Shuffle Products in Higher K-Theory

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Abstract. We construct shuffle products in higher K-theory. The fundamental observation for this is that the following assignments can be melted into one another in an entirely natural fashion. On the one hand to any locally free modules $V_1, \ldots, V_k$ we assign the module $\oplus_\sigma (\otimes_{r=1}^p V_{\sigma(r)}) \otimes (\otimes_{r=p+1}^k V_{\sigma(r)})$ and on the other hand to any chain $V_1 \hookrightarrow \ldots \hookrightarrow V_k =: V$ of admissible monomorphisms we assign the submodule $\sum_\sigma (\Lambda_{r=1}^p V_{\sigma(r)}) \otimes (\Lambda_{r=p+1}^k V_{\sigma(r)})$ of $\Lambda^p V \otimes \Lambda^{k-p} V$. In both cases the (direct) sum is taken over all $(p, k-p)$-shuffles σ. By means of these shuffle products we show that the exterior power operations in higher K-theory defined by D. Grayson are compatible (already on the simplicial level) with the direct sum and with the symmetric power operations in the expected way. Furthermore we investigate the connection between the shuffle products and the usual products in higher K-theory.

Introduction

The λ-structure on the higher K-theory of a scheme X is the fundamental prerequisite to formulate and to prove theorems of Riemann-Roch-type in higher K-theory (for instance, see [14], [15], [8]) and to define motivic cohomology (for instance, see [12]). The content of this paper is to study the λ-structure not only on the higher K-groups $K_q(X)$, but already on the topological space (respectively on the simplicial set) whose homotopy groups are the higher K-groups.

On the Grothendieck ring $K_0(X)$ of the category of locally free $\mathcal{O}_X$-modules the λ-operation

$$\lambda^k : K_0(X) \to K_0(X)$$

is essentially the map $\mathcal{E} \mapsto \Lambda^k \mathcal{E}$ where $\Lambda^k \mathcal{E}$ is the k-th exterior power of the locally free

$\mathcal{O}_X$-module $\mathcal{E}$. On higher K-groups for affine X the λ-operations

$$\lambda^k : K_q(X) \to K_q(X), \quad k \geq 1,$$

were first defined by Ch. Kratzer ([10]) and by D. Quillen (exposed by H. L. Hiller in [7]). In [4] D. Grayson gives a new, more general and more natural construction. For this in [3] a simplicial set G was constructed whose q-th homotopy group is the q-th K-group of X (without index shifting $q \leftrightarrow q+1$ as in [11]). By taking into consideration the so-called Koszul filtration of exterior powers, D. Grayson generalizes G to a certain simplicial set $\mathcal{H}^k$ (see section 1) and realizes the λ-operation λ^k already on the topological level, namely as the composition

$$|G| \xrightarrow{|\Lambda^k|} |\mathcal{H}^k| \xrightarrow{\Xi^k} |G|$$

of the simplicial map Λ^k (induced by taking exterior powers) and a certain continuous map Ξ^k.

The aim of this paper is to prove two of the three axioms of a λ-structure (see [2]): We will prove the rule $\lambda^k(x+y) = \sum_{p=0}^k (\lambda^p x) \cdot (\lambda^{k-p} y)$ on the simplicial level (see section 5) and we will prove the rule $\lambda^k(x \cdot y) = P_k(\lambda^1 x, \ldots, \lambda^k x, \lambda^1 y, \ldots, \lambda^k y)$ at least on the level of K-groups (see section 7). Furthermore we show the expected connection $\sum_{p=0}^k (-1)^p (\lambda^p x) \cdot (s^{k-p} x) = 0$ between the exterior powers λ^k and the symmetric powers s^k (see section 6) on the simplicial level.

The essential device for this and so the central part of this paper is the construction of certain simplicial maps

$$\otimes : \mathcal{H}^p \times \mathcal{H}^{k-p} \to \mathcal{H}^k, \quad k \geq p \geq 1,$$

(see section 3) which will serve us as products in higher K-theory and which we will call shuffle products because of their similarity to the shuffle products in (co)homology.

We conjecture that these shuffle products are compatible with the (usual) products defined in [3] in the following sense: The diagram

$$\begin{array}{ccc} |\mathcal{H}^p| \times |\mathcal{H}^{k-p}| & \xrightarrow{|\otimes|} & |\mathcal{H}^k| \\ \downarrow \Xi^p \times \Xi^{k-p} & & \downarrow \Xi^k \\ |G| \times |G| & \xrightarrow{|\otimes|} & |G| \end{array}$$

of continuous maps commutes up to homotopy. In section 4 we will prove this conjecture in the case $k = 2$, $p = 1$, and in the case $k > 2$ we will prove it after restricting the diagram to certain subspaces.

In order to get a quick impression of the structure of the shuffle products the reader should take a look at the objects in the exact sequences (for the cases $k = 2$ and $k = 3$) listed in section 6.

I would like to thank J. Stienstra and F. Kouwenhoven for stimulating discussions on this topic during a short stay at the university of Utrecht.

1. The Work of H. Gillet and D. Grayson

The content of this section is to recall the notations of [3] and [4] used in the succeeding sections (for the reader's convenience) and to introduce some new notations (for example: $\mathcal{H}^k$).

Let $\mathcal{M}$ be an exact category in the sense of [11]. At first we will recall the axiomatic definition of exterior power operations on $\mathcal{M}$ given by D. Grayson in section 7 of [4].

For any $k \in \mathbb{N}$ let $F_k(\mathcal{M})$ be the category of chains $V_1 \hookrightarrow \ldots \hookrightarrow V_k$ of admissible monomorphisms in $\mathcal{M}$. We assume we are given a bi-exact, associative functor

$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

and functors

$$\begin{array}{ccc} F_k(\mathcal{M}) & \to & \mathcal{M} \\ V_1 \hookrightarrow \ldots \hookrightarrow V_k & \mapsto & V_1 \wedge \ldots \wedge V_k \end{array}$$

which satisfy the following axioms:

For any chain $V \hookrightarrow \ldots \hookrightarrow W \hookrightarrow X \hookrightarrow \ldots \hookrightarrow Y$ there are natural maps

(E1) $$(V \wedge \ldots \wedge W) \otimes (X \wedge \ldots \wedge Y) \to V \wedge \ldots \wedge W \wedge X \wedge \ldots \wedge Y$$

and

(E2) $$V \wedge \ldots \wedge W \wedge X \wedge \ldots \wedge Y \to (V \wedge \ldots \wedge W) \otimes (\tfrac{X}{W} \wedge \ldots \wedge \tfrac{Y}{W})$$

which are associative in the obvious sense and which satisfy the following compatibility conditions: For any chain $U \hookrightarrow \ldots \hookrightarrow V \hookrightarrow W \hookrightarrow \ldots \hookrightarrow X \hookrightarrow Y \hookrightarrow \ldots \hookrightarrow Z$ the following diagrams commute:

$$(U\wedge\ldots\wedge V\wedge W\wedge\ldots\wedge X)\otimes(Y\wedge\ldots\wedge Z) \quad \stackrel{\text{(E1)}}{\longrightarrow} \quad U\wedge\ldots\wedge V\wedge W\wedge\ldots\wedge X\wedge Y\wedge\ldots\wedge Z$$

(E3) $$\downarrow \text{(E2)} \qquad\qquad\qquad\qquad \downarrow \text{(E2)}$$

$$(U\wedge\ldots\wedge V)\otimes(\tfrac{W}{V}\wedge\ldots\wedge\tfrac{X}{V})\otimes(\tfrac{Y}{V}\wedge\ldots\wedge\tfrac{Z}{V}) \quad \stackrel{\text{(E1)}}{\longrightarrow} \quad (U\wedge\ldots\wedge V)\otimes(\tfrac{W}{V}\wedge\ldots\wedge\tfrac{X}{V}\wedge\tfrac{Y}{V}\wedge\ldots\wedge\tfrac{Z}{V})$$

and

$$(U\wedge\ldots\wedge V)\otimes(W\wedge\ldots\wedge X\wedge Y\wedge\ldots\wedge Z) \quad \stackrel{\text{(E1)}}{\longrightarrow} \quad U\wedge\ldots\wedge V\wedge W\wedge\ldots\wedge X\wedge Y\wedge\ldots\wedge Z$$

(E4) $$\downarrow \text{(E2)} \qquad\qquad\qquad\qquad \downarrow \text{(E2)}$$

$$(U\wedge\ldots\wedge V)\otimes(W\wedge\ldots\wedge X)\otimes(\tfrac{Y}{X}\wedge\ldots\wedge\tfrac{Z}{X}) \quad \stackrel{\text{(E1)}}{\longrightarrow} \quad (U\wedge\ldots\wedge V\wedge W\wedge\ldots\wedge X)\otimes(\tfrac{Y}{X}\wedge\ldots\wedge\tfrac{Z}{X})$$

(E5) Given $U\hookrightarrow\ldots\hookrightarrow V\hookrightarrow W'\hookrightarrow W\hookrightarrow X\hookrightarrow\ldots\hookrightarrow Y$ the sequence

$$0\to U\wedge\ldots\wedge V\wedge W'\wedge X\wedge\ldots\wedge Y\to U\wedge\ldots\wedge V\wedge W\wedge X\wedge\ldots\wedge Y\to(U\wedge\ldots\wedge V)\otimes(\frac{W}{W'}\wedge\frac{X}{W'}\wedge\ldots\wedge\frac{Y}{W'})\to 0$$

is an exact sequence.

We will call an exact category with these properties an *exact category with power operations*.

Example: The most important example for $\mathcal{M}$ is the category of locally free $\mathcal{O}_X$-modules of finite presentation on a locally ringed space X. Then besides the exterior power operations also the symmetric power operations satisfy the above axioms and the module $V_1\wedge\ldots\wedge V_k$ is defined to be

$$\text{Image}(V_1\otimes\ldots\otimes V_k \stackrel{\text{(E1)}}{\to} \Lambda^k V_k) \quad \text{resp.} \quad \text{Image}(V_1\otimes\ldots\otimes V_k \stackrel{\text{(E1)}}{\to} S^k V_k).$$

As we already mentioned in the introduction the λ-operations on $K_q(\mathcal{M})$ are realized as simplicial maps between certain simplicial sets. We will now recall these simplicial sets.

