elements) between transverse 2-spheres in M or by identifying $\pi_2(M) \cong H_2(M; \mathbb{Z}\pi)$ via the Hurewicz homomorphism and setting

$$\lambda_M(x, y) := \langle PD^{-1}(y), x \rangle \quad \text{for } x, y \in H_2(M; \mathbb{Z}\pi),$$

where $PD: H^2(M; \mathbb{Z}\pi) \rightarrow H_2(M; \mathbb{Z}\pi)$ is the Poincaré duality isomorphism. The intersection form λ_M is *sesquilinear*, meaning it is additive in each variable and $\lambda_M(ax, by) = a\lambda_M(x, y)\bar{b}$ holds for all $a, b \in \mathbb{Z}\pi$, where the involution $a \mapsto \bar{a}$ on the group ring $\mathbb{Z}\pi$ is determined by $\bar{g} := g^{-1}$ for $g \in \pi$. Note that λ_M is frequently singular. In fact, its kernel and cokernel are determined by the fundamental group π because by the universal coefficient spectral sequence, its adjoint fits into the exact sequence

$$\text{(UCSS)} \quad 0 \rightarrow H^2(\pi; \mathbb{Z}\pi) \rightarrow \pi_2(M) \xrightarrow{\lambda_M^{\text{ad}}} \text{Hom}_{\mathbb{Z}\pi}(\pi_2(M), \mathbb{Z}\pi) \rightarrow H^3(\pi; \mathbb{Z}\pi) \rightarrow 0.$$

The intersection form λ_M is also hermitian in the following sense. The dual λ^* of a sesquilinear form λ is given by

$$\lambda^*(x, y) := \overline{\lambda(y, x)}.$$

and λ is called *hermitian* if $\lambda^* = \lambda$. We say that λ is *even* if $\lambda = q + q^*$ for some sesquilinear form q on $\pi_2(M)$.

Remark 1.9. Such a form q , evaluated on the diagonal $x \mapsto \hat{q}(x) := q(x, x)$, defines a *quadratic refinement* $\hat{q}: \pi_2(M) \rightarrow \mathbb{Z}\pi$ of λ_M in the sense that for $x, y \in \pi_2(M)$ we have

$$\text{(qr)} \quad \lambda_M(x, x) = q(x, x) + q^*(x, x) = \hat{q}(x) + \overline{\hat{q}(x)}.$$

It is important to point out that for \widetilde{M} spin, every class $x \in \pi_2(M)$ can be represented by a generically immersed 2-sphere $f: S^2 \looparrowright M$ with trivial normal bundle. Since $w_2(x) = 0$, the Euler number e of the normal bundle to f is even and hence can be changed to zero by a non-regular homotopy of f , performing

$e/2$ cusp homotopies. As a consequence, the intersection form λ_M has a quadratic refinement

$$\mu_M: \pi_2(M) \rightarrow \mathbb{Z}\pi / \{a - \bar{a} \mid a \in \mathbb{Z}\pi\}$$

given by counting self-intersections (with group elements and signs) of a generic immersion $f: S^2 \looparrowright M$ with $e(f) = 0$. The expression $\mu_M(x) + \overline{\mu_M(x)}$ can be lifted canonically to $\mathbb{Z}\pi$ and then equals $\lambda_M(x, x)$. It is not hard to show algebraically that μ_M , if it exists, is completely determined from λ_M by this property. As a consequence, we shall not keep μ_M in our notation but will remember that λ_M is a *weakly even hermitian* form, meaning by definition that a quadratic refinement exists in this sense.

Remark 1.10. Our normalisation of $\mu(x)$ at $1 \in \pi$ implies that $\lambda_M(x, x) = \mu(x) + \overline{\mu(x)}$, which can be useful for a relation to quadratic L -theory. It works the Euler number $e(\nu\Sigma)$ is even for every Σ , or equivalently if $w_2(\widetilde{M}) = 0$. Using cusp homotopies it is always possible to change Σ so that the self-intersection number of Σ at 1 is trivial, even if the universal covering of M is not spin. This gives an element $\mu'(x) \in \mathbb{Z}\pi/g \sim g^{-1}$ which again only depends on the homotopy class of x . Using this convention, $\mu'(x)$ is an obstruction to representing x by an embedded sphere. In the setting of this paper, our 4-manifolds are spin and we usually assume that $\lambda_M(x, x) = 0$. In that case, the two conventions agree and $\mu(x) = \mu'(x) = 0$.

In general, weakly even forms are not even. If $\pi_2(M)$ happens to be a free $\mathbb{Z}\pi$ -module on an ordered basis $\{e_i\}$ then weakly even forms λ are even, as can be seen by lifting $\mu(e_i)$ to $q(e_i, e_i) \in \mathbb{Z}\pi$ and setting $q(e_i, e_j) := \lambda(e_i, e_j), q(e_j, e_i) = 0$ for $i < j$. But as we shall see, for many spin 4-manifolds M the intersection form λ_M is not even. To make this precise, consider a closed, connected spin 4-manifold M with fundamental group π and $\text{pri}(M) = 0$. In the notation of Theorem 1.6, where we chose a 2-complex K , also choose a splitting

$$s = (s_1, s_2): H_K := H^2(K; \mathbb{Z}\pi) \rightarrow (\mathbb{Z}\pi)^r \oplus \pi_2(M)$$

of the short exact sequence $0 \rightarrow \pi_2(K) \rightarrow (\mathbb{Z}\pi)^r \oplus \pi_2(M) \rightarrow H^2(K; \mathbb{Z}\pi) \rightarrow 0$. Consider the sesquilinear form $\lambda_{M,s}: H_K \times H_K \rightarrow \mathbb{Z}\pi$ on H_K induced by s_2 , $\lambda_M(s_2(-), s_2(-))$. We write $\text{Sesq}(H)$ for the group of $\mathbb{Z}\pi$ -sesquilinear forms on the $\mathbb{Z}\pi$ -module H . Sending a form to its hermitian conjugate induces a $\mathbb{Z}/2 = \langle T \rangle$ action on $\text{Sesq}(H)$, and we consider the corresponding Tate group

$$\begin{aligned} \widehat{H}^0(\text{Sesq}(H)) &:= \widehat{H}^0(\mathbb{Z}/2; \text{Sesq}(H)) = \ker(1 - T) / \text{im}(1 + T) \\ &= \{\text{Hermitian forms}\} / \{\text{even forms}\}. \end{aligned}$$

In Definition 3.18 we will construct, purely algebraically, a homomorphism

$$A_K: H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H_K)).$$

For every map $\varphi: K \rightarrow K'$ inducing the given identification with π on fundamental groups, the diagram

$$\begin{array}{ccc} H_3(\pi; \mathbb{Z}/2) & \xrightarrow{A_K} & \widehat{H}^0(\text{Sesq}(H_K)) \\ & \searrow A_{K'} & \downarrow \varphi_* \\ & & \widehat{H}^0(\text{Sesq}(H_{K'})) \end{array}$$

commutes by Lemma 3.19. We obtain a group

$$\text{Tate}(\pi) := H_3(\pi; \mathbb{Z}/2) / \ker(A_K) \cong \text{im } A_K,$$

which is independent of the choice of K . We write the projection as

$$A_\pi: H_3(\pi; \mathbb{Z}/2) \twoheadrightarrow \text{Tate}(\pi).$$

Although it will not play a role for this article, note that $\text{im } A_K$ lies in the subgroup of weakly even forms on H_K by Lemma 3.23. One might hope to identify $\text{Tate}(\pi)$ with a particular class of such weakly even forms on H_K .

We will show in Section 3.4 that for every spin 4-manifold M with fundamental group π and $\text{pri}(M) = 0$, a splitting $s: H_K \rightarrow (\mathbb{Z}\pi)^r \oplus \pi_2(M)$ determines a map $f_s: M \rightarrow (B\pi)^{(3)}$, which in turn determines an element $\text{sec}(M, f_s) \in H_3(\pi; \mathbb{Z}/2)$, lifting the element $\text{sec}(M) \in H_3(\pi; \mathbb{Z}/2) / \text{im}(d_2)$. We will show that the two elements $\lambda_{M,s}$ and $A_\pi(\text{sec}(M, f_s))$ agree in $\text{Tate}(\pi) \subseteq \widehat{H}^0(\text{Sesq}(H_K))$.

Secondary Property 1.11. *We say that a group π has the Secondary Property if for all closed, connected spin 4-manifolds M with fundamental group π and $\text{pri}(M) = 0$, $\lambda_{M,s} \in \text{Tate}(\pi)$ is independent of the splitting s and $\text{sec}(M)$ is determined by $\lambda_{M,s} \in \text{Tate}(\pi)$.*

The Secondary Property of a group π follows from the purely algebraic assertion that the differential d_2 in the AHSS makes the following sequence exact:

$$(\text{sec}_\pi) \quad H_5(\pi; \mathbb{Z}) \xrightarrow{d_2} H_3(\pi; \mathbb{Z}/2) \xrightarrow{A_\pi} \text{Tate}(\pi).$$

In Sections 6.1 and 6.2 we will prove this exactness (and hence the Secondary Property) for the following classes of groups, proving the secondary part of Theorem 1.5.

Theorem 1.12. *The Secondary Property 1.11 holds for the following classes of groups.*

- Groups π with a finite 3-dimensional model for $B\pi$.
- Finitely generated abelian groups and right angled Artin groups.
- Finite groups whose 2-Sylow subgroups are abelian or quaternion.

When π has geometric dimension at most two, the Secondary Property holds since $H_3(\pi; \mathbb{Z}/2) = 0$. If π is the fundamental group of a closed, oriented, aspherical 3-manifold then $H_5(\pi; \mathbb{Z}) = 0$ and $H_K \cong_s I\pi$, the augmentation ideal in $\mathbb{Z}\pi$. We showed in [KLPT17] that A_π is an isomorphism between two groups of order 2.

The case of finite groups with quaternion 2-Sylow subgroups was proven in [Tei92, Theorem 6.4.1]; see also our Section 6.1.3. Emboldened by our success, we make the following conjecture cf. [Tei92, Conjecture B].

Conjecture 1.13. *The sequence sec_π is exact for every finitely presented group π .*

1.3.3. The algebraic tertiary obstruction. We start by giving a brief description of the *Kervaire–Milnor* invariant τ that appears in the Tertiary Property. Recall that in the smooth category, a generic map of a surface to a 4-manifold looks locally like a linear subspace $\mathbb{R}^2 \subset \mathbb{R}^4$ or like a transverse double point $\mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^4$. We know that, stably, every spin manifold with signature zero has a smooth structure, so we may assume that all our surfaces are smoothly embedded with respect to such a structure.

Let $f: S^2 \looparrowright M$ be a generically immersed 2-sphere with vanishing algebraic self-intersection, that is $\mu_M(f) = 0$. Then all the double points of f can be paired

by a family of generically immersed Whitney discs $\{W_i\}$. They can be chosen to be disjointly embedded, framed, and to intersect f transversely, by boundary twisting to fix the framing and then pushing any intersections down to f . We define

$$\tau(f, \{W_i\}) := \sum_i |\dot{W}_i \cap f| \pmod{2}.$$

This count depends a priori on the choice of Whitney discs, so one has to be careful.

Recall that for a closed, connected 4-manifold M , since the $\mathbb{Z}/2$ intersection form is nonsingular the homomorphism

$$H_2(M; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2, \quad x \mapsto x \cdot x := \langle PD^{-1}(x), x \rangle$$

can be written as $x \cdot x = c \cdot x$ for a unique *characteristic* element $c \in H_2(M; \mathbb{Z}/2)$. This motivates the terminology in the next definition.

Definition 1.14. For $\alpha \in H_2(M; \mathbb{Z}/2)$, if $x \cdot x = \alpha \cdot x$ for all *spherical* x , meaning those x represented by a map $S^2 \rightarrow M$, we call α *S^2 -characteristic*.

Similarly we call α *$\mathbb{R}P^2$ -characteristic* if the same equation holds for all classes x represented by a map $\mathbb{R}P^2 \rightarrow M$. Note that $\mathbb{R}P^2$ -characteristic implies S^2 -characteristic.

We say that an element $\beta \in \pi_2(M)$ is *S^2 -* or *$\mathbb{R}P^2$ -characteristic* if its image under the modulo 2 Hurewicz map $\pi_2(M) \rightarrow H_2(M; \mathbb{Z}/2)$ is S^2 - or $\mathbb{R}P^2$ -characteristic respectively.

It was shown in [Sto94] and [ST01] that $\tau(f, \{W_i\})$ is independent of the choice of $\{W_i\}$ exactly when f is $\mathbb{R}P^2$ -characteristic. Moreover, the resulting invariant

$$\tau([f]) := \tau(f, \{W_i\}) \in \mathbb{Z}/2$$

only depends on the homotopy class $[f] \in \pi_2(M)$. It was called the Kervaire-Milnor invariant by Freedman-Quinn, who gave a slightly incomplete account of it in [FQ90, Section 10]. This was later resolved by Stong [Sto94].

Let us return to our obstruction theory for closed, connected spin 4-manifolds M with $\mathfrak{pri}(M) = 0 = \mathfrak{sec}(M)$. For the algebraic secondary invariant, we chose a 2-complex K and a stable splitting $s: H_K \rightarrow (\mathbb{Z}\pi)^r \oplus \pi_2(M)$ of the short exact sequence from Theorem 1.6 and defined the hermitian form $\lambda_{M,s}$ on H_K as the composition of λ_M with $s_2: H_K \rightarrow \pi_2(M)$. Up to even forms, this did not depend K nor on the splitting s and there are splittings s for which $\lambda_{M,s}$ actually vanishes; see Lemma 4.7.

We want to define τ on elements in the image of such splittings s . Start with an element $x \in H^2(\pi; \mathbb{Z}/2)$, restrict it using $K \subseteq B\pi$ to $H^2(K; \mathbb{Z}/2)$ and lift that to $\tilde{x} \in H_K = H^2(K; \mathbb{Z}\pi)$, using that $H^3(K; -) = 0$, so the Bockstein potentially obstructing this lifting vanishes. Given a splitting s with $\lambda_{M,s} \equiv 0$, let $y := s_2(\tilde{x}) \in \pi_2(M)$, and then compute $\tau_M(y) \in \mathbb{Z}/2$. This is well defined if y is $\mathbb{R}P^2$ -characteristic.

We will show in Definition 4.5 that the above construction determines a well-defined map

$$\tau_{M,s}: h^2(\pi) \rightarrow \mathbb{Z}/2$$

on the set

$$h^2(\pi) := \ker(Sq^2: H^2(\pi; \mathbb{Z}/2) \rightarrow H^4(\pi; \mathbb{Z}/2)).$$

Here $h^2(\pi)$ is dual to the $E_{2,2}^3$ -term in our AHSS, since

$$Sq^2 = (d_{4,1}^2)^*: H^2(\pi; \mathbb{Z}/2) \rightarrow H^4(\pi; \mathbb{Z}/2)$$

is the dual of the $d_{4,1}^2$ differential. The duals of the other relevant differentials in the AHSS for $\Omega_4^{\text{Spin}}(B\pi)$ are

$$(d_{5,0}^2)^* : H^3(\pi; \mathbb{Z}/2) \rightarrow \text{Hom}_{\mathbb{Z}}(H_5(\pi; \mathbb{Z}), \mathbb{Z}) \quad \text{and}$$

$$(d_{5,0}^3)^* : h^2(\pi) = \ker(d_{4,1}^2)^* \rightarrow \ker(d_{5,0}^2)^*.$$

There is a canonical isomorphism

$$\begin{aligned} \omega : E_{2,2} = H_2(\pi; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3) &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}/2}(H^2(\pi; \mathbb{Z}/2), \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3) \\ &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}/2}(\ker(d_{5,0}^3)^*, \mathbb{Z}/2). \end{aligned}$$

Since $\ker(d_{5,0}^3)^* \subseteq h^2(\pi)$, we can restrict $\tau_{M,s}$ to a map $\tau_{M,s}| : \ker(d_{5,0}^3)^* \rightarrow \mathbb{Z}/2$. We want to show that this coincides with the image of $\text{ter}(M)$ under ω . Again, we give the statement in the form of a property of a group.

Tertiary Property 1.15. *We say that π has the Tertiary Property if $\text{ter}(M) \in E_{2,2}$ is sent to $\tau_{M,s}|$ via ω for all closed, connected, spin 4-manifolds M with fundamental group π and $\text{pri}(M) = 0 = \text{sec}(M)$.*

We remark that $\tau_{M,s}(0) = 0$ because $0 \in h^2(\pi)$ leads to computing τ_M on $0 \in \pi_2(M)$, which is represented by a trivial (embedded) sphere. So in the case that $\ker(d_{5,0}^3)^* = 0$, the Tertiary Property automatically holds. We are unable to show in general that $\tau_{M,s}|$ is a homomorphism. This follows from the Tertiary Property, when it holds, since $\text{ter}(M)$ maps to a homomorphism.

In Proposition 4.10 we will show that the Tertiary Property holds whenever the splitting s is induced by a map $f : M \rightarrow K$, in the sense that

$$s_2 = PD \circ f^* : H^2(K; \mathbb{Z}\pi) \rightarrow H^2(M; \mathbb{Z}\pi) \rightarrow H_2(M; \mathbb{Z}\pi) \cong \pi_2(M)$$

for K a 2-complex with $f_* : \pi_1(M) \rightarrow \pi_1(K)$ an isomorphism. In Theorem 4.16 we give a condition under which every splitting is realised in this way, and we use this to deduce the following theorem.

Theorem 1.16. *The Tertiary Property holds for the following classes of groups.*

- Groups π with a finite 3-dimensional model for $B\pi$.
- Finitely generated abelian groups and right angled Artin groups.
- Finite groups whose 2-Sylow subgroups are abelian or quaternion.

Our verification of Theorem 1.16 in the case of abelian groups relies on computations of Whitehead's Γ -groups [Whi50] made by the first and second authors with Ben Ruppik [KPR20]. In the special case that π is the fundamental group of a closed, oriented, aspherical 3-manifold, we proved in [KLPT17] that π has the Tertiary Property 1.15. To do this, we computed $\text{ter}(M)$ and $\tau_{M,s}$ on sufficiently many concrete examples for which the primary and secondary obstructions vanish. We close this section with the following conjecture, analogous to Secondary Property 1.11.

Conjecture 1.17. *Every group has the Tertiary Property.*

1.4. Context for our work

The stable classification of 4-manifolds was reduced to a bordism computation by Kreck in [Kre99]. 4-manifolds with finite fundamental group were studied by Hambleton and Kreck in [HK88], as well as in the PhD thesis of the third author [Tei92]. The case of geometrically 2-dimensional groups was solved by Hambleton, Kreck and the last author in [HKT09]. Groups of cohomological dimension at most 3, and in particular right angled Artin groups with this property, were studied by Hambleton and Hildum [HH19] in the case that the equivariant intersection form is even, which by our results in the current paper is equivalent to the secondary obstruction vanishing. Previous work on the stable classification question also includes work of Cavicchioli, Hegenbarth and Repovš [CHR95], Spaggiari [Spa03] and Davis [Dav05].

The papers [HK88], [HKT09] and [HH19] focussed on the normal 2-type – roughly this means they looked at the entire intersection form of the 4-manifold – also obtaining unstable classification results. On the other hand, the algebraic invariants that we study are tailored to the stable question, so we are able to handle more fundamental groups with more easily computable obstructions.

1.4.1. When stable homeomorphism implies homeomorphism. By combining our results with results of Hambleton-Kreck, Khan and Crowley-Sixt, we obtain results on the unstable homeomorphism classification. If we consider manifolds that are already sufficiently stabilised, in a sense to be made precise presently, then it turns out that stable homeomorphism implies homeomorphism.

Recall that a finitely presented group π is *polycyclic-by-finite* if it has a subnormal series where the quotients are either cyclic or finite. The number of infinite cyclic quotients in such a subnormal series turns out to be an invariant of π , called the *Hirsch number* $\mathfrak{h}(\pi)$. Define $\mathfrak{h}'(\pi) = 1$ if π is finite, and define $\mathfrak{h}'(\pi) = \mathfrak{h}(\pi) + 3$ if π is infinite. The following theorem is due to Hambleton and Kreck [HK93, Theorem B] for π finite, and due to Crowley and Sixt [CS11, Theorem 1.1] for π infinite.

Theorem 1.18 ([HK93, CS11]). *Let M and N be closed, connected 4-manifolds with polycyclic-by-finite fundamental group π such that $\chi(N) + 2k = \chi(M)$ for some $k \geq \mathfrak{h}'(\pi)$. If M and N are stably homeomorphic then M is homeomorphic to $N \# k(S^2 \times S^2)$.*

Note that [CS11, Theorem 1.1] is only stated in the smooth category, but as remarked at the beginning of [CS11, Section 2.1], the theorem also holds in the topological category since polycyclic-by-finite groups are good in the sense of Freedman [FQ90, Section 2.9 and Theorem 5.1A].

For π virtually abelian and infinite, there is a similar statement to [CS11, Theorem 1.1] by Khan [Kha17, Corollary 2.4], with a slightly better bound on the number of stabilisations needed.

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CHAPTER 2

Background

2.1. Stable diffeomorphism, bordism groups, and spectral sequences

Throughout the paper M denotes a smooth, closed, compact, connected, oriented 4-manifold with fundamental group $\pi_1(M) \cong \pi$. The universal cover of M will be denoted \widetilde{M} . If we fix an identification of $\pi_1(M)$ with π , the identification determines a homotopy class of maps $c: M \rightarrow B\pi$ classifying \widetilde{M} .

Two smooth 4-manifolds M and N are called *stably diffeomorphic* if there exist integers $m, n \in \mathbb{N}_0$ such that stabilising M and N with copies of $S^2 \times S^2$ yields diffeomorphic manifolds

$$M \# m(S^2 \times S^2) \cong N \# n(S^2 \times S^2).$$

We require that the diffeomorphism respects orientations. Note that, unlike elsewhere in the literature, we do not require that $m = n$.

2.1.1. Bordism groups. The starting point for our investigation, coming from Kreck's modified surgery, is that stable diffeomorphism can be understood in terms of bordism over the normal 1-type.

Definition 2.1. Let M be a closed oriented manifold of dimension n . A *normal 1-type* is a fibration over BSO , denoted by $\xi: B \rightarrow BSO$, through which the stable normal bundle $\nu_M: M \rightarrow BSO$ factors as:

$$\begin{array}{ccc} & \tilde{\nu}_M \curvearrowright & B \\ & & \downarrow \xi \\ M & \xrightarrow{\nu_M} & BSO \end{array}$$

such that $\tilde{\nu}_M$ is 2-connected and ξ is 2-coconnected. A choice of a map $\tilde{\nu}_M$ is called a *normal 1-smoothing* of M .

The different normal 1-types of a fixed M are fibre homotopy equivalent to one another.

Theorem 2.2 ([Kre99, Theorem C]). *Two closed 4-dimensional manifolds with the same Euler characteristic and the same normal 1-type $\xi: B \rightarrow BSO$, admitting bordant normal 1-smoothings, are diffeomorphic after connected sum of both with r copies of $S^2 \times S^2$ for some r .*

Here is a summary of the proof. Given a bordism over the normal 1-type, one can perform surgery below the middle dimension until the bordism becomes 1-connected. Excise 2-spheres by tubing a neighbourhood to the boundary. This makes the bordism into an s -cobordism, which is stably a product, at the expense of connect summing the boundary 4-manifolds with copies of $S^2 \times S^2$. A key point to check is that the normal smoothing data allows us to excise 2-spheres with a

framed normal bundle, resulting in connect sums with $S^2 \times S^2$ and not the twisted bundle $S^2 \widetilde{\times} S^2$. The converse, that stably diffeomorphic manifolds admit bordant 1-smoothings, can be found in Crowley-Sixt [CS11, Lemma 2.3(ii)].

By taking account of the different choices of normal 1-smoothing, and allowing different numbers of $S^2 \times S^2$ to be added to either side to allow the Euler characteristics of the initial manifolds to differ, we obtain the following.

Theorem 2.3. *The stable diffeomorphism classes of 4-manifolds with normal 1-type ξ are in one-to-one correspondence with $\Omega_4(\xi)/\text{Aut}(\xi)$.*

Next we need to understand the normal 1-type. The following lemma, determining the normal 1-type of spin 4-manifolds, is well-known to the experts. We refer to [KLPT17, Lemma 3.5] for a proof.

Lemma 2.4. *Let π be a finitely presented group. A normal 1-type of a spin 4-manifold with fundamental group π is given by*

$$B\pi \times BSpin \xrightarrow{\gamma \circ \text{pr}_2} BSO,$$

where pr_2 is the projection onto $BSpin$ and γ is the canonical map $BSpin \rightarrow BSO$.

This article gives algebraic invariants that obstruct null-bordism in $\Omega_4(\xi)$ in the above two cases.

2.1.2. Spectral sequences. The main tool for understanding the bordism groups is the James spectral sequence. However, as explained in [KLPT17, Section 2], this is isomorphic to the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{\text{Spin}}(B\pi)$.

Let h_* be a generalised homology theory (for us spin bordism Ω_*^{Spin}), and let X be a CW complex. The Atiyah-Hirzebruch spectral sequence arising from the homology Leray-Serre spectral sequence for the fibration $\text{pt} \rightarrow X \rightarrow X$ has E_2 page given by

$$E_{p,q}^2 := H_p(X; h_q(\text{pt})).$$

The d^r differential has (p, q) -bidegree $(-r, r-1)$, and the sequence converges to $h_*(X)$. We denote the differentials by

$$d_{p,q}^r: H_p(X; h_q(\text{pt})) \rightarrow H_{p-r}(X; h_{q+r-1}(\text{pt})).$$

Denote the filtration on the abutment of an Atiyah-Hirzebruch spectral sequence by

$$0 \subset F_{0,n} \subset F_{1,n-1} \subset \cdots \subset F_{n-q,q} \subset \cdots \subset F_{n,0} = h_n(X) = \Omega_n(\xi).$$

Recall that $F_{n-q,q}/F_{n-q-1,q+1} \cong E_{n-q,q}^\infty$.

Let $B := B\pi \times BSpin$. Denote the restriction of $\text{pr}_1: B \rightarrow B\pi$ to the inverse image of the p -skeleton of $B\pi$ by $B|_p$, and let $\xi|_p: B|_p \rightarrow BSO$ be the restriction of ξ to $B|_p$. An element of $\Omega_n(\xi)$ lies in $F_{p,n-p}$ if and only if it is in the image of the map $\Omega_n(\xi|_p) \rightarrow \Omega_n(\xi)$. That is, if the element lies in the image of $\Omega_n^{\text{Spin}}(B\pi^{(p)})$. This follows from the naturality of the spectral sequence applied to the map of fibrations induced by the inclusion of $B\pi^{(p)} \rightarrow B\pi$.

In our situation, being in $F_{p,4-p}$ means that a 4-manifold M together with its classifying map $c: M \rightarrow B\pi$ is spin bordant to a 4-manifold M' over $B\pi$ where the map to $B\pi$ factors through the p -skeleton $B\pi^{(p)}$ of $B\pi$. For $2 \leq p \leq 4$ we can perform surgeries on M' to convert the map $M' \rightarrow B\pi^{(p)}$ to a map $M'' \rightarrow B\pi^{(p)}$ that induces an isomorphism on fundamental groups. By Kreck's modified surgery

(Theorem 2.2), M and M'' are stably diffeomorphic, and thus after connected sums with copies of $S^2 \times S^2$, c is homotopic to a map factoring through $B\pi^{(p)}$.

The converse also holds, that is if one can find a map $f: M \rightarrow B\pi^{(p)}$ for $2 \leq p \leq 4$ inducing the chosen isomorphism on fundamental groups, then (M, c) lies in $F_{p,4-p}$.

Now, to state the next lemma, let X be any CW complex and let $X^{(p)}$ be its p -skeleton. Denote the barycentres of the p -cells $\{e_i^p\}$ of X by $\{b_i^p\}_{i \in I}$. Given an element $[M \rightarrow X^{(p)}] \in \Omega_n^{\text{Spin}}(X^{(p)})$, denote the regular preimage of the barycentre $\{b_i^p\} \in X^{(p)}$ by $N_i \subset M$. Note that $[N_i] \in \Omega_{n-p}^{\text{Spin}}$, since the normal bundle of N_i in M is trivial, and hence N_i inherits a spin structure from M .

The following lemma is a consequence of [KLPT17, Lemma 2.5].

Lemma 2.5. *The canonical map $\Omega_n^{\text{Spin}}(X^{(p)}) \rightarrow H_p(X^{(p)}; \Omega_{n-p}^{\text{Spin}})$ that comes from the spectral sequence (see below) coincides with the map*

$$\begin{aligned} \Omega_n^{\text{Spin}}(X^{(p)}) &\rightarrow H_p(X^{(p)}; \Omega_{n-p}^{\text{Spin}}) \\ [M \rightarrow X^{(p)}] &\mapsto \left[\sum_{i \in I} [N_i] \cdot e_i^p \right]. \end{aligned}$$

The same holds with oriented bordism replacing spin bordism.

The map in the statement of the lemma $\Omega_n^{\text{Spin}}(X^{(p)}) \rightarrow H_p(X^{(p)}; \Omega_{n-p}^{\text{Spin}})$ is sometimes called an edge homomorphism. It arises as follows. The abutment of the Atiyah-Hirzebruch spectral sequence $\Omega_n(\xi|_{X^{(p)}}) = F_{n,0}$ maps to its quotient by the first filtration step $F_{p,n-p}$ that differs from $F_{n,0}$. This term is indeed $F_{p,n-p}$, since the homology of $X^{(p)}$ vanishes in degrees greater than p , thus $E_{s,t}^2 = E_{s,t}^\infty = 0$ for all $s > p$. We have $F_{n,0}/F_{p,n-p} \cong E_{p,n-p}^\infty$. The target, $H_p(X^{(p)}; \Omega_{n-p}^{\text{Spin}})$, is the $E_{p,n-p}^2$ term of the spectral sequence. Since no differentials have image in $E_{p,n-p}^2$, we have that $E_{p,n-p}^\infty \subseteq E_{p,n-p}^2 = H_p(X^{(p)}; \Omega_{n-p}^{\text{Spin}})$, and so the composition

$$\Omega_n^{\text{Spin}}(X^{(p)}) = F_{n,0} \rightarrow F_{n,0}/F_{p,n-p} \xrightarrow{\cong} E_{p,n-p}^\infty \rightarrow E_{p,n-p}^2 = H_p(X^{(p)}; \Omega_{n-p}^{\text{Spin}})$$

gives the desired map.

We will need the following standard fact, so we give the argument here separately. Recall that π can be any finitely presented group.

Lemma 2.6. *The subgroup $F_{0,4} \subset \Omega_4^{\text{Spin}}(B\pi)$ is isomorphic to Ω_4^{Spin} , and is a direct summand of $\Omega_4^{\text{Spin}}(B\pi)$.*

PROOF. The term $E_{0,4}^2 \cong H_0(B\pi; \Omega_4^{\text{Spin}}) \cong \Omega_4^{\text{Spin}}$. The maps $\text{pt} \rightarrow B\pi$ and $B\pi \rightarrow \text{pt}$ induce maps between the Atiyah-Hirzebruch spectral sequences. But the spectral sequence for a point is supported in a single column, so there are maps $\Omega_4^{\text{Spin}} \rightarrow \Omega_4^{\text{Spin}}(B\pi) \rightarrow \Omega_4^{\text{Spin}}$. The composition $\text{pt} \rightarrow B\pi \rightarrow \text{pt}$ is the identity, and maps between the spectral sequences are natural, so the map $\Omega_4^{\text{Spin}} \rightarrow \Omega_4^{\text{Spin}}(B\pi)$ splits. Thus Ω_4^{Spin} is a direct summand as claimed. \square

We consider the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{\text{Spin}}(B\pi)$.

Proposition 2.7. *There is a filtration*

$$0 \subseteq F_3 \subseteq F_2 \subseteq F_1 \subseteq \Omega_4^{\text{Spin}}(B\pi)$$

by subgroups such that the following holds.

(1) The quotient $\Omega_4^{\text{Spin}}(B\pi)/F_1$ is isomorphic to

$$\ker(d_{4,0}^3: \ker(\text{Sq}_2 \circ \text{red}_2: H_4(\pi; \mathbb{Z}) \rightarrow H_2(\pi; \mathbb{Z}/2)) \rightarrow H_1(\pi; \mathbb{Z}/2)).$$

The image of an element (M, c) in this subgroup of $H_4(B\pi; \mathbb{Z})$ is $c_*([M])$, and is denoted $\mathbf{pri}(M)$. Here $\text{Sq}_2: H_4(\pi; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2)$ denotes the map on homology that is dual to the usual cohomology Steenrod operation $\text{Sq}^2: H^2(\pi; \mathbb{Z}/2) \rightarrow H^4(\pi; \mathbb{Z}/2)$ and red_2 is the reduction modulo two of the coefficients.

(2) The quotient F_1/F_2 is isomorphic to $H_3(\pi; \mathbb{Z}/2)/\text{im}(d_{5,0}^2)$, where

$$d_{5,0}^2 = \text{Sq}_2 \circ \text{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_3(\pi; \mathbb{Z}/2).$$

The image of an element (M, c) is denoted $\mathbf{sec}(M)$. If $\mathbf{pri}(M) = 0$, then $\mathbf{sec}(M)$ can be computed by Lemma 2.5.

(3) The quotient F_2/F_3 is isomorphic to $H_2(\pi; \mathbb{Z}/2)/\text{im}(d_{4,1}^2, d_{5,0}^3)$, where

$$d_{4,1}^2 = \text{Sq}_2: H_4(\pi; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2).$$

If $\mathbf{pri}(M) = \mathbf{sec}(M) = 0$, then $\mathbf{ter}(M)$ can be computed by Lemma 2.5.

(4) The subgroup F_3 is a direct summand isomorphic to \mathbb{Z} , and the map to \mathbb{Z} is given by $\sigma/16$, that is take the signature and divide by 16.

PROOF. The proposition follows from computing using the Atiyah-Hirzebruch spectral sequence, with $E_{p,q}^2 = H_p(B\pi; \Omega_q^{\text{Spin}})$. We know from Lemma 2.6 that the coefficients split, so recalling Rochlin's theorem that $\Omega_4^{\text{Spin}} \cong 16\mathbb{Z}$, this proves the last item.

Spin bordism in the lower dimensions is given by $\Omega_3^{\text{Spin}} = 0$, $\Omega_2^{\text{Spin}} = \mathbb{Z}/2$, $\Omega_1^{\text{Spin}} = \mathbb{Z}/2$ and $\Omega_0^{\text{Spin}} \cong \mathbb{Z}$ [Mil63], [ABP67].

For the spin bordism Atiyah-Hirzebruch spectral sequence, the E^2 -page differential d_2 is dual to the Steenrod square Sq^2 , together with reduction of the coefficients mod 2 if necessary, according to [Tei92, Theorem 3.1.3]. This implies that the differential vanishes when the codomain is $H_p(B\pi; \Omega_q^{\text{Spin}})$ and $p < 2$, since $\text{Sq}^r: H^n \rightarrow H^{n+r}$ is zero whenever $n < r$.

There are a couple of potentially nontrivial d^2 and d^3 differentials, that are recorded in the proposition. \square

Now, computing the secondary and tertiary invariants as inverse images is not a very feasible task. A particular difficulty is that one typically has to find a bordant manifold such that the map to $B\pi$ can be homotoped onto the p -skeleton, where $p = 3$ for \mathbf{sec} and $p = 2$ for \mathbf{ter} . Then one takes inverse images with framings. The goal of this paper, as explained in the introduction, is to give algebraic criteria that decide whether these invariants vanish. A particularly nice feature is that we show that the algebraic secondary obstruction does not depend on the way in which the primary invariant vanishes. That is, while it needs the primary obstruction to vanish in order to be well-defined, it does not depend on the choice of vanishing i.e. of splitting of the short exact sequence. Similarly, the algebraic tertiary obstruction needs the first two obstructions to vanish in order to be well-defined, but it does not depend on the way in which they vanish. These features increase the computability of our obstructions. Chapter 7 provides an explicit instance of this phenomenon.

2.2. Topological 4-manifolds and stable homeomorphism

Our results apply to the stable homeomorphism classification of topological spin 4-manifolds, as we explain in this section. First we point out that the invariants from Theorem 1.1 apply in the topological category, as asserted in the introduction. Note that the signature $\sigma(M)$, $\pi_2(M)$, and λ_M are all homotopy invariant, so give rise to stable homeomorphism invariants. The Kervaire-Milnor invariant τ_M is defined using intersections: by topological transversality [FQ90, Theorem 9.5A], such intersections can be counted in topological manifolds. Theorem 1.1 requires that we compute these invariants on all possible 1-skeleton connected sums $(N_1, \alpha_1) \#_1 (N_2, -\alpha_2)$. To avoid handle structures, one can instead take the connected sum $N_1 \# -N_2$ and then perform surgery on circles, framed using the α_i , to make the fundamental group isomorphic to π , and in this way construct the test manifolds on which we claim one must compute our algebraic invariants in order to decide stable homeomorphism.

Now we explain why the classifications coincide. To start, we have forgetful map

$$\mathbb{Z} \cong \Omega_4^{\text{Spin}} \xrightarrow{\cdot 2} \Omega_4^{\text{TOPSpin}} \cong \mathbb{Z}.$$

Recall that the isomorphism with \mathbb{Z} is given by the signature, taking account of its divisibility. That is, we have isomorphisms $\sigma: \Omega_4^{\text{SO}} \xrightarrow{\cong} \mathbb{Z}$, $\sigma/16: \Omega_4^{\text{Spin}} \xrightarrow{\cong} \mathbb{Z}$ and $\sigma/8: \Omega_4^{\text{TOPSpin}} \xrightarrow{\cong} \mathbb{Z}$. The cokernel

$$\text{coker}(\Omega_4^{\text{Spin}} \rightarrow \Omega_4^{\text{TOPSpin}}) \cong \mathbb{Z}/2$$

is detected by the Kirby-Siebenmann invariant. Thus in particular the Kirby-Siebenmann invariants must coincide for stably homeomorphic 4-manifolds, but for spin manifolds the Kirby-Siebenmann invariant is determined by the signature.

We want to apply our stable classification, but for this we need smooth manifolds. Consider two 4-manifolds M and M' that are spin and have nonvanishing Kirby-Siebenmann invariant. Suppose that $\pi_1(M) \cong \pi_1(M')$ has the Secondary and Tertiary Properties. We want to know if the two manifolds are stably homeomorphic. Let W be the E_8 -manifold. The manifolds $M \# W$ and $M' \# W$ have vanishing Kirby-Siebenmann invariants, and so are stably smoothable by [FQ90, Section 8.6]. We then have smooth manifolds, and so we can apply our classification programme to decide whether they are stably diffeomorphic. This involves algebraic invariants that are independent of the smooth structure. Suppose that we discover $M \# W$ and $M' \# W$ to be stably diffeomorphic. Then $M \# W \# -W$ is stably homeomorphic to $M' \# W \# -W$. But then $W \# -W$ is B -null bordant, so is stably homeomorphic to S^4 . It follows that M and M' are stably homeomorphic. Thus the same programme as in the smooth case determines whether two closed, spin topological 4-manifolds, whose fundamental groups satisfy Secondary Property 1.11 and Tertiary Property 1.15, are stably homeomorphic.

2.3. Whitehead's certain exact sequence and the Γ functor

In this section, we give some background on the ‘‘certain exact sequence’’ of Whitehead [Whi50], material that we will make extensive use of in our investigation of the Secondary Property, and its proof for some families of groups.

Let Y be a simply connected CW complex. Let $C_n := \pi_n(Y^{(n)}, Y^{(n-1)})$ be the n th cellular chain group. Let $d_n: C_n \rightarrow C_{n-1}$ denote the canonical boundary map

arising from the long exact sequence of the pair. Let $i_n: \pi_n(Y^{(n)}) \rightarrow C_n$ be the map induced by the inclusion of pairs. Let $Z_n := \ker(d_n)$ and let $B_n := \text{im}(d_{n+1})$. Then of course $H_n(Y; \mathbb{Z}) \cong Z_n/B_n$. Let $h_n: \pi_n(Y) \rightarrow Z_n/B_n$ be the Hurewicz map, and let $b_n: Z_n/B_n \rightarrow \ker(i_{n-1})$ be induced from the boundary map $\pi_n(Y^{(n)}, Y^{(n-1)}) \rightarrow \pi_{n-1}(Y^{(n-1)})$. Furthermore, let $\iota_n: \ker(i_n) \rightarrow \pi_n(Y)$ be induced by the inclusion $Y^{(n)} \rightarrow Y$. These maps enable us to formulate the ‘‘certain exact sequence’’ of J.H.C. Whitehead, which describes the kernel and cokernel of the Hurewicz map.

Theorem 2.8 ([Whi50, Section 10]). *There is a long exact sequence*

$$\cdots \rightarrow \pi_4(Y) \xrightarrow{h_4} H_4(C_n, d_n) \xrightarrow{b_4} \ker(i_3) \xrightarrow{\iota_3} \pi_3(Y) \xrightarrow{h_3} H_3(C_n, d_n) \rightarrow 0.$$

The extra ingredient that makes this sequence extremely useful is a description of $\ker(i_3)$ in terms of $\pi_2(Y)$. This uses the Γ group, which we will now define.

Definition 2.9. Let A be an abelian group. Then $\Gamma(A)$ is an abelian group with generators the elements of A . We write a as $v(a)$ when we consider it as an element of $\Gamma(A)$. The group $\Gamma(A)$ has the following relations:

$$\{v(-a) - v(a) \mid a \in A\} \quad \text{and} \\ \{v(a + b + c) - v(b + c) - v(c + a) - v(a + b) + v(a) + v(b) + v(c) \mid a, b, c \in A\}.$$

We remark that the symbol v has no meaning on its own, rather it is used to differentiate the generating set for $\Gamma(A)$ from the generating set for A .

The functor Γ is the universal quadratic functor, in the following sense. A map $f: A \rightarrow B$ of abelian groups is called quadratic if $f(a) = f(-a)$ for all $a \in A$ and for all $a, a' \in A$ the map

$$A \times A \rightarrow B, (a, a') \mapsto f(a + a') - f(a) - f(a')$$

is bilinear. The map $j: A \rightarrow \Gamma(A)$ sending a to $v(a)$ is quadratic. The functor Γ satisfies the universal property that for every quadratic map $f: A \rightarrow B$, there is a unique homomorphism $\Gamma(f): \Gamma(A) \rightarrow B$ with $f = \Gamma(f) \circ j$.

Lemma 2.10 ([Whi50, p. 62]). *If A is free abelian with basis \mathcal{B} , then $\Gamma(A)$ is free abelian with basis*

$$\{v(b), v(b + b') - v(b) - v(b') \mid b, b' \in \mathcal{B}\}.$$

In particular, if A is free abelian, then sending $v(a)$ to $a \otimes a$ defines an isomorphism between $\Gamma(A)$ and the subgroup of symmetric tensors of $A \otimes_{\mathbb{Z}} A$, that is, the subgroup generated by $\{a \otimes a \mid a \in A\}$.

Theorem 2.11 ([Whi50, Sections 10 and 13]). *Let $\eta: S^3 \rightarrow S^2$ be the Hopf map. The map $\widehat{\eta}: \Gamma(\pi_2(Y)) \rightarrow \ker(i_3: \pi_3(Y^{(3)}) \rightarrow \pi_3(Y^{(3)}, Y^{(2)}))$ given by $v(\alpha) \mapsto \alpha \circ \eta$ is an isomorphism. In particular, for every simply-connected CW complex Y we have the exact sequence:*

$$\cdots \pi_4(Y) \rightarrow H_4(Y; \mathbb{Z}) \xrightarrow{\widehat{\eta}^{-1} \circ b_4} \Gamma(\pi_2(Y)) \xrightarrow{\iota_3 \circ \widehat{\eta}} \pi_3(Y) \rightarrow H_3(Y; \mathbb{Z}) \rightarrow 0.$$

Now let L be any CW complex and as above let $\eta: S^3 \rightarrow S^2$ be the Hopf map.

