THE LEFSCHETZ THEOREM IN HIGHER EQUIVARIANT K-THEORY

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Introduction

In this paper we prove a Lefschetz-Grothendieck-Riemann-Roch formula in higher equivariant K-theory of nonsingular projective varieties over an algebraically closed field k:

Let X be a variety equipped with an action of a group or more generally of a monoid G. By a G-module on X we mean an $\mathcal{O}_X$-module $\mathcal{E}$ together with in $g \in G$ functorial homomorphisms

$$g : g^*\mathcal{E} \to \mathcal{E}.$$

We denote the exact category of locally free G-modules on X by $\mathcal{P}(G, X)$ and define the q-th equivariant K-group to be

$$K_q(G, X) := K_q(\mathcal{P}(G, X))$$

(see [10]). Let $K(G, X)$ be the direct sum $\oplus_{q \geq 0} K_q(G, X)$. The Lefschetz-Riemann-Roch problem is to compute the covariant map

$$f_* : K(G, X) \to K(G, Y)$$

for any projective G-morphism $f : X \to Y$. For example, if $f : X \to \mathrm{Spec}(k)$ is the structure morphism of X, in case $q = 0$ this means for any locally free G-module $\mathcal{E}$ on X to compute the alternating sum

$$\sum_{i\geq 0}(-1)^i[H^i(X,\mathcal{E})]$$

of the virtual representations $[H^i(X,\mathcal{E})]$ of G on the cohomology groups $H^i(X,\mathcal{E})$, the socalled Lefschetz trace.

Lefschetz theorem: If the action of G on X and Y factorizes through a fixed finite abelian group G', after a suitable localization, the following commtative diagram exists:

$$\begin{array}{ccc} K(G,X) & \stackrel{\lambda_{-1}(\mathcal{N}_X)^{-1}\cdot i_X^*}{\longrightarrow} & K(G,X^G) \\ f_* \downarrow & & \downarrow f_* \\ K(G,Y) & \stackrel{\lambda_{-1}(\mathcal{N}_Y)^{-1}\cdot i_Y^*}{\longrightarrow} & K(G,Y^G); \end{array}$$

here $i_X : X^G \to X$ and $i_Y : Y^G \to G$ denote the embeddings of the fixed point varieties and $\mathcal{N}_X$ and $\mathcal{N}_Y$ denote the conormal sheaves of these embeddings.

The proof (see section 3) essentially consists of two ingredients: firstly the excess intersection formula (see section 2) and secondly the computation of higher equivariant K-theory of a projective G-fibre-bundle (see [6] or [14] or [1]).

By combining the Lefschetz theorem with the Grothendieck-Riemann-Roch theorem for higher K-theory (see [13] or [12]) we obtain the Lefschetz-Grothendieck-Riemann-Roch formula mentioned in the beginning.

That formula generalizes the result of Donovan ([2]) to higher equivariant K-theory. Further Thomason's theorem ([15]) for étale-topological K-theory and our Lefschetz formula overlap in case of a finite abelian group G.

1. Equivariant K-Theory

Let G be a monoid, i. e. a semigroup with 1, and let X be a noetherian scheme equipped with an action of G (i. e., a G-scheme). An $\mathcal{O}_X$-module $\mathcal{E}$ together with in $g \in G$ functorial maps

$$g : g^*\mathcal{E} \to \mathcal{E}$$

is called a G-module on X. The category $\mathcal{M}(G,X)$ respectively $\mathcal{P}(G,X)$ of coherent respectively locally free G-modules on X is an abelian respectively exact category in the sense of [10]. For any $q \geq 0$ let

$$K'_q(G,X) := K_q(\mathcal{M}(G,X))$$

respectively

$$K_q(G,X) := K_q(\mathcal{P}(G,X))$$

denote the corresponding q-th K-group according to [10]. The tensor product of G-modules makes $K_0(G,X)$ a commutative ring with identity element $[\mathcal{O}_X]$ and makes $K_q(G,X)$ and $K'_q(G,X)$ a $K_0(G,X)$-module. Defining the product of two homogenous elements of positive degree in

$$K(G,X) := \bigoplus_{q\geq 0} K_q(G,X)$$

to be zero K(G,X) becomes a graded $K_0(G,X)$-algebra. Obviously $K(G,X)$ is contravariant functorial under G-morphisms.

