On Stability of Collaborative Supplier Selection

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Abstract

This note discusses the possibility of fair gain sharing in cooperative situations where players optimally partition themselves across a number of alternative channels. An example is group purchasing among a set of buyers facing with a range of suppliers. We introduce channel selection games as a new class of cooperative games and give a representation of their cores. With two channels (suppliers), the game has a non-empty core if the gain functions across every individual channel is supermodular.

Keywords: Game Theory, Supply Chain Management, Procurement, Gain Sharing

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1. Introduction

In some collaborative situations, the participants (players) organize themselves across a set of alternative channels to maximize their gain. The alternative channels may represent different suppliers, logistics service providers, or adopted technologies. As a special instance, consider buyers in a group purchasing organization who collaboratively select their suppliers for different products that they procure. An example of such a situation was recently studied in Hezarkhani et al. [8] where buyers optimize their purchases across two alternative channels: an intermediary and an original equipment manufacturer.

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The supplier selection problem has been the subject of extensive study in the literature (see for example Ghadimi et al. [5], Yu and Wong [16], Mohammaditabar et al. [9] and references therein). The advantages of collaboration in this context can be intuitive—combining bargaining powers or deliveries results in savings. Yet, the mere existence of economies of scale does not necessarily grants the formation and sustenance of a collaborative organization. From a cooperative game theory point of view, the possibility of sharing the obtained savings among the players in a "fair" manner is a crucially important requirement. A widely adopted notion of fairness in the literature requires that each subgroup of players receives at least as much as they could accrue on their own. Accordingly, an allocation of gains among the players is called stable if it satisfies the latter condition. The core of a cooperative game [13] contains all stable allocations which divide total savings among the players. It is worth mentioning that although the literature often associates the definition of the core to Gillies [6], it was Shapley [13] who first defined the core in its current form [17].

In this note, we introduce channel selection games where coalitions of players optimally partition themselves across a set of given channels. The underlying optimization problems in these games are the same as the winner-determination problem in combinatorial auctions (see for example Sandholm [10]) where a set of products are distributed among a set of bidders with different valuation functions to maximize the sum of all bidders' valuations. However, to the best of our knowledge, previous literature does not study cooperative games with the same structure as ours. Unlike partition games (e.g. Deng et al. [3]), players in channel selection games choose among a fixed number of distinct options with non-homogeneous costs, and dissimilar to assignment games (e.g. Shapley and Shubik [15]), here multiple players can be assigned to a single channel.

We give a representation of the cores of channel selection games in terms of intersections of extended contra-polymatroids associated with gain functions across all channels (e.g. savings obtained by joint purchasing from each supplier), and provide two main observations regarding the existence of allocations in the core of channel selection games. First, if the number of channels is two and the gain function across every channel is supermodular, then the core of the associated game is always non-empty. We prove this result using the generalization of Edmond's matroid intersection theorem [4]. This closes an open problem in Hezarkhani et al. [8] regarding the non-emptiness of the core of games associated with collaborative replenishment situations in the presence of intermediaries. Second, if the number of channels is more than two, then the core of a channel selection game may be empty, even under supermodularity conditions.

2. Preliminaries

Let N be a non-empty finite set, and let $v : 2^N \to \mathbb{R}$ be a set function defined on N. The set function v is:

- *monotonic* if for every $S, T \subseteq N$, $S \subset T$ we have $v(S) \leq v(T)$,
- *additive* if for every $S \subseteq N$, we have $v(S) = \sum_{i \in S} v(\{i\})$,
- *superadditive* if for every $S, T \subseteq N$, $S \cap T = \emptyset$, we have $v(S) + v(T) \leq v(S \cup T)$, and
- *supermodular* if for every $S, T \subseteq N$ we have $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$.

A supermodular set function is also superadditive and a non-negative superadditive set function is also monotonic.

