Higher K'-Groups of Integral Group Rings

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Abstract: For any finite group G, which is a split extension with a nilpotent group, we prove a splitting formula for $K_q'(\mathbb{Z}[G])$. Applying it to the group of upper (3×3)-matrices over a finite field we obtain the formula conjectured by Hambleton, Taylor and Williams in [4].

Key words: group ring, nilpotent group, group of triangular matrices over a finite field.

Introduction

In this note we give a new approach towards a calculation of higher K'-groups of integral group rings initiated for K_0' by the fundamental work [5] of H. Lenstra. The idea in its simplest nontrivial form is the following:

The spectrum of the group ring $\mathbb{Z}[G] \cong \mathbb{Z}[X]/(X^p-1)$ associated with the cyclic group $G=\mathbb{Z}/p\mathbb{Z}$ is the disjoint union of the closed subset $\mathrm{Spec}(\mathbb{Z}) \cong \mathrm{Spec}\,(\mathbb{Z}[X]/(X-1))$ and of the open subset

$$\mathrm{Spec}\left(\mathbb{Z}[X]\left[(X-1)^{-1}\right]/(X^p-1)\right) \cong \mathrm{Spec}\left(\mathbb{Z}[\frac{1}{p}][\zeta_p]\right).$$

Because of the splitting $\mathbb{Z}\to\mathbb{Z}[G]\to\mathbb{Z}$, the localization sequence associated with this situation splits and we obtain the well-known formula: $K_q'(\mathbb{Z}[G]) \cong K_q'(\mathbb{Z}) \oplus K_q'(\mathbb{Z}[\frac{1}{p}][\zeta_p])$.

We will generalize this observation to the case of a finite group $G=\pi\rtimes\Gamma$, which is a split extension of an arbitrary group Γ with a nilpotent group π, thereby obtaining the result of T. Mitsuda ([7]) for $q=0$ and the results of D. Webb ([13], [14], [16]) and of I. Hambleton, L. R. Taylor and E. B. Williams ([4]) for nilpotent groups. The theorem we will prove decomposes the K'-groups of $\mathbb{Z}[G]$ into a direct sum of K'-groups of twisted group rings $\mathbb{Z}{<}\Gamma\rho{>}\#\Gamma$ as follows:

$$K_q'(\mathbb{Z}[\pi\rtimes\Gamma]) \cong \bigoplus_{\Gamma\rho} K_q'(\mathbb{Z}{<}\Gamma\rho{>}\#\Gamma);$$

here the direct sum is indexed by the orbits $\Gamma\rho$ of the action of Γ on rational representations of π and the ring $\mathbb{Z}{<}\Gamma\rho{>}$ is defined to be the product $\prod_{\rho'\in\Gamma\rho}\mathbb{Z}{<}\rho'{>}$ of the maximal $\mathbb{Z}[\frac{\#ker(\rho)}{\#\pi}]$-orders $\mathbb{Z}{<}\rho'{>}$ in the Wedderburn component $M_{n_{\rho'}}(K_{\rho'})$ of $\mathbb{Q}[\pi]$.

Finally applying the theorem to the group B of upper triangular (3×3)-matrices over the finite field $\mathbb{F}_p$ we obtain:

$$K_q'(\mathbb{Z}[B]) \cong \bigoplus_{d|p-1}\left[K_q'\left(\mathbb{Z}[\frac{1}{d}][\zeta_d]\right)^{m_{3,d}} \oplus K_q'\left(\mathbb{Z}[\frac{1}{p\cdot d}][\zeta_d]\right)^{1+3m_{2,d}}\right];$$

here $m_{2,d}$ respectively $m_{3,d}$ denotes the number of cyclic subgroups of order d of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ respectively $\mathbb{F}_p^\times \times \mathbb{F}_p^\times \times \mathbb{F}_p^\times$.

§1 Notations and some General Facts

Let R be a (not necessary commutative) ring with 1 and let G be a (mulitplicative) finite group. We will denote the group ring associated with R and G by $R[G]$ and the canonical basis elements of $R[G]$ by $[g]$, $g \in G$. For any normal subgroup N of G the ideal I_N is defined to be the kernel of the canonical ring epimorphism $R[G] \to R[G/N]$.

Lemma 1: The two-sided ideal I_N is generated by the elements $[g]-[1]$, $g \in G$, as left- and as right ideal. In particular: If N_1, N_2 are elementwise commuting normal subgroups of G, then: $I_{N_1} \cdot I_{N_2} = I_{N_2} \cdot I_{N_1}$.

