# Chow Motif and Higher Chow Theory of $G/P$

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## Introduction

In this note we generalize Manin's calculation of the Chow motif of projective fibre bundles (see [10]) to homogenous spaces of the form $G/P$:

**Theorem:** Let $G$ be a $k$-split reductive linear algebraic group defined over a field $k$ and let $P$ be a parabolic subgroup of G. Then the Chow motif $\tilde{Y} := (Y, \Delta_Y)$ of the smooth projective $k$-variety $Y := G/P$ decomposes in a direct sum of twisted Tate motifs as follows:

$$\tilde{Y} \cong \bigoplus_{w} L^{\otimes \dim w};$$

here $w$ runs through the set of cells of $Y$.

In particular we obtain the Chow motif of the Grassmann variety $G_d$ of $d$-planes in the vector space $k^n$. More generally by using Manin's identity principle we get the Chow motif of the Grassmann bundle $G_d(E)$ for any vector bundle $E$ on any smooth projective $k$-variety $X$.

Being a first approach towards a universal cohomology theory for algebraic varieties the theory of Chow motifs in particular yields a calculation of higher Chow groups. Furthermore by using the Riemann-Roch theorem (see [1]) we get analogous statements for higher $K$-theory with denominators. For example we have:

$$K_q(G_d(E))_{\mathbb{Q}} \cong K_q(X)_{\mathbb{Q}}^N;$$

here $N$ is the number of cells of the Grassmann variety $G_d$.

For any variety $Y$, which possesses a cellular decomposition, the higher Chow- and $K$-groups of $Y$ can also be obtained by using the localization sequence and the homotopy theorem (see appendix).

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## 1 Bruhat Decomposition and classical Chow Theory of G/P

Let $k$ be a field and let $G$ be a $k$-split reductive linear algebraic group defined over $k$. We fix a maximal $k$-split $k$-torus $T$ in $G$ and a Borel subgroup $B$ of $G$ containing $T$ and defined over $k$. We consider the set of simple $B$-positive roots and let denote $S$ the corresponding set of reflections in the Weylgroup $W$. Then the pair $(W, S)$ is a Coxeter system in the sense of Bourbaki ([3]). Let

$$l : W \to \mathbb{N}_0$$

be the length function relative to the system $S$ of generators of $W$.

Futhermore we fix a subset $\theta$ of $S$ and let denote $W_\theta$ the subgroup of $W$ generated by $\theta$. Let $P_\theta := BW_\theta B$ be the corresponding parabolic subgroup of $G$ and let $Y$ be the projective smooth $k$-variety $G/P$.

Subsequently we recall the Bruhat decomposition of $Y$ and deduce the classical Chow theory of $Y$ from the one of $X := G/B$. For this $W^\theta$ is defined to be following subset of $W$:

$$W^\theta := \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$$

**(1.1) Proposition:** The map

$$\begin{array}{ccc} W^\theta \times W_\theta & \to & W \\ (w, w_\theta) & \mapsto & ww_\theta \end{array}$$

is bijective. Furthermore we have for any $w \in W^\theta$ and any $w_\theta \in W_\theta$:

$$l(ww_\theta) = l(w) + l(w_\theta).$$

In particular $W^\theta$ is precisely the elements of smallest length in each coset $wW_\theta$.

**Proof:** [7], theorem (5.3), p. 43.

For any $w \in W$ let $\overset{\circ}{X}_w$ be the locally closed Schubert cell $BwB/B$ in $X$ and let $X_w$ be the closure of $\overset{\circ}{X}_w$ in $X$. Similarly for any $w \in W^\theta$ let $\overset{\circ}{Y}_w$ respecively $Y_w$ denote the cell $BwP/P$ respectively the closure of $BwP/P$ in $Y$.

**(1.2) Lemma:** Let $p : X \to Y$ be the canonical projection. Then:
a) The morphism $p$ is smooth.
b) For any $w \in W^\theta$ we have

$$p^{-1}(\overset{\circ}{Y}_w) = \bigcup_{w_\theta \in W_\theta} \overset{\circ}{X}_{ww_\theta} .$$

c) For any $w \in W^\theta$ the projection

$$p : \overset{\circ}{X}_w \to \overset{\circ}{Y}_w$$

is an isomorphism of varieties.