Let Ord be the category of totally ordered finite sets and let A be an object in Ord. Then $\gamma(A)$ is defined to be the disjoint union $\{L,R\}\cup A$ being ordered in such a way that A is an ordered subset of $\gamma(A)$ and that $L<a$ and $R<a$ for all $a\in A$. The elements L and R are not comparable. For any $k\in\mathbb{N}$ let $\Gamma^k(A)$ be the set of collections

$$\alpha=(\frac{i_1}{l_1},*_2,\frac{i_2}{l_2},*_3,\ldots,*_k,\frac{i_k}{l_k})$$

where for each r we have

(A1) $i_r\in A$, $l_r\in\gamma(A)$, $*_r\in\{\wedge,\otimes\}$

(A2) $l_r \leq i_r$

(A3) if $r > 1$ and $*_r = \wedge$, then $l_{r-1} = l_r$ and $i_{r-1} \leq i_r$.

We will sometimes denote i_r by $i_r(\alpha)$, l_r by $l_r(\alpha)$ and $*_r$ by $*_r(\alpha)$. We define an order on $\Gamma^k(A)$ in the following way: There is an arrow $\alpha \to \alpha'$, iff for each r we have

(B1) $i_r \leq i'_r$

(B2) $l_r \leq l'_r$

(B3) if $*_r = \wedge$ and $*'_r = \otimes$ then $i_{r-1} \leq l'_r$.

Remark:

For $A = \{0 < \ldots < n\}$ we have the following picture of the category $\Gamma(A) := \Gamma^1(A)$:

$$
\begin{array}{ccccccccc}
 & \frac{0}{R} & \to & \frac{1}{R} & \to & \cdots & \to & \frac{n}{R} \\
 & & & & & & & \\
\frac{0}{L} & \to & \frac{1}{L} & \to & \cdots & \to & \frac{n}{L} & \\
 & & & & & & & \\
 & \frac{0}{0} & \to & \frac{1}{0} & \to & \cdots & \to & \frac{n}{0} \\
 & & & \downarrow & & & & \downarrow \\
 & & & \frac{1}{1} & \to & \cdots & \to & \frac{n}{1} \\
 & & & & \ddots & & & \downarrow \\
 & & & & & & & \vdots \\
 & & & & & & \ddots & \downarrow \\
 & & & & & & & \frac{n}{n}
\end{array}
$$

Finally we call a sequence $\alpha' \to \alpha \to \alpha''$ in $\Gamma^k(A)$ *exact*, iff there are integers $p \leq s$ such that

(C1) for any r with $r < p$ or $s < r$ we have: $i'_r = i_r = i''_r$, $l'_r = l_r = l''_r$, $*'_r = *_r = *''_r$

(C2) for any r satisfying $p < r \leq s$ we have: $*'_r = *_r = *''_r = \wedge$, $i'_r = i_r = i''_r$

(C3) $l_p = l'_p \leq i'_p = l''_p \leq i''_p = i'_p$, $*'_p = *_p$ and $*''_p = \otimes$.

More generally for any $A_1, \ldots, A_n \in \mathrm{Ord}$ and for any $k_1, \ldots, k_n \in \mathbb{N}$ we call a sequence $\alpha' \to \alpha \to \alpha''$ in $\Gamma^{k_1}(A_1) \times \ldots \times \Gamma^{k_n}(A_n)$ *exact*, iff there is an index $h \in \{1, \ldots, n\}$ such that $\alpha'_i = \alpha_i = \alpha''_i$ for all $i \neq h$ and such that the sequence $\alpha'_h \to \alpha_h \to \alpha''_h$ is exact.

For any exact category $\mathcal{M}$ and any $k_1, \ldots, k_n \in \mathbb{N}$ we define the multisimplicial set $\mathcal{H}^{k_1,\ldots,k_n}\mathcal{M}$

by

$$(\mathcal{H}^{k_1,\ldots,k_n}\mathcal{M})(A_1,\ldots,A_n) := \mathrm{Exact}(\Gamma^{k_1}(A_1)\times\ldots\times\Gamma^{k_n}(A_n),\mathcal{M})$$

($A_1,\ldots,A_n \in \mathrm{Ord}$), a functor from $\Gamma^{k_1}(A_1)\times\ldots\times\Gamma^{k_n}(A_n)$ to $\mathcal{M}$ being *exact*, iff it transforms exact sequences into short exact sequences of $\mathcal{M}$ and if also any α of the form $(\ldots,\frac{i}{i},\ldots)$ is mapped to a previously chosen zero object 0 of $\mathcal{M}$. If $k_1=\ldots=k_n=1$ we write $G^n\mathcal{M}$ for $\mathcal{H}^{1,\ldots,1}\mathcal{M}$ and if in addition $n=1$ we simply write $G\mathcal{M}$.

Theorem.

a) There is a natural homotopy equivalence

$$|G\mathcal{M}| \simeq \Omega|Q\mathcal{M}|$$

between the geometric realization $|G\mathcal{M}|$ of $G\mathcal{M}$ and the loopspace $\Omega|Q\mathcal{M}|$ of the classifying space $|Q\mathcal{M}|$ of the Quillen category $Q\mathcal{M}$ (see [11]). The identification $\pi_0(|G\mathcal{M}|)\cong K_0(\mathcal{M})$ is induced by the map

$$\begin{array}{rcl} G\mathcal{M}(\{0\}) & \to & K_0(\mathcal{M}) \\ M & \mapsto & [M(\frac{0}{L})]-[M(\frac{0}{R})]. \end{array}$$

b) For each $k\in\mathbb{N}$ the k-simplicial maps ($p=1,\ldots,k$)

$$\begin{array}{rrcl} j_p: & pr_p^*(G\mathcal{M}) & \to & G^k\mathcal{M} \\ & M & \mapsto & \left((\frac{i_1}{j_1},\ldots,\frac{i_k}{j_k}) \mapsto \begin{cases} M(\frac{i_p}{j_p}) & \text{, if } j_r = L \text{ for all } r\neq p \\ 0 & \text{, else} \end{cases} \right) \end{array}$$

induce homotopy equivalences between the geometric realizations and on homotopy groups they all induce the same isomorphism (The k-simplicial set $pr_p^*(G\mathcal{M})$ is given by $pr_p^*(G\mathcal{M})(A_1,\ldots,A_k)$ $:= G\mathcal{M}(A_p)$ for $A_1,\ldots,A_k\in\mathrm{Ord}$).

Proof. See [3].

According to this theorem we will always identify the K-groups of $\mathcal{M}$ with the homotopy groups of $|G\mathcal{M}|$ and more generally with the homotopy groups of $|G^k\mathcal{M}|$ for each $k\in\mathbb{N}$. Note that changing the index p in b) does *not* change the sign of the identification though claimed in [3].

Now we fix $k\geq 1$. To define the λ-operation λ^k on higher K-groups we need one more construction: The *simplicial subdivision* yields a canonical procedure to transform a simplicial set $X:\mathrm{Ord}\to\mathrm{Ens}$ into a k-simplicial set.

For this the *concatenation* $A_1 \cdot \ldots \cdot A_k$ of the totally ordered sets $A_1, \ldots, A_k$ is defined to be the disjoint union $A_1 \dot\cup \ldots \dot\cup A_k$ equipped with the total order determined by the following conditions: Each set A_r is an ordered subset of $A_1 \cdot \ldots \cdot A_k$ and $a < b$ whenever $a \in A_r$ and $b \in A_s$ with $r < s$. Then the multisimplicial set $\mathrm{Sub}_k X$ is defined by

$$\mathrm{Sub}_k X(A_1, \ldots, A_k) := X(A_1 \cdot \ldots \cdot A_k)$$

($A_1, \ldots, A_k \in \mathrm{Ord}$).

Theorem. *There is a natural homeomorphism*

$$|\mathrm{Sub}_k X| \cong |X|$$

between the geometric realizations of $Sub_k X$ *and* X *.*

Proof. See [4].

According to this theorem we will always identify the topological space $|\mathrm{Sub}_k X|$ with $|X|$.

For any $A_1, \ldots, A_k \in \mathrm{Ord}$ we have the following exact functor

$$\begin{array}{rccc} \Xi^k: & \mathit{\Gamma}(A_1) \times \ldots \times \mathit{\Gamma}(A_k) & \to & \mathit{\Gamma}^k(A_1 \cdot \ldots \cdot A_k) \\ & (\frac{i_1}{j_1}, \ldots, \frac{i_k}{j_k}) & \mapsto & (\frac{i_1}{l_1}, *_2, \ldots, *_k, \frac{i_k}{l_k}) \end{array}$$

where we define $l_1 := j_1$ and then inductively for $r > 1$ we declare:

(D1) if $j_r = L$ then $*_r := \wedge$ and $l_r := l_{r-1}$

(D2) if $j_r \neq L$ then $*_r := \otimes$ and $l_r := j_r$.