Proof: well-known.

The $(\#G)$-fold product $\prod_{g\in G} R$ of R is denoted by $R^{(G)}$ and the canonical basis elements of $R^{(G)}$ are denoted by e_g, $g \in G$. If G acts on a ring S by ring automorphisms, then $S\#G$ denotes the twisted group ring. Here the group G is embedded by $g \mapsto \{g\}$ and the multiplication is defined by $(s_1 \cdot \{g_1\}) \cdot (s_2 \cdot \{g_2\}) := s_1 \cdot g_1(s_2) \cdot \{g_1 \cdot g_2\}$.

We recall that a ring extension S/R is called a **Galois extension with group** G if S is a finitely generated projective R-module and if G acts on S by ring automorphisms, such that the fixed ring is R and the canonical map $S\#G \to \mathrm{End}_R(S)$ is bijective.

Examples:
a) If the prime p is a unit in R, then the extension $R[\zeta_p]$ of R by a primitive p-th root of unity is a Galois extension with Galois group $\mathbb{F}_p^\times$. In particular the twisted group ring $R[\zeta_p]\#\mathbb{F}_p^\times$ is isomorphic to the ring $M_{p-1}(R)$ of $(p-1)\times(p-1)$-matrices.
b) The ring $R^{(G)}$ is a Galois extension of R (diagonally embedded) with Galois group G. In particular we have $R^{(G)}\#G \cong M_{\#G}(R)$.

Lemma 2: Let G be a finite commutative group and Γ a finite group, which acts on G by group automorphisms. Then the map

$$\begin{array}{ccc} \Gamma \times R^{(G)}\#G & \to & R^{(G)}\#G \\ (\gamma, e_g\{h\}) & \mapsto & e_{\gamma g}\{\gamma h\} \end{array}$$

defines an action of Γ on $R^{(G)}\#G$ by ring automorphisms and there is a canonical isomorphism

$$(R^{(G)}\#G)\#\Gamma \cong R[\Gamma]^{(G)}\#G$$

of R-algebras.

Proof: The first statement is clear. For the second statement we write G additively and Γ mulitplicatively. We define an R-module-homomorphism $\alpha : R[\Gamma]^{(G)}\#G \to (R^{(G)}\#G)\#\Gamma$ by $[\gamma]e_g\{h\} \mapsto e_g\{\gamma h - \gamma g + g\}\{\gamma\}$. Then obviously α takes an R-basis to an R-basis and hence α is bijective. Furthermore we have

$$\alpha(1) = \alpha(\sum_{g\in G}[1]e_g\{0\}) = \sum_{g\in G} e_g\{0\}\{1\} = 1$$

Finally the following calculation shows that α preserves products:

$$\alpha([\gamma_1]e_{g_1}\{h_1\}) \cdot \alpha([\gamma_2]e_{g_2}\{h_2\})$$

$$
\begin{aligned}
&= (e_{g_1}\{\gamma_1 h_1 - \gamma_1 g_1 + g_1\}\{\gamma_1\}) \cdot (e_{g_2}\{\gamma_2 h_2 - \gamma_2 g_2 + g_2\}\{\gamma_2\}) \\
&= e_{g_1}\{\gamma_1 h_1 - \gamma_1 g_1 + g_1\} e_{\gamma_1 g_2}\{\gamma_1\gamma_2 h_2 - \gamma_1\gamma_2 g_2 + \gamma_1 g_2\}\{\gamma_1\}\{\gamma_2\} \\
&= e_{g_1} e_{\gamma_1 g_2 + \gamma_1 h_1 - \gamma_1 g_1 + g_1}\{\gamma_1 h_1 - \gamma_1 g_1 + g_1\}\{\gamma_1\gamma_2 h_2 - \gamma_1\gamma_2 g_2 + \gamma_1 g_2\}\{\gamma_1\gamma_2\} \\
&= \delta_{g_1, g_2 + h_1} e_{g_1}\{\gamma_1 h_1 - \gamma_1 g_1 + g_1 + \gamma_1\gamma_2 h_2 - \gamma_1\gamma_2 g_1 + \gamma_1\gamma_2 h_1 + \gamma_1 g_1 - \gamma_1 h_1\}\{\gamma_1\gamma_2\} \\
&= \delta_{g_1, g_2 + h_1} e_{g_1}\{\gamma_1\gamma_2(h_1 + h_2) - \gamma_1\gamma_2 g_1 + g_1\}\{\gamma_1\gamma_2\} \\
&= \alpha(\delta_{g_1, g_2 + h_1}[\gamma_1 \cdot \gamma_2] e_{g_1}\{h_1 + h_2\}) \\
&= \alpha([\gamma_1] e_{g_1}[\gamma_2] e_{g_2 + h_1}\{h_1\}\{h_2\}) \\
&= \alpha([\gamma_1] e_{g_1}\{h_1\} \cdot [\gamma_2] e_{g_2}\{h_2\}).
\end{aligned}
$$