**Proof:** By faithfully flat descent we can assume, that $k$ is algebraically closed.
a) By definition of quotient varieties (see [8]) the map $p : X \to Y$ is separable, i. e. generically smooth. By homogeneity the claim follows.
b) For any $w \in W^\theta$, $w_\theta \in W_\theta$ we obtain from [8], Lemma A (a) in 29.3, p. 177 and from proposition (1.1):
$(*)$
$$wBw_\theta \subseteq Bww_\theta B.$$
Hence for any $w \in W^\theta$ we have

$$p^{-1}(\overset{\circ}{Y}_w) = BwP/B = (\bigcup_{w_\theta \in W_\theta} BwBw_\theta B)/B =$$

$$= (\bigcup_{w_\theta \in W_\theta} Bww_\theta B)/B = \bigcup_{w_\theta \in W_\theta} \overset{\circ}{X}_{ww_\theta} .$$

Last union is a disjoint union of cells of $X$ (see [4]).
c) By similar arguments as in a) the surjective morphism

$$p : \overset{\circ}{X}_w \to \overset{\circ}{Y}_w$$

is smooth. So it suffices to show injectivity: Let $\overline{bw} \in BwP/P = \overset{\circ}{Y}_w$. Then

$$\begin{array}{rll} p^{-1}(\overline{bw}) & = bwP/B \cap BwB/B & \\ & = \{bwB/B\} & ((*) + \text{Bruhat decomposition for } G/B, \text{ see [4]}) \end{array}$$

**(1.3) Proposition (Bruhat decomposition for G/P):**
a) $Y = \cup_{w\in W^\theta} \overset{\circ}{Y}_w$ .
b) For any $w \in W^\theta$ the cell $\overset{\circ}{Y}_w$ is $k$-isomorphic to the affine space $A_k^{l(w)}$ of dimension $l(w)$.

**Proof:** For example this follows from (1.2) and from Bruhat decomposition for $X$ (see [4]).

Let $w_0 \in W$ respectively $v_0 \in W_\theta$ be the element of maximal length in $W$ respectively $W_\theta$. Then by (1.1) and (1.3) $l(w_0)$ respectively $l(w_0) - l(v_0)$ is the dimension of $X$ respectively $Y$.

**(1.4) Proposition (Orthogonality relations in CH$^*$(G/P)):** Let $S$ be a smooth $k$-variety and let $w$, $w'$ be two elements of $W^\theta$ with $l(w) + l(w') \le l(w_0) - l(v_0)$. Then the intersection of the cycle $[Y_w \times S]$ with the cycle $[Y_{w'} \times S]$ in $CH^*(Y \times S)$ is $\delta_{w,w_0w'v_0} \cdot [Y_1 \times S]$. ($Y_1$ is the single point $\{\bar{1}\}$!)

**Proof:** We can assume $S = \mathrm{Spec}(k)$. Let $p : X \to Y$ denote the projection. Then:

$$\begin{array}{rlll} [Y_w] \cdot [Y_{w'}] & = [Y_w] \cdot p_*[X_{w'}] & & \text{(Lemma (1.2)c))} \\ & = p_*(p^*[Y_w] \cdot [X_{w'}]) & & \text{(Projection formula)} \\ & = p_*([X_{wv_0}] \cdot [X_{w'}]) & & \text{(Lemma (1.2)b))} \\ & = p_*(\delta_{wv_0,w_0w'} \cdot [X_1]) & & \text{([5], Proposition 1(a), p. 69)} \\ & = \delta_{w,w_0w'v_0} \cdot [Y_1] & & \text{(because } v_0^2 = 1) \end{array}$$

**(1.5) Corollary (Chow theory of G/P):** Let $S$ be a smooth $k$-variety. Then the map

$$\begin{array}{ccc} \bigoplus_{w\in W^\theta} CH^*(S) & \to & CH^*(Y \times S) \\ (\alpha_w)_{w\in W^\theta} & \mapsto & \sum_{w\in W^\theta} [Y_w \times \alpha_w] \end{array}$$

is an isomorphism of groups. In particular:
a) $CH(Y) \cong \oplus_{w\in W^\theta} \mathbb{Z}[Y_w]$
b) The ring homomorphism

$$\begin{array}{ccc} CH^*(Y) \otimes CH^*(Y) & \to & CH^*(Y \times Y) \\ \alpha \otimes \beta & \mapsto & \alpha \times \beta \end{array}$$

is an isomorphism.