For any exact category $\mathcal{M}$ these functors induce a multisimplicial map

$$\Xi^k : \mathrm{Sub}_k \mathcal{H}^k \mathcal{M} \to G^k \mathcal{M}.$$

Furthermore for any exact category $\mathcal{M}$ with power operations we obtain a simplicial map

$$\begin{array}{rccc} \mathit{\Lambda}^k: & G\mathcal{M} & \to & \mathcal{H}^k \mathcal{M} \\ & M & \mapsto & \left((\frac{i_1}{l_1}, *_2, \ldots, *_k, \frac{i_k}{l_k}) \mapsto M(\frac{i_1}{l_1}) *_2 \ldots *_k M(\frac{i_k}{l_k}) \right) \end{array}$$

in the obvious way and by composing with Ξ^k we obtain the λ*-operation*

$$\lambda^k : \mathrm{Sub}_k G\mathcal{M} \stackrel{\mathrm{Sub}_k \mathit{\Lambda}^k}{\longrightarrow} \mathrm{Sub}_k \mathcal{H}^k \mathcal{M} \stackrel{\Xi^k}{\to} G^k \mathcal{M}$$

which induces on homotopy groups the desired λ-operation

$$\lambda^k : K_q(\mathcal{M}) \to K_q(\mathcal{M}).$$

Finally we recall the additive and multiplicative structure on $|G\mathcal{M}|$ (see [3]):

For each $k \geq 1$ the multisimplicial map

$$\begin{array}{rcl} G^k\mathcal{M} \times G^k\mathcal{M} & \to & G^k\mathcal{M} \\ (M,N) & \mapsto & \Big[M \oplus N := \Big((\tfrac{i_1}{j_1}, \ldots, \tfrac{i_k}{j_k}) \mapsto M(\tfrac{i_1}{j_1}, \ldots, \tfrac{i_k}{j_k}) \oplus N(\tfrac{i_1}{j_1}, \ldots, \tfrac{i_k}{j_k}) \Big) \Big] \end{array}$$

defines an H-space structure on $|G^k\mathcal{M}|$ ("*addition*") which makes $|G^k\mathcal{M}|$ into a group object in the homotopy category. The identifications $\Omega|Q\mathcal{M}| \simeq |G\mathcal{M}| \simeq |G^k\mathcal{M}|$ are compatible with these group structures.

For any $k \geq p \geq 1$ the k-simplicial map

$$\begin{array}{rcl} G^p\mathcal{M} \times G^{k-p}\mathcal{M} & \to & G^k\mathcal{M} \\ (M,N) & \mapsto & \Big[(M \otimes N) := \Big((\tfrac{i_1}{j_1}, \ldots, \tfrac{i_k}{j_k}) \mapsto M(\tfrac{i_1}{j_1}, \ldots, \tfrac{i_p}{j_p}) \otimes N(\tfrac{i_{p+1}}{j_{p+1}}, \ldots, \tfrac{i_k}{j_k}) \Big) \Big] \end{array}$$

defines an H-space structure on $|G\mathcal{M}|$ which on homotopy groups does not depend on (p,k) ("*multiplication*").

2. Intersection and Sum of Subobjects

Let $\mathcal{M}$ be an exact category. The aim of this section is to prove that for any subobjects $V_1, \ldots, V_m$, $W_1, \ldots, W_m \in \mathcal{M}$ of some $Z \in \mathcal{M}$ occuring as values of an exact functor $M : \Gamma^{k_1}(A_1) \times \ldots \times \Gamma^{k_n}(A_n) \to \mathcal{M}$ we have the elementary formula (in general a freshman's dream):

$$(V_1 + \ldots + V_m) \cap (W_1 + \ldots + W_m) = \sum_{i,j=1}^{m} (V_i \cap W_j).$$

(See proposition 1 for an exact formulation.)

In addition to the axioms of an exact category we assume (for simplicity) that for any sequence $V \hookrightarrow W \hookrightarrow X$ of monomorphisms in $\mathcal{M}$ with $V \hookrightarrow X$ and $W \hookrightarrow X$ admissible also $V \hookrightarrow W$ is admissible. In order to ensure that sum and intersection of subobjects always exist, we furthermore assume that we have fixed an abelian category $\mathcal{A}$ such that $\mathcal{M}$ is a full subcategory of $\mathcal{A}$ closed under extensions in $\mathcal{A}$ and such that a short sequence in $\mathcal{M}$ is exact, iff it is exact in $\mathcal{A}$. The above additional axiom means that $\mathcal{M}$ is also closed under taking kernel of epimorphisms.

Definition. A set $\mathcal{U}$ of admissible subobjects of an object $X \in \mathcal{M}$ is called *stable*, iff for any $V_1, V_2, W \in \mathcal{U}$ we have: $V_1 \cap V_2$ and $V_1 + V_2$ are in $\mathcal{U}$ and $(V_1 + V_2) \cap W = (V_1 \cap W) + (V_2 \cap W)$.

Examples:

a) For each chain $V_1 \hookrightarrow \ldots \hookrightarrow V_n$ of admissible subobjects of an object $X \in \mathcal{M}$ the set $\{V_1, \ldots, V_n\}$ is stable.

b) Let M be a finite set and let U be a set of subsets of M stable under $\cap$ and $\cup$. For any ring A let $A[M]$ denote the free A-module with basis M. Then the set $\mathcal{U} := \{A[N] : N \in U\}$ is a stable set of admissible subobjects of $A[M]$.

c) Let $\mathcal{U}$ be a stable set. Then for any $U \in \mathcal{U}$ also the set $\{V \in \mathcal{U} : V \subseteq U\}$ and the set $\{\frac{V}{U} : V \in \mathcal{U} \text{ and } V \supseteq U\}$ are stable.

Lemma 1. *Let $\mathcal{U}$ be a stable set of admissible subobjects of an object $X \in \mathcal{M}$ and let $V_1, \ldots, V_n$ be in $\mathcal{U}$. Then:*

a) The sequence

$$\begin{array}{ccccccc} \bigoplus\limits_{i<j} (V_i \cap V_j) & \to & \bigoplus\limits_{i=1}^{n} V_i & \to & \sum\limits_{i=1}^{n} V_i & \to & 0 \\ (a_{ij})_{i<j} & \mapsto & (\sum\limits_{i<j} a_{ij} - \sum\limits_{i>j} a_{ji})_{i=1,\ldots,n} & & & & \end{array}$$

is exact.

b) For any $U \in \mathcal{U}$ the sequence

$$0 \to \sum_{i=1}^{n} (V_i \cap U) \to \sum_{i=1}^{n} V_i \to \sum_{i=1}^{n} \frac{V_i + U}{U} \to 0$$

is exact.

Remark. The sequence in a) can be continued to the left in a natural fashion (see [13]).

Proof.

a) Consider the following diagram with the obvious maps:

$$\begin{array}{ccccccc} 0 & & 0 & & & & \\ \downarrow & & \downarrow & & & & \\ \bigoplus\limits_{i<j\leq n-1} V_i \cap V_j & = & \bigoplus\limits_{i<j\leq n-1} V_i \cap V_j & & & & \\ \downarrow & & \downarrow & & & & \\ \bigoplus\limits_{i<j\leq n} V_i \cap V_j & \to & (\bigoplus\limits_{i=1}^{n-1} V_i) \oplus V_n & \to & \sum\limits_{i=1}^{n} V_i & \to & 0 \\ \downarrow & & \downarrow & & \| & & \\ \bigoplus\limits_{i=1}^{n-1} V_i \cap V_n & \to & (\sum\limits_{i=1}^{n-1} V_i) \oplus V_n & \to & \sum\limits_{i=1}^{n} V_i & \to & 0 \\ \downarrow & & \downarrow & & & & \\ 0 & & 0 & & & & \end{array}$$

Because of $(\sum_{i=1}^{n-1} V_i) \cap V_n = \sum_{i=1}^{n-1} V_i \cap V_n$, the lower row is exact. Obviously the left column is exact and by induction on n we may assume that the middle column is exact. By diagram chasing we obtain that the middle row is exact.

b) This follows from the following equality:

$$\sum_{i=1}^{n} \frac{V_i + U}{U} = \frac{\sum_{i=1}^{n} V_i}{(\sum_{i=1}^{n} V_i) \cap U} = \frac{\sum_{i=1}^{n} V_i}{\sum_{i=1}^{n} (V_i \cap U)}.$$

The following lemma has a similar shape as lemma 1, but has nothing to do with the notion "stable".

Lemma 2. *Let X be an object in $\mathcal{M}$ and let $U_i \hookrightarrow V_i$ subobjects of X $(i = 1, \ldots, n)$. If the canonical map $\oplus_{i=1}^{n} V_i \to \oplus_{i=1}^{n} \frac{V_i}{U_i}$ factorizes through $\oplus_{i=1}^{n} V_i \to \sum_{i=1}^{n} V_i$, the sequence*

$$0 \to \sum_{i=1}^{n} U_i \to \sum_{i=1}^{n} V_i \to \bigoplus_{i=1}^{n} \frac{V_i}{U_i} \to 0$$

is exact.

Proof. Let $K := \ker(\sum_{i=1}^{n} V_i \to \oplus_{i=1}^{n} \frac{V_i}{U_i})$. Then the snake lemma, applied to the following commutative diagram with exact rows, shows the lemma:

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{i=1}^{n} U_i & \to & \bigoplus_{i=1}^{n} V_i & \to & \bigoplus_{i=1}^{n} \frac{V_i}{U_i} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K & \to & \sum_{i=1}^{n} V_i & \to & \bigoplus_{i=1}^{n} \frac{V_i}{U_i} & \to & 0 \end{array}$$

Next we are going to introduce stable sets of admissible subobjects which arise from an exact functor $\Gamma^k(A) \to \mathcal{M}$ $(k \in \mathbb{N},\ A \in \mathrm{Ord})$. To begin with we define some notions concerning $\Gamma^k(A)$.