Lemma 2.12. [Whi50, Section 13] *The map*

$$\Gamma(\eta): \Gamma(\pi_2(L)) \rightarrow \pi_3(L); v(\alpha) \mapsto \alpha \circ \eta$$

yields a well defined homomorphism.

For later use we restate the Whitehead exact sequence in the form that we will need it, namely for general CW complexes that need not be simply connected.

Theorem 2.13. *For a CW complex L , the following sequence is exact:*

$$H_4(\tilde{L}; \mathbb{Z}) \rightarrow \Gamma(\pi_2(L)) \xrightarrow{\Gamma(\eta)} \pi_3(L) \rightarrow H_3(\tilde{L}; \mathbb{Z}) \rightarrow 0.$$

Each of the terms is functorial in L and the maps in the sequence above are natural.

To finish the section, we record a couple of preliminary facts on the Γ groups that we will use throughout the rest of the paper.

Corollary 2.14. *Let K be a 2-complex. Then the map $\Gamma(\eta): \Gamma(\pi_2(K)) \rightarrow \pi_3(K)$ is an isomorphism.*

PROOF. For K a 2-complex, both the third and fourth homology groups of \tilde{K} vanish, so $\Gamma(\eta)$ is an isomorphism by Theorem 2.13. \square

Corollary 2.15. *Let K be a 2-complex. Every element in $\pi_3(K)$ is a sum of elements of the form $\beta \circ \eta$ with $\beta \in \pi_2(K)$.*

PROOF. Every element of $\Gamma(\pi_2(K))$ is a sum of elements $v(a)$ with $a \in \pi_2(K)$ by definition. Since $\Gamma(\eta)$ is a surjection and $\Gamma(\eta)(v(a)) = a \circ \eta$, the corollary follows. \square

Let

$$\begin{aligned} T: N \otimes_{\mathbb{Z}} N' &\rightarrow N' \otimes_{\mathbb{Z}} N \\ n \otimes n' &\mapsto n' \otimes n \end{aligned}$$

be the transposition map.

Lemma 2.16. *Let N, N' be free \mathbb{Z} -modules. Then*

$$\Gamma(N \oplus N') \cong \Gamma(N) \oplus (N \otimes N') \oplus \Gamma(N').$$

Moreover the inclusion into

$$(N \oplus N') \otimes_{\mathbb{Z}} (N \oplus N') \cong (N \otimes_{\mathbb{Z}} N) \oplus (N \otimes_{\mathbb{Z}} N') \oplus (N' \otimes_{\mathbb{Z}} N) \oplus (N' \otimes_{\mathbb{Z}} N')$$

is given by the direct sum of the inclusion $\Gamma(N) \rightarrow N \otimes_{\mathbb{Z}} N$, the diagonal map $(1, T): N \otimes_{\mathbb{Z}} N' \rightarrow (N \otimes_{\mathbb{Z}} N') \oplus (N' \otimes_{\mathbb{Z}} N)$ and the inclusion $\Gamma(N') \rightarrow N' \otimes_{\mathbb{Z}} N'$.

PROOF. Since $N \oplus N'$ is free, $\Gamma(N \oplus N')$ is the symmetric tensors. Embed $\Gamma(N \oplus N')$ into $(N \oplus N') \otimes (N \oplus N')$ and observe that the subgroup of symmetric tensors is isomorphic to $\Gamma(N) \oplus N \otimes N' \oplus \Gamma(N')$ with

$$(n_1 \otimes n_2, m \otimes m', \ell'_1 \otimes \ell'_2) \mapsto n_1 \otimes n_2 + m \otimes m' + m' \otimes m + \ell'_1 \otimes \ell'_2$$

the inverse of the embedding. \square

CHAPTER 3

The secondary obstruction

Here is an outline of this chapter. First we discuss the equivariant intersection form and some of its different guises. In Section 3.2 we associate an element of Tate cohomology to the intersection form of a 4-manifold with vanishing primary obstruction. The Tate group measures whether or not the intersection form λ_M is even, when restricted to the $H^2(K; \mathbb{Z}\pi)$ summand of the stabilised second homotopy group $\pi_2(M) \oplus \mathbb{Z}\pi^n$. The Secondary Property asserts that this element of the Tate group detects $\mathbf{sec}(M)$.

In Condition 3.20, we give a condition that we then prove in the remainder of the section is sufficient for the Secondary Property 1.11 to hold for a group π . In Chapter 6 we will use this condition to show that many families of groups have the Secondary Property 1.11.

3.1. The equivariant intersection form

Let M be a smooth, closed, spin, based 4-manifold together with an identification $\pi_1(M) \xrightarrow{\cong} \pi$. The equivariant intersection form

$$\lambda_M: \pi_2(M) \times \pi_2(M) \rightarrow \mathbb{Z}\pi$$

is defined as follows. Identify $\pi_2(M) \cong H_2(M; \mathbb{Z}\pi)$ via the Hurewicz theorem. Then for classes $x, y \in H_2(M; \mathbb{Z}\pi)$, we have by definition

$$\lambda_M(x, y) = \langle PD^{-1}(y), x \rangle,$$

where $PD: H^2(M; \mathbb{Z}\pi) \rightarrow H_2(M; \mathbb{Z}\pi)$ is the Poincaré duality isomorphism given by cap product with the fundamental class $[M] \in H_4(M; \mathbb{Z})$. Here, and indeed throughout the article, we use the involution in the definition of cohomology $H^n(M; \mathbb{Z}; \pi) := H^n(\mathrm{Hom}_{\mathbb{Z}\pi}(C_*(M; \mathbb{Z}\pi)^t, \mathbb{Z}\pi))$ to consider chains as a right $\mathbb{Z}\pi$ -module $C_*(M; \mathbb{Z}\pi)^t$, so that cohomology still carries a left $\mathbb{Z}\pi$ -module structure.

We identify the equivariant intersection form on M with $\mathbb{Z}\pi$ coefficients with the intersection form on \widetilde{M} . Pick a lift of each cell of M to obtain an identification $\theta: C_*(M; \mathbb{Z}\pi) \xrightarrow{\cong} C_*(\widetilde{M}; \mathbb{Z})$. We also have an isomorphism

$$\begin{aligned} \Psi: C_{cs}^*(\widetilde{M}; \mathbb{Z}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{M}; \mathbb{Z}), \mathbb{Z}\pi) \\ f &\mapsto (a \mapsto \sum_{g \in \pi} f(g^{-1}a) \cdot g) \end{aligned}$$

for $a \in C_*(\widetilde{M}; \mathbb{Z})$. The inverse is given by sending $\varphi \in \mathrm{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{M}; \mathbb{Z}), \mathbb{Z}\pi)$ to the homomorphism that maps $x \in C_*(\widetilde{M}; \mathbb{Z})$ to the coefficient of the neutral group element of $\varphi(x) \in \mathbb{Z}\pi$. These two maps induce an isomorphism $\theta^* \circ \Psi_*: H_{cs}^*(\widetilde{M}; \mathbb{Z}) \xrightarrow{\cong}$

$H^*(M; \mathbb{Z}\pi)$, which fits into the following commuting diagram.

$$\begin{array}{ccccc}
H_2(M; \mathbb{Z}\pi) & \xrightarrow[\cong]{PD^{-1}} & H^2(M; \mathbb{Z}\pi) & \xrightarrow{\text{ev}} & \text{Hom}_{\mathbb{Z}\pi}(H_2(M; \mathbb{Z}\pi), \mathbb{Z}\pi) \\
\theta \downarrow \cong & & \theta^* \circ \Psi \uparrow \cong & & \theta^* \uparrow \cong \\
H_2(\widetilde{M}; \mathbb{Z}) & \xrightarrow[\cong]{PD^{-1}} & H_{cs}^2(\widetilde{M}; \mathbb{Z}) & \xrightarrow{\text{ev} \circ \Psi} & \text{Hom}_{\mathbb{Z}\pi}(H_2(\widetilde{M}; \mathbb{Z}), \mathbb{Z}\pi)
\end{array}$$

In the diagram we also write $PD: H_{cs}^2(\widetilde{M}; \mathbb{Z}) \rightarrow H_2(\widetilde{M}; \mathbb{Z})$ for Poincaré duality in \widetilde{M} .

Thus, by taking the two routes from $H_2(M; \mathbb{Z}\pi)$ to $\text{Hom}_{\mathbb{Z}\pi}(H_2(M; \mathbb{Z}\pi), \mathbb{Z}\pi)$, we see that for $x, y \in H_2(M; \mathbb{Z}\pi)$, we have

$$\lambda_M(x, y) = \langle PD^{-1}(y), x \rangle = \sum_{g \in \pi} \langle PD^{-1}(\theta(y)), g^{-1}\theta(x) \rangle \cdot g.$$

Now we will consider $w = \theta(x)$ and $u = \theta(y)$ in $H_2(\widetilde{M}; \mathbb{Z})$. Note that $H_0(\widetilde{M}; \mathbb{Z}) \cong H_{cs}^4(\widetilde{M}; \mathbb{Z}) \cong \mathbb{Z}$. In the final expression in the equation above, we write $PD: H_{cs}^2(\widetilde{M}; \mathbb{Z}) \rightarrow H_2(\widetilde{M}; \mathbb{Z})$ for Poincaré duality. Then since the cup product is signed commutative, we have, for $u, w \in H_2(\widetilde{M}; \mathbb{Z})$:

$$\begin{aligned}
\langle PD^{-1}(u), g^{-1}w \rangle &= PD^{-1}(u) \cap g^{-1}w \\
&= PD^{-1}(u) \cup (g^{-1}PD^{-1}(w) \cap [M]) \\
&= (gPD^{-1}(u) \cup PD^{-1}(w)) \cap [M] \\
&= (PD^{-1}(w) \cup gPD^{-1}(u)) \cap [M].
\end{aligned}$$

We record the outcome in the following proposition.

Proposition 3.1. *The map $\theta_*: H_2(M; \mathbb{Z}\pi) \rightarrow H_2(\widetilde{M}; \mathbb{Z})$ induces an isometry of the equivariant intersection form $\lambda_M(x, y) = \langle PD^{-1}(y), x \rangle$ with the form*

$$\lambda_{\widetilde{M}}(\theta_*(x), \theta_*(y)) = \lambda_{\widetilde{M}}(w, u) = \sum_{g \in \pi} ((PD^{-1}(w) \cup gPD^{-1}(u)) \cap [M])g \in \mathbb{Z}\pi.$$

It is well-known that $(PD^{-1}(x) \cup PD^{-1}(y)) \cap [M] \in H_0(\widetilde{M}; \mathbb{Z}) \cong \mathbb{Z}$ agrees with the geometric intersection of S_x and S_y , where S_x and S_y are transverse, generically immersed surfaces in \widetilde{M} representing x and y respectively.

Here each intersection point p is counted with a sign depending on whether the orientation of $T_p S_x \oplus T_p S_y$ agrees with $T_p M$ or not.

Let x_0 be the chosen base point of M and let \tilde{x}_0 be a chosen lift of x_0 in \widetilde{M} . Given two transverse, generically immersed spheres $\alpha, \beta \in \pi_2(M, x_0)$, any intersection point p determines an element of π by choosing a path in α from x_0 to p and concatenating it with a path from p to x_0 in β . This element is g precisely if the lifts $\tilde{\alpha} \in \pi_2(\widetilde{M}, \tilde{x}_0)$ and $g\tilde{\beta} \in \pi_2(\widetilde{M}, g\tilde{x}_0)$ intersect at a lift \tilde{p} of p . Hence we have the following statement, which we will use later in our discussion of the tertiary invariant.

Proposition 3.2. *The above geometric count of intersections in $\mathbb{Z}\pi$ computes the equivariant intersection form.*

Using the description

$$\lambda_M(x, y) = \sum_{g \in \pi} ((PD^{-1}(x) \cup gPD^{-1}(y)) \cap [M])g,$$

it is easy to see that the form λ_M is sesquilinear and hermitian. But it is often not nonsingular or even nondegenerate, as can be computed by the universal coefficient spectral sequence [Lev77, Theorem 2.3].

Proposition 3.3. *There is an exact sequence*

$$0 \rightarrow H^2(\pi; \mathbb{Z}\pi) \rightarrow H_2(M; \mathbb{Z}\pi) \rightarrow \text{Hom}_{\mathbb{Z}\pi}(H_2(M; \mathbb{Z}\pi), \mathbb{Z}\pi) \rightarrow H^3(\pi; \mathbb{Z}\pi) \rightarrow 0$$

where the middle map is the adjoint of the intersection form.

PROOF. Recall that the universal coefficient spectral sequence has E^2 page $E_2^{p,q} \cong \text{Ext}_{\mathbb{Z}\pi}^q(H_p(M; \mathbb{Z}\pi), \mathbb{Z}\pi)$, differential d^r of degree $(1-r, r)$, and the sequence converges to $H^{p+q}(M; \mathbb{Z}\pi)$. Since $H_1(M; \mathbb{Z}\pi) = 0$, the spectral sequence yields a filtration $0 \subseteq F_{0,2} \subseteq H^2(M; \mathbb{Z}\pi)$ with $F_{0,2} \cong \text{Ext}_{\mathbb{Z}\pi}^2(H_0(M; \mathbb{Z}\pi), \mathbb{Z}\pi)$ and

$$H^2(M; \mathbb{Z}\pi)/F_{0,2} \cong \ker(\text{Hom}_{\mathbb{Z}\pi}(H_2(M; \mathbb{Z}\pi), \mathbb{Z}\pi) \rightarrow \text{Ext}_{\mathbb{Z}\pi}^3(H_0(M; \mathbb{Z}\pi), \mathbb{Z}\pi)).$$

Note that $\text{Ext}_{\mathbb{Z}\pi}^i(H_0(M; \mathbb{Z}\pi), \mathbb{Z}\pi) = H^i(\pi; \mathbb{Z}\pi)$ to obtain the exact sequence claimed. \square

3.2. Sesquilinear forms and Tate cohomology

In this section, and indeed throughout the rest of the article, when not specified every $\mathbb{Z}\pi$ -module is assumed by default to be a left $\mathbb{Z}\pi$ -module.

Let N be a left $\mathbb{Z}\pi$ -module. We write N^* for $\text{Hom}_{\mathbb{Z}\pi}(N, \mathbb{Z}\pi)$. This would a priori be a right $\mathbb{Z}\pi$ -module, where $\mathbb{Z}\pi$ acts by right multiplication on the target, but we turn it into a left module using the involution on $\mathbb{Z}\pi$.

Below, and indeed also throughout the article, when we consider the tensor product $N \otimes_{\mathbb{Z}\pi} N'$ of two left $\mathbb{Z}\pi$ -modules N, N' , we use the involution to turn N into a right $\mathbb{Z}\pi$ -module, so that the tensor product makes sense.

We denote the group of sesquilinear forms on N by $\text{Sesq}(N)$, that is the group of maps

$$\lambda: N \times N \rightarrow \mathbb{Z}\pi$$

with $\lambda(am, bn) = a\lambda(m, n)\bar{b}$ for $a, b \in \mathbb{Z}\pi$ and $m, n \in N$. The group operation is the obvious addition, defined by $(\lambda + \lambda')(m, n) = \lambda(m, n) + \lambda'(m, n)$. Equivalently $\text{Sesq}(N) \cong \text{Hom}_{\mathbb{Z}\pi}(N, N^*)$ by $\lambda(n)(m) := \lambda(m, n)$. This is a contravariant functor, where a map $f: N \rightarrow N'$ is sent to the map $f^*: \text{Sesq}(N') \rightarrow \text{Sesq}(N)$ with $(f^*\lambda)(m, m') = \lambda(f(m), f(m'))$. The group $\mathbb{Z}/2$ acts on $\text{Sesq}(N)$ via $(T\lambda)(m, n) = \overline{\lambda(n, m)}$. Thus we can form the *Tate cohomology group*

$$\widehat{H}^0(\text{Sesq}(N)) := \widehat{H}^0(\mathbb{Z}/2; \text{Sesq}(N)) = \ker(1 - T)/\text{im}(1 + T).$$

Note that $\overline{\ker(1 - T)}$ is precisely the hermitian forms, that is those for which $\lambda(m, n) = \overline{\lambda(n, m)}$ for all $m, n \in N$. In addition, every hermitian form has order two in the Tate cohomology, since $\lambda = T\lambda$ implies that $2\lambda = \lambda + T\lambda = (1 + T)\lambda$.

Definition 3.4. A hermitian form is called *even* if it is in the image of $1 + T: \text{Sesq}(N) \rightarrow \text{Sesq}(N)$, that is if it vanishes in $\widehat{H}^0(\text{Sesq}(N))$.

We therefore obtain a contravariant functor $\widehat{H}^0(\text{Sesq}(-))$ from $\mathbb{Z}\pi$ -modules to $\mathbb{Z}/2$ -modules. Let $(-)^* \otimes_{\mathbb{Z}\pi} (-)^*$ be the contravariant functor sending N to $N^* \otimes_{\mathbb{Z}\pi} N^*$, where we consider the first N^* as a right $\mathbb{Z}\pi$ -module as described above; equivalently, we do not use the involution on this N^* .

Definition 3.5. For left $\mathbb{Z}\pi$ -modules N and N' , define

$$\begin{aligned} \Phi_{N,N'} : N^* \otimes_{\mathbb{Z}\pi} N' &\rightarrow \text{Hom}_{\mathbb{Z}\pi}(N, N') \\ f \otimes n' &\mapsto (n \mapsto f(n)n'). \end{aligned}$$

We also define

$$\begin{aligned} \Phi_N := \Phi_{N,N^*} : N^* \otimes_{\mathbb{Z}\pi} N^* &\rightarrow \text{Hom}_{\mathbb{Z}\pi}(N, N^*) \cong \text{Sesq}(N). \\ f_1 \otimes f_2 &\mapsto ((n_1, n_2) \mapsto f_1(n_1)\overline{f_2(n_2)}). \end{aligned}$$

The next lemma follows directly from the definition.

Lemma 3.6. For two left $\mathbb{Z}\pi$ -modules N and N' , the map $\Phi_{N \oplus N'}$ is equivalent to the direct sum $\Phi_N \oplus \Phi_{N'} \oplus \Phi_{N,(N')^*} \oplus \Phi_{N',N^*}$, under the obvious isomorphisms of the domains and the codomains.

Lemma 3.7. The map

$$\Phi_N : N^* \otimes_{\mathbb{Z}\pi} N^* \rightarrow \text{Sesq}(N)$$

defines a natural transformation

$$\Phi : (-)^* \otimes_{\mathbb{Z}\pi} (-)^* \Rightarrow \text{Sesq}(-).$$

The proof of Lemma 3.7 is straightforward and we omit the details.

Lemma 3.8. The map

$$\begin{aligned} \Theta_N : \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} N^* &\rightarrow \widehat{H}^0(N^* \otimes_{\mathbb{Z}\pi} N^*) \\ 1 \otimes f &\mapsto [f \otimes f] \end{aligned}$$

determines a natural isomorphism

$$\Theta : \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} (-)^* \Rightarrow \widehat{H}^0((-)^* \otimes_{\mathbb{Z}\pi} (-)^*).$$

PROOF. First we show that Θ_N is well-defined. For all $f, f' \in N^*$ and for all $g \in \pi$, we have

$$\Theta_N(1 \otimes gf) = [gf \otimes gf] = [fg^{-1} \otimes gf] = [f \otimes f] = \Theta_N(1 \otimes f)$$

and

$$\begin{aligned} \Theta_N(1 \otimes (f + f')) &= [(f + f') \otimes (f + f')] \\ &= [f \otimes f + f' \otimes f' + (1 + T)(f \otimes f')] \\ &= [f \otimes f] + [f' \otimes f'] = \Theta_N(1 \otimes f) + \Theta_N(1 \otimes f') \end{aligned}$$

This shows that Θ_N is well-defined as desired.

It is easy to see that Θ is natural, so it remains to prove that it is an isomorphism. For this we will define an inverse. Let $N^*/2 := \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} N^*$. The projection $N^* \rightarrow N^*/2$ defines a map $N^* \otimes_{\mathbb{Z}\pi} N^* \rightarrow N^*/2 \otimes_{\mathbb{Z}/2} N^*/2$, and then applying \widehat{H}^0 we obtain a map

$$\widehat{H}^0(N^* \otimes_{\mathbb{Z}\pi} N^*) \rightarrow \widehat{H}^0(N^*/2 \otimes_{\mathbb{Z}/2} N^*/2).$$

Since $N^*/2$ is a $\mathbb{Z}/2$ -vector space, we have an isomorphism $\widehat{H}^0(N^*/2 \otimes_{\mathbb{Z}/2} N^*/2) \cong N^*/2$ given by sending $[[f] \otimes [f]] \rightarrow [f]$. To see that this map makes sense and is

an isomorphism, observe that symmetric modulo even forms are determined by the diagonal entries. So every element in $\widehat{H}^0(N^*/2 \otimes_{\mathbb{Z}/2} N^*/2)$ is a sum of elements of the form $[[f] \otimes [f]]$. The composition

$$\widehat{H}^0(N^* \otimes_{\mathbb{Z}\pi} N^*) \rightarrow \widehat{H}^0(N^*/2 \otimes_{\mathbb{Z}/2} N^*/2) \xrightarrow{\cong} N^*/2$$

sends $[f \otimes f] \in \widehat{H}^0(N^* \otimes_{\mathbb{Z}\pi} N^*)$ to $[f] = 1 \otimes f \in N^*/2 = \mathbb{Z}/2 \otimes N^*$, and is therefore the desired inverse to Θ_N . \square

Definition 3.9. Define the natural transformation

$$\Psi := \widehat{H}^0(\Phi) \circ \Theta: \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} (-)^* \Rightarrow \widehat{H}^0(\text{Sesq}(-)).$$

Lemma 3.10. *Let P, N be left $\mathbb{Z}\pi$ -modules, and suppose in addition that P is finitely generated projective. Then:*

- (i) $\Phi_{P,N}$ and $\Phi_{N,P}$ are isomorphisms.
- (ii) $\Psi_P: \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} P^* \rightarrow \widehat{H}^0(\text{Sesq}(P))$ is an isomorphism.
- (iii) Φ_P is an isomorphism.

PROOF.

- (i) First suppose that $P \cong \mathbb{Z}\pi$. In this case $\Phi_{\mathbb{Z}\pi, N}$ and $\Phi_{N, \mathbb{Z}\pi}$ are obviously isomorphisms. Since $\Phi_{N, \mathbb{Z}\pi^n}$ is the direct sum of n times $\Phi_{N, \mathbb{Z}\pi}$, it is an isomorphism. The same holds for $\Phi_{\mathbb{Z}\pi^n, N}$.

Now let P be finitely generated projective and let Q be such that $P \oplus Q$ is finitely generated free. Then the isomorphism $\Phi_{N, P \oplus Q}$ is the direct sum of $\Phi_{N, P}$ and $\Phi_{N, Q}$. Thus both these maps have to be isomorphisms as well. By the same argument $\Phi_{P, N}$ is an isomorphism.

- (ii) Since Φ_P is an isomorphism by part (i), so is $\widehat{H}^0(\Phi_P)$. Furthermore, Θ is a natural isomorphism by Lemma 3.8, and thus also the composition $\Psi_P = \widehat{H}^0(\Phi_P) \circ \Theta_P$ is an isomorphism.
- (iii) Immediate from (i). \square

Corollary 3.11. *Let N be a left $\mathbb{Z}\pi$ -module. Then the inclusion $N \rightarrow N \oplus \mathbb{Z}\pi$ induces isomorphisms*

$$\ker \Phi_N^{\mathbb{Z}/2} \xrightarrow{\cong} \ker \Phi_{N \oplus \mathbb{Z}\pi}^{\mathbb{Z}/2} \quad \text{and} \quad \ker \widehat{H}^0(\Phi_N) \xrightarrow{\cong} \ker \widehat{H}^0(\Phi_{N \oplus \mathbb{Z}\pi}),$$

where $\Phi_N^{\mathbb{Z}/2}$ is the map induced on $\mathbb{Z}/2$ -fixed points by Φ_N .

This corollary will be used in the proof of Lemma 3.19.

PROOF. We have a commutative diagram

$$\begin{array}{ccc} ((N^* \oplus \mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} (N^* \oplus \mathbb{Z}\pi))^{\mathbb{Z}/2} & \xrightarrow{\cong} & (N^* \otimes_{\mathbb{Z}\pi} N^*)^{\mathbb{Z}/2} \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} N^*) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi)^{\mathbb{Z}/2} \\ \Phi_{N \oplus \mathbb{Z}\pi} \downarrow & & \downarrow \\ \text{Sesq}(N \oplus \mathbb{Z}\pi)^{\mathbb{Z}/2} & \xrightarrow{\cong} & \text{Sesq}(N)^{\mathbb{Z}/2} \oplus \text{Hom}_{\mathbb{Z}\pi}(\mathbb{Z}\pi, N^*) \oplus \text{Sesq}(\mathbb{Z}\pi)^{\mathbb{Z}/2}. \end{array}$$

In the two terms in the right hand column, the middle summands are isomorphic, as are the third summands by Lemma 3.10 (i). The map between the first summands is $\Phi_N^{\mathbb{Z}/2}$. Since the horizontal maps are isomorphisms, the first part of the lemma follows from the commutativity of the diagram.

The diagram descends to a similar diagram on \widehat{H}^0 groups. By Lemma 3.10 (ii) and Lemma 3.8, the analogous statements hold, namely the maps induced by the

right hand vertical map in the diagram (with $\mathbb{Z}/2$ fixed points replaced by \widehat{H}^0) splits along the direct summands, and the maps on the second and third summands are isomorphisms. The second part of the lemma then also follows from commutativity. \square

3.3. Relating $H_3(\pi; \mathbb{Z}/2)$ with symmetric modulo even forms

Let K be a connected finite 2-complex with $\pi_1(K) \cong \pi$. Let (D_*, d_*) be the cellular $\mathbb{Z}\pi$ -module chain complex of K . In this subsection we establish a map $H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$ and we use this map to formulate Condition 3.20. This condition will then later be shown to imply the Secondary Property 1.11.

Given $x \in \ker d_2 \cong \pi_2(K)$, we define $f_x: \mathbb{Z}\pi \rightarrow \ker d_2$ by $f_x(a) = ax$. By dualizing we obtain a map $f_x^*: H^2(K; \mathbb{Z}\pi) \rightarrow \mathbb{Z}\pi^* \cong \mathbb{Z}\pi$, where the map $\mathbb{Z}\pi^* \rightarrow \mathbb{Z}\pi$ is given by $f \mapsto \overline{f(1)}$, which is a left $\mathbb{Z}\pi$ -module homomorphism. Hence f_x^* is a left $\mathbb{Z}\pi$ -module homomorphism. Note that for $\varphi \in H^2(K; \mathbb{Z}\pi)$ we have $f_x^*(\varphi) = \overline{\varphi(x)}$.

Lemma 3.12. *Let K be a connected finite 2-complex with $\pi_1(K) \cong \pi$. The canonical map*

$$I: \pi_2(K) \rightarrow \text{Hom}_{\mathbb{Z}\pi}(H^2(K; \mathbb{Z}\pi), \mathbb{Z}\pi) \\ x \mapsto f_x^*$$

is an isomorphism.

PROOF. To see that I is a left $\mathbb{Z}\pi$ -module homomorphism, let $g \in \pi$. We compute that

$$I(gx)(\varphi) = f_{gx}^*(\varphi) = \overline{\varphi(gx)} = \overline{g \cdot \varphi(x)} = \overline{\varphi(x)} \cdot \overline{g} = f_x^*(\varphi) \cdot \overline{g} = (g \cdot f_x^*)(\varphi) = gI(x)(\varphi).$$

Now let (D_*, d_*) denote the cellular $\mathbb{Z}\pi$ -module chain complex of K . Consider the exact sequence

$$D^1 \xrightarrow{d^2} D^2 \rightarrow H^2(K; \mathbb{Z}\pi) \rightarrow 0.$$

Since the functor $\text{Hom}_{\mathbb{Z}\pi}(-, \mathbb{Z}\pi)$ is left exact, we obtain an exact sequence

$$0 \rightarrow \text{Hom}_{\mathbb{Z}\pi}(H^2(K; \mathbb{Z}\pi), \mathbb{Z}\pi) \rightarrow \text{Hom}_{\mathbb{Z}\pi}(D^2, \mathbb{Z}\pi) \xrightarrow{(d^2)^*} \text{Hom}_{\mathbb{Z}\pi}(D^1, \mathbb{Z}\pi)$$

and since D_1 and D_2 are finitely generated, free $\mathbb{Z}\pi$ -modules, this is isomorphic to the sequence

$$0 \rightarrow \text{Hom}_{\mathbb{Z}\pi}(H^2(K; \mathbb{Z}\pi), \mathbb{Z}\pi) \rightarrow D_2 \xrightarrow{d_2} D_1.$$

Hence $\text{Hom}_{\mathbb{Z}\pi}(H^2(K; \mathbb{Z}\pi), \mathbb{Z}\pi)$ is isomorphic to $\ker d_2 = H_2(K; \mathbb{Z}\pi) \cong \pi_2(K)$. The isomorphism $\text{Hom}(D^2, \mathbb{Z}\pi) \rightarrow D_2$ sends f_x^* , considered as a function $D^2 \rightarrow \mathbb{Z}\pi$, to $f(1) = x$, considered as an element in D_2 . \square

On the other hand, if we start with $\text{coker } d^2 = H^2(K; \mathbb{Z}\pi)$, the evaluation map is not necessarily an isomorphism.

Lemma 3.13. *Let (C_*, d_*) be a free resolution of \mathbb{Z} as a $\mathbb{Z}\pi$ -module. There is an exact sequence*

$$0 \rightarrow H^2(\pi; \mathbb{Z}\pi) \rightarrow \text{coker } d^2 \xrightarrow{\text{ev}} \text{Hom}_{\mathbb{Z}\pi}(\ker d_2; \mathbb{Z}\pi) \rightarrow H^3(\pi; \mathbb{Z}\pi) \rightarrow 0$$

where the map ev sends $[f] \in \text{coker } d^2$ with $f \in C^2$ to $(x \mapsto f(x))$.

PROOF. Consider the dual chain complex (C^*, d^*) . We have an exact sequence

$$(3.14) \quad 0 \rightarrow H^2(\pi; \mathbb{Z}\pi) \rightarrow \text{coker } d^2 \xrightarrow{d^3} \ker d^4 \rightarrow H^3(\pi; \mathbb{Z}\pi) \rightarrow 0.$$

From the exactness of (C_*, d_*) it follows that $\ker d_2 = \text{im } d_3$ and that $C_4 \xrightarrow{d_4} C_3 \xrightarrow{d_3} \text{im } d_3 \rightarrow 0$ is exact. By left exactness of $\text{Hom}_{\mathbb{Z}\pi}(-, \mathbb{Z}\pi)$, we deduce that $d^3: \text{Hom}_{\mathbb{Z}\pi}(\text{im } d_3, \mathbb{Z}\pi) \xrightarrow{\cong} \ker d^4$. Therefore $\ker d^4 \cong \text{Hom}_{\mathbb{Z}\pi}(\ker d_2, \mathbb{Z}\pi)$. Substituting into (3.14) we obtain the exact sequence

$$0 \rightarrow H^2(\pi; \mathbb{Z}\pi) \rightarrow \text{coker } d^2 \rightarrow \text{Hom}_{\mathbb{Z}\pi}(\ker d_2; \mathbb{Z}\pi) \rightarrow H^3(\pi; \mathbb{Z}\pi) \rightarrow 0.$$

The middle map now used d^3 and $(d^3)^{-1}$, so is given simply by the evaluation as claimed. \square

Remark 3.15. Let M be a closed oriented 4-manifold with $\pi_1(M) = \pi$ and let (C_*^M, d_*^M) be a $\mathbb{Z}\pi$ -chain complex arising from a handle decomposition. Let $C_*^M[0, 2]$ be the 2-dimensional truncation of C_*^M and let $j: C_*^M[0, 2] \rightarrow C_*^M$ be the inclusion. Then the exact sequences from Lemma 3.13 and from Proposition 3.3 fit into the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \rightarrow & H^2(\pi; \mathbb{Z}\pi) & \rightarrow & H_2(M; \mathbb{Z}\pi) & \xrightarrow{\lambda_M} & \text{Hom}_{\mathbb{Z}\pi}(H_2(M; \mathbb{Z}\pi), \mathbb{Z}\pi) & \rightarrow & H^3(\pi; \mathbb{Z}\pi) & \rightarrow & 0 \\ & & \downarrow = & & \downarrow j^* \circ PD^{-1} & \searrow PD^{-1} & \downarrow (j_*)^* & & \downarrow = & & \\ & & H^2(\pi; \mathbb{Z}\pi) & \rightarrow & \text{coker } d_M^2 & \xrightarrow{\text{ev}} & \text{Hom}_{\mathbb{Z}\pi}(\ker d_M^2, \mathbb{Z}\pi) & \rightarrow & H^3(\pi; \mathbb{Z}\pi) & \rightarrow & 0 \end{array}$$

This identifies the failure of the equivariant intersection form to be nonsingular with the failure of the map ev from Lemma 3.13 to be an isomorphism.

As above, consider a CW-complex model for $B\pi$ with finite 2-skeleton K . Now we define a map ι , which will be important for relating the secondary invariant to pairings.

Definition 3.16. Consider the Leray-Serre spectral sequence computing $H_*(K; \mathbb{Z}/2)$ from the fibration $\tilde{K} \rightarrow K \rightarrow B\pi$ with E^2 -term $E_{p,q}^2 := H_p(\pi; H_q(\tilde{K}; \mathbb{Z}/2))$. Identify $H_2(\tilde{K}; \mathbb{Z}/2)$ with $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \pi_2(K)$. Then $H_0(\pi; H_2(\tilde{K}; \mathbb{Z}/2)) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K)$. Since $H_1(\tilde{K}; \mathbb{Z}/2) = 0$, the entire $q = 1$ row of the E^2 page vanishes, so the d_3 differential gives a map:

$$\iota := d_3: H_3(\pi; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K).$$

Lemma 3.17. *The map $\iota: H_3(\pi; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K)$ is injective.*

PROOF. In the spectral sequence, $\ker \iota \cong E_{3,0}^\infty$. Since $E_{3,0}^\infty$ is isomorphic to the quotient of $H_3(K; \mathbb{Z}/2)$ by largest subgroup in the filtration of it determined by the Leray-Serre spectral sequence, we obtain a surjection $H_3(K; \mathbb{Z}/2) \rightarrow \ker \iota$. Since K is 2-dimensional, we have $H_3(K; \mathbb{Z}/2) = 0$ and thus ι is injective. \square

Definition 3.18. Let $H := H^2(K; \mathbb{Z}\pi)$. Define

$$A: H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H))$$

to be the composition $\Psi_H \circ (\text{Id} \times I) \circ \iota$.

Lemma 3.19. *For any other choice K' as the 2-skeleton of $B\pi$ and every map $\varphi: K \rightarrow K'$ that induces the identity on fundamental groups, the diagram*

$$\begin{array}{ccc} H_3(\pi; \mathbb{Z}/2) & \xrightarrow{A} & \widehat{H}^0(\text{Sesq}(H)) \\ & \searrow^{A'} & \downarrow \varphi_* \\ & & \widehat{H}^0(\text{Sesq}(H')) \end{array}$$

commutes, where A' and H' are the analogous definitions of A and H for K' instead of K . In particular, the kernel of A does not depend on the choice of finite 2-complex K with $\pi_1(K) = \pi$, but only on π .

PROOF. Consider the following diagram where ι' denotes the map ι for K' instead of K .

$$\begin{array}{ccccc} H_3(\pi; \mathbb{Z}/2) & \xrightarrow{\iota} & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K) & \xrightarrow{\Psi_H} & \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))) \\ & \searrow^{\iota'} & \downarrow & & \downarrow \\ & & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K') & \xrightarrow{\Psi_{H'}} & \widehat{H}^0(\text{Sesq}(H^2(K'; \mathbb{Z}\pi))) \end{array}$$

Here we use the isomorphism $I: \pi_2(K) \cong H^*$ from Lemma 3.12 in order to consider Ψ_H as a map from $\mathbb{Z}/2 \otimes \pi_2(K)$ to $\widehat{H}^0(\text{Sesq}(H))$, and similarly for $H' := H^2(K'; \mathbb{Z}\pi)$. The triangle commutes by naturality of the spectral sequence defining ι , and the square commutes by naturality of Ψ .

It follows that the kernel of A is contained in the kernel of A' . By switching the rôles of K and K' , the kernel of A is independent of the choice of K . \square

In light of the previous lemma, $\text{Tate}(\pi) := H_3(\pi; \mathbb{Z}/2) / \ker(A)$ is independent of the choice of K and we can formulate the following condition.

Condition 3.20. The sequence

$$H_5(\pi; \mathbb{Z}) \xrightarrow{\text{Sq}_2^{\text{ored}_2}} H_3(\pi; \mathbb{Z}/2) \xrightarrow{A_\pi} \text{Tate}(\pi)$$

is exact at $H_3(\pi; \mathbb{Z}/2)$, where A_π is the projection.

Remark 3.21. By definition, Condition 3.20 is equivalent to the exactness of the sequence

$$H_5(\pi; \mathbb{Z}) \xrightarrow{\text{Sq}_2^{\text{ored}_2}} H_3(\pi; \mathbb{Z}/2) \xrightarrow{A} \widehat{H}^0(\text{Sesq}(H))$$

for any choice of 2-complex K and $H := H^2(K; \mathbb{Z}\pi)$ as in Definition 3.18.

By the end of this section we will have proven the following theorem.

Theorem 3.22. *If Condition 3.20 holds for π , then π has the Secondary Property 1.11.*

Thus we will have reduced the Secondary Property, for a given group π , to verifying Condition 3.20 for that group. The classes of group for which we have been able to show that Condition 3.20 holds are described in Sections 6.1 and 6.2. In [KLPT17], we proved the Secondary Property by constructing model manifolds and computing their intersection forms and their secondary invariant in $H_3(\pi; \mathbb{Z}/2)$. Here we cannot construct model manifolds, but as long as we can

compute $H_3(\pi; \mathbb{Z}/2)$ and the map $\text{Sq}_2 \circ \text{red}_2$, then by 3.22 we can test the Secondary Property on “model” intersection forms produced by the map A .

The following lemma will not be used in this section, but it may become useful in the future because, unlike the Tate group $\widehat{H}^0(\text{Sesq}(H))$, the subgroup of *weakly even* hermitian forms modulo even forms on a $\mathbb{Z}\pi$ -module H is unchanged under adding projective modules $H \mapsto H \oplus P$.

Recall the exact sequence from the Leray-Serre spectral sequence for $\widetilde{K} \rightarrow K \rightarrow B\pi$

$$0 \longrightarrow H_3(\pi; \mathbb{Z}/2) \xrightarrow{\iota} \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} H_2(K; \mathbb{Z}\pi) \xrightarrow{H_2(\varepsilon_2)} H_2(K; \mathbb{Z}/2),$$

where $\varepsilon_2: \mathbb{Z}\pi \rightarrow \mathbb{Z}/2$ is the nontrivial ring homomorphism, inducing the map on coefficients in the final map. Also recall that $H := H^2(K; \mathbb{Z}\pi)$ has dual module $H^* \cong H_2(K; \mathbb{Z}\pi) \cong \pi_2(K)$ and that we defined a map $\Psi_H: \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} H^* \rightarrow \widehat{H}^0(\text{Sesq}(H))$ via

$$\Psi_H(1 \otimes f)(h_1, h_2) = f(h_1) \cdot \overline{f(h_2)} \text{ for } f \in H^*, h_i \in H.$$

Lemma 3.23. *For every $f \in H^*$, the hermitian form $\Psi_H(1 \otimes f)$ is weakly even if and only if $1 \otimes f$ is in the image of ι , or equivalently, if and only if $H_2(\varepsilon_2)(1 \otimes f) = 0$.*

PROOF. For $h \in H$, write $f(h) = \sum_{i=1}^n m_i g_i$ and compute in $\mathbb{Z}\pi$

$$f(h) \cdot \overline{f(h)} = \sum_{i,j=1}^n m_i m_j (g_i \bar{g}_j) = \sum_{i=1}^n (m_i)^2 + \sum_{i < j} m_i m_j (g_i \bar{g}_j + g_j \bar{g}_i).$$

The second sum is of the form $a + \bar{a}$ and the first sum is just a multiple of the unit. That multiple $\sum_{i=1}^n (m_i)^2$ is modulo 2 the same as $\sum_{i=1}^n m_i \equiv \varepsilon_2(f(h))$, so we conclude that the above element is of the form $a + \bar{a}$ if and only if $\varepsilon_2(f(h)) = 0$.

The property of $\Psi_H(1 \otimes f)$ being weakly even means that $\Psi_H(1 \otimes f)(h, h)$ lies in the subgroup $\{a + \bar{a} \mid a \in \mathbb{Z}\pi\}$ for all $h \in H$. By the above, this is equivalent to the vanishing of the composition $\varepsilon_2 \circ f: H \rightarrow \mathbb{Z}\pi \rightarrow \mathbb{Z}/2$. Finally, we use that $H^2(\varepsilon_2): H^2(K; \mathbb{Z}\pi) \rightarrow H^2(K; \mathbb{Z}/2)$ is surjective because the next term in the Bockstein sequence is $H^3(K; \ker(\varepsilon_2)) = 0$. Hence the vanishing of $\varepsilon_2 \circ f$ is equivalent to the vanishing of $H_2(\varepsilon_2)(f) = H_2(\varepsilon_2)(1 \otimes f)$, which is our original claim. \square

3.4. Proof of the Secondary Property Theorem

Our aim for this section is to prove Theorem 3.22. However, before we start the proof, we switch to a dual picture that we find convenient for the statements and the proofs of some imminent lemmas. By [KPT18, Proposition 1.7], the short exact sequence

$$0 \rightarrow \ker d_2 \rightarrow C_2 \oplus H_2(M; \mathbb{Z}\pi) \rightarrow \text{coker } d_3 \rightarrow 0.$$

splits if and only if the primary obstruction vanishes. Recall that we write $C_* := C_*(M; \mathbb{Z}\pi)$ for the handle chain complex coming from a choice of handle decomposition of M , with boundary maps d_i .

In the discussion of the secondary obstruction we work with a slightly different “dual” formulation. For this, let (C'_*, d'_*) denote the $\mathbb{Z}\pi$ -module chain complex coming from a dual handle decomposition of M . We denote M endowed with the dual handle decomposition by M^d . The above sequence for M^d has the form

$$0 \rightarrow \ker d'_2 \rightarrow C'_2 \oplus H_2(M^d; \mathbb{Z}\pi) \rightarrow \text{coker } d'_3 \rightarrow 0.$$

We can identify the chain complex of the dual handle decomposition with the dual of the chain complex of the first handle decomposition, i.e. $C'_* \cong C^{4-*}$. Apply this to the above sequence to yield

$$0 \rightarrow \ker d^3 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi) \rightarrow \operatorname{coker} d^2 \rightarrow 0.$$

Since the primary invariant is independent of the chosen handle decomposition of M , the last sequence splits if and only if $\mathbf{pri}(M) = 0$. Note that $\operatorname{coker} d^2 \cong H^2(M^{(2)}; \mathbb{Z}\pi)$.

Next, we show that every splitting of this dual short exact sequence can be realised by a geometric map $M^{(3)} \rightarrow M^{(2)}$.