**Proof:** This follows from (1.3) and (1.4) (compare the proof of the corollary on page 69 in [5]).

## 2 The Chow motif of G/P

Let $\mathcal{V}$ be the category of (connected) smooth projective varieties over a fixed ground field $k$. Then the additive category $\mathcal{CV}^0$ of Chow correspondences of degree zero is defined as follows: The objects are the objects of $\mathcal{V}$ and the morphisms in $\mathcal{CV}^0$ are defined by $\mathrm{Hom}(X, Y) := CH^{\dim X}(X \times Y)$. The composition of two correspondences $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, Z)$ is given by

$$g \circ f := (p_{13})_*(p_{12}^*(f) \cdot p_{23}^*(g)) \in \mathrm{Hom}(X, Z),$$

where notation of the type $p_{13} : X \times Y \times Z \to X \times Z$ means the projection onto the product of the first and third factors. The category of Chow motifs arises from $\mathcal{CV}^0$ by adding formally cernels of projectors (see [10]): The objects are pairs $(X, p)$ with $X \in \mathcal{V}$ and $p \in \mathrm{End}(X) = CH^{\dim X}(X \times X)$ with $p \circ p = p$; the morphisms are defined by

$$\mathrm{Hom}((X,p),(Y,q)) := \{f \in \mathrm{Hom}_{\mathcal{CV}^0}(X,Y) : fp = qf\}/\{f : fp = qf = 0\}.$$

The $n$-fold twist $L^{\otimes n}$ of the Tate motif $L$ is the pair $(\mathbb{P}^n, [\mathbb{P}^n \times \mathrm{pt}])$ (see [10]).

Let $\tilde{Y} := (Y, \Delta_Y)$ be the motif of the smooth projective $k$-variety $Y = G/P$ of section 1.

**(2.1) Theorem:** The motif $\tilde{Y}$ decomposes in a direct sum of twisted Tate motifs as follows:

$$\tilde{Y} = \bigoplus_{w \in W^\theta} L^{\otimes l(w)}$$

The proof of (2.1) is based on following lemma:

**(2.2) Lemma:**

a) $\mathrm{End}(\tilde{Y}) = \bigoplus_{w,w' \in W^\theta,\, l(w)+l(w')=l(w_0)-l(v_0)} \mathbb{Z}[Y_w \times Y_{w'}]$

b) For any two elements $[Y_w \times Y_{w'}]$, $[Y_v \times Y_{v'}]$ in $\mathrm{End}(\tilde{Y})$ we have:

$$[Y_w \times Y_{w'}] \circ [Y_v \times Y_{v'}] = \delta_{v', w_0 w v_0}[Y_v \times Y_{w'}]$$

**Proof:** Statement a) is an immediate consequence of corollary (1.5). For b) let

$$p_{12}, p_{23}, p_{13} : Y \times Y \times Y \to Y \times Y$$

denote the canonical projections. Then:

$$\begin{aligned}
&[Y_w \times Y_{w'}] \circ [Y_v \times Y_{v'}] = \\
&= (p_{13})_*((p_{12})^*[Y_v \times Y_{v'}] \cdot (p_{23})^*[Y_w \times Y_{w'}]) && \text{(by definition of } \circ) \\
&= (p_{13})_*([Y_v \times Y_{v'} \times Y] \cdot [Y \times Y_w \times Y_{w'}]) \\
&= (p_{13})_*([Y_v \times (Y_{v'} \cdot Y_w) \times Y_{w'}]) && \text{([6], Example 8.1.4)} \\
&= (p_{13})_*((p_{13})^*[Y_v \times Y_{w'}] \cdot [Y \times (Y_{v'} \cdot Y_w) \times Y]) && \text{([6], Example 8.1.4)} \\
&= [Y_v \times Y_{w'}] \cdot (p_{13})_*[Y \times (Y_{v'} \cdot Y_w) \times Y] && \text{(Projection formula)} \\
&= \delta_{v', w_0 w v_0} \cdot [Y_v \times Y_{w'}]
\end{aligned}$$