A *cooperative game* is a pair (N, v) with *player set* N and *characteristic function* v, which is a set function defined on N such that $v(\emptyset) = 0$. A pivotal question in cooperative game theory concerns finding appropriate ways to allocate the gains of the grand coalition among the players. Given the game (N, v) an *allocation* $x \in \mathbb{R}^N$ is called *efficient* if $\sum_{i \in N} x_i = v(N)$, and *stable* if for every $S \subseteq N$ we have $\sum_{i \in S} x_i \geq v(S)$. Denote the set of stable allocations for (N, v) with $Q(N, v)$, that is

$$Q(N, v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \forall S \subseteq N : \sum_{i \in S} x_i \geq v(S) \right\}.$$

In polyhedral combinatorics, the set $Q(N, v)$ is referred to as the *extended contra-polymatroid* associated with set function v. The *core* of a cooperative game is the set of all efficient and stable allocations for that game. The core of (N, v) is defined via:

$$\mathcal{C}(N, v) = \left\{ x \in Q(N, v) \,\middle|\, \sum_{i \in N} x_i = v(N) \right\}.$$

With an allocation in the core the value of the grand coalition can be distributed among the players such that each coalition of players receives at least as much as its endogenous value. The core of a cooperative game can be empty. A cooperative game with a supermodular characteristic function is called *convex*. The core of a convex game is always non-empty [12].

3. Channel Selection Games

Suppose a non-empty finite set of channels M exists to select from. For a channel $j \in M$, let $w_j : 2^N \to \mathbb{R}_+$ be channel j's gain function which gives the non-negative and finite gain obtained by every coalition of players that select channel j such that $w_j(\emptyset) = 0$. With the interpretation of channels as suppliers, a gain function specifies per coalition the cost savings from joint purchasing if all players in the coalition buy via this channel. A *channel selection situation* is $(N, M, (w_j)_{j \in M})$ with its elements being defined previously.

For every $S \subseteq N$, let $(T_j)_{j \in M}$ be a partition of S over the channels. That is, for every $j, k \in M$ we have $T_j \cap T_k = \emptyset$ and $\bigcup_{j \in M} T_j = S$. Let $\mathcal{S}$ the set of all such partitions. The characteristic function of the *channel selection game* (N, v) assigns to each coalition $S \subseteq N$ the value:

$$v(S) = \max_{(T_j)_{j \in M} \in \mathcal{S}} \sum_{j \in M} w_j(T_j).$$

Let $(T_j^S)_{j \in M}$ be an optimal partition of S over the channels.

If all w_j are supermodular functions, it is known that the optimization problem above for $|M| = 2$ is solvable in polynomial time [7], but for $|M| \geq 3$ the optimization problem becomes NP-hard [2]. Nevertheless, if all gain functions are supermodular then there would be economies of scale in cooperation and the characteristic function of the associated cooperative channel selection game would at least be superadditive.

Given the definition of gain functions and games, a channel selection game can be regarded as a combination of cooperative games associated with its individual channels. Let (N, w_j) be channel j's individual cooperative game. The first result of this note gives a representation of the core of a channel selection game with respect to the stable allocations of the individual cooperative games associated with all its channels. Define

$$\hat{\mathcal{C}}(N,v)=\left\{x\in\mathbb{R}^N \,\middle|\, \sum_{i\in N}x_i=v(N), x\in\bigcap_{j\in M}Q(N,w_j)\right\}.$$

The set $\hat{\mathcal{C}}(N,v)$ contains all efficient allocations for (N,v) which are stable for all channels' individual games.

Theorem 1. *Given a channel selection situation $(N,M,(w_j)_{j\in M})$ we have $\mathcal{C}(N,v)=\hat{\mathcal{C}}(N,v)$.*

Proof. We proceed in two steps:

[Step 1]: $\mathcal{C}(N,v)\subseteq\hat{\mathcal{C}}(N,v)$: Let $x\in\mathcal{C}(N,v)$ and suppose that for $j\in M$, $x\notin Q(N,w_j)$. This means that there is $S\subseteq N$ such that $\sum_{i\in S}x_i<w_j(S)$. Given the definition of v it follows immediately that for every $j\in M$ and every $S\subseteq N$ we have $v(S)\geq w_j(S)$. Thus, it must be that $\sum_{i\in S}x_i<v(S)$ which contradicts the assumption. Therefore, if $x\in\mathcal{C}(N,v)$ then $x\in\bigcap_{j\in M}Q(N,w_j)$. Since $\sum_{i\in N}x_i=v(N)$ it follows that $\mathcal{C}(N,v)\subseteq\hat{\mathcal{C}}(N,v)$.