Proposition 3.24. *For every splitting*

$$s = (s_1, s_2): H^2(M^{(2)}; \mathbb{Z}\pi) \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi),$$

there is a map $f: M^{(3)} \rightarrow M^{(2)}$ such that

$$f^* = s_2: H^2(M^{(2)}; \mathbb{Z}\pi) \rightarrow H^2(M^{(3)}; \mathbb{Z}\pi) \cong H^2(M; \mathbb{Z}\pi).$$

PROOF. Let $j = (j_1, j_2): C^2 \oplus H^2(M; \mathbb{Z}\pi) \rightarrow H^2(M^{(2)}; \mathbb{Z}\pi)$ be the map from the short exact sequence above, so j is split by $s = (s_1, s_2)$. That is, j_1 is the projection $C^2 \rightarrow C^2 / \operatorname{im} d^2$, and $j_2: \ker d^3 / \operatorname{im} d^2 \hookrightarrow C^2 / \operatorname{im} d^2$ is the inclusion.

Since s is a splitting, we have

$$j_1(c) = j_1 s_1 j_1(c) + j_2 s_2 j_1(c), \text{ so that } j_1(c - s_1 j_1(c)) = j_2 s_2 j_1(c).$$

Thus $j_1(c - s_1 j_1(c)) \in \operatorname{im} j_2$, and so in particular $c - s_1 j_1(c) \in \ker d^3$.

Define a map $\rho: C^2 \rightarrow \ker d^3 \subset C^2$ by $c \mapsto c - s_1 j_1(c)$. Furthermore, for all $e \in C^1$,

$$(\rho \circ d^2)(e) = d^2(e) - s_1 j_1 d^2(e) = d^2(e) - s_1(0) = d^2(e).$$

Therefore, we can consider the following commutative diagram, in which we use the dual ρ^* of $\rho: C^2 \rightarrow C^2$, composed with the identification of C_i with its double dual.

$$\begin{array}{ccccccc} & & & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 \\ & & & \uparrow \rho^* & & \uparrow \operatorname{Id} & & \uparrow \operatorname{Id} \\ & & 0 & \nearrow & & & & \\ C_3 & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 \end{array}$$

We will show that this chain map can be realised as a map $f: M^{(3)} \rightarrow M^{(2)}$. To see this, start with the identity map $M^{(2)} \rightarrow M^{(2)}$. Since $d_2 \circ \rho^* - d_2 = (\rho \circ d^2)^* - d_2 = (d^2)^* - d_2 = 0$, we can change this map on the 2-cells by elements of $\pi_2(M^{(2)})$ according to $\rho^* - \operatorname{Id}$, i.e. pinch off a 2-sphere from each 2-cell of $M^{(2)}$, and map this 2-sphere to the image of the 2-cell under $\rho^* - \operatorname{Id}: C_2 \rightarrow \ker d_2 = \pi_2(M^{(2)})$. Since $\rho^* \circ d_3 = (d^3 \circ \rho)^* = 0$, the attaching maps of the 3-cells are null homotopic under the new map $M^{(2)} \rightarrow M^{(2)}$. Therefore it can be extended to a map $f: M^{(3)} \rightarrow M^{(2)}$, as desired.

It remains to show that the map f^* induced by f on second cohomology is s_2 . Consider the following diagram.

$$\begin{array}{ccccc}
C^2/\text{im } d^2 & \xrightarrow{s_2} & \ker d^3/\text{im } d^2 & & \\
\uparrow j_1 & & & \searrow j_2 & \\
C^2 & \xrightarrow{\rho=\text{Id}-s_1j_1} & \ker d^3 \subset C^2 & \xrightarrow{j_1} & C^2/\text{im } d^2 \\
\downarrow j_1 & & \downarrow & \nearrow j_2 & \\
C^2/\text{im } d^2 & \xrightarrow{f^*} & \ker d^3/\text{im } d^2 & &
\end{array}$$

The bottom left square commutes since f was constructed to realise the chain level map ρ^* . The top trapezium commutes by the formula $j_1(c - s_1j_1(c)) = j_2s_2j_1(c)$ from above. It is straightforward to see that the bottom right triangle commutes. Since j_1 is surjective and j_2 injective, it follows from the equality of the top and bottom routes that $f^* = s_2: H^2(M^{(2)}; \mathbb{Z}\pi) \rightarrow H^2(M^{(3)}; \mathbb{Z}\pi) \cong H^2(M; \mathbb{Z}\pi)$. \square

Let X be a CW complex model for $B\pi$ with finite 2-skeleton K . Let $i: K \rightarrow X^{(3)}$ denote the inclusion of the 2-skeleton into the 3-skeleton. Let $\alpha \in \pi_3(M^{(3)})$ denote the class of the attaching map of the (unique) 4-cell of M .

Lemma 3.25. *The map $i: K \rightarrow X^{(3)}$ induces the trivial map $\pi_3(K) \xrightarrow{0} \pi_3(X^{(3)})$. In particular, for every map $f: M^{(3)} \rightarrow K$ we have $i_*f_*(\alpha) = 0 \in \pi_3(K)$, and therefore the map $i \circ f$ can be extended to a map $\hat{f}: M \rightarrow X^{(3)}$.*

PROOF. Since the maps in Whitehead's sequence (Theorem 2.13) are natural, we have a commutative diagram:

$$\begin{array}{ccccccc}
0 = H_4(\tilde{K}; \mathbb{Z}) & \longrightarrow & \Gamma(\pi_2(K)) & \xrightarrow{\cong} & \pi_3(K) & \longrightarrow & H_3(\tilde{K}; \mathbb{Z}) = 0 \\
\downarrow & & \downarrow i_* & & \downarrow i_* & & \downarrow \\
0 = H_4(\tilde{X}^{(3)}; \mathbb{Z}) & \longrightarrow & \Gamma(\pi_2(X^{(3)})) & \longrightarrow & \pi_3(X^{(3)}) & \longrightarrow & H_3(\tilde{X}^{(3)}; \mathbb{Z}).
\end{array}$$

But $\pi_2(X^{(3)}) = 0$, and hence $i_*: \pi_3(K) \rightarrow \pi_3(X^{(3)})$ is trivial. \square

We obtain the following corollary to Proposition 3.24.

Corollary 3.26. *There exists a map $f: M^{(3)} \rightarrow K$ that induces an isomorphism $\pi_1(M^{(3)}) \rightarrow \pi_1(K)$ if and only if $\text{pri}(M) = 0$.*

PROOF. Recall that all 2-complexes with the same fundamental group become homotopy equivalent after wedging with sufficiently many 2-spheres, by Lemma 1.8. Let K and K' be two such 2-complexes. We have a sequence of maps

$$K' \rightarrow K' \vee \bigvee^m S^2 \xrightarrow{\cong} K \vee \bigvee^{m'} S^2 \rightarrow K,$$

where the first map is inclusion, the second map is a homotopy equivalence, and the final map is a collapse map. The composition induces an isomorphism on fundamental groups, so it is enough to show that there is a map $M^{(3)} \rightarrow K'$ for one 2-complex K' with $\pi_1(K') \cong \pi$. We will use $K' = M^{(2)}$. If $\text{pri}(M) = 0$, then the sequence

$$0 \rightarrow \ker d^3 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi) \rightarrow \text{coker } d^2 = H^2(M^{(2)}; \mathbb{Z}\pi) \rightarrow 0$$

splits and there is a map $f: M^{(3)} \rightarrow M^{(2)}$ by Proposition 3.24, as required.

On the other hand, if there is a map $f: M^{(3)} \rightarrow K$, then we can extend f to a map $\widehat{f}: M \rightarrow X^{(3)}$ by Lemma 3.25. If f is an isomorphism on fundamental groups, then so is \widehat{f} . By Proposition 2.7 and the discussion preceding it, this implies $\mathfrak{pri}(M) = 0$. \square

From now on in this section we will now only consider manifolds M with $\mathfrak{pri}(M) = 0$. Let $f: M^{(3)} \rightarrow K$ be a fixed map that is an isomorphism on fundamental groups, which exists by Corollary 3.26. Let $\alpha \in \pi_3(M^{(3)})$ denote the class of the attaching map of the 4-cell of M (we may and will assume that M has a unique 4-cell).

By Lemma 3.25, f can be extended to a map $\widehat{f}: M \rightarrow X^{(3)}$. The invariant

$$\mathfrak{sec}(M) \in H_3(\pi; \mathbb{Z}/2)/\text{im}(d_{5,0}^2)$$

from Proposition 2.7 is independent of the choice of homotopy of the classifying map $M \rightarrow B\pi$ to a map $M \rightarrow B\pi^{(3)} = X^{(3)}$. If we fix a map $\widehat{f}: M \rightarrow X^{(3)}$, then we can consider an element $\mathfrak{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$, defined as in Lemma 2.5. Note that $\mathfrak{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$ is independent of the homotopy class of \widehat{f} viewed as a map $M \rightarrow X^{(4)}$.

We briefly recall the construction here, since we will need the details in the near future. Take a regular preimage F_i of a barycentre for every 3-cell e_i of X . A framing of the normal bundle of this point pulls back to a framing of the normal bundle of the regular preimage $F_i \subseteq D^4 \subseteq M$ and thus gives the preimage a framing of the normal bundle. Thus we can consider $[F_i] \in \Omega_1^{\text{fr}} \cong \Omega_1^{\text{Spin}} \cong \mathbb{Z}/2$. Then $\sum_i [F_i]e_i \in \mathbb{Z}/2 \otimes C_3(X)$ lies in $\ker(\mathbb{Z}/2 \otimes C_3(X) \rightarrow \mathbb{Z}/2 \otimes C_2(X))$, and so defines a homology class $\mathfrak{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$.

Lemma 3.27. *The element $\mathfrak{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$, where $\widehat{f}: M \rightarrow X^{(3)}$ is an extension of $i \circ f: M^{(3)} \rightarrow K \rightarrow X^{(3)}$, only depends on the map f .*

PROOF. Note that any two extensions of f to \widehat{f} only differ by an element of $\pi_4(X^{(3)})$, that is obtained from the two choices of extension on the unique 4-cell of M . Consider the diagram

$$\begin{array}{ccc} \pi_4(X^{(3)}) & \longrightarrow & \pi_4(X^{(4)}) \\ \downarrow h & & \cong \downarrow h \\ H_4(X^{(3)}; \mathbb{Z}\pi) = 0 & \longrightarrow & H_4(X^{(4)}; \mathbb{Z}\pi), \end{array}$$

where h denotes Hurewicz maps, which is commutative by naturality of the Hurewicz homomorphism. It follows that $\pi_4(X^{(3)}) \rightarrow \pi_4(X^{(4)})$ is the trivial map, so any two choices of extension \widehat{f} are homotopic over $X^{(4)}$. Hence $\mathfrak{sec}(M, \widehat{f})$ only depends on f , and not on the choice of \widehat{f} . \square

Definition 3.28. We define

$$\mathfrak{sec}(M, f) := \mathfrak{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2),$$

for an extension $\widehat{f}: M \rightarrow X^{(3)}$ of $i \circ f: M^{(3)} \rightarrow X^{(3)}$. Such an extension exists by Lemma 3.25 and $\mathfrak{sec}(M, f)$ is independent of the choice of extension by Lemma 3.27.

Consider the following diagram, which commutes by functoriality of Tate cohomology $\widehat{H}^0(-)$, and by the definition of $A: H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$. We will explain more about the maps in the diagram below. The proof of Theorem 3.22, that Condition 3.20 implies the Secondary Property, will be based on this diagram.

(3.29)

$$\begin{array}{ccccc}
\mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(3)}) & & H_3(\pi; \mathbb{Z}/2) & \xleftarrow{\iota} & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K) \\
\downarrow f_* & & \searrow A & & \downarrow \Theta_{H^2(K; \mathbb{Z}\pi)} \\
\mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(K) & \xrightarrow[\cong]{S} & \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K)) & \longrightarrow & \widehat{H}^0(\pi_2(K) \otimes_{\mathbb{Z}\pi} \pi_2(K)) \\
& & \downarrow \Phi_{H^2(K; \mathbb{Z}\pi)} & & \downarrow \widehat{H}^0(\Phi_{H^2(K; \mathbb{Z}\pi)}) \\
& & (\text{Sesq}(H^2(K; \mathbb{Z}\pi)))^{\mathbb{Z}/2} & \longrightarrow & \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))) \\
& & \uparrow \text{Sesq}(f^*) & & \uparrow \widehat{H}^0(\text{Sesq}(f^*)) \\
& & (\text{Sesq}(H^2(M; \mathbb{Z}\pi)))^{\mathbb{Z}/2} & \longrightarrow & \widehat{H}^0(\text{Sesq}(H^2(M; \mathbb{Z}\pi)))
\end{array}$$

As promised, some remarks on the diagram are in order.

- (1) In the bottom row, we could have written $M^{(3)}$ instead of M , since $M^{(3)}$ is the domain of the map f . But attaching a 4-cell makes no difference to second cohomology, so we allow ourselves this slight abuse.
- (2) Recall that for a set X and a group G that acts on X , X^G denotes the fixed points. Thus $(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))^{\mathbb{Z}/2}$ is the $\mathbb{Z}/2$ -fixed points, that is the kernel of the $\mathbb{Z}[\mathbb{Z}/2]$ -module homomorphism $1 - T: \text{Sesq}(H^2(K; \mathbb{Z}\pi)) \rightarrow \text{Sesq}(H^2(K; \mathbb{Z}\pi))$. In other words, the hermitian forms.
- (3) Also recall that, by Lemma 3.12, there is a canonical isomorphism

$$I: \pi_2(K) \xrightarrow{\cong} (H^2(K; \mathbb{Z}\pi))^*,$$

so that we can consider the map

$$\Phi_{H^2(K; \mathbb{Z}\pi)}: \pi_2(K) \otimes_{\mathbb{Z}\pi} \pi_2(K) \rightarrow \text{Sesq}(H^2(K; \mathbb{Z}\pi)).$$

In the diagram, we see the map induced on Tate cohomology by $\Phi_{H^2(K; \mathbb{Z}\pi)}$ in the right hand column, and we see the map restricted to the Γ group in the middle column.

- (4) The map $S := \text{Id}_{\mathbb{Z}} \otimes \Gamma(\eta)$ is an isomorphism by Corollary 2.14, where we use that $H_4(\widetilde{K}) = H_3(\widetilde{K}) = 0$.
- (5) The map $\iota: H_3(\pi; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K)$ is an injection by Lemma 3.17.

We will show that the image of $\text{sec}(M, f) \in H_3(\pi; \mathbb{Z}/2)$ in $\widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$ agrees with the image of $\lambda_M \in (\text{Sesq}(H^2(M; \mathbb{Z}\pi)))^{\mathbb{Z}/2}$ in $\widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$. Here the image of $\text{sec}(M, f)$ is $A(\text{sec}(M, f))$ (the map A was introduced in Definition 3.18). The image of λ_M is the intersection form restricted to $H^2(K; \mathbb{Z}\pi)$ along f^* , modulo even forms. To show that the images coincide we will show that both of them agree with a third element of $\widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$, namely the image of $[\alpha] \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(3)})$ in $\widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$; recall that $\alpha \in \pi_3(M^{(3)})$ is the attaching map of the 4-cell of M . We abuse notation and write α for $1 \otimes \alpha \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(3)})$.

Lemma 3.30. *We have an equality*

$$\Theta_{H^2(K; \mathbb{Z}\pi)} \circ \iota(\mathbf{sec}(M, f)) = [S \circ f_*(\alpha)] \in \widehat{H}^0(\pi_2(K) \otimes_{\mathbb{Z}\pi} \pi_2(K)).$$

PROOF. We have that $(M\#(S^2 \times S^2))^{(3)} \simeq M^{(3)} \vee S^2 \vee S^2$, and under this stabilisation the attaching map of the 4-cell changes by the Whitehead product of two identity maps $S^2 \rightarrow S^2$. Let $\alpha' \in \pi_3(M^{(3)} \vee S^2 \vee S^2)$ denote the new attaching map of the 4-cell. For any two elements $a, b \in \pi_2(K)$, we can consider the map $f \vee a \vee b: M^{(3)} \vee S^2 \vee S^2 \rightarrow K$ and obtain

$$S \circ (f \vee a \vee b)_* \alpha' = S \circ f_* \alpha + a \otimes b + b \otimes a \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K)).$$

Hence

$$[S \circ (f \vee a \vee b)_* \alpha'] = [S \circ f_* \alpha] \in \widehat{H}^0(\pi_2(K) \otimes_{\mathbb{Z}\pi} \pi_2(K)).$$

On the other hand, if we extend f and $f \vee a \vee b$ to $X^{(3)}$ via the inclusion $K \rightarrow X^{(3)}$, then they become bordant maps over $X^{(3)}$, since $X^{(3)}$ is built from K by adding 3-cells to kill $\pi_2(K)$. It follows that $\mathbf{sec}(M, f) = \mathbf{sec}(M\#(S^2 \times S^2), f \vee a \vee b)$. Thus the lemma holds for (M, f) if and only if it holds true for $(M\#(S^2 \times S^2), f \vee a \vee b)$.

Up to elements of the form $a \otimes b + b \otimes a$, every element in $\Gamma(\pi_2(K))$ can be written as $\beta \otimes \beta$ for some $\beta \in \pi_2(K)$. Thus by the previous paragraph we can and will assume that the attaching map of the 4-cell $\alpha \in \pi_3(M^{(3)})$ is mapped to $\beta \otimes \beta$ under the composition

$$\Gamma(\eta)^{-1} \circ f_*: \pi_3(M^{(3)}) \rightarrow \pi_3(K) \rightarrow \Gamma(\pi_2(K)).$$

It follows from Lemma 2.12 that $f_*(\alpha) = \beta \circ \eta$, where $\eta: S^3 \rightarrow S^2$ is the Hopf map.

Then f extends to a map $\widehat{f}: M \rightarrow K \cup_{\beta} D^3$, where the map from the top cell of M to D^3 is the cone on $\eta: S^3 \rightarrow S^2$.

Attach further 3-cells to turn $K \cup_{\beta} D^3$ into the 3-skeleton $B\pi^{(3)}$ of a model for $B\pi$. Therefore, we obtain $\mathbf{sec}(M, f) = \mathbf{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$ by the construction given in the proof of Lemma 3.27, where technically speaking we extend \widehat{f} further by the inclusion $\widehat{f}: M \rightarrow K \cup_{\beta} D^3 \rightarrow B\pi^{(3)}$. This is a particular choice of extension \widehat{f} , as in the proof of Lemma 3.27. Recall that for the definition of $\mathbf{sec}(M, f)$, we take a regular preimage F_i of a barycentre for every 3-cell e_i of $X^{(3)}$, where $X \simeq B\pi$, and then $\sum_i [F_i] e_i$ determines $\mathbf{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$, where $[F_i]$ is the framed or spin bordism class of F_i .

We assert that in the above construction $[F_i] = 1$ if and only if $e_i = e_{\beta}$, the first cell attached along β . If $e_i \neq e_{\beta}$, then F_i is empty since \widehat{f} factors through $K \cup_{\beta} D^3 \rightarrow B\pi^{(3)}$. To see that $[F_{\beta}] = 1$, note that it was a regular preimage of $C\eta: D^4 \rightarrow D^3$, and thus of $\Sigma\eta$, since $\Sigma\eta: S^4 \rightarrow S^3$ arises from gluing together two copies of $C\eta: D^4 \rightarrow D^3$, and we can assume the point whose preimage we take to be in one of the $C\eta$ halves. Hence $[F_{\beta}] = 1$ follows from

$$\Sigma\eta = 1 \in \pi_1^{st} \cong \Omega_1^{fr} \cong \Omega_1^{Spin} \cong \mathbb{Z}/2.$$

Hence $\mathbf{sec}(M, f) = [e_{\beta}] \in H_3(\pi; \mathbb{Z}/2)$. Consider the diagram

$$\begin{array}{ccccc} H_3(K; \mathbb{Z}/2) & \longrightarrow & H_3(\pi; \mathbb{Z}/2) & \xrightarrow{\iota} & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K) \\ \downarrow & & \downarrow & & \downarrow \\ H_3(K \cup_{\beta} D^3; \mathbb{Z}/2) & \longrightarrow & H_3(\pi; \mathbb{Z}/2) & \xrightarrow{\iota} & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K \cup_{\beta} D^3) \end{array}$$

Since $[e_\beta]$ comes from $H_3(K \cup_\beta D^3; \mathbb{Z}/2)$ it maps to zero in $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K \cup_\beta D^3)$ and thus $\iota([e_\beta]) \in \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K)$ lies in the kernel of the map to $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K \cup_\beta D^3)$. This kernel precisely consists of $1 \otimes \beta$. Hence $\iota(\mathbf{sec}(M, f)) = 1 \otimes \beta$ and thus

$$\Theta_{H^2(K; \mathbb{Z}\pi)} \circ \iota(\mathbf{sec}(M, f)) = [\beta \otimes \beta] = [S \circ f_*(\alpha)]. \quad \square$$

Lemma 3.31. *Let $\alpha \in \pi_3(M^{(3)})$ be the attaching map of the 4-cell of M , and let $f: M^{(3)} \rightarrow K$ be a map that induces an isomorphism on fundamental groups. We have an equality:*

$$[\Phi_{H^2(K; \mathbb{Z}\pi)}(S \circ f_*(\alpha))] = [\text{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))).$$

PROOF. Arguing as in the first two paragraphs of the proof of Lemma 3.30, it suffices to consider the case $f_*(\alpha) = \beta \circ \eta$ for some $\beta \in \pi_2(K)$. Denote the extension of $f: M^{(3)} \rightarrow K$ to $M \rightarrow K \cup_{\beta \circ \eta} D^4$ again by f . Then for any $x, y \in H^2(K; \mathbb{Z}\pi)$ we have

$$\begin{aligned} \lambda_M(PD(f^*(x)), PD(f^*(y))) &= \langle f^*(y), PD(f^*(x)) \rangle \\ &= \langle y, f_*(f^*(x) \cap [M]) \rangle = \langle y, x \cap f_*[M] \rangle. \end{aligned}$$

Denote the extension of $\beta: S^2 \rightarrow K$ by $\widehat{\beta}: \mathbb{C}\mathbb{P}^2 \rightarrow K \cup_{\beta \circ \eta} D^4$. Then $\widehat{\beta}_*[\mathbb{C}\mathbb{P}^2] = f_*[M]$ and we have

$$\langle y, x \cap \widehat{\beta}_*[\mathbb{C}\mathbb{P}^2] \rangle = \langle \widehat{\beta}^*(y), \widehat{\beta}^*(x) \cap [\mathbb{C}\mathbb{P}^2] \rangle = \widehat{\beta}^*(x) \langle 1, 1 \cap [\mathbb{C}\mathbb{P}^2] \rangle \overline{\widehat{\beta}^*(y)},$$

where $1 \in H^2(\mathbb{C}\mathbb{P}^2; \mathbb{Z}\pi) \cong \mathbb{Z}\pi$ denotes a chosen generator. For the second equation we used the sesquilinearity of the evaluation. By the cup product form of $\mathbb{C}\mathbb{P}^2$ we have $\langle 1, 1 \cap [\mathbb{C}\mathbb{P}^2] \rangle = 1$ and since $\widehat{\beta}$ is given by β when restricted to the 2-skeleton, we obtain

$$\widehat{\beta}^*(x) \langle 1, 1 \cap [\mathbb{C}\mathbb{P}^2] \rangle \overline{\widehat{\beta}^*(y)} = \beta^*(x) \overline{\beta^*(y)} = \Phi_{H^2(K; \mathbb{Z}\pi)}(\beta^* \otimes \beta^*)(x \otimes y).$$

Combining the above equations we obtain

$$\begin{aligned} \lambda_M(PD(f^*(x)), PD(f^*(y))) &= \Phi_{H^2(K; \mathbb{Z}\pi)}(\beta^* \otimes \beta^*)(x \otimes y) \\ &= \Phi_{H^2(K; \mathbb{Z}\pi)}(S \circ (\beta \circ \eta))(x \otimes y). \end{aligned}$$

as claimed. \square

Theorem 3.32. *We have an equality*

$$[\text{Sesq}(f^*)(\lambda_M)] = A(\mathbf{sec}(M, f)) \in \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))).$$

PROOF. The theorem follows from combining Lemma 3.30 and Lemma 3.31, together with a straightforward diagram chase in diagram (3.29). \square

Corollary 3.33.

- (1) *If $\ker A \subseteq \text{im}(Sq^2 \circ \text{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_3(\pi; \mathbb{Z}/2))$, then $\mathbf{sec}(M) = 0$ if and only if there exists $f: M^{(3)} \rightarrow K$ as above with*

$$0 = [\text{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))).$$

- (2) *If $\text{im}(Sq^2 \circ \text{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_3(\pi; \mathbb{Z}/2)) \subseteq \ker A$, then the class*

$$[\text{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$$

is independent of the choice of f .

PROOF.

- (1) If $\mathbf{sec}(M) = 0$, then there exists an f with $\text{Sesq}(f^*)(\lambda_M) = 0$. This is because $\mathbf{sec}(M) = 0$ implies that there is a map $f: M \rightarrow B\pi^{(2)}$, and we can take K for $B\pi^{(2)}$. The intersection form vanishes on $f^*(H^2(K; \mathbb{Z}\pi))$ by naturality of the evaluation, because $H_*(K; \mathbb{Z}) = 0$.

Conversely, if there exists a map f with

$$0 = [\text{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))),$$

then by Theorem 3.32, $\mathbf{sec}(M, f) \in \ker A \subseteq \text{im}(Sq^2 \circ \text{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_3(\pi; \mathbb{Z}/2))$. Since $Sq^2 \circ \text{red}_2$ is the second differential in the Atiyah-Hirzebruch spectral sequence computing $\Omega_4^{\text{Spin}}(B\pi)$ (see Section 2.1), we have $\mathbf{sec}(M) = 0 \in H_3(\pi; \mathbb{Z}/2)/\text{im}(Sq^2 \circ \text{red}_2)$.

- (2) Given two choices $f, f': M^{(3)} \rightarrow K$, then there exists an $x \in \text{im}(Sq^2 \circ \text{red}_2)$ with $\mathbf{sec}(M, f) = \mathbf{sec}(M, f') + x$ since $Sq^2 \circ \text{red}_2$ is the second differential of the Atiyah-Hirzebruch spectral sequence and $\mathbf{sec}(M)$ is well-defined in $H_3(\pi; \mathbb{Z}/2)/\text{im}(S^2 \circ \text{red}_2)$. Thus by Theorem 3.32 we have

$$\begin{aligned} [\text{Sesq}(f^*)(\lambda_M)] &= A(\mathbf{sec}(M, f)) = A(\mathbf{sec}(M, f')) + A(x) = A(\mathbf{sec}(M, f')) \\ &= [\text{Sesq}((f')^*)(\lambda_M)]. \end{aligned}$$

Here we used that A is a homomorphism, and $A(x) = 0$ by assumption. \square

We can now apply Corollary 3.33 to prove Theorem 3.22, which says that under Condition 3.20 the Secondary Property 1.11 holds, which we recall here in an expanded form for the convenience of the reader:

If M is a spin manifold with $\pi_1(M) \xrightarrow{\cong} \pi$ and $\mathbf{pri}(M) = 0$, then $\mathbf{sec}(M) = 0$ if and only if there exists a splitting

$$s = (s_1, s_2): \text{coker } d_3 \rightarrow C_2 \oplus H_2(M; \mathbb{Z}\pi)$$

of the extension from Theorem 1.6 such that λ_M restricted to the image of s_2 is even. The restriction of λ_M for one splitting s is even if and only if it is even for every splitting.

Moreover, for two such manifolds M, M' with the same fundamental group we have $\mathbf{sec}(M) = \mathbf{sec}(M')$ if and only if the restrictions of λ_M and $\lambda_{M'}$ are isomorphic modulo even forms for some splittings s and s' .

PROOF OF THEOREM 3.22. Condition 3.20 says precisely that the hypotheses of Corollary 3.33 (1) and (2) hold. If $\mathbf{pri}(M) = 0$, then there is a splitting $s: \text{coker } d_3 \rightarrow C_2 \oplus H_2(M; \mathbb{Z}\pi)$.

We want to apply Proposition 3.24 to obtain a map $f: M^{(3)} \rightarrow K$ for a 2-complex K with $\pi_1(K) \cong \pi$. However note that Proposition 3.24 was formulated for the dual exact sequence

$$0 \rightarrow \ker d^3 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi) \rightarrow \text{coker } d^2 \rightarrow 0.$$

So to remedy this, for a splitting $s = (s_1, s_2): \text{coker } d_3 \rightarrow C_2 \oplus H_2(M; \mathbb{Z}\pi)$ of the sequence

$$0 \rightarrow \ker d_2 \rightarrow C_2 \oplus H_2(M; \mathbb{Z}\pi) \rightarrow \text{coker } d_3 \rightarrow 0$$

from Theorem 1.6, we apply Proposition 3.24 with the dual handle decomposition of M . Recall that M^d denotes M with the dual handle decomposition. We obtain a map $f: (M^d)^{(3)} \rightarrow (M^d)^{(2)}$ with $s_2 = f^*: H^2((M^d)^{(2)}; \mathbb{Z}\pi) \rightarrow H^2(M^d; \mathbb{Z}\pi)$

by Proposition 3.24, and use the identifications $H^2((M^d)^{(2)}; \mathbb{Z}\pi) \cong \text{coker } d_3$ and $H^2(M^d; \mathbb{Z}\pi) \cong H_2(M; \mathbb{Z}\pi)$.

Now by Corollary 3.33 (1), $\text{Sesq}(f^*)(\lambda_M) = \lambda_M|_{s_2(\text{coker } d_3)}$ is even (i.e. zero in the Tate cohomology $\widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$) if and only if $\mathfrak{sec}(M) = \mathfrak{sec}(M^d) = 0$. By Corollary 3.33 (2), $\text{Sesq}(f^*)(\lambda_M)$ is independent of the choice of f , and is therefore independent of the choice of splitting s .

The last sentence of the Secondary Property, that $\mathfrak{sec}(M) = \mathfrak{sec}(M')$ if and only if the restrictions of λ_M and $\lambda_{M'}$ are equal for some choices of splitting s and s' , also follows from Theorem 3.32. If $\mathfrak{sec}(M) = \mathfrak{sec}(M')$, then the difference lies in $\text{im}(Sq^2 \circ \text{red}_2) = \ker A$, so for some choice of f, f' we have $A(\mathfrak{sec}(M, f)) = A(\mathfrak{sec}(M', f'))$. Thus $[\text{Sesq}(f^*)(\lambda_M)] = [\text{Sesq}((f')^*)(\lambda_{M'})]$ by Theorem 3.32. On the other hand, $[\text{Sesq}(f^*)(\lambda_M)] = [\text{Sesq}((f')^*)(\lambda_{M'})]$ implies that $A(\mathfrak{sec}(M, f)) = A(\mathfrak{sec}(M', f'))$, so $\mathfrak{sec}(M, f) - \mathfrak{sec}(M', f') \in \ker A = \text{im}(Sq^2 \circ \text{red}_2)$. Thus $\mathfrak{sec}(M) = \mathfrak{sec}(M') \in H_3(\pi; \mathbb{Z}/2) / \text{im } d_{5,0}^2$. \square

The tertiary obstruction

Let M be a spin 4-manifold with an identification $\pi_1(M) \xrightarrow{\cong} \pi$, such that $\mathbf{pri}(M) = 0$ and $\mathbf{sec}(M) = 0$. The identification determines a classifying map $c: M \rightarrow B\pi$ uniquely up to homotopy.

In this chapter, more precisely in Theorem 4.16, we will give an algebraic criterion on π , under which we can identify the tertiary obstruction $\mathbf{ter}(M)$ with a τ invariant defined on a subset of $\pi_2(M)$. The reader should recall the definition of the Kervaire–Milnor invariant τ from Section 1.3.3.

4.1. Relating the tertiary obstruction with the Kervaire–Milnor invariant

For the well-definedness of the upcoming Definition 4.5, we have to recall the following statements from [KPT21] and [KLPT17]. Let K be a finite connected 2-complex homotopy equivalent to $M^{(2)}$. We view K as a 2-skeleton for $B\pi$ and denote the inclusion by $i: K \rightarrow B\pi$.

Lemma 4.1 ([KPT21, Lemma 4.2]). *Let $f: M^{(3)} \rightarrow K$ be a map that induces the given isomorphism on fundamental groups. Let $j: M^{(3)} \rightarrow M$ be the inclusion of the 3-skeleton. For every $\varphi \in \ker \mathrm{Sq}^2 \subseteq H^2(B\pi; \mathbb{Z}/2) = H^2(B\pi; \mathbb{Z}/2)$ and every lift $\varphi' \in H^2(K; \mathbb{Z}\pi)$ of $i^*\varphi \in H^2(K; \mathbb{Z}/2)$, the element $PD \circ (j^*)^{-1} \circ f^*(\varphi') \in H_2(M; \mathbb{Z}\pi) \cong \pi_2(M)$ is $\mathbb{R}\mathbb{P}^2$ -characteristic.*

In the statement of Lemma 4.1 we used the following sequence of maps:

$$\varphi' \in H^2(K; \mathbb{Z}\pi) \xrightarrow{f^*} H^2(M^{(3)}; \mathbb{Z}\pi) \xrightarrow{(j^*)^{-1}} H^2(M; \mathbb{Z}\pi) \xrightarrow{PD} H_2(M; \mathbb{Z}\pi) \cong \pi_2(M).$$

Lemma 4.2 ([KLPT17, Lemma 8.3 and Lemma 8.4]). *(1) Let $x, y \in \pi_2(M)$ be such that $\lambda(x, y) = 0$, $\mu(x)$ and $\mu(y)$ are trivial, and x is $\mathbb{R}\mathbb{P}^2$ -characteristic. Then for every element $\kappa \in \ker(\mathbb{Z}\pi \rightarrow \mathbb{Z}/2)$, we have $\tau(x) = \tau(x + \kappa y) \in \mathbb{Z}/2$.*

(2) Let Y be a finite 2-dimensional CW complex with fundamental group π . Every element in the kernel of $H^2(Y; \mathbb{Z}\pi) \rightarrow H^2(Y; \mathbb{Z}/2)$ can be written as $\sum_{i=1}^n \kappa_i x_i$ with $\kappa_i \in \ker(\mathbb{Z}\pi \rightarrow \mathbb{Z}/2)$ and $x_i \in H^2(Y; \mathbb{Z}\pi)$.

Remark 4.3. In [KLPT17] we asserted that the invariant τ_M is well-defined, and the results of the next lemma, for S^2 -characteristic spheres. But we ought to have required the stronger $\mathbb{R}\mathbb{P}^2$ -characteristic. The results of that paper are for torsion-free groups, where the two notions coincide. But the proof of the previous lemma goes through unchanged.

As in Chapter 3, in this section we will consider the dual sequence

$$(4.4) \quad 0 \rightarrow \ker d^3 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi) \rightarrow \mathrm{coker} d^2 \rightarrow 0$$

to the sequence in Theorem 1.6. As noted previously, this can be obtained from the exact sequence in Theorem 1.6 by working with the dual handle decomposition M^d . Recall that we say a map $f: M^{(3)} \rightarrow M^{(2)} \simeq K$ realises a splitting $s: \text{coker } d^2 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi)$ if $s_2: \text{coker } d^2 \rightarrow H^2(M; \mathbb{Z}\pi)$ coincides with the map

$$\text{coker } d^2 \cong H^2(K; \mathbb{Z}\pi) \xrightarrow{f^*} H^2(M^{(3)}; \mathbb{Z}\pi) \cong H^2(M; \mathbb{Z}\pi).$$

Recall that $\mathcal{RC} \subseteq \pi_2(M) \cong H_2(M; \mathbb{Z}\pi)$ denotes the subset of $\mathbb{R}\mathbb{P}^2$ -characteristic elements α with $\mu(\alpha) = 0$. The following diagram should help to read the upcoming definition, whose aim is to define a map $\tau_{M,s}: \ker \text{Sq}^2 \rightarrow \mathbb{Z}/2$.

$$\begin{array}{ccccc}
H^2(K; \mathbb{Z}\pi) & \xrightarrow{\cong} & \text{coker } d^2 & \xrightarrow{s_2} & H^2(M; \mathbb{Z}\pi) & \xrightarrow{PD} & H_2(M; \mathbb{Z}\pi) \\
\downarrow p & & \uparrow & & \downarrow & & \downarrow \tau \\
& & (PD \circ s_2)^{-1}(\mathcal{RC}) & \xrightarrow{(PD \circ s_2)|} & \mathcal{RC} & & \\
& & \downarrow p| & & \downarrow & & \\
H^2(K; \mathbb{Z}/2) & \xleftarrow{p} & p((PD \circ s_2)^{-1}(\mathcal{RC})) & \xrightarrow{\tau'_{M,s}} & \mathbb{Z}/2 & & \\
\uparrow i^* & & \uparrow i^* & & \nearrow \tau_{M,s} & & \\
H^2(\pi; \mathbb{Z}/2) & \xleftarrow{} & \ker \text{Sq}^2 & & & &
\end{array}$$

Definition 4.5. Let $s = (s_1, s_2): \text{coker } d^2 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi)$ be a section of the sequence from Theorem 1.6 such that λ_M vanishes on $\text{im } s_2$ (see Lemma 4.7 below for why such a splitting exists stably whenever $\mathbf{sc}(M) = 0$). It follows from Lemma 4.2 that the map

$$\tau \circ PD \circ s_2: (PD \circ s_2)^{-1}\mathcal{RC} \rightarrow \mathbb{Z}/2$$

factors through the restriction of $p: H^2(K; \mathbb{Z}\pi) \rightarrow H^2(K; \mathbb{Z}/2)$ to $(PD \circ s_2)^{-1}(\mathcal{RC})$. Denote the map $p((PD \circ s_2)^{-1}\mathcal{RC}) \rightarrow \mathbb{Z}/2$ arising in this factorisation by $\tau'_{M,s}$. Let $i^*: H^2(\pi; \mathbb{Z}/2) \rightarrow H^2(K; \mathbb{Z}/2)$ be the map induced by inclusion of the 2-skeleton. Note that

$$i^*(\ker \text{Sq}^2) \subseteq p((PD \circ s_2)^{-1}(\mathcal{RC}))$$

by Lemma 4.1, since by Proposition 3.24 every splitting s can be realised by a map $f: M^{(3)} \rightarrow M^{(2)}$. Denote

$$\tau_{M,s} := \tau'_{M,s} \circ i^*: \ker \text{Sq}^2 \rightarrow \mathbb{Z}/2.$$

We quickly make the following observation.

Lemma 4.6. *We have that $\tau_{M,s}(0) = 0$ for any section s .*

PROOF. Computing $\tau_{M,s}(0)$ involves computing $\tau(S)$, where S is an embedded sphere contained in a single chart. But there are no intersections, therefore no Whitney discs, and so τ evidently vanishes. \square

For the convenience of the reader, we recall that the *Tertiary Property 1.15* requires that for every spin 4-manifold with $\pi_1(M) \xrightarrow{\cong} \pi$, and $\mathbf{pri}(M) = \mathbf{sc}(M) = 0$, we have the following, where we have switched to the dual version of the short exact sequence from Theorem 1.6 (see the preamble to Section 3.4, just before Proposition 3.24).

For every section

$$s = (s_1, s_2): \operatorname{coker} d^2 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi)$$

of the (dual version of the) short exact sequence from Theorem 1.6 with $\lambda_M|_{\operatorname{im} PD \circ s_2} \equiv 0$, the map $\tau_{M,s}|_{\ker(d_{5,0}^3)^*}$ is a homomorphism, and $\omega(\mathbf{tet}(M)) = \tau_{M,s}|_{\ker(d_{5,0}^3)^*}$, where ω is the composition:

$$\begin{aligned} \omega: H_2(\pi; \mathbb{Z}/2)/\operatorname{im}(d_{4,1}^2, d_{5,0}^3) &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}/2}(H^2(\pi; \mathbb{Z}/2), \mathbb{Z}/2)/\operatorname{im}(d_{4,1}^2, d_{5,0}^3) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}/2}(\ker(d_{5,0}^3)^*, \mathbb{Z}/2). \end{aligned}$$

Lemma 4.7. *Assume that $\mathbf{pri}(M) = \operatorname{sec}(M) = 0$. Then there exists a section $s = (s_1, s_2): \operatorname{coker} d^2 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi)$ of (4.4) such that λ_M restricted to the image of $PD \circ s_2$ is trivial.*

This lemma tells us that a section of the form required for the formulation of the Tertiary Property exists.

PROOF. We view $M^{(2)}$ as a 2-skeleton for $B\pi$. Since $\mathbf{pri}(M) = \operatorname{sec}(M) = 0$, by Section 2.1 there exists, after stabilisation, a map $f: M \rightarrow M^{(2)}$ inducing an isomorphism on fundamental groups. Since f is an isomorphism on π_1 , we can homotopy it on the 1-skeleton to be the identity. Extending this homotopy to all of M , we can assume that the map f is the identity on the 1-skeleton. On cochains the map f then induces a map $f^2: C^2 \rightarrow \ker d^3 \subseteq C^2$ such that $f^2 \circ d^2 = d^2$. Let $s'_1 := (\operatorname{Id} - f^2): C^2 \rightarrow C^2$ and $s'_2: C^2 \xrightarrow{f^2} \ker d^3 \rightarrow H^2(M; \mathbb{Z}\pi)$. Since $f^2 \circ d^2 = d^2$, the maps s'_1 and s'_2 factor through $\operatorname{coker} d^2$ inducing a map $s = (s_1, s_2): \operatorname{coker} d^2 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi)$. Then s is a section since $(\operatorname{Id} - f^2) + f^2 = \operatorname{Id}$. It remains to show that λ_M restricted to the image of $PD \circ s_2$ is trivial. By definition, $s_2 = f^2: H^2(M^{(2)}; \mathbb{Z}\pi) \rightarrow H^2(M; \mathbb{Z}\pi)$. For $x, y \in H^2(M^{(2)}; \mathbb{Z}\pi)$ we have

$$\begin{aligned} \lambda_M(PD \circ f^*(x), PD \circ f^*(y)) &= \langle f^*(y), f^*(x) \cap [M] \rangle = \langle y, f_*(f^*(x) \cap [M]) \rangle \\ &= \langle y, x \cap f_*[M] \rangle. \end{aligned}$$

Since $M^{(2)}$ is 2-dimensional, $f_*[M] = 0$ and thus λ_M restricted to the image of $PD \circ s_2$ is trivial as needed. \square

Remark 4.8. Lemma 4.6 implies that the Tertiary Property 1.15 holds when $\operatorname{Hom}_{\mathbb{Z}/2}(\ker(d_{5,0}^3)^*, \mathbb{Z}/2) = 0$.

Remark 4.9. In order to prove the Tertiary Property 1.15 when the group in which $\mathbf{tet}(M)$ lives, $\operatorname{Hom}_{\mathbb{Z}/2}(\ker(d_{5,0}^3)^*, \mathbb{Z}/2)$, is nontrivial, we will show the following. Recall that $d_{4,1}^2$ is dual to $\operatorname{Sq}^2: H^2(\pi; \mathbb{Z}/2) \rightarrow H^4(\pi; \mathbb{Z}/2)$.

- (1) The map $\tau_{M,s}$ defines a homomorphism $\ker \operatorname{Sq}^2 \rightarrow \mathbb{Z}/2$.
- (2) The image of $\tau_{M,s}$ under the map

$$\operatorname{Hom}(\ker \operatorname{Sq}^2; \mathbb{Z}/2) \cong H_2(\pi; \mathbb{Z}/2)/\operatorname{im}(d_{4,1}^2) \rightarrow H_2(\pi; \mathbb{Z}/2)/\operatorname{im}(d_{4,1}^2, d_{5,0}^3)$$

is independent of s .

- (3) The image of $\tau_{M,s}$ under the above map agrees with $\mathbf{tet}(M)$.

We will need [KPT21, Theorem 1.5], which we recall next, for convenience. This shows that the above statements hold, in the case that the splitting s comes from a map $M \rightarrow M^{(2)}$. Theorem 4.12 gives an algebraic condition that implies s is geometrically realised in this manner.