Now let $\mathcal{M}$ be an abelian category and let $\mathcal{M}_1 \subseteq \mathcal{M}$ be a nonempty full subcategory. We recall that $\mathcal{M}_1$ is called a **Serre subcategory**, if for each exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{M}$ the following condition is fulfilled: the object M is in $\mathcal{M}_1$ if and only if the objects M' and M'' are in $\mathcal{M}_1$. For any Serre subcategory $\mathcal{M}_1$ of $\mathcal{M}$ the **quotient category** $\mathcal{M}/\mathcal{M}_1$ is defined as follows (see [3]): Its objects are those of $\mathcal{M}$ and the homomorphisms from M to N are defined to be

$$\lim_{M',N'} \mathrm{Hom}_{\mathcal{M}}(M', N/N'),$$

the inductive limit being taken over all subobjects $M' \subseteq M$ respectively $N' \subseteq N$ such that $M/M' \in \mathcal{M}_1$ respectively $N' \in \mathcal{M}_1$.

Lemma 3:
a) Let $\mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ be Serre subcategories. Then $\mathcal{M}_1/\mathcal{M}_2$ is a Serre subcategory of $\mathcal{M}/\mathcal{M}_2$ and the canonical functor

$$(\mathcal{M}/\mathcal{M}_2) \,/\, (\mathcal{M}_1/\mathcal{M}_2) \stackrel{\text{can}}{\to} \mathcal{M}/\mathcal{M}_1$$

is an equivalence of categories.
b) Let $\mathcal{M}_1 \subseteq \mathcal{M}$ be a Serre subcategory and let $\mathcal{N} \subseteq \mathcal{M}$ be a nonempty, full subcategory, closed under taking subobjects, quotients, and finite products (in particular $\mathcal{N}$ is an abelian subcategory). Then $\mathcal{N} \cap \mathcal{M}_1$ is a Serre subcategory of $\mathcal{N}$ and $\mathcal{N}/\mathcal{N} \cap \mathcal{M}_1$ canonically is a full subcategory of $\mathcal{M}/\mathcal{M}_1$ being closed under taking subobjects, quotients, and finite products.

Proof: straightforward.

§2 The Theorem

Let π be a finite group. We will denote the set of isomorphism classes of simple $\mathbb{Q}[\pi]$-modules by $C(\pi)$. Let

$$\mathbb{Q}[\pi] = \prod_{\rho \in C(\pi)} M_{n_\rho}(K_\rho)$$

be the decomposition of $\mathbb{Q}[\pi]$ into simple algebras according to the structure theorem of Wedderburn-Artin, and for any $\rho \in C(\pi)$ let $e_\rho \in \mathbb{Q}[\pi]$ be the idempotent associated with this decomposition. It is well-known that e_ρ is an element of the subring $\mathbb{Z}[\frac{1}{\#\pi}][\pi]$ (see for instance [2]). So, if $\omega_\rho \subseteq \pi$ denotes the kernel of the representation ρ, the ring

$$\mathbb{Z}\langle\rho\rangle := \mathbb{Z}[\frac{\#\omega_\rho}{\#\pi}][\pi/\omega_\rho] e_\rho$$

is well-defined. For any subset $C \subseteq C(\pi)$ we set

$$\mathbb{Z}{<}C{>} := \prod_{\rho \in C} \mathbb{Z}{<}\rho{>}$$

and for any ring R we set

$$R{<}C{>} := R \otimes_{\mathbb{Z}} \mathbb{Z}{<}C{>} = \prod_{\rho \in C} R[\frac{\#\omega_\rho}{\#\pi}][\pi/\omega_\rho] e_\rho .$$

