Solving heterogeneous-belief asset pricing models with short selling constraints and many agents

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Abstract

Short-selling constraints are common in financial markets, while physical assets such as housing often lack markets for short-selling altogether. As a result, investment decisions are often restricted by such constraints. This paper studies asset prices in behavioural heterogeneous-belief models with short-selling constraints. We provide expressions for price and demands in a market with arbitrarily many belief types, plus efficient solution algorithms, relevant for a wide range of models. Extensions include *conditional* short-selling constraints, multiple asset markets, and a market maker. An application studies how an alternative uptick rule, as in the United States, affects price and wealth distribution in a market with many belief types in evolutionary competition.

Keywords: Asset pricing, heterogeneous beliefs, short-selling constraints, updating.

JEL-Classification: C63, D84, G12, G18, G40.

1 Introduction

The practice of short-selling is common in financial markets but is also widely regulated. When investors go short, they borrow and immediately sell a financial asset before repurchasing and returning the asset to the lender, closing their position. Whereas a long position can be thought of as a bet that asset prices will increase, short-selling allows investors to bet on a fall in asset prices. It has been argued that such betting may increase volatility in financial markets. A common policy response among regulators has been to restrict short-selling; for example, during the 2008-9 financial crisis many countries introduced short-selling bans following sharp declines in asset prices. Similar short-selling bans were reinstated in some

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European economies during the 2011-12 sovereign debt crisis and the Covid-19 outbreak (see Siciliano and Ventoruzzo, 2020). It is therefore important that researchers be able to solve asset pricing models with short-selling constraints in an efficient manner.

In this paper, we show how to efficiently solve behavioural asset pricing models with short-selling constraints and arbitrarily many heterogeneous beliefs.[1] We are thinking here of dynamic, discrete time heterogeneous-belief asset pricing models such as Brock and Hommes (1998), LeBaron et al. (1999), and Westerhoff (2004); for instance, the population shares of different types may be endogenously determined by an *evolutionary competition* mechanism. For the case of such dynamic models, we derive expressions for the market-clearing price and demands and show how these results can be used to construct computationally-efficient solution algorithms. Our algorithm enables researchers to incorporate short-selling constraints in models with many agents or belief types – as in real-world asset markets – and is supported by analytical results that do not appear to have been documented previously.

We provide results for a benchmark asset pricing model, as well as several other cases studied in the literature: *conditional* short-selling constraints, the case of multiple asset markets with short-selling constraints; and the market-maker approach to price determination. We also indicate how the benchmark results can be related to certain physical investment assets, such as housing, or to models in which beliefs are determined by social networks.

Our analysis is built around a behavioural heterogeneous-beliefs asset pricing model. In particular, we allow arbitrarily many different belief types whose population shares may be exogenous or determined by an evolutionary competition mechanism as in the asset pricing model of Brock and Hommes (1998). The Brock and Hommes (1998) model with evolutionary competition has been studied in the many-types case by Brock et al. (2005) who allow short-selling by investors, whereas we show how asset prices and demands depend on *belief dispersion* when investors face short-selling constraints and provide computationally-efficient solution algorithms that exploit the analytical results.

The difficulty in the many-types case results from the demand functions being *piecewise-linear*, such that the market-clearing price depends on how many types are short-selling constrained. For a market with a large number of investor types, it is computationally intensive to solve for a price and demands. To overcome this problem, we exploit the fact that types who are short-selling constrained in a given period must be *more pessimistic* than those agents who were unconstrained, such that ranking types in terms of optimism is useful. As a result, it becomes computationally feasible to simulate models with large numbers of belief types over many periods while retaining solution accuracy.

We provide analytical and numerical examples, as well as a policy application that studies an *alternative uptick rule*, as currently in place in the United States, in a model with a large number of belief types whose population shares are determined by *evolutionary competition*. The alternative uptick rule is a 'circuit breaker' that bans short-selling if prices fall 10% or more in the previous trading period; surprisingly, there do not appear to be any previous

[1] Short-selling constraints appear to have first been studied, in a static model, by Miller (1977).

assessments of this rule in the literature. The results indicate that an alternative uptick rule may attenuate (or prevent) falls in price; however, we also find that such rules can hinder price discovery, increase price volatility and lead to explosive price paths. In addition, we find an alternative uptick rule can have substantive distributional (i.e. wealth) implications.

The closest papers in the literature are Anufriev and Tuinstra (2013) and Dercole and Radi (2020). Anufriev and Tuinstra (2013) add trading costs for short-selling into a two-type asset pricing model and find that this leads to additional (non-fundamental) steady states as beliefs are updated more aggressively; in a similar vein, but with the addition of a leverage constraint, see in't Veld (2016). By comparison, Dercole and Radi (2020) study the original 'uptick rule' in the United States from 1938–2007, which banned shorting at lower prices, and find that there is no clear-cut impact on price volatility. There is also a wider literature on non-smooth asset pricing models (see e.g. Tramontana et al., 2010); short-selling constraints can be thought of as a specific application that gives rise to such models.

The above papers all consider a small number of investor types and solve for prices and demands in specific cases. The present paper contributes to the literature by solving for price and demands when there are arbitrarily many belief types with general price predictors, and by providing computationally-efficient algorithms. We make our results accessible by considering a general form of beliefs and several cases studied in the literature, such as *conditional* short-selling constraints, multiple risky assets, and the market-maker approach.

Our paper is part of a growing literature studying heterogeneous beliefs, asset prices and the effectiveness of regulatory policies in financial markets (Westerhoff, 2016). In financial market models it is known that differences in beliefs combined with short-selling constraints can lead to price bubbles (see e.g. Scheinkman and Xiong, 2003), but such regulations could also aid market stability as noted above. There has also been interest in the impact of short-selling restrictions in markets for *physical* investment assets like housing which are subject to boom and bust (see Shiller, 2015; Fabozzi et al., 2020). Our results are thus of potential relevance for physical as well as financial assets, as we explain using housing as an example.

The paper proceeds as follows. Section 2 presents a baseline model for which analytical results are presented in Section 3. Section 4 presents three extensions of the baseline model, and Section 5 presents our policy application. Finally, Section 6 concludes.

2 Model

Consider a finite set of myopic, risk-averse investor types $\mathcal{H} = \{h_1, ..., h_H\}$. At each date $t \in \mathbb{N}_+$, each type $h \in \mathcal{H}$ chooses a portfolio of a risky asset $z_{t,h}$ and a riskless bond paying $\tilde{r} > 0$ to maximize a mean-variance utility function over future wealth with risk-aversion parameter $a > 0$. The risky asset has current price p_t, future price p_{t+1}, and pays stochastic dividends d_{t+1}, which are exogenous. Investors form subjective expectations of the future price and future dividends of the risky asset as described below. The underlying model follows Brock and Hommes (1998), except the risky asset is in positive net supply $\overline{Z} > 0$ and short-selling is ruled out by a short-selling constraint of the form $z_{t,h} \geq 0$ for all t and h.

2.1 Asset demand

We denote the subjective expectation of type h at date t by $\tilde{E}_{t,h}[.]$, and the subjective variance by $V\tilde{a}r_{t,h}[.]$. The optimal portfolio choice of type h solves the problem:[2]

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2} V\tilde{a}r_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad z_{t,h} \geq 0 \tag{1}$$

where $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1 + \tilde{r})(w_{t,h} - p_t z_{t,h})$ is the future wealth, $w_{t,h} - p_t z_{t,h}$ is holdings of the risk-free asset, and we assume $V\tilde{a}r_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$, with $\sigma^2 > 0$.

Given short-selling constraints, the demand of each investor type $h \in \mathcal{H}$ is:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}. \end{cases} \tag{2}$$

If the price p_t is small enough, then type h's short-selling constraint is slack and their demand for the risky asset decreases with the price; this is the standard demand function that arises in Brock and Hommes (1998) where short-selling constraints are absent. However, if the price is high enough to make the expected excess return of type h *negative*, then the short-selling constraint will bind on type h and their position in the risky asset is zero.

Dividends follow an IID process $d_t = \overline{d} + \epsilon_t$, where $\overline{d} > 0$ and ϵ_t is a zero-mean shock with constant variance. We assume all investor types know the dividend process, such that $\tilde{E}_{t,h}[d_{t+1}] = \overline{d}$ for all t and h; there is *no loss of generality* as our solution nests a generic specification of $\tilde{E}_{t,h}[d_{t+1}]$ at no extra cost.[3] Note from (2) that the short-selling constraint is more likely to bind on type h the more *pessimistic* their price expectation $\tilde{E}_{t,h}[p_{t+1}]$.

2.2 Price beliefs

We consider generic price beliefs which are boundedly-rational and may depend linearly on the current price p_t; in particular, the current price may be a common 'reference point'.

Assumption 1 *All price beliefs are of the form:*

$$\tilde{E}_{t,h}[p_{t+1}] = \overline{c} p_t + \tilde{f}_{t,h} \tag{3}$$

where $\overline{c} \in [0, 1 + \tilde{r})$ and $\tilde{f}_{t,h} \in \mathbb{R}$ is a generic forecast that cannot depend on current price p_t.

[2] We assume (as is standard) that these operators satisfy some basic properties of conditional expectation operators, namely, $\tilde{E}_{t,h}[y_t] = y_t$ and $V\tilde{a}r_{t,h}[y_t] = 0$ for any variable y_t that is determined at date t; $\tilde{E}_{t,h}[x_{t+1} + y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$ for any variables x and y; and $V\tilde{a}r_{t,h}[x_t y_{t+1}] = x_t^2 V\tilde{a}r_{t,h}[y_{t+1}]$.

[3] See the definition of $f_{t,h}$ in (4), which potentially allows $\tilde{E}_{t,h}[d_{t+1}]$ to vary across time and types.

Assumption 1 allows a wide range of boundedly-rational beliefs. The coefficient $\bar{c}$ allows linear dependence of price expectations on the current price; for example, investors may extrapolate on top of the current price (Barberis et al., 2018) or use the current price as a reference point (see LeBaron et al., 1999; Westerhoff, 2004). We assume $\bar{c} < 1 + \tilde{r}$ to ensure that individual demands are decreasing in the current price p_t. *Time-varying or heterogeneous values of* $\bar{c}$ are discussed as an extension in Section 3.2.

Only linear dependence on the current price p_t is allowed, but the generic forecast $\tilde{f}_{t,h}$ (which can differ across types and over time) permits a potentially non-linear response to past prices (e.g. trend-following) as in Yang (2009). In addition, a wide range of idiosyncrasies are allowed; for example, forecasts $\tilde{f}_{t,h}$ may contain type-specific 'fixed effects', be subject to random disturbances, or be influenced by social networks as in Yang (2009) or Panchenko et al. (2013). Assumption 1 in Brock and Hommes (1998) is nested when $\bar{c} = 0$ and $\tilde{f}_{t,h} = E_t[p^*_{t+1}] + g_h(x_{t-1}, ..., x_{t-L_h})$, where $g_h : \mathbb{R}^{L_h} \to \mathbb{R}$ is a function that can differ across types, L_h is the lag of type h and $x_t = p_t - p^*_t$ is the price deviation from the fundamental price p^*_t.

For convenience, let $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2\overline{Z}$ and $r := \tilde{r} - \bar{c}$. Given IID dividends, $f_{t,h} = \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and the demands in (2) can be written as

$$z_{t,h} = \begin{cases} \frac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } p_t > \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r}. \end{cases} \tag{4}$$

Using the adjusted price forecast $f_{t,h}$ is convenient because it does *not* depend on the current price p_t and allows us to add the term $a\sigma^2\overline{Z}$ in the numerator of the demand function (see top line of (4)), which simplifies the algebra of the price solution; see Proposition 1. Writing demands this way is also consistent with a 'deviation from fundamentals' representation; see Brock and Hommes (1998) and several other papers in the related literature.[4]

2.3 Population shares

Aggregate demand for the risky asset is $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h}$, where $n_{t,h}$ is the population share of type h at date t. We allow the population shares $n_{t,h}$ to be endogenous and time-varying, but we rule out dependence on the contemporaneous price p_t (see Assumption 2).