The last equality results from proposition (1.4):

$$p_*(Y_{v'} \cdot Y_w) = \delta_{v', w_0 w v_0} \cdot 1 \quad \text{in} \quad CH^*(\mathrm{pt}) \cong \mathbb{Z}$$

hence by [6], Proposition 1.7, p. 18:

$$(p_{13})_*[Y \times (Y_{v'} \cdot Y_w) \times Y] = \delta_{v', w_0 w v_0} \cdot 1 \quad \text{in} \quad CH^*(Y \times Y).$$

**Proof of (2.1):** By proposition (1.1) for any $w \in W^\theta$ the element $w_0 w v_0$ is in $W^\theta$, too. So by Lemma (2.2) the correspondences

$$p_w := [Y_w \times Y_{w_0 w v_0}],\ w \in W^\theta$$

are pairwise orthogonal projectors in $\mathrm{End}(\tilde{Y})$ with $\sum_{w \in W^\theta} p_w = 1$. Hence:

$$\tilde{Y} = \bigoplus_{w \in W^\theta} (Y, [Y_w \times Y_{w_0 w v_0}])$$

One checks easily, that the homomorphisms $[Y_w \times \mathrm{pt}] \in CH^{\dim Y}(Y \times \mathbb{P}^{l(w)})$ and $[\mathbb{P}^{l(w)} \times Y_{w_0 w v_0}] \in CH^{l(w)}(\mathbb{P}^{l(w)} \times Y)$ between $(Y, [Y_w \times Y_{w_0 w v_0}])$ and $(\mathbb{P}^{l(w)}, [\mathbb{P}^{l(w)} \times \mathrm{pt}]) = L^{\otimes\, l(w)}$ are inverse to each other. Now the claim follows.

Let

$$H^i(-,j) : \mathcal{V} \to (\text{Abelian groups}); \quad i,j \in \mathbb{Z}$$

be a twisted cohomology theory in the sense of [2], Definition 1.1, i. e. a sequence of contravariant functors indexed by $i,j \in \mathbb{Z}$. If $H^i(-,j)$ factorizes through the category of Chow motifs

$$\begin{array}{ccc} \mathcal{V} & \stackrel{H^i(-,j)}{\longrightarrow} & (\text{Abelian groups}) \\ \downarrow & & \\ (\text{Chow motifs}) & & , \end{array}$$

then we conclude from theorem (2.1):

$$H^i(Y,j) = \bigoplus_{w \in W^\theta} H^i(L^{\otimes\, l(w)}, j).$$

For instance we obtain a twisted cohomology theory by means of higher Chow groups (see [1]):

$$H^i(X,j) := CH^j(X, 2j-i).$$

Associating a correspondence $\alpha$ of degree zero the homomorphism $\beta \mapsto (p_2)_*(p_1^*(\beta)\cdot\alpha)$ between the corresponding Chow groups we get a factorization of $H^i(-,j)$ through $\mathcal{CV}^0$ and therefore through the category of Chow motifs. In this case by the projective bundle theorem (see [1], theorem (7.1)) we have $H^i(L^{\otimes\, n}, j) = H^{i-2n}(\mathrm{Spec}(k), j-n)$ and hence

$$H^i(Y,j) = \bigoplus_{w \in W^\theta} H^{i-2l(w)}(\mathrm{Spec}(k), j - l(w))$$

This conclusion generalizes the corollary of proposition 1 in [5] on p. 69 to higher Chow groups. We will deduce it in the appendix again by an argument, which is appropriate for higher $K'$-groups, too.