Proposition 4.10 ([KPT21, Theorem 1.5]). *Suppose that there is a map $f: M \rightarrow K$ to a 2-complex K that is an isomorphism on fundamental groups. Let $i: K \rightarrow B\pi$ be a classifying map.*

- (i) *For each $\varphi \in \ker(\text{Sq}^2: H^2(\pi; \mathbb{Z}/2) \rightarrow H^4(\pi; \mathbb{Z}/2))$, there exists a lift $\widehat{\varphi} \in H^2(K; \mathbb{Z}\pi)$ of $i^*(\varphi) \in H^2(K; \mathbb{Z}/2)$.*
- (ii) *The element $PD(f^*(\widehat{\varphi})) \in H_2(M; \mathbb{Z}\pi) \cong \pi_2(M)$ is $\mathbb{R}\mathbb{P}^2$ -characteristic and has trivial self-intersection number, so that the Kervaire-Milnor invariant $\tau(PD \circ f^*(\widehat{\varphi})) \in \mathbb{Z}/2$ is well-defined.*
- (iii) *The map*

$$\begin{aligned} \tau_{M,f}: \ker \text{Sq}^2 &\rightarrow \mathbb{Z}/2 \\ \varphi &\mapsto \tau(PD \circ f^*(\widehat{\varphi})) \end{aligned}$$

is well-defined (i.e. is independent of the choice of $\widehat{\varphi}$) and is a homomorphism.

- (iv) *Under the map*

$$\text{Hom}(\ker \text{Sq}^2, \mathbb{Z}/2) \xrightarrow{\cong} H_2(\pi; \mathbb{Z}/2) / \text{im} \text{Sq}_2 \rightarrow H_2(\pi; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3),$$

$\tau_{M,f}$ is sent to $\text{ter}(M)$. In particular the image of $\tau_{M,f}$ under this map is independent of the choices of f and K .

This shows that, as we are aiming for, $\tau_{M,s}$ agrees with $\text{ter}(M)$, when there is a map $M \rightarrow K$ inducing the given splitting. Now we give an algebraic criterion guaranteeing that such a map $M \rightarrow K$ exists. Note that if we did not want to show independence on the choice of splitting, we would be done. But for our obstructions to be computationally useful, it ought to be possible to choose any splitting.

Definition 4.11. Let N be a $\mathbb{Z}\pi$ -module that is free as a \mathbb{Z} -module. Then

$$\varpi_N: \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(N) \rightarrow N \otimes_{\mathbb{Z}\pi} N$$

denotes the map induced from the inclusion $\omega: \Gamma(N) \rightarrow N \otimes_{\mathbb{Z}} N$ by tensoring with \mathbb{Z} over $\mathbb{Z}\pi$. As per our convention, we view the first N as a right $\mathbb{Z}\pi$ -module using the involution. Here $\mathbb{Z} \otimes_{\mathbb{Z}\pi} (N \otimes_{\mathbb{Z}} N) \cong N \otimes_{\mathbb{Z}\pi} N$, since $\mathbb{Z}\pi$ acts on $N \otimes_{\mathbb{Z}} N$ on the left by the diagonal action.

Theorem 4.12. *Let M be a spin 4-manifold with $\text{pri}(M) = 0 = \text{sec}(M)$, and let $s: \text{coker } d^2 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi)$ be a splitting such that $\lambda_M|_{\text{im}(PD \circ s_2)} \equiv 0$ vanishes. If*

$$\Phi_{H^2(M^{(2)}; \mathbb{Z}\pi)} \circ \varpi_{\pi_2(M^{(2)})}: \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(M^{(2)})) \rightarrow \text{Sesq}(H^2(M^{(2)}; \mathbb{Z}\pi))$$

is injective, then the map $\tau_{M,s}$ is a homomorphism, and maps to $\text{ter}(M)$ under

$$\text{Hom}(\ker(\text{Sq}^2), \mathbb{Z}/2) \cong \text{coker}(\text{Sq}_2) \cong E_{2,2}^3 \rightarrow E_{2,2}^\infty = H_2(\pi; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3).$$

Moreover, the image of $\tau_{M,s}$ in $E_{2,2}^\infty$ does not depend on the choice of s .

PROOF. By Proposition 3.24, the splitting $s: \text{coker } d^2 \rightarrow C^2 \oplus H^2(M; \mathbb{Z}\pi)$ can be realised as the splitting induced from a map $f: M^{(3)} \rightarrow M^{(2)}$, where f_* is the identity on π_1 . Here a splitting is said to be realised by f if $s_2 = f^*: H^2(M^{(2)}; \mathbb{Z}\pi) \rightarrow H^2(M; \mathbb{Z}\pi)$ after identifying $H^2(M^{(3)}; \mathbb{Z}\pi) = H^2(M; \mathbb{Z}\pi)$.

The right hand side of the equation in Lemma 3.31 vanishes, since the intersection form vanishes on the coker $d^2 = H^2(M^{(2)}; \mathbb{Z}\pi)$ summand of $\pi_2(M)$. Thus the left hand side of the equation in Lemma 3.31 vanishes, and then by injectivity of $\Phi_{H^2(M^{(2)}; \mathbb{Z}\pi)} \circ \varpi_{\pi_2(M^{(2)})}$, we have that $f_*(\alpha) = 0 \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)})$; recall that α denotes the attaching map of the 4-handle of M . Here we also use the identification $S: \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)}) \xrightarrow{\cong} \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(M^{(2)}))$ from diagram 3.29.

We claim that the images $x_n \in \mathbb{Z}\pi$ of $1 \in \mathbb{Z}\pi$ under $\mathbb{Z}\pi \cong C_4 \xrightarrow{d_4} C_3 \cong \mathbb{Z}\pi^n \xrightarrow{p_n} \mathbb{Z}\pi$, where $p_n: \mathbb{Z}\pi^n \rightarrow \mathbb{Z}\pi$ are the projections, generate the augmentation ideal $I\pi \subseteq \mathbb{Z}\pi$. This can be seen as follows. By Poincaré duality the cokernel of the dual map $C^3 \xrightarrow{d^4} C^4$ is \mathbb{Z} and hence the image of d^4 is the augmentation ideal. The image is generated by $\{\overline{x_n}\}$ (when we view C^4 as a left module using the involution on $\mathbb{Z}\pi$). By dualising, we obtain the claim.

By changing $f: M^{(3)} \rightarrow M^{(2)}$ on the 3-cells, it follows from the claim that we can change $f_*(\alpha) \in \pi_3(M^{(2)})$ by elements of $I\pi \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)})$. Thus an extension $f': M \rightarrow M^{(2)}$ exists if and only if $f_*(\alpha) = 0 \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)})$. But that is exactly what we have, so such a map $f': M \rightarrow M^{(2)}$ indeed exists.

Therefore s is realised geometrically, not just in the sense that we have $s_2 = f^*: H^2(M^{(2)}; \mathbb{Z}\pi) \rightarrow H^2(M^{(3)}; \mathbb{Z}\pi)$, but now in the stronger sense that s_2 is induced from a map $f: M \rightarrow M^{(2)}$.

By Proposition 4.10, we therefore have $\tau_{M,s}$ is independent of s and is a homomorphism, i.e. is an element of $\text{Hom}(\ker \text{Sq}^2, \mathbb{Z}/2)$. Moreover $\tau_{M,s}$ maps to $\text{tr}(M)$ under the composition in the statement of the theorem, completing the proof. \square

Next, we want to express the injectivity condition of Theorem 4.12 as a property of the group π alone.

Lemma 4.13. *Let N, N' be $\mathbb{Z}\pi$ -modules that are free as \mathbb{Z} -modules. Then $\varpi_{N \oplus N'}$ is injective if and only if ϖ_N and $\varpi_{N'}$ are both injective.*

PROOF. By Lemma 2.16, the inclusion $\Gamma(N \oplus N') \rightarrow (N \oplus N') \otimes_{\mathbb{Z}} (N \oplus N')$ is the direct sum of the inclusion $\Gamma(N) \rightarrow N \otimes_{\mathbb{Z}} N$, the diagonal map $(1+T): N \otimes_{\mathbb{Z}} N' \rightarrow (N \otimes_{\mathbb{Z}} N') \oplus (N' \otimes_{\mathbb{Z}} N)$ and the inclusion $\Gamma(N') \rightarrow N' \otimes_{\mathbb{Z}} N'$. Thus, $\varpi_{N \oplus N'}$ is the direct sum of ϖ_N , $\varpi_{N'}$ and the diagonal map $(1+T): N \otimes_{\mathbb{Z}\pi} N' \rightarrow (N \otimes_{\mathbb{Z}\pi} N') \oplus (N' \otimes_{\mathbb{Z}\pi} N)$. Since the diagonal map is always injective, the lemma follows. \square

Lemma 4.14. *Let P be a finitely generated projective left $\mathbb{Z}\pi$ -module. Then ϖ_P is injective.*

PROOF. We begin with the case $P = \mathbb{Z}\pi$. The module $\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi$ is a free $\mathbb{Z}\pi$ -module with basis $\{1 \otimes g\}_{g \in \pi}$. The action of $\mathbb{Z}\pi$ is the diagonal action. The module $\Gamma(\mathbb{Z}\pi)$ is generated as a subset of $\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi$ by the elements $1 \otimes 1$ and $1 \otimes g + g \otimes 1$. Since $1 \otimes g + g \otimes 1 = 1 \otimes g + g(1 \otimes g^{-1})$ it follows that there is a homomorphism

$$\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi \rightarrow \bigoplus_{g \in \pi, g^2 \neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g \neq 1} \mathbb{Z}\pi/(1+g).$$

The homomorphism above sends $1 \otimes g$ to the generator of the g summand, with kernel the image of $\Gamma(\mathbb{Z}\pi)$. If $g^2 = 1$, then $1 \otimes g + g \otimes 1 = (1+g)(1 \otimes g)$, whence the quotient by $(1+g)$. Here we think of $(1+g)$ as a left ideal of the ring $\mathbb{Z}\pi$, and

form the quotient *ring*. Then we consider the ring $\mathbb{Z}\pi/(1+g)$ as a left $\mathbb{Z}\pi$ -module. Therefore we have a short exact sequence

$$0 \rightarrow \Gamma(\mathbb{Z}\pi) \rightarrow \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi \rightarrow \bigoplus_{g \in \pi, g^2 \neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g \neq 1} \mathbb{Z}\pi/(1+g) \rightarrow 0.$$

Tensor this with \mathbb{Z} over $\mathbb{Z}\pi$ and apply the 6-term exact sequence to obtain

$$\begin{aligned} \mathrm{Tor}_1^{\mathbb{Z}\pi} \left(\mathbb{Z}, \bigoplus_{g \in \pi, g^2 \neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g \neq 1} \mathbb{Z}\pi/(1+g) \right) &\rightarrow \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\mathbb{Z}\pi) \\ \xrightarrow{\varpi_{\mathbb{Z}\pi}} \mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi &\rightarrow \bigoplus_{g \in \pi, g^2 \neq 1} \mathbb{Z} \oplus \bigoplus_{g^2=1, g \neq 1} \mathbb{Z}/2 \rightarrow 0. \end{aligned}$$

A free $\mathbb{Z}\pi$ -module resolution of $\mathbb{Z}\pi/(1+g)$, where g has order two, is given by

$$\dots \xrightarrow{1+g} \mathbb{Z}\pi \xrightarrow{1-g} \mathbb{Z}\pi \xrightarrow{1+g} \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/(1+g) \rightarrow 0.$$

Tensor this with \mathbb{Z} to obtain

$$\dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

The first homology of the corresponding deleted resolution vanishes. As $\mathrm{Tor}_1^{\mathbb{Z}\pi}(-, -)$ commutes with direct sums in the second factor, we obtain

$$\mathrm{Tor}_1^{\mathbb{Z}\pi} \left(\mathbb{Z}, \bigoplus_{g \in \pi, g^2 \neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g \neq 1} \mathbb{Z}\pi/(1+g) \right) = 0.$$

This concludes the proof in the case $P = \mathbb{Z}\pi$. For a general finitely generated free $\mathbb{Z}\pi$ -module the lemma follows from this and Lemma 4.13 by induction on the rank.

If P is finitely generated projective, there exists P' such that $P \oplus P'$ is finitely generated free. So the injectivity of ϖ_P follows from the injectivity of $\varpi_{P \oplus P'}$, again using Lemma 4.13. \square

Corollary 4.15. *Let N be a left $\mathbb{Z}\pi$ -module that is free as a \mathbb{Z} -module. Then the map $\Phi_N \circ \varpi_{N^*}$ is injective if and only if $\Phi_{N \oplus \mathbb{Z}\pi} \circ \varpi_{N^* \oplus \mathbb{Z}\pi^*}$ is injective.*

PROOF. By Lemma 3.6 and the definition of ϖ , the map $\Phi_{N \oplus \mathbb{Z}\pi} \circ \varpi_{N^* \oplus \mathbb{Z}\pi^*}$ is the direct sum of the maps $\Phi_N \circ \varpi_{N^*}$, $\Phi_{\mathbb{Z}\pi} \circ \varpi_{\mathbb{Z}\pi^*}$ and $(\Phi_{N, \mathbb{Z}\pi^*} \oplus \Phi_{\mathbb{Z}\pi, N^*}) \circ (1+T)$. It suffices to argue that the last two maps are injective. Note that $(1+T): N^* \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^* \rightarrow (N^* \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^*) \oplus (\mathbb{Z}\pi^* \otimes_{\mathbb{Z}\pi} N^*)$ is injective, and $\varpi_{\mathbb{Z}\pi^*}$ is injective by Lemma 4.14. Moreover Lemma 3.10 says that $\Phi_{\mathbb{Z}\pi}$, $\Phi_{N, \mathbb{Z}\pi^*}$ and $\Phi_{\mathbb{Z}\pi, N^*}$ are injective. It follows that $\Phi_{\mathbb{Z}\pi} \circ \varpi_{\mathbb{Z}\pi^*}$ and $(\Phi_{N, \mathbb{Z}\pi^*} \oplus \Phi_{\mathbb{Z}\pi, N^*}) \circ (1+T)$ are injective, as required. \square

We can now give the desired statement of a condition implying the Tertiary Property 1.15 that depends only on the group π .

Theorem 4.16. *Let K be a finite 2-complex with fundamental group π . Suppose that*

$$\Phi_{H^2(K; \mathbb{Z}\pi)} \circ \varpi_{\pi_2(K)}: \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K)) \rightarrow \mathrm{Sesq}(H^2(K; \mathbb{Z}\pi))$$

is injective. Then π has the Tertiary Property 1.15.

PROOF. The theorem with $K = M^{(2)}$ follows from Theorem 4.12. Since any two choices of K are homotopy equivalent after wedging with enough copies of S^2 , it remains to check that the injectivity condition is preserved under wedging with S^2 . But this follows from Corollary 4.15. Therefore the injectivity condition is independent of the choice of 2-complex K , as desired. \square

4.2. The Tertiary Property for finite groups

We will prove the following corollary of Theorem 4.16, which will enable us to verify that many finite groups have the Tertiary Property 1.15.

Corollary 4.17. *Let π be a finite group and let K be a finite 2-dimensional CW complex with fundamental group π . If $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$ is torsion-free, then π has the Tertiary Property 1.15.*

For example, computations showing that $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$ is torsion-free have been done for finite 2-generator abelian groups [KPR20], dihedral groups [KNR20], and groups with 4-periodic cohomology [HK88]. We will make use of some of these in Chapter 6.

We will prove the corollary via a couple of lemmas.

Let π be a finite group and let M be a left $\mathbb{Z}\pi$ -module. Consider the map of abelian groups

$$F_M: M^* \otimes_{\mathbb{Z}} M^* \rightarrow \text{Hom}_{\mathbb{Z}\pi}(M, \text{Hom}_{\mathbb{Z}\pi}(M, \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi))$$

$$f \otimes h \mapsto (m \mapsto (n \mapsto f(n) \otimes \overline{h(m)})).$$

Here we view $\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi$ as a left $\mathbb{Z}\pi$ -module via left multiplication in the first factor. The group $\text{Hom}_{\mathbb{Z}\pi}(M, \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi)$ is a right $\mathbb{Z}\pi$ -module via right multiplication in the second factor of $\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi$ and as usual we view it as a left $\mathbb{Z}\pi$ -module using the involution.

Lemma 4.18. *Let π be a finite group and let M be a finitely generated left $\mathbb{Z}\pi$ -module. Then the map F_M is injective.*

PROOF. Consider the abelian group homomorphism $\text{ev}_1: M^* \rightarrow \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ given by evaluation at the neutral group element. This map has an inverse given by sending f to the map $(x \mapsto \sum_{g \in \pi} f(g^{-1}x)g)$. Consider the commutative diagram

$$\begin{array}{ccc} M^* \otimes M^* & \xrightarrow{F_M} & \text{Hom}_{\mathbb{Z}\pi}(M, \text{Hom}_{\mathbb{Z}\pi}(M, \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi)) \\ \cong \downarrow \text{ev}_1 \otimes \text{ev}_1 & & \downarrow \text{ev}_1 \otimes 1 \\ \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(M, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})). \end{array}$$

Since $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ is a free \mathbb{Z} -module, the lower horizontal map is an isomorphism. Hence F_M is injective. \square

Recall that we have an inclusion map $\omega: \Gamma(M^*) \rightarrow M^* \otimes_{\mathbb{Z}} M^*$, and that $\varpi: \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(M^*) \rightarrow M^* \otimes_{\mathbb{Z}\pi} M^*$ denotes the map induced after tensoring with \mathbb{Z} . In the codomain, as per the conventions stipulated at the start of Section 3.2, the first M^* in the tensor product is a right $\mathbb{Z}\pi$ -module and the second M^* is a left $\mathbb{Z}\pi$ -module. We obtain the following commutative diagram, where the right-hand vertical map is induced by the $(\mathbb{Z}\pi, \mathbb{Z}\pi)$ -bimodule homomorphism $\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi$, $x \otimes y \mapsto xy$, where as above the left $\mathbb{Z}\pi$ action is via left multiplication in the first factor.

$$\begin{array}{ccccc} \Gamma(M^*) & \xrightarrow{\omega} & M^* \otimes_{\mathbb{Z}} M^* & \xrightarrow{F_M} & \text{Hom}_{\mathbb{Z}\pi}(M, \text{Hom}_{\mathbb{Z}\pi}(M, \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(M^*) & \xrightarrow{\varpi} & M^* \otimes_{\mathbb{Z}\pi} M^* & \xrightarrow{\Phi_M} & \text{Sesq}(M) \end{array}$$

Let $N_\pi \in \mathbb{Z}\pi$ be the norm element $N_\pi := \sum_{g \in \pi} g$. It is easy to see that the map

$$\mathrm{tr}_M: \mathbb{Z} \otimes_{\mathbb{Z}\pi} M \rightarrow M, \quad 1 \otimes m \mapsto N_\pi \cdot m$$

is well-defined. We obtain a map

$$\mathrm{tr}_{\mathrm{Sesq}}: \mathrm{Sesq}(M) \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(M, \mathrm{Hom}_{\mathbb{Z}\pi}(M, \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi))$$

by applying $\mathrm{tr}: \mathbb{Z}\pi \cong \mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi$. Then we also have that

$$F_M \circ \omega \circ \mathrm{tr}_{\Gamma(M^*)} = \mathrm{tr}_{\mathrm{Sesq}} \circ \varpi \circ \Phi_M.$$

Lemma 4.19. *Let K be a connected, finite 2-complex with finite fundamental group π . Then the kernel of the map*

$$\Phi_{H^2(K; \mathbb{Z}\pi)} \circ \varpi: \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K)) \rightarrow \mathrm{Sesq}(H^2(K; \mathbb{Z}\pi))$$

is the torsion in $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$.

PROOF. We will apply the discussion above with $M = H^2(K; \mathbb{Z}\pi) \cong \pi_2(K)^*$, so that $M^* = \pi_2(K)$ by Lemma 3.12. Since $\mathrm{Sesq}(H^2(K; \mathbb{Z}\pi))$ is torsion free, all torsion elements have to lie in the kernel. Now suppose $a \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$ lies in the kernel of $\Phi_{H^2(K; \mathbb{Z}\pi)} \circ \varpi$. Then

$$\begin{aligned} 0 &= \mathrm{tr}_{\mathrm{Sesq}} \circ \Phi_{H^2(K; \mathbb{Z}\pi)} \circ \varpi(a) \\ &= F_{H^2(K; \mathbb{Z}\pi)} \circ \omega \circ \mathrm{tr}_{\Gamma(\pi_2(K))}(a). \end{aligned}$$

Since $F_{H^2(K; \mathbb{Z}\pi)}$ and ω are injective, this implies that $N_\pi a = 0$ and thus $0 = |\pi|a \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$. In particular, a is a torsion element. \square

Corollary 4.17 now follows immediately from Lemma 4.19 and Theorem 4.16.

Inheritance results

In this chapter we give two inheritance results for each of the secondary and tertiary properties. The last two show that the properties are inherited from odd index subgroups, so it suffices to prove the properties on these. The inheritance results from this chapter will be used in the next chapter to prove the corresponding properties for families of groups.

5.1. Secondary property inheritance

We will actually prove an inheritance result for Condition 3.20 instead of the Secondary Property 1.11 itself.

As above, let K be a finite, connected 2-complex with $\pi_1(K) = \pi$. Let (C_*, d_*) be the cellular chain complex $C_*(K; \mathbb{Z}\pi)$, let (C^*, d^*) be the dual complex, and let $H := H^2(K; \mathbb{Z}\pi)$.

Definition 5.1. If $\varphi: \pi \rightarrow G$ is a group homomorphism and K_G a finite connected 2-complex with $\pi_1(K_G) = G$, we obtain a homomorphism

$$\varphi_*: \widehat{H}^0(\text{Sesq}(H)) \rightarrow \widehat{H}^0(\text{Sesq}(H^2(K_G; \mathbb{Z}G)))$$

as follows. First construct a homomorphism

$$\begin{aligned} \widehat{H}^0(\text{Sesq}(H)) &\rightarrow \widehat{H}^0(\text{Sesq}(\mathbb{Z}G \otimes C^2 / \text{im}(\text{Id}_{\mathbb{Z}G} \otimes d^2))) \\ \lambda &\mapsto [a \otimes x] \otimes [a' \otimes x'] \mapsto a\varphi(\lambda(x, x'))\overline{a'}. \end{aligned}$$

Then compose this with the map on $\widehat{H}^0(\text{Sesq}(-))$ induced by $H^2(K_G; \mathbb{Z}G) \rightarrow H^2(K; \mathbb{Z}G) \cong \mathbb{Z}G \otimes C^2 / \text{im}(\text{Id}_{\mathbb{Z}G} \otimes d^2)$.

Recall that for a finitely presented group π we defined a map $A: H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H^2(K_\pi; \mathbb{Z}\pi)))$ in Definition 3.18.

Lemma 5.2. *The following square commutes*

$$\begin{array}{ccc} H_3(\pi; \mathbb{Z}/2) & \xrightarrow{\varphi_*} & H_3(G; \mathbb{Z}/2) \\ \downarrow A & & \downarrow A \\ \widehat{H}^0(\text{Sesq}(H)) & \xrightarrow{\varphi_*} & \widehat{H}^0(\text{Sesq}(H^2(K_G; \mathbb{Z}G))) \end{array}$$

PROOF. By tracing the definition of the map $A: H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$, via the maps ι , I , Θ and Φ , it is not too hard to see, for $c \in C_3(B\pi; \mathbb{Z}\pi)$, $m, m' \in C^2 = C^2(K; \mathbb{Z}\pi)$ representatives of cohomology classes in $H^2(K; \mathbb{Z}\pi)$, that

$$A([1 \otimes c]) = (([m], [m']) \mapsto \overline{m(\partial_3(c))}m'(\partial_3(c)) \in \mathbb{Z}\pi).$$

Here $\partial_3: C_3(B\pi; \mathbb{Z}\pi) \rightarrow C_2(B\pi; \mathbb{Z}\pi)$ is the boundary map. The map

$$\varphi_*: C_*(B\pi; \mathbb{Z}\pi) \rightarrow \mathbb{Z}G \otimes_{\mathbb{Z}\pi} C_*(B\pi; \mathbb{Z}\pi) = C_*(B\pi; \mathbb{Z}G) \rightarrow C_*(BG; \mathbb{Z}G)$$

is given by reducing coefficients followed by the chain map induced by the map of spaces $B\pi \rightarrow BG$. In the following formulae, we also use the induced map

$$\varphi^*: C^2(BG; \mathbb{Z}G) \rightarrow C^2(B\pi; \mathbb{Z}G) = \text{Hom}_{\mathbb{Z}\pi}(C_2(B\pi; \mathbb{Z}\pi), \mathbb{Z}G) \cong \mathbb{Z}G \otimes_{\mathbb{Z}\pi} C^2(B\pi; \mathbb{Z}\pi).$$

Then we compute, with $n, n' \in C^2(K_G; \mathbb{Z}G) = C^2(BG; \mathbb{Z}G)$, as follows.

$$\begin{aligned} A(\varphi_*([1 \otimes c])) &= A([1 \otimes \varphi_*(c)]) \\ &= (([n], [n'] \mapsto \overline{n(\partial_3(\varphi_*(c)))}n'(\partial_3(\varphi_*(c)))) \\ &= (([n], [n'] \mapsto \overline{n(\varphi_*(\partial_3(c)))}n'(\varphi_*(\partial_3(c)))) \\ &= (([n], [n'] \mapsto \overline{\varphi^*(n)(\partial_3(c))}\varphi^*(n')(\partial_3(c)))) \\ &= \varphi_*([m], [m'] \mapsto \overline{m(\partial_3(c))}m'(\partial_3(c))) \\ &= \varphi_*(A([1 \otimes c])). \end{aligned} \quad \square$$

Here is our first secondary property inheritance result.

Theorem 5.3. *Let G, G' be finitely presented groups satisfying Condition 3.20.*

- (1) *Let $p: \pi \rightarrow G'$ be a homomorphism, and let $x \in H_3(\pi; \mathbb{Z}/2)$ be such that $p_*(x)$ does not lie in the image of $\text{Sq}_2 \circ \text{red}_2$. Then $A(x) \neq 0$.*
- (2) *Let $i: G \rightarrow \pi$ be a homomorphism and let $x \in H_3(G; \mathbb{Z}/2)$ be in the image of $\text{Sq}_2 \circ \text{red}_2$. Then $A(i_*x) = 0$.*

Moreover, if $\pi \cong N \rtimes G$ for some normal subgroup N of π and $x \in H_3(G; \mathbb{Z}/2)$, then $i_*(x) \in H_3(\pi; \mathbb{Z}/2)$ is in the kernel of A if and only if it is in the image of $\text{Sq}_2 \circ \text{red}_2$. Here $i: G \rightarrow \pi$ denotes the inclusion of G into π that sends $g \mapsto (1_N, g)$.

PROOF. Let $K, K_{G'}$ and K_G be finite 2-complexes with fundamental groups π, G' and G respectively. Consider the following diagram

$$\begin{array}{ccccc} H_5(G; \mathbb{Z}) & \xrightarrow{i_*} & H_5(\pi; \mathbb{Z}) & \xrightarrow{p_*} & H_5(G'; \mathbb{Z}) \\ \downarrow \text{Sq}_2 \circ \text{red}_2 & & \downarrow \text{Sq}_2 \circ \text{red}_2 & & \downarrow \text{Sq}_2 \circ \text{red}_2 \\ H_3(G; \mathbb{Z}/2) & \xrightarrow{i_*} & H_3(\pi; \mathbb{Z}/2) & \xrightarrow{p_*} & H_3(G'; \mathbb{Z}/2) \\ \downarrow A & & \downarrow A & & \downarrow A \\ \widehat{H}^0(\text{Sesq}(H^2(K_G; \mathbb{Z}G))) & \xrightarrow{i_*} & \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))) & \xrightarrow{p_*} & \widehat{H}^0(\text{Sesq}(H^2(K_{G'}; \mathbb{Z}G'))) \end{array}$$

The bottom two squares commute by Lemma 5.2, and the top two squares commute by naturality of Sq_2 and red_2 . The left and right columns are exact by assumption, but the middle column need not be. The enumerated items in the theorem now follow from Theorem 3.22 and a diagram chase.

To see the last part, note that the homomorphism $i: G \rightarrow \pi = N \rtimes G$ is a splitting for the quotient map $Q: \pi \rightarrow \pi/N \cong G$, that is $Q \circ i = \text{Id}_G$. The map Q

induces a diagram similar to the diagram above.

$$\begin{array}{ccc}
H_5(G; \mathbb{Z}) & \begin{array}{c} \xrightarrow{i_*} \\ \xleftarrow{Q_*} \end{array} & H_5(\pi; \mathbb{Z}) \\
\downarrow \text{Sq}_2 \circ \text{red}_2 & & \downarrow \text{Sq}_2 \circ \text{red}_2 \\
H_3(G; \mathbb{Z}/2) & \begin{array}{c} \xrightarrow{i_*} \\ \xleftarrow{Q_*} \end{array} & H_3(\pi; \mathbb{Z}/2) \\
\downarrow A & & \downarrow A \\
\widehat{H}^0(\text{Sesq}(H^2(K_G; \mathbb{Z}G))) & \begin{array}{c} \xrightarrow{i_*} \\ \xleftarrow{Q_*} \end{array} & \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))
\end{array}$$

The squares commute with the horizontal arrows going in both directions. The left hand column is exact. Also note that $Q_* \circ i_* = \text{Id}$, but be warned that we say nothing about $i_* \circ Q_*$. Another diagram chase now shows that $i_*(x) \in H_3(\pi; \mathbb{Z}/2)$ is in the kernel of A if and only if it is in the image of $\text{Sq}_2 \circ \text{red}_2$, as required. \square

Corollary 5.4. *Condition 3.20 holds for all finitely generated abelian groups if and only if it holds for all abelian groups with at most three generators.*

PROOF. Let E be an abelian group. For every $y \in H_3(E; \mathbb{Z}/2)$, there exists a decomposition $E \cong E' \oplus B$, such that E' has at most 3 generators and y is in the image of i_* , where $i: E' \rightarrow E$ is the inclusion. Hence the sequence

$$H_5(E; \mathbb{Z}) \xrightarrow{\text{Sq}_2 \circ \text{red}_2} H_3(E; \mathbb{Z}/2) \xrightarrow{A} \widehat{H}^0(\text{Sesq}(H^2(K_E; \mathbb{Z}E)))$$

is “exact at y ,” that is y lies in the kernel of A if and only if it lies in the image of $\text{Sq}_2 \circ \text{red}_2$, by Theorem 5.3. The corollary now follows from Theorem 3.22. \square

5.2. Tertiary property inheritance

Before we state our tertiary inheritance theorem, we have a couple of lemmas. The first is a well known fact from group theory.

Lemma 5.5. *Let $p: \pi \rightarrow G$ be a surjective homomorphism between finitely generated groups π and G . If G is finitely presented, then $\ker(p)$ is finitely normally generated.*

PROOF. Let $\pi \cong \langle X = \{x_i\}_{i=1}^n \mid R \rangle$, where R is some set of relations. Let $G \cong \langle X' = \{x'_i\}_{i=1}^{n'} \mid R' = \{w'_j(X')\}_{j=1}^{m'} \rangle$, where w'_i is a word in n' variables and $w'_i(X')$ denotes the word given by inserting x'_i for the i th variable. Let $y_i \in \pi$ be a preimage of x'_i for all $i = 1, \dots, n'$, $Y = \{y_i\}_{i=1}^{n'}$ and let $y_i = v_i(X)$ for some words v_i . Then

$$\pi \cong \langle X, Y \mid R, \{y_i^{-1}v_i(X)\}_{i=1}^{n'} \rangle.$$

Let $p(x_i) = v'_i(X')$ for some words v'_i . Then

$$G \cong \langle X', p(X) \mid R', \{p(x_i)^{-1}v'_i(X')\}_{i=1}^n \rangle.$$

Since $p(Y) = X'$ by definition and the relations of π hold in G we also have

$$G \cong \langle X, Y \mid R, \{y_i^{-1}v_i(X)\}_{i=1}^{n'}, \{w'_j(Y)\}_{j=1}^{m'}, \{x_i^{-1}v'_i(Y)\}_{i=1}^n \rangle.$$

To obtain this presentation for G , substitute $p(Y) = X'$ and then remove all the instances of p . In particular, the kernel of p is normally generated by the elements $\{w'_j(X')\}_{j=1}^{m'}$ and $\{x_i^{-1}v'_i(Y)\}_{i=1}^n$. \square

Lemma 5.6. *Let N be a codimension zero submanifold (possibly with boundary) of a spin 4-manifold M and let $i: N \rightarrow M$ be the inclusion. Suppose $\alpha \in \pi_2(N)$ has $\mu(\alpha) = 0$ and $i_*(\alpha) \in \pi_2(M)$ is $\mathbb{R}\mathbb{P}^2$ -characteristic. Then $\tau(\alpha)$ and $\tau(i_*\alpha)$ are well-defined and equal.*

PROOF. From $\mu(i_*\alpha) = i_*(\mu(\alpha)) = 0$, it follows that $\tau(i_*\alpha) \in \mathbb{Z}/2$ is well-defined. The number of transverse intersection points between α and a generically immersed $\mathbb{R}\mathbb{P}^2$ β is the same as those between $i_*\alpha$ and $i_*\beta$. Therefore α is $\mathbb{R}\mathbb{P}^2$ -characteristic and $\tau(\alpha) \in \mathbb{Z}/2$ is well-defined.

Pair up the self-intersections of α with Whitney discs in such a way that the intersection of these discs with α computes $\tau(\alpha)$. Then the same Whitney discs can be used to compute $\tau(i_*\alpha)$ inside the bigger manifold M . Hence $\tau(\alpha) = \tau(i_*\alpha)$. \square

Here is our first tertiary property inheritance result.

Theorem 5.7. *Let π be a finitely presented group and assume that there are surjections $p_i: \pi \rightarrow G_i$ for some finitely presented groups G_1, \dots, G_n , such that*

$$\prod_{i=1}^n (p_i)_*: H_2(\pi; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3) \rightarrow \prod_{i=1}^n H_2(G_i; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3)$$

is an injection. Furthermore, assume that all the G_i have the Tertiary Property 1.15. Then π has the Tertiary Property 1.15.

PROOF. Let M be a spin manifold with $\pi_1(M) = \pi$ and $c_*([M]) = 0 \in H_4(B\pi; \mathbb{Z})$. Choose a model for $B\pi$ with finite 2-skeleton. Let $f: M^{(3)} \rightarrow B\pi^{(2)}$ be a map that induces an isomorphism on fundamental groups, and such that the equivariant intersection form vanishes on the image of

$$f^*: H^2(B\pi^{(2)}; \mathbb{Z}\pi) \rightarrow H^2(M^{(3)}; \mathbb{Z}\pi) \cong H^2(M; \mathbb{Z}\pi) \cong H_2(M; \mathbb{Z}\pi).$$

In particular, we assume that the primary and secondary obstructions vanish for M .

Recall that $\mathcal{RC} \subset \pi_2(M) \cong H_2(M; \mathbb{Z}\pi)$ denotes the subset of $\mathbb{R}\mathbb{P}^2$ -characteristic elements on which μ vanishes. Consider the following diagram, in which $H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}}$ denotes the subset $\mathcal{RC} \cap \ker \mu$ of $H^2(M; \mathbb{Z}\pi)$, $H^2(B\pi^{(2)}; \mathbb{Z}\pi)_{\mathcal{RC}}$ denotes the preimage of $H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}}$, and $H^2(B\pi^{(2)}; \mathbb{Z}/2)_{\mathcal{RC}}$ denotes the image of $H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}}$. By Lemma 4.1, the maps land in the subsets claimed.

$$\begin{array}{ccccc} & & H^2(B\pi^{(2)}; \mathbb{Z}\pi)_{\mathcal{RC}} & \xrightarrow{f^*} & H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}} & \xrightarrow{\tau} & \mathbb{Z}/2 \\ & & \downarrow & & & & \\ H^2(B\pi; \mathbb{Z}/2) & \longrightarrow & H^2(B\pi^{(2)}; \mathbb{Z}/2)_{\mathcal{RC}} & & & & \end{array}$$

Make a choice of lift, as shown by the dashed arrow; that is, for each element of $H^2(B\pi^{(2)}; \mathbb{Z}/2)_{\mathcal{RC}}$ choose an element in $H^2(B\pi^{(2)}; \mathbb{Z}\pi)_{\mathcal{RC}}$ that maps to it.

We want to show that

$$\kappa(\tau_f|_{\ker(d_{5,0}^3)^*}) = \mathbf{tr}(M) \in H_2(B\pi; \mathbb{Z}/2) / (\text{im } d_{4,1}^2, \text{im } d_{5,0}^3)$$

where the isomorphism $\text{Hom}(H^2(B\pi; \mathbb{Z}/2), \mathbb{Z}/2) \cong H_2(B\pi; \mathbb{Z}/2)$ induces an identification

$$\kappa: \text{Hom}(\ker(d_{5,0}^3)^*, \mathbb{Z}/2) \xrightarrow{\cong} H_2(B\pi; \mathbb{Z}/2) / (\text{im } d_{4,1}^2, \text{im } d_{5,0}^3).$$

Note that this includes showing that $\tau_f|_{\ker(d_{5,0}^3)^*}$ is a homomorphism.

We will also write κ for the corresponding identification with $B\pi$ replaced by BG . Observe that it suffices, by the injectivity hypothesis of Theorem 5.7, to show that for each surjective homomorphism $p: \pi \rightarrow G$, where G has the Tertiary property, that $p_*\kappa(\tau_f|_{\ker(d_{5,0}^3)^*}) = p_*\mathbf{ter}(M)$. The remainder of the proof verifies this equality.

The element $p_*\mathbf{ter}(M)$ can be computed via the bordism class $[M \xrightarrow{c} B\pi \xrightarrow{p} BG] \in \Omega_4^{\text{Spin}}(BG)$. We want to identify a class in $H_2(BG; \mathbb{Z}/2)$, from the E_2 -page of the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{\text{Spin}}(BG)$, with a τ invariant, using that G has the Tertiary property. For this, we need a manifold with fundamental group G .

Perform surgery on M along a normal generating set of curves $\coprod S^1 \subset M$ for $\ker(p: \pi \rightarrow G)$; this is a finite set by Lemma 5.5. This removes, for each surgery, a copy of $D^2 \times S^2$. Note that we will allow ourselves in the future to modify the representative circles normally generating $\ker p$ within their homotopy classes. Let

$$M' := M \setminus \left(\coprod S^1 \times D^3 \right) \cup_{\coprod S^1 \times S^2} \coprod D^2 \times S^2.$$

The gluing map in the surgery arises from the framing of the normal bundle of each S^1 i.e. the identification of a regular neighbourhood of S^1 with $S^1 \times D^3$. We use the unique (up to homotopy) identification for which the spin structure of M extends over M' .

Note that we can arrange for the surgery data $\coprod S^1 \times D^3$ to be contained in $M^{(3)}$. Choose a model for BG with finite 2-skeleton. We obtain a commutative diagram

$$\begin{array}{ccccc} M^{(3)} & \xleftarrow{i_M} & M^{(3)} \setminus \left(\coprod S^1 \times D^3 \right) & \xrightarrow{i_{M'}} & (M')^{(3)} \\ & \searrow p \circ f & \downarrow p \circ f = f' & \swarrow f' & \\ & & BG^{(2)} & & \end{array}$$

We will view $M^{(3)} \setminus \left(\coprod S^1 \times D^3 \right)$ as a manifold with boundary, by only removing the interiors of the $S^1 \times D^3$. In the following diagram, the boundary will be denoted by ∂ .

The proof will be based on the diagram in Figure 1, which we will explain in detail below.

The strategy of the proof is as follows. The passage from $H^2(BG; \mathbb{Z}/2)$ (including a choice of lift for the top left red_2 map), that goes down the left hand side of the diagram, computes $\tau_{f'}$. Note that

$$\tau_{f'}|_{\ker(d_{5,0}^3(G))^*} = \kappa^{-1}(\mathbf{ter}(M')) = \kappa^{-1}(p_*(\mathbf{ter}(M))).$$

The passage along the right hand side of the diagram computes $\tau_f \circ p^*$. If we show that $\tau_{f'}$ and $\tau_f \circ p^*$ are equal, this will imply that τ_f is a group homomorphism on the image of p^* . The injectivity statement in Theorem 5.7 dualises to a surjectivity statement $\prod_{i=1}^n p_i^*: \prod_{i=1}^n \ker(d_{5,0}^3(G_i))^* \rightarrow \ker(d_{5,0}^3(\pi))^*$ and hence we will obtain that $\tau_f|_{\ker(d_{5,0}^3(\pi))^*}$ is a homomorphism. Here $d_{5,0}^3(\pi)$ and $d_{5,0}^3(G_i)$ denote the differentials of the Atiyah-Hirzebruch spectral sequences for $B\pi$ and BG_i respectively. By naturality of the spectral sequences, $p^*: H^2(G; \mathbb{Z}/2) \rightarrow H^2(\pi; \mathbb{Z}/2)$ restricts to a homomorphism $p^*: \ker(d_{5,0}^3(G))^* \rightarrow \ker(d_{5,0}^3(\pi))^*$.

$$\begin{array}{ccccccc}
H^2(BG^{(2)}; \mathbb{Z}/2) & \xleftarrow{i^*} & H^2(BG; \mathbb{Z}/2) & & & & \\
\uparrow \text{red}_2 & & & & & & \\
H^2(BG^{(2)}; \mathbb{Z}G) & \xrightarrow{p^*} & H^2(B\pi^{(2)}; \mathbb{Z}G) & \xleftarrow{\text{red}} & H^2(B\pi^{(2)}; \mathbb{Z}\pi) & & \\
\downarrow (f')^* & \searrow (f')^* & \downarrow f^* & & \downarrow f^* & & \\
H^2((M')^{(3)}; \mathbb{Z}G) & \xleftarrow{\text{ex}} & H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G) & \xleftarrow{\text{red}} & H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}\pi) & \xrightarrow{\text{ex}} & H^2(M^{(3)}; \mathbb{Z}\pi) \\
\cong \uparrow i^* & & \cong \uparrow i^* & & \cong \uparrow i^* & & \cong \uparrow i^* \\
H^2(M'; \mathbb{Z}G) & \xleftarrow{\text{ex}} & H^2(M \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G) & \xleftarrow{\text{red}} & H^2(M \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}\pi) & \xrightarrow{\text{ex}} & H^2(M; \mathbb{Z}\pi) \\
\downarrow PD & & \downarrow PD & & \downarrow PD & & \downarrow PD \\
H_2(M'; \mathbb{Z}G) & \xleftarrow{i_*} & H_2(M \setminus \coprod S^1 \times D^3; \mathbb{Z}G) & \xleftarrow{\text{red}} & H_2(M \setminus \coprod S^1 \times D^3; \mathbb{Z}\pi) & \xrightarrow{i_*} & H_2(M; \mathbb{Z}\pi) \\
\downarrow h^{-1} & & \downarrow h^{-1} & & \downarrow h^{-1} & & \downarrow h^{-1} \\
\pi_2(M') & \xleftarrow{i_*} & \pi_2(M \setminus \coprod S^1 \times D^3) & \xrightarrow{i_*} & \pi_2(M) & & \\
& & \downarrow \tau & & \downarrow \tau & & \\
& & \mathbb{Z}/2 & & \mathbb{Z}/2 & &
\end{array}$$

FIGURE 1. The proof of tertiary inheritance.