Assumption 2 *We consider generic population shares $n_{t,h}$ that satisfy $n_{t,h} \in (0, 1)$ $\forall t, h$ and $\sum_{h\in\mathcal{H}} n_{t,h} = 1$. The shares may be exogenous or endogenously determined, but we rule out dependence of $n_{t,h}$ on the current price p_t (though not on lagged prices p_{t-1}, p_{t-2} etc.).*

Assumption 2 is quite general. For instance, population shares may be determined *endogenously* through an evolutionary competition-type mechanism as in Brock and Hommes

[4]Given our assumptions, the fundamental price is $p^*_t = \overline{p} := (\overline{d} - a\sigma^2\overline{Z})/\tilde{r}$, so $f_{t,h} = \tilde{f}_{t,h} + \tilde{r}\overline{p}$ (see (4)). Thus, $f_{t,h} - (1+r)p_t = \hat{E}_{t,h}[x_{t+1}] - (1+r)x_t$, where $r = \tilde{r} - \bar{c}$, $x_t := p_t - \overline{p}$, and $\hat{E}_{t,h}[x_{t+1}] := \tilde{f}_{t,h} - (1-\bar{c})\overline{p}$. Demands follow (4), with x_t replacing p_t and $\hat{E}_{t,h}[x_{t+1}]$ replacing $f_{t,h}$. For an applied example, see Sec. 5.

(1997, 1998). Following Brock and Hommes (1997), a popular approach is a discrete choice logistic model $n_{t+1,h} = \frac{\exp(\beta U_{t,h})}{\sum_{h\in\mathcal{H}}\exp(\beta U_{t,h})}$, where the intensity of choice $\beta \geq 0$ determines how fast agents switch to better-performing predictors. Various measures of fitness $U_{t,h}$ are used in the literature, including realized profits net of a predictor cost (Brock and Hommes, 1998) and measures of forecast accuracy (e.g. Ap Gwilym, 2010), which may also be used to derive a risk-adjusted measure of profits (De Grauwe and Grimaldi, 2006).

Fixed population shares $n_{t,h} = 1/H$ are relevant for agent-based or social network models where types are *individuals*, whereas exogenous time-varying population shares, as in the herding models of Kirman (1991, 1993), are also straightforward to implement.

3 Benchmark results

The asset market clears when $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ subject to (4) and Assumptions 1–2. Given positive outside supply $\overline{Z} > 0$, there exists a unique market-clearing price p_t (see Anufriev and Tuinstra, 2013, Proposition 2.1). We now characterize the price and demands.

Proposition 1 *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$ and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H}\setminus\mathcal{B}_t$) be the set of unconstrained types (constrained types). Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{1+r} := p_t^* \tag{5}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$, at least one type is short-selling constrained and $\exists$ unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}$, $\mathcal{S}_t^*$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\})$, and the price and demands are given by

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{6}$$

and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

Proof. See the Appendix. ■

Proposition 1 gives the market-clearing price and demands for an arbitrarily large set of belief types; since the proposition applies at an arbitrary date $t \in \mathbb{N}_+$, it allows us to find a solution for $t = 1, 2, ...$, starting from period 1. Note that the asset price depends on *all beliefs* if no types are short-selling constrained (see (5)); however, only the beliefs of the unconstrained (i.e. 'buyers') matter when short-selling constraints are binding (see (6)).

Part (i) gives a simple condition that can be used to check, in a single computation, whether short-selling constraints are slack for all types. If so, then the price is given by p_t^* in (5), which amounts to the usual expression in the heterogeneous-beliefs asset pricing model in the absence of short-selling constraints (see e.g. Brock and Hommes, 1998).

Whether short-selling constraints bind depends on *belief dispersion* relative to the most pessimistic type, i.e. $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\})$. If belief dispersion is small enough relative to the (risk-adjusted) outside supply $\overline{Z}$, then no types are short-selling constrained at date t. Otherwise, we are in Part (ii) of Proposition 1, such that at least one type (and at most $H-1$ types) are short-selling constrained. In this case, the sets of unconstrained and short-selling constrained types ($\mathcal{B}_t^*$, $\mathcal{S}_t^*$) are determined by 'cut-off' conditions which require that for unconstrained types $h \in \mathcal{B}_t^*$, the average belief dispersion *within the group* is sufficiently small relative to outside supply, whereas for the short-selling constrained types $h \in \mathcal{S}_t^*$ this condition of sufficiently small belief dispersion is not met for any type in the set.

Finally, note that when one or more types are short-selling constrained, the market-clearing price p_t is higher than the (hypothetical) price p_t^* if short-selling constraints were absent – i.e. short-selling constraints raise the asset price, as argued by Miller (1977).

We now illustrate these results using a simple two-type example.

Example 1 *Consider two types h_1, h_2 with population shares $n_{t,h_1} \in (0,1)$, $n_{t,h_2} = 1 - n_{t,h_1}$ and generic beliefs f_{t,h_1}, f_{t,h_2} that satisfy Assumption 1 and have the form $f_{t,h} = \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ for $h = h_1, h_2$; see (4) By Proposition 1, if $\sum_{h\in\{h_1,h_2\}} n_{t,h}(f_{t,h} - \min\{f_{t,h_1}, f_{t,h_2}\}) \le a\sigma^2\overline{Z}$ neither type is short-selling constrained, and $p_t = \sum_{h\in\{h_1,h_2\}} n_{t,h} f_{t,h}/(1+r)$ by (5). If the above condition is not met, then either $f_{t,h_1} - f_{t,h_2} > a\sigma^2\overline{Z}/n_{t,h_1}$ (if h_1 is more optimistic) or $f_{t,h_2} - f_{t,h_1} > a\sigma^2\overline{Z}/n_{t,h_2}$ (if h_2 is more optimistic). In the former case, $\mathcal{B}_t^* = \{h_1\}$, $\mathcal{S}_t^* = \{h_2\}$, and by (6) the market-clearing price is $p_t = [(1+r)n_{t,h_1}]^{-1}(n_{t,h_1} f_{t,h_1} - (1-n_{t,h_1})a\sigma^2\overline{Z})$, with demands $z_{t,h_1} = \overline{Z}/n_{t,h_1}$, $z_{t,h_2} = 0$. In the latter case, $\mathcal{B}_t^* = \{h_2\}$, $\mathcal{S}_t^* = \{h_1\}$, so $p_t = [(1+r)n_{t,h_2}]^{-1}(n_{t,h_2} f_{t,h_2} - (1-n_{t,h_2})a\sigma^2\overline{Z})$ and $z_{t,h_1} = 0$, $z_{t,h_2} = \overline{Z}/n_{t,h_2}$.*

Suppose that beliefs follow the two-type Brock and Hommes (1998) model, where $\overline{c} = 0$ such that $r = \tilde{r}$. Type h_1 is a fundamentalist with $\tilde{E}_{t,h_1}[p_{t+1}] = \overline{p}$, where $\overline{p} = (\overline{d} - a\sigma^2\overline{Z})/r$ is the fundamental price, and h_2 is a 1-lag chartist: $\tilde{E}_{t,h_2}[p_{t+1}] = \overline{p} + \overline{g}(p_{t-1} - \overline{p})$, where $\overline{g} > 0$. Note that these beliefs imply that $f_{t,h_1} = (1+r)\overline{p}$ and $f_{t,h_2} = (1+r)\overline{p} + \overline{g}(p_{t-1} - \overline{p})$; see (4). Assuming $p_{t-1} > \overline{p}$, the chartist is more optimistic at date t, and hence by Proposition 1:

$$p_t = \begin{cases} \overline{p} + \dfrac{n_{t,h_2}\overline{g}(p_{t-1} - \overline{p})}{1+r} & \text{if } \overline{g}(p_{t-1} - \overline{p}) \le a\sigma^2\overline{Z}/n_{t,h_2} \\ \overline{p} + \dfrac{n_{t,h_2}\overline{g}(p_{t-1} - \overline{p}) - (1-n_{t,h_2})a\sigma^2\overline{Z}}{n_{t,h_2}(1+r)} & \text{if } \overline{g}(p_{t-1} - \overline{p}) > a\sigma^2\overline{Z}/n_{t,h_2}. \end{cases} \tag{7}$$

The above two-type example is especially simple: if belief dispersion is large enough that some type is constrained, then ranking types by optimism immediately determines the sets of unconstrained types ($\mathcal{B}_t^*$) and short-selling constrained types ($\mathcal{S}_t^*$). In a general setting with many types, however, there are many candidates for the sets $\mathcal{B}_t^*$, $\mathcal{S}_t^*$, and this number increases *exponentially* as the number of types H is increased. In fact, including the case where short-selling constraints are slack for all types, there are $2^H - 1$ different candidates for $\mathcal{B}_t^*$, $\mathcal{S}_t^*$.[5] As a result, the task of finding the price is *computationally intensive* when there are a large number of types H, as seems plausible in many real-world asset markets.

To overcome this problem, we now set out a version of Proposition 1 that reduces the number of candidates that need to be checked and hence is useful for computational purposes. We use the fact that types who are short-selling constrained in a given period t must be *more pessimistic* than those who were unconstrained (see (4)), such that ranking types in terms of optimism is useful. In fact, we have already seen the usefulness of ranking types by optimism in Example 1: knowing that the chartist type was more optimistic allowed us to narrow down to 2 cases for the price rather than 3 ($= 2^2 - 1$) if beliefs were left unordered. We now show how this principle can be applied in a general setting with many types.

Consider the function $\tilde{h}_t : \mathcal{H} \to \tilde{\mathcal{H}}_t$, where $\tilde{\mathcal{H}}_t := \{1, ..., \tilde{H}_t\}$ is an adjusted set of types with the property that the most optimistic type(s) in $\mathcal{H}$ get label $\tilde{H}_t$, the next most optimistic type(s) gets label $\tilde{H}_t - 1$, and so on, down to the least optimistic type(s) in $\mathcal{H}$ with label 1. Types with equal optimism get the *same* label, so $\tilde{H}_t \leq H$, which implies that $|\tilde{\mathcal{H}}_t| \leq |\mathcal{H}|$. In the case of ties, the period t population share of the 'group' is the sum of the population shares of the individual types. We first present a corollary based on the adjusted set of types $\tilde{\mathcal{H}}_t$, before presenting a computationally-efficient algorithm.

Corollary 1 *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set defined above, such that beliefs are ordered as $\tilde{E}_{t,1}[p_{t+1}] < \tilde{E}_{t,2}[p_{t+1}] < ... < \tilde{E}_{t,\tilde{H}_t}[p_{t+1}]$, or equivalently $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$. Let $disp_{t,k} := \sum_{h=k+1}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,k})$, where $k \in \{1, ..., \tilde{H}_t - 1\}$. Then we have the following:*

$$p_t = \begin{cases} \dfrac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{1+r} := p_t^* & \text{if } disp_{t,1} \leq a\sigma^2 \overline{Z} \\ \dfrac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} - n_{t,1} a\sigma^2 \overline{Z}}{(1-n_{t,1})(1+r)} := p_t^{(1)} & \text{if } disp_{t,2} \leq a\sigma^2 \overline{Z} < disp_{t,1} \\ \dfrac{\sum_{h=3}^{\tilde{H}_t} n_{t,h} f_{t,h} - (n_{t,1}+n_{t,2}) a\sigma^2 \overline{Z}}{(1-n_{t,1}-n_{t,2})(1+r)} := p_t^{(2)} & \text{if } disp_{t,3} \leq a\sigma^2 \overline{Z} < disp_{t,2} \\ \vdots & \vdots \\ \dfrac{n_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t-1} n_{t,h}) a\sigma^2 \overline{Z}}{(1-\sum_{h=1}^{\tilde{H}_t-1} n_{t,h})(1+r)} := p_t^{(\tilde{H}_t-1)} & \text{if } disp_{t,\tilde{H}_t-1} > a\sigma^2 \overline{Z} \end{cases} \tag{8}$$

where $p_t^{(k^*)}$ is the price if types $1, ..., k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), and

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{9}$$

[5] The number of candidate sets corresponds to the number of members of the power set of $\mathcal{H}$ minus 1. Intuitively, the power set of $\mathcal{H}$ is the set of all subsets of $\mathcal{H}$, including the empty set. The 'minus 1' correction arises because the asset market cannot clear if $\mathcal{B}_t^*$ were an empty set (i.e. if no agent held the asset).