Definition.

a) We will call an arrow $\alpha' \to \alpha$ in $\Gamma^k(A)$ a *monomorphism*, iff for each r we have $*'_r = *_r$, $l'_r = l_r$ and $i'_r \le i_r$ (for short: $\alpha' \hookrightarrow \alpha$). If in addition there is an index $p \in \{1, \ldots, k\}$ such that $i'_r = i_r$ for all $r \ne p$ we will call $\alpha' \to \alpha$ a *simple monomorphism* (for short: $\alpha' \overset{p}{\hookrightarrow} \alpha$).

b) For any simple monomorphism $\alpha' \overset{p}{\hookrightarrow} \alpha$ in $\Gamma^k(A)$ we define the *quotient* $\frac{\alpha}{\alpha'} \in \Gamma^k(A)$ as follows: Put

$$s := s(\alpha' \overset{p}{\hookrightarrow} \alpha) := \begin{cases} p & , \text{if } *_{p+1} = \otimes \\ \max\{p < r : *_p = \ldots = *_r = \wedge\} & , \text{else} \end{cases}$$

$$*_r(\tfrac{\alpha}{\alpha'}) := \begin{cases} *_r & , \text{if } r \neq p \\ \otimes & , \text{else} \end{cases} \qquad (r = 2, \ldots, k)$$

$$i_r(\tfrac{\alpha}{\alpha'}) := i_r \qquad (r = 1, \ldots, k)$$

$$l_r(\tfrac{\alpha}{\alpha'}) := \begin{cases} l_r & , \text{if } r < p \text{ or } s < r \\ i'_p & , \text{if } p \leq r \leq s \end{cases} \qquad (r = 1, \ldots, k)$$

c) Let $\alpha \hookrightarrow \gamma$ and $\beta \hookrightarrow \gamma$ be two monomorphisms in $\Gamma^k(A)$. Then the element $\alpha \cap \beta$ of $\Gamma^k(A)$ is defined by

$$*_r(\alpha \cap \beta) := *_r(\gamma); \quad i_r(\alpha \cap \beta) := \min(i_r(\alpha),\, i_r(\beta)); \quad l_r(\alpha \cap \beta) := l_r(\gamma).$$

More generally let $k := (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $A_1, \ldots, A_n \in \text{Ord}$ and put $\Gamma^k(A) := \Gamma^{k_1}(A_1) \times \ldots \times \Gamma^{k_n}(A_n)$

Definition.
a) We will call an arrow $\alpha' = (\alpha'_1, \ldots, \alpha'_n) \to \alpha = (\alpha_1, \ldots, \alpha_n)$ in $\Gamma^k(A)$ a *monomorphism*, iff $\alpha'_i \hookrightarrow \alpha_i$ is a monomorphism for all i. If in addition there is an index $h \in \{1, \ldots, n\}$ such that $\alpha'_i = \alpha_i$ for all $i \neq h$ and such that $\alpha'_h \overset{p}{\hookrightarrow} \alpha_h$ is a simple monomorphism in $\Gamma^{k_h}(A_h)$, we will call $\alpha' \hookrightarrow \alpha$ a *simple monomorphism* (for short: $\alpha' \overset{(h,p)}{\hookrightarrow} \alpha$).
b) For any simple monomorphism $\alpha' \overset{(h,p)}{\hookrightarrow} \alpha$ in $\Gamma^k(A)$ the *quotient* $\frac{\alpha}{\alpha'} \in \Gamma^k(A)$ is defined as follows:

$$\frac{\alpha}{\alpha'} := (\alpha_1, \ldots, \alpha_{h-1}, \frac{\alpha_h}{\alpha'_h}, \alpha_{h+1}, \ldots, \alpha_n).$$

c) For any $\alpha \hookrightarrow \gamma$, $\beta \hookrightarrow \gamma$ in $\Gamma^k(A)$ we put

$$\alpha \cap \beta := (\alpha_1 \cap \beta_1, \ldots, \alpha_n \cap \beta_n) \in \Gamma^k(A).$$

Obviously any monomorphism in $\Gamma^k(A)$ can be expressed as a composition of simple monomorphisms, the element $\frac{\alpha}{\alpha'}$ is well-defined (see axioms (A1), (A2), (A3)) and the sequence $\alpha' \to \alpha \to \frac{\alpha}{\alpha'}$ is an exact sequence in $\Gamma^k(A)$ (see axioms (C1), (C2), (C3)).

Lemma 3 (Homomorphism theorem). *Let $\alpha, \beta, \gamma \in \Gamma^k(A)$ such that we have a monomorphism $\beta \hookrightarrow \gamma$ and a simple monomorphism $\alpha \overset{(h,p)}{\hookrightarrow} \gamma$. Then $\alpha \cap \beta \overset{(h,p)}{\hookrightarrow} \beta$ is a simple monomorphism, and we have a monomorphism $\frac{\beta}{\alpha\cap\beta} \hookrightarrow \frac{\gamma}{\alpha}$ or we have $\alpha \cap \beta = \beta$.*

Proof. The first claim is clear, because $\alpha \cap \beta$ differs from β at most at the place (h,p). For the second claim we may assume that $n=1$. Furthermore, we may assume $i_p(\alpha) < i_p(\beta)$ (Otherwise $\alpha \cap \beta \to \beta$ is the identity). Then for each r we have

$$*_r(\tfrac{\beta}{\alpha\cap\beta}) = \left\{ \begin{array}{cl} *_r(\beta) = *_r(\gamma) & \text{, if } r \neq p \\ \otimes & \text{, else} \end{array} \right\} = *_r(\tfrac{\gamma}{\alpha})$$

$$i_r(\tfrac{\beta}{\alpha\cap\beta}) = i_r(\beta) \leq i_r(\gamma) = i_r(\tfrac{\gamma}{\alpha})$$

$$s(\alpha\cap\beta \overset{p}{\hookrightarrow} \beta) = s(\alpha \overset{p}{\hookrightarrow} \gamma) =: s$$

$$l_r(\tfrac{\beta}{\alpha\cap\beta}) = \left\{ \begin{array}{cl} l_r(\beta) = l_r(\gamma) & \text{, if } r<p \text{ or } s<r \\ i_p(\alpha\cap\beta) = i_p(\alpha) & \text{, if } p \leq r \leq s \end{array} \right\} = l_r(\tfrac{\gamma}{\alpha}).$$

This shows the second claim (see definition a)).

In the following α_i does not denote a component of α, but α_i itself is an element of $\Gamma^k(A) = \Gamma^{k_1}(A_1) \times \ldots \times \Gamma^{k_n}(A_n)$. Furthermore the number n of factors in $\Gamma^{k_1}(A_1) \times \ldots \times \Gamma^{k_n}(A_n)$ will not occur explicitly and the index n will have a new meaning.

Proposition 1. *Let $M : \Gamma^k(A) \to \mathcal{M}$ be an exact functor. Then for each $\gamma \in \Gamma^k(A)$ the set*

$$\left\{ \sum_{i=1}^{n} M(\alpha_i) : \quad n \geq 1;\ \alpha_i \hookrightarrow \gamma \text{ monomorphism in } \Gamma^k(A) \text{ for all } i = 1,\ldots,n \right\}$$

is a stable set of admissible subobjects of $M(\gamma)$.

Proof. This immediately follows from the following two claims: For each $n \in \mathbb{N}$ and $\gamma \in \Gamma^k(A)$ we have:

a) Given monomorphisms $\alpha_i \hookrightarrow \gamma$ ($i=1,\ldots,n$) in $\Gamma^k(A)$ the subobject $\sum_{i=1}^{n} M(\alpha_i)$ of $M(\gamma)$ is admissible.

b) Given $l,m \geq 1$ with $l+m = n+1$ and given monomorphisms $\alpha_i \hookrightarrow \gamma$ ($i = 1,\ldots,l$) and $\beta_j \hookrightarrow \gamma$ ($j=1,\ldots,m$) in $\Gamma^k(A)$ we have

$$\left[\sum_{i=1}^{l} M(\alpha_i)\right] \cap \left[\sum_{j=1}^{m} M(\beta_j)\right] = \sum_{i=1}^{l}\sum_{j=1}^{m} M(\alpha_i \cap \beta_j).$$

In the case $n=1$ claim a) is clear, because any monomorphism in $\Gamma^k(A)$ can be decomposed into simple monomorphisms. For claim b) we have to show $M(\alpha) \cap M(\beta) = M(\alpha\cap\beta)$. For this by the same argument we may assume that $\alpha \hookrightarrow \gamma$ is simple. Then we have the following diagram with exact rows and with vertical monomorphisms by lemma 3:

$$\begin{array}{ccccccccc}
0 & \to & M(\alpha\cap\beta) & \to & M(\beta) & \to & M(\frac{\beta}{\alpha\cap\beta}) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M(\alpha) & \to & M(\gamma) & \to & M(\frac{\gamma}{\alpha}) & \to & 0
\end{array}$$

This shows the claim b) in the case $n=1$.

For the general case we proceed by induction on n:

a) We decompose $\alpha_n \hookrightarrow \beta$ into simple monomorphisms $\alpha_n \hookrightarrow \alpha'_n \hookrightarrow \ldots \hookrightarrow \gamma$. Then by induction on the length of this decomposition we may assume that the claim is already proved for α'_n instead of α_n. By lemma 3 for each $i=1,\ldots,n-1$ the monomorphism $\alpha_i\cap\alpha_n \hookrightarrow \alpha_i\cap\alpha'_n$ is simple and we have a monomorphism $\frac{\alpha_i\cap\alpha'_n}{\alpha_i\cap\alpha_n} \hookrightarrow \frac{\alpha'_n}{\alpha_n}$ or $\alpha_i\cap\alpha_n = \alpha_i\cap\alpha'_n$. We consider now the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & \frac{\sum_{i=1}^{n-1}(M(\alpha_i)\cap M(\alpha'_n))+M(\alpha_n)}{M(\alpha_n)} & \to & \frac{\sum_{i=1}^{n} M(\alpha_i)}{M(\alpha_n)} & \to & \frac{\sum_{i=1}^{n-1} M(\alpha_i)+M(\alpha'_n)}{M(\alpha'_n)} & \to & 0\\
 & & \| & & & & & & \\
 & & \sum_{i=1}^{n-1} M(\frac{\alpha_i\cap\alpha'_n}{\alpha_i\cap\alpha_n}) & & \downarrow & & \downarrow & & \\
 & & \downarrow & & & & & & \\
0 & \to & M(\frac{\alpha'_n}{\alpha_n}) & \to & \frac{M(\gamma)}{M(\alpha_n)} & \to & \frac{M(\gamma)}{M(\alpha'_n)} & \to & 0
\end{array}$$