Now we know that $\tau_f \circ p^*|_{\ker(d_{5,0}^3(G))^*}$ is a homomorphism, and we can apply κ to the equality $\kappa^{-1}(p_*(\mathbf{ter}(M))) = \tau_{f'}|_{\ker(d_{5,0}^3(G))^*} = \tau_f \circ p^*|_{\ker(d_{5,0}^3(G))^*}$ to obtain

$$\begin{aligned}
p_*(\mathbf{ter}(M)) &= \kappa(\tau_{f'}|_{\ker(d_{5,0}^3(G))^*}) = \kappa((\tau_f \circ p^*)|_{\ker(d_{5,0}^3(G))^*}) \\
&= \kappa((p^*)^*(\tau_f|_{\ker(d_{5,0}^3(\pi))^*})) = p_*(\kappa(\tau_f|_{\ker(d_{5,0}^3(\pi))^*})).
\end{aligned}$$

This equality for all the surjective group homomorphisms $p_i: \pi \rightarrow G_i$, together with the injectivity assumption in Theorem 5.7, will then imply that $\mathbf{ter}(M) = \kappa(\tau_f|_{\ker(d_{5,0}^3(G))^*})$ as desired. Once we have explained the diagram and shown that it commutes, it will follow easily that $\tau_{f'}$ and $\tau_f \circ p^*$ are equal.

Next we explain the maps in the diagram in Figure 1.

- Arrows labelled with $f^*, p^*, (f')^*$ are the maps induced by $f: M^{(3)} \rightarrow B\pi^{(2)}$, $p: \pi \rightarrow G$ and $f': (M')^{(2)} \rightarrow BG^{(2)}$ respectively.
- Arrows labelled with red are reduction of the coefficients.
- Arrows labelled with ex are given by the inverse of excision

$$H^2(M, \coprod S^1 \times D^3) \xrightarrow{\cong} H^2(M \setminus \coprod S^1 \times D^3, \partial)$$

composed with the map of from the long exact sequence of a pairs

$$H^2(M, \coprod S^1 \times D^3) \rightarrow H^2(M)$$

with the stated coefficients or by the analogous maps for $M^{(3)}$ or M' instead of M . The fact that $M \setminus \coprod S^1 \times D^3 \cong M' \setminus \coprod S^1 \times D^3$ is also used in the definition of the left-hand ex maps.

- Arrows labelled with i^* or i_* are maps induced by inclusions of the 3-skeleta.

- Arrows labelled with PD are the isomorphisms from Poincaré duality.
- Arrows labelled with h^{-1} are the inverses of the Hurewicz isomorphism.
- Arrows labelled with τ are the τ invariant. These arrows are dashed since the τ invariant is only defined on a subset.

More precisely, the map

$$(f')^*: H^2(BG^{(2)}; \mathbb{Z}G) \rightarrow H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G)$$

is given as follows.

Consider the cohomology long exact sequence of the pair $(M^{(3)}, \coprod D^2 \times S^2)$:

$$0 \rightarrow H^2(M^{(3)}, \coprod D^2 \times S^2; \mathbb{Z}G) \rightarrow H^2(M^{(3)}; \mathbb{Z}G) \rightarrow H^2(\coprod D^2 \times S^2; \mathbb{Z}G)$$

Since f' restricted to the attached copies of $D^2 \times S^2$ is null homotopic, the map $f': H^2(BG^{(2)}; \mathbb{Z}G) \rightarrow H^2(M^{(3)}; \mathbb{Z}G)$ lands in the image of $H^2(M^{(3)}, \coprod D^2 \times S^2; \mathbb{Z}G)$. We can compose this $(f')^*$ further with the map induced by the map of pairs

$$H^2(M^{(3)}, \coprod D^2 \times S^2; \mathbb{Z}G) \rightarrow H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G)$$

(by excision this is an isomorphism) to obtain the diagonal $(f')^*$ map in the big diagram

$$(f')^*: H^2(BG^{(2)}; \mathbb{Z}G) \rightarrow H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G).$$

All quadrilaterals in the diagram commute by naturality of the involved maps. Note that one of the quadrilaterals looks like a triangle at first glance. The commutativity of the two triangles at the bottom (when they are defined) follows from Lemma 5.6.

Let $\alpha \in H^2(BG^{(2)}; \mathbb{Z}G)$ be given and let $\tilde{\alpha} \in H^2(B\pi^{(2)}; \mathbb{Z}\pi)$ be a lift of $p^*\alpha$.

CLAIM. There exists $y \in H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}\pi)$ with $\text{red}(y) = (f')^*\alpha$ and $\text{ex}(y) = f^*\tilde{\alpha}$.

The claim can be seen as follows. Consider the following commutative diagram with exact rows coming from the long exact sequences of pairs.

(5.8)

$$\begin{array}{ccccccc} H^1(\coprod S^1 \times D^3; \mathbb{Z}\pi) & \longrightarrow & H^2(M^{(3)}, \coprod S^1 \times D^3; \mathbb{Z}\pi) & \xrightarrow{j} & H^2(M^{(3)}; \mathbb{Z}\pi) & \longrightarrow & 0 \\ \downarrow \text{red} & & \downarrow \text{red} & & \downarrow \text{red} & & \\ H^1(\coprod S^1 \times D^3; \mathbb{Z}G) & \longrightarrow & H^2(M^{(3)}, \coprod S^1 \times D^3; \mathbb{Z}G) & \longrightarrow & H^2(M^{(3)}; \mathbb{Z}G) & \longrightarrow & 0 \end{array}$$

The left hand vertical map is surjective, as we now argue. The circles in the left hand column represent elements of π that normally generate $\ker(\pi \rightarrow G)$. Let us denote these elements of π by g_1, \dots, g_n . We have that $H^1(\coprod S^1 \times D^3; \mathbb{Z}G) \cong \bigoplus^n \mathbb{Z}G$, with one summand per copy of $S^1 \times D^3$. On the other hand, $H^1(\coprod S^1 \times D^3; \mathbb{Z}\pi) \cong \bigoplus_{i=1}^n \mathbb{Z}\pi/(g_i - 1)$. The reduction map simply adds the relations $g_j - 1$ to each summand, for $j \neq i$. This shows that the left hand vertical map is surjective.

Using excision and commutativity of the big diagram from Figure 1, it not too hard to see, by taking $a_1 = f^*(\tilde{\alpha})$ and $a_2 = (f')^*(\alpha)$, that the claim follows if we can show the following: for every $a_1 \in H^2(M^{(3)}; \mathbb{Z}\pi)$ and for every $a_2 \in$

$H^2(M^{(3)}, \coprod S^1 \times D^3; \mathbb{Z}G)$ with the same image in $H^2(M^{(3)}; \mathbb{Z}G)$, there exists $y \in H^2(M^{(3)}, \coprod S^1 \times D^3; \mathbb{Z}\pi)$ with $j(y) = a_1$ and $\text{red}(y) = a_2$. The map

$$j: H^2(M^{(3)}, \coprod S^1 \times D^3; \mathbb{Z}\pi) \rightarrow H^2(M^{(3)}; \mathbb{Z}\pi)$$

is identified with the map

$$H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}\pi) \rightarrow H^2(M^{(3)}; \mathbb{Z}\pi)$$

using excision. The existence of such an element y follows from a diagram chase in the above diagram (5.8). Lift a_1 to $z \in H^2(M^{(3)}, \coprod S^1 \times D^3; \mathbb{Z}\pi)$. The element z might not map to a_2 under red . Take $\text{red}(z) - a_2$, lift it to the top left corner and map it to $w \in H^2(M^{(3)}, \coprod S^1 \times D^3; \mathbb{Z}\pi)$. Define $y = z - w$. It is now straightforward to see that y has the desired properties. This completes the proof of the claim.

It follows from the claim that for any two elements $\alpha, \alpha' \in H^2(BG^{(2)}; \mathbb{Z}G)$, there are elements

$$PD(y), PD(y') \in \pi_2(M \setminus \coprod S^1 \times D^3)$$

that map to $PD((f')^*\alpha)$, $PD((f')^*\alpha')$ and $PD(f^*\tilde{\alpha})$, $PD(f^*\tilde{\alpha}')$ under the respective inclusions. Since the equivariant intersection form vanishes on $PD \circ f^*$, we see that $0 = \lambda(PD(f^*\tilde{\alpha}), PD(f^*\tilde{\alpha}')) = \lambda(PD(y), PD(y')) = p(\lambda(PD((f')^*\alpha), PD((f')^*\alpha'))$.

Hence the equivariant intersection form of M' vanishes on $PD \circ (f')^*$. For every $x \in H^2(BG; \mathbb{Z}/2)$ and every lift $\hat{x} \in H^2(BG^{(2)}; \mathbb{Z}G)$ of i^*x we have

$$\tau_{f'}(x) = \tau(PD((f')^*\hat{x})).$$

For every lift $\tilde{x} \in H^2(B\pi^{(2)}; \mathbb{Z}\pi)$ of $p^*\hat{x}$ and every $y \in H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}\pi)$ with $\text{red}(y) = (f')^*\tilde{x}$ and $\text{ex}(y) = f^*\tilde{x}$ we have

$$\tau_f(p^*x) = \tau(PD(f^*\tilde{x})) = \tau(PD(y)) = \tau(PD((f')^*\hat{x})) = \tau_{f'}(x).$$

This completes the proof of the assertion that $\tau_f \circ p^* = \tau_{f'}$ and therefore completes the proof of the inheritance theorem for the Tertiary property. \square

Corollary 5.9. *The Tertiary Property 1.15 holds for all finitely generated abelian groups if and only if it holds for all abelian groups with at most two generators.*

PROOF. Assume that the Tertiary Property 1.15 holds for all abelian groups with at most two generators.

Let $\pi = \bigoplus_{i=1}^n C_i$ with C_i cyclic. Let $p_i: \pi \rightarrow C_i$ be the projection homomorphism and let $s_i: C_i \rightarrow \pi$ be the inclusion homomorphism. For $i \neq j$ let $G_{ij} := C_i \oplus C_j$, let $p_{ij}: G_{ij} \rightarrow \pi$ be the projection homomorphism and let $s_{ij}: G_{ij} \rightarrow \pi$ be the inclusion homomorphism. By the Künneth theorem

$$H_2(\pi; \mathbb{Z}/2) \cong \bigoplus_{1 \leq i \leq n} H_2(C_i; \mathbb{Z}/2) \oplus \bigoplus_{1 \leq i < j \leq n} H_1(C_i; \mathbb{Z}/2) \otimes H_1(C_j; \mathbb{Z}/2).$$

It follows, that

$$p := \bigoplus_{1 \leq i < j \leq n} (p_{ij})_*: H_2(\pi; \mathbb{Z}/2) \rightarrow \bigoplus_{1 \leq i < j \leq n} H_2(G_{ij}; \mathbb{Z}/2)$$

is injective. Let

$$s := \bigoplus_{1 \leq i < j \leq n} (s_{ij})_*: \bigoplus_{1 \leq i < j \leq n} H_2(G_{ij}; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2).$$

Depending on n , the composition $s \circ p$ might not be the identity since any nontrivial elements from $H_2(C_i; \mathbb{Z}/2)$ (i.e. whenever C_i has even order) appear $(n-1)$ times in the terms $H_2(G_{ij}; \mathbb{Z}/2)$ and hence get multiplied by $(n-1)$. For n even this requires no change, but for n odd this needs a remedy. We give a unified treatment. Define the $\mathbb{Z}/2$ -module

$$\Lambda_n := \begin{cases} \bigoplus_{1 \leq i < j \leq n} H_2(G_{ij}; \mathbb{Z}/2) \oplus \bigoplus_{1 \leq i \leq n} H_2(C_i; \mathbb{Z}/2) & n \text{ odd} \\ \bigoplus_{1 \leq i < j \leq n} H_2(G_{ij}; \mathbb{Z}/2) \oplus \{0\} & n \text{ even.} \end{cases}$$

For n odd, define

$$p'_n := \bigoplus_{1 \leq i \leq n} (p_i)_* : H_2(\pi; \mathbb{Z}/2) \rightarrow \bigoplus_{1 \leq i \leq n} H_2(C_i; \mathbb{Z}/2)$$

and

$$s'_n := \bigoplus_{1 \leq i \leq n} (s_i)_* : \bigoplus_{1 \leq i \leq n} H_2(C_i; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2).$$

For n even, take $p'_n : H_2(\pi; \mathbb{Z}/2) \rightarrow \{0\}$ and $s'_n : \{0\} \rightarrow H_2(\pi; \mathbb{Z}/2)$ to be the zero maps. Then

$$(s \oplus s'_n) \circ (p \oplus p'_n) : H_2(\pi; \mathbb{Z}/2) \rightarrow \Lambda_n \rightarrow H_2(\pi; \mathbb{Z}/2)$$

is the identity map. It follows by naturality of the Atiyah Hirzebruch spectral sequence that this composition still induces the identity map after modding out the $d_{4,1}^2$ and $d_{5,0}^3$ differentials.

In particular, the induced map

$$p \oplus p'_n : H_2(\pi; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3) \rightarrow \Lambda_n / \text{im}(d_{4,1}^2, d_{5,0}^3)$$

is injective. We can therefore apply the inheritance result Theorem 5.7 above, with G_i , in the notation of that theorem, as all the abelian groups G_{ij} or C_i with either one or two generators involved in the sum Λ_n . We conclude that π has the Tertiary Property 1.15 as asserted. \square

5.3. Reduction to odd index subgroups

This section will show that if either the Secondary Property 1.11 or the Tertiary Property 1.15 holds for a finite index subgroup $P \leq \pi$ where the index of P in π is odd, then the corresponding property also holds for π .

Let $P \leq \pi$ be a subgroup of a finitely presented group π . Let X be a CW complex with fundamental group π . There is a covering space $p: \widehat{X} \rightarrow X$ with fundamental group P . Both spaces have the same universal cover \widetilde{X} , and we can express the cohomology of the universal cover as $H^*(X; \mathbb{Z}\pi) \cong H^*(\widehat{X}; \mathbb{Z}P)$, where the isomorphism is of $\mathbb{Z}P$ -modules. Similarly, there are isomorphisms of the homology groups $H_*(X; \mathbb{Z}\pi) \cong H_*(\widehat{X}; \mathbb{Z}P)$, where again the $\mathbb{Z}\pi$ homology is thought of as a $\mathbb{Z}P$ module by restriction. We will make use of this simple observation several times in this section.

Now suppose that $P \leq \pi$ is a finite index subgroup. If M is a right $\mathbb{Z}\pi$ -module we can restrict the action to P and consider the projection

$$M \otimes_{\mathbb{Z}P} C_* \xrightarrow{p} M \otimes_{\mathbb{Z}\pi} C_*$$

inducing

$$p_* : H_*(\widehat{X}; M) \rightarrow H_*(X; M).$$

On the chain level we obtain a *transfer map* in the other direction by

$$\mathrm{tr}: M \otimes_{\mathbb{Z}\pi} C_* \rightarrow M \otimes_{\mathbb{Z}P} C_*, \quad m \otimes c \mapsto \sum_{Pg \in P \setminus \pi} mg^{-1} \otimes gc.$$

This map induces a map

$$\mathrm{tr}_*: H_*(X; M) \rightarrow H_*(\widehat{X}; M).$$

Similarly one constructs a map

$$\mathrm{tr}^*: H^*(\widehat{X}; M) \rightarrow H^*(X; M).$$

The transfer maps have the key property that $p_* \circ \mathrm{tr}_*$ and $\mathrm{tr}^* \circ p^*$ are equal to multiplication by the index $[\pi : P]$ of P in π .

Let K be a finite 2-complex with $\pi_1(K) = \pi$. Then there is a finite cover K_P of K with fundamental group P .

Start with a map $f: H^2(K; \mathbb{Z}\pi) \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(H^2(K; \mathbb{Z}\pi), \mathbb{Z}\pi)$, and define

$$\mathrm{tr}(f) \in \mathrm{Sesq}(H^2(K_P; \mathbb{Z}\pi))$$

to be the composition

$$\begin{aligned} H^2(K_P; \mathbb{Z}\pi) &\xrightarrow{\mathrm{tr}^*} H^2(K; \mathbb{Z}\pi) \xrightarrow{f} \mathrm{Hom}_{\mathbb{Z}\pi}(H^2(K; \mathbb{Z}\pi), \mathbb{Z}\pi) \\ &\xrightarrow{(\mathrm{tr}^*)^*} \mathrm{Hom}_{\mathbb{Z}\pi}(H^2(K_P; \mathbb{Z}\pi), \mathbb{Z}\pi). \end{aligned}$$

Post-compose with the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}\pi}(H^2(K_P; \mathbb{Z}\pi), \mathbb{Z}\pi) \cong \mathrm{Hom}_{\mathbb{Z}P}(H^2(K_P; \mathbb{Z}\pi), \mathbb{Z}P),$$

given by post-composition with the map

$$\begin{aligned} \mathrm{ev}_{\mathbb{Z}P}: \mathbb{Z}\pi &\rightarrow \mathbb{Z}P \\ \sum_{g \in \pi} n_g g &\mapsto \sum_{g \in P} n_g g, \end{aligned}$$

and pre-compose with the map $H^2(K_P; \mathbb{Z}P) \rightarrow H^2(K_P; \mathbb{Z}\pi)$ given by the inclusion $\mathbb{Z}P \rightarrow \mathbb{Z}\pi$, to restrict $\mathrm{tr}(f)$ to an element of $\mathrm{Sesq}(H^2(K_P; \mathbb{Z}P))$ i.e. a map $H^2(K_P; \mathbb{Z}P) \rightarrow \mathrm{Hom}_{\mathbb{Z}P}(H^2(K_P; \mathbb{Z}P), \mathbb{Z}P)$. This construction commutes with the transposition T and so defines a transfer map

$$\mathrm{tr}_*: \widehat{H}^0(\mathrm{Sesq}(H^2(K; \mathbb{Z}\pi))) \rightarrow \widehat{H}^0(\mathrm{Sesq}(H^2(K_P; \mathbb{Z}P))).$$

Lemma 5.10. *The following diagram commutes:*

$$\begin{array}{ccc} H_3(\pi; \mathbb{Z}/2) & \xrightarrow{A} & \widehat{H}^0(\mathrm{Sesq}(H^2(K; \mathbb{Z}\pi))) \\ \downarrow \mathrm{tr}_* & & \downarrow \mathrm{tr}_* \\ H_3(P; \mathbb{Z}/2) & \xrightarrow{A} & \widehat{H}^0(\mathrm{Sesq}(H^2(K_P; \mathbb{Z}P))) \end{array}$$

PROOF. By tracing the definition of the map $A: H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\mathrm{Sesq}(H^2(K; \mathbb{Z}\pi)))$, via the maps ι , I , Θ and Φ , it is not too hard to see, for $c \in C_3(B\pi; \mathbb{Z}\pi)$, $m, m' \in C^2 = C^2(K; \mathbb{Z}\pi)$ representatives of cohomology classes in $H^2(K; \mathbb{Z}\pi)$, that

$$A([1 \otimes c]) = (([m], [m']) \mapsto \overline{m(\partial_3(c))} m'(\partial_3(c)) \in \mathbb{Z}\pi).$$

Here $\partial_3: C_3(B\pi; \mathbb{Z}\pi) \rightarrow C_2(B\pi; \mathbb{Z}\pi)$ is the boundary map.

Similarly to the proof that the above definition of $A([1 \otimes c])$ does not depend on the preimage $c \in C_3$ of $[1 \otimes c]$, one checks that for $[c'] \in H_3(P; \mathbb{Z}/2)$ and for any

choice of a preimage $\tilde{c} \in \mathbb{Z}\pi \otimes_{\mathbb{Z}P} C_3(BP; \mathbb{Z}P)$, the $\widehat{H}^0(\text{Sesq } H^2(K_P; \mathbb{Z}P))$ class of the map

$$([m], [m']) \mapsto \text{ev}_{\mathbb{Z}P} \left(\overline{(1 \otimes m)(\partial_3(\tilde{c}))} (1 \otimes m')(\partial_3(\tilde{c})) \right) \in \mathbb{Z}\pi$$

agrees with $A([c'])$.

For convenience we provide the details. For all $g, g' \in \pi$ and all $c \in C_3(BP; \mathbb{Z}P)$ we have the following equality in $\widehat{H}^0(\text{Sesq}(H^2(K_P; \mathbb{Z}P)))$.

$$\begin{aligned} & \left((m, m') \mapsto \text{ev}_{\mathbb{Z}P} \left(\overline{(1 \otimes m)((g \pm g') \otimes \partial_3(c))} (1 \otimes m')((g \pm g') \otimes \partial_3(c)) \right) \right) \\ &= \left((m, m') \mapsto \text{ev}_{\mathbb{Z}P} \left(\overline{m(\partial_3(c))} (\overline{(g \pm g')}(g \pm g')) m'(\partial_3(c)) \right) \right) \\ &= \left((m, m') \mapsto \text{ev}_{\mathbb{Z}P} \left(\overline{m(\partial_3(c))} (2 \pm (g^{-1}g' + \overline{g^{-1}g'}) m'(\partial_3(c))) \right) \right) \\ &= \left((m, m') \mapsto \text{ev}_{\mathbb{Z}P} \left(\overline{m(\partial_3(c))} (g^{-1}g' + \overline{g^{-1}g'}) m'(\partial_3(c)) \right) \right) \\ &= 0 \end{aligned}$$

Since any two choices of preimage of $[c'] \in H_3(P; \mathbb{Z}/2)$ in $\mathbb{Z}\pi \otimes_{\mathbb{Z}P} C_3(BP; \mathbb{Z}P)$ differ by a sum of elements of the form $(g \pm g') \otimes c$, this shows that the element in $\widehat{H}^0(\text{Sesq}(H^2(K_P; \mathbb{Z}P)))$ is independent of the choice of preimage. Hence,

$$\begin{aligned} A(\text{tr}_*[1 \otimes c]) &= \text{ev}_{\mathbb{Z}P} \left(([m], [m']) \mapsto \overline{(1 \otimes m)(\partial_3(\text{tr}_* c))} (1 \otimes m')(\partial_3(\text{tr}_* c)) \right) \\ &= \text{ev}_{\mathbb{Z}P} \left(([m], [m']) \mapsto \overline{\text{tr}^*(1 \otimes m)(\partial_3(c))} \text{tr}^*(1 \otimes m')(\partial_3(c)) \right) \\ &= \text{tr}_* A([1 \otimes c]). \end{aligned} \quad \square$$

5.3.1. Secondary Property inheritance from odd index subgroups.

Like the Secondary inheritance from Section 5, we will not actually show that the Secondary Property 1.11 is inherited from finite odd index subgroups, but instead show inheritance for Condition 3.20. A group can have the Secondary Property without necessarily satisfying Condition 3.20, although Condition 3.20 implies the Secondary Property 1.11 by Theorem 3.22.

Theorem 5.11. *Let π be a finitely presented group and let P be a finite odd index subgroup of π . Then Condition 3.20 holds for π if it holds for P .*

PROOF. Let K be a finite 2-complex with $\pi_1(K) = \pi$. Then there is a finite cover K_P of K with fundamental group P . In particular, $H^2(K_P; \mathbb{Z}P) \cong H^2(K; \mathbb{Z}\pi)$, by the observation made at the beginning of this section.

Consider the following diagram. Since $i_* \circ \text{tr}_*$ is multiplication by the index $[\pi : P]$, which is odd, the middle vertical composition is the identity.

$$\begin{array}{ccccc} H_5(\pi; \mathbb{Z}) & \xrightarrow{Sq_2 \text{ored}_2} & H_3(\pi; \mathbb{Z}/2) & \xrightarrow{A} & \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))) \\ \downarrow \text{tr}_* & & \downarrow \text{tr}_* & & \downarrow \text{tr}_* \\ H_5(P; \mathbb{Z}) & \xrightarrow{Sq_2 \text{ored}_2} & H_3(P; \mathbb{Z}/2) & \xrightarrow{A} & \widehat{H}^0(\text{Sesq}(H^2(K_P; \mathbb{Z}P))) \\ \downarrow i_* & & \downarrow i_* & & \downarrow i_* \\ H_5(\pi; \mathbb{Z}) & \xrightarrow{Sq_2 \text{ored}_2} & H_3(\pi; \mathbb{Z}/2) & \xrightarrow{A} & \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi))) \end{array}$$

Exactness of the top and bottom rows (which are equal) now follows from the exactness of the middle row by a diagram chase using the fact that $i_* \circ \text{tr}_* = \text{Id}_{H_3(\pi; \mathbb{Z}/2)}$. \square

5.3.2. Tertiary property inheritance from odd index subgroups.

Theorem 5.12. *Let π be a finitely presented group, and let P be a finite index subgroup of odd index. Then the Tertiary Property 1.15 holds for π if it holds for P .*

For the proof we first need a couple of lemmas.

Lemma 5.13. *Let X be a finite CW complex with fundamental group π . The composition*

$$H^2(\widehat{X}; \mathbb{Z}P) \xrightarrow{\text{red}} H^2(\widehat{X}; \mathbb{Z}/2) \xrightarrow{\text{tr}^*} H^2(X; \mathbb{Z}/2)$$

agrees with the reduction of coefficients

$$H^2(X; \mathbb{Z}\pi) \xrightarrow{\text{red}} H^2(X; \mathbb{Z}/2)$$

under the identification of $H^2(\widehat{X}; \mathbb{Z}P)$ and $H^2(X; \mathbb{Z}\pi)$ given at the beginning of the subsection.

PROOF. Let C_* denote the cellular $\mathbb{Z}\pi$ -chain complex of X . On the chain level, the identification of $H^2(X; \mathbb{Z}\pi)$ with $H^2(\widehat{X}; \mathbb{Z}P)$ is given by

$$\text{ev}_{\mathbb{Z}P}: \text{Hom}_{\mathbb{Z}\pi}(C_2, \mathbb{Z}\pi) \rightarrow \text{Hom}_{\mathbb{Z}P}(C_2, \mathbb{Z}P).$$

Here $\text{ev}_{\mathbb{Z}P}: \mathbb{Z}\pi \rightarrow \mathbb{Z}P$, sending $\sum_{g \in \pi} n_g g$ to $\sum_{g \in P} n_g g$, induces the map on Hom modules above by post-composition; we abuse notation and also denote the map on Hom modules by $\text{ev}_{\mathbb{Z}P}$.

The reduction of coefficients is given by post-composing with the augmentations $\varepsilon: \mathbb{Z}\pi \rightarrow \mathbb{Z}/2$ and $\varepsilon: \mathbb{Z}P \rightarrow \mathbb{Z}/2$ respectively. The lemma now follows from the straightforward computation that the following diagram commutes.

$$\begin{array}{ccc} \text{Hom}_{\mathbb{Z}\pi}(C_2, \mathbb{Z}\pi) & \xrightarrow{\text{ev}_{\mathbb{Z}P}} & \text{Hom}_{\mathbb{Z}P}(C_2; \mathbb{Z}P) \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ \text{Hom}_{\mathbb{Z}\pi}(C_2, \mathbb{Z}/2) & \xleftarrow{\text{tr}^*} & \text{Hom}_{\mathbb{Z}P}(C_2, \mathbb{Z}/2) \end{array}$$

\square

Lemma 5.14. *Let $i: P \rightarrow \pi$ be the inclusion. Let M be a spin manifold with fundamental group π and such that $\text{pri}(M) = \text{sec}(M) = 0$. Then*

$$i_*: H_2(P; \mathbb{Z}/2) / \text{im}(d_{5,0}^3, d_{4,1}^2) \rightarrow H_2(\pi; \mathbb{Z}/2) / \text{im}(d_{5,0}^3, d_{4,1}^2)$$

maps $\text{ter}(\widehat{M})$ to $\text{ter}(M)$, where $p: \widehat{M} \rightarrow M$ is the cover associated to $P \leq \pi$.

PROOF. Stabilisation of M by a single $S^2 \times S^2$ corresponds to stabilisation of \widehat{M} with $[\pi: P]$ copies of $S^2 \times S^2$. Since stabilisation does not change the ter invariant, and since stably the map $c: M \rightarrow B\pi$ factors through $B\pi^{(2)}$ up to homotopy (because $\text{pri}(M) = \text{sec}(M) = 0$), we will assume that we are in the situation that $c: M \rightarrow B\pi$ maps to the 2-skeleton $B\pi^{(2)} \subset B\pi$, and therefore also that $\widehat{c} = c \circ p: \widehat{M} \rightarrow M \rightarrow B\pi^{(2)}$ factors as $M \rightarrow BP^{(2)} \rightarrow B\pi^{(2)}$, through the 2-skeleton of BP . By Lemma 2.5, $\text{ter}(M)$ is given as follows. Choose a point e_i in each 2-cell

c_i of $B\pi^{(2)}$, and let F_i be the regular preimage of e_i under c . The spin structure of M induces a spin structure on F_i and we have $\mathbf{ter}(M, c) = [\sum_i \text{Arf}(F_i)c_i]$. A regular preimage of e_i under $c \circ p: \widehat{M} \rightarrow B\pi^{(2)}$ consists of $[\pi : P]$ copies of F_i , each with the same spin structure. Therefore, when we view \widehat{M} as an element of $\Omega_4^{\text{Spin}}(B\pi)$, we get

$$\mathbf{ter}(\widehat{M}, c \circ p) = [\pi : P] \left[\sum_i \text{Arf}(F_i)c_i \right] = [\pi : P] \mathbf{ter}(M, c) = \mathbf{ter}(M, c),$$

where the last equality uses that the index $[\pi : P]$ is odd. Since $\mathbf{ter}(\widehat{M}, c \circ p) = i_* \mathbf{ter}(\widehat{M}, \widehat{c})$, the lemma follows. \square

PROOF OF THEOREM 5.12. Let K be a finite 2-complex with fundamental group π , and let \widehat{K} be the covering space corresponding to P .

Every splitting s of $H^2(M; \mathbb{Z}\pi) \rightarrow H^2(K; \mathbb{Z}\pi)$ also is a splitting s_P of $H^2(\widehat{M}; \mathbb{Z}P) \rightarrow H^2(\widehat{K}; \mathbb{Z}P)$, because $H^2(\widehat{M}; \mathbb{Z}P) = H^2(M; \mathbb{Z}\pi)$ and $H^2(\widehat{K}; \mathbb{Z}P) = H^2(K; \mathbb{Z}\pi)$, with the latter considered as $\mathbb{Z}P$ -modules.

By Lemma 5.14 it suffices to show that $i_* \tau_{s_P} = \tau_s$, since then $\tau_s = i_* \tau_{s_P} = i_* \mathbf{ter}(\widehat{M}) = \mathbf{ter}(M)$. Here for the middle equality we used the assumption that P has the Tertiary Property 1.15.

Recall that the map $\tau_{s_P}: H^2(B\pi; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$ is defined as follows: restrict $x \in H^2(BP; \mathbb{Z}/2)$ to an element of $H^2(\widehat{K}; \mathbb{Z}/2)$, choose a preimage $x' \in H^2(\widehat{K}; \mathbb{Z}P)$ and then apply $\tau \circ PD \circ s_P$. By Lemma 5.13 with $X = K$, $x' \in H^2(\widehat{K}; \mathbb{Z}P) \cong H^2(K; \mathbb{Z}\pi)$ is a preimage of the restriction of $\text{tr}^*(x) \in H^2(B\pi; \mathbb{Z}/2)$ to $H^2(K; \mathbb{Z}/2)$. Hence $\tau_s(\text{tr}^*(x)) = \tau(PD(s(x'))) = \tau_{s_P}(x)$ and we obtain that $\tau_s \circ \text{tr}^* = \tau_{s_P}$. Equivalently, we have $\text{tr}_* \tau_s = \tau_{s_P}$, when we view τ_s and τ_{s_P} as elements of $H_2(\pi; \mathbb{Z}/2) \cong \text{Hom}(H^2(\pi; \mathbb{Z}/2), \mathbb{Z}/2)$ and $H_2(P; \mathbb{Z}/2) \cong \text{Hom}(H^2(P; \mathbb{Z}/2), \mathbb{Z}/2)$ respectively. Apply i_* to obtain

$$i_* \tau_{s_P} = i_* \text{tr}_* \tau_s = [\pi : P] \tau_s = \tau_s,$$

where the last equality follows from the assumption that the index of P in π is odd. \square

CHAPTER 6

Examples

In this chapter we prove that several families of groups possess the Secondary and Tertiary properties, thus classifying, at least modulo the limitations discussed in the introduction, spin 4-manifolds with these fundamental groups up to stable diffeomorphism. In particular we prove that both properties hold for all finitely generated abelian groups.

6.1. The secondary and tertiary properties for three families of groups

We begin by considering the following three families of groups: cohomologically 3-dimensional groups, right-angled Artin groups, and generalised quaternion groups.

6.1.1. Cohomologically 3-dimensional groups. A group G is said to have cohomological dimension at most n if \mathbb{Z} , thought of as a $\mathbb{Z}G$ -module via the augmentation map $\mathbb{Z}G \rightarrow \mathbb{Z}$, admits a $\mathbb{Z}G$ -module projective resolution of length n . A group G is said to be of type FP_n if \mathbb{Z} has a projective resolution such that the first n terms are finitely generated. A group G has type FP_3 and cohomological dimension at most three if and only if there exists a finite 2-complex K with $\pi_1(K) \cong G$ such that $\pi_2(K)$ is finitely generated projective as a $\mathbb{Z}G$ -module.

Theorem 6.1. *The Secondary Property 1.11 and the Tertiary Property 1.15 hold for groups of type FP_3 that have cohomological dimension at most three.*

Under the assumption that the equivariant intersection form is even, Hambleton-Hildum [HH19] give an alternative classification to ours, using the stable quadratic 2-type instead of the τ invariant. By Theorem 6.1, the intersection form being even is equivalent to $\mathfrak{sec}(M) = 0$. As discussed in the introduction, Hambleton and Hildum use the whole intersection form. This means that their results also apply to the unstable homeomorphism problem, unlike ours. On the other hand, the τ -invariant on a summand of $\pi_2(M)$ can often be an easier invariant to compute.

The proof of Theorem 6.1 is split into two parts, one for each property. Each property is dealt with in its own subsection below.

Three dimensional groups have the secondary property. For the convenience of the reader, we recall that Condition 3.20 requires the sequence

$$H_5(\pi; \mathbb{Z}) \xrightarrow{\text{Sq}_2^{\text{ored}_2}} H_3(\pi; \mathbb{Z}/2) \xrightarrow{A} \widehat{H}^0(\text{Sesq}(H))$$

be exact at $H_3(\pi; \mathbb{Z}/2)$.

For a cohomologically 3-dimensional group π of type FP_3 , choose a 2-dimensional complex K with $\pi_1(K) = \pi$ and such that $\pi_2(K)$ is a finitely generated projective $\mathbb{Z}\pi$ -module. Let $H := H^2(K; \mathbb{Z}\pi)$. Recall from Lemma 3.12 that $\pi_2(K) \cong H^*$. Then the double dual H^{**} is the dual of $\pi_2(K)$ and hence is a finitely generated

projective $\mathbb{Z}\pi$ -module. We can consider the natural evaluation map $e_H: H \rightarrow H^{**}$. Since $H^* \cong \pi_2(K)$ is finitely generated projective, the map $e_{H^*}: H^* \rightarrow H^{***}$ is an isomorphism. It is straightforward to verify that $e_H^* \circ e_{H^*}: H^* \rightarrow H^*$ is the identity on H^* . Thus the map $e_H^*: H^{***} \rightarrow H^*$ is an isomorphism.

For a $\mathbb{Z}\pi$ -module M and a projective $\mathbb{Z}\pi$ -module P , the canonical map

$$\mathrm{Hom}_{\mathbb{Z}\pi}(M, \mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} P \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(M, P)$$

is an isomorphism. Hence the map $\mathrm{Hom}_{\mathbb{Z}\pi}(H^{**}, H^*) \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(H, H^*)$ given by precomposing with e_H is an isomorphism, because e_H^* is an isomorphism, and the given map can be identified with $e_H^* \otimes \mathrm{Id}_{H^*}$. This implies that

$$\begin{aligned} \mathrm{Sesq}(e_H): \mathrm{Sesq}(H^{**}) &\cong \mathrm{Hom}_{\mathbb{Z}\pi}(H^{**}, H^{***}) \xrightarrow{\mathrm{Hom}(-, e_H^*)} \\ &\mathrm{Hom}_{\mathbb{Z}\pi}(H^{**}, H^*) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}\pi}(H, H^*) \cong \mathrm{Sesq}(H) \end{aligned}$$

is also an isomorphism.

Consider the diagram

$$\begin{array}{ccc} H^{***} \otimes_{\mathbb{Z}\pi} H^{***} & \xrightarrow[\cong]{e_H^* \otimes e_H^*} & H^* \otimes_{\mathbb{Z}\pi} H^* \\ \downarrow \Phi_{H^{**}} & & \downarrow \Phi_H \\ \mathrm{Sesq}(H^{**}) & \xrightarrow[\cong]{\mathrm{Sesq}(e_H)} & \mathrm{Sesq}(H) \end{array}$$

Now the map $\Phi_{H^{**}}$ is an isomorphism by Lemma 3.10, and hence Φ_H is also an isomorphism. It follows that $\Psi_H = \widehat{H}^0(\Phi_H) \circ \Theta_H$ is an isomorphism as well, since Θ_H is an isomorphism by Lemma 3.8. The map $A: H_3(\pi; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\mathrm{Sesq}(H))$, which by definition is a composite $\Psi_H \circ (\mathrm{Id} \times I) \circ \iota$, where I is an isomorphism by Lemma 3.12 and ι is injective by Lemma 4.14, is therefore injective. Since $H_5(\pi; \mathbb{Z}) = 0$, this implies that Condition 3.20 holds, and therefore that π has the Secondary Property 1.11.

Three dimensional groups have the tertiary property. Since Φ_H is injective by the argument in Section 6.1.1, and ι_{H^*} is injective by Lemma 4.14, it follows from Theorem 4.12 that π has the Tertiary Property 1.15.

6.1.2. Right angled Artin Groups. Let (V, E) be a finite graph, with $V = \{v_i\}_{i \in I}$ the set of vertices and $E = \{e_j = \{v_j^1, v_j^2\}\}_{j \in J}$ the set of edges, where e_j is an edge between $v_j^1, v_j^2 \in V$. The corresponding right-angled Artin group (or RAAG for short) is $\langle V \mid \{[v_j^1, v_j^2]\}_{j \in J} \rangle$. An n -clique is a subset of n vertices $v_1, \dots, v_n \in V$, such that for every pair of vertices v_i, v_j , with $i \neq j \in \{1, \dots, n\}$, we have $\{v_i, v_j\} \in E$.

Theorem 6.2. *The Secondary Property 1.11 and the Tertiary Property 1.15 hold for right-angled Artin groups.*

We will use the following well-known theorem.

Theorem 6.3 ([CD95, Corollary 3.2.2]). *Let π be the RAAG associated to a graph (V, E) . The dimension of $H_n(\pi; \mathbb{Z}/2)$ is the number of n -cliques in (V, E) .*

RAAGs have the secondary property. For each n -clique in (V, E) with vertices v'_k , $k = 1, \dots, n$, there is an inclusion $\mathbb{Z}^n = \langle x_k \mid [x_k, x_{k'}] \rangle \rightarrow \pi$ given by $x_k \mapsto v'_k$, and a projection $\pi \rightarrow \mathbb{Z}^n$ given by $v'_k \mapsto x_k$ and $v_i \mapsto 0$ for all $v_i \notin \{v'_k\}_{k=1, \dots, n}$. In particular, π is a semi-direct product $N \rtimes \mathbb{Z}^n$ of \mathbb{Z}^n and a group N .

We apply this with $n = 3$, for each 3-clique. By Theorem 6.3, the rank of $H_3(\pi; \mathbb{Z}/2)$ is the same as the number of 3-cliques. We can find a basis for $H_3(\pi; \mathbb{Z})$ such that each basis element is the image $i_*(x)$ for some $x \in H_3(\mathbb{Z}^3; \mathbb{Z}/2) = \mathbb{Z}/2$. We know that \mathbb{Z}^3 has the Secondary property by Theorem 6.1. Then the “moreover” part of Theorem 5.3 says that $i_*(x)$ lies in the kernel of A if and only if it lies in the image of $\text{Sq}_2 \circ \text{red}_2$. Apply this argument at each 3-clique, and therefore for each basis element of $H_3(\pi; \mathbb{Z}/2)$, to see that Condition 3.20 holds for π . Therefore π has the Secondary Property 1.11.

RAAGs have the tertiary property. By Theorem 6.3 and the discussion at the beginning of the proof of the Secondary Property in Section 6.1.2, it follows that there are surjections $p_j: \pi \rightarrow \mathbb{Z}^2$ such that $H_2(\pi; \mathbb{Z}/2) \rightarrow H_2(\oplus_{j \in J} \mathbb{Z}^2; \mathbb{Z}/2)$ is injective, where as above J indexes the edges of the graph defining the RAAG π . Since all differentials in the spectral sequence for $\Omega_4^{\text{Spin}}(B\mathbb{Z}^2)$ are trivial, this map is still injective after dividing out the image of the differentials. Hence the Tertiary Property 1.15 for π follows from Theorem 5.7 and the fact that we know the property holds for \mathbb{Z}^2 (since \mathbb{Z}^2 has cohomological dimension two).

6.1.3. Generalised quaternion groups. By [Bro82, Theorem VI 9.3], every 2-group with periodic cohomology is either a cyclic group or a generalised quaternion group. In the next section, we will show that all finitely generated abelian groups have the Secondary Property 1.11 and the Tertiary Property 1.15. Thus by Theorem 5.11 and Theorem 5.12 below, every finite group whose 2-Sylow subgroup has periodic cohomology has both properties once we have shown them for all generalised quaternion groups. This is the purpose of this subsection. In combination with our other results, Theorem 6.4 is therefore part of the stable diffeomorphism classification of spin 4-manifolds with finite fundamental group, whose 2-Sylow subgroup has periodic cohomology.

Note that the Secondary Property 1.11 for finite groups with quaternion 2-Sylow subgroup was already proved in [Tei92, Theorem 6.4.1]. Since the statement there is slightly different, we reprove it here for completeness, as an instructive illustration of the use of Condition 3.20.

Let n be a power of two. A presentation of the generalised quaternion group with $8n$ elements is given by

$$Q_{8n} = \langle x, y \mid x^{2n}y^{-2}, xyxy^{-1} \rangle.$$

(The quaternion group with 4 elements is omitted because it is cyclic.) Note that $xyx = y$ implies $x^{2n}yx^{2n} = y$, so that $y^4 = 1$ and therefore $x^{4n} = 1$. In the case $n = 1$, sending $x \mapsto i$, $y \mapsto j$ gives an isomorphism with the usual presentation of Q_8 given by $\langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$.

Theorem 6.4. *The generalised quaternion group Q_{8n} has the Secondary Property 1.11 and the Tertiary Property 1.15.*

PROOF. By [Tei92, Proposition 4.2.1], the differential

$$d_{5,0}^3: H_5(Q_{8n}; \mathbb{Z}) \rightarrow H_2(Q_{8n}; \mathbb{Z}/2)$$

is an isomorphism. In particular, the ter invariant lies in the trivial group. Moreover, this implies that $d_{5,0}^2: H_5(Q_{8n}; \mathbb{Z}) \rightarrow H_3(Q_{8n}; \mathbb{Z}/2)$ is the zero map. It follows immediately from the fact that $H_2(Q_{8n}; \mathbb{Z}/2)/\text{im}(d_{5,0}^3) = 0$ that Q_{8n} has the Tertiary Property 1.15.