Proof. See the Appendix. ■

Corollary 1 streamlines the task of finding the market-clearing price. In Proposition 1, where beliefs are unordered, there are $2^H - 1$ cases (regions) to check, as compared to $\tilde{H}_t \leq H$ when belief types are ordered as in Corollary 1. Clearly, this amounts to a substantial reduction in computational burden in models with a large number of types H. For example, with only 15 distinct beliefs (types) at date t, there are $2^{15} - 1 = 32767$ candidates for the sets $\mathcal{B}_t^*, \mathcal{S}_t^*$ if types are not ordered by optimism. However, if we rank types from least to most optimistic and construct the set $\tilde{\mathcal{H}}_t = \{1, , ..., 15\}$, then there are only 15 candidates for the sets and the market-clearing price, corresponding to Corollary 1 when $\tilde{H}_t = 15$.

The final part of Corollary 1 states, first, that the market price when one or more short-selling constraints are binding is higher than the (hypothetical) price p_t^* if short-selling constraint are absent; and, second, that $p_t^* < p_t^{(1)}$ and $p_t^{(1)} < p_t^{(2)} < ... < p_t^{(k^*-1)} < p_t^{(k^*)}$, i.e. the price is smaller in value the fewer short-selling constraints are assumed to be binding. These properties are useful because we can use the unconstrained solution p_t^* to obtain a *lower bound* $\underline{k}$ for the actual number of types k^* who are short-selling constrained, by counting the number of negative (unconstrained) demands at price p_t^*. In a similar way, counting the number of negative (unconstrained) demands at prices $p_t^{(k)}$, for $k < k^*$, will give an improved estimate of k^* when it lies above the lower bound $\underline{k}$.

We now present a computational algorithm which efficiently finds the number of short-selling constrained types k^* and hence the market-clearing price and demands.

3.1 Computational algorithm

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering beliefs as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$ and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,1})$. If $disp_{t,1} \leq a\sigma^2\overline{Z}$, accept $p_t = p_t^*$ as the date t price solution and move to period $t+1$. Otherwise, move to Step 3.

3. Set $p_t^{guess} = p_t^*$ and find the largest k such that $z_{t,k}^{guess} = \frac{f_{t,k} + a\sigma^2\overline{Z} - (1+r)p_t^{guess}}{a\sigma^2} < 0$, and denote this value $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a k^* is found such that $disp_{t,k^*+1} \leq a\sigma^2\overline{Z} < disp_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h}\right] a\sigma^2\overline{Z}}{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}(1+r)}$, and move to period $t+1$.

The above algorithm is efficient for two reasons. First, if the condition in Step 2 is met, no computation time is wasted checking cases where short-selling constraints are binding. Second, if that condition is not met, then using the unconstrained solution p_t^* as a guess gives a lower bound $\underline{k}$ on the number of short-selling constrained types k^*, and hence all cases $k < \underline{k}$ need not be checked. Note that $\underline{k}$ is a lower bound for k^* since $p_t^{(k)} > p_t^*$ for all $k \leq k^*$ (see Corollary 1); that is, the presence of (binding) short-selling constraints raises price relative to the counterfactual scenario where short-selling constraints are absent. Therefore, if types $1, ..., k$ would like to short-sell at price p_t^*, they *must* be short-selling constrained at price $p_t^{(k^*)} > p_t^*$ also (see (4)), implying that that $k^* \geq \underline{k}$.

In practice, we have found that the speed of the computational algorithm can be improved using an iterative procedure. In particular, rather than increasing k in steps of 1 from the initial value $\underline{k}$ as in Step 3 (whenever $\underline{k}$ is not a solution), the algorithm will 'jump' closer to the true number of constrained types k^* by repeatedly replacing p_t^{guess} with $p_t^{(k)}$ (i.e. price based on the current guess of k) in Step 3 and recomputing an updated value of k, say $k = k'$, that equals the number of negative (unconstrained) demands at this price. In other words, we exploit the property that $p_t^{(1)} < ... < p_t^{(k^*-1)} < p_t^{(k^*)}$ to find k^* faster. Our simulations suggest that with a large number of types such as several thousand or more, there is a considerable speed-up with 5-10 iterations of this procedure.[6]

Example 2 *Suppose there are H = 3,000 belief types. We think of each type as an individual with a fixed population share $n_{t,h} = 1/H$ for all h. There are three main groups of investors consisting of 1,000 types each; in each group the same forecast method is used, but the exact forecasts of investors (i.e. beliefs) differ. Trend-followers expect the future change in price to be linked to past changes in price (up to two lags); contrarians believe the recent trend in prices will be reversed; and arbitrageurs expect any deviation between the current price and a perceived fundamental value to be eliminated next period. The first 1,000 types are trend-followers, types 1,001–2,000 are contrarians, and types 2,001–3,000 are arbitrageurs.*

We assume all types use the current price as a reference point (i.e. $\bar{c} = 1$) and have an idiosyncratic random component to beliefs $u_{t,h}$. Beliefs of trend-followers have the form $\tilde{E}_{t,h}[p_{t+1}] = p_t + g_h^1 \Delta p_{t-1} + g_h^2 \Delta p_{t-2} + u_{t,h}$, where $g_h^1, g_h^2 > 0$ and $\Delta p_t = p_t - p_{t-1}$. Contrarians have beliefs $\tilde{E}_{t,h}[p_{t+1}] = p_t + g_h^3 \Delta p_{t-1} + g_h^4 \Delta p_{t-2} + u_{t,h}$, where $g_h^3, g_h^4 < 0$. For arbitrageurs, $\tilde{E}_{t,h}[p_{t+1}] = p_t - g_h^5(p_{t-1} - \overline{p}) + u_{t,h}$, where $g_h^5 > 0$ and $\overline{p}$ is the fundamental price. We set $d_t = \overline{d} = 1.1$, $\tilde{E}_{t,h}[d_{t+1}] = \overline{d}$ for all h, $\tilde{r} = 0.1$, $a = \sigma^2 = 1$ and $\overline{Z} = 0.1$. The fundamental price is therefore $\overline{p} = \frac{\overline{d} - a\sigma^2 \overline{Z}}{\tilde{r}} = 10$. Prior to period 1, the parameters g_h^1 and g_h^2 are drawn from uniform distributions on (0,0.5) and (0,0.2), g_h^3, g_h^4 are drawn from a uniform distribution on (-0.1,0), and g_h^5 is drawn from a uniform distribution on (0.2,0.8). The idiosyncratic shocks $u_{t,h}$ are set at zero in periods 1–10 and are drawn from a normal distribution $\mathcal{N}(0, 0.04^2)$ in all later periods. Finally, all initial prices are set at $\overline{p} + 0.6 = 10.6$.

Figure 1 shows the asset price and the number of short-selling constrained types in a particular simulation of $T = 500$ periods, of which the first 20 periods are shown. The inability to short-sell softens the initial drop in price in period 1 (see left panel), consistent with Proposition 1. Price remains higher in several subsequent periods because many types are

[6] Note that this procedure will not overshoot k^* because the guessed price will remain below the market-clearing price. If $k' = k_{prev}$ or if k^* is reached, the iterations are terminated early using a 'break' command.

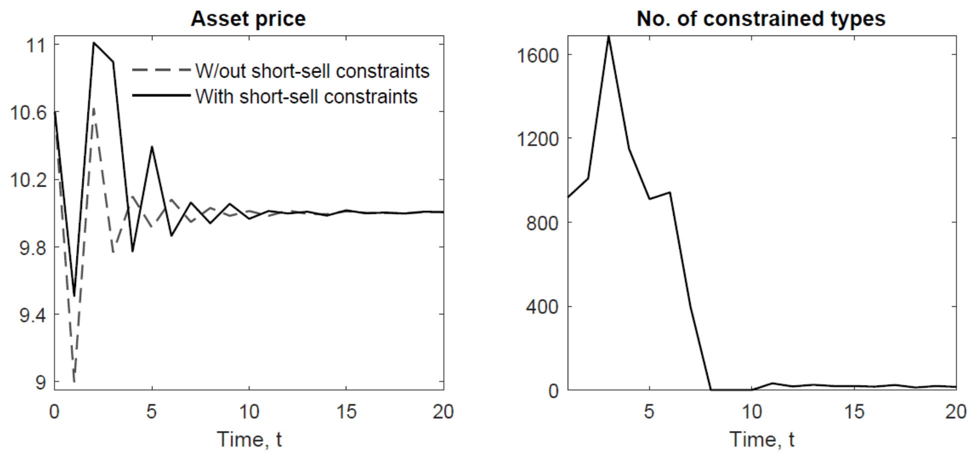

Figure 1: Simulation plotted over the first 20 periods ($H = 3,000$ types)

short-selling constrained until the price dynamics settle. Due to the idiosyncratic belief shocks that 'kick in' after period 10, a small number of types are short-selling constrained in each period even once the price has stabilized (see right panel). The results in Table 1 show that the solution with short-selling constraints is computed quickly using our algorithm, with both computation time and accuracy being comparable to the case where short-selling constraints are absent (which is found without search using the known solution $p_t^* = \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}}{1+r}$*).*[7] *Note that our measure of accuracy is based on the excess demand at the computed market-clearing price in each period, i.e.* $Error_t := |\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$ *for* $t = 1, ..., T$.

Table 1: Computation times and accuracy: $T = 500$ periods, $H = 3,000$ types

Case	Time (s)	Bind freq.	$\max(Error_t)$
W/out short-selling constraints	0.09	-	3.8e-16
With short-selling constraints	0.16	497/500	4.3e-14

Note: $\max(Error_t) = \max\{Error_1, , ..., Error_T\}$, $Error_t := |\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$.

3.2 Discussion

We have derived benchmark analytical results for the case of an arbitrarily large number of belief types subject to unconditional short-selling constraints $z_{t,h} \geq 0$ for all t, h. In addition, we showed that ordering belief types by optimism gives a computationally-efficient solution algorithm. Before turning to extensions, we briefly discuss some cases which are nested by the above results. Any formal results appear in the *Supplementary Appendix*.[8]

[7] The simulations were run in Matlab 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz processor and 8GB of RAM.

[8] The Supplementary Appendix is available at: https://github.com/MCHatcher.

3.2.1 Generalizations and nested cases

First, note the types $h_1, h_2, ..., h_H$ may be interpreted as *individual investors* with population shares $n_{t,h} = 1/H$ in Proposition 1 and Corollary 1. This interpretation is relevant for asset pricing models with many agents that differ in beliefs; for example, agent-based models as in LeBaron et al. (1999) or the *social network* model in Hatcher and Hellmann (2022). An alternative social network model is Panchenko et al. (2013), where updating follows the Brock and Hommes (1998) model except that only the types (and performance) of investors in an agent's *social network* can be observed and adopted in type updating. This case is also nested by the benchmark results; see Panchenko et al. (2013, Eq. 10) and Assumption 2. In these cases, our results apply because the demand schedules in these papers have the same functional form as in (2) and (4), and the beliefs $\tilde{E}_{t,h}[p_{t+1}]$ are nested by Assumption 1.

Second, there has been some interest in the inability to short housing as a possible explanation for rising house prices and market volatility (Shiller, 2015; Fabozzi et al., 2020). One simple approach to model housing as an investment asset is to replace the exogenous expected dividends with exogenous imputed rents from housing (see Bolt et al., 2019). In this case, the analytical results are essentially unchanged as we just require a re-labelling of variables, as shown in Section 2.1 of the *Supplementary Appendix*.

Third, several other cases of interest are essentially nested by the benchmark model, including heterogeneity in expected dividends $\tilde{E}_{t,h}[d_{t+1}]$, which are nested by defining the forecast as $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2\overline{Z}$; a time-varying response of beliefs to the current price, such that $\bar{c}$ is replaced with some $\bar{c}_t \in [0, 1+\tilde{r})$ (see (3)), which may be exogenous or endogenous but cannot depend on the current price p_t;[9] and short-selling constraints of the form $z_{t,h} \geq L$, where $L \leq 0$, such that negative positions are permitted up to some limit. We show how the case $z_{t,h} \geq L$ is nested in Section 2.2 of the *Supplementary Appendix*.