[Step 2]: $\hat{\mathcal{C}}(N,v)\subseteq\mathcal{C}(N,v)$: Let $x\in\hat{\mathcal{C}}(N,v)$. By definition we have $\sum_{i\in N}x_i=v(N)$. Let $S\subseteq N$ be an arbitrary coalition and $(T_j^S)_{j\in M}$ an optimal partitioning of S across channels. For every $j\in M$, as $x\in Q(N,w_j)$ we have $\sum_{i\in T_j^S}x_i\geq w_j(T_j^S)$. Hence, we have $\sum_{i\in S}x_i\geq\sum_{j\in M}w_j(T_j^S)=v(S)$ which implies that $x\in\mathcal{C}(N,v)$. Therefore $\hat{\mathcal{C}}(N,v)\subseteq\mathcal{C}(N,v)$.

Combining these two steps concludes that $\mathcal{C}(N,v)=\hat{\mathcal{C}}(N,v)$. □

As the above theorem indicates, the core of a channel selection game is exactly the set of efficient allocations which are situated at the intersection of extended contra-polymatroids associated with its channels' gain functions. In the remainder of this note, we examine nonemptiness of the core of a channel selection game.

4. Case of Two Channels

For the case with two channels, the characteristic function of the game with player set N and channel gain functions w_1 and w_2 can be written as

$$v(S)=\max_{T\subseteq S}\{w_1(T)+w_2(S\setminus T)\}.$$

Theorem 2 in Hezarkhani et al. [8] proves that in two-channel situations if one of the gain functions is supermodular and the other is additive, then the characteristic function of the associated game is supermodular so the game is convex and thus its core is non-empty. The first result of this note extends the latter by presenting a more general sufficient condition for nonemptiness of the cores of channel selection games with two channels.

Theorem 2. *Given a two-channel situation with supermodular gain functions, the core of the associated channel selection game is non-empty.*

Proof. Let $(N, \{1,2\}, (w_1, w_2))$ be a channel selection game and assume that w_1 and w_2 are supermodular. By Theorem 1, to prove the nonemptiness of the core it suffices to show that $\hat{\mathcal{C}}(N, v) \neq \emptyset$.

First we introduce some definitions. A system $Ax \leq b$ in n dimensions is called *totally dual integral* if A and b are rational and for each $c \in \mathbb{Z}^n$, the dual of the program $\max\{c^\intercal x | Ax \leq b\}$, i.e., $\min\{y^\intercal b | y \geq \mathbf{0}, y^\intercal A = c^\intercal\}$, has an integer optimum solution y if it is finite (Schrijver [11], p. 76). A system $Ax \leq b$ is called *box-totally dual integral* if the system $d \leq x \leq e$, $Ax \leq b$ is totally dual integral for each choice of vectors $d, e \in \mathbb{R}^n$ (Schrijver [11], p. 83). A box-totally dual integral system is also totally dual integral.

The following is due to Schrijver [11] (Corollary 46.1d): if w_1 and w_2 are supermodular set functions, then for the intersection of their associated extended contra-polymatroids, i.e.

$$Q(N, w_1) \cap Q(N, w_2) = \left\{ x \in \mathbb{R}^N \middle| \forall S \subseteq N : \sum_{i \in S} x_i \geq w_1(S) \text{ and } \sum_{i \in S} x_i \geq w_2(S) \right\}$$

the underlying system is box-totally dual integral. Note that by changing the sign of the objective function in the original definition, box-totally dual integral property can be expressed in terms of a minimization primal and a maximization dual. This means that given the optimization program:

$$\min_{x \in \mathbb{R}^N} \left\{ \sum_{i \in N} x_i \middle| x \in Q(N, w_1) \cap Q(N, w_2) \right\},$$

the associated dual has integer optimal solutions (note that since w_1 and w_2 are finite, the program above has a finite optimal value.). The dual to the above program is

$$\max_{y_1,y_2\in\mathbb{R}_+^{2^N\setminus\emptyset}}\left\{\sum_{S\subseteq N,S\neq\emptyset} y_1^S w_1(S)+y_2^S w_2(S)\middle|\forall i\in N:\sum_{S\subseteq N,S\ni i} y_1^S+y_2^S=1\right\}.$$