## 3 Grassmann bundles

In this section we consider the general linear group $Gl_n$ over a fixed ground field $k$. It is a $k$-split reductive linear algebraic group, whose Weylgroup is the symmetric group $\Sigma_n$. In $\Sigma_n$ we fix the set

$$S := \{<1,2>, <2,3>, \ldots, <n-1,n>\}$$

of transpositions. Then the length relative to $S$ of an element $\tau \in \Sigma_n$ is given by

$$l(\tau) = \sum_{j=1}^{n-1} \#\{i > j : \tau(i) < \tau(j)\}$$

(see [7], example (1.3a), p. 9).

We fix an integer $d \in \{1, \ldots, n-1\}$ and let $\theta_d$ denote the subset

$$\theta_d := S \backslash \{<d, d+1>\}$$

of $S$. Then the subgroup $W_d$ of $\Sigma_n$ generated by $\theta_d$ is $\Sigma_d \times \Sigma_{n-d}$. The element of maximal length in $W$ respectively $W_d$ is

$$w_0 := \begin{pmatrix} 1 & \dots & n \\ n & \dots & 1 \end{pmatrix} \quad \text{respectively} \quad v_0 := \begin{pmatrix} 1 & \dots & d & d+1 & \dots & n \\ d & \dots & 1 & n & \dots & d+1 \end{pmatrix}.$$

Let $B$ be the Borel subgroup of $Gl_n$ consisting of upper triangular matrices and let $P_d$ be the parabolic subgroup $BW_dB$ of $Gl_n$; it consists of the matrices of the form $\begin{pmatrix} Gl_d & * \\ 0 & Gl_{n-d} \end{pmatrix}$ (see [7], p. 126).

In the case of Grassmann bundles the set

$$W^d := \{(\lambda_1, \dots, \lambda_d) \in \mathbb{N}_0^d : n-d \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0\}$$

is a more appropriate description of the subset $W^{\theta_d}$ of $\Sigma_n$ defined in section 1. For any $\lambda \in W^d$ we set $|\lambda| := \sum_{i=1}^d \lambda_i$ and $\lambda^{\mathrm{op}} := (n-d-\lambda_{d-i+1})_{i=1,\dots,d} \in W^d$.

**(3.1) Lemma:** The map

$$\begin{array}{cccc} \alpha : & W^d & \to & \Sigma_n \\ & (\lambda_1, \dots, \lambda_d) & \mapsto & \begin{pmatrix} 1 & \dots & d & d+1 \quad \dots \quad n \\ \lambda_d+1 & \dots & \lambda_1+d & \text{increasing enumeration of complement} \end{pmatrix} \end{array}$$

induces a 1–1–correspondence between $W^d$ and the indexset $W^{\theta_d} \subseteq \Sigma_n$ defined in section 1. Furthermore we have $l(\alpha(\lambda)) = |\lambda|$ and $\alpha(\lambda^{\mathrm{op}}) = w_0 \alpha(\lambda) v_0$.

**Proof:** This follows from [7], Lemma 5.3, p.158.

Let $G_d$ be the Grassmann variety of $d$-planes in the vector space $k^n$. The stabilizer of the $d$-plane $< e_1, \dots, e_d >$ under the natural (transitive) action $Gl_n \times G_d \to G_d$ is the parabolic subgroup $P_d$; hence we obtain a canonical isomorphisms

$$Gl_n / P_d \tilde{\to} G_d$$

(see [7], p. 127).

So theorem (2.1) gives us following calculation of the Chow motif $\tilde{G_d}$ of the Grassmann variety $G_d$:

$$\tilde{G_d} \cong \bigoplus_{\lambda \in W^d} L^{\otimes |\lambda|}.$$

Subsequently we will generalize this fact to arbitrary Grassmann bundles $G_d(E)$. So in the rest of this section $E$ is a vector bundle of rank $n$ on a smooth projective $k$-variety $X$ and

$$\pi : G := G_d(E) \to X$$

is the corresponding Grassmann bundle of $d$-planes in $E$.