3.2.2 Additional heterogeneity

Assumption 1 allows a common response of beliefs to price via the term $\bar{c}p_t$ in (3). Allowing time-variation in $\bar{c}$ is straightforward (see above), but for the case of heterogeneity across types some extra care is needed. In this case, price beliefs are

$$\tilde{E}_{t,h}[p_{t+1}] = \bar{c}_h p_t + \tilde{f}_{t,h} \tag{10}$$

where $\bar{c}_h \in [0, 1+\tilde{r})$ for all h. Defining $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and $r_h := \tilde{r} - \bar{c}_h$, demands are

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r_h)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h} \end{cases} \tag{11}$$

where the only difference relative to (4) is that r is now *type-specific*.

[9] For example, if agents used adaptive learning, $\bar{c}_t$ would be updated and r is replaced by $r_t = \tilde{r} - \bar{c}_t$.

Equation (11) shows that optimism is no longer determined solely by $f_{t,h}$; however, we can distinguish least and most optimistic types by looking at the term $\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}$, since a given type h will short-sell only if this term is sufficiently small. As a result, we can give an amended version of Proposition 1 in which the sets of unconstrained and short-selling constrained types $\mathcal{B}_t^*, \mathcal{S}_t^*$ depend on $\min_{h\in\mathcal{B}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$ and $\max_{h\in\mathcal{S}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$, rather than $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}$ and $\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ as in the benchmark case. In a similar way, we can give an amended version of Corollary 1 for the case of heterogeneous $\overline{c}_h$ and amend the computational algorithm (Section 3.1) for this case. We provide analytical results, an updated algorithm and a numerical example in Section 4.1 of the *Supplementary Appendix*.

A similar approach can be used when there is heterogeneity in *subjective return variances*; see (1). In this case, the terms $a\sigma^2$ in the denominator of the demand function (2) become type-specific, i.e. $a\sigma_h^2$, and it is convenient to define $\tilde{a}_h := (a\sigma_h^2)^{-1}$ and $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - \overline{Z}/\tilde{a}_h$. The demands of types $h \in \mathcal{H}$ can then be written as

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) & \text{if } p_t \leq \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r} \end{cases} \tag{12}$$

where $r = \tilde{r} - \overline{c}$ as before.

In this case, optimism depends on $f_{t,h} + \overline{Z}/\tilde{a}_h$, i.e. types who short-sell must have lower values of $f_{t,h}+\overline{Z}/\tilde{a}_h$ than those who do not. As a result, it is easy to provide amended versions of Proposition 1 and Corollary 1 and to adjust the computational algorithm. We include these results, and a numerical example, in Section 4.2 of the *Supplementary Appendix*.

4 Extensions

We now present several extensions of the baseline model set out above, including *conditional* short-selling constraints; the case of multiple risky assets; and the case where the price is determined by a market-maker who adjusts price in response to excess demand.

4.1 Conditional short-selling constraints

Thus far we studied *unconditional* short-selling constraints: $z_{t,h} \geq 0$ for all t and h. Such constraints are unconditional because they apply in *all periods* regardless of the evolution of the price. However, in practice, many short-selling restrictions are *conditional* in the sense that short-selling is banned at date t only if a certain price condition was met in the recent past. For instance, the United States had an 'uptick rule' from 1938-2007, which banned short-selling if price fell in the last trading interval; in 2010 this was replaced by an alternative uptick rule which bans short-selling if the price falls 10% or more in a day.

To model *conditional* short-selling constraints, let $g(p_{t-1}, ..., p_{t-K})$ be the 'trigger' for the short-selling constraint, with K being the longest lag in the price that is considered. If $g(p_{t-1}, ..., p_{t-K}) \leq 0$ the short-selling constraint is present at date t; if $g(p_{t-1}, ..., p_{t-K}) > 0$, the short-selling constraint is lifted. Sticking with our two examples, the uptick rule has the form $g^{UR}(p_{t-1}, p_{t-2}) = p_{t-1} - p_{t-2}$, whereas the alternative uptick rule can be written as $g^{AUR}(p_{t-1}, p_{t-2}) = p_{t-1} - p_{t-2} + \kappa p_{t-2} = p_{t-1} - (1-\kappa)p_{t-2}$, where $\kappa = 0.1$ (i.e. 10%).

To nest generic rules, we introduce an indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K}) \leq 0\}}$ which is equal to 1 if the short-selling constraint is present at date t (i.e. if $g(p_{t-1}, ..., p_{t-K}) \leq 0$), and equal to 0 otherwise. The problem of type $h \in \mathcal{H}$ at date t is thus amended from (1) to

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2} V\tilde{a}r_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad \mathbb{1}_t z_{t,h} \geq 0 \tag{13}$$

where $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1+\tilde{r})(w_{t,h} - p_t z_{t,h})$ as before.

With this formulation, investors may take negative positions in periods where the indicator variable is zero (short-selling constraint absent) but are restricted to non-negative positions in periods where the indicator variable is 1 (short-selling constraint present).[10]

The demand of type $h \in \mathcal{H}$ is thus given by

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d} - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d}}{1+\tilde{r}} \text{ or } \mathbb{1}_t = 0 \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d}}{1+\tilde{r}} \text{ and } \mathbb{1}_t = 1. \end{cases} \tag{14}$$

Equivalently, with $r := \tilde{r} - \overline{c}$ and $f_{t,h}$ defined as in (4),

$$z_{t,h} = \begin{cases} \frac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ or } \mathbb{1}_t = 0 \\ 0 & \text{if } p_t > \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ and } \mathbb{1}_t = 1. \end{cases} \tag{15}$$

Note that the only difference in the demand schedules relative to (2) is compound 'if-or' and 'if-and' statements, whose second part depends on the value of the indicator variable. As a result, it is straightforward to apply the same approach as in Proposition 1 and Corollary 1 to find the market-clearing price and demands, as explained in the following remark.

Remark 1 *In the above model with a conditional short-selling constraint, the market-clearing price and demands follow Proposition 1, except that in part (i) the 'if...' statement is replaced by 'if...or $\mathbb{1}_t = 0$', and in part (ii) the 'if...' statement is replaced by 'if...and $\mathbb{1}_t = 1$'. A proposition and proof are provided in Section 3.1 of the Supplementary Appendix.*

[10] When $\mathbb{1}_t = 1$, condition $\mathbb{1}_t z_{t,h} \geq 0$ simplifies to $z_{t,h} \geq 0$ as in (1). However, when $\mathbb{1}_t = 0$, then $\mathbb{1}_t z_{t,h} \geq 0$ collapses to $0 \geq 0$ (which is always satisfied) and hence $z_{t,h}$ is not constrained in periods where $\mathbb{1}_t = 0$. For more details, see Section 1.2 of the Supplementary Appendix.

4.2 Multiple asset markets

Suppose there are multiple risky assets $M \geq 2$ in positive net supply. Let $z_{t,h}^m$ be the date t demand of type h for asset $m \in \{1, ..., M\}$. Following Westerhoff (2004), we assume type h's demand for asset m depends not only on the expected excess return on asset m, but also on the attractiveness of that asset market relative to others. In particular, suppose a fraction w_t^m of each investor type participates in a given market m, with this fraction determined by comparison with all other markets (see below). Differently from Westerhoff, we allow all M asset markets to have unconditional *short-selling constraints*, such that $z_{t,h}^m \geq 0$ for all m. Each asset market m is assumed to have IID dividends $d_t^m = \overline{d}^m + \epsilon_t^m$, so $\tilde{E}_{t,h}[d_{t+1}^m] = \overline{d}^m > 0$.

Analogous to (2), the demand of type $h \in \mathcal{H}$ in market $m \in \{1, ..., M\}$ is

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{\tilde{E}_{t,h}\left[p_{t+1}^m\right] + \overline{d}^m - (1+\tilde{r})p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{\tilde{E}_{t,h}\left[p_{t+1}^m\right] + \overline{d}^m}{1+\tilde{r}} \\ 0 & \text{if } p_t^m > \frac{\tilde{E}_{t,h}\left[p_{t+1}^m\right] + \overline{d}^m}{1+\tilde{r}} \end{cases} \tag{16}$$

where p_t^m is price in market m and σ_m^2 is the conditional return variance (assumed constant).

The demand function (16) has the same form as in the benchmark case (see (2)), except for the scaling by the share w_t^m that participates in the market. As in Westerhoff (2004), we assume the participation shares w_t^m depend on relative attractiveness of each market A_t^m:

$$w_{t+1}^m = \frac{\exp(\beta A_t^m)}{\sum_{m=1}^M \exp(\beta A_t^m)}, \qquad A_t^m = f([p_t^m - \overline{p}^m]) \tag{17}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function with $f(0) = 0$ and $\beta \geq 0$ is the intensity of choice.

Equation (17) states that the participation shares w_t^m are determined by *evolutionary competition*, with the fitness of a market A_t^m depending on the deviation of the market price from the fundamental price $\overline{p}^m$.[11] The fundamental price in market m is the price that would clear the market if all types $h \in \mathcal{H}$ had common rational expectations $E_t[.]$. Given equations (16)–(17), the fundamental price in market $m \in \{1, ..., M\}$ is:[12]

$$\overline{p}^m = \frac{\overline{d}^m - a\sigma_m^2 M \overline{Z}_m}{\tilde{r}} \tag{18}$$

where $\overline{Z}_m > 0$ is the fixed supply of asset m per investor.

Belief types in each market m follow a market-specific version of (3):

$$\tilde{E}_{t,h}\left[p_{t+1}^m\right] = \overline{c}^m p_t^m + \tilde{f}_{t,h}^m \tag{19}$$

[11] Westerhoff (2004) sets $f([p_t^m - \overline{p}^m]) = \ln\left[(1 + c[p_t^m - \overline{p}^m]^2)^{-1}\right]$, where $c > 0$, such that attractiveness declines with distance from the fundamental price due to the risk of being caught in a bubble that collapses.

[12] If all investors are fundamentalists, then $A_t^m = f(0) = 0 \ \forall m$, such that $w_t^m = 1/M$ for all m. Using this result in conjunction with the demands (16), common expectations $E_t[p_{t+1}^m]$ and market-clearing leads to the equation $p_t^m = (1+\tilde{r})^{-1}[E_t[p_{t+1}^m] + \overline{d}^m - a\sigma_m^2 M \overline{Z}_m]$, which can be solved forwards to give (18).

where $\overline{c}^m \in [0, 1+\tilde{r})$ and $\tilde{f}_{t,h}^m$ is a generic price forecast of type h in market m that does not depend on the current price p_t^m.

Let $f_{t,h}^m := \tilde{f}_{t,h}^m + \overline{d}^m - a\sigma_m^2 \overline{Z}_m / w_t^m$ and $r^m := \tilde{r} - \overline{c}^m$ (see (3)–(4)). Then the demand of type h in market m can be written as

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1+r^m) p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m} \\ 0 & \text{if } p_t^m > \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m}. \end{cases} \tag{20}$$

We assume the population shares in each market m are determined by Assumption 2. Market-clearing in each market is given by

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \overline{Z}_m / w_t^m, \qquad \text{where } \tilde{z}_{t,h}^m := z_{t,h}^m / w_t^m. \tag{21}$$

With the change in variables in (21), the market-clearing condition has the same form as in the benchmark model (except a scaling of supply by $1/w_t^m$). Hence, we have the following.