The fact that an integer optimal solution y exists for the above program implies that (a) for every $S\subseteq N, S\neq\emptyset$, we have $y_1^S, y_2^S\in\{0,1\}$; (b) for every $S\subseteq N, S\neq\emptyset$, it holds that $y_1^S y_2^S=0$; and (c) for every $i\in N$ there exists exactly one $S\subseteq N, S\neq\emptyset$, $S\ni i$, such that either $y_1^S=1$ or $y_2^S=1$. The latter means that for every non-empty subsets $S,S^{'}\subset N$, $S\neq S^{'}$, such that $y_k^S=y_k^{S'}=1$, $k\in\{1,2\}$, it must be that $S\cap S'=\emptyset$.

Next, we show that there exists an optimal solution y to the dual wherein for every $k\in\{1,2\}$ there is at most one $S\subseteq N, S\neq\emptyset$, such that $y_k^S\neq 0$. To see this, assume the contrary and consider non-empty pair of subsets $S,S'\subset N$, $S\cap S'=\emptyset$, such that with the optimal solution (y_1,y_2), we have w.l.o.g. $y_1^S=y_1^{S'}=1$. Consider the alternative solution $(\dot{y}_1,y_2)$ which is identical to (y_1,y_2) except that $\dot{y}_1^S=\dot{y}_1^{S'}=0$ and $\dot{y}_1^{S\cup S'}=1$. Since w_1 is supermodular, we have $\sum_{S\subseteq N}\dot{y}_1^S w_1(S)+y_2^S w_2(S)\geq\sum_{S\subseteq N} y_1^S w_1(S)+y_2^S w_2(S)$. Therefore there always exists an optimal dual solution such that for some $T\subseteq N$ we have $y_1^T=1$ and $y_1^S=0$ for all $S\subseteq N$, $S\neq T$, and $y_2^{N\setminus T}=1$ and $y_2^S=0$ for all $S\subseteq N$, $S\neq N\setminus T$. Hence, the optimal value of the dual program is the solution to $\max_{T\subseteq N} w_1(T)+w_2(N\setminus T)=v(N)$. Using the strong duality theorem we get

$$\min_{x\in\mathbb{R}^N}\left\{\sum_{i\in N} x_i\middle|x\in Q(N,w_1)\cap Q(N,w_2)\right\}=v(N).$$

The above implies that there exists $x\in\mathbb{R}^N$ such that $x\in Q(N,w_1)\cap Q(N,w_2)$ and $\sum_{i\in N} x_i=v(N)$. Hence $\hat{\mathcal{C}}(N,v)\neq\emptyset$. □

The proof of Theorem 2 draws upon the generalization of Edmond's matroid intersection theorem [4] to establish the nonemptiness of the cores of channel selection games with two channels whose gain functions are supermodular.

We provide two further remarks with regard to the relationships between the properties of channels' gain functions and their associated channel selection games. Our first observation indicates that non-emptiness of the cores

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
w_1	0	3	0	0	0	3	3	3	0	0	0	3	3	3	0	3
w_2	0	2	0	0	0	4	4	3	4	3	3	6	5	5	4	7
v	0	3	0	0	0	4	4	3	4	3	3	7	6	6	4	7

Table 1: Situation in Example 1

of cooperative games associated with individual channels does not guarantee the same for the corresponding channel selection game because the intersection of stable allocations for the individual channel games may be empty. Before providing a counterexample we present a condition for emptiness of the core.