Now we work on showing that the Secondary Property holds for Q_{8n} . Let $N \in \mathbb{Z}Q_{8n}$ be the norm element $N = \sum_{g \in Q_{8n}} g$ and let $\varepsilon: \mathbb{Z}Q_{8n} \rightarrow \mathbb{Z}$ be the augmentation. By [CE56, page 253] the beginning of a free resolution of \mathbb{Z} as a $\mathbb{Z}Q_{8n}$ -module is given as follows:

$$\begin{array}{c}
C_4 \\
\downarrow d_4 \\
C_3 \\
\downarrow d_3 \\
C_2 \\
\downarrow d_2 \\
C_1 \\
\downarrow d_1 \\
C_0
\end{array}
=
\begin{array}{c}
\mathbb{Z}Q_{8n} \\
\downarrow N \\
\mathbb{Z}Q_{8n} \\
\begin{array}{ccc}
\swarrow x-1 & & \searrow 1-xy \\
\mathbb{Z}Q_{8n} & \oplus & \mathbb{Z}Q_{8n} \\
\downarrow \sum_{i=0}^{2n-1} x^i & \swarrow -y-1 & \searrow xy+1 \\
\mathbb{Z}Q_{8n} & \oplus & \mathbb{Z}Q_{8n} \\
\downarrow x-1 & \swarrow x-1 & \searrow y-1 \\
\mathbb{Z}Q_{8n} \\
\downarrow \varepsilon \\
\mathbb{Z}
\end{array}
\end{array}$$

Let K denote the 2-dimensional CW complex determined by the start of this resolution (which is the same as the presentation complex for the above presentation of Q_{8n}).

One easily computes that $H_3(Q_{8n}; \mathbb{Z}/2) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}Q_{8n}} \pi_2(K) \cong \mathbb{Z}/2$. Recall from the proof of the Tertiary Property for these groups, that $d_{5,0}^2$ is the zero map. To verify Condition 3.20 and therefore show that the Secondary Property 1.11 holds, we therefore have to show that the map $A: H_3(Q_{8n}; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}\pi)))$ is injective.

The nontrivial element in $\mathbb{Z}/2 \otimes_{\mathbb{Z}Q_{8n}} \pi_2(K) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}Q_{8n}} \ker d_2$ is represented by $1 \otimes d_3(1)$. By Lemma 3.12 and the definition of A (Definition 3.18), the form $\lambda := A(1 \otimes d_3(1))$ on

$$H^2(K; \mathbb{Z}Q_{8n}) \cong \text{coker } d^2 = (\mathbb{Z}Q_{8n})^2 / \left\langle \left(\sum_{i=0}^{2n-1} x^{-i}, (xy)^{-1} + 1 \right), \left(-y^{-1} - 1, x^{-1} - 1 \right) \right\rangle$$

is given by the matrix

$$L := \begin{pmatrix} (x^{-1} - 1)(x - 1) & (x^{-1} - 1)(1 - xy) \\ (1 - (xy)^{-1})(x - 1) & (1 - (xy)^{-1})(1 - xy) \end{pmatrix}.$$

Note that we used the involution on $\mathbb{Z}Q_{8n}$ to view $\text{coker } d^2$ as a left module. The entries of the matrix are obtained from the third boundary map d_3 in the above resolution. This matrix a priori defines a pairing on the free module $\mathbb{Z}Q_{8n}^2$. However the matrix L determines a well-defined pairing on $H^2(K; \mathbb{Z}Q_{8n}) \cong \text{coker } d^2$, due to

the fact that $d_2 \circ d_3 = 0$ in the resolution above. Throughout our verification of the Secondary Property, we will often write forms on quotient modules as matrices defining forms on free modules, and it will always be necessary that these descend to well defined maps on the quotient modules.

CLAIM. The form $\lambda := A(1 \otimes d_3(1))$ is odd.

To see the claim, we investigate possible forms q that might possibly exhibit λ as even, and show that no such q can exist. A possible q with $q + q^* = \lambda$ can be written as

$$\begin{pmatrix} 1 - x + z_1 & (x^{-1} - 1)(1 - xy) + z_2 \\ -\overline{z_2} & 1 - xy + z_3 \end{pmatrix}$$

for some $z_1, z_2, z_3 \in \mathbb{Z}Q_{8n}$ satisfying $z_1 = -\overline{z_1}$, $z_3 = -\overline{z_3}$. Since

$$[-y^{-1} - 1, x^{-1} - 1], \left[\sum x^{-i}, (xy)^{-1} + 1 \right] = 0 \in \text{coker } d^2,$$

the form q has to satisfy the relations

$$\begin{aligned} 0 &= q((1, 0), (-y^{-1} - 1, x^{-1} - 1)) \\ &= (1 - x + z_1)(-y - 1) + (x^{-1} - 1)(1 - xy)(x - 1) + z_2(x - 1) \end{aligned}$$

and

$$0 = q((0, 1), \left(\sum x^{-i}, (xy)^{-1} + 1 \right)) = -\overline{z_2} \left(\sum x^i \right) + (1 - xy + z_3)(xy + 1).$$

To derive a contradiction, we will show that these equations cannot be solved after passing to the abelianisation $(Q_{8n})_{ab} \cong (\mathbb{Z}/2)^2$. We get that $z_1 = -\overline{z_1}$ implies $2z_1 = 0$ in $\mathbb{Z}[(\mathbb{Z}/2)^2]$, so $z_1 = 0$. Similarly $z_3 = 0$. Also $x^2 = 1$ implies that $\sum_{i=1}^{2n-1} x^i = n(1 + x)$. The first relation, with $z_1 = 0$ substituted, gives

$$0 = (1-x)(-y-1) + 2(1-x)(1-xy) + z_2(x-1) = (1-x)(1+y) + z_2(x-1) \in \mathbb{Z}[(\mathbb{Z}/2)^2].$$

It follows that $z_2 = (1+y) - a(1+x)$ for some $a \in \mathbb{Z}(Q_{8n})_{ab}$, since $(1+x)$ divides any element of $\mathbb{Z}[(\mathbb{Z}/2)^2]$ that annihilates $(1-x)$. Insert $z_2 = (1+y) - a(1+x)$ into the second relation to obtain

$$\begin{aligned} 0 &= -n(1+x)(1+y-a-ax) + (1-xy)(1+xy) \\ &= -n(1+x)(1+y-a-ax) = 2na(1+x) - nN, \end{aligned}$$

where N now denotes the norm element in $\mathbb{Z}[(\mathbb{Z}/2)^2]$ of $Q_{8n}^{ab} = (\mathbb{Z}/2)^2$. But the first summand has all coefficients divisible by $2n$, while the second summand has all coefficients only divisible by n . Hence this element does not vanish in $\mathbb{Z}[(\mathbb{Z}/2)^2]$. Hence there are no solutions for z_1, z_2, z_3 that define a q as desired and so λ cannot be even as claimed. This completes the proof of the claim, so that Condition 3.20 holds for Q_{8n} , completing the proof of Theorem 6.4. \square

6.2. The secondary and tertiary properties for abelian groups

In this section we will focus on finitely generated abelian groups. The goal is to prove the next theorem, classifying spin 4-manifolds with abelian fundamental groups up to stable diffeomorphism.

Theorem 6.5. *The Secondary Property 1.11 and the Tertiary Property 1.15 hold for all finitely generated abelian groups.*

The proof breaks up into stages, each of which is considered in one of the next few subsections. First we consider cyclic groups, then we consider abelian groups with at most two generators, then at most three generators, and finally we use the inheritance properties from Chapter 5 to deduce that the properties hold for any finitely generated abelian group.

We will need the following structure of certain cohomology rings of finite cyclic groups.

Theorem 6.6 ([Hat02, Example 3.41]). *Let C be a cyclic group of order $n = 2k$. Then the cohomology ring with \mathbb{Z}/n coefficients is a quotient of the polynomial ring $(\mathbb{Z}/n)[\alpha, \beta]$, as follows:*

$$H^*(C; \mathbb{Z}/n) \cong (\mathbb{Z}/n)[\alpha, \beta]/(\alpha^2 - k\beta),$$

where $|\alpha| = 1$ and $|\beta| = 2$.

Corollary 6.7. *Let C be a cyclic group of order $n = 2k$. Then the cohomology ring with $\mathbb{Z}/2$ coefficients is a quotient of the polynomial ring $(\mathbb{Z}/2)[\alpha, \beta]$, as follows:*

$$H^*(C; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2[\alpha, \beta]/\alpha^2 & \text{if } k = 2m \\ \mathbb{Z}/2[\alpha] & \text{if } k = 2m + 1 \end{cases}$$

where $|\alpha| = 1$ and $|\beta| = 2$.

Note that the corollary is not obtained simply by setting $n = 2$. Rather, $\alpha^2 = k\beta$ becomes either $\alpha^2 = 0$ or $\alpha^2 = \beta$, when $k = 2m$ or $k = 2m + 1$ respectively.

6.2.1. Cyclic groups.

Lemma 6.8. *The Secondary Property 1.11 and the Tertiary Property 1.15 hold for all cyclic groups.*

Cyclic groups have the secondary property. Let G be a cyclic group. If the cyclic group G is of odd order or infinite order, then $H_3(G; \mathbb{Z}/2) = 0$, and there is nothing further to show.

If $G = \langle T \rangle$ is cyclic of even order $2k$, then we need to check that Condition 3.20. Note that

$$\text{red}_2: \mathbb{Z}/|G| \cong H_5(G; \mathbb{Z}) \rightarrow H_5(G; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is the projection. The generator is dual to $\alpha\beta^2$ in the notation of Corollary 6.7, if $k = 2m$ for some m , and the generator is dual to α^5 if $k = 2m + 1$. Similarly the generator of $H_3(G; \mathbb{Z}/2)$ is dual to $\alpha\beta$ when $k = 2m$ and α^3 when $k = 2m + 1$ in both cases

$$Sq^2: H^3(G; \mathbb{Z}/2) \rightarrow H^5(G; \mathbb{Z}/2)$$

is an isomorphism by Corollary 6.7, together with a straightforward computation using the axioms of the Steenrod operations. In particular recall the Cartan formula $Sq^n(xy) = \sum_{p+q=n} Sq^p(x)Sq^q(y)$. Thus

$$Sq_2 \circ \text{red}_2: H_5(G; \mathbb{Z}) \rightarrow H_3(G; \mathbb{Z}/2)$$

is surjective.

Therefore, to verify Condition 3.20, we have to show that

$$A: H_3(G; \mathbb{Z}/2) \rightarrow \widehat{H}^0(\text{Sesq}(H^2(K; \mathbb{Z}G)))$$

is trivial. A free resolution of \mathbb{Z} as a $\mathbb{Z}G$ module is given by

$$\dots \rightarrow \mathbb{Z}G \xrightarrow{N_T} \mathbb{Z}G \xrightarrow{1-T} \mathbb{Z}G \xrightarrow{N_T} \mathbb{Z}G \xrightarrow{1-T} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z},$$

where $N_T = \sum_{i=0}^{n-1} T^i$. Let K be the corresponding 2-complex with $\pi_1(K) \cong G$. Then $\mathbb{Z}/2 \otimes_{\mathbb{Z}G} \pi_2(K) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}G} \ker(N_T)$ is generated by the image of 1 under

$$d_3 = (1 - T): \mathbb{Z}G \rightarrow \ker(N_T) \rightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}G} \ker(N_T).$$

Hence by Lemma 3.12, the form λ on $H^2(K; \mathbb{Z}G)$ that we have to consider is given by

$$\lambda(x, y) = x(1 - T^{-1})(1 - T)\bar{y},$$

for $x, y \in \mathbb{Z}G/(\overline{N_T}) = \mathbb{Z}G/(N_T)$. For q given by $q(x, y) = x(1 - T)\bar{y}$, we have $\lambda = q + q^*$. Thus λ is even and so the map A is trivial as required. It follows that the sequence of Condition 3.20 is exact as required, so that the cyclic group G has the Secondary Property 1.11.

Cyclic groups have the tertiary property. By Corollary 6.7, we have that $d_{4,1}^2 = Sq_2: H_4(G; \mathbb{Z}/2) \rightarrow H_2(G; \mathbb{Z}/2)$ is an isomorphism. Hence the group in which the ter invariant resides is trivial, and the Tertiary Property 1.15 trivially holds for cyclic groups.

6.2.2. Abelian groups with at most two generators. This section proves the following lemma.

Lemma 6.9. *The Secondary Property 1.11 and the Tertiary Property 1.15 hold for all abelian groups with at most two generators.*

The proof of this lemma will require about eight pages. We consider the secondary property first.

Two generator abelian groups have the secondary property.

CLAIM. To prove the secondary property for two generator abelian groups, it suffices to consider the groups $\pi \cong \mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2} = \langle a, b \mid a^{2^{k_1}}, b^{2^{k_2}}, [a, b] \rangle$, for some $k_1, k_2 \geq 1$, and the groups $\pi \cong \mathbb{Z} \times \mathbb{Z}/2^k$ for some $k \geq 1$.

The claim follows from the fact that we can pass to a finite odd index subgroup by Theorem 5.11, and the fact that $\mathbb{Z} \times \mathbb{Z}$ has cohomological dimension two, which we already know to have the Secondary property by Theorem 6.1. Every abelian group with two generators (as the minimal number of generators) has a finite odd index subgroup that belongs to the list in the claim. We already proved that cyclic groups have the Secondary property in Section 6.2.1. This completes the proof of the claim. \square

For each of the groups in the claim, we need to check Condition 3.20. This will occupy the next few pages.

The groups $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$.

We begin by considering the groups $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$, with $k_1, k_2 \geq 1$. In this case $H_3(\pi; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^4$. If $\alpha \in H_3(\pi; \mathbb{Z}/2)$ is in the image of the inclusion of one of the two factors, then we know exactness at α by the last subsection. Thus by symmetry we only have to consider the nontrivial element $\gamma \in H_3(\pi; \mathbb{Z}/2)$ coming from the generator of $H_1(\mathbb{Z}/2^{k_1}; \mathbb{Z}/2) \otimes H_2(\mathbb{Z}/2^{k_2}; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

CLAIM. The element γ lies in $\text{im}(Sq_2 \circ \text{red}_2)$ if and only if $k_1 \leq k_2$.

The proof of this claim will take the next page and a half. We will show, in the case that $k_1 < k_2$, that γ lies in the image of $Sq_2 \circ \text{red}_2$, and that the nontrivial element $\bar{\gamma} \in H_2(\mathbb{Z}/2^{k_1}; \mathbb{Z}/2) \otimes H_1(\mathbb{Z}/2^{k_2}; \mathbb{Z}/2) \subseteq H_3(\pi; \mathbb{Z}/2)$ does not lie in the

image. In the case that $k_1 = k_2$, we will show that one, and hence both, of γ and $\bar{\gamma}$ lies in $\text{im}(Sq_2 \circ \text{red}_2)$.

First we compute the image of the $Sq^2: H^3(\pi; \mathbb{Z}/2) \rightarrow H^5(\pi; \mathbb{Z}/2)$. A preliminary computation that we will soon need is that $Sq^1(\beta_i) = 0$, with $\beta_i \in H^2(\mathbb{Z}/2^{k_i}; \mathbb{Z}/2)$ as in Corollary 6.7, and $k_i > 1$. To see this, we use the fact that Sq^1 coincides with the Bockstein homomorphism $BS = Sq^1: H^2(\mathbb{Z}/2^{k_i}; \mathbb{Z}/2) \rightarrow H^3(\mathbb{Z}/2^{k_i}; \mathbb{Z}/2)$ associated to the coefficient sequence $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$ (see [MT68, Chapter 3, Theorem 1]). We have that $H^2(\mathbb{Z}/2^{k_i}; \mathbb{Z}/4) = \mathbb{Z}/4 = H^3(\mathbb{Z}/2^{k_i}; \mathbb{Z}/4)$. The differentials in the tensored down resolutions all vanish, so maps in the Bockstein long exact sequence coincide with the maps in the short exact sequence $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$ that induces it. Thus the connecting homomorphism vanishes. This completes the proof that $Sq^1(\beta_i) = 0$.

To compute the image of Sq^2 , we start with the case that $1 < k_1 \leq k_2$. Then $H^3(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}; \mathbb{Z}/2)$ is generated by elements of the form $\alpha_i \beta_j$, in the notation of Corollary 6.7 with $i, j \in \{1, 2\}$. We compute:

$$Sq^2(\alpha_i \beta_j) = Sq^2(\alpha_i) \beta_j + Sq^1(\alpha_i) Sq^1(\beta_j) + \alpha_i Sq^2(\beta_j) = \alpha_i \beta_j^2.$$

Next, we consider the case that $k_1 = 1 < k_2$. Now $H^3(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}; \mathbb{Z}/2)$ is generated by elements $\alpha_1 \beta_2$, $\alpha_2 \beta_2$, α_1^3 , and $\alpha_1^2 \alpha_2$. The image of the first two cases are $\alpha_1 \beta_2^2$ and $\alpha_2 \beta_2^2$ by the computation above. We also have $Sq^2(\alpha_1^3) = \alpha_1^5$ and $Sq^2(\alpha_1^2 \alpha_2) = \alpha_1^4 \alpha_2$.

Finally we consider the case $k_1 = k_2 = 1$. In this case $H^3(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2)$ is generated by elements $\alpha_1^i \alpha_2^j$ with $i+j = 3$. An inductive argument using the Cartan formula shows that $Sq^2(\alpha^i)$ is equal to α^{i+2} if $i \equiv 2, 3 \pmod{4}$ and $Sq^2(\alpha^i) = 0$ if $i \equiv 0, 1 \pmod{4}$. Thus we compute

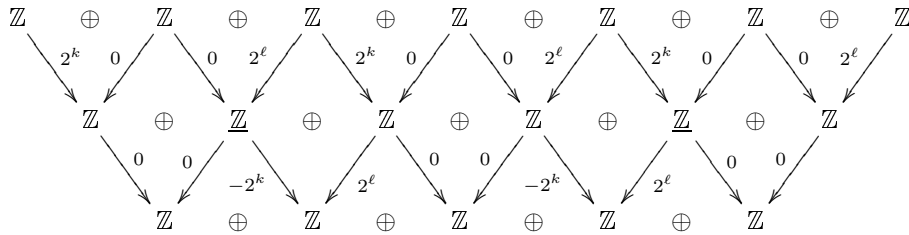
$$Sq^2(\alpha_1^2 \alpha_2) = \alpha_1^4 \alpha_2; \quad Sq^2(\alpha_i^3) = \alpha_i^5; \quad \text{and} \quad Sq^2(\alpha_1 \alpha_2^2) = \alpha_1 \alpha_2^4.$$

It follows that the map $Sq_2: H_5(\pi; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$ is onto in all cases. A resolution of \mathbb{Z} by free $\mathbb{Z}\pi$ -modules is given below. To complete the proof of the claim, one has to check whether the elements of $H_5(\pi; \mathbb{Z}/2) = H_5(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}; \mathbb{Z}/2)$, that hit the generators of $H_3(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}; \mathbb{Z}/2)$ under the dual of the Steenrod square, are in the image of the reduction modulo two map red_2 .

First we consider the case that $k_1 < k_2$. $(\alpha_1 \beta_2^2)^* \in H_5(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}; \mathbb{Z}/2)$ maps to γ and $(\alpha_2 \beta_1^2)^*$ maps to $\bar{\gamma}$. The terms in degree 4, 5 and 6 of the $\mathbb{Z}\pi$ -module resolution of \mathbb{Z} , tensored down over \mathbb{Z} , are shown in the diagram below. Let $k := k_1$ and let $\ell := k_2$, so that $k \leq \ell$ and $\pi = \mathbb{Z}/2^k \times \mathbb{Z}/2^\ell$. Also denote $C_* := C_*(\pi; \mathbb{Z}\pi)$. The next diagram shows

$$\mathbb{Z} \otimes C_6 \cong \mathbb{Z}^7 \rightarrow \mathbb{Z} \otimes C_5 \cong \mathbb{Z}^6 \rightarrow \mathbb{Z} \otimes C_4 \cong \mathbb{Z}^5.$$

Ignore the underlining of two \mathbb{Z} summands for now.



In $\mathbb{Z}/2 \otimes C_5$, we have $e_2 = (\alpha_2 \beta_1^2)^*$, which maps to $\bar{\gamma} = (\alpha_2 \beta_1)^* \in H_3(\pi; \mathbb{Z}/2)$. Similarly $e_5 = (\alpha_1 \beta_2^2)^*$, which maps to $\gamma = (\alpha_1 \beta_2)^* \in H_3(\pi; \mathbb{Z}/2)$. The summands generated by e_2 and e_5 are underlined in the diagram above. In the case that one or both of k, ℓ are equal to 1, replace β_i by α_i^2 in the above statements. Otherwise the computation is the same.

The relevant generator of $H_5(\pi; \mathbb{Z})$ for $\bar{\gamma}$ is $2^{k-\ell}e_2 + e_3$. If $k < \ell$ then this maps to e_3 in $\mathbb{Z}/2 \otimes C_5$, and so $\bar{\gamma}$ is not hit by $\text{Sq}_2 \circ \text{red}_2$. On the other hand, the relevant generator for $H_5(\pi; \mathbb{Z})$ for γ is $e_5 + 2^{k-\ell}e_4$, which maps to e_5 in $\mathbb{Z}/2 \otimes C_5$. Then e_5 maps to γ under $\text{Sq}_2: H_5(\pi; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$. Thus when $k < \ell$, we see that γ lies in the image of $\text{Sq}_2 \circ \text{red}_2$. This proves the claim in the case that $k \neq \ell$, that is $k_1 \neq k_2$.

Now consider the case that $k_1 = k_2$, that is $k = \ell$. We just need to show that γ does lie in the image of $\text{Sq}_2 \circ \text{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_3(\pi; \mathbb{Z}/2)$. The relevant generator of the \mathbb{Z} -homology is $e_4 + e_5$. This maps to $e_4 + e_5$ in $\mathbb{Z}/2 \otimes C_5$. Now, e_4 is dual to $\beta_1 \beta_2 \alpha_2$ if $k = \ell \geq 2$, or $\alpha_1^2 \alpha_2^3$ if $k_1 = k_2 = 1$. From the computation of the Steenrod square $\text{Sq}^2: H^3(\pi; \mathbb{Z}/2) \rightarrow H^5(\pi; \mathbb{Z}/2)$ that we made not long ago, it follows that $e_4 + e_5 \mapsto (\alpha_1 \beta_2)^* = \gamma$ under the dual map $\text{Sq}_2: H_5(\pi; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$, as required. Thus the element γ lies in $\text{im}(\text{Sq}_2 \circ \text{red}_2)$ if and only if $k_1 \leq k_2$. This completes the proof of the claim. \square

Now we need to show that the cases in which $\gamma \in \text{im}(\text{Sq}_2 \circ \text{red}_2)$ correspond to the cases in which $A(\gamma)$ is even. Let $n := 2^{k_1} = 2^k$ and $m := 2^{k_2} = 2^\ell$. Let $N_a := \sum_{i=0}^{n-1} a^i$ and $N_b := \sum_{i=0}^{m-1} b^i$. From the $\mathbb{Z}\pi$ -chain complex

$$\begin{array}{ccccccc}
 \mathbb{Z}\pi & & \oplus & \mathbb{Z}\pi & & \oplus & \mathbb{Z}\pi & & \oplus & \mathbb{Z}\pi & & \oplus & \mathbb{Z}\pi \\
 & \searrow & & \swarrow & & \swarrow & \searrow & & \swarrow & \searrow & & \swarrow & \searrow \\
 & & \mathbb{Z}\pi & & \mathbb{Z}\pi & & \mathbb{Z}\pi & & \mathbb{Z}\pi & & & & \mathbb{Z}\pi \\
 & & \swarrow & & \swarrow & & \swarrow & & \swarrow & & & & \swarrow \\
 & & & \mathbb{Z}\pi & & \mathbb{Z}\pi & & \mathbb{Z}\pi & & & & & \mathbb{Z}\pi \\
 & & & \swarrow & & \swarrow & & \swarrow & & & & & \swarrow \\
 & & & & & \mathbb{Z}\pi & & & & & & & \mathbb{Z}\pi
 \end{array}$$

$\begin{matrix} \text{1-a} & \text{1-b} & \text{-N}_a & \text{N}_b & \text{1-a} & \text{1-b} \\ \text{1-a} & \text{1-b} & \text{1-b} & \text{a-1} & \text{1-b} \\ \text{N}_a & \text{1-b} & \text{a-1} & \text{N}_b \\ \text{1-a} & \text{1-b} \end{matrix}$

one computes $H^2(K; \mathbb{Z}\pi) \cong (\mathbb{Z}\pi)^3 / \langle (N_a, 1-b^{-1}, 0), (0, a^{-1}-1, N_b) \rangle$, and that $A(\gamma)$ is the form represented by:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & mN_b & (1-a)N_b \\ 0 & (1-a^{-1})N_b & 2-a-a^{-1} \end{pmatrix}.$$

CLAIM. For $k_1 > k_2$, that is $n > m$, the form $A(\gamma)$ is not even.

A possible q with $A(\gamma) = q + q^*$ has to have the following shape:

$$\begin{pmatrix} u & v & w \\ -\bar{v} & \frac{m}{2}N_b + y & -aN_b + x \\ -\bar{w} & N_b - \bar{x} & 1-a+z \end{pmatrix}$$

with $u, v, w, x, y, z \in \mathbb{Z}\pi$, $u + \bar{u} = y + \bar{y} = z + \bar{z} = 0$. In order to determine a form on $H^2(K; \mathbb{Z}\pi)$ it has to satisfy:

$$0 = (N_b - \bar{x})(a-1) + (1-a+z)N_b = (1-a)\bar{x} + N_b z$$

and

$$0 = \frac{m}{2}(a-1)N_b + (a-1)y - amN_b + xN_b = -\frac{m}{2}(a+1)N_b + (a-1)y + xN_b.$$

Apply the involution to the second equation and then multiply by $(1-a)$ to yield

$$\begin{aligned} 0 &= -\frac{m}{2}(1-a)(a^{-1}+1)N_b - (1-a)(a^{-1}-1)y + (1-a)\bar{x}N_b \\ &= \frac{m}{2}(a-a^{-1})N_b + (2-a-a^{-1})y + (1-a)\bar{x}N_b \\ &= \frac{m}{2}(a-a^{-1})N_b + (2-a-a^{-1})y - mzN_b. \end{aligned}$$

Here we used $\bar{y} = -y$, while for the last equality we used $(1-a)\bar{x} = -N_bz$ as well as $N_b^2 = mN_b$. Reduce coefficients modulo m and take the coefficient of a to obtain:

$$0 = \frac{m}{2} + 1y_a - y_0 - y_{a^2} \pmod{m}.$$

Since $y = -\bar{y}$ we have $y_0 = 0$ and thus $y_{a^2} = 2y_a + \frac{m}{2}$. Take the coefficient of a^k for $1 < k \leq \frac{n}{2}$ to obtain:

$$0 = 2y_{a^k} - y_{a^{k-1}} - y_{a^{k+1}} \pmod{m}.$$

Thus by induction we have:

$$(6.10) \quad y_{a^k} = ky_a + (-1)^{\delta_{4|k}} \delta_{2|k} \frac{m}{2} \pmod{m},$$

where $\delta_{2|k}$ and $\delta_{4|k}$ are the characteristic functions of the Boolean statements $2|k$ and $4|k$ respectively. That is, for example, $\delta_{2|k} = 1$ if 2 divides k , and $\delta_{2|k} = 0$ otherwise. If $n = 2^k m$ for $k \geq 1$, then $m|\frac{n}{2}$ and

$$y_{a^{n/2}} = \pm \frac{m}{2} \pmod{m}$$

But since $y = -\bar{y}$, the coefficient $y_{a^{n/2}}$ has to be zero. This shows the claim that the form $A(\gamma)$ cannot be even if $n = 2^k m$ for some $k \geq 1$, i.e. if $k_1 > k_2$. \square

CLAIM. The form $A(\gamma)$ is even if $k_1 \leq k_2$.

First we suppose that $k_1 = k_2$, that is $n = m$. Let y be such that $y_{a^k b^l} = k - \frac{n}{2}$ for $k \neq 0$. First note that $y_{a^{-k} b^{-l}} = (n-k) - \frac{n}{2} = -(k - \frac{n}{2}) = -y_{a^k b^l}$. Therefore, $\bar{y} = -y$. Consider the form q given by

$$\begin{pmatrix} 0 & 0 & -(1-b^{-1}) \\ 0 & \frac{m}{2}N_b + y & -aN_b + N_a \\ (1-b) & N_b - N_a & 1-a \end{pmatrix}$$

It follows from $N_a = \overline{N_a}$ and $\bar{y} = -y$ that $q + q^* = A(\gamma)$. It remains to show that q is indeed a form on $H^2(K; \mathbb{Z}\pi)$. For this it has to satisfy the equations:

$$(6.11) \quad \begin{aligned} 0 &= 0 \cdot N_a + 0 \cdot (1-b) \\ 0 &= 0 \cdot N_a + (\frac{m}{2}N_b + y)(1-b) \\ 0 &= (1-b) \cdot N_a + (N_b - N_a)(1-b) \end{aligned}$$

$$(6.12) \quad \begin{aligned} 0 &= 0 \cdot (a-1) - (1-b^{-1})N_b \\ 0 &= (\frac{m}{2}N_b + y)(a-1) - (aN_b + N_a)N_b \\ 0 &= N_b(a-1) + (1-a)N_b \end{aligned}$$

Equation (6.11) is true since y is a multiple of N_b by definition. All but Equation (6.12) are satisfied, using $(1-b)N_b = (1-b^{-1})N_b = 0$. To check (6.12), we have to show that

$$0 = \left(\frac{m}{2}N_b + y\right)(a-1) - (aN_b + N_a)N_b = -\frac{m}{2}(a+1)N_b + (a-1)y - N_aN_b.$$

Since each term is a multiple of N_b , it suffices to show that this holds for the coefficients of a^k , for every $k \geq 0$. For a^0 we have

$$0 = -\frac{m}{2} + y_{a^{-1}} - y_0 + 1 = -\frac{m}{2} + m - 1 - \frac{m}{2} - 0 + 1 = 0.$$

For a^1 we get

$$0 = -\frac{m}{2} + y_0 - y_a + 1 = -\frac{m}{2} + 0 - 1 + \frac{m}{2} + 1 = 0.$$

Finally for a^k with $k > 1$, we have

$$0 = 0 + y_{a^{k-1}} - y_{a^k} + 1 = k - 1 - \frac{m}{2} - k + \frac{m}{2} + 1 = 0.$$

This completes the verification of Equation (6.12). This completes the proof of the claim that $A(\gamma)$ is even when $k_1 = k_2$.

If $k_1 < k_2$, consider the projection $\mathbb{Z}/2^{k_2} \times \mathbb{Z}/2^{k_2} \rightarrow \mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$ applied to the form q , to see that the form $A(\gamma)$ also has to be even in this case. This completes the proof of the claim that $A(\gamma)$ is even when $k_1 \leq k_2$, completing the proof that the groups $\mathbb{Z}/s^{k_1} \times \mathbb{Z}/2^{k_2}$ have the Secondary Property 1.11. \square

The groups $\mathbb{Z} \times \mathbb{Z}/2^k$.

Next, we consider the group $\pi = \mathbb{Z} \times \mathbb{Z}/2^k = \langle a, b \mid [a, b], b^{2^k} \rangle$. Once again we consider the element $\gamma \in H_3(\pi; \mathbb{Z}/2)$ coming from $H_1(\mathbb{Z}; \mathbb{Z}/2) \otimes H_2(\mathbb{Z}/2^k; \mathbb{Z}/2)$.

CLAIM. The form $A(\gamma)$ is not even.

A resolution of \mathbb{Z} by $\mathbb{Z}\pi$ modules can be obtained from the tensor product

$$(\mathbb{Z}\pi \xrightarrow{1-a} \mathbb{Z}\pi) \otimes (\mathbb{Z}\pi \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_*(B\mathbb{Z}/2; \mathbb{Z}[\mathbb{Z}/2])).$$

This is a subcomplex of the resolution depicted above in the case $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$, obtained by deleting everything apart from the two right-most summands of each chain group. We compute that $H^2(K; \mathbb{Z}\pi) \cong (\mathbb{Z}\pi)^2 / \langle (1-b^{-1}, 0), (a^{-1}-1, N_b) \rangle$ and that $A(\gamma)$ is the form

$$\begin{pmatrix} 2^k N_b & (1-a)N_b \\ (1-a^{-1})N_b & 2-a-a^{-1} \end{pmatrix}.$$

Now let $m = 2^k$. A possible q with $A(\gamma) = q + q^*$ has to have the following shape:

$$\begin{pmatrix} \frac{m}{2}N_b + y & -aN_b + x \\ N_b - \bar{x} & 1-a+z \end{pmatrix}$$

with $x, y, z \in \mathbb{Z}\pi$, $y + \bar{y} = z + \bar{z} = 0$.

A verbatim repetition of the argument for $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$ in the case $k_1 > k_2$, starting at ‘‘In order to determine a form on $H^2(K; \mathbb{Z}\pi)$ it has to satisfy,’’ shows that (6.10) also has to hold in the case $\mathbb{Z} \times \mathbb{Z}/m = \mathbb{Z} \times \mathbb{Z}/2^k$. Since in this case a has infinite order, (6.10) implies that infinitely many coefficients of y are non-zero. This is a contradiction to y being an element of the group ring. Hence the form $A(\gamma)$ cannot be even for $\mathbb{Z} \times \mathbb{Z}/2^k$. This completes the proof of the claim that $A(\gamma)$ is not even. \square

CLAIM. The element γ is not in the image of $\text{Sq}_2 \circ \text{red}_2$.

It is straightforward to compute that the map

$$\text{red}_2: H_5(\mathbb{Z} \times \mathbb{Z}/2^k; \mathbb{Z}) \cong \mathbb{Z}/2^k \rightarrow H_5(\mathbb{Z} \times \mathbb{Z}/2^k; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is given by $(0, \text{pr})$, where $\text{pr}: \mathbb{Z}/2^k \rightarrow \mathbb{Z}/2$ is the reduction modulo two. The generator of the image of this map is dual to $\alpha_2\beta_2^2$. For $k > 1$, the cohomology group $H^3(\mathbb{Z}/2^k; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by $\alpha_2\beta_2$ and $\alpha_1\beta_2$; recall that $H^*(\mathbb{Z}/2^k; \mathbb{Z}/2) = (\mathbb{Z}/2)[\alpha_2, \beta_2]/\alpha_2^2$, and note that $H^*(\mathbb{Z}; \mathbb{Z}/2) = \mathbb{Z}[\alpha_1]/\alpha_1^2$, where α_1 has degree 1. But $\text{Sq}^2(\alpha_2\beta_2) = \alpha_2\beta_2^2$ and $\text{Sq}^2(\alpha_1\beta_2) = \alpha_1\beta_2^2$. It follows that

$$\text{Sq}_2((\alpha_2\beta_2^2)^*) = \alpha_2\beta_2 \neq \alpha_1\beta_2 = \gamma.$$

The same argument goes through in the case that $k = 1$ if we replace β_2 by α_2^2 in the above computations. This completes proof of the claim that γ is not in the image of $\text{Sq}_2 \circ \text{red}_2$. \square

We have therefore completed the proof that $\mathbb{Z} \times \mathbb{Z}/2^k$ satisfies Condition 3.20 and therefore has the Secondary Property. Since this is the last case we had to check, this completes the proof that abelian groups with at most two generators have the Secondary Property 1.11. \square

Two generator abelian groups have the tertiary property. Let us now consider the Tertiary Property 1.15. From the discussion of the Tertiary property for cyclic groups (Section 6.2.1), it follows that the ter invariant of an abelian group with two generators either lives in the trivial group or in $\mathbb{Z}/2$. In more detail, consider a group $\pi := C_1 \times C_2$, where C_1 and C_2 are cyclic groups. We have that

$$H_2(\pi; \mathbb{Z}/2) \cong \bigoplus_{i=0}^2 H_i(C_1; \mathbb{Z}/2) \otimes H_{2-i}(C_2; \mathbb{Z}/2).$$

However the parts with $i = 0$ and $i = 2$ lie in the image of $\text{Sq}_2 = d_{4,0}^2: H_4(\pi; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2)$ by the structure of the cohomology ring of cyclic groups (Corollary 6.7). Therefore if the ter invariant is nontrivial, it lives in $H_1(C_1; \mathbb{Z}/2) \otimes H_1(C_2; \mathbb{Z}/2) \cong \mathbb{Z}/2$. In the case that the ter invariant lives in $\mathbb{Z}/2$, the nontrivial element corresponds to the commutator relation (i.e. to the 2-cell in a model for the classifying space inducing that relation).

By [KPR20, Proposition 6.1], $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$ is torsion-free if π is a finite abelian group with two generators. Hence the Tertiary Property 1.15 holds for these groups by Corollary 4.17.

Any infinite abelian group with two generators admits a surjection onto $\mathbb{Z}/8 \times \mathbb{Z}/2$, mapping the commutator relation to the commutator relation. By the discussion at the beginning of this subsection, this surjection induces an injection on the ter invariant if the ter invariant for $\mathbb{Z}/8 \times \mathbb{Z}/2$ is nontrivial. Hence by the Inheritance Theorem 5.7, it suffices to show that the ter invariant for $\mathbb{Z}/8 \times \mathbb{Z}/2$ is nontrivial.

The commutator relation is dual to the product $\alpha_1\alpha_2 \in H^2(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2)$ where α_1 and α_2 are the generators of H^1 of the two cyclic subgroups. Since $\mathbb{Z}/8$ has order higher than two, α_1^2 is trivial. In particular, $\text{Sq}^2(\alpha_1\alpha_2) = \alpha_1^2\alpha_2^2 = 0$, and the commutator relation is not in the image of the dual of Sq^2 .

Now consider the commutative square
(6.13)

$$\begin{array}{ccc} \mathbb{Z}/2 \cong H_2(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2) / \text{im } d_{4,1}^2 & \xrightarrow[p_*]{\cong} & \mathbb{Z}/2 \cong H_2(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}/2) / \text{im } d_{4,1}^2 \\ \uparrow d_{5,0}^3 & & \uparrow d_{5,0}^3 \\ \ker d_{5,0}^2 \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) & \xrightarrow[p_*]{} & \ker d_{5,0}^2 \subseteq H_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}), \end{array}$$

where $p: \mathbb{Z}/8 \times \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \times \mathbb{Z}/2$ is the projection.

CLAIM. The map p_* in the bottom row of the diagram (6.13) is trivial on $\ker d_{5,0}^2 \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})$.

Using the claim, it follows by commutativity that $d_{5,0}^3: \ker d_{5,0}^2 \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \rightarrow \mathbb{Z}/2 \cong H_2(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2) / \text{im } d_{4,1}^2$ is trivial. Thus the **ter** invariant for $\mathbb{Z}/8 \times \mathbb{Z}/2$ is nontrivial. This completes the proof that abelian groups with at most two generators have the Tertiary Property 1.15, modulo the proof of the claim.

Before we prove the claim we give a computation of the maps between group homology that is needed. Consider the projection map $p: \mathbb{Z}/2^n \rightarrow \mathbb{Z}/2^{n-1}$, and view $\mathbb{Z}[\mathbb{Z}/2^{n-1}]$ as a module over $\mathbb{Z}[\mathbb{Z}/2^n]$ via this projection. Write T for the generator of $\mathbb{Z}/2^n$ and U for the generator of $\mathbb{Z}/2^{n-1}$, and note that $p_*(T) = U$. The induced map on resolutions, shown in degrees up to four, is as follows:

$$\begin{array}{ccccccccc} \mathbb{Z}[\mathbb{Z}/2^n] & \xrightarrow{N_{2^n}} & \mathbb{Z}[\mathbb{Z}/2^n] & \xrightarrow{1-T} & \mathbb{Z}[\mathbb{Z}/2^n] & \xrightarrow{N_{2^n}} & \mathbb{Z}[\mathbb{Z}/2^n] & \xrightarrow{1-T} & \mathbb{Z}[\mathbb{Z}/2^n] \\ \downarrow 4 \cdot p_* & & \downarrow 2 \cdot p_* & & \downarrow 2 \cdot p_* & & \downarrow p_* & & \downarrow p_* \\ \mathbb{Z}[\mathbb{Z}/2^{n-1}] & \xrightarrow{N_{2^{n-1}}} & \mathbb{Z}[\mathbb{Z}/2^{n-1}] & \xrightarrow{1-U} & \mathbb{Z}[\mathbb{Z}/2^{n-1}] & \xrightarrow{N_{2^{n-1}}} & \mathbb{Z}[\mathbb{Z}/2^{n-1}] & \xrightarrow{1-U} & \mathbb{Z}[\mathbb{Z}/2^{n-1}], \end{array}$$

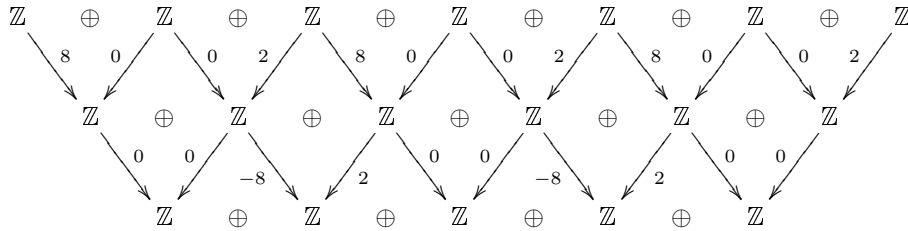
where $N_{2^j} = 1 + S + \dots + S^{2^j - 1}$ is the norm element of $\mathbb{Z}/2^j$ for $(S, j) \in \{(T, n), (U, n-1)\}$. This can be seen by starting with the identity map on C_0 , and then extending in order to make the diagram commute. After tensoring with \mathbb{Z} , the induced vertical maps are unchanged: multiplication by 2^i in degrees $2i$ and $2i + 1$.

Proof of the claim.

Now we embark upon the proof of the claim, which is elementary but giving a detailed computation will require most of the next two pages. We need to show that $p_*: H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \rightarrow H_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z})$ is trivial on $\ker d_{5,0}^2 \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})$. Here we recall the part of the chain complex

$$C_6(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^7 \rightarrow C_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^6 \rightarrow C_4(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^5$$

relevant to computing the fifth homology:



Let e_0, \dots, e_5 be the basis for $C_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})$ generating the summands shown in order. So $e_i \in C_{5-i}(\mathbb{Z}/8; \mathbb{Z}) \otimes C_i(\mathbb{Z}/2; \mathbb{Z})$. We see that

$$\ker(C_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \rightarrow C_4(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})) = \langle e_0, e_1 + 4e_2, e_3 + 4e_4, e_5 \rangle.$$

Meanwhile

$$\text{im}(C_6(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \rightarrow C_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})) = \langle 8e_0, 2e_1 + 8e_2, 2e_3 + 8e_4, 2e_5 \rangle.$$

Therefore

$$H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3 = \langle e_0, e_1 + 4e_2, e_3 + 4e_4, e_5 \rangle.$$

On the other hand part of the chain complex

$$C_6(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^7 \rightarrow C_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^6 \rightarrow C_4(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^5$$

relevant to computing the fifth homology is:

$$\begin{array}{cccccccccccc}
 \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} \\
 \swarrow 4 & \searrow 0 & \swarrow 0 & \searrow 2 & \swarrow 4 & \searrow 0 & \swarrow 0 & \searrow 2 & \swarrow 4 & \searrow 0 & \swarrow 0 & \searrow 2 & \swarrow 0 & \searrow 2 \\
 & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} \\
 & \swarrow 0 & \searrow 0 & \swarrow -4 & \searrow 2 & \swarrow 0 & \searrow 0 & \swarrow -4 & \searrow 2 & \swarrow 0 & \searrow 0 & \swarrow 0 & \searrow 0 & \swarrow 0 & \searrow 0 \\
 & & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z}
 \end{array}$$

Let f_0, \dots, f_5 be the basis for $C_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z})$ generating the summands shown in order. We see that

$$\ker(C_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}) \rightarrow C_4(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z})) = \langle f_0, f_1 + 2f_2, f_3 + 2f_4, f_5 \rangle.$$

We also have

$$\text{im}(C_6(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}) \rightarrow C_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z})) = \langle 4f_0, 2f_1 + 4f_2, 2f_3 + 4f_4, 2f_5 \rangle.$$

Therefore

$$H_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3 = \langle f_0, f_1 + 2f_2, f_3 + 2f_4, f_5 \rangle.$$

The induced map on group homology sends $e_i \mapsto 4f_i$ for $i = 0, 1$, $e_i \mapsto 2f_i$ for $i = 2, 3$, and $e_i \mapsto f_i$ for $i = 4, 5$. We therefore compute that the induced map

$$p_*: H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3 \rightarrow H_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3$$

has kernel $\langle e_0, e_1 + 4e_2, e_3 + 4e_4 \rangle \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$, and is an isomorphism between the last $\mathbb{Z}/2$ summands of each, so $e_5 \mapsto f_5$.