Example 1

Let G be the cyclic monoid $\mathbb{N}$ and let X be a projective variety over a field k with trivial G-action. In this case $\mathcal{P}(G,X)$ respectively $\mathcal{M}(G,X)$ is the category of pairs $(\mathcal{E},g)$ consisting of a locally free respectively coherent $\mathcal{O}_X$-module $\mathcal{E}$ and an $\mathcal{O}_X$-endomorphism g of $\mathcal{E}$. For any irreducible polynomial $\pi \in k[T]$ let k_π be the field $k[T]/(\pi)$ and let X_π be the base extension $X \otimes_k k_\pi$.

Proposition 1: There is a natural isomorphism of groups

$$K'_q(G,X) \cong \bigoplus_{\pi\in k[T] \text{ irreducible}} K'_q(X_\pi).$$

In particular, if k is algebraically closed, the ring homomorphism

$$\begin{array}{ccc} \mathbb{Z}[k] & \to & K_0(G,\mathrm{Spec}(k)) \\ \sum n_i \cdot [a_i] & \mapsto & \sum n_i \cdot [(k,\mu_{a_i})] \end{array}$$

from the semigroup ring $\mathbb{Z}[k]$ corresponding to the multiplicative monoid k to $K_0(G,\mathrm{Spec}(k))$ is an isomorphism and induces for any smooth X a ringisomorphism

$$K(X)\otimes \mathbb{Z}[k] \tilde{\to} K(G,X).$$

Proof: For any irreducible polynomial $\pi \in k[T]$ let $\mathcal{M}_\pi(G,X)$ be the full subcategory of $\mathcal{M}(G,X)$ consisting of pairs $(\mathcal{E},g)$ with $\pi^n(g)=0$ in $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ for one $n\in\mathbb{N}$. Then the canonical functor

$$\bigoplus_{\pi\in k[T] \text{ irreducible}} \mathcal{M}_\pi(G,X) \tilde{\to} \mathcal{M}(G,X)$$

is an equivalence of categories:
For any pair $(\mathcal{E},g)\in\mathcal{M}(G,X)$ there is a polynomial $\alpha\in k[T]$ with $\alpha(g)=0$, because $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) = \Gamma(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ is a finite dimensional k-algebra. Let

$\alpha = \pi_1^{m_1} \cdots \pi_r^{m_r}$ be the decomposition of α into irreducible polynomials and for any π_i let $\mathcal{E}(\pi_i) := \ker(\pi_i^{m_i}(g)) \subseteq \mathcal{E}$ be the π_i-primary component of $\mathcal{E}$ relative to g. Then the canonical map

$$(\bigoplus_{i=1}^{r} \mathcal{E}(\pi_i), \bigoplus_{i=1}^{r} g|_{\mathcal{E}(\pi_i)}) \tilde{\to} (\mathcal{E}, g)$$

is an isomorphism. This shows that the considered functor is essentially surjective. It is full, because any morphism $(\mathcal{E}, g) \to (\mathcal{F}, h)$ in $\mathcal{M}(G, X)$ maps each primary component of $\mathcal{E}$ relative to g to the corresponding one of $\mathcal{F}$ relative to h.

Now let $\pi \in k[T]$ be an irreducible polynomial. The above equivalence of categories in particular yields that $\mathcal{M}_\pi(G, X)$ is a Serre subcategory of $\mathcal{M}(G, X)$, hence $\mathcal{M}_\pi(G, X)$ is an abelian category. Let $\mathcal{M}_{\pi,1}(G, X) \subseteq \mathcal{M}_\pi(G, X)$ be the full subcategory consisting of pairs $(\mathcal{E}, g)$ with $\pi(g) = 0$. Obviously $\mathcal{M}_{\pi,1}(G, X)$ is nonempty and closed under taking subobjects, quotients and finite products in $\mathcal{M}_\pi(G, X)$, hence again an abelian category. Because any object $(\mathcal{E}, g)$ in $\mathcal{M}_\pi(G, X)$ possesses the finite filtration