Theorem: Let π be a finite nilpotent group and let Γ be an arbitrary finite group, which acts on π by group automorphisms. Then for any ring R and any $q \geq 0$ there is a group isomorphism

$$K'_q(R[\pi \rtimes \Gamma]) \cong \bigoplus_{\rho \in C(\pi)/\Gamma} K'_q(R{<}\Gamma\rho{>}\#\Gamma).$$

(Here $\pi \rtimes \Gamma$ denotes the semidirect product, $\Gamma\rho$ denotes the orbit of $\rho \in C(\pi)$ under the induced action of Γ on $C(\pi)$ and $C(\pi)/\Gamma$ denotes the set of orbits.)

Proof: Let $\pi = \oplus_p \pi_p$ be the decomposition of π into its p-Sylow-subgroups and let $P(\pi)$ be the set of primes p with $\pi_p \neq 0$. For any $l \in P(\pi)$ we set ${}_l\pi := \oplus_{p \neq l} \pi_p \cong \pi/\pi_l$ and

$$I_l := I_l(R,\pi) := \ker(R[\pi \rtimes \Gamma] \stackrel{\text{can}}{\to} R[{}_l\pi \rtimes \Gamma]).$$

Further for any subset $L \subseteq P(\pi)$ the ideal $I_L = I_L(R,\pi)$ is defined to be $\prod_{l \in L} I_l(R,\pi)$, which is well-defined by Lemma 1.
Let $\mathcal{M} = \mathcal{M}(R,\pi)$ be the abelian category of finitely generated $R[\pi \rtimes \Gamma]$-modules and let $\mathcal{M}_L$ be the Serre subcategory of $\mathcal{M}$ consisting of those objects which are annihilated by some power of I_L. We define $\mathcal{C}_L$ to be the quotient category $\mathcal{M}/\mathcal{M}_L$. Now the theorem is the special case $L = \emptyset$ of the more general

Theorem': For any subset L of $P(\pi)$ and any $q \geq 0$ there is a group isomorphism

$$K'_q(\mathcal{C}_L) \cong \bigoplus_{\rho \in (C(\pi) - \cup_{l \in L} C({}_l\pi))/\Gamma} K'_q(R{<}\Gamma\rho{>}\#\Gamma)$$

Lemma 4: For any subset L of $P(\pi)$ the canonical functor

$$\mathcal{C}_L(R,\pi) \to \mathcal{C}_L(R[\frac{1}{l}, l \in L], \pi)$$

is an equivalence of categories.

Proof: Let $\mathcal{H}_L$ be the Serre subcategory of $\mathcal{M}$ consisting of those objects which are annihilated by some power of $\prod_{l \in L} l$. We claim that $\mathcal{H}_L$ is a subcategory of $\mathcal{M}_L$. Then

$$\mathcal{C}_L(R,\pi) \stackrel{\text{def}}{=} \mathcal{M}(R,\pi)/\mathcal{M}_L(R,\pi) =$$

$$\cong (\mathcal{M}(R,\pi)/\mathcal{H}_L(R,\pi)) \,/\, (\mathcal{M}_L(R,\pi)/\mathcal{H}_L(R,\pi)) \qquad \text{(Lemma 3a)}$$

$$\cong \mathcal{M}(R[\tfrac{1}{l}, l \in L], \pi)/\mathcal{M}_L(R[\tfrac{1}{l}, l \in L], \pi) \stackrel{\text{def}}{=} \mathcal{C}_L(R[\tfrac{1}{l}, l \in L], \pi).$$

For the last equation note that the well-known equivalence $\mathcal{M}/\mathcal{H}_L \cong \mathcal{M}(R[\frac{1}{l}, l \in L], \pi)$ induces an equivalence $\mathcal{M}_L/\mathcal{H}_L \cong \mathcal{M}_L(R[\frac{1}{l}, l \in L], \pi)$, because the ideal $I_L(R[\frac{1}{l}, l \in L], \pi)$ arises from