**(3.2) Theorem:** The Chow motif $\tilde{G} := (G, \Delta_G)$ of $G = G_d(E)$ decomposes in a direct sum as follows:

$$\tilde{G} = \bigoplus_{\lambda \in W^d} \tilde{X} \otimes L^{\otimes |\lambda|}.$$

**Proof:** Let

$$0 \to S \to \pi^* E \to Q \to 0$$

be the universal exact sequence of vector bundles on $G_d(E)$. For any $i \in \mathbb{N}_0$ let $c_i := c_i(Q - \pi^*E) \in CH^i(G_d(E))$ be the $i$-th Chern class of $Q - \pi^*E$. For any $\lambda \in W^d$ we define

$$\Delta_\lambda := \Delta_\lambda(c) := \det(c_{\lambda_i+j-i})_{i,j=1,\ldots,d} \in CH^{|\lambda|}(G_d(E)).$$

By [6], 14.6 we have following generalizations of (1.4) respectively (1.5):

**Duality theorem:** Let $\lambda, \mu \in W^d$ be elements of $W^d$ with $|\lambda| + |\mu| \leq d(n-d)$ and let $\alpha \in CH^*(X)$. Then

$$\pi_*(\Delta_\lambda \cdot \Delta_\mu \cap \pi^*\alpha) = \delta_{\lambda^{\mathrm{op}},\mu} \cdot \alpha.$$

**Basis theorem:** The map

$$\begin{array}{ccc} \bigoplus_{\lambda \in W^d} CH^*(X) & \to & CH^*(G_d(E)) \\ (\alpha_\lambda)_{\lambda \in W^d} & \mapsto & \sum_{\lambda \in W^d} \Delta_\lambda \cap \pi^*(\alpha_\lambda) \end{array}$$

is an isomorphism of groups.

Using this two theorems theorem (3.2) will be proved by a generalization of Manin's calculation of the Chow motif of projective fibre bundles (see [10], §7):

We choose a total ordering "$\leq$" on $W^d$, such that

(*) $\qquad \lambda \leq \lambda' \Rightarrow |\lambda| \leq |\lambda'|.$

We define correspondences $p_\lambda \in CH^*(G \times G)$, $\lambda \in W^d$, by a downward induction on $\lambda$ as follows:

$$p_\lambda := c(\Delta_\lambda) \circ c(\pi) \circ c(\pi)^t \circ c(\Delta_{\lambda^{\mathrm{op}}}) \circ (1 - \sum_{\mu \in W^d,\, \mu > \lambda} p_\mu).$$

In this expression the various parts have following meaning:

1) $\sum_{\mu>\lambda} p_\mu := 0$, if $\lambda$ is maximal.

2) $c(\Delta_\lambda) := (\Delta_G)_*(\Delta_\lambda) \in CH^{|\lambda|+d(n-d)+\dim X}(G \times G)$, where $\Delta_G : G \to G \times G$ is the diagonal embedding.

3) $c(\pi) := (\Gamma_\pi)_*(1) \in CH^{\dim X}(X \times G)$, where $\Gamma_\pi : G \to G \times X$ is the graph of $\pi$.

4) $c(\pi)^t := (\Gamma_\pi)_*(1) \in CH^{\dim X}(G \times X)$.

Clearly the correspondences $p_\lambda$, $\lambda \in W^d$ are of degree zero. We will show in Lemma (3.3), that they form a complete system of pairwise orthogonal projectors. So we have

$$\tilde{G} = \bigoplus_{\lambda \in W^d} (G, p_\lambda).$$

One checks easily, that for any two vector bundles $E', E''$ on $X$ of the same rank the homomorphism

$$h'_\lambda := c(\Delta''_\lambda) \circ c(\pi'') \circ c(\pi')^t \circ c(\Delta'_{\lambda^{\mathrm{op}}}) \circ (1 - \sum_{\mu>\lambda} p'_\mu) \in CH^*(G_d(E') \times G_d(E''))$$

and the analougously defined homomorphism $h''_\lambda \in CH^*(G_d(E'') \times G_d(E'))$ between $(G_d(E'), p'_\lambda)$ and $(G_d(E''), p''_\lambda)$ are inverse to each other. In particular the Chow motif $\tilde{G}$ of $G := G_d(E)$ is isomorphic to the motif of $G_d(A^n) \cong \oplus_{\lambda \in W^d} \tilde{X} \otimes L^{\otimes |\lambda|}$ (Theorem (2.1) + Lemma (3.1)).