Next we need to compute the kernel of $d_{5,0}^2 = \text{Sq}_2 \circ \text{red}_2$. We start by looking at $\text{red}_2: H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \rightarrow H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2)$. After tensoring with $\mathbb{Z}/2$, all the maps in the chain complex above become zero maps. Therefore $H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^6$. Denote the basis elements by \tilde{e}_i , for $i = 0, \dots, 6$. The map $\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3 \rightarrow (\mathbb{Z}/2)^6$ is given by projection $\mathbb{Z}/8 \rightarrow \mathbb{Z}/2$ on the first summand, and maps the second, third and fourth summands isomorphically to the second, fourth and sixth $\mathbb{Z}/2$ summands of $H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^6$ respectively. The kernel is thus $2e_0$. It remains to compute the dual of the Steedrod square. First we look at the map

$$\text{Sq}^2: H^3(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^4 \rightarrow H^5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2),$$

then we take its dual. Recall that $H^*(\mathbb{Z}/8; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha_1, \beta_1]/(\alpha_1^2)$ and $H^*(\mathbb{Z}/2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha_2]$, where α_1 and α_2 have degree one and β_1 has degree two. We computed earlier that:

$$\begin{aligned} \text{Sq}^2(\alpha_2^3) &= \alpha_2^5 = \tilde{e}_5^* \\ \text{Sq}^2(\alpha_2^2\alpha_1) &= \alpha_2^4\alpha_1 = \tilde{e}_4^* \\ \text{Sq}^2(\alpha_2\beta_1) &= \alpha_2\beta_1^2 = \tilde{e}_1^* \\ \text{Sq}^2(\alpha_1\beta_1) &= \alpha_1\beta_1^2 = \tilde{e}_0^*. \end{aligned}$$

Therefore

$$\ker(\text{Sq}_2: (\mathbb{Z}/2)^6 \rightarrow (\mathbb{Z}/2)^4) = \langle \tilde{e}_2, \tilde{e}_3 \rangle.$$

It follows from combining the maps Sq_2 and red_2 that

$$\ker d_{4,0}^2 = \langle 2e_0, e_3 + 4e_4 \rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \subset \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3,$$

where the $\mathbb{Z}/2$ in the domain includes into the second $\mathbb{Z}/2$ summand in the codomain. This is contained in $\ker p_*$ as claimed. This completes the proof of the claim: the map p_* in the bottom row of diagram (6.13) is trivial on $\ker d_{5,0}^2 \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})$. \square

We have therefore completed the proof that abelian groups with at most two generators have the Tertiary Property 1.15, which completes the proof of Lemma 6.9. \square

6.2.3. Abelian groups with at most 3 generators.

Lemma 6.14. *The Secondary Property 1.11 and the Tertiary Property 1.15 hold for all abelian groups with at most three generators.*

For the Tertiary property, Corollary 5.9 implies that knowing the property for abelian groups with at most two generators is sufficient. Therefore we do not need to consider the Tertiary property in this subsection. The proof for the Secondary property will take about five pages.

Three generator abelian groups have the secondary property. Once again, we can pass to finite odd index subgroups by Theorem 5.11. Thus it suffices to consider groups G of the form \mathbb{Z}^3 , $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2^k$, $\mathbb{Z} \times \mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$ or $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2} \times \mathbb{Z}/2^{k_3}$. We already know the Secondary property for \mathbb{Z}^3 by Theorem 6.1. We will compute that the Secondary Property 1.11 holds for the groups $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$, in a fashion similar to the computation in Section 6.2.2, and then we will deduce that this implies the Secondary property for all the groups G just listed.

The group $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

If $\pi \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$, in order to check Condition 3.20, we only have to consider the element $\gamma \in H_3(\pi; \mathbb{Z}/2)$ coming from the generator of $H_1(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H_1(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H_1(\mathbb{Z}/2; \mathbb{Z}/2)$, since all other elements come from a direct summand with at most 2 generators.

CLAIM. The element γ is in the image of $\text{Sq}_2 \circ \text{red}_2$.

The $\mathbb{Z}/2$ -coefficient cohomology of $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ is

$$\begin{aligned} H^*(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{Z}/2) &\cong H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^*(\mathbb{Z}/2; \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2[\alpha_1] \otimes \mathbb{Z}/2[\alpha_2] \otimes \mathbb{Z}/2[\alpha_3], \end{aligned}$$

where α_i has degree one for $i = 1, 2, 3$. Computing with Steenrod squares, we obtain:

$$\begin{aligned} \text{Sq}^2(\alpha_1\alpha_2\alpha_3) &= \text{Sq}^2(\alpha_1\alpha_2)\alpha_3 + \text{Sq}^1(\alpha_1\alpha_2)\text{Sq}^1(\alpha_3) \\ &= \alpha_1^2\alpha_2^2\alpha_3 + \text{Sq}^1(\alpha_1)\alpha_2\alpha_3^2 + \alpha_1\text{Sq}^1(\alpha_2)\alpha_3^2 \\ &= \alpha_1^2\alpha_2^2\alpha_3 + \alpha_1^2\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2 \in H^5(\pi; \mathbb{Z}/2). \end{aligned}$$

Let $a_i := \alpha_i^* \in H_1(\mathbb{Z}/2; \mathbb{Z}/2)$ be the dual to α_i and let $a_i^2 = (\alpha_i^2)^* \in H_2(\mathbb{Z}/2; \mathbb{Z}/2)$ be the dual to α_i^2 . From the computation of Sq^2 above, we see that the element $\gamma = a_1 \otimes a_2 \otimes a_3$ lies in the image of $\text{Sq}_2: H_5(\pi; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$. We therefore suffice to show that

$$(\alpha_1^2\alpha_2^2\alpha_3 + \alpha_1^2\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2)^* = a_1^2 \otimes a_2^2 \otimes a_3 + a_1^2 \otimes a_2 \otimes a_3^2 + a_1 \otimes a_2^2 \otimes a_3^2$$

lies in the image of $\text{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_5(\pi; \mathbb{Z}/2)$. Recall that a $\mathbb{Z}[\mathbb{Z}/2]$ -module resolution of \mathbb{Z} , tensored down over \mathbb{Z} , is given by

$$\mathbb{Z}(6) \xrightarrow{2} \mathbb{Z}(5) \xrightarrow{0} \mathbb{Z}(4) \xrightarrow{2} \mathbb{Z}(3) \xrightarrow{0} \mathbb{Z}(2) \xrightarrow{2} \mathbb{Z}(1) \xrightarrow{0} \mathbb{Z}(0),$$

where the parenthetical numbers indicate the grading. It is then straightforward to compute that this element is a cycle with integer coefficients. Here we will also use the notation a_i and a_i^2 as above, but now this notation represents the corresponding integral chains in $C_*(\mathbb{Z}/2; \mathbb{Z})$ i.e. generators of $\mathbb{Z}(1)$ and $\mathbb{Z}(2)$ respectively. We have the following computation in $C_*(\pi; \mathbb{Z}) = \bigotimes^3 C_*(\mathbb{Z}/2; \mathbb{Z})$.

$$\begin{aligned} \partial(a_1^2 \otimes a_2^2 \otimes a_3 + a_1^2 \otimes a_2 \otimes a_3^2 + a_1 \otimes a_2^2 \otimes a_3^2) &= 2a_1 \otimes a_2^2 \otimes a_3 + 2a_1^2 \otimes a_2 \otimes a_3 \\ &\quad + 2a_1 \otimes a_2 \otimes a_3^2 - 2a_1^2 \otimes a_2 \otimes a_3 \\ &\quad - 2a_1 \otimes a_2 \otimes a_3^2 - 2a_1 \otimes a_2^2 \otimes a_3 \\ &= 0. \end{aligned}$$

If $a_1^2 \otimes a_2^2 \otimes a_3 + a_1^2 \otimes a_2 \otimes a_3^2 + a_1 \otimes a_2^2 \otimes a_3^2$ were a boundary over \mathbb{Z} , then it would be a boundary over $\mathbb{Z}/2$ as well. Therefore it represents an element of $H_5(\pi; \mathbb{Z})$, that maps to $(\alpha_1^2\alpha_2^2\alpha_3 + \alpha_1^2\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2)^*$ under the reduction modulo two. As we have shown above that this element maps to γ under Sq_2 , this completes the proof of the claim that γ is in the image of $\text{Sq}_2 \circ \text{red}_2$. \square

CLAIM. The form $A(\gamma)$ is even.

We have

$$\begin{aligned} H^2(K; \mathbb{Z}\pi) \cong (\mathbb{Z}\pi)^6 / \langle (1-b, c-1, 0, 1+a, 0, 0), (a-1, 0, 1-c, 0, 1+b, 0), \\ (0, 1-b, a-1, 0, 0, 1+c) \rangle \end{aligned}$$

and the form $A(\gamma)$ is given by

$$\begin{pmatrix} 2(1-c) & (1-c)(1-b) & (1-c)(1-a) & 0 & 0 & 0 \\ (1-b)(1-c) & 2(1-b) & (1-a)(1-c) & 0 & 0 & 0 \\ (1-a)(1-c) & (1-a)(1-b) & 2(1-a) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For q given by

$$\begin{pmatrix} 1-c & -(c+b) & -(c+a) & 0 & 0 & 0 \\ 1+cb & 1-b & -(b+a) & 0 & 0 & 0 \\ 1+ca & 1+ab & 1-a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have $\lambda = q + q^*$. This completes the proof of the claim that $A(\gamma)$ is even. \square

This therefore completes the proof of the Secondary Property 1.11 for $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

The group $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$.

Now we consider $\pi := \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$. Verifying Condition 3.20 for this group will occupy the next two and a half pages.

CLAIM. The element $\gamma \in H_3(\pi; \mathbb{Z}/2)$ coming from the generator of

$$H_1(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H_1(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H_1(\mathbb{Z}/4; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is not in the image of $\text{Sq}_2 \circ \text{red}_2$.

The $\mathbb{Z}/2$ -coefficient cohomology of $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$ is

$$\begin{aligned} H^*(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4; \mathbb{Z}/2) &\cong H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^*(\mathbb{Z}/4; \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2[\alpha_1] \otimes \mathbb{Z}/2[\alpha_2] \otimes \mathbb{Z}/2[\alpha_3, \beta_3]/\alpha_3^2, \end{aligned}$$

where α_i has degree one for $i = 1, 2, 3$ and β_3 has degree two. Note that $\gamma = (\alpha_1\alpha_2\alpha_3)^*$. We first compute the image of the Steenrod square map $\text{Sq}^2: H^3(\pi; \mathbb{Z}/2) \rightarrow H^5(\pi; \mathbb{Z}/2)$. First we have

$$\begin{aligned} \text{Sq}^2(\alpha_1\alpha_2\alpha_3) &= \text{Sq}^2(\alpha_1)\alpha_2\alpha_3 + \alpha_1^2\text{Sq}^1(\alpha_2\alpha_3) + \alpha_1\text{Sq}^2(\alpha_2\alpha_3) \\ &= \alpha_1^2\alpha_2^2\alpha_3 + \alpha_1\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2 \\ &= \alpha_1^2\alpha_2^2\alpha_3 \end{aligned}$$

since $\alpha_3^2 = 0$. We also compute:

$$\text{Sq}^2(\alpha_1^2\alpha_2) = \alpha_1^4\alpha_2; \quad \text{Sq}^2(\alpha_1\alpha_2^2) = \alpha_1\alpha_2^4; \quad \text{Sq}^2(\alpha_i^2\alpha_3) = \alpha_i^4\alpha_3$$

and

$$\text{Sq}^2(\alpha_i\beta_3) = \alpha_i\beta_3^2; \quad \text{Sq}^2(\alpha_i^3) = \alpha_i^5.$$

We can thus compute the dual map $\text{Sq}_2: H_5(\pi; \mathbb{Z}/2) \rightarrow H_3(\pi; \mathbb{Z}/2)$. We need to show that the element $(\alpha_1^2\alpha_2^2\alpha_3)^* \in H_5(\pi; \mathbb{Z}/2)$ does not lie in the image of the reduction modulo two.

For the chain complex $C_*(\mathbb{Z}/2; \mathbb{Z})$ we will again use a_i^j to denote a generator of $C_j(\mathbb{Z}/2; \mathbb{Z})$ in the i th copy, for $i = 1, 2$. Recall that a $\mathbb{Z}[\mathbb{Z}/4]$ -module resolution of \mathbb{Z} , tensored down over \mathbb{Z} , is given by

$$\mathbb{Z}(6) \xrightarrow{4} \mathbb{Z}(5) \xrightarrow{0} \mathbb{Z}(4) \xrightarrow{4} \mathbb{Z}(3) \xrightarrow{0} \mathbb{Z}(2) \xrightarrow{4} \mathbb{Z}(1) \xrightarrow{0} \mathbb{Z}(0),$$

where the parenthetical numbers indicate the grading. We will use the notation $a_3^j b_3^k$ for the generator of $\mathbb{Z}(j+2k)$, an integral lift of the dual element to $\alpha_3^j \beta_3^k$. In $C_*(\pi; \mathbb{Z}) = C_*(\mathbb{Z}/2; \mathbb{Z}) \otimes C_*(\mathbb{Z}/2; \mathbb{Z}) \otimes C_*(\mathbb{Z}/4; \mathbb{Z})$, we compute:

$$\partial(a_1^2 \otimes a_2^2 \otimes a_3) = 2a_1 \otimes a_2^2 \otimes a_3 + 2a_1^2 \otimes a_2 \otimes a_3.$$

So the obvious dual element to $\alpha_1^2 \alpha_2^2 \alpha_3$ is not a cycle with \mathbb{Z} coefficients. To show that γ does not lie in the image of $\text{Sq}_2 \circ \text{red}_2$ however, we need to argue that there is no way to add other elements of $C_5(\pi; \mathbb{Z})$ to make $a_1^2 \otimes a_2^2 \otimes a_3$ into a cycle, in such a way that we preserve the correct image γ of the reduction modulo two in $H_3(\pi; \mathbb{Z}/2)$. We can try adding linear combinations of the chains

$$a_1^i \otimes a_2^j \otimes a_3; a_k \otimes b_3^2; a_s^3 \otimes b_3; a_t^5 \text{ or } a_\ell \otimes a_m^2 \otimes b_3,$$

for some i, j with $i + j = 4$, for some $k, s, t = 1, 2$, and for some nonempty $\{\ell, m\} \subseteq \{1, 2\}$. Then compute

$$\partial(a_1^3 \otimes a_2 \otimes a_3) = \partial(a_1 \otimes a_2^3 \otimes a_3) = \partial(a_1^5) = \partial(a_2^5) = 0;$$

$$\partial(a_i^4 \otimes a_3) = 2a_i^3 \otimes a_3; \partial(a_j^3 \otimes b_3) = -4a_j^3 \otimes a_3$$

for $i, j = 1, 2$;

$$\partial(a_1 \otimes a_2^2 \otimes b_3) = -2a_1 \otimes a_2 \otimes b_3 - 4a_1 \otimes a_2^2 \otimes a_3;$$

$$\partial(a_1^2 \otimes a_2 \otimes b_3) = 2a_1 \otimes a_2 \otimes b_3 - 4a_1^2 \otimes a_2 \otimes a_3;$$

and

$$\partial(a_k \otimes b_3^2) = -4a_k \otimes a_3 b_3.$$

Since $\alpha_i \beta_3^2$ and $\alpha_j^4 \alpha_3$ are in the image of Sq^2 , for any $i, j = 1, 2$, any occurrence of their dual terms $a_i \otimes b_3^2$ or $a_j^4 \otimes a_3$ in a putative linear combination must have even coefficient, or this occurrence would alter the image under Sq_2 , causing it to deviate from being $\gamma = (\alpha_1 \alpha_2 \alpha_3)^* = a_1 \otimes a_2 \otimes a_3$. Therefore the boundary of every term in our linear combination must have a term with coefficient divisible by 4. On the other hand the boundary we are trying to cancel is

$$\partial(a_1^2 \otimes a_2^2 \otimes a_3) = 2a_1 \otimes a_2^2 \otimes a_3 + 2a_1^2 \otimes a_2 \otimes a_3,$$

in which both terms are only divisible by 2. We see that there is no way to cancel the terms $-4a_1 \otimes a_2^2 \otimes b_3$ and $-4a_1^2 \otimes a_2 \otimes b_3$ while still having the necessary odd coefficient of $a_1^2 \otimes a_2^2 \otimes a_3$. It follows that $\gamma = a_1 \otimes a_2 \otimes a_3$ does not lie in the image of $\text{Sq}_2 \circ \text{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_3(\pi; \mathbb{Z}/2)$ as claimed. For interest, we remark that $2a_1^2 \otimes a_2^2 \otimes a_3 + a_1 \otimes a_2^2 \otimes b_3 + a_1^2 \otimes a_2 \otimes b_3$ represents a homology class in $H_5(\pi; \mathbb{Z})$, but of course this does not map to γ . This completes the proof of the claim. \square

We write $\pi := \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 = \langle a, b, c \mid [a, b], [b, c], [c, a], a^2, b^2, c^4 \rangle$.

CLAIM. The form $A(\gamma)$ is not even.

$$H^2(K; \mathbb{Z}\pi) \cong (\mathbb{Z}\pi)^6 / \langle (1-b, c^{-1}-1, 0, 1+a, 0, 0), (a-1, 0, 1-c^{-1}, 0, 1+b, 0), (0, 1-b, a-1, 0, 0, 1+c+c^2+c^3) \rangle$$

and the form $A(\gamma)$ is given by

$$\begin{pmatrix} (1-c)(1-c^{-1}) & (1-c^{-1})(1-b) & (1-c^{-1})(1-a) & 0 & 0 & 0 \\ (1-b)(1-c) & 2(1-b) & (1-a)(1-b) & 0 & 0 & 0 \\ (1-a)(1-c) & (1-a)(1-b) & 2(1-a) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Assume there exists q with $q + q^* = A(\gamma)$. Then q has to be of the form

$$\begin{pmatrix} (1-c) + z & (1-b) + x & * \\ -c(1-b) - \bar{x} & (1-b) + y & * \\ * & * & * \end{pmatrix}$$

with $z + \bar{z} = y + \bar{y} = 0$. The stars mean that we only specify the four entries in the upper left. The proof will show that this matrix cannot represent any form on $H^2(K; \mathbb{Z}\pi)$, no matter how we fill in the rest of the matrix.

Now work modulo 2 and under the projection $a = 1$. To represent a form on $H^2(K; \mathbb{Z}\pi)$, the following equations have to be satisfied:

$$0 = (1-c+z)(1-b) + (1-b+x)(c-1) = (1-b)z + (c-1)x$$

$$0 = (-c(1-b) - \bar{x})(1-b) + (1-b+y)(c-1).$$

Since $z + \bar{z} = 0$, there are $z_1, z_2 \in \mathbb{Z}$ with $z = (z_1 + z_2b)(c - c^3)$. Thus, for $k := z_1 - z_2$, we have $(1-b)z = (k - kb)(c - c^3)$. It follows from the first equation that $x = (k - kb)(c + c^2) + x'N_c$ for some element $x' \in \mathbb{Z}\pi$. Since $(1-b)(1-b) = 2(1-b)$ and we are working modulo 2, if we replace x in the second equation above by $(k - kb)(c + c^2) + x'N_c$, we see that

$$0 = -\bar{x}'N_c(1-b) + (1-b+y)(c-1).$$

Multiply by $(1+c)$, to obtain

$$0 = (1-b+y)(1+c^2).$$

Evaluate at the neutral group element to yield

$$0 = 1 + y_0 + y_{c^2} = 1,$$

since $y_0 = y_{c^2} = 0$. This is a contradiction, and it follows that $A(\gamma)$ cannot be even as claimed. \square

Therefore, the Secondary Property 1.11 holds for $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$.

Any 3-generator abelian group.

Now we deduce the Secondary Property for G one of the groups on the list from the start of this subsection. For $G \cong G_1 \times G_2 \times G_3$, with G_i cyclic and $|G_3| \geq 4$, the element $\gamma' \in H_3(G; \mathbb{Z}/2)$ coming from the generator of

$$H_1(G_1; \mathbb{Z}/2) \otimes H_1(G_2; \mathbb{Z}/2) \otimes H_1(G_3; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is also not in the image of $\text{Sq}_2 \circ \text{red}_2$. This can be seen by considering the projection $\varphi: G \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 = \pi$, under which γ' is mapped to γ . By Lemma 5.2, we have a commutative diagram

$$\begin{array}{ccc} H_3(G; \mathbb{Z}/2) & \xrightarrow{\varphi_*} & H_3(\pi; \mathbb{Z}/2) \\ \downarrow A & & \downarrow A \\ \widehat{H}^0(\text{Sesq}(H^2(K_G; \mathbb{Z}G))) & \xrightarrow{\varphi_*} & \widehat{H}^0(\text{Sesq}(H^2(K_\pi; \mathbb{Z}\pi))) \end{array}$$

Recall that $\gamma' \in H_3(G; \mathbb{Z}/2)$ is such that $\varphi_*(\gamma') = \gamma$, and we just showed that $A(\gamma) \neq 0$. It follows that $A(\gamma') \neq 0$. Therefore the Secondary Property 1.11 also holds for G . This completes the proof of Lemma 6.14. \square

6.2.4. Abelian groups with any finite number of generators. Theorem 6.5 now follows by combining the work we have done above.

PROOF OF THEOREM 6.5. The Secondary Property 1.11 holds for all finitely generated abelian groups, since by the previous subsections it holds for all abelian groups with at most three generators, and Corollary 5.4 tells us that this suffices to prove the conjecture for all finitely generated abelian groups.

The Tertiary Property 1.15 holds for all finitely generated abelian groups, since by the previous subsections it holds for all abelian groups with at most two generators, and Corollary 5.9 applies to show that the conjecture holds for all abelian groups. \square

We also have the following result on the Tertiary Property.

Corollary 6.15. *If a finitely presented group G has the Tertiary Property 1.15, then so does $G \times \mathbb{Z}$.*

PROOF. Let G^{ab} be the abelianisation of G and let $G^{ab} \cong \bigoplus_{i=1}^m C_i$ be a decomposition into cyclic groups C_i . Consider the projection $p_0: G \times \mathbb{Z} \rightarrow G =: G_0$ and the surjections

$$p_i: G \times \mathbb{Z} \rightarrow G^{ab} \times \mathbb{Z} \rightarrow C_i \times \mathbb{Z} =: G_i,$$

for $i = 1, \dots, m$. To apply Theorem 5.7, we need to see that the induced map

$$\prod_{i=0}^m (p_i)_*: H_2(G \times \mathbb{Z}; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3) \rightarrow \prod_{i=0}^m H_2(G_i; \mathbb{Z}/2) / \text{im}(d_{4,1}^2, d_{5,0}^3)$$

is an injection. This is obvious for elements of $H_2(G \times \mathbb{Z}; \mathbb{Z}/2)$ coming from $H_2(G; \mathbb{Z}/2)$, by considering the image under $(p_0)_*$. For elements of $H_2(G \times \mathbb{Z}; \mathbb{Z}/2)$ coming from $H_1(G; \mathbb{Z}/2) \otimes_{\mathbb{Z}} H_1(\mathbb{Z}; \mathbb{Z}/2)$ in the Künneth theorem, we consider the appropriate map $(p_i)_*$, $i = 1, \dots, m$, and use that for every cyclic group C (of even or infinite order, so that $H_1(C; \mathbb{Z}/2)$ is nontrivial), the nontrivial element in $H_2(C \times \mathbb{Z}; \mathbb{Z}/2)$ coming from $H_1(C; \mathbb{Z}/2) \otimes_{\mathbb{Z}} H_1(\mathbb{Z}; \mathbb{Z}/2)$ does not lie in the image of the boundary maps $d_{4,1}^2, d_{5,0}^3$, as we showed in Section 6.2.2. Thus the corollary indeed follows from Theorem 5.7. \square

Corollary 6.16. *Let π be a finite group whose 2-Sylow subgroup is abelian or has periodic cohomology. Then the Secondary Property 1.11 and the Tertiary Property 1.15 hold for π . In particular, both properties hold for every finite group with periodic cohomology.*

PROOF. First, we can reduce the verification of the properties to 2-Sylow subgroups, by Theorems 5.11 and 5.12. By [Bro82, Theorem VI 9.3], every finite 2-group with periodic cohomology is either abelian or generalised quaternion. The corollary therefore follows from the statement for abelian groups, Theorem 6.5, together with Theorem 6.4 on the generalised quaternion groups.

By [Bro82, Theorem VI 9.5], for every finite group with periodic cohomology its 2-Sylow subgroup also has periodic cohomology. This implies the last statement. \square

The stable classification for fundamental group $\mathbb{Z} \times \mathbb{Z}/2$

We apply our results to the case of smooth, closed, connected, spin 4-manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}/2$, and prove Theorem 1.2. We will recall the statement later, after some preliminary lemmas.

For the following, let $\pi := \mathbb{Z} \times \mathbb{Z}/2 = \langle T, t \mid [T, t], T^2 \rangle$. Let $I := I\pi$ denote the augmentation ideal of $\mathbb{Z}\pi$, generated by $1 - t$ and $1 - T$. We start by collecting some basic results about group homology and cohomology that we shall need.

Lemma 7.1. *Let $\pi := \mathbb{Z} \times \mathbb{Z}/2 = \langle t, T \mid [t, T], T^2 \rangle$. We have*

$$H_i(\pi; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & i = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i \geq 1 \end{cases}$$

$$H_i(\pi; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & i = 1 \\ \mathbb{Z}/2 & i \geq 2. \end{cases}$$

The cohomology ring with $\mathbb{Z}/2$ coefficients is

$$H^*(\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2[t, T]/(t^2),$$

where we abuse notation and use t, T for elements of $H^1(\pi; \mathbb{Z}/2)$ that are dual to the corresponding generators of π . Moreover, we have the Steenrod operations:

$$\begin{array}{ccc} H^3(\pi; \mathbb{Z}/2) & \xrightarrow{\text{Sq}^2} & H^5(\pi; \mathbb{Z}/2) \\ \cong \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \cong \downarrow \\ \mathbb{Z}/2\langle T^3 \rangle \oplus \mathbb{Z}/2\langle T^2t \rangle & \xrightarrow{\quad} & \mathbb{Z}/2\langle T^5 \rangle \oplus \mathbb{Z}/2\langle T^4t \rangle \end{array}$$

$$\begin{array}{ccc} H^2(\pi; \mathbb{Z}/2) & \xrightarrow{\text{Sq}^2} & H^4(\pi; \mathbb{Z}/2) \\ \cong \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \cong \downarrow \\ \mathbb{Z}/2\langle T^2 \rangle \oplus \mathbb{Z}/2\langle Tt \rangle & \xrightarrow{\quad} & \mathbb{Z}/2\langle T^4 \rangle \oplus \mathbb{Z}/2\langle T^3t \rangle. \end{array}$$

The reduction modulo 2 maps are given by:

$$\text{red}_2: H_5(\pi; \mathbb{Z}) \cong \mathbb{Z}/2 \xrightarrow{(1,0)} H_5(\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2\langle (T^5)^* \rangle \oplus \mathbb{Z}/2\langle (T^4t)^* \rangle,$$

$$\text{red}_2: H_4(\pi; \mathbb{Z}) \cong \mathbb{Z}/2 \xrightarrow{(0,1)} H_4(\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2\langle (T^4)^* \rangle \oplus \mathbb{Z}/2\langle (T^3t)^* \rangle.$$

It follows that

$$\text{Sq}_2 \circ \text{red}_2: H_5(\pi; \mathbb{Z}) \cong \mathbb{Z}/2 \rightarrow H_3(\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is injective, while

$$\mathrm{Sq}_2 \circ \mathrm{red}_2: H_4(\pi; \mathbb{Z}) \cong \mathbb{Z}/2 \rightarrow H_2(\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is the zero map.

PROOF. The computations of the homology and the cohomology ring with $\mathbb{Z}/2$ -coefficients follow from the Künneth theorem. The Steenrod square computations then follow from the axioms, as in Section 6.2. \square

We can now calculate the bordism group $\Omega_4^{\mathrm{Spin}}(B\pi)$.

Lemma 7.2. *We have an isomorphism $\Omega_4^{\mathrm{Spin}}(B\pi) \cong 16\mathbb{Z} \oplus \mathbb{Z}/8$ given by the signature and the Brown invariant sending $f: M \rightarrow S^1 \times \mathbb{RP}^3 \subseteq B\pi$ to $f^{-1}(\{\mathrm{pt}\} \times \mathbb{RP}^2) \in \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$.*

PROOF. Recall that $\Omega_4^{\mathrm{Spin}} \cong 16\mathbb{Z}$, where the isomorphism is given by the signature. As usual, there is a splitting $\Omega_4^{\mathrm{Spin}}(X) \cong \tilde{\Omega}_4^{\mathrm{Spin}}(X) \oplus 16\mathbb{Z}$, as explained in Proposition 2.7. So it suffices to show that $\tilde{\Omega}_4^{\mathrm{Spin}}(B(\mathbb{Z} \times \mathbb{Z}/2)) \cong \mathbb{Z}/8$.

We have the following chain of isomorphisms

$$\tilde{\Omega}_4^{\mathrm{Spin}}(B\pi) \cong \tilde{\Omega}_4^{\mathrm{Spin}}(B\mathbb{Z}/2) \oplus \Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong \Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8.$$

The second isomorphism uses that $\tilde{\Omega}_4^{\mathrm{Spin}}(B\mathbb{Z}/2) = 0$, as the calculations in Section 6.2.1 showed. The fourth isomorphism $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ was computed by Kirby and Taylor [KT90]. Recall that $B\pi \simeq S^1 \times \mathbb{RP}^\infty$. The composite of the first two isomorphisms is given on $[M \xrightarrow{f} S^1 \times \mathbb{RP}^\infty]$ by making f transverse to $\{s_0\} \times \mathbb{RP}^\infty$ and then sending $[M, f]$ to $[f^{-1}(\{s_0\} \times \mathbb{RP}^\infty) \rightarrow \mathbb{RP}^\infty]$. The third isomorphism, the Smith map, is given by homotoping the map to \mathbb{RP}^∞ to the 3-skeleton, making f transverse to \mathbb{RP}^2 , and then sending $[M \xrightarrow{f} \mathbb{RP}^3]$ to $[f^{-1}(\mathbb{RP}^2)]$. The spin structure on M induces a Pin^- structure on $f^{-1}(\mathbb{RP}^2)$. Generators of the bordism groups are represented by $S^1 \times \mathbb{RP}^3 \in \tilde{\Omega}_4^{\mathrm{Spin}}(B(\mathbb{Z} \times \mathbb{Z}/2))$, $\mathbb{RP}^3 \in \Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2)$ and $\mathbb{RP}^2 \in \Omega_2^{\mathrm{Pin}^-}$.

We argue that the third map $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong \Omega_2^{\mathrm{Pin}^-}$ is indeed an isomorphism, as we were unable to find an explicit reference. As described above [KT90, pp. 212–3], the map arises by representing a class $[M, c] \in \Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2)$ by a map $c: M \rightarrow \mathbb{RP}^3$, taking the inverse image $c^{-1}(\mathbb{RP}^2)$. A standard transversality argument shows that this is well-defined, and a homomorphism. The map is surjective since \mathbb{RP}^2 with one of its Pin^- structures represents a generator of $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ [KT90, Section 3], and the map applied to $\mathrm{Id}: \mathbb{RP}^3 \rightarrow \mathbb{RP}^3$ yields \mathbb{RP}^2 . Therefore $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2)$ surjects to $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$. We will now show that $\Omega_3^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong \tilde{\Omega}_4^{\mathrm{Spin}}(B(\mathbb{Z} \times \mathbb{Z}/2))$ has at most eight elements and thus this surjection is an isomorphism. Write $\pi \cong \mathbb{Z} \times \mathbb{Z}/2$. By Proposition 2.7 there is a filtration $F_2 \subseteq F_1 \subseteq \tilde{\Omega}_4^{\mathrm{Spin}}(B\pi)$, arising from the Atiyah-Hirzebruch spectral sequence, with quotients:

- (1) $\tilde{\Omega}_4^{\mathrm{Spin}}(B\pi)/F_1$, a subgroup of $H_4(\pi; \mathbb{Z}) \cong \mathbb{Z}/2$;
- (2) F_1/F_2 , isomorphic to the cokernel of $\mathrm{Sq}_2 \circ \mathrm{red}_2: H_5(\pi; \mathbb{Z}) \rightarrow H_3(\pi; \mathbb{Z}/2)$, which is $\mathbb{Z}/2$ using the computation from Lemma 7.1;
- (3) F_2 a quotient of the cokernel of $\mathrm{Sq}_2: H_4(\pi; \mathbb{Z}/2) \rightarrow H_2(\pi; \mathbb{Z}/2)$. The latter is isomorphic to $\mathbb{Z}/2$ using the computation from Lemma 7.1.

So $\widetilde{\Omega}_4^{\text{Spin}}(B\pi)$ has at most eight elements as claimed. It follows that in fact these three copies of $\mathbb{Z}/2$ survive to the E^∞ page and assemble to give $\widetilde{\Omega}_4^{\text{Spin}}(B(\mathbb{Z} \times \mathbb{Z}/2)) \cong \mathbb{Z}/8$, i.e. the filtration is $0 \subseteq \mathbb{Z}/2 \subseteq \mathbb{Z}/4 \subseteq \mathbb{Z}/8$. \square

We remark that $(n, m) \in \mathbb{Z} \oplus \mathbb{Z}/8 \cong \Omega_4^{\text{Spin}}(B(\mathbb{Z} \times \mathbb{Z}/2))$ is realised by the 4-manifold $\#^n K3 \# ((\#^m \mathbb{R}P^3) \times S^1)$. We also make the following observations for later use:

- (i) $\text{pri}(M) = 0$ if and only if $\beta(M)$ is even;
- (ii) if $\text{pri}(M) = 0$, then $\text{sec}(M) = 0$ if and only if $\beta(M) \in \{0, 4\}$.

Lemma 7.3. *Changing the spin structure by $T \in H^1(\pi; \mathbb{Z}/2)$ acts on $\widetilde{\Omega}_4^{\text{Spin}}(B\pi) \cong \mathbb{Z}/8$ by multiplication with -1 .*

PROOF. The isomorphism $\widetilde{\Omega}_4^{\text{Spin}}(B\pi) \cong \Omega_3^{\text{Spin}}(B\mathbb{Z}/2)$ is equivariant with respect to changing the spin structure by T . Pick a fixed orientation and spin structure for $\mathbb{R}P^3$. Since $\mathbb{R}P^3$ generates $\Omega_3^{\text{Spin}}(B\mathbb{Z}/2)$, it suffices to see that changing the spin structure of $\mathbb{R}P^3$ yields $-\llbracket \mathbb{R}P^3 \rrbracket \in \Omega_3^{\text{Spin}}(B\mathbb{Z}/2)$. Use that $\mathbb{R}P^3$ admits an orientation reversing diffeomorphism to see that $-\llbracket \mathbb{R}P^3 \rrbracket$ equals either $\llbracket \mathbb{R}P^3 \rrbracket$ or $\llbracket T^*\mathbb{R}P^3 \rrbracket$, where $T^*\mathbb{R}P^3$ denotes $\mathbb{R}P^3$ with the other spin structure. But $\llbracket \mathbb{R}P^3 \rrbracket$ is a generator of $\Omega_3^{\text{Spin}}(B\mathbb{Z}/2) \cong \mathbb{Z}/8$, so $\llbracket \mathbb{R}P^3 \rrbracket \neq -\llbracket \mathbb{R}P^3 \rrbracket$. It follows that $-\llbracket \mathbb{R}P^3 \rrbracket = \llbracket T^*\mathbb{R}P^3 \rrbracket$ as claimed. \square

Using Lemmas 7.2 and 7.3, Kreck's modified surgery implies that smooth, closed, connected, spin 4-manifolds with fundamental group $\pi = \mathbb{Z} \times \mathbb{Z}/2$ are classified by their signature and the Brown invariant in $(\mathbb{Z}/8)/\pm 1$. As promised in the introduction, we now apply our classification results to obtain an intrinsic description of the Brown invariant in this case. For convenience we repeat the statement of Theorem 1.2.

Theorem 7.4. *For a smooth, closed, connected, spin 4-manifold with fundamental group π , the Brown invariant $\beta(M) \in \mathbb{Z}/8$ is detected as follows:*

- (1) $\beta(M) = \pm 1$ if and only if $\pi_2(M)$ is stably free and λ_M is stably isometric to a form induced up to $\mathbb{Z}\pi$ from a form over \mathbb{Z} ;
- (2) $\beta(M) = \pm 3$ if and only if $\pi_2(M)$ is stably free but λ_M is not stably induced from a form over \mathbb{Z} ;
- (3) $\beta(M) = \pm 2$ if and only if $\pi_2(M)$ is not stably free and λ_M is odd;
- (4) $\beta(M) \in \{0, 4\}$ if and only if $\pi_2(M)$ is not stably free and λ_M is even.

In the last case, $\pi_2(M)$ is stably isomorphic to $I \oplus I^*$, where $I := I\pi$ is the augmentation ideal, and there exists such a stable isomorphism for which the restriction of λ_M to I^* is trivial. Let $x \in \pi_2(M)$ be the image of $\varphi \in I^*$ under such an isomorphism, where $\varphi \in I^*$ is determined by $\varphi(1-t) = 1+T$ and $\varphi(1-T) = 0$. Then:

- (5) $\beta(M) = 0$ if and only if $\tau_M(x) = 0$, where τ_M is the Kervaire–Milnor invariant.

The rest of this section consists of the proof of this theorem. We will assume that all 4-manifolds are smooth, closed, connected, and spin without further comment. First we show that the stable isomorphism class of $\pi_2(M)$ determines the parity of $\beta(M)$. Since the map from $\Omega_4^{\text{Spin}}(B\pi) \rightarrow H_4(\pi; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is given by reducing the Brown invariant modulo two, this is implied by the following lemma.

Lemma 7.5. *Let M be a spin 4-manifold with $\pi_1(M) = \mathbb{Z} \times \mathbb{Z}/2$. The image $c_*[M] \in H_4(\pi; \mathbb{Z}) \cong \mathbb{Z}/2$ of the fundamental class is trivial if and only if $\pi_2(M)$ is stably free as a $\mathbb{Z}\pi$ -module.*

PROOF. By [KPT18], the image $c_*[M] \in H_4(B(\mathbb{Z} \times \mathbb{Z}/2); \mathbb{Z}) \cong \mathbb{Z}/2$ of the fundamental class of M , stably determines $\pi_2(M)$ as a $\mathbb{Z}\pi$ -module. Since $\pi_2(S^1 \times \mathbb{R}\mathbb{P}^3) = 0$, this implies that $\pi_2(M)$ is stably free if $c_*[M]$ is nontrivial. Let (C_*, d_*) be a resolution of \mathbb{Z} as a $\mathbb{Z}\pi$ -module by finitely generated free modules. If $c_*[M]$ is trivial, then $\pi_2(M)$ is stably isomorphic to $\ker d_2 \oplus \operatorname{coker} d_2$, which is not stably free. To see this, note that if $\ker d_2$ were projective, then there would be a finite length projective resolution of \mathbb{Z} as a $\mathbb{Z} \times \mathbb{Z}/2$ -module, which cannot happen since $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2]$ has homology in all nonnegative degrees. \square

We have shown that $\operatorname{pri}(M)$, i.e. whether $\beta(M) \in \{\pm 1, \pm 3\}$ or $\beta(M) \in \{0, \pm 2, 4\}$, is detected by whether or not $\pi_2(M)$ is stably free.

7.1. The case $\beta(M) \in \{\pm 1, \pm 3\}$

This section distinguishes between the cases (1) and (2) of Theorem 7.4. A generator of $\widetilde{\Omega}_4^{\operatorname{Spin}}(B\pi) \cong \mathbb{Z}/8$ is given by $S^1 \times \mathbb{R}\mathbb{P}^3$, which has trivial $\pi_2(M)$. It follows that $\beta(M) = \pm 1$ implies that λ_M is stably isometric to a form over $\mathbb{Z}\pi$ on a f.g. free module induced from a form over \mathbb{Z} .

It therefore remains to show that, for a fixed signature, the stable isometry class of the equivariant intersection form distinguishes between the two stable diffeomorphism classes with nontrivial image of the fundamental class. Since the signature determines the stable isometry class of forms over \mathbb{Z} , we deduce that if λ_M is stably isometric to a form induced from \mathbb{Z} , then $\beta(M) = \pm 1$.

Let M_1 and M_2 be two spin 4-manifolds with nontrivial $c_*[M_1] = c_*[M_2] \in H_4(\pi; \mathbb{Z}) \cong \mathbb{Z}/2$. Fix a spin structure on each of M_1 and M_2 .

Lemma 7.6. *After changing the spin structure on M_1 if necessary, we can assume that the secondary invariant of the 1-skeleton sum $M_1 \#_1 M_2$ in $H_3(\pi; \mathbb{Z}/2)/d_2$ is trivial.*

PROOF. The signature is unaffected by the change in spin structure and changing the signature by adding a simply-connected manifold does not change the secondary invariant. Hence we can consider the image of $M_1 \#_1 M_2$ in reduced bordism $\widetilde{\Omega}_4^{\operatorname{Spin}}(B\pi)$.

By Lemma 7.3, we can change the Brown invariant of M_1 by a sign by changing the spin structure on M_1 . Since $\operatorname{pri}(M_i) \neq 0$, we know that $\beta(M_i) \in \{\pm 1, \pm 3\}$ for $i = 1, 2$. It follows that $\beta(M_1 \#_1 M_2) = \beta(M_1) + \beta(M_2)$ is even, so $\operatorname{pri}(M_1 \#_1 M_2) = 0$. But in addition, for each $x \in \{1, 3\}$, and for every possible $\beta(M_2) \in \{\pm 1, \pm 3\}$, there is a choice of sign \pm such that $\pm x + \beta(M_2) \in \{0, 4\}$. The lemma follows. \square

Next we will show that the tertiary invariant of $M_1 \#_1 M_2$ in $H_2(\pi; \mathbb{Z}/2)/d_2 \cong \mathbb{Z}/2$ is determined by $[\lambda_{M_1}], [\lambda_{M_2}] \in L_4(\mathbb{Z}\pi)$. Let $\mathbb{L}\langle 1 \rangle$ denote the 1-connective cover of the quadratic L -theory spectrum \mathbb{L} . Recall that

$$\mathbb{L}\langle 1 \rangle_4(X) \cong H_4(X; \mathbb{L}\langle 1 \rangle_\bullet) \cong \mathcal{N}_{\operatorname{TOP}}(X)$$

for a 4-dimensional Poincaré complex X .

Lemma 7.7. *The assembly map $\alpha: \mathbb{L}\langle 1 \rangle_4(S^1 \times \mathbb{R}\mathbb{P}^3) \rightarrow L_4(\mathbb{Z}\pi)$ has domain isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, codomain isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$, and the torsion element $(0, 0, 1)$ in the codomain lies in the image of α .*

PROOF. From a spectral sequence computation using the Atiyah-Hirzebruch sequence $H_p(S^1 \times \mathbb{R}\mathbb{P}^3; \mathbb{L}\langle 1 \rangle_q) \Rightarrow \mathbb{L}\langle 1 \rangle_{p+q}(S^1 \times \mathbb{R}\mathbb{P}^3)$, and noting that $\mathbb{L}\langle 1 \rangle_q = 0$ for $q \leq 1$ and $q = 3$, $\mathbb{L}\langle 1 \rangle_4 \cong \mathbb{Z}$ and $\mathbb{L}\langle 1 \rangle_2 \cong \mathbb{Z}/2$, it follows from $H_0(S^1 \times \mathbb{R}\mathbb{P}^3; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(S^1 \times \mathbb{R}\mathbb{P}^3; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ that

$$\mathbb{L}\langle 1 \rangle_4(S^1 \times \mathbb{R}\mathbb{P}^3) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

since there are no nontrivial differentials going in or out of the 4-line. The only potentially nonzero map is $H_3(S^1 \times \mathbb{R}\mathbb{P}^3; \mathbb{Z}/2) \rightarrow H_0(S^1 \times \mathbb{R}\mathbb{P}^3; \mathbb{Z}) \cong \mathbb{Z}$, but there can be no nontrivial homomorphism since the domain is torsion.