$I_L(R,\pi)$ by inverting the primes $l \in L$.
There remains to prove the above claim. Let M be an object in $\mathcal{H}_L$. We may assume that $l \cdot M = 0$ for some $l \in L$, because obviously M possesses a filtration by submodules, such that the successive quotients are annihilated by some prime $l \in L$. Now we proceed by induction on $\#\pi_l$. The case $\pi_l = \{1\}$ being clear, we may assume that the center $Z(\pi_l)$ of π_l is not trivial. By Lemma 1 the ideal $I_{Z(\pi)} \stackrel{\text{def}}{=} \ker(R[\pi \rtimes \Gamma] \to R[\pi/Z(\pi_l) \rtimes \Gamma])$ is generated by the elements $[\sigma]-[1]$, $\sigma \in Z(\pi_l)$. Hence some power of $I_{Z(\pi)}$ is contained in the principal ideal (l) and therefore by the inductive hypothesis

$$M \supseteq I_{Z(\pi_l)}M \supseteq \ldots \supseteq I^n_{Z(\pi_l)}M \supseteq 0 \qquad (n >> 0)$$

is a filtration of M by submodules, such that the successive quotients are annihilated by some power of $I_l/I_{Z(\pi_l)}$ and in particular by some power of $I_L \subseteq I_l$.

Proof of Theorem': If the order of π is a unit in R, then for each $\rho \in C(\pi)$ the component $R[\pi]e_\rho$ of $R[\pi]$ is isomorphic to $R{<}\rho{>}$ and the category $\mathcal{C}_L$ obviously is equivalent to the category of $(\prod_{\rho \in (C(\pi) - \cup_{l \in L} C({}_l\pi))/\Gamma} R{<}\Gamma\rho{>}\#\Gamma)$-modules. This shows the theorem in this trivial case.
For the general case we proceed by induction on $\#P(\pi)$. If $\#P(\pi) = 0$, π is trivial and there is nothing to prove. If $\#P(\pi) > 0$, we consider first the case $L = P(\pi)$. By Lemma 4 the category $\mathcal{C}_{P(\pi)}(R,\pi)$ is equivalent to the category $\mathcal{C}_{P(\pi)}(R[\frac{1}{\#\pi}],\pi)$. Further for each $\rho \in C(\pi) - \cup_{l \in P(\pi)} C({}_l\pi)$ we have $P(\pi/\omega_\rho) = P(\pi)$ and in particular $R{<}\Gamma\rho{>} = R[\frac{1}{\#\pi}]{<}\Gamma\rho{>}$. Now the above special case proves the theorem for $L = P(\pi)$. If L is a proper subset of $P(\pi)$, we can choose a prime $p \in P(\pi) - L$ and by descending induction on $\#L$ we can assume that the theorem is already proved for $L \cup \{p\}$.
By Lemma 3a we have

$$\mathcal{C}_{L\cup\{p\}} = \mathcal{C}_L/(\mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L)$$

and by Quillen's localization sequence (see [10]) there is a long exact sequence of groups

$$\ldots \to K'_q(\mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L) \xrightarrow{i_*} K'_q(\mathcal{C}_L) \xrightarrow{j^*} K'_q(\mathcal{C}_{L\cup\{p\}}) \to K'_{q-1}(\mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L) \to \ldots$$

Now the theorem easily follows from the two inductive hypotheses and the following two claims:
a) There is an isomorphism of groups

$$K'_q(\mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L) \cong K'_q(\mathcal{C}_L(R,\ {}_p\pi))$$

b) For each $q \geq 0$ there is a homomorphism

$$\varepsilon_* : K'_q(\mathcal{C}_L) \to K'_q(\mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L)$$

such that $\varepsilon_* \circ i_* = \mathrm{id}$.

To prove claim a) let $\mathcal{N}$ be the full subcategory of $\mathcal{M}_{L\cup\{p\}}$ consisting of those objects M such that $I_pM = 0$. Then the equivalence $\mathcal{M}(R,\ {}_p\pi) \tilde{\to} \mathcal{N}$ induces an equivalence

$$\mathcal{C}_L(R,\ {}_p\pi) \tilde{\to} \mathcal{N}/(\mathcal{N} \cap \mathcal{M}_L),$$

because $I_L(R,\ {}_p\pi) = (I_L(R,\pi) + I_p(R,\pi))/I_p(R,\pi)$. By Lemma 3b) $\mathcal{N}/(\mathcal{N} \cap \mathcal{M}_L)$ is a full subcategory of $\mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L$ closed under taking subobjects, quotients, and finite products. Further for each $M \in \mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L$

$$M \supset I_pM \supset \ldots \supset I_p^nM = 0 \qquad (n >> 0)$$

is a finite filtration of M by submodules, such that the successive quotients are in $\mathcal{N}/(\mathcal{N}\cap\mathcal{M}_L)$. Now Quillen's devissage theorem ([10]) yields the desired isomorphism

$$K'_q(\mathcal{C}_L(R,\, {}_p\pi)) \cong K'_q(\mathcal{N}/(\mathcal{N}\cap\mathcal{M}_L)) \tilde{\to} K'_q(\mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L).$$