$$0 \subseteq (\ker \pi(g), g) \subseteq (\ker \pi^2(g), g) \subseteq \ldots \subseteq (\ker \pi^n(g), g) = (\mathcal{E}, g),$$

whose successive quotients are in $\mathcal{M}_{\pi,1}(G, X)$, by Quillen's devissage theorem ([10], theorem 4, p. 112) the canonical map

$$K_q(\mathcal{M}_{\pi,1}(G, X)) \tilde{\to} K_q(\mathcal{M}_\pi(G, X))$$

is an isomorphism.

Finally we obtain an equivalence of categories

$$\mathcal{M}(X_\pi) \tilde{\to} \mathcal{M}_{\pi,1}(G, X)$$

by mapping an $\mathcal{O}_{X_\pi}$-module $\mathcal{M}$ to the pair $((\alpha_\pi)_*(\mathcal{M}), (\alpha_\pi)_*(\overline{T}))$, where α_π denotes the canonical projection $X_\pi \to X$ and $\overline{T} \in k[T]/(\pi)$ is considered as multiplication endomorphism of $\mathcal{M}$.

Now the first claim of the proposition is proved. For the remaining claims observe that for any smooth X equivariant K'-theory equals equivariant K-theory (see for instance [6]).

2. Excess Intersection Formula

Let G' be a finite group and let $\mathcal{C}$ be the category of projective G'-varieties over a fixed field k. In order to study not only isomorphisms of finite order but even homomorphisms on $\mathcal{O}_X$-modules we furthermore fix a monoid G together with a monoidhomomorphism $G \to G'$ and consider all G'-objects as G-objects, too.

By [6] for any G'-morphism $f: X \to Y$ in $\mathcal{C}$ of complete intersection there is a covariant homomorphism

$$f_*: K(G,X) \to K(G,Y).$$

For example, if $f: X \to \mathrm{Spec}(k)$ is the structure morphism and if $\mathcal{E}$ is a locally free G-module on X, then $f_*[\mathcal{E}]$ is the alternating sum $\sum_i (-1)^{-1}[H^i(X,\mathcal{E})]$ of the virtual representations $[H^i(X,\mathcal{E})]$ of G on the cohomology groups $H^i(X,\mathcal{E})$ of $\mathcal{E}$.

Now let

$$(*) \qquad \begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \psi \downarrow & & \downarrow \phi \\ X & \xrightarrow{f} & Y \end{array}$$

be a fibre square of G'-varieties in $\mathcal{C}$, where f, f_1 are morphisms of complete intersection. We choose a factorization

$$f: X \overset{i}{\hookrightarrow} \mathbb{P}_Y(\mathcal{F}) \xrightarrow{p} Y$$

of f into a regular closed G'-embedding i and into the structure morphism p of a projective G'-fibre-bundle $\mathbb{P}_Y(\mathcal{F})$ (see [6] for existence). This decomposition induces the fibre diagram

$$(**) \qquad \begin{array}{ccccc} X_1 & \overset{i_1}{\hookrightarrow} & \mathbb{P}_{Y_1}(\mathcal{F}_1) & \xrightarrow{p_1} & Y_1 \\ \psi \downarrow & & \downarrow \phi_{\mathbb{P}} & & \downarrow \phi \\ X & \overset{i}{\hookrightarrow} & \mathbb{P}_Y(\mathcal{F}) & \xrightarrow{p} & Y, \end{array}$$

where i and i_1 are regular G'-embeddings. We define the G'-excess-conormal-sheaf $\mathcal{E}$ of diagram $(*)$ to be the G'-excess-conormal-sheaf of the left square in diagram $(**)$, i. e.

$$\mathcal{E} := \ker(\psi^*\mathcal{N} \to \mathcal{N}_1),$$

where $\mathcal{N}$ respectively $\mathcal{N}_1$ is the G'-conormal-sheaf of i respectively i_1. One checks easily that the element $[\mathcal{E}]$ in $K_0(G,X_1)$ doesn't depend on the factorization of f (compare [3]).