Remark 2 *In the above model with M risky assets subject to short-selling constraints, the expressions for the market-clearing prices p_t^m and the demands $z_{t,h}^m$ $\forall h \in \mathcal{H}$ in each market $m \in \{1, ..., M\}$ are given by Proposition 1, except that p_t, $f_{t,h}$, r, $\overline{Z}$ must be replaced by p_t^m, $f_{t,h}^m$, r^m, $\overline{Z}_m / w_t^m$, and the demands $z_{t,h}$ are replaced by market-specific demands $z_{t,h}^m$ in (20). A proposition is provided in Section 3.2 of the Supplementary Appendix.*

4.3 Market-maker approach

We now return to the case of one risky asset and let price be determined by a market-maker rather than market-clearing; see e.g. Beja and Goldman (1980), Chiarella (1992), Farmer and Joshi (2002) and Westerhoff (2003). As is standard in the literature, we consider price impact functions which are linear in excess demand. We allow the price to potentially depend on both current and past excess demand as follows:[13]

$$p_t = p_{t-1} + \mu[\lambda(Z_t - \overline{Z}) + (1-\lambda)(Z_{t-1} - \overline{Z})] \tag{22}$$

where $\mu > 0$, $\lambda \in (0, 1]$ and $Z_t := \sum_{h \in \mathcal{H}} n_{t,h} z_{t,h}$ is aggregate demand per investor at date t, such that $Z_t - \overline{Z}$ can be interpreted as (average) excess demand per investor.

When $\lambda \in (0, 1)$, past demand matters for the current price, whereas if $\lambda = 1$ only current demand $Z_t = \sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h}$ matters. We stick with the beliefs $\tilde{E}_{t,h}[p_{t+1}] = \overline{c} p_t + \tilde{f}_{t,h}$ in Assumption 1 and consider two specifications of asset demands. In the first case we work with the demands considered thus far; see (2) and (4). In the second case we allow a different demand specification as in some models that use the market-maker approach.

[13] Allowing price to be a nonlinear function of *past* excess demand $Z_{t-1} - \overline{Z}$ does not pose any difficulty as this variable is predetermined at date t; however, we use linearity in current excess demand to solve for p_t.

4.3.1 Benchmark demand specification

We first consider demands as in (4), with $f_{t,h} = \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and $r = \tilde{r} - \overline{c}$. We can easily solve for the price and demands in this case as shown in Proposition 2.

Proposition 2 *Let p_t be the price given by (22) at $t \in \mathbb{N}_+$ and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained (short-selling constrained) types. Then the following holds:*

1. If $p_{t-1} - \frac{1}{1+r}\min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$, $z_{t,h} \geq 0\ \forall h$) and price is given by

$$p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}}.$$

2. If $p_{t-1} - \frac{1}{1+r}\min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} > (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then one or more types are short-selling constrained with $z_{t,h} = 0$ and we have the following:

(i) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\min\limits_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1} < \frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{S}_t^*} n_{t,h}(f_{t,h} - \max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, price is

$$p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h}}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^*$.

(ii) Else, $\exists \mathcal{B}_t^* = \emptyset, \mathcal{S}_t^* = \mathcal{H}$ such that $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z}$, all types are constrained ($z_{t,h} = 0\ \forall h$), and price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

Proof. See the Appendix. ■

There are *three* distinct cases in Proposition 2, since it is possible that *all types* will be short-selling constrained at the price set by the market maker. By contrast, with market-clearing at least one type must buy the risky asset (see Proposition 1).

Analogous to the results in Proposition 1, cases 2(i) and 2(ii) in Proposition 2 imply a higher price than when short-selling constraints are absent (in which case price is given by the expression in Part 1 of Proposition 2). The intuition for this result is simple: looking at (22) we see that, given predetermined past excess demand $Z_{t-1} - \overline{Z}$ and price p_{t-1}, the current price increases with current excess demand, $Z_t - \overline{Z}$. Since excess demand is unambiguously smaller if short-selling is permitted, the current price is also smaller.

The benchmark *computational algorithm* can easily be amended to fit the market-maker approach. In particular, to check for short-sellers in period t, we obtain the set $\tilde{\mathcal{H}}_t$ and then Step 2 of the algorithm is amended to $disp_{t,1} \leq (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1}$, where $disp_{t,1} := \frac{\mu\lambda}{a\sigma^2} \sum_{h \in \tilde{\mathcal{H}}_t} n_{t,h}(f_{t,h} - f_{t,1}) - \frac{1}{1+r} f_{t,1}$ (see Proposition 2, Part 1). If this condition is satisfied, then the price follows Proposition 2 Part 1.

If the above condition is not satisfied, then further steps are needed. Following Proposition 2 Part 2(i), we search for a $k^* \in \{1, ..., \tilde{H}_t - 1\}$ such that $disp_{t,k^*+1} \leq (\mu + (1+r)^{-1} a\sigma^2)\, \overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1} < disp_{t,k^*}$, where $disp_{t,k} := \frac{\mu\lambda}{a\sigma^2} \sum_{h>k} n_{t,h}(f_{t,h} - f_{t,k}) - \frac{1}{1+r} f_{t,k}$. If such a k^* exists, the price p_t is given by the expression in Proposition 2 Part 2(i) .

Finally, if there is no k^* that satisfies the above condition, then *all types* are short-selling constrained in period t. By Proposition 2, Part 2(ii), this is the case if $disp_{t,\tilde{H}_t} > (\mu + (1+r)^{-1} a\sigma^2)\, \overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1}$ and the price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

4.3.2 Alternative demand specification

We now consider an alternative demand specification, as used in several papers in the market-maker literature; this section describes the solution for the price and demands in this case.

A common specification for demand is $\tilde{a}_h(\tilde{E}_{t,h}[p_{t+1}] - p_t)$, where $\tilde{a}_h > 0$; see, for example, Westerhoff (2004). With a short-selling constraint $z_{t,h} \geq 0$, the demands are adjusted to:

$$z_{t,h} = \begin{cases} \tilde{a}_h(\tilde{E}_{t,h}[p_{t+1}] - p_t) & \text{if } p_t \leq \tilde{E}_{t,h}[p_{t+1}] \\ 0 & \text{if } p_t > \tilde{E}_{t,h}[p_{t+1}]. \end{cases} \tag{23}$$

The key difference relative to (2) is that demand is scaled by the *type-specific* coefficient $\tilde{a}_h$. Note that more pessimistic types – i.e. those with lower expectations $\tilde{E}_{t,h}[p_{t+1}]$ – are more likely to be short-selling constrained at a given price p_t set by the market-maker.

We let $f_{t,h} := \tilde{f}_{t,h} = \tilde{E}_{t,h}[p_{t+1}] - \bar{c} p_t$ (see (3)) and write the demands in (23) as:

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} - (1+r)p_t) & \text{if } p_t \leq \dfrac{f_{t,h}}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h}}{1+r} \end{cases} \tag{24}$$

where $r := -\bar{c}$ and we assume $\bar{c} \in [0, 1)$ to ensure demands are decreasing in the price.

The demands in (24) match (4), except the 'intercept' $a\sigma^2\overline{Z}$ is absent, the expected dividend $\overline{d}$ is absent, and the scaling $\tilde{a}_h$ is type-specific. As a result, the price solution and demands are similar to Proposition 2, as summarized in the following remark.

Remark 3 *When demands are given by* (24) *and a market-maker sets price following* (22)*, the price solution and demands follow Proposition 2, except $n_{t,h}$ is replaced by $\tilde{n}_{t,h} := n_{t,h}\tilde{a}_h$ (for all h), terms in $a\sigma^2$ are set at 1, and the coefficient μ is multiplied by $\overline{Z}$ in all expressions. A proposition and proof are given in Section 3.3 of the Supplementary Appendix, along with an amended version of Corollary 1 and the computational algorithm.*

5 Application: Alternative uptick rule

We now consider an application based on an alternative uptick rule, as currently in place in the United States. Under the rule, short-selling is banned following price falls of 10% or more. This contrasts with the original uptick rule in place from 1938 to 2007, which banned short-selling of shares after any fall in prices (regardless of the magnitude). We work with a version of the Brock and Hommes (1998) model with a large number of types and an alternative uptick rule; this case does not seem to have been studied previously.[14]

Since the alternative uptick rule bans short-selling following price falls of 10% or more (but *not* otherwise) it is a *conditional* short-selling constraint; see Section 4.1. Accordingly, the indicator variable has the form $\mathbb{1}_t = \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\leq 0\}}$ for $\kappa = 0.1$, such that the short-selling constraint is present only if the price fell 10% or more in the previous period, and the solution is described by Remark 1. Demands of types $h \in \mathcal{H}$ are given by (14):

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d} - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d}}{1+\tilde{r}} \text{ or } \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\leq 0\}} = 0 \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d}}{1+\tilde{r}} \text{ and } \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\leq 0\}} = 1. \end{cases} \tag{25}$$

where we have assumed IID dividends $d_t = \overline{d} + \epsilon_t$ with $\tilde{E}_{t,h}[d_{t+1}] = \overline{d}\ \forall t, h$.

Equation (25) shows that short-selling is banned in period t only if $p_{t-1} \leq (1-\kappa)p_{t-2}$. Following Brock and Hommes (1998), we consider linear predictors of the form:

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{p} + b_h + g_h(p_{t-1} - \overline{p}) \tag{26}$$

where $b_h \in \mathbb{R}$ and $g_h \geq 0$.

Equation (27) is a standard specification in the literature. The intercept term consists of the (expected) fundamental price $\overline{p}$ plus 'bias' b_h in the price forecast of type h (relative to the fundamental benchmark), whereas the g_h parameter represents the degree of trend-following of type h. Type h is a pure fundamentalist investor if $b_h = g_h = 0$, while larger values of g_h or $|b_h|$ imply, respectively, stronger trend-following and forecast bias.

The fundamental price $\overline{p}$ is the unique fundamental solution under common rational expectations; see Brock and Hommes (1998). Given that the risky asset is in positive net supply $\overline{Z} > 0$, the fundamental price is $\overline{p} = (\overline{d} - a\sigma^2\overline{Z})/r$, where $r = \tilde{r}$ is the interest rate on the riskless asset.[15] Writing the predictor in (26) in price deviations $x_t := p_t - \overline{p}$ gives:

$$\hat{E}_{t,h}\left[x_{t+1}\right] = b_h + g_h x_{t-1} \tag{27}$$

where $\hat{E}_{t,h}\left[x_{t+1}\right] := \tilde{E}_{t,h}\left[p_{t+1}\right] - \overline{p}$.

In a similar vein, the indicator variable can be written in terms of price deviations as

[14] The original uptick rule has been studied for a small number of types (see Dercole and Radi, 2020).

[15] Recall that $r := \tilde{r} - \overline{c}$; see (4). For the predictors in (26), $r = \tilde{r}$ as there is no response to p_t (i.e. $\overline{c} = 0$).

$\mathbb{1}_t = \mathbb{1}_{\{x_{t-1}+\kappa\overline{p}\leq(1-\kappa)x_{t-2}\}}$. Therefore, demands can be written as:[16]

$$z_{t,h} = \begin{cases} \dfrac{\hat{E}_{t,h}[x_{t+1}] - (1+r)x_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t \leq \dfrac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \vee \mathbb{1}_{\{x_{t-1}+\kappa\overline{p}\leq(1-\kappa)x_{t-2}\}} = 0 \\ 0 & \text{if } x_t > \dfrac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \wedge \mathbb{1}_{\{x_{t-1}+\kappa\overline{p}\leq(1-\kappa)x_{t-2}\}} = 1. \end{cases} \tag{28}$$

We consider a standard specification for fitness $U_{t,h}$ whereby performance is a linear function of past profits net of predictor costs $C_h \geq 0$. Profits are given by scaling the realized excess return $R_t := p_t + d_t - (1+r)p_{t-1} = x_t + a\sigma^2\overline{Z} - (1+r)x_{t-1} + \epsilon_t$ by demand $z_{t-1,h}$, where ϵ_t is the IID dividend shock, and we abstract from memory of past performance. For all $t \geq 1$ fitness and population shares are given by

$$U_{t,h} = R_t z_{t-1,h} - C_h, \qquad n_{t+1,h} = \frac{\exp(\beta U_{t,h})}{\sum_{h\in\mathcal{H}} \exp(\beta U_{t,h})}, \tag{29}$$

where $\beta \geq 0$ and $n_{t,h} \in (0,1)$ is given by a logistic updating equation.