Obviously the lower row is an exact sequence. Futhermore we have $[\sum_{i=1}^{n} M(\alpha_i)] \cap M(\alpha'_n) = \left[\sum_{i=1}^{n-1} M(\alpha_i)\right] \cap M(\alpha'_n) + M(\alpha_n)$. This equals $\sum_{i=1}^{n-1}(M(\alpha_i)\cap M(\alpha'_n)) + M(\alpha_n)$ by the inductive hypothesis b). Hence the upper row is an exact sequence. By applying the inductive hypothesis a) to $\gamma^{\text{new}} := \frac{\alpha'_n}{\alpha_n}$, $\alpha_i^{\text{new}} := \frac{\alpha_i\cap\alpha'_n}{\alpha_i\cap\alpha_n}$ ($i=1,\ldots,n-1$) we obtain that the left vertical monomorphism is admissible. The right vertical monomorphism is admissible by assumption. Now the snake lemma shows that the middle vertical monomorphism is also admissible which is equivalent to claim a).

b) We may assume $m\geq 2$ by symmetry. At first we consider the case that $\beta_m \hookrightarrow \gamma$ is simple. We may assume that $\alpha_i\cap\beta_m \neq \alpha_i$ for $i=1,\ldots,l$ and $\beta_j\cap\beta_m \neq \beta_j$ for $j=1,\ldots,m-1$ (Otherwise the assertion immediatiely follows from the inductive hypothesis b)). Then by lemma 3 for each $i=1,\ldots,l$ and $j=1,\ldots,m-1$ the monomorphisms $\alpha_i\cap\beta_m \hookrightarrow \alpha_i$ and $\beta_j\cap\beta_m \hookrightarrow \beta$ are simple and we have the monomorphisms

$$\frac{\alpha_i}{\alpha_i\cap\beta_m} \hookrightarrow \frac{\gamma}{\beta_m} \quad \text{and} \quad \frac{\beta_j}{\beta_j\cap\beta_m} \hookrightarrow \frac{\gamma}{\beta_m}.$$

By applying the inductive hypothesis b) we obtain the following equality of subobjects of

$M(\frac{\gamma}{\beta_m})$:

$$\frac{\sum_{i=1}^{l} M(\alpha_i) + M(\beta_m)}{M(\beta_m)} \cap \frac{\sum_{j=1}^{m} M(\beta_j)}{M(\beta_m)}$$
$$= \sum_{i=1}^{l} M(\frac{\alpha_i}{\alpha_i \cap \beta_m}) \cap \sum_{j=1}^{m-1} M(\frac{\beta_j}{\beta_j \cap \beta_m}) = \sum_{i=1}^{l} \sum_{j=1}^{m-1} M(\frac{\alpha_i}{\alpha_i \cap \beta_m} \cap \frac{\beta_j}{\beta_j \cap \beta_m})$$
$$= \sum_{i=1}^{l} \sum_{j=1}^{m-1} M(\frac{\alpha_i \cap \beta_j}{\alpha_i \cap \beta_j \cap \beta_m}) = \frac{\sum_{i=1}^{l} \sum_{j=1}^{m-1} M(\alpha_i \cap \beta_j) + M(\beta_m)}{M(\beta_m)}.$$

In particular the numerators of this equation are equal and intersecting with $\sum_{i=1}^{l} M(\alpha_i)$ gives the desired equality by using the inductive hypothesis b) once more:

$$\left(\sum_{i=1}^{l} M(\alpha_i)\right) \cap \left(\sum_{j=1}^{m} M(\beta_j)\right)$$
$$= \left(\sum_{i=1}^{l} M(\alpha_i)\right) \cap \left(\sum_{i=1}^{l} M(\alpha_i) + M(\beta_m)\right) \cap \left(\sum_{j=1}^{m} M(\beta_j)\right)$$
$$= \left(\sum_{i=1}^{l} M(\alpha_i)\right) \cap \left(\sum_{i=1}^{l} \sum_{j=1}^{m-1} M(\alpha_i \cap \beta_j) + M(\beta_m)\right)$$
$$= \sum_{i=1}^{l} \sum_{j=1}^{m} M(\alpha_i \cap \beta_j).$$

(For the last equality note that the left term in the right brackets is contained in the left brackets).

For the general case we decompose $\beta_m \hookrightarrow \gamma$ into simple monomorphisms $\beta_m \hookrightarrow \beta'_m \hookrightarrow \ldots \hookrightarrow \gamma$ and may assume that claim b) is already proved for β'_m instead of β_m. We put $V := \sum_{i=1}^{l} M(\alpha_i)$, $W := \sum_{j=1}^{m-1} M(\beta_j)$, $W_m := M(\beta_m)$, $W'_m := M(\beta'_m)$. Then applying the case considered above to $\alpha_i^{\text{new}} := \alpha_i \cap \beta'_m$, $\beta_j^{\text{new}} := \beta_j \cap \beta'_m$ $\gamma^{\text{new}} := \beta'_m$ we obtain the desired equality:

$$\left(\sum_{i=1}^{l} M(\alpha_i)\right) \cap \left(\sum_{j=1}^{m} M(\beta_j)\right)$$
$$= V \cap (W + W_m) = V \cap (W + W'_m) \cap (W + W_m)$$
$$= (V \cap W + V \cap W'_m) \cap (W + W_m) = V \cap W + (V \cap W'_m) \cap (W + W_m)$$
$$= V \cap W + (V \cap W'_m \cap W + V \cap W_m) = V \cap W + V \cap W_m$$
$$= \sum_{i=1}^{l} \sum_{j=1}^{m} M(\alpha_i \cap \beta_j).$$

Such as the preceding proposition also the following lemma is based on the notion "exact sequence in $\Gamma^k(A)$".

Lemma 4. *Let* $A \in Ord$, $k \in \mathbb{N}$ *and* $\alpha \to \tilde{\alpha}$ *an arrow in* $\Gamma^k(A)$ *such that* $i_p(\alpha) \leq l_p(\tilde{\alpha})$ *for some* p. *Then for each exact functor* $M : \Gamma^k(A) \to \mathcal{M}$ *the map* $M(\alpha) \to M(\tilde{\alpha})$ *is the zero map.*

Proof. We set

$$\tilde{s} := \begin{cases} p & \text{, if } \tilde{*}_{p+1} = \otimes \\ \max\{p < r : \tilde{*}_{p+1} = \ldots = \tilde{*}_r = \wedge\} & \text{, else} \end{cases}$$

and decompose $\alpha \to \tilde{\alpha}$ as follows:

$$\begin{array}{rl}
\alpha = & (\frac{i_1}{l_1}, \quad *_2, \ldots \quad , *_p, \frac{i_p}{l_p}, \quad *_{p+1} \quad , \ldots \qquad\qquad\qquad\qquad \ldots, *_k, \frac{i_k}{l_k}) \\
& \qquad\qquad\qquad\qquad \downarrow \\
& (\qquad\qquad\qquad , \frac{i_p}{l_p}, \quad \wedge \quad , \frac{\tilde{i}_{p+1}}{l_p}, \wedge, \ldots, \wedge, \frac{\tilde{i}_{\tilde{s}}}{l_p}, \quad \otimes, \frac{\tilde{i}_{\tilde{s}+1}}{\tilde{l}_{\tilde{s}+1}}, \tilde{*}_{\tilde{s}+2}, \ldots, \tilde{*}_k, \frac{\tilde{i}_k}{\tilde{l}_k}) \\
& \qquad\qquad\qquad\qquad \downarrow \\
& (\qquad\qquad\qquad , \frac{\tilde{i}_p}{l_p}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad) \\
& \qquad\qquad\qquad\qquad \downarrow \\
& (\qquad\qquad , \otimes, \frac{\tilde{i}_p}{i_p}, \quad \wedge \quad , \frac{\tilde{i}_{p+1}}{i_p}, \wedge, \ldots, \wedge, \frac{\tilde{i}_{\tilde{s}}}{i_p}, \qquad\qquad\qquad\qquad) \\
& \qquad\qquad\qquad\qquad \downarrow \\
\tilde{\alpha} = & (\frac{\tilde{i}_1}{\tilde{l}_1}, \quad \tilde{*}_2, \ldots \quad , \tilde{*}_p, \frac{\tilde{i}_p}{\tilde{l}_p}, \quad \tilde{*}_{p+1} \quad , \ldots \qquad\qquad\qquad\qquad \ldots, \tilde{*}_k, \frac{\tilde{i}_k}{\tilde{l}_k}).
\end{array}$$

(The empty space means that nothing changes). Then the three middle terms form an exact sequence in $\Gamma^k(A)$ and the claim follows.

3. Construction of Shuffle Products $\mathcal{H}^p \times \mathcal{H}^{k-p} \to \mathcal{H}^k$

Let $\mathcal{M}$ be an exact category as in section 2 and let $k \geq p \geq 1$ be fixed integers. We are going to construct a simplicial map

$$\mathrm{Sym} := \mathrm{Sym}^{p,k-p} : \mathcal{H}^{p,k-p}\mathcal{M} \to \mathcal{H}^k\mathcal{M}$$

which we will call *shuffle operation*. (We will use the same notation for the bi-simplicial set $\mathcal{H}^{p,k-p}\mathcal{M}$ as for the associated 1-simplicial set defined by $\mathcal{H}^{p,k-p}\mathcal{M}(A) := \mathcal{H}^{p,k-p}\mathcal{M}(A, A)$ for $A \in \mathrm{Ord}$.)

We recall: A $(p, k-p)$-*shuffle* is a permutation σ of $\{1, \ldots, k\}$ with $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(k)$. The assignment $\sigma \mapsto \{\sigma(1), \ldots, \sigma(p)\}$ defines a bijection between the set of $(p,k-p)$-shuffles and the set $\mathcal{P}(k,p) := \{R \subseteq \{1,\dots,k\} : |R| = p\}$. We will use the latter interpretation of $(p,k-p)$-shuffles.