Proposition 2 (Excess Intersection Formula): The following diagram of K-groups commutes:

$$\begin{array}{ccc} K(G,X_1) & \xrightarrow{(f_1)_*} & K(G,Y_1) \\ \lambda_{-1}(\mathcal{E}) \cdot \psi^* \uparrow & & \uparrow \phi^* \\ K(G,X) & \xrightarrow{f_*} & K(G,Y). \end{array}$$

Proof: If all maps ψ, ϕ, f, f_1 are regular embeddings and if the excess dimension is zero, the intersection formula is proved in [6]. By equivariant deformation to normal bundle (see [6] or [1]) the intersection formula follows from this in the case that f and f_1 are regular embeddings (compare [3]). The general case can now be proved in the same way as in [3].

3. Lefschetz Theorem

We assume the same situation as in section 2. In addition we assume that G' is abelian and that the ground field k is algebraically closed and that the characteristic of k is prime to the order of G'.

For any X in $\mathcal{C}$ let X^G be the fixed point variety (we omit the apostrophe), i. e. X^G represents the functor

$$\begin{array}{ccc}(\text{Varieties}/k) & \to & \text{Ens} \\ T & \mapsto & \text{Mor}_k(T,X)^{G'}.\end{array}$$

It is the intersection $\cap_{g\in G'} X^g$ of the fixed point varieties X^g defined in EGA I, Proposition 0(1.4.10). Let $i_X : X^G \hookrightarrow X$ denote the corresponding closed embedding and let $\mathcal{N}_X$ denote the conormal sheaf of i_X. By [5] for any smooth X in $\mathcal{C}$ the fixed point variety is smooth, too. In particular $\mathcal{N}_X$ is locally free.

Example 2: Let E be a finite dimensional G'-vector-space over k and let

$$E = \bigoplus_{\chi\in \text{Hom}(G',k^\times)} E_\chi$$

be the decomposition of E into simultanous eigenspaces $E_\chi := \{x \in E : g(x) = \chi(g)\cdot x \text{ for all } g \in G'\}$. Then the canonical surjections $E \to E_\chi$ induce an isomorphism

$$\coprod_\chi \mathbb{P}(E_\chi) \tilde{\to} \mathbb{P}(E)^G$$

between the disjoint union of the projective spaces $\mathbb{P}(E_\chi)$ and the fixed point variety of the action of G' on the projective space $\mathbb{P}(E)$. Furthermore the fixed module $\mathcal{N}^G_{\mathbb{P}(E)}$ of the action of G' on the conormal sheaf $\mathcal{N}_{\mathbb{P}(E)}$ vanishes.

Proof: The first statement is well known. For the second claim we choose a k-basis $x_i, i \in I$, of the vector space E consisting of simultanous eigenvectors. Then G' acts on each affine subspace $D(x_i)$ by linear automorphisms. Now it is clear that for any $i \in I$ the sheaf $\mathcal{N}^G_{D(x_i)}$ vanishes, hence $\mathcal{N}^G_{\mathbb{P}(E)}$ vanishes, too.

Now we introduce the following set of denominators, which will be used in the formulation of the Lefschetz theorem. For this we consider the map

$$\begin{array}{ccc}\text{Hom}(G',k^\times) & \to & K_0(G',k) \\ \chi & \mapsto & [\chi],\end{array}$$

which maps a character χ to the class $[\chi]$ of the onedimensionsal G'-vector-space k equipped with the action of G' induced by χ. Let S denote the set

$$S := \{1 - [\chi] : \chi \in \text{Hom}(G',k^\times)\backslash\{1\}\}$$

and let $K_0(G',k)[S^{-1}]$ denote the localization of $K_0(G',k)$ relative to the multiplicative monoid $<S>$ generated by S. Obviously all co- and contravariant homomorphisms f_* and f^* can be extended to

$$K(G,X)[S^{-1}] := K(G,X) \underset{K_0(G',k)}{\otimes} K_0(G',k)[S^{-1}].$$

Lemma 1: For any smooth X in $\mathcal{C}$ the element

$$\lambda_{-1}(\mathcal{N}_X) := \sum_{i\geq 0}(-1)^{-1}[\Lambda^i\mathcal{N}_X]$$

is a unit in $K_0(G,X^G)[S^{-1}]$.