Since $\text{Wh}(\mathbb{Z} \times \mathbb{Z}/2) = 0 = \text{Wh}(\mathbb{Z}/2)$ by [LS00, Theorem 3.2 (d-iii) and (e)] we have that

$$L_4^h(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2]) \cong L_4^s(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2]) \text{ and } L_k^h(\mathbb{Z}[\mathbb{Z}/2]) \cong L_k^s(\mathbb{Z}[\mathbb{Z}/2])$$

by [Sha69, Proposition 4.1], for $k \geq 0$. Therefore combining this with Shaneson splitting [Sha69, Theorem 5.1] we have

$$L_4^h(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2]) \cong L_4^s(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2]) \cong L_4^s(\mathbb{Z}[\mathbb{Z}/2]) \oplus L_3^h(\mathbb{Z}[\mathbb{Z}/2]).$$

Now since $L_4^h(\mathbb{Z}[\mathbb{Z}/2]) \cong L_4^s(\mathbb{Z}[\mathbb{Z}/2])$, by [Wal99, Chapter 13A.1] we have

$$L_4^s(\mathbb{Z}[\mathbb{Z}/2]) \oplus L_3^h(\mathbb{Z}[\mathbb{Z}/2]) \cong L_4^h(\mathbb{Z}[\mathbb{Z}/2]) \oplus L_3^h(\mathbb{Z}[\mathbb{Z}/2]) \cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z}/2.$$

We have therefore identified the assembly map $\alpha: \mathbb{L}\langle 1 \rangle_4(S^1 \times \mathbb{R}\mathbb{P}^3) \rightarrow L_4(\mathbb{Z}\pi)$ with a map $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$. We show that $(0, 0, 1)$ lies in the image. We start with the commuting diagram which says that Shaneson splitting of quadratic L groups arises from taking the product of a degree one normal map with the given surgery obstruction with S^1 (see also [Ran86]):

$$\begin{array}{ccc} \mathbb{L}\langle 1 \rangle_3(\mathbb{R}\mathbb{P}^3) & \xrightarrow{\times S^1} & \mathbb{L}\langle 1 \rangle_3(S^1 \times \mathbb{R}\mathbb{P}^3) \\ \downarrow \alpha_{\mathbb{R}\mathbb{P}^3} & & \downarrow \alpha_{S^1 \times \mathbb{R}\mathbb{P}^3} \\ L_3(\mathbb{Z}[\mathbb{Z}/2]) & \hookrightarrow & L_4(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2]) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z}/2 & \hookrightarrow & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2. \end{array}$$

In order to show that $(0, 0, 1)$ lies in the image of the right hand surgery obstruction map $\alpha_{S^1 \times \mathbb{R}\mathbb{P}^3}$, it suffices to show that the left hand $\alpha_{\mathbb{R}\mathbb{P}^3}$ is surjective. To see this we consider the following commutative diagram of \mathbb{L} -homology groups and assembly maps. The key input is that the vertical map $L_3(\mathbb{Z}[\mathbb{Z}]) \rightarrow L_3(\mathbb{Z}[\mathbb{Z}/2])$ induced by the group homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}/2$ is an isomorphism by [Wal99, Lemma 13A.9]. We also use that the assembly map $\alpha: L_3(S^1) \rightarrow L_3(\mathbb{Z}[\mathbb{Z}])$ is an isomorphism.

$$\begin{array}{ccccccccc} \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{L}\langle 1 \rangle_3(S^1) & \xrightarrow{\cong} & L_3(S^1) & \xrightarrow[\alpha]{\cong} & L_3(\mathbb{Z}[\mathbb{Z}]) & \xrightarrow{\cong} & \mathbb{Z}/2 \\ \downarrow = & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow = \\ \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{L}\langle 1 \rangle_3(\mathbb{R}\mathbb{P}^3) & \rightarrow & \mathbb{L}\langle 1 \rangle_3(\mathbb{R}\mathbb{P}^\infty) & \rightarrow & L_3(\mathbb{Z}[\mathbb{Z}/2]) & \xrightarrow{\cong} & \mathbb{Z}/2 \end{array}$$

It now follows from commutativity that $\alpha_{\mathbb{RP}^3} : \mathbb{L}\langle 1 \rangle_3(\mathbb{RP}^3) \rightarrow L_3(\mathbb{Z}[\mathbb{Z}/2])$, which in the diagram we factored through $\mathbb{L}\langle 1 \rangle_3(\mathbb{RP}^\infty) \rightarrow L_3(\mathbb{RP}^\infty)$, is an isomorphism and so is in particular surjective. \square

This assembly map can be identified with the surgery obstruction map and we can interpret elements of $\mathbb{L}\langle 1 \rangle_4(S^1 \times \mathbb{RP}^3)$ as degree 1 normal maps over $S^1 \times \mathbb{RP}^3$.

Lemma 7.8. *Suppose that $f: M \rightarrow S^1 \times \mathbb{RP}^3$ is a degree 1 normal map. Then M is spin.*

PROOF. The existence of a degree 1 normal map implies that there is a stable vector bundle ξ over $N := S^1 \times \mathbb{RP}^3$, which reduces to the Spivak normal fibration η of N . Moreover ξ pulls back to the stable normal bundle ν_M of M . Let ν_N be the stable normal bundle of N , which also reduces to the Spivak normal fibration of N . Since M and N are orientable, $w_1(M) = 0 = w_1(N)$. By the Whitney sum formula, we therefore have $w_2(\nu_N) = w_2(N)$ and $w_2(\nu_M) = w_2(M)$. Since N is spin $w_2(N) = 0$. Since there is an analogue of the Thom isomorphism theorem, Stiefel-Whitney class are defined for spherical fibrations, and we have that $w_2(\xi) = w_2(\eta) = w_2(\nu_N) = 0$. Thus $0 = f^*(0) = f^*(w_2(\nu_N)) = w_2(\nu_M) = w_2(M)$. Therefore M is spin as asserted. \square

Lemma 7.9. *Let M_1 and M_2 be spin 4-manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}/2$ and $c_*[M_i] \neq 0$ for $i = 1, 2$. Then M_1 and M_2 are stably diffeomorphic if and only if $[\lambda_{M_1}] = [\lambda_{M_2}] \in L_4(\mathbb{Z}\pi)$.*

PROOF. Let $\Omega_{4, \deg 1}^{\text{Spin}}(S^1 \times \mathbb{RP}^3)$ be the subset of $\Omega_4^{\text{Spin}}(S^1 \times \mathbb{RP}^3)$ of elements of degree 1. On both sets, $H^1(\pi; \mathbb{Z}/2)$ acts by changing the spin structure.

Similarly to Lemma 7.8, a degree 1 normal bordism gives rise to a spin bordism. Hence we can define a map (of sets)

$$u : \mathbb{L}\langle 1 \rangle_4(S^1 \times \mathbb{RP}^3) \rightarrow \Omega_{4, \deg 1}^{\text{Spin}}(S^1 \times \mathbb{RP}^3)/H^1(\pi; \mathbb{Z}/2).$$

Also let $\Omega_{4, \deg 1}^{\text{Spin}}(B\pi)$ denote the subset of $\Omega_4^{\text{Spin}}(B\pi)$ of elements $([M, f: M \rightarrow S^1 \times \mathbb{RP}^3])$ with $c_*[M] \neq 0$. Define a map

$$\mathcal{O} : \Omega_{4, \deg 1}^{\text{Spin}}(B\pi)/H^1(\pi; \mathbb{Z}/2) \rightarrow L_4(\mathbb{Z}\pi)$$

by sending $[M \xrightarrow{f} S^1 \times \mathbb{RP}^3]$, represented by a map f that is an isomorphism on fundamental groups, to $(\pi_2(M), \lambda_M) \in L_4(\mathbb{Z}\pi)$. This map is defined, since $\pi_2(M)$ is stably free and λ_M admits a quadratic refinement because M is spin. The map \mathcal{O} is well-defined because two 4-manifolds with fundamental group π representing the same spin bordism class are stably diffeomorphic and the action of $H^1(B\pi; \mathbb{Z}/2)$ does not change the intersection form. Now consider the following diagram, obtained from restricting the above maps to reduced (generalised) homology.

$$\begin{array}{ccc} \tilde{\mathbb{L}}\langle 1 \rangle_4(S^1 \times \mathbb{RP}^3) & \xrightarrow{u} & \tilde{\Omega}_{4, \deg 1}^{\text{Spin}}(S^1 \times \mathbb{RP}^3)/H^1(\pi; \mathbb{Z}/2) \\ \downarrow \alpha & & \downarrow \\ \tilde{L}_4(\mathbb{Z}\pi) & \xleftarrow{\mathcal{O}} & \tilde{\Omega}_{4, \deg 1}^{\text{Spin}}(B\pi)/H^1(B\pi; \mathbb{Z}/2) \end{array}$$

The diagram commutes since $\pi_2(S^1 \times \mathbb{R}\mathbb{P}^3) = 0$ and thus the surgery obstruction of a degree one map $M \rightarrow S^1 \times \mathbb{R}\mathbb{P}^3$ that is an isomorphism on fundamental groups is precisely $(\pi_2(M), \lambda_M)$.

Since $\tilde{\Omega}_{4, \deg 1}^{\text{Spin}}(B\pi)/H^1(B\pi; \mathbb{Z}/2)$ has two elements by Lemma 7.3, the map \mathcal{O} to $L_4(\mathbb{Z}\pi)$ is either trivial or not: we know that $S^1 \times \mathbb{R}\mathbb{P}^3$ represents a bordism class mapping to $0 \in \tilde{L}_4(\mathbb{Z}\pi)$ under \mathcal{O} , again since $\pi_2(S^1 \times \mathbb{R}\mathbb{P}^3) = 0$. By Lemma 7.7, the assembly map α is nontrivial. Thus the map $\mathcal{O}: \tilde{\Omega}_{4, \deg 1}^{\text{Spin}}(B\pi)/H^1(B\pi; \mathbb{Z}/2) \rightarrow \tilde{L}_4(\mathbb{Z}\pi)$ is injective. It follows that spin manifolds M_1 and M_2 with the same signature, fundamental group $\pi = \mathbb{Z} \times \mathbb{Z}/2$, and $0 \neq c_*[M_i] \in H_4(\pi; \mathbb{Z})$ are stably diffeomorphic if and only if $[\lambda_{M_1}] = [\lambda_{M_2}] \in \tilde{L}_4(\mathbb{Z}\pi)$. Finally, note that the signature is detected by the image of the intersection forms of M_1 and M_2 in the unreduced L -group $L_4(\mathbb{Z}\pi)$. It follows that spin 4-manifolds M_1 and M_2 with fundamental group π and $c_*[M_i] \neq 0$ are stably diffeomorphic if and only if $[\lambda_{M_1}] = [\lambda_{M_2}] \in L_4(\mathbb{Z}\pi)$. This completes the proof of the lemma. \square

For the case that $\pi_2(M)$ is stably free, i.e. $\text{pri}(M) \neq 0$, we have shown that whether or not the stable isometry class of the equivariant intersection form is induced from a form over \mathbb{Z} determines whether $\beta(M) \in \{\pm 1\}$ or $\beta(M) \in \{\pm 3\}$.

7.2. The case $\beta(M) \in \{0, \pm 2, 4\}$

This covers cases (3) to (5) of Theorem 7.4. The elements in $\tilde{\Omega}_4^{\text{Spin}}(B\pi) \cong \mathbb{Z}/8$ with trivial image in $H_4(\pi; \mathbb{Z})$ but nontrivial secondary invariant correspond to $\beta(M) = \pm 2$.

Let (C_*, d_*) be the 2-periodic free $\mathbb{Z}\pi$ -module resolution of \mathbb{Z} :

$$\begin{array}{ccccccc} \cdots & \xrightarrow{1-T} & \mathbb{Z}\pi & \xrightarrow{1+T} & \mathbb{Z}\pi & \xrightarrow{1-T} & \mathbb{Z}\pi \\ & \nearrow d_3 & & \nearrow d_2 & & \nearrow d_1 & \\ & 1-t & \oplus & 1-t & \oplus & 1-t & \\ \cdots & \xrightarrow{1+T} & \mathbb{Z}\pi & \xrightarrow{1-T} & \mathbb{Z}\pi & & \end{array}$$

Lemma 7.10. *We have that $\ker d_2 \cong I\pi =: I$ and $\text{coker } d^2 \cong I^*$.*

PROOF. We have $\ker d_2 \cong \text{im } d_3 \cong \text{coker } d_4 \cong \text{coker } d_2 \cong \text{im } d_1 \cong I\pi =: I$. By Shapiro's lemma we have $H^2(\pi; \mathbb{Z}\pi) = H^2(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}]) = 0$ and $H^3(\pi; \mathbb{Z}\pi) = H^3(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}]) = 0$. Then it follows from Lemma 3.13 that $\text{coker } d^2 \cong I^*$. \square

Recall that $c_*[M] = 0$ implies that the stable extension

$$0 \rightarrow \ker d_2 \rightarrow C_2^M \oplus \pi_2(M) \xrightarrow{(p_1, p_2)} \text{coker } d^2 \rightarrow 0$$

splits, and thus by Lemma 7.10 we see that $\pi_2(M)$ is stably isomorphic to $I \oplus I^*$. Let \mathcal{H}_I be the hyperbolic form on $I \oplus I^*$.

To identify the intersection form on the null-bordant element, we need the following lemma, which is well-known to the experts.

Lemma 7.11. *Let N be a 4-dimensional thickening of a 2-complex K . Then the double $D(N) := N \cup_{\partial N} N$ of N has*

$$\pi_2(D(N)) \cong H_2(K; \mathbb{Z}\pi) \oplus H^2(K; \mathbb{Z}\pi)$$

and the following equivariant intersection form.

- (i) Restricted to $H^2(K; \mathbb{Z}\pi)$, $\lambda_{D(N)}$ is trivial.

- (ii) Restricted to $H_2(K; \mathbb{Z}\pi)$, $\lambda_{D(N)}$ is isometric to the equivariant intersection form λ_N on $\pi_2(N)$.
- (iii) For $x \in H^2(K; \mathbb{Z}\pi)$, $y \in H_2(K; \mathbb{Z}\pi)$ we have $\lambda_{D(N)}(x, y) = \langle x, y \rangle$.

PROOF. Let $j: N \rightarrow D(N)$ be the inclusion and let $i: D(N) \rightarrow N \times I$ be the inclusion given by identifying $D(N)$ with the boundary of $N \times I$. Then $i \circ j$ is a homotopy equivalence and both i and j are isomorphisms on fundamental groups. We have $H_3(N \times I; \mathbb{Z}\pi) = 0$ and by Poincaré-Lefschetz duality $H_2(N \times I, D(N); \mathbb{Z}\pi) \cong H^3(N \times I; \mathbb{Z}\pi) = 0$. Thus we have a short exact sequence from the long exact sequence of the pair.

$$0 \rightarrow H_3(N \times I, D(N); \mathbb{Z}\pi) \xrightarrow{\partial} H_2(D(N); \mathbb{Z}\pi) \xrightarrow{i_*} H_2(N \times I; \mathbb{Z}\pi) \rightarrow 0.$$

Again by Poincaré-Lefschetz duality, the boundary map ∂ is identified with the map $i^*: H^2(N \times I; \mathbb{Z}\pi) \rightarrow H^2(D(N); \mathbb{Z}\pi)$. Identifying $H_2(N; \mathbb{Z}\pi) \cong H_2(N \times I; \mathbb{Z}\pi)$ via $i_* \circ j_*$, the map $j_*: H_2(N; \mathbb{Z}\pi) \rightarrow H_2(D(N); \mathbb{Z}\pi)$ provides a splitting of i_* . Thus we have an isomorphism

$$H_2(N; \mathbb{Z}\pi) \oplus H^2(N \times I; \mathbb{Z}\pi) \xrightarrow{(j_*, PD \circ i^*)} H_2(D(N); \mathbb{Z}\pi) \cong \pi_2(N).$$

The first part of the lemma then follows from the homotopy equivalence $N \simeq K$.

Under the isomorphism $H^2(D(N); \mathbb{Z}\pi) \xrightarrow{PD; \cong} H_2(D(N); \mathbb{Z}\pi) \cong \pi_2(D(N))$, the equivariant intersection form pulls back to the form $\langle - \cup -, [D(N)] \rangle$. On the image of $i^*: H^2(N \times I; \mathbb{Z}\pi) \rightarrow H^2(D(N); \mathbb{Z}\pi)$ we have

$$\langle i^*x \cup i^*y, [D(N)] \rangle = \langle x \cup y, i_*[D(N)] \rangle = \langle x \cup y, 0 \rangle = 0.$$

On the image of $i_*: H_2(N; \mathbb{Z}\pi) \rightarrow H_2(D(N); \mathbb{Z}\pi)$ we use the geometric computation of the intersection form to see that $\lambda_{D(N)}$ pulls back to λ_N as claimed.

Given $x \in H^2(K; \mathbb{Z}\pi)$, $y \in H_2(K; \mathbb{Z}\pi)$ we have

$$\lambda_{D(N)}(x, y) = \langle i^*(x) \cup PD^{-1}(j_*y), [M] \rangle = \langle i_*x, j_*y \rangle = \langle j^*i^*x, y \rangle.$$

Since we used j^*i^* for the identification of the cohomology of $N \times I$ with that of N , this completes the proof of the lemma. \square

Lemma 7.12. *The equivariant intersection form on the double of the 2-skeleton of $S^1 \times \mathbb{R}\mathbb{P}^3$, representing the trivial element of $\Omega_4^{\text{Spin}}(B(\mathbb{Z} \times \mathbb{Z}/2))$, is given by the even form $(I \oplus I^*, \mathcal{H}_I)$.*

PROOF. Taking the standard CW-structure on $S^1 \times \mathbb{R}\mathbb{P}^3$, the 2-skeleton has $H_2((S^1 \times \mathbb{R}\mathbb{P}^3)^{(2)}; \mathbb{Z}\pi) \cong I$ and $H^2((S^1 \times \mathbb{R}\mathbb{P}^3)^{(2)}; \mathbb{Z}\pi) \cong I^*$ by the above calculations. We thicken the 2-skeleton to obtain a manifold N with boundary. Consider immersed spheres α and β in N that intersect transversely. We can view α and β as elements of $\pi_2(S^1 \times \mathbb{R}\mathbb{P}^3)$, and then since $\pi_2(S^1 \times \mathbb{R}\mathbb{P}^3) = 0$, the intersection product must be trivial. It follows that the intersection form on $\pi_2(N)$ is trivial. The lemma now follows from Lemma 7.11. \square

Let $(s_1, s_2): \text{coker } d^2 \rightarrow C_2 \oplus \pi_2(M)$ be some splitting. Since the Secondary Property 1.11 holds for all abelian groups by Section 6.2, it holds for $\mathbb{Z} \times \mathbb{Z}/2$. Therefore $\text{sec}(M) = 0$ if and only if the equivariant intersection form λ_M restricted along s_2 is even. We have already seen that on the stably diffeomorphism class given by $\text{sec} = \text{ter} = 0$, corresponding to the null-bordant element considered in Lemma 7.12, the equivariant intersection form is even. Also taking connected sum with a simply-connected spin manifold in order to change the signature does not

affect whether the intersection form is even. It follows that λ_M is even if and only if $\mathfrak{sec}(M) = 0$. This covers the cases (3) and (4) of Theorem 7.4. That is, we have shown that the parity of the intersection form distinguishes between $\beta(M) \in \{\pm 2\}$ and $\beta(M) \in \{0, 4\}$.

7.3. The case $\beta(M) \in \{0, 4\}$

It remains to relate the τ invariant in (5) to the \mathfrak{tet} invariant from the Atiyah-Hirzebruch spectral sequence. It will be useful to identify I^* with I^w , where I^w is the submodule of $\mathbb{Z}\pi$ generated by $1 - t$ and $1 + T$.

Lemma 7.13. *The following map is an isomorphism*

$$\begin{aligned} \theta_{I^w} : I^w &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}\pi}(I, \mathbb{Z}\pi) \\ 1 - t &\mapsto \left\{ \begin{array}{ll} 1 - t & \mapsto 1 - t \\ 1 - T & \mapsto 1 - T \end{array} \right\} \\ 1 + T &\mapsto \left\{ \begin{array}{ll} 1 - t & \mapsto 1 + T \\ 1 - T & \mapsto 0 \end{array} \right\}. \end{aligned}$$

PROOF. First note that the composition

$$I^w \xrightarrow{\theta_{I^w}} \text{Hom}_{\mathbb{Z}\pi}(I, \mathbb{Z}\pi) \xrightarrow{\text{ev}_{1-t}} \mathbb{Z}\pi,$$

where $\text{ev}_{1-t}(f) := f(1 - t)$, is the standard inclusion and thus θ_{I^w} is injective.

Let $f \in \text{Hom}_{\mathbb{Z}\pi}(I, \mathbb{Z}\pi)$. Then f is determined by $f(1 - t)$ and $f(1 - T)$. We have $(1 - t)f(1 - T) = f((1 - t)(1 - T)) = (1 - T)f(1 - t)$. As multiplication with $(1 - t)$ is injective as a $\mathbb{Z}\pi$ -module homomorphism $\mathbb{Z}\pi \rightarrow \mathbb{Z}\pi$, $f(1 - T)$ is determined by $f(1 - t)$. Hence $\text{Hom}_{\mathbb{Z}\pi}(I, \mathbb{Z}\pi) \xrightarrow{\text{ev}_{1-t}} \mathbb{Z}\pi$ is injective. Thus to show that θ_{I^w} is surjective, it suffices to show that the image of ev_{1-t} is contained in I^w . For then we have a factorisation of $\text{ev}_{1-t} \circ \theta_{I^w}$ as

$$I^w \xrightarrow{\theta_{I^w}} \text{Hom}_{\mathbb{Z}\pi}(I, \mathbb{Z}\pi) \xrightarrow{\text{ev}_{1-t}} I^w \subseteq \mathbb{Z}\pi,$$

and the composition $I^w \rightarrow I^w$ is the identity map. It follows that ev_{1-t} is onto I^w , so is an isomorphism onto I^w , and therefore θ_{I^w} is an isomorphism.

To show that image of ev_{1-t} is contained in I^w , let \mathbb{Z}^w denote \mathbb{Z} considered as a $\mathbb{Z}\pi$ -module, with the $\mathbb{Z}\pi$ -action given as follows: t acts trivially and T acts by multiplication by -1 . Let $\varepsilon^w : \mathbb{Z}\pi \rightarrow \mathbb{Z}^w$ be the map sending t to 1 and T to -1 . In \mathbb{Z}^w we have

$$2\varepsilon^w(f(1 - t)) = \varepsilon^w((1 - T)f(1 - t)) = \varepsilon^w((1 - t)f(1 - T)) = 0$$

and thus $f(1 - t) \in I^w$ as needed. Note that we have shown that the evaluation map $\text{ev}_{1-t} : \text{Hom}_{\mathbb{Z}\pi}(I, \mathbb{Z}\pi) \rightarrow I^w$ is an inverse of θ_{I^w} . \square

If the secondary invariant of M is trivial, then by Lemma 4.7 there are $k, n \in \mathbb{N}$ and an isomorphism $\psi : I \oplus I^w \oplus \mathbb{Z}\pi^n \rightarrow \pi_2(M \#^k(S^2 \times S^2))$, such that $\lambda_{M \#^k(S^2 \times S^2)}$ restricted to I^w along ψ is trivial.

Lemma 7.14. *We have that $\mu(\psi(0, 1+T, 0)) = 0$ and $\psi(0, 1+T, 0)$ is r -characteristic. Furthermore, $\tau(\psi(0, 1+T, 0)) = 0 \in \mathbb{Z}/2$ if and only if $\mathfrak{tet}(M) = 0$. In particular, $\tau(\psi(0, 1+T, 0))$ is independent of the choice of ψ .*

This will complete the proof of (5) of Theorem 7.4, using that φ from Theorem 7.4 agrees with $\theta_{I^w}(1+T)$.

We need some lemmas on Tate cohomology before starting the proof of Lemma 7.14. Let $\mathbb{Z}/2$ act on a $\mathbb{Z}\pi$ -module P by multiplication by T . The Tate cohomology groups are 2-periodic and

$$\begin{aligned}\widehat{H}^0(\mathbb{Z}/2; P) &= \ker(1-T)/\operatorname{im}(1+T) \\ \widehat{H}^1(\mathbb{Z}/2; P) &= \ker(1+T)/\operatorname{im}(1-T).\end{aligned}$$

The next lemma follows from the definition.

Lemma 7.15. *The Tate cohomology has the following properties:*

- (i) $\widehat{H}^{2i}(\mathbb{Z}/2; I^w) \cong \widehat{H}^{2i+1}(\mathbb{Z}/2; I) \cong \mathbb{Z}/2$ generated by $1+T$ and $1-T$ respectively;
- (ii) $\widehat{H}^{2i}(\mathbb{Z}/2; \mathbb{Z}\pi) = \widehat{H}^{2i+1}(\mathbb{Z}/2; \mathbb{Z}\pi) = \widehat{H}^{2i}(\mathbb{Z}/2; I) = \widehat{H}^{2i+1}(\mathbb{Z}/2; I^w) = 0$; and
- (iii) $\widehat{H}^{2i}(\mathbb{Z}/2; P \oplus P') \cong \widehat{H}^{2i}(\mathbb{Z}/2; P) \oplus \widehat{H}^{2i}(\mathbb{Z}/2; P')$.

Lemma 7.16. *Let X be a CW model for $B\pi$ with 2-skeleton K . Let $i: K \rightarrow X$ be the inclusion map. Then any element $\alpha \in H^2(K; \mathbb{Z}\pi)$ generating $\widehat{H}^0(\mathbb{Z}/2; H^2(K; \mathbb{Z}\pi)) \cong \mathbb{Z}/2$ maps to $i^*(Tt) \in H^2(K; \mathbb{Z}/2)$ under the reduction map.*

PROOF. We first prove the statement in the case where $X = S^1 \times \mathbb{R}\mathbb{P}^\infty$ with the standard CW-structure and a single choice of generator α in $H^2(K; \mathbb{Z}\pi)$, where K is the 2-skeleton of X . The cellular cochain complex of K is given by:

$$\begin{array}{ccccc} \mathbb{Z}\pi & \xrightarrow{1-T} & \mathbb{Z}\pi & \xrightarrow{1+T} & \mathbb{Z}\pi \\ & \searrow^{1-t^{-1}} & \oplus & \searrow^{1-t^{-1}} & \oplus \\ & & \mathbb{Z}\pi & \xrightarrow{1-T} & \mathbb{Z}\pi. \end{array}$$

The generator A of bottom right $\mathbb{Z}\pi$ summand, in $C^2(K; \mathbb{Z}\pi)$, maps to $i^*(Tt)$ under reduction to $\mathbb{Z}/2$. We need to show that A can be taken as our choice of α , that is A generates $\widehat{H}^0(\mathbb{Z}/2; H^2(K; \mathbb{Z}\pi)) \cong \widehat{H}^0(\mathbb{Z}/2; I^w) \cong \mathbb{Z}/2$. We can extend $C^*(K; \mathbb{Z}\pi)$ to a resolution for \mathbb{Z}^w as follows:

$$\begin{array}{ccccccc} \mathbb{Z}\pi & \xrightarrow{1-T} & \mathbb{Z}\pi & \xrightarrow{1+T} & \mathbb{Z}\pi & \xrightarrow{1-t^{-1}} & \mathbb{Z}\pi \\ & \searrow^{1-t^{-1}} & \oplus & \searrow^{1-t^{-1}} & \oplus & \searrow^{1-t^{-1}} & \\ & & \mathbb{Z}\pi & \xrightarrow{1-T} & \mathbb{Z}\pi & \xrightarrow{1+T} & \mathbb{Z}\pi \xrightarrow{\epsilon^w} \mathbb{Z}^w. \end{array}$$

Write $\delta = (1-t^{-1}, 1+T): \mathbb{Z}\pi^{\oplus 2} \rightarrow \mathbb{Z}\pi$. The identification

$$H^2(K; \mathbb{Z}\pi) \cong C^2(K; \mathbb{Z}\pi)/\operatorname{im} d^2 \cong C^2/\ker \delta \cong \operatorname{im} \delta \cong \ker \epsilon^w \cong I^w$$

sends A to $1+T$, which generates $\widehat{H}^0(\mathbb{Z}/2; I^w)$ by Lemma 7.15. This completes the proof for the fixed CW model and our choice α .

For the general case note that any two generators of $\widehat{H}^0(\mathbb{Z}/2; H^2(K; \mathbb{Z}\pi))$ differ by $(1+T)x$ for some $x \in H^2(K; \mathbb{Z}\pi)$. Under the reduction map $H^2(K; \mathbb{Z}\pi) \rightarrow H^2(K; \mathbb{Z}/2)$ any element of the form $(1+T)x$ is mapped to zero. Hence the statement is true for all generators if and only if it is true for one given generator. Since any two 2-skeleta K, K' become homotopy equivalent after taking a wedge product with sufficiently many copies of S^2 , the general case follows from the case $X = S^1 \times \mathbb{R}\mathbb{P}^\infty$. \square

PROOF OF LEMMA 7.14. Let $N := M \#^k(S^2 \times S^2)$ and let

$$\psi: I \oplus I^w \oplus \mathbb{Z}\pi^{\oplus n} \rightarrow \pi_2(N)$$

be an isomorphism such that λ_N restricted to I^w along ψ is trivial as assumed for the statement of Lemma 7.14. Consider the extension

$$(7.17) \quad 0 \rightarrow \ker d_2^N \xrightarrow{(i_1, i_2)} C_2^N \oplus \pi_2(N) \xrightarrow{\rho_1 + \rho_2} \text{coker } d_3^N \rightarrow 0.$$

Let F, F' be free modules such that there is an isomorphism

$$\theta: \text{coker } d_3^N \oplus F \rightarrow I^w \oplus F'.$$

Let $\iota: \text{coker } d_3^N \rightarrow \text{coker } d_3^N \oplus F$ and $j: I^w \rightarrow I \oplus I^w \oplus \mathbb{Z}\pi^{\oplus 2n}$ be the inclusions and let $p: I^w \oplus F' \rightarrow I^w$ be the projection. Consider the composition

$$s_2 := \psi \circ j \circ p \circ \theta \circ \iota: \text{coker } d_3^N \xrightarrow{\iota} \text{coker } d_3^N \oplus F \xrightarrow{\theta} I^w \oplus F' \xrightarrow{p} I^w \xrightarrow{j} I \oplus I^w \oplus \mathbb{Z}\pi^{\oplus 2n} \xrightarrow{\psi} \pi_2(N).$$

We want to define a map $s_1: \text{coker } d_3^N \rightarrow C_2^N$ such that $(s_1, s_2): \text{coker } d_3^N \rightarrow C_2^N \oplus \pi_2(N)$ is a section of the extension (7.17).

It will be helpful to consider the extensions in the next diagram, which we may identify using the isomorphism $\theta: \text{coker } d_3^N \oplus F \rightarrow I^w \oplus F'$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker d_2^N & \longrightarrow & C_2^N \oplus \pi_2(N) \oplus F & \xrightarrow{(\rho_1 + \rho_2) \oplus \text{Id}_F} & \text{coker } d_3^N \oplus F \longrightarrow 0 \\ & & \downarrow = & & \downarrow = & & \downarrow \cong \theta \\ 0 & \longrightarrow & \ker d_2^N & \longrightarrow & C_2^N \oplus \pi_2(N) \oplus F & \xrightarrow{\rho'_1 + \rho'_2 + \rho'_3} & I^w \oplus F' \longrightarrow 0 \end{array}$$

Here ρ'_k , for $k \in \{1, 2, 3\}$ denote the resulting maps defined by the diagram, with $\rho'_1: C_2^N \rightarrow I^w \oplus F'$, $\rho'_2: \pi_2(N) \rightarrow I^w \oplus F'$, and $\rho'_3: F \rightarrow I^w \oplus F'$. Now define a map s'_2 by

$$s'_2 := \psi \circ j \circ p: I^w \oplus F' \xrightarrow{p} I^w \xrightarrow{j} I \oplus I^w \oplus \mathbb{Z}\pi^{\oplus 2n} \xrightarrow{\psi} \pi_2(N).$$

The maps s_2 and s'_2 fit into the following commutative diagram:

$$\begin{array}{ccc} \text{coker } d_3^N & \xrightarrow{\iota} & \text{coker } d_3^N \oplus F \\ s_2 \downarrow & & \downarrow \cong \theta \\ \pi_2(N) & \xleftarrow{s'_2} & I^w \oplus F' \\ \psi \uparrow \cong & & \downarrow p \\ I \oplus I^w \oplus \mathbb{Z}\pi^{\oplus 2n} & \xleftarrow{j} & I^w \end{array}$$

As above let $\mathbb{Z}/2$ act on $I \oplus I^w \oplus \mathbb{Z}\pi^{\oplus 2n}$ by multiplication with T . Then by Lemma 7.15 we have:

$$\begin{aligned} \widehat{H}^0(\mathbb{Z}/2; I \oplus I^w \oplus \mathbb{Z}\pi^{\oplus 2n}) &= \ker(1 - T) / \text{im}(1 + T) \\ &\cong \widehat{H}^0(\mathbb{Z}/2; I) \oplus \widehat{H}^0(\mathbb{Z}/2; I^w) \oplus \widehat{H}^0(\mathbb{Z}/2; \mathbb{Z}\pi^{\oplus 2n}) \cong 0 \oplus \mathbb{Z}/2 \oplus 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \widehat{H}^0(\mathbb{Z}/2; \operatorname{coker} d_3^N) &\cong \widehat{H}^0(\mathbb{Z}/2; \operatorname{coker} d_3^N \oplus F) \cong \widehat{H}^0(\mathbb{Z}/2; I^w \oplus F') \\ &\cong \widehat{H}^0(\mathbb{Z}/2; I^w) \oplus \widehat{H}^0(\mathbb{Z}/2; F') \cong \mathbb{Z}/2 \oplus 0. \end{aligned}$$

Since $\ker d_2^N$ is stably isomorphic to I , we have $\widehat{H}^0(\mathbb{Z}/2; \ker d_2^N) = 0$. Thus the map $C_2^N \oplus \pi_2(N) \rightarrow \operatorname{coker} d_3^N$ induces an isomorphism

$$\mathbb{Z}/2 \cong \widehat{H}^0(\mathbb{Z}/2; \pi_2(N)) \rightarrow \widehat{H}^0(\mathbb{Z}/2; \operatorname{coker} d_3^N) \cong \mathbb{Z}/2.$$

These groups are generated by $\psi(0, 1+T, 0) \in \pi_2(N)$ and $\theta^{-1}(1+T, 0) \in I^w \oplus F'$ respectively. It follows that $\rho'_2 \circ s'_2(1+T, 0) \in I^w \oplus F'$ differs from $(1+T, 0)$ by an element of the form $(1+T)(x_1, x_2)$ for some $(x_1, x_2) \in I^w \oplus F'$. That is:

$$(7.18) \quad \rho'_2 \circ s'_2(1+T, 0) = (1+T, 0) + (1+T)(x_1, x_2) \in I^w \oplus F'.$$

Multiply both sides by $(1-t)$ and apply linearity of $\rho'_2 \circ s'_2$ to deduce that

$$(1+T)\rho'_2 \circ s'_2(1-t, 0) = (1+T)(1-t, 0) + (1+T)(1-t)(x_1, x_2) \in I^w \oplus F'.$$

Therefore $\rho'_2 \circ s'_2(1-t, 0) - (1-t, 0) - (1-t)(x_1, x_2) \in \ker(1+T)$. Combining this with $\widehat{H}^1(\mathbb{Z}/2; I^w \oplus F') = \ker(1+T)/\operatorname{im}(1-T) = 0$ from Lemma 7.15, it follows that

$$(7.19) \quad \rho'_2 \circ s'_2(1-t, 0) = (1-t, 0) + (1-t)(x_1, x_2) + (1-T)(x'_1, x'_2) \in I^w \oplus F'$$

for some $(x'_1, x'_2) \in I^w \oplus F'$.

Let $q: C_2^N \rightarrow \operatorname{coker} d_3^N$ be the quotient map. The diagram

$$\begin{array}{ccc} C_2^N \oplus F & \xrightarrow{\rho'_1 + \rho'_3} & I^w \oplus F' \\ & \searrow q \oplus \operatorname{Id} & \nearrow \theta \\ & \operatorname{coker} d_3^N \oplus F & \cong \end{array}$$

commutes. Therefore $\rho'_1 + \rho'_3: C_2^N \oplus F \rightarrow I^w \oplus F'$ is surjective, so we can pick preimages $(y_1, y_2), (y'_1, y'_2) \in C_2^N \oplus F$ of $(x_1, x_2), (x'_1, x'_2)$, respectively, so that in $I^w \oplus F'$ we have:

$$(7.20) \quad \rho'_1(y_1) + \rho'_3(y_2) = (x_1, x_2) \text{ and } \rho'_1(y'_1) + \rho'_3(y'_2) = (x'_1, x'_2).$$

Also pick a lift $r: F' \rightarrow C_2^N \oplus F$ of the inclusion $i_2: F' \rightarrow I^w \oplus F'$, satisfying $(\rho'_1 + \rho'_3) \circ r = i_2$.

Define

$$(7.21) \quad \begin{aligned} s'_1: I^w \oplus F' &\rightarrow C_2^N \oplus F \\ (1+T, 0) &\mapsto -(1+T)(y_1, y_2) \\ (1-t, 0) &\mapsto -(1-t)(y_1, y_2) - (1-T)(y'_1, y'_2) \\ (0, x) &\mapsto r(x). \end{aligned}$$

Then it follows from a straightforward computation using (7.18), (7.19), (7.20), and (7.21) that $(s'_1, s'_2): I^w \oplus F' \rightarrow C_2^N \oplus \pi_2(N) \oplus F$ is a section of $\rho'_1 + \rho'_2 + \rho'_3$. Let

$$s_1: \operatorname{coker} d_3^N \rightarrow C_2^N$$

be $s'_1 \circ \theta \circ \iota$ composed with the projection $C_2^N \oplus F \rightarrow C_2^N$. Then we claim that

$$(s_1, s_2): \operatorname{coker} d_3 \rightarrow C_2^N \oplus \pi_2(N)$$

is a section of $\rho_1 + \rho_2$ and the restriction of λ_N to the image of $s_2: \text{coker } d_3^N \rightarrow \pi_2(N)$ is trivial.

To see that (s_1, s_2) is a section of $\rho_1 + \rho_2$ we consider the following diagram of maps.

$$\begin{array}{ccccc}
0 & \longrightarrow & \text{coker } d_3^N & \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{p'} \end{array} & \text{coker } d_3^N \oplus F & \xrightarrow[\cong]{\theta} & I^w \oplus F' \\
& & \downarrow (s_1, s_2) & & \downarrow (\rho_1 + \rho_2) \oplus \text{Id}_F & \nearrow (s'_1, s'_2) & \nearrow \rho'_1 + \rho'_2 + \rho'_3 \\
0 & \longrightarrow & C_2^N \oplus \pi_2(N) & \xrightarrow{\iota'} & C_2^N \oplus \pi_2(N) \oplus F & &
\end{array}$$

Here ι' is the inclusion map and p' is the projection. Ignoring (s_1, s_2) , (s'_1, s'_2) , and p' , the diagram commutes. It also commutes if we ignore $\rho_1 + \rho_2$, $(\rho_1 + \rho_2) \oplus \text{Id}_F$, $\rho'_1 + \rho'_2 + \rho'_3$, and p' . In addition $p' \circ \iota = \text{Id}$ and $(\rho'_1 + \rho'_2 + \rho'_3) \circ (s'_1, s'_2) = \text{Id}$. Therefore we have:

$$\text{Id} = (\rho'_1 + \rho'_2 + \rho'_3) \circ (s'_1, s'_2) = \theta \circ ((\rho_1 + \rho_2) \oplus \text{Id}_F) \circ (s'_1, s'_2)$$

where the second equation uses the right hand triangle. Conjugating with θ yields:

$$(7.22) \quad \text{Id} = ((\rho_1 + \rho_2) \oplus \text{Id}_F) \circ (s'_1, s'_2) \circ \theta: \text{coker } d_3^N \oplus F \rightarrow \text{coker } d_3^N \oplus F.$$

Thus

$$\begin{aligned}
(\rho_1 + \rho_2) \circ (s_1, s_2) &= p' \circ \iota \circ (\rho_1 + \rho_2) \circ (s_1, s_2) \\
&= p' \circ ((\rho_1 + \rho_2) \oplus \text{Id}_F) \circ \iota' \circ (s_1, s_2) \\
&= p' \circ ((\rho_1 + \rho_2) \oplus \text{Id}_F) \circ (s'_1, s'_2) \circ \theta \circ \iota \\
&= p' \circ \iota = \text{Id}.
\end{aligned}$$

We used $p' \circ \iota = \text{Id}$, the left hand square, commutativity of the trapezium involving (s_1, s_2) and (s'_1, s'_2) , (7.22), and finally $p' \circ \iota = \text{Id}$ again. This completes the proof that (s_1, s_2) is a section of $\rho_1 + \rho_2$. The image of $s_2: \text{coker } d_3^N \rightarrow \pi_2(N)$ is contained in the image $\psi(I^w)$ on which λ_N vanishes by assumption. It follows that λ_N vanishes on $s_2(\text{coker } d_3^N)$ as claimed.

Now that we have found an appropriate section, we complete the proof of Lemma 7.14. Let $\alpha \in \text{coker } d_3^N$ be an element generating $\widehat{H}^0(\mathbb{Z}/2; \text{coker } d_3^N)$. Then $s_2(\alpha) \in \pi_2(M)$ has trivial self-intersection number and is r -characteristic by Proposition 4.10. Hence we can compute $\tau(s_2(\alpha)) \in \mathbb{Z}/2$. The tertiary property holds for $\pi = \mathbb{Z} \times \mathbb{Z}/2$, and the group $E_{2,2}^3 \cong H_2(\pi; \mathbb{Z}/2)/d_2 = \mathbb{Z}/2$ in which the tertiary invariant lives is generated by $(Tt)^*$. Thus using Lemma 7.16 and the construction of $\tau_{M,s}$, we see that $\text{ter}(M) = 0$ if and only if $\tau_{M,s}(Tt) = \tau(s_2(\alpha)) = 0$.

Note that $\theta \circ \iota(\alpha) = (1 + T, 0) + (1 + T)(x, y)$ for some $(x, y) \in I^w \oplus F'$, and thus $s_2(\alpha) = \psi(0, 1 + T, 0) + (1 + T)\psi(0, x, 0)$. Since λ_M vanishes on the image of $\psi|_{I^w}$, $\tau(s_2(\alpha)) = \tau(\psi(0, 1 + T, 0))$ by Lemma 4.2. Since the τ -invariant on $s_2(\alpha)$ is well-defined in $\mathbb{Z}/2$, it follows that also $\psi(0, 1 + T, 0)$ has to be r -characteristic. By the previous paragraph we deduce that $\tau(\psi(0, 1 + T, 0)) = 0 \in \mathbb{Z}/2$ if and only if $\text{ter}(M) = 0$. \square

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