**Lemma (3.3):**

a) For any $\lambda, \mu \in W^d$ we have: $p_\lambda \circ p_\mu = \delta_{\lambda,\mu} \cdot p_\lambda$.

b) $\sum_{\lambda \in W^d} p_\lambda = 1$.

**Proof:** Let $\alpha \in \mathrm{Hom}(Y_1, Y_2) = CH^{\dim Y_1}(Y_1 \times Y_2)$ be a morphism in $\mathcal{CV}^0$. Then for any $T \in \mathcal{V}$ the correspondence $\alpha$ induces a map

$$\begin{array}{ccc} \mathrm{Hom}(T, Y_1) & \xrightarrow{\alpha_T} & \mathrm{Hom}(T, Y_2) \\ \| & & \| \\ CH^{\dim T}(T \times Y_1) & \xrightarrow{\alpha_T} & CH^{\dim T}(T \times Y_2) \end{array}$$

denoted by $\alpha_T$.
By Manin's identity principle it suffices to show, that for any $T \in \mathcal{V}$ the equations

$$(p_\lambda)_T \circ (p_\mu)_T = \delta_{\lambda,\mu}(p_\lambda)_T \quad \text{and} \quad \sum_{\lambda \in W^d} (p_\lambda)_T = 1$$

hold.By changing the base scheme $X$ to $X \times T$ and $E$ to $E_{(X \times T)}$ we can assume $T = e := \mathrm{Spec}(k)$. In this case the above equations follows from the basis theorem and from following statement: For any $(\alpha_\mu)_{\mu \in W^d} \in \oplus_{\mu \in W^d} CH^*(X)$ and for any $\lambda \in W^d$ we have

$$(p_\lambda)_e(\sum_{\mu \in W^d} \Delta_\lambda \cdot \pi^*(\alpha_\lambda)) = \Delta_\lambda \cdot \pi^*(\alpha_\lambda).$$

By [10], Lemma on p. 449 we have

$$\begin{aligned} (c(\Delta_\lambda))_e &= \text{multiplication with } \Delta_\lambda \\ (c(\pi))_e &= \pi^* \\ (c(\pi)^t)_e &= \pi_*. \end{aligned}$$

Hence by downward induction on $\lambda$:

$$\begin{aligned} &(p_\lambda)_e(\textstyle\sum_{\mu \in W^d} \Delta_\lambda \cdot \pi^*(\alpha_\lambda)) = \\ &= \Delta_\lambda \cdot \pi^*\pi_*\, \Delta_{\lambda^{\mathrm{op}}} \cdot (1 - \textstyle\sum_{\mu > \lambda}(p_\lambda)_e)(\textstyle\sum_{\mu \in W^d} \Delta_\lambda \cdot \pi^*(\alpha_\mu)) \\ &= \Delta_\lambda \cdot \pi^*\pi_*\, \Delta_{\lambda^{\mathrm{op}}} \cdot (\textstyle\sum_{\mu \le \lambda} \Delta_\mu \cdot \pi^*(\alpha_\mu)) \\ &= \Delta_\lambda \cdot \pi^*\alpha_\lambda \qquad\qquad \text{(Duality theorem + (*))} \end{aligned}$$

As in section 2 we obtain the higher Chow groups $H^i(G, j) := CH^j(G, 2j - i)$ of $G = G_d(E)$ from theorem (3.2):

$$H^i(G, j) \cong \bigoplus_{\lambda \in W^d} H^{i-2|\lambda|}(X, j - |\lambda|).$$

Using the Riemann-Roch theorem (see [1], Theorem (9.1)) we get furthermore the higher $K'$-groups with denominators of $G$:

$$K'(G_d(E))_{\mathbb{Q}} \cong \bigoplus_{\lambda \in W^d} K'(X)_{\mathbb{Q}}.$$