Proof: We prove first that in the decomposition

$$\mathcal{N}_X = \sum_{\chi}\mathcal{E}_\chi$$

of $\mathcal{N}_X$ into simualtanous eigensheaves $\mathcal{E}_\chi$ (see example 1) the fixed module $\mathcal{E}_1 = \mathcal{N}_X^G$ vanishes. For this we choose an equivariant embedding $i : X \hookrightarrow \mathbb{P}^n$ of X into a projective space $\mathbb{P}^n$ equipped with a linear G'-action (see [6] for existence). Then the diagram of fixed point varieties

$$\begin{array}{ccc} X^G & \to & X \\ \downarrow i & & \downarrow i \\ (\mathbb{P}^n)^G & \to & \mathbb{P}^n \end{array}$$

is a fibre square; hence the corresponding G'-homomorphism $i^*\mathcal{N}_{\mathbb{P}^n} \to \mathcal{N}_X$ is surjective. By taking means we see that the induced homomorphism

$$i^*\mathcal{N}_{\mathbb{P}^n}^G \to \mathcal{N}_X^G$$

between the fixed modules remains surjective. Hence by example 2 the fixed module $\mathcal{N}_X^G$ vanishes.

Now by multiplicativity of λ_{-1} it suffices to show that for any character $\chi \neq 1$ the element $\lambda_{-1}(\mathcal{E}_\chi)$ is a unit in $K_0(G',X^G)[S^{-1}]$. For this let $[\mathcal{E}]$ denote the class in $K_0(X^G)$ of the underlying $\mathcal{O}_X$-module $\mathcal{E}$ of $\mathcal{E}_\chi$ and let n denote the rank of $\mathcal{E}$. Then the element

$$\begin{aligned} \lambda_{-1}(\mathcal{E}_\chi) &= \lambda_{-1}([\mathcal{E}]\cdot[\chi]) \\ &= \lambda_{-1}(n\cdot[\chi]) - (\lambda_{-1}(n\cdot[\chi]) - \lambda_{-1}([\mathcal{E}]\cdot[\chi])) \\ &= (1-[\chi])^n - \sum_i(-1)^i(\tbinom{n}{i} - [\Lambda^i\mathcal{E}])[\chi^i] \end{aligned}$$

is a unit in $K_0(G',X^G)[S^{-1}]$, because $\binom{n}{i} - [\Lambda^i\mathcal{E}]$ is nilpotent in $K_0(X^G)$ for any i (see [3], Proposition 1.5, page 52).

Lefschetz Theorem: Let $f : X \to Y$ be a G'-morphism of smooth projective G'-varieties over k. Then the following diagram commutes:

$$\begin{array}{ccc} K(G,X) & \stackrel{\lambda_{-1}(\mathcal{N}_X)^{-1}\cdot i_X^*}{\longrightarrow} & K(G,X^G)[S^{-1}] \\ \downarrow f_* & & \downarrow f_* \\ K(G,Y) & \stackrel{\lambda_{-1}(\mathcal{N}_Y)^{-1}\cdot i_Y^*}{\longrightarrow} & K(G,Y^G)[S^{-1}] \end{array}$$

In other words: The **Lefschetz map**

$$L_X := \lambda_{-1}(\mathcal{N}_X)^{-1} \cdot i_X^* : K(G,X) \to K(G,X^G)[S^{-1}]$$

is natural for G'-morphisms between smooth projective G'-varieties over k.