Finally the canonical injection ${}_p\pi \to \pi$ yields an exact functor $\varepsilon_* : \mathcal{C}_L(R,\pi) \to \mathcal{C}_L(R,\, {}_p\pi)$, such that the composition

$$\mathcal{C}_L(R,\, {}_p\pi) \tilde{\to} \mathcal{N}/(\mathcal{N}\cap\mathcal{M}_L) \subseteq \mathcal{M}_{L\cup\{p\}}/\mathcal{M}_L \stackrel{i_*}{\tilde{\to}} \mathcal{C}_L(R,\pi) \stackrel{\varepsilon_*}{\tilde{\to}} \mathcal{C}_L(R,\, {}_p\pi)$$

is isomorphic to the identical functor. This proves claim b).

Remarks:
1) If Γ is trivial and π is a p-group the above approach is a slight simplification of [4].
2) If π is abelian and if the Γ-action on π stabilizes each cocyclic subgroup of π, the theorem was established by D. Webb in Proposition (3.1) of [14].
3) In the same way an analogous result for the K'-groups of the category of coherent $(\pi \rtimes \Gamma)$-modules on a (noetherian) scheme can be proved.
4) In order to generalize the above proof to more general groups G note that it relies on following facts:
a) The existence of a normal subgroup of prime power order gives the first step in the induction.
b) If there are several normal p-subgroups commuting elementwise, the induction can be continued.
c) To obtain a decomposition of $K'(\mathbb{Z}[G])$ into a direct sum, a decomposition of G into a product is necessary.
If only assumption a) or b) is fulfilled, we obtain one or more localization sequences, which indeed yield some information about $K'(\mathbb{Z}[G])$ (for instance Lemma 4 remains true) but which perhaps don't split.

§3 Example

In this section we apply the theorem of §2 to the group of upper triangular (3×3)-matrices over a finite field $\mathbb{F}_p$. So we fix a prime p and define B to be the group

$$B := B_3(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & x & y \\ & b & z \\ & & c \end{pmatrix} : a,\, b,\, c \in \mathbb{F}_p^\times,\; x,\, y,\, z \in \mathbb{F}_p \right\}.$$

For any divisor d of $p-1$ and for any $k>0$ let $m_{k,d}$ be the number of cyclic subgroups of order d of $(\mathbb{F}_p^\times)^k$. An easy calculation shows that

$$m_{k,d} = \prod_i q_i^{(k-1)(r_i-1)} \cdot \left(\frac{q_i^k - 1}{q_i - 1}\right),$$

if $d = \prod_i q_i^{r_i}$ is the decomposition of d into prime factors.

Theorem: For any $q \geq 0$ there is a group isomorphism

$$K'_q(\mathbb{Z}[B]) \cong \bigoplus_{d|p-1} \left[K'_q\left(\mathbb{Z}[\frac{1}{d}][\zeta_d]\right)^{m_{3,d}} \oplus K'_q\left(\mathbb{Z}[\frac{1}{p\cdot d}][\zeta_d]\right)^{1+3m_{2,d}} \right].$$

Proof: Let $T \cong (\mathbb{F}_p^\times)^3$ be the subgroup of B consisting of diagonal matrices and let

$$U := \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} : x,\ y,\ z \in \mathbb{F}_p \right\} \subset B$$

be the unipotent subgroup. Then B is the semidirect product $U \rtimes T$ and

$$N := \left\{ \begin{pmatrix} 1 & 0 & y \\ & 1 & 0 \\ & & 1 \end{pmatrix} : y \in \mathbb{F}_p \right\} \cong \mathbb{F}_p$$

is a T-equivariant normal subgroup of U, such that U/N is isomorphic to $\mathbb{F}_p \times \mathbb{F}_p$. Let ε, $\rho_0, \dots \rho_p$, ω be the following simple $\mathbb{Q}[U]$-algebras (ζ_p denotes a primitive p-th root of unity):