Note that the fitness levels $U_{t,h}$ determine the population shares $n_{t+1,h}$ of each type according to the discrete-choice logistic model with intensity of choice β. The intensity of choice determines how fast agents switch toward the better-performing predictors, i.e. those with higher past profit net of predictor costs. In the special case $\beta = 0$ no switching occurs; increasing β implies more switching to profitable predictors. Following Brock and Hommes (1997, 1998), this *evolutionary competition* mechanism has been widely used.

We give the model the same parameters as in Section 3.1 of Anufriev and Tuinstra (2013): $\overline{Z} = 0.1$, $a\sigma^2 = 1$, $r = 0.1$, and we set $\overline{d} = 0.6$, giving a fundamental price $\overline{p} = \frac{\overline{d}-a\sigma^2\overline{Z}}{r} = 5$. In their model there are two types: a fundamentalist type with $\hat{E}_{t,f}[x_{t+1}] = 0$ and cost parameter $C = 1$, and a chartist type with $\hat{E}_{t,c}[x_{t+1}] = \overline{g}x_{t-1}$, where $\overline{g} = 1.2$, and cost 0. We consider a large number of investor types $H = 1{,}000$, with predictors described by (27) and predictor costs depending on the 'closeness' of beliefs to a pure fundamentalist.

5.1 Benchmark exercise

We first perform a sanity check by giving 500 types the same (pure) fundamental predictor (at cost $C = 1$) and the remaining 500 types the same chartist predictor $\overline{g} = 1.2$ at no cost. In this case there are two 'groups' in the population whose (aggregate) population shares are endogenously determined based on fitness. As a result, we should replicate the numerical bifurcation results in Anufriev and Tuinstra (2013) for the case of a *two-type* model in which the price deviation x_t is studied as intensity of choice parameter β is increased. Following

[16] Since $\overline{c} = 0$, $f_{t,h} = \tilde{E}_{t,h}[p_{t+1}] + \overline{d} - a\sigma^2\overline{Z} = \tilde{E}_{t,h}[p_{t+1}] + r\overline{p}$, where $\overline{p} = (\overline{d} - a\sigma^2\overline{Z})/r$ is used. Since $\overline{d} - r\overline{p} = a\sigma^2\overline{Z}$, substituting $\tilde{E}_{t,h}[p_{t+1}] = f_{t,h} - r\overline{p}$ in (25) gives $(a\sigma^2)^{-1}(f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z})$. Adding and subtracting $(1+r)\overline{p}$ gives $(a\sigma^2)^{-1}(\hat{E}_{t,h}[x_{t+1}] - (1+r)x_t + a\sigma^2\overline{Z})$ as in (28), where $\hat{E}_{t,h}[x_{t+1}] := \tilde{E}_{t,h}[p_{t+1}] - \overline{p}$ and $x_t = p_t - \overline{p}$. This expression is in the form of (4) except p_t, $f_{t,h}$ are replaced by x_t, $\hat{E}_{t,h}[x_{t+1}]$.

Anufriev and Tuinstra (2013, Fig. 5), we study the deterministic skeleton with $d_t = \overline{d}$ and the case of no short-selling constraint (we do not present a separate diagram for an alternative uptick rule as we found no substantive difference in the attractors).

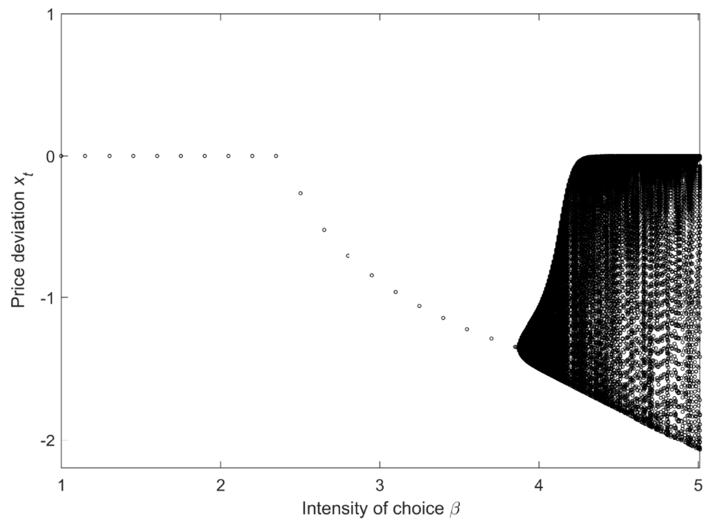

Figure 2: Bifurcation diagram in the absence of short-selling constraints. For each β, we plot 300 points following a transitory of 3,000 periods from given initial values $x_0 \in (-4, 0)$.

Figure 2 shows that for sufficiently low values of the intensity of choice, the fundamental steady state $x = 0$ is the unique price attractor. Intuitively, this is because we are in the case $(1 + r) < \overline{g} < 2(1 + r)$ and positive outside supply, for which Anufriev and Tuinstra (2013, Proposition 3.1) show that the fundamental steady state is globally stable for sufficiently small values of the intensity of choice, β. Once a critical value of β is exceeded, there initially exist two non-fundamental steady states in addition to the fundamental steady state, which is locally stable. As β is increased further, however, the fundamental steady state becomes unstable, while the non-fundamental steady states are locally stable if β is not too large.

Given negative initial price, only the non-fundamental steady state with $x < 0$ is an attractor for the price dynamics at intermediate values of β; this amounts to the lower 'fork' seen for β between (approx.) 2.4 and 3.8 in Figure 2. Increasing β further causes the non-fundamental steady states to lose their stability through a Neimark-Sacker bifurcation, leading to an invariant closed curve and (quasi-)periodic dynamics. The results in Figure 2 (top panel) are consistent with those in Anufriev and Tuinstra (2013) for the same parameter values. Note that while we obtained the above diagram using $H = 1,000$ types rather than two, we effectively have a two-type model as groups are homogeneous.

5.2 Simulated time series: four scenarios

We now introduce heterogeneity proper by having many different types and present some simulated time series generated by the model. We consider four different scenarios where the initial price x_0 is held fixed and only the intensity of choice β or the degree of heterogeneity (in g_h, b_h and C_h) are changed.[17] We first present simulated price series (without noise) in four scenarios, and we then report some results on computation speed and accuracy when stochastic dividend shocks are present. Finally, we consider some distributional implications of an alternative uptick rule by simulating the wealth distribution across types.

5.2.1 Four price simulations

The simulated price series in the four scenarios (S1–S4) are presented in Figure 3. All four time series are started from the same initial price $x_0 = 3$ and we assume deterministic dividends $d_t = \overline{d} = 0.6$ for all t in order to focus on the underlying dynamics. The four scenarios correspond to: heterogeneity among the 500 fundamental types with $g_h = 0$ due to bias b_h which is linearly-spaced on the interval $[-0.2, 0.2]$ and predictor costs $C_h = 1 - |b_h|$ for such types (S1); the same setting as S1 except that heterogeneity is increased such that $b_h \in [-0.4, 0.4]$ (S2); the same setting as S1 except that the intensity of choice is increased from $\beta = 3$ to $\beta = 4.5$ (S3); and the same setting as S3 except that chartists are also heterogeneous with g_h drawn from a uniform distribution on the interval $(1, 1.4)$.

We see that the price paths in these cases are quite different even though the additional heterogeneities are fairly small. In Scenario 1 (Figure 3, top left), we see that if short-selling constraints are absent, then the price quickly falls towards its fundamental value and then slowly converges on a non-fundamental steady state $x < 0$ (black line). Under an alternative uptick rule, by comparison, the initial drop in price is halted because the short-selling constraint binds; the price then oscillates around this higher value before converging on a non-fundamental steady state with $x > 0$. Thus, the alternative uptick rule leads to a different long run outcome and convergence to quite a different steady state − in this case one where the asset is somewhat overvalued. In Scenario 2, only the degree of bias among fundamentalists is increased, but this is enough to ensure that price converges on the same non-fundamental steady state in both cases (top right). Thus, long run price implications of the alternative uptick rule are absent in this case, though in the short run the drop in price is less severe and price stays higher under the alternative uptick rule (dashed line).

In Scenario 3 (bottom left), the intensity of choice β is set at 4.5 rather than at 3, and this is the only difference relative to Scenario 1. In this case the dynamics settle on permanent price oscillations. However, the short run price dynamics under an alternative uptick rule are quite different, with an initial price spike after the short-selling constraint first binds, which arises because the short-selling constraint binds on many types simultaneously. Lastly, in Scenario 4, heterogeneity in chartists is added on top of Scenario 3. In this case the reversal in price under an alternative uptick rule is reinforced by trend-following into a permanent price 'bubble' where the asset price diverges to $+\infty$. By comparison, price converges on a non-fundamental steady state $x < 0$ when short-selling constraints are absent, and hence the explosive price dynamics can be attributed to the short-selling regulation.

[17] All other parameters are the same as in the previous section, so $a\sigma^2 = 1$, $\overline{d} = 0.6$, $r = 0.1$ and $\overline{Z} = 0.1$.

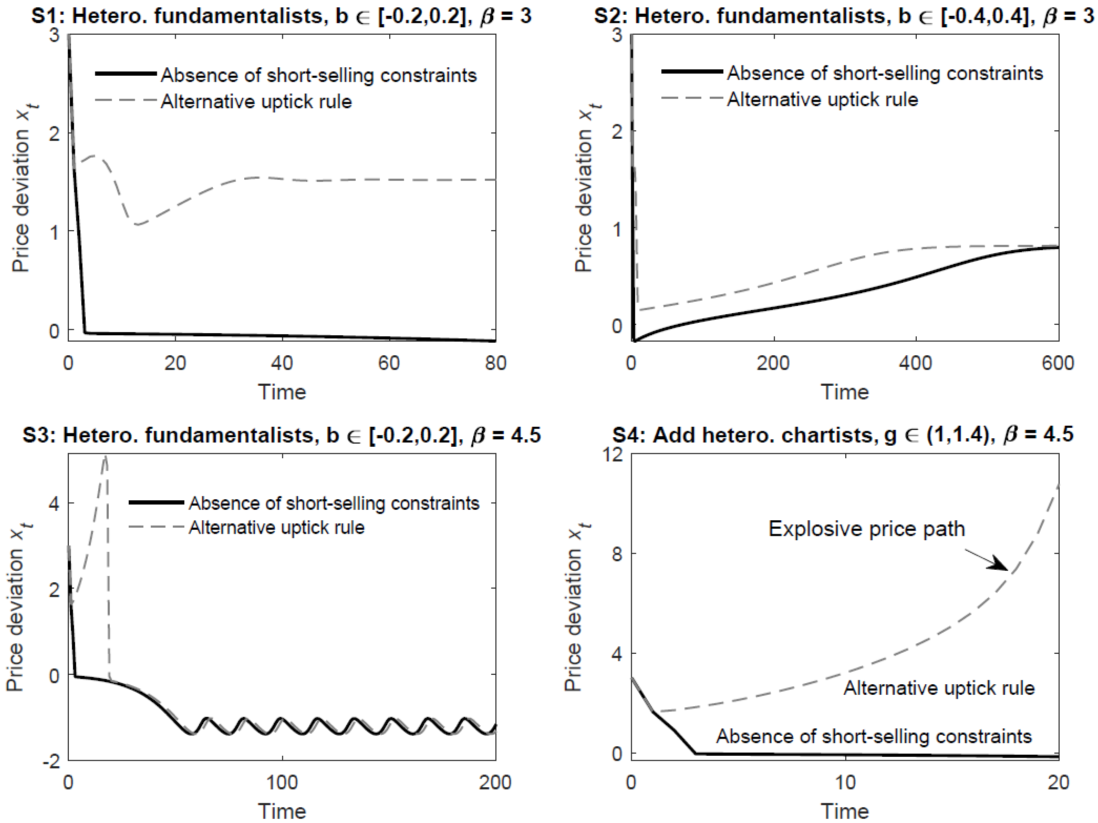

Figure 3: Simulated price series in four scenarios from an initial value $x_0 = 3$.