Proof. If f is a closed embedding the diagram

$$\begin{array}{rcl} X^G & \stackrel{i_X}{\longrightarrow} & X \\ f \downarrow & & \downarrow f \\ Y^G & \stackrel{i_Y}{\longrightarrow} & Y \end{array}$$

is a fibre square and the class $[\mathcal{E}]$ in $K_0(G,X^G)$ of the excess conormal sheaf $\mathcal{E}$ of this square can be described in two ways:

$$[\mathcal{E}] = f^*[\mathcal{N}_Y] - [\mathcal{N}_X] = i_X^*[\mathcal{N}_{X/Y}] - [\mathcal{N}_{X^G/Y^G}]$$

Now the following computation proves the Lefschetz formula in this case:

$$\begin{array}{ll} f_*(\lambda_{-1}(\mathcal{N}_X)^{-1} \cdot i_X^*) & \\ = f_*(\lambda_{-1}(\mathcal{E}) \cdot f^*(\lambda_{-1}(\mathcal{N}_Y)^{-1}) \cdot i_X^*) & \\ = \lambda_{-1}(\mathcal{N}_Y)^{-1} \cdot f_*(\lambda_{-1}(\mathcal{E}) \cdot i_X^*) & \text{(Projection formula)} \\ = \lambda_{-1}(\mathcal{N}_Y)^{-1} \cdot i_Y^* \circ f_* & \text{(Excess intersection formula).} \end{array}$$

For the general case we choose an equivariant G'-embedding $X \hookrightarrow \mathbb{P}(E)$, where E is a finite dimensional G'-vector-space (see [6] for existence). It induces a factorization

$$X \stackrel{i}{\hookrightarrow} \mathbb{P}(E) \times Y \stackrel{p}{\to} Y$$

of f into a closed immersion i and the G'-projection p. Now the Lefschetz theorem amounts to prove that following diagram commutes:

$$\begin{array}{rcl} K(G,X) & \stackrel{\lambda_{-1}(\mathcal{N}_X)^{-1}\cdot i_X^*}{\longrightarrow} & K(G,X^G)[S^{-1}] \\ i_* \downarrow & & \downarrow i_* \\ K(G,\mathbb{P}(E)\times Y) & \stackrel{\lambda_{-1}(\mathcal{N}_{\mathbb{P}(E)\times Y})^{-1}\cdot i_{\mathbb{P}(E)\times Y}^*}{\longrightarrow} & K(G,\mathbb{P}(E)^G \times Y^G)[S^{-1}] \\ \| & & \downarrow (i_{\mathbb{P}(E)} \times 1)_* \\ K(G,\mathbb{P}(E)\times Y) & \stackrel{p^*\lambda_{-1}(\mathcal{N}_Y)^{-1}\cdot(1\times i_Y)^*}{\longrightarrow} & K(G,\mathbb{P}(E) \times Y^G)[S^{-1}] \\ p_* \downarrow & & \downarrow p_* \\ K(G,Y) & \stackrel{\lambda_{-1}(\mathcal{N}_Y)^{-1}\cdot i_Y^*}{\longrightarrow} & K(G,Y^G)[S^{-1}] \end{array}$$

In this diagram the upper square is commutative by the case considered above. The excess intersection formula (for elementary projections) yields the commutativity of the lower square. For the middle square notice that $\mathcal{N}_{\mathbb{P}(E)\times Y}$ is the sum of the preimages of $\mathcal{N}_{\mathbb{P}(E)}$ and $\mathcal{N}_Y$. Therefore using the multiplicativity of λ_{-1} and the projection formula we may assume that $Y = \mathrm{Spec}(k)$. Then the commutativity of the middle square follows from the formula

$$(*) \qquad (i_{\mathbb{P}(E)})_*(\lambda_{-1}(\mathcal{N}_{\mathbb{P}(E)})^{-1}) = 1 \text{ in } K_0(G, \mathbb{P}(E))[S^{-1}]$$

by means of the projection formula. We prove the equation $(*)$ by generalizing [1] to our situation:
The composition

$$K_0(G, \mathbb{P}(E)^G)[S^{-1}] \xrightarrow{i_*} K_0(G, \mathbb{P}(E))[S^{-1}] \xrightarrow{i^*} K_0(G, \mathbb{P}(E)^G)[S^{-1}]$$

is multiplication with $\lambda_{-1}(\mathcal{N}_{\mathbb{P}(E)})$ by the excess intersection formula, hence an isomorphism by lemma 1. Further both $K_0(G, \mathbb{P}(E))$ and $K_0(G, \mathbb{P}(E)^G)$ are free $K_0(G, \mathrm{Spec}(k))$-modules of rank $\dim(E)$ (see example 2 and for instance [6]), so i^* is injective and formula $(*)$ follows from $i^* i_*(\lambda_{-1}(\mathcal{N}_{\mathbb{P}(E)})^{-1}) = 1$.