$$\begin{array}{llllll} \varepsilon : & \mathbb{Q}[U] & \overset{\text{can}}{\to} & \mathbb{Q} & & \\ \rho_\alpha : & \mathbb{Q}[U] & \overset{\text{can}}{\to} & \mathbb{Q}[U/N] & \to & \mathbb{Q}(\zeta_p) \quad (\alpha \in \{0, \dots, p-1\}) \\ & & & [(x,z)] & \mapsto & \zeta_p^{x-\alpha z} \\ \rho_p : & \mathbb{Q}[U] & \overset{\text{can}}{\to} & \mathbb{Q}[U/N] & \to & \mathbb{Q}(\zeta_p) \\ & & & [(x,z)] & \mapsto & \zeta_p^{z} \\ \omega : & \mathbb{Q}[U] & \to & \mathbb{Q}(\zeta_p)^{(\mathbb{F}_p)} \# \mathbb{F}_p & \cong & M_p(\mathbb{Q}(\zeta_p)) \\ & [\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}] & \mapsto & \zeta_p^y \{z\} \sum\limits_{x' \in \mathbb{F}_p} \zeta_p^{x'x} e_{x'} & & \end{array}$$

One easily checks that ω is multiplicative and surjective and that $\varepsilon,\ \rho_0, \dots, \rho_p,\ \omega$ are the Wedderburn components of $\mathbb{Q}[U]$, i. e. $C(U) \cong \{\varepsilon, \rho_0, \dots, \rho_p, \omega\}$.

Lemma 5:
1) $C(U)/T = \{\{\varepsilon\}, \{\rho_0\}, \{\rho_1, \dots, \rho_{p-1}\}, \{\rho_p\}, \{\omega\}\}$
2) One has the following five isomorphisms of rings:

$$a)\mathbb{Z}{<}\varepsilon{>}\#T \cong \mathbb{Z}[\mathbb{F}_p^\times \times \mathbb{F}_p^\times \times \mathbb{F}_p^\times]$$

$$b)\mathbb{Z}{<}\rho_0{>}\#T \cong M_{p-1}\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times \times \mathbb{F}_p^\times]\right)$$

$$c)\mathbb{Z}{<}\{\rho_1, \dots, \rho_{p-1}\}{>}\#T \cong M_{(p-1)^2}\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times]\right)$$

$$d)\mathbb{Z}{<}\rho_p{>}\#T \cong M_{p-1}\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times \times \mathbb{F}_p^\times]\right)$$

$$e)\mathbb{Z}{<}\omega{>}\#T \cong M_{p(p-1)}\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times \times \mathbb{F}_p^\times]\right)$$

Applying the theorem of §2 to $B = U \rtimes T$ and using Lemma 5 and Morita equivalence we obtain

$$K_q'(\mathbb{Z}[B]) \cong K_q'\left(\mathbb{Z}[\mathbb{F}_p^\times \times \mathbb{F}_p^\times \times \mathbb{F}_p^\times]\right) \oplus K_q'\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times \times \mathbb{F}_p^\times]\right)^3 \oplus K_q'\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times]\right).$$

Applying the theorem of §2 again we obtain the desired calculation of $K'_q(\mathbb{Z}[B])$.

Proof of Lemma 5: Obviously T fixes ε and ω. Using the identity

$$(*) \qquad \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \cdot \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ab^{-1}x & ac^{-1}y \\ & 1 & bc^{-1}z \\ & & 1 \end{pmatrix},$$

we easily see that T fixes ρ_0 and ρ_p and permutes $\rho_1, \dots, \rho_{p-1}$ transitively. This proves the first statement.

Ad a): This is clear.

Ad b): By definition we have $\mathbb{Z}{<}\rho_0{>} \cong \mathbb{Z}[\frac{1}{p}][\zeta_p]$ and because of (*) the element $(a,b,c) \in T$ acts on $\mathbb{Z}{<}\rho_0{>}$ by $\zeta_p \mapsto \zeta_p^{ab^{-1}}$. Hence the subgroup $T_1 := \{(a,1,1) : a \in \mathbb{F}_p^\times\} \subset T$ acts as Galois group on $\mathbb{Z}[\frac{1}{p}][\zeta_p]$ and $T_{2,3} := \{(b,b,c) : b,c \in \mathbb{F}_p^\times\}$ acts trivially. This yields the desired isomorphism

$$\mathbb{Z}{<}\rho_0{>}\#T = \mathbb{Z}[\frac{1}{p}][\zeta_p]\#(T_1 \times T_{2,3}) \cong M_{p-1}\left(\mathbb{Z}[\frac{1}{p}]\right)[T_{2,3}] \cong M_{p-1}\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times \times \mathbb{F}_p^\times]\right).$$