Computation speed and accuracy. To give an idea of computation speed and accuracy, Table 2 reports simulation times for Scenario 3 when the simulation length is 500 periods and the number of types increased from $H = 1,000$ to $H = 10,000$ and then to $H = 50,000$. We simulate with stochastic dividend shocks $d_t = \overline{d} + \epsilon_t$ and allow the coefficient κ to also take on the value of $\kappa = 0$ (original uptick rule), so that short-selling constraints bind more frequently.[18] We also include a measure of accuracy based on the absolute difference between demand and supply at the computed price, i.e. $Error_t := |\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}|$.

The results show that the solution algorithm is fast and accurate. The final column confirms that excess demand is essentially zero in all simulations, and the accuracy here is similar to when short-selling constraints are absent (top rows), in which case the standard analytical solution $x_t = (1+r)^{-1}\sum_{h\in\mathcal{H}} n_{t,h}\hat{E}_{t,h}[x_{t+1}]$ is used to compute the price and the simulation error. Simulation times are below one second in all cases, increase with the number of types H, and are higher under the original uptick rule (where $\kappa = 0$), since this

[18]Dividend shocks ϵ_t are drawn from a truncated-normal distribution with mean zero, standard deviation $\sigma_d = 0.01$ and support $[-\overline{d}, \overline{d}]$. We use the same draws of shocks in each simulation in Table 2. Simulations were run in Matlab 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz processor and 8GB of RAM.

Table 2: Computation times and accuracy in Scenario 3

No. of types	Regime	Time (s)	Bind freq.	$\max(Error_t)$
$H = 1,000$	No short-sell constraints	0.02	-	2.4e-14
	Alt. uptick rule: $\kappa = 0.1$	0.03	1/500	3.2e-14
	Orig. uptick rule: $\kappa = 0$	0.05	34/500	4.8e-14
$H = 10,000$	No short-sell constraints	0.17	-	2.3e-13
	Alt. uptick: $\kappa = 0.1$	0.18	1/500	6.3e-13
	Orig. uptick: $\kappa = 0$	0.25	43/500	3.0e-13
$H = 50,000$	No short-sell constraints	0.82	-	1.2e-12
	Alt. uptick: $\kappa = 0.1$	0.84	1/500	2.3e-12
	Orig. uptick: $\kappa = 0$	0.94	36/500	1.8e-12

Notes: Simulation length $T = 500$ periods. $\max(Error_t) = \max\{Error_1, ..., Error_T\}$, where we define the date t simulation error as $Error_t = |\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$.

causes the short-selling constraint to bind in a much larger number of periods, as shown in the fourth column.[19] Even when the short-selling constraint binds much more frequently, computation times do not increase much and accuracy of the solution is preserved.

5.2.2 Distributional implications

We now consider some distributional effects of an alternative uptick rule. Recall that the evolution of wealth of type h is $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1 + r)(w_{t,h} - p_t z_{t,h})$, such that an alternative uptick rule will affect wealth distribution though its impact on price and demands $z_{t,h}$. Further, note that if the short-selling constraint binds on type h at date t, then $z_{t,h} = 0$ and hence their wealth evolves as $w_{t+1,h} = (1 + r)w_{t,h}$. By being out of the market in period t, type h foregoes potential returns but also avoids the possibility of losses; hence the overall implications for their wealth will depend on whether they would have on average made returns or losses in the absence of an alternative uptick rule.

We stick with the same four scenarios as in Figure 3 but we focus on a measure of wealth inequality across different types.[20] In particular, we plot the Gini coefficient of the wealth distribution across investor types at each date t. We assume all investor types have equal initial wealth, which we set equal to 50. The results are shown in Figure 4.

An alternative uptick rule has mixed effects on wealth inequality across types. In Scenario 1 (Figure 4, top left), the Gini coefficient initially increases and then settles, but there is a smaller increase in inequality if the alternative uptick rule is present because price does not

[19] Recall that $\kappa = 0.1$ implies that the short-selling constraint is not present in period t unless the price fell by 10% or more in the previous period. For $\kappa = 0$, the short-selling constraint will be present following any previous fall in price (regardless of the magnitude), and short-selling is permitted only on an uptick.

[20] That is, we do not describe the wealth distribution of the population, but rather differentials in wealth due to differences in performance of different forecasting strategies.

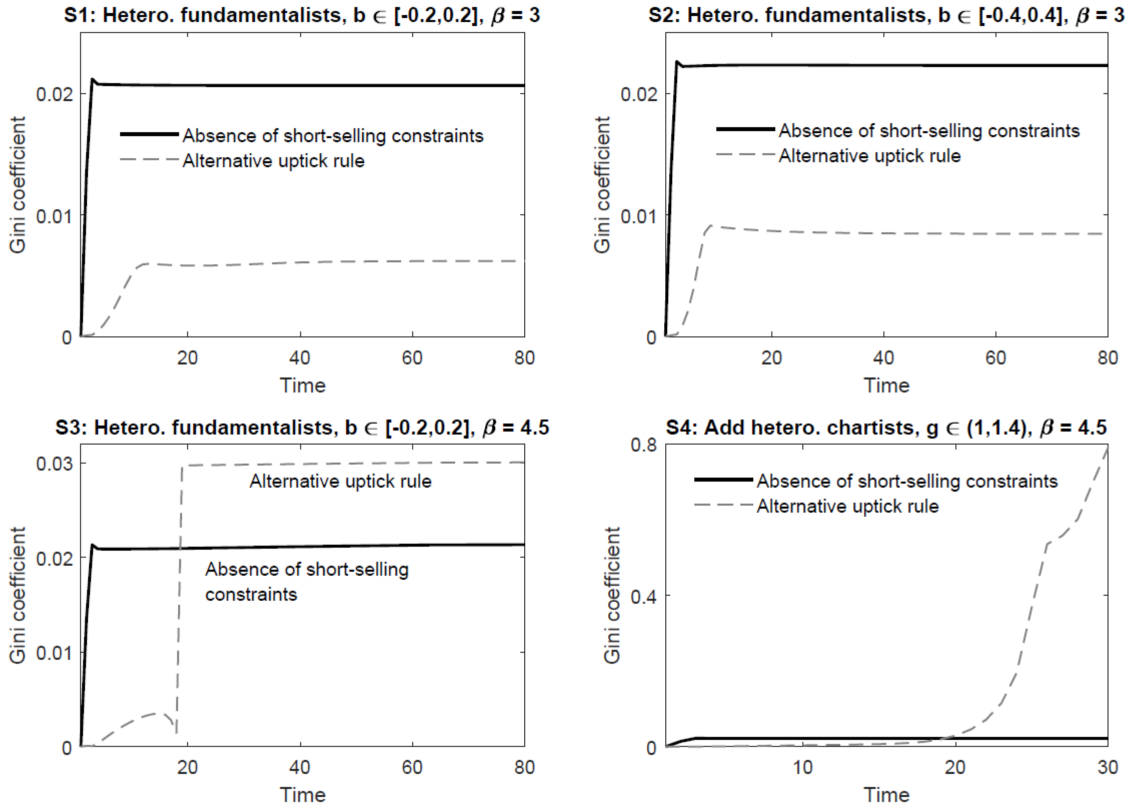

Figure 4: Simulated Gini coefficient of wealth in Scenarios 1 to 4.

fall sharply for several periods (see Figure 3, top left), which benefits fundamental types at the expense of chartist types. Such redistribution is smaller and more gradual under the alternative uptick rule because the fall in asset prices is smaller and, since price stabilizes, inequality remains lower in the long run. The dynamics are similar in Scenario 2 (top right) because the price paths are quite similar to those in Scenario 1.

In Scenario 3, wealth inequality is initially muted under an alternative uptick rule because price rises rather than falls (Figure 3, bottom left). However, this initial period is followed by a severe drop in price, such that more fundamental types outperform more chartist types, and wealth inequality increases before stabilizing (see Figure 4, bottom left). As a result, wealth inequality across types is initially lower under an alternative uptick rule but ends up higher in the long run. Finally, in Scenario 4 (Figure 4, bottom right), wealth inequality across types is initially lower under an alternative uptick rule since the initial period of falling prices is ended as in Scenario 1 (Figure 3, bottom right). However, because price then explodes, chartist types earn profits and fundamental types make losses, such that wealth inequality across types increases dramatically and the Gini coefficient is around 0.8 by period 30.[21]

[21] The 'kink' in period 26 arises because we assume that types that hit negative wealth (in this case more fundamental types) have it reset to zero, and period 26 is the first period in which this rule is triggered.

To better understand the wealth dynamics in Scenario 4, Figure 5 plots the wealth distribution across types in periods $t = 3$, $t = 6$ and $t = 24$ under both unrestricted short-selling (top panel) and an alternative uptick rule (bottom panel). We see that wealth inequalities appear rather quickly under unrestricted short-selling, but not under an alternative uptick rule where the initial fall in price is halted. However, as time increases, the price bubble under the alternative uptick rule soon leads to much greater inequality than if short-selling constraints are absent, and by period 24 an extremely large number of types have wealth levels that are a small fraction of the highest wealth type. These results are consistent with the rapid and sustained increase in the Gini coefficient that is observed in Figure 4.

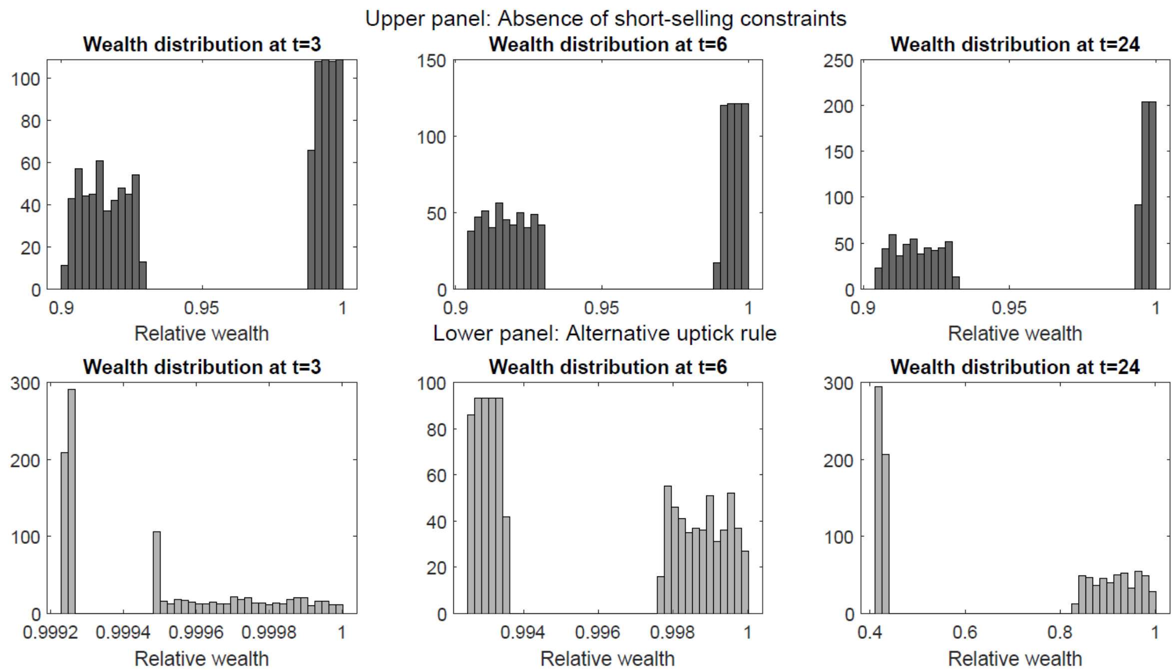

Figure 5: Simulated wealth distribution across types in Scenario 4

6 Conclusion

This paper has studied asset pricing in dynamic, behavioural heterogeneous-belief models with short-selling constraints and many belief types. Our results provide analytical expressions for asset prices along with conditions on beliefs such that short-selling constraints bind on different types, allowing us to construct computationally-efficient solution algorithms. The analysis is built around a version of the Brock and Hommes (1998) model with short-selling constraints and admits a wide range of beliefs; we also presented extensions for conditional short-selling constraints, multiple risky assets, and pricing by a market-maker.