Now we fix an object $A \in \mathrm{Ord}$ and an element $\alpha \in \Gamma^k(A)$ and introduce some notations: Let $\sim_\alpha$ be the following equivalence relation on $\{1,\dots,k\}$:

$$r \sim_\alpha s :\Leftrightarrow \begin{cases} *_{r+1}(\alpha) = \ldots = *_s(\alpha) = \wedge & \text{, if } r \le s \\ *_{s+1}(\alpha) = \ldots = *_r(\alpha) = \wedge & \text{, if } s \le r \end{cases}$$

and let $\{1,\dots,k\} = Z_1 \cdot \ldots \cdot Z_{n(\alpha)}$ be the representation of $\{1,\dots,k\}$ as the concatenation of the equivalence classes $Z_n, n = 1,\dots,n(\alpha)$ of $\{1,\dots,k\}$ corresponding to this equivalence relation. We define an equivalence relation $\sim_\alpha$ on $\mathcal{P}(k,p)$ as follows:

$$\begin{aligned} R = \{r_1 < \ldots < r_p\} \sim_\alpha S = \{s_1 < \ldots < s_p\} &:\Leftrightarrow r_t \sim_\alpha s_t \text{ for all } t = 1,\dots,p \\ &\Leftrightarrow |R \cap Z_1 \cdot \ldots \cdot Z_n| = |S \cap Z_1 \cdot \ldots \cdot Z_n| \text{ for all } n = 1,\dots,n(\alpha) \\ &\Leftrightarrow |R \cap Z_n| = |S \cap Z_n| \text{ for all } n = 1,\dots,n(\alpha). \end{aligned}$$

Remark.
a) The equivalence relation $\sim_\alpha$ depends only on the symbols $*_r(\alpha)$, $r = 2,\dots,k$.
b) For any $R, S \in \mathcal{P}(k,p)$ with $R \sim_\alpha S$ we have also $\bar{R} \sim_\alpha \bar{S}$ in $\mathcal{P}(k,k-p)$ where $\bar{R}, \bar{S}$ denotes the complement of R, S in $\{1,\dots,k\}$, respectively. This immediately follows from the previous definition.

For any $R = \{r_1 < \ldots < r_p\} \in \mathcal{P}(k,p)$ we define $\alpha^R \in \Gamma^p(A)$ by

$$\begin{aligned} i_t(\alpha^R) &:= i_{r_t}(\alpha) \quad (t = 1,\dots,p) \\ l_t(\alpha^R) &:= l_{r_t}(\alpha) \quad (t = 1,\dots,p) \\ *_t(\alpha^R) &:= \begin{cases} \wedge & \text{, if } r_{t-1} \sim_\alpha r_t \\ \otimes & \text{, else} \end{cases} \quad (t = 2,\dots,p) \end{aligned}$$

and in a similar way $\alpha^{\bar{R}} \in \Gamma^{k-p}(A)$ is defined (For short: $*_t(\alpha^R)$ is defined to be $\wedge$, if only $\wedge$'s occur in α between the places r_{t-1} and r_t, and $\otimes$, else). For any $\alpha, \beta \in \Gamma^k(A)$ with $*_r(\alpha) = *_r(\beta)$ and $l_r(\alpha) = l_r(\beta)$ for all r the element $\alpha \cup \beta \in \Gamma^k(A)$ is given by

$$*_r(\alpha \cup \beta) := *_r(\alpha); \quad l_r(\alpha \cup \beta) := l_r(\alpha); \quad i_r(\alpha \cup \beta) := \max(i_r(\alpha), i_r(\beta)).$$

Note that for any $R, S \in \mathcal{P}(k,p)$ with $R \sim_\alpha S$ we have $*_t(\alpha^R) = *_t(\alpha^S)$ and $l_t(\alpha^R) = l_t(\alpha^S)$ for all t (by (A3)) and hence the element $\alpha^R \cup \alpha^S \in \Gamma^p(A)$ is defined.

Lemma 5. *Let* $\alpha \to \beta$ *be an arrow in* $\Gamma^k(A)$. *Then for each* $R \in \mathcal{P}(k,p)$ *we have an arrow* $\alpha^R \to \beta^R$. *In particular for any* $\mathcal{R} \in \mathcal{P}(k,p)/\sim_\alpha$, $\mathcal{S} \in \mathcal{P}(k,p)/\sim_\beta$ *and any subset* $\mathcal{T} \subseteq \mathcal{R} \cap \mathcal{S}$ *we have an arrow* $\cup_{R\in\mathcal{T}}\alpha^R \to \cup_{R\in\mathcal{T}}\beta^R$.

Proof. Let $R = \{r_1 < \ldots < r_p\}$ and $t \in \{1, \ldots, p\}$. Then:

$$i_t(\alpha^R) = i_{r_t}(\alpha) \leq i_{r_t}(\beta) = i_t(\beta^R) \text{ and } l_t(\alpha^R) = l_{r_t}(\alpha) \leq l_{r_t}(\beta) = l_t(\beta^R).$$

If $t > 1$ and $*_t(\alpha^R) = \wedge$ and $*_t(\beta^R) = \otimes$, there is an index $z \in \{r_{t-1}+1, \ldots, r_t\}$ such that $*_z(\alpha) = \wedge$ and $*_z(\beta) = \otimes$. Let $z \in \{r_{t-1}+1, \ldots, r_t\}$ be maximal with this property. Then:

$$i_{t-1}(\alpha^R) = i_{r_{t-1}}(\alpha) \leq i_{r_{t-1}+1}(\alpha) \leq \ldots \leq i_{z-1}(\alpha) \leq l_z(\beta) \leq l_{z+1}(\beta) \leq \ldots \leq l_{r_t}(\beta) = l_t(\beta^R).$$

This shows axiom (B3).

Now the shuffle operation

$$\mathrm{Sym} := \mathrm{Sym}^{p,k-p} : \mathcal{H}^{p,k-p}\mathcal{M}(A) \to \mathcal{H}^k\mathcal{M}(A)$$

is defined as follows: Let $M \in \mathcal{H}^{p,k-p}\mathcal{M}(A) = \mathrm{Exact}(\Gamma^p(A) \times \Gamma^{k-p}(A), \mathcal{M})$.

1. Definition of Sym$(M)(\alpha)$

Let $\alpha \in \Gamma^k(A)$. Then by proposition 1 for each $\mathcal{R} \in \mathcal{P}(k,p)/\sim_\alpha$ the sum $\sum_{R\in\mathcal{R}} M(\alpha^R, \alpha^{\bar{R}})$ is an admissible subobject of $M(\cup_{R\in\mathcal{R}}\alpha^R, \cup_{R\in\mathcal{R}}\alpha^{\bar{R}})$. We set

$$\mathrm{Sym}(M)(\alpha) := \bigoplus_{\mathcal{R}\in\mathcal{P}(k,p)/\sim_\alpha} \sum_{R\in\mathcal{R}} M(\alpha^R, \alpha^{\bar{R}}).$$

Note: Regarding this mixture of the direct sum $\oplus$ and the sum $\sum$ of subobjects as one sum "$\sum$" the sum "$\sum$" is taken over all $(p, k-p)$-shuffles $R \in \mathcal{P}(k,p)$. This explains the name shuffle operation.

2. Definition of Sym$(M)(\alpha \to \beta)$

Let $\alpha \to \beta$ be an arrow in $\Gamma^k(A)$. Then by lemma 5 for each $R \in \mathcal{P}(k,p)$ we have $\alpha^R \to \beta^R$ and $\alpha^{\bar{R}} \to \beta^{\bar{R}}$. We claim that there is a morphism $f : \mathrm{Sym}(M)(\alpha) \to \mathrm{Sym}(M)(\beta)$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_{R\in\mathcal{P}(k,p)} M(\alpha^R, \alpha^{\bar{R}}) & \xrightarrow{\bigoplus_{R\in\mathcal{P}(k,p)} M(\alpha^R\to\beta^R, \alpha^{\bar{R}}\to\beta^{\bar{R}})} & \bigoplus_{R\in\mathcal{P}(k,p)} M(\beta^R, \beta^{\bar{R}}) \\ \downarrow & & \downarrow \\ \mathrm{Sym}(M)(\alpha) & \xrightarrow{f} & \mathrm{Sym}(M)(\beta) \end{array}$$

Obviously the morphism f is uniquely determined by this property and we set $\mathrm{Sym}(M)(\alpha \to \beta) := f$. This defines a functor

$$\begin{array}{ccc} \Gamma^k(A) & \to & \mathcal{M} \\ \alpha & \mapsto & \mathrm{Sym}(M)(\alpha) \\ [\alpha \to \beta] & \mapsto & [\mathrm{Sym}(M)(\alpha \to \beta)] \,. \end{array}$$

To prove the existence of f, by lemma 1a) it suffices to show that for all $R, S \in \mathcal{P}(k,p)$ with $R \sim_\alpha S$ the following diagram commutes:

$$\begin{array}{ccc} M(\alpha^R \cap \alpha^S, \alpha^{\bar{R}} \cap \alpha^{\bar{S}}) & \to & \left\{ \begin{array}{cc} 0 & \text{, if } R \not\sim_\beta S \\ M(\beta^R \cap \beta^S, \beta^{\bar{R}} \cap \beta^{\bar{S}}) & \text{, if } R \sim_\beta S \end{array} \right\} \\ \downarrow \Delta & & \downarrow \Delta \\ M(\alpha^R, \alpha^{\bar{R}}) \oplus M(\alpha^S, \alpha^{\bar{S}}) & \to & M(\beta^R, \beta^{\bar{R}}) \oplus M(\beta^S, \beta^{\bar{S}}) \end{array}$$

(Δ is the diagonal map.)