Example 3: Let X be a projective k-variety and let g be an automorphism of X of finite order being prime to char(k). We assume that g fixes only one point, say x. Then $\mathcal{N}_X$ is the cotangent space $\check{T}_x$ in x. Writing $[\check{T}_x] = \sum_i n_i[a_i]$ in $K_0(\mathbb{N}, k) \cong \mathbb{Z}[k]$ according to example 1 we have

$$\lambda_{-1}(\mathcal{N}_X) = \prod_i (1 - [a_i])^{n_i}$$

and in particular we obtain for any locally free $\mathcal{O}_X$-module $\mathcal{E}$ equipped with an $\mathcal{O}_X$-homomorphism $g : g^*\mathcal{E} \to \mathcal{E}$:

$$\sum_{i\geq 0} (-1)^i \mathrm{trace}(H^i(g)|H^i(X, \mathcal{E})) = \frac{\mathrm{trace}(g_x|\mathcal{E}_x)}{\det(1 - g|\check{T}_x)};$$

here $\mathcal{E}_x$ denotes the fibre of $\mathcal{E}$ in x. Obviously this formula can be generalized to the case of several isolated fixed points.

Finally we combine the Lefschetz formula with the Grothendieck-Riemann-Roch formula, which we first recall (see [13] or [12]):

For any smooth k-variety X let $K(X) := \sum_{q\geq 0} K_q(X)$ denote the "Grothendieck ring" of X. It carries a λ-ring structure augmented by the binomial ring $H^0(X, \mathbb{Z})$. Let $Gr^{\cdot}K(X)$ denote the induced graded ring and let

$$ch : K(X) \to Gr^{\cdot}K(X)_{\mathbb{Q}}$$

be the Chern character. Then for any projective morphism $f : X \to Y$

between smooth k-varieties X, Y the following diagram commutes:

$$\begin{array}{ccc} K(X) & \stackrel{Td(T_X)\cdot ch}{\longrightarrow} & Gr^{\cdot}K(X)_{\mathbb{Q}} \\ \downarrow f_* & & \downarrow f_* \\ K(Y) & \stackrel{Td(T_Y)\cdot ch}{\longrightarrow} & Gr^{\cdot}K(Y)_{\mathbb{Q}}; \end{array}$$

here $Td(T_X)$ respectively $Td(T_Y)$ denotes the Todd class of the tangent element T_X respectively T_Y of X respectively Y.

Corollary (Lefschetz-Grothendieck-Riemann-Roch Theorem): Let G be the cyclic monoid $\mathbb{N}$ and let G' be the cyclic group $\mathbb{Z}/(n)$ (where n is prime to char(k)) and let $G \to G'$ be the canonical monoidhomomorphism. Then for any G'-morphism $f : X \to Y$ between projective smooth G'-varieties over k the following diagram commutes:

$$\begin{array}{ccc} K(G,X) & \stackrel{Td(T_{X^G})\cdot(ch\otimes 1)\circ(\lambda_{-1}(\mathcal{N}_X)^{-1}\cdot i_X^*)}{\longrightarrow} & Gr^{\cdot}K(X^G)\otimes\mathbb{Q}[k][S^{-1}] \\ \downarrow f_* & & \downarrow f_*\otimes 1 \\ K(G,Y) & \stackrel{Td(T_{Y^G})\cdot(ch\otimes 1)\circ(\lambda_{-1}(\mathcal{N}_Y)^{-1}\cdot i_Y^*)}{\longrightarrow} & Gr^{\cdot}K(Y^G)\otimes\mathbb{Q}[k][S^{-1}] \end{array}$$

Proof. Note that $K(G, X^G) = K(X^G) \otimes \mathbb{Z}[k]$ by Proposition 1.

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