The utility of these results was illustrated using examples and a numerical application that studied an alternative uptick rule, as currently in place in the United States, in a market with a large number of belief types in evolutionary competition. The results highlight a complicated relationship between the design of short-selling regulations and their implications for asset price stabilization, as judged by mispricing relative to fundamentals. We also saw that belief heterogeneity can affect the impact of short-selling constraints on price stability, and that such regulations can have non-trivial distributional (wealth) implications.

There are several promising avenues for future research. First, it would be of interest to investigate whether adding short-selling restrictions in models with many belief types improves the ability of models to reproduce empirical stylized facts, especially during times of market turmoil, when such constraints are more likely to be active. In a similar vein, it may be feasible to estimate such models in order to evaluate the relative empirical contribution of adding short-selling restrictions. Second, from a policy perspective, there has been interest in whether short-selling restrictions lead to mispricing and might cause or exacerbate price bubbles, both in the context of financial markets and other important asset classes such as housing (Shiller, 2015; Fabozzi et al., 2020). The main focus has been price volatility, but it would also be of interest to investigate the distributional implications of short-selling restrictions for income and wealth inequality in models with many agents.

Finally, from a technical perspective, there are some modelling specifications of interest which are not covered by the results presented in this paper. For instance, one could confront a large number of investor types with additional restrictions such as a leverage constraint (see in't Veld, 2016), the elimination of investors who hit low or negative wealth, or margin calls that prevent a short position being maintained in future periods. These approaches might have important implications not just for the price effects of short-selling restrictions, but also their distributional implications that have received little attention so far.

References

Anufriev, M. and Tuinstra, J. (2013). The impact of short-selling constraints on financial market stability in a heterogeneous agents model. *Journal of Economic Dynamics and Control*, 37(8):1523–1543.

Ap Gwilym, R. (2010). Can behavioral finance models account for historical asset prices? *Economics Letters*, 108(2):187–189.

Barberis, N., Greenwood, R., Jin, L., and Shleifer, A. (2018). Extrapolation and bubbles. *Journal of Financial Economics*, 129(2):203–227.

Beja, A. and Goldman, M. B. (1980). On the dynamic behavior of prices in disequilibrium. *Journal of Finance*, 35(2):235–248.

Bolt, W., Demertzis, M., Diks, C., Hommes, C., and Van Der Leij, M. (2019). Identifying booms and busts in house prices under heterogeneous expectations. *Journal of Economic Dynamics and Control*, 103:234–259.

Brock, W. A. and Hommes, C. H. (1997). A rational route to randomness. *Econometrica: Journal of the Econometric Society*, pages 1059–1095.

Brock, W. A. and Hommes, C. H. (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, 22(8-9):1235–1274.

Brock, W. A., Hommes, C. H., and Wagener, F. O. (2005). Evolutionary dynamics in markets with many trader types. *Journal of Mathematical Economics*, 41(1-2):7–42.

Chiarella, C. (1992). The dynamics of speculative behaviour. *Annals of Operations Research*, 37(1):101–123.

De Grauwe, P. and Grimaldi, M. (2006). Exchange rate puzzles: a tale of switching attractors. *European Economic Review*, 50(1):1–33.

Dercole, F. and Radi, D. (2020). Does the "uptick rule" stabilize the stock market? insights from adaptive rational equilibrium dynamics. *Chaos, Solitons & Fractals*, 130:109426.

Fabozzi, F. J., Shiller, R. J., and Tunaru, R. S. (2020). A 30-year perspective on property derivatives: What can be done to tame property price risk? *Journal of Economic Perspectives*, 34(4):121–45.

Farmer, J. D. and Joshi, S. (2002). The price dynamics of common trading strategies. *Journal of Economic Behavior & Organization*, 49(2):149–171.

Hatcher, M. and Hellmann, T. (2022). Networks, beliefs, and asset prices. *SSRN WP*.

in't Veld, D. (2016). Adverse effects of leverage and short-selling constraints in a financial market model with heterogeneous agents. *Journal of Economic Dynamics and Control*, 69:45–67.

Kirman, A. (1991). Epidemics of opinion and speculative bubbles in financial markets. *Money and financial market, Macmillan.*

Kirman, A. (1993). Ants, rationality, and recruitment. *The Quarterly Journal of Economics*, 108(1):137–156.

LeBaron, B., Arthur, W. B., and Palmer, R. (1999). Time series properties of an artificial stock market. *Journal of Economic Dynamics and control*, 23(9-10):1487–1516.

Miller, E. M. (1977). Risk, uncertainty, and divergence of opinion. *The Journal of Finance*, 32(4):1151–1168.

Panchenko, V., Gerasymchuk, S., and Pavlov, O. V. (2013). Asset price dynamics with heterogeneous beliefs and local network interactions. *Journal of Economic Dynamics and Control*, 37(12):2623–2642.

Scheinkman, J. A. and Xiong, W. (2003). Overconfidence and speculative bubbles. *Journal of Political Economy*, 111(6):1183–1220.

Shiller, R. J. (2015). The housing market still isn't rational. *The Upshot, New York Times.*

Siciliano, G. and Ventoruzzo, M. (2020). Banning cassandra from the market? an empirical analysis of short-selling bans during the covid-19 crisis. *European Company and Financial Law Review*, 17(3-4):386–418.

Tramontana, F., Westerhoff, F., and Gardini, L. (2010). On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. *Journal of Economic Behavior & Organization*, 74(3):187–205.

Westerhoff, F. (2003). Heterogeneous traders and the tobin tax. *Journal of Evolutionary Economics*, 13(1):53–70.

Westerhoff, F. H. (2004). Multiasset market dynamics. *Macroeconomic Dynamics*, 8(5):596–616.

Westerhoff, F. H. (2016). The use of agent-based financial market models to test the effectiveness of regulatory policies. In *Agent Based Models for Economic Policy Advice*, pages 195–227. De Gruyter Oldenbourg.

Yang, J.-H. S. (2009). Social network influence and market instability. *Journal of Mathematical Economics*, 45(3-4):257–276.

Appendix

Proof of Proposition 1

A unique market-clearing price exists by Proposition 2.1 in Anufriev and Tuinstra (2013).

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ that $p_t = p_t^* := \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{1+r}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^* \geq 0 \ \forall h \in \mathcal{H}$, which amounts to $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} \leq \min_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z}$. Given $\sum_{h\in\mathcal{H}} n_{t,h} = 1$, the above inequality simplifies to $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$, as stated in Proposition 1.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0 \ \forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h} z_{t,h}$ and hence the market-clearing condition is $\sum_{h\in\mathcal{B}_t^*} n_{t,h} z_{t,h} = \overline{Z}$, which gives $p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^{\mathcal{B}_t^*} \geq 0 \ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^{\mathcal{B}_t^*} < 0 \ \forall h \in \mathcal{S}_t^*$, i.e. iff $(f_{t,h} + a\sigma^2\overline{Z})\sum_{h\in\mathcal{B}_t^*} n_{t,h} \geq (<) \sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z} \ \forall h \in \mathcal{B}_t^* \ (\forall h \in \mathcal{S}_t^*)$, which simplify to $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\})$.

It remains to show $p_t^{\mathcal{B}_t^*} > p_t^* = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{1+r}$, where p_t^* is the price if short-selling constraints are absent. Note $(1+r)(p_t^{\mathcal{B}_t^*} - p_t^*) = (1 - \frac{1}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}}) a\sigma^2\overline{Z} + \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}$ and $\sum_{h\in\mathcal{B}_t^*} n_{t,h} = 1 - \sum_{h\in\mathcal{S}_t^*} n_{t,h}$. Since $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} + \sum_{h\in\mathcal{S}_t^*} n_{t,h} f_{t,h}$, we obtain:

$$(1+r)(p_t^{\mathcal{B}_t^*} - p_t^*) = (\sum_{h\in\mathcal{S}_t^*} n_{t,h}) \left[\frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \frac{\sum_{h\in\mathcal{S}_t^*} n_{t,h} f_{t,h}}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}} \right] > 0$$

where $\sum_{h\in\mathcal{S}_t^*} \frac{n_{t,h}}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}} f_{t,h} \leq \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ and $\frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ is implied by the condition $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) > a\sigma^2\overline{Z}$ above. ■

Proof of Corollary 1

The first 'if' statement follows from Proposition 1 as $\sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,1}) \leq a\sigma^2\overline{Z}$ is equivalent to $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$. The other cases follow as there are $\tilde{H}_t - 1$ candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, i.e. $\mathcal{S}_t = \{1\}, \mathcal{B}_t = \{2, .., \tilde{H}_t - 1\}$; $\mathcal{S}_t = \{1, 2\}, \mathcal{B}_t = \{3, .., \tilde{H}_t - 1\}$;...$\mathcal{S}_t = \{1, ..., \tilde{H}_t - 1\}, \mathcal{B}_t = \{\tilde{H}_t\}$. For arbitrary non-empty sets $\mathcal{S}_t = \{1, ..., k\}$, $B_t = \{k+1, ..., \tilde{H}_t\}$, where $k \in \{1, .., \tilde{H}_t - 1\}$, by market-clearing $p_t = \frac{\sum_{h>k} n_{t,h} f_{t,h} - \left[\sum_{h=1}^{k} n_{t,h}\right] a\sigma^2\overline{Z}}{(1-\sum_{h=1}^{k} n_{t,h})(1+r)} := p_t^{(k)}$ and by Proposition 1 the guess is verified iff $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$. Note that $p_t^{(k^*)} > p_t^* = \frac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{1+r}$ for any $k^* \in \{1, ..., \tilde{H}_t - 1\}$ and $\mathcal{B}_t^*, \mathcal{S}_t^*$ is shown in the Proposition 1 proof.

It remains to show $p_t^* < p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}$ $\forall k < k^*$. If $k^* = 1$ there is nothing to show, and $k^* = 2$ is nested for $k^* > 2$, so let $k^* \geq 3$. Note $p_t^{(1)}$ solves $\sum_{h>1} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(1)}) = a\sigma^2\overline{Z}$ and p_t^* solves $a\sigma^2\overline{Z} = \sum_{h>1} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^*) + n_{t,1}(f_{t,1} + a\sigma^2\overline{z} - (1+r)p_t^*)$, where the last term is < 0 since p_t^* is not verified. So $p_t^{(1)} > p_t^*$. Note $p_t^{(2)}$ solves $\sum_{h>2} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(2)}) = a\sigma^2\overline{Z}$ and $p_t^{(1)}$ solves $a\sigma^2\overline{Z} = \sum_{h>2} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(1)}) + n_{t,2}(f_{t,2} + a\sigma^2\overline{z} - (1+r)p_t^{(1)})$, where the last term is < 0 since $p_t^{(1)}$ is not verified. So $p_t^{(2)} > p_t^{(1)}$. Inductive arguments give $p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}$ $\forall k < k^*$. ■

Proof of Proposition 2

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0$ $\forall h \in \mathcal{H}$, which implies by the price equation that $p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1+\mu\lambda(1+r)(a\sigma^2)^{-1}} := p_t^*$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^* \geq 0$ $\forall h \in \mathcal{H}$, which requires $\left(\frac{1}{1+r} + \frac{\mu\lambda}{a\sigma^2}\right)(a\sigma^2\overline{Z} + \min_{h\in\mathcal{H}}\{f_{t,h}\}) \geq p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})$, giving the inequality in Proposition 2 Part 1.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0$ $\forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0$ $\forall h \in \mathcal{S}_t^* = \mathcal{H} \setminus \mathcal{B}_t^*$, so $p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1+\mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h}}$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t \geq 0$ $\forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t < 0$ $\forall h \in \mathcal{S}_t^*$, which requires $(\frac{1}{1+r} + \mu\lambda(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h})(a\sigma^2\overline{Z} + f_{t,h}) \geq p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}] \geq 0$ (< 0) $\forall h \in \mathcal{B}_t^*$ $(\forall h \in \mathcal{S}_t^*)$, giving the inequality in Proposition 2(i).

Case 2(ii): Short-selling constraint binds for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = 0$ $\forall h \in \mathcal{H}$, which implies that $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t < 0$ $\forall h \in \mathcal{H}$, i.e. iff $\max_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z} < (1+r)(p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}])$, which is the inequality in Proposition 2 Part 2(ii). ■