If $R \sim_\beta S$ we have $\alpha^R \cap \alpha^S \to \beta^R \cap \beta^S$ and $\alpha^{\bar{R}} \cap \alpha^{\bar{S}} \to \beta^{\bar{R}} \cap \beta^{\bar{S}}$ and the claim follows. If $R = \{r_1 < \ldots < r_p\} \not\sim_\beta S = \{s_1 < \ldots < s_p\}$ there is an index $t \in \{1, \ldots, p\}$ such that $r_t \not\sim_\beta s_t$. By symmetry we may assume $r_t < s_t$. There is an index $z \in \{r_t + 1, \ldots, s_t\}$ with $*_z(\beta) = \otimes$. Let $z \in \{r_t + 1, \ldots, s_t\}$ be maximal with this property. Then:

$$\begin{aligned} i_t(\alpha^R \cap \alpha^S) &= \min(i_{r_t}(\alpha), i_{s_t}(\alpha)) = i_{r_t}(\alpha) \\ &\leq i_{r_t+1}(\alpha) \leq \ldots \leq i_{z-1}(\alpha) \leq l_z(\beta) = l_{z+1}(\beta) = \ldots = l_{s_t}(\beta) = l_t(\beta^S) \end{aligned}$$

and by lemma 4 the map $M(\alpha^R \cap \alpha^S \to \beta^S, \alpha^{\bar{R}} \cap \alpha^{\bar{S}} \to \beta^{\bar{S}})$ is the zero map. For the complements $\bar{R} = \{\bar{r}_1 < \ldots < \bar{r}_{k-p}\}$ and $\bar{S} = \{\bar{s}_1 < \ldots < \bar{s}_{k-p}\}$ there is an element $t \in \{1, \ldots, k-p\}$ with $\bar{r}_t \not\sim_\beta \bar{s}_t$ and $\bar{s}_t < \bar{r}_t$. By the same argumentation we obtain that $M(\alpha^R \cap \alpha^S \to \beta^R, \alpha^{\bar{R}} \cap \alpha^{\bar{S}} \to \beta^{\bar{R}})$ is the zero map. This shows the above assertion.

3. The functor $\mathrm{Sym}(M)$ is exact

There remains to prove: For any exact sequence $\alpha \to \beta \to \gamma$ in $\Gamma^k(A)$ the sequence

$$0 \to \mathrm{Sym}(M)(\alpha) \to \mathrm{Sym}(M)(\beta) \to \mathrm{Sym}(M)(\gamma) \to 0$$

is an exact sequence in $\mathcal{M}$. Because the equivalence relation $\sim_\gamma$ is finer than $\sim_\alpha = \sim_\beta$, by lemma 2 it is enough to show that for each $\mathcal{R} \in \mathcal{P}(k,p)/\sim_\gamma$ the sequence

$$0 \to \sum_{R \in \mathcal{R}} M(\alpha^R, \alpha^{\bar{R}}) \to \sum_{R \in \mathcal{R}} M(\beta^R, \beta^{\bar{R}}) \to \sum_{R \in \mathcal{R}} M(\gamma^R, \gamma^{\bar{R}}) \to 0$$

is exact.

We may assume $\alpha \neq \beta$, i. e. there is a unique element $p_0 \in \{1, \ldots, k\}$ with $i_{p_0}(\alpha) < i_{p_0}(\beta)$. We put

$$s_0 := \begin{cases} p_0 & , \text{if } *_{p_0+1} = \otimes \\ \max\{p_0 < r : *_{p_0+1}(\alpha) = \ldots = *_r(\alpha) = \wedge\} & , \text{else.} \end{cases}$$

First case: For one (and then for each) $R \in \mathcal{R}$ we have $\bar{R} \cap \{p_0, \ldots, s_0\} = \emptyset$.
Then for each $R \in \mathcal{R}$ the sequence $\alpha^R \to \beta^R \to \gamma^R$ in $\Gamma^p(A)$ is exact and we have $\alpha^{\bar{R}} = \beta^{\bar{R}} = \gamma^{\bar{R}}$ in $\Gamma^{k-p}(A)$. Hence the sequence

$$0 \to M(\alpha^R, \alpha^{\bar{R}}) \to M(\beta^R, \beta^{\bar{R}}) \to M(\gamma^R, \gamma^{\bar{R}}) \to 0$$

in $\mathcal{M}$ is exact and by (the proof of) proposition 1 we have

$$M(\beta^R, \beta^{\bar{R}}) \cap M(\cup_{R\in\mathcal{R}}\alpha^R, \cup_{R\in\mathcal{R}}\beta^{\bar{R}}) = M(\alpha^R, \beta^{\bar{R}}) = M(\alpha^R, \alpha^{\bar{R}}).$$

By proposition 1 the objects $X := M(\cup_{R\in\mathcal{R}}\beta^R, \cup_{R\in\mathcal{R}}\beta^{\bar{R}})$, $V_R := M(\beta^R, \beta^{\bar{R}})$ and $U := M(\cup_{R\in\mathcal{R}}\alpha^R, \cup_{R\in\mathcal{R}}\beta^{\bar{R}})$ satisfy the assumptions of lemma 1b) and we obtain the above assertion.

Second case: For one (and then for each) $R \in \mathcal{R}$ we have $R \cap \{p_0, \ldots, s_0\} = \emptyset$.
This can be proved in the same way as the first case.

Third case: For one (and then for each) $R \in \mathcal{R}$ we have $R \cap \{p_0, \ldots, s_0\} \neq \emptyset$ and $\bar{R} \cap \{p_0, \ldots, s_0\} \neq \emptyset$.
We put

$$\begin{aligned} Rp_0 &:= R \backslash \{\min(R \cap \{p_0, \ldots, s_0\})\} \cup \{p_0\} \in \mathcal{R} \\ \bar{R}p_0 &:= \bar{R} \backslash \{\min(\bar{R} \cap \{p_0, \ldots, s_0\})\} \cup \{p_0\} \in \{\bar{R} : R \in \mathcal{R}\}. \end{aligned}$$

Then for each $R \in \mathcal{R}$ the sequences

$$\alpha^{Rp_0} \to \beta^R \to \gamma^R \quad \text{and} \quad \alpha^{\bar{R}p_0} \to \beta^{\bar{R}} \to \gamma^{\bar{R}}$$

are exact. Hence the sequence

$$0 \to M(\alpha^{Rp_0}, \beta^{\bar{R}}) + M(\beta^R, \alpha^{\bar{R}p_0}) \to M(\beta^R, \beta^{\bar{R}}) \to M(\gamma^R, \gamma^{\bar{R}}) \to 0$$

in $\mathcal{M}$ is exact and by (the proof of) proposition 1 we have

$$M(\beta^R, \beta^{\bar{R}}) \cap \Big[M(\cup_{R\in\mathcal{R}}\alpha^{Rp_0}, \cup_{R\in\mathcal{R}}\beta^{\bar{R}}) + M(\cup_{R\in\mathcal{R}}\beta^R, \cup_{R\in\mathcal{R}}\alpha^{\bar{R}p_0})\Big]$$

$$= M(\alpha^{Rp_0}, \beta^{\bar{R}}) + M(\beta^{R}, \alpha^{\bar{R}p_0})$$
$$\subseteq M(\alpha^{Rp_0}, \alpha^{\overline{Rp_0}}) + M(\alpha^{\overline{\bar{R}p_0}}, \alpha^{\bar{R}p_0})$$
$$\subseteq \sum_{R\in\mathcal{R}} M(\alpha^{R}, \alpha^{\bar{R}}).$$

By proposition 1 the objects $X := M(\cup_{R\in\mathcal{R}}\beta^{R}, \cup_{R\in\mathcal{R}}\beta^{\bar{R}})$, $V_R := M(\beta^{R}, \beta^{\bar{R}})$ and $U := M(\cup_{R\in\mathcal{R}}\alpha^{Rp_0}, \cup_{R\in\mathcal{R}}\beta^{\bar{R}}) + M(\cup_{R\in\mathcal{R}}\beta^{R}, \cup_{R\in\mathcal{R}}\alpha^{\bar{R}p_0})$ satisfy the assumptions of lemma 1b) and we obtain the above assertion.

Now we in addition assume that the exact category $\mathcal{M}$ is equipped with a bi-exact tensor product

$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}.$$

This defines a simplicial map

$$\otimes : \mathcal{H}^{p}\mathcal{M} \times \mathcal{H}^{k-p}\mathcal{M} \to \mathcal{H}^{p,k-p}\mathcal{M}$$

in a natural way (compare the definition of $\otimes$ at the end of section 1).

Definition. The composition

$$\mathcal{H}^{p} \times \mathcal{H}^{k-p} \xrightarrow{\otimes} \mathcal{H}^{p,k-p} \xrightarrow{\mathrm{Sym}} \mathcal{H}^{k}$$

of simplicial maps is called $(p, k-p)$-*shuffle product*. We will again write $\otimes$ for this composition.

Remark. One easily checks that the shuffle operations $\mathcal{H}^{p,k-p} \to \mathcal{H}^{k}$ are associative in the obvious sense. In particular we obtain shuffle operations

$$\mathrm{Sym}^{k_1,\ldots,k_n} : \mathcal{H}^{k_1,\ldots,k_n} \to \mathcal{H}^{k_1+\ldots+k_n}$$

for any $k_1, \ldots, k_n \in \mathbb{N}$ in a natural way. If in addition the tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is associative, we obtain shuffle products

$$\otimes : \mathcal{H}^{k_1} \times \ldots \times \mathcal{H}^{k_n} \to \mathcal{H}^{k_1+\ldots+k_n}.$$

For instance, in the case $k_1 = \ldots = k_n = 1$ the shuffle operation $\mathrm{Sym}^{1,\ldots,1} : G^{n} \to \mathcal{H}^{n}$ is given by

$$M \mapsto \left[(\frac{i_1}{l_1}, *_2, \ldots, *_k, \frac{i_k}{l_k}) \mapsto \text{“} \sum_{\sigma\in\Sigma_n} \text{”} M(\frac{i_{\sigma(1)}}{l_{\sigma(1)}}, \ldots, \frac{i_{\sigma(n)}}{l_{\sigma(n)}}) \right]$$

where “$\sum$” is a certain mixture of $\oplus$ and $\sum$ and σ is running through the whole symmetric group Σ_n.