Ad c): By definition we have $\mathbb{Z}{<}\{\rho_1, \dots, \rho_{p-1}\}{>} \cong \left(\mathbb{Z}[\frac{1}{p}][\zeta_p]\right)^{(\mathbb{F}_p^\times)}$ and because of (*) the element $(a,b,c) \in T$ acts on $\mathbb{Z}{<}\{\rho_1, \dots, \rho_{p-1}\}{>}$ by $\zeta_p e_\alpha \mapsto \zeta_p^{ab^{-1}} e_{ab^{-2}c\alpha}$. Hence the subgroup $\Delta := \{(a,a,a) : a \in \mathbb{F}_p^\times\}$ acts trivially, $T_2 := \{(1,b,b^2) : b \in \mathbb{F}_p^\times\}$ acts as Galois group on each factor and $T_3 := \{(1,1,c) : c \in \mathbb{F}_p^\times\}$ acts by permuting the factors. This yields the desired isomorphism:

$$\mathbb{Z}{<}\{\rho_1, \dots, \rho_{p-1}\}{>}\#T = \left(\mathbb{Z}[\frac{1}{p}][\zeta_p]^{(\mathbb{F}_p^\times)}\right)\#(T_2 \times T_3 \times \Delta)$$

$$\cong \left(M_{p-1}\left(\mathbb{Z}[\frac{1}{p}]\right)^{(\mathbb{F}_p^\times)}\right)\#(T_3 \times \Delta) \cong M_{p-1}\left(M_{p-1}\left(\mathbb{Z}[\frac{1}{p}]\right)\right)[\Delta] \cong M_{(p-1)^2}\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times]\right)$$

Ad d): This can be proved in the same way as b).

Ad e): Because $\det\left((\zeta_p^{xx'})_{x,x' \in \mathbb{F}_p}\right)$ is a unit in $\mathbb{Z}[\frac{1}{p}][\zeta_p]$ (Vandermonde), the restricted map $\omega : \mathbb{Z}[\frac{1}{p}][U] \to \mathbb{Z}[\frac{1}{p}][\zeta_p]^{(\mathbb{F}_p)}\#\mathbb{F}_p$ remains surjective and we have $\mathbb{Z}{<}\omega{>} \cong \mathbb{Z}[\frac{1}{p}][\zeta_p]^{(\mathbb{F}_p)}\#\mathbb{F}_p$. Because of (*) the element $(a,b,c) \in T$ acts on $\mathbb{Z}{<}\omega{>}$ by $\zeta_p e_x\{z\} \mapsto \zeta_p^{ac^{-1}} e_{bc^{-1}}\{bc^{-1}z\}$. Hence the subgroup Δ acts trivially, T_1 acts as Galois group on $\mathbb{Z}[\frac{1}{p}][\zeta_p]$ and the element $(c^{-1}, 1, c^{-1}) \in T'_3 := \{(c,1,c) : c \in \mathbb{F}_p^\times\}$ acts by $\zeta_p e_x\{z\} \mapsto \zeta_p e_{cx}\{cz\}$. By Lemma 2 this yields the desired isomorphism:

$$\mathbb{Z}{<}\omega{>}\#T \cong \left(\mathbb{Z}[\frac{1}{p}][\zeta_p]^{(\mathbb{F}_p)}\#\mathbb{F}_p\right)\#(T_1 \times T'_3 \times \Delta) \cong \left(\left(M_{p-1}\left(\mathbb{Z}[\frac{1}{p}]\right)\right)^{(\mathbb{F}_p)}\#\mathbb{F}_p\right)\#(T'_3 \times \Delta)$$

$$\cong \left(\left(M_{p-1}\left(\mathbb{Z}[\frac{1}{p}]\right)[T'_3]\right)^{(\mathbb{F}_p)}\#\mathbb{F}_p\right)[\Delta] \cong M_{p(p-1)}\left(\mathbb{Z}[\frac{1}{p}][\mathbb{F}_p^\times \times \mathbb{F}_p^\times]\right).$$

Remark: One easily checks that this example confirms the conjecture of Hambleton, Taylor and Williams ([4]).

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