## Appendix

Let $k$ be a field and let $\mathcal{C}$ be the category of quasiprojective $k$-schemes. Let

$$H_a(-, b) : \mathcal{C} \to (\text{Abelian groups}),\ a, b \in \mathbb{Z}$$

be a twisted homology theory in the sense of [2], Definition (1.2). We assume, that $\oplus_{a,b} H_a(-, b)$ is contravariant functorial under flat morphisms $f : Y \to X$ in $\mathcal{C}$ with constant relative dimension $m := \dim Y - \dim X$:

$$f^* : H_a(X, b) \to H_{a+2m}(Y, b + m)$$

and, that the projection formula holds; furthermore we assume following homotopy axiom:

**Homotopy axiom:** For any $X \in \mathcal{C}$ and any $n \geq 1$ the contravariant map

$$H_a(X,b) \to H_{a+2m}(X \times A^m, b+m)$$

is an isomorphism.

We obtain examples by using higher $K'$-theory due to Quillen ([12]) or higher Chow theory due to Bloch ([1]):

$$H_a(X,b) := K'_{a-2b}(X) \quad \text{respectively} \quad H_a(X,b) := CH_b(X, a-2b)$$

(In $CH_b(X, a-2b)$ the subscript $b$ denotes dimension of cycles!)

**Theorem (Higher K′-theory and higher Chow theory of varieties with cellular decomposition):** Let $X$ be an object in $\mathcal{C}$ and let

$$\emptyset = Y_{-1} \subset Y_0 \subset \ldots \subset Y_n =: Y$$

be an increasing sequence of closed embeddings of flat quasiprojective $X$-schemes $\pi_k : Y_k \to X$. Assume, that for any $k \in \{0, \ldots, n\}$ the open complement $U_k := Y_k \backslash Y_{k-1}$ is $X$-isomorphic to an affine space $A_X^{m_k}$ of relative dimension $m_k$. Let $i_k$ denote the closed embedding $Y_k \hookrightarrow Y$. Then for any $a, b \in \mathbb{Z}$ the map

$$\begin{array}{ccc} \bigoplus_{k=0}^{n} H_{a-2m_k}(X, b-m_k) & \to & H_a(Y,b) \\ (\alpha_0, \ldots, \alpha_n) & \mapsto & \sum_{k=0}^{n} (i_k)_*(\pi_k)^* \alpha_k \end{array}$$

is an isomorphism.

**Proof:** By induction on $n$ we can assume, that the left vertical arrow in the commutative diagram with exact rows

$$\begin{array}{ccccccccc} \ldots \to & H_a(Y_{n-1},b) & \overset{(i_{n-1})_*}{\to} & H_a(Y_n,b) & \overset{j^*}{\to} & H_a(U_n,b) & \to \ldots \\ & \uparrow \sum (i'_k)_*(\pi_k)^* & & \uparrow \sum (i_k)_*(\pi_k)^* & & \uparrow \pi^* & \\ 0 \to & \bigoplus_{k=0}^{n-1} H_{a\text{-}2m_k}(X, b\text{-}m_k) & \to & \bigoplus_{k=0}^{n} H_{a\text{-}2m_k}(X, b\text{-}m_k) & \to & H_{a\text{-}2m_n}(X, b\text{-}m_n) & \to 0 \\ & (\alpha_0, \ldots, \alpha_{n-1}) & \mapsto & (\alpha_0, \ldots, \alpha_{n-1}, 0) & & & \\ & & & (\alpha_0, \ldots, \alpha_n) & \mapsto & \alpha_n & \end{array}$$

is an isomorphism. By the homotopy axiom $\pi^*$ is an ismomorphism and hence $j^*$ is surjective. The localization sequence shows, that the map $(i_{n-1})_*$ is injective. Now the claim follows.

For instance in the case of the variety $Y = G/P$ (see section 1) we obtain for any $X \in \mathcal{C}$ and any $q \geq 0$:

$$K'_q(Y \times X) = \bigoplus_{w \in W^\theta} K'_q(X).$$

This fact generalizes proposition 7 of [11] to higher $K$-theory. Whereas Marlin uses information about intersecting cells in $G/P$, the cellular decomposition of $G/P$ is the only geometrical ingredient in our proof.
Analogous statements can also be obtained by using equivariant $K'$-groups (see [9] or [13] for a definition of equivariant $K'$-groups).

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