Lemma 1. *Let (N, v) be a game. The core of (N, v) is empty whenever $\sum_{i \in N} v(N \setminus \{i\}) > (|N| - 1)v(N)$.*

Proof. Let x be an allocation in the core. Given $i \in N$, the stability condition for coalition $N \setminus \{i\}$ requires that $\sum_{j \in N \setminus \{i\}} x_i \geq v(N \setminus \{i\})$. Combined with the efficiency condition, it must be that $x_i \leq v(N) - v(N \setminus \{i\})$ for all $i \in N$. Summing the later inequalities combined with efficiency yields $\sum_{i \in N} v(N \setminus \{i\}) \leq (|N| - 1)v(N)$ as a necessary condition for the existence of allocations in the core. □

The condition in Lemma 1 can also be derived as a special case of Bondavera-Shapley theorem [1, 14] on balanced games. Consider the following example.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $M = \{1, 2\}$. Table 1 gives the values of channels' gain functions w_1 and w_2 (which also correspond to characteristic functions of cooperative games (N, w_1) and (N, w_2) associated with individual channels), as well as the characteristic function v of their associated channel selection game (N, v). Since $(3, 0, 0, 0)$ and $(2, 2, 2, 1)$ are core allocations in games (N, w_1) and (N, w_2) respectively, both individual channel games have non-empty cores. However, the core of (N, v) is empty. In this example we have $v(\{1, 2, 3\}) + v(\{1, 2, 4\}) + v(\{1, 3, 4\}) + v(\{2, 3, 4\}) = 23 > 21 = 3v(N)$. By Lemma 1 the core of (N, v) must be empty. △

Accordingly, we have the following remark.

Remark 1. *Consider the channel selection situation $(N, \{1, 2\}, (w_1, w_2))$. If the games (N, w_1) and (N, w_2) have non-empty cores, (N, v) does not necessarily have a non-empty core.*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
w_1	0	0	0	1	0	0	1
w_2	0	0	0	0	1	0	1
v	0	0	0	1	1	0	1

Table 2: Situation in Example 2

The second observation states that supermodularity of channels' gain functions is not a sufficient condition for convexity of the associated channel selection games. Consider the following example.

Example 2. Let $N = \{1, 2, 3\}$ and $M = \{1, 2\}$. Table 2 gives the values of the channels' gain functions w_1 and w_2 as well as the characteristic function of their associated game in this example. It is easily verifiable that the gain functions over both channels are supermodular. In order for v to be supermodular we must have $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. To test the supermodularity of v, consider the coalitions $S = \{1, 2\}$ and $T = \{1, 3\}$. Observe that the condition for supermodularity is not preserved with the latter choice of subcoalitions. Thus v is not supermodular and the game is not convex. △

The following remark expresses this observation.

Remark 2. *Consider the channel selection situation* $(N, \{1, 2\}, (w_1, w_2))$. *If* w_1 *and* w_2 *are supermodular,* v *is not necessarily supermodular so* (N, v) *is not necessarily convex.*

5. Case of Three (or More) Channels

We provide an example to show that with three channels the cores of channel selection games can be empty, even if the gain functions across all channels are supermodular.

Example 3. Let $N = \{1, 2, 3\}$ and $M = \{1, 2, 3\}$. Table 3 gives the values of channels' gain functions as well as the characteristic function of their associated game. In this example all gain functions are supermodular. In this example we have $v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) = 3 > 2 = 2v(N)$. By Lemma 1 the core of (N, v) must be empty. △

In light of the the above example, we have our last remark.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
w_1	0	0	0	0	0	1	1
w_2	0	0	0	1	0	0	1
w_3	0	0	0	0	1	0	1
v	0	0	0	1	1	1	1

Table 3: Situation in Example 3

Remark 3. *The core of a channel selection game with more than three channels can be empty, even if all channels' gain functions are supermodular.*

The above result reveals a fundamental obstacle regarding finding fair allocations in channel selection games when more than two channels exists. In the context of collaborative supplier selection in group purchasing, this means that buyers who aim for organizing their purchases optimally across three or more suppliers may find it impossible to find allocations in the core for the grand coalition—even under the supermodularity of savings obtained from cooperative purchasing via individual suppliers.

A natural direction for future research is to find conditions, perhaps stronger than supermodularity, that guarantee the non-emptiness of cores of channel selection games with three or more channels.

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