4. Connection between Shuffle Products and Classical Products

Let $\mathcal{M}$ be an exact category as in section 2, equipped with an bi-exact tensor product $\otimes$, and let $k \geq p \geq 1$ be fixed integers.

Conjecture. *The shuffle product* $\otimes : \mathcal{H}^p \times \mathcal{H}^{k-p} \to \mathcal{H}^k$ *is compatible with the products* $\otimes : G^p \times G^{k-p} \to G^k$ *defined by Gillet and Grayson in [3] (see section 1) in the following sense: The diagram*

$$(*) \qquad \begin{array}{ccc} |\mathcal{H}^p| \times |\mathcal{H}^{k-p}| & \stackrel{\otimes}{\longrightarrow} & |\mathcal{H}^k| \\ \downarrow {\scriptstyle |\Xi^p| \times |\Xi^{k-p}|} & & \downarrow {\scriptstyle |\Xi^k|} \\ |G^p| \times |G^{k-p}| & \stackrel{\otimes}{\longrightarrow} & |G^k| \end{array}$$

of continuous maps commutes up to homotopy. Here $|\Xi^k|$ *denotes the composition of the homeomorphism* $|\mathcal{H}^k| \tilde{\to} |\mathrm{Sub}_k \mathcal{H}^k|$ *(given by Grayson's theorem, see section 1) and the realization of the* k*-simplicial map* $\Xi^k : \mathrm{Sub}_k \mathcal{H}^k \to G^k$ *(see section 1).*

It is possible to define a map $|\Xi^{p,k-p}| : |\mathcal{H}^{p,k-p}| \to |G^k|$ similar to the map $|\Xi^k| \times |\Xi^{k-p}|$ which, however, we won't carry out. Using this map we can strengthen the above conjecture in the following way:

Conjecture'. *The diagram*

$$\begin{array}{ccc} |\mathcal{H}^{p,k-p}| & \stackrel{|\mathrm{Sym}^{p,k-p}|}{\longrightarrow} & |\mathcal{H}^k| \\ {\scriptstyle |\Xi^{p,k-p}|} \searrow & & \downarrow {\scriptstyle |\Xi^k|} \\ & & |G^k| \end{array}$$

of continuous maps commutes up to homotopy (Note that in this formulation no tensor product is involved).

Assuming the above conjecture we in particular obtain that for any exact category $\mathcal{M}$ with power operations the continuous map

$$|G| \times |G| \stackrel{|\lambda^p| \times |\lambda^{k-p}|}{\to} |G^p| \times |G^{k-p}| \stackrel{\otimes}{\longrightarrow} |G^k|$$

is homotopic to the composition

$$|G| \times |G| \stackrel{|\Lambda^p| \times |\Lambda^{k-p}|}{\longrightarrow} |\mathcal{H}^p| \times |\mathcal{H}^{k-p}| \stackrel{\otimes}{\longrightarrow} |\mathcal{H}^k| \stackrel{|\Xi^k|}{\longrightarrow} |G^k|.$$

We are going to prove that for some special points $x \in |\mathcal{H}^p|$ the diagram (*) commutes up to homotopy after replacing $|\mathcal{H}^p| \times |\mathcal{H}^{k-p}|$ by the subspace $\{x\} \times |\mathcal{H}^{k-p}|$. This will in particular give an affirmative answer to the conjecture' in the case $k = 2,\ p = 1$.

To begin with we recall the construction and some well-known facts about the realization of a multisimplicial map $X : (\mathrm{Ord}^{\mathrm{op}})^k \to \mathrm{Ens}$. For any object $A \in \mathrm{Ord}$ let

$$\Delta(A) := \left\{ f : A \to [0,1] : \quad \sum_{a \in A} f(a) = 1 \right\}$$

be the associated standard simplex equipped with the Euclidean topology. For any $A_1, \ldots, A_k \in \mathrm{Ord}$ give $X(A_1, \ldots, A_k)$ the discrete topology. The relations

$$(x, s_*(\underline{f})) \sim (s^*(x), \underline{f}) \quad (s \in \mathrm{Mor}(\mathrm{Ord}^k))$$

generate on $\coprod_{A_1,\ldots,A_k \in \mathrm{Ord}} X(A_1, \ldots, A_k) \times \Delta(A_1) \times \ldots \times \Delta(A_k)$ an equivalence relation $\sim$.

Definition. The quotient space

$$X := \coprod_{A_1,\ldots,A_k \in \mathrm{Ord}} X(A_1, \ldots, A_k) \times \Delta(A_1) \times \ldots \times \Delta(A_k) \Big/ \sim$$

is called *geometric realization of* X.

Now let X be a 1-simplicial set. We denote the k-simplicial map $(\mathrm{Ord}^{\mathrm{op}})^k \stackrel{\mathrm{pr}_p}{\longrightarrow} \mathrm{Ord}^{\mathrm{op}} \stackrel{X}{\longrightarrow} \mathrm{Ens}$ by $\mathrm{pr}_p^* X$. It is well-known that the p-th projection

$$\mathrm{pr}_p^* X(A_1, \ldots, A_k) \times \Delta(A_1) \times \ldots \times \Delta(A_k) \longrightarrow X(A_p) \times \Delta(A_p)$$

($A_1, \ldots, A_k \in \mathrm{Ord}$) induces a homeomorphism

$$\pi_p : |\mathrm{pr}_p^* X| \stackrel{\sim}{\longrightarrow} |X|.$$

Grayson's homeomorphism

$$\psi : |\mathrm{Sub}_k X| \stackrel{\sim}{\longrightarrow} |X|$$

(see section 1) is given by ($A_1, \ldots, A_k \in \mathrm{Ord}$)

$$\begin{array}{ccc} \mathrm{Sub}_k X(A_1, \ldots, A_k) \times \Delta(A_1) \times \ldots \times \Delta(A_k) & \to & X(A_1 \cdot \ldots \cdot A_k) \times \Delta(A_1 \cdot \ldots \cdot A_k) \\ (x, f_1, \ldots, f_k) & \mapsto & (x; \quad a \mapsto \frac{1}{k} f_r(a),\ \text{if } a \in A_r) \end{array}$$

(see [4]).

Furthermore we define the k-simplicial map

$$\alpha_p : \mathrm{Sub}_k X \to \mathrm{pr}_p^* X$$

by ($A_1, \ldots, A_k \in \mathrm{Ord}$):

$$\begin{array}{ccc} X(A_1 \cdot \ldots \cdot A_k) & \to & X(A_p) \\ x & \mapsto & i_p^*(x) \end{array}$$

where $i_p : A_p \hookrightarrow A_1 \cdot \ldots \cdot A_k$ denotes the canonical inclusion. The following lemma shows that Grayson's homeomorphism ψ modulo homotopy may be replaced by the realization of α_p.

Lemma 6. *For each* $p \in \{1, \ldots, k\}$ *we have*

$$\psi \simeq \pi_p \circ |\alpha_p|.$$

Proof. For any $A_1, \ldots, A_k \in \mathrm{Ord}$ let $h_p := h_p(A_1, \ldots, A_k)$ be the map

$$\begin{array}{rccc} h_p : & [0,1] \times \Delta(A_1) \times \ldots \times \Delta(A_k) & \to & \Delta(A_1 \cdot \ldots \cdot A_k) \\ & (t, f_1, \ldots, f_k) & \mapsto & \left(a \mapsto \begin{cases} \frac{1+(k-1)t}{k} \cdot f_p(a) \text{ , if } a \in A_p \\ \frac{1-t}{k} \cdot f_r(a) \text{ , if } a \in A_r \text{ with } r \neq p \end{cases} \right). \end{array}$$

The map h_p is well-defined, because

$$\begin{aligned} \sum_{a \in A_1 \cdot \ldots \cdot A_k} h_p(t, f_1, \ldots, f_k)(a) &= \frac{1+(k-1)t}{k} \cdot \sum_{a \in A_p} f_p(a) + \frac{1-t}{k} \cdot \sum_{r \neq p} \sum_{a \in A_r} f_r(a) \\ &= \frac{1+(k-1)t}{k} + \frac{(1-t)(k-1)}{k} = 1 \end{aligned}$$

and obviously h_p is continuous. Furthermore we have

$$\begin{aligned} h_p(0, f_1, \ldots, f_k)(a) &= \tfrac{1}{k} f_r(a), \text{ if } a \in A_r \\ h_p(1, f_1, \ldots, f_k)(a) &= \left\{ \begin{array}{ll} f_p(a) & \text{, if } a \in A_p \\ 0 & \text{, else} \end{array} \right\} = \left[(i_p)_*(f_p) \right](a) \end{aligned}$$

and h_p is compatible with the morphisms $s \in (\mathrm{Ord}^{\mathrm{op}})^k$. Hence h_p induces a homotopy between ψ and the map $|\mathrm{Sub}_k X| \to |X|$ given by ($A_1, \ldots, A_k \in \mathrm{Ord}$)

$$\begin{array}{ccc} \mathrm{Sub}_k X(A_1, \ldots, A_k) \times \Delta(A_1) \times \ldots \times \Delta(A_k) & \to & X(A_1 \cdot \ldots \cdot A_k) \times \Delta(A_1 \cdot \ldots \cdot A_k) \\ (x, f_1, \ldots, f_k) & \mapsto & (x, (i_p)_*(f_p)). \end{array}$$

The latter is the map $\pi_p \circ |\alpha_p|$, because $(x, (i_p)_*(f_p)) \sim (i_p^*(x), f_p)$. This shows the lemma.

Now we fix an element $M \in \mathcal{H}^p([0])$ which has the following property: $M(\alpha) = 0$, if $l_t(\alpha) \neq L$ for some $t \in \{1, \dots, p\}$ and we view M as a point in $|\mathcal{H}^p|$ (For instance, we may take $M = \Lambda^p N$, if $N \in G^1([0])$ with $N(\frac{0}{R}) = 0$).

Theorem 1. *The following diagram of continuous maps commutes up to homotopy:*

$$\begin{array}{ccccc} \{M\} \times |\mathcal{H}^{k-p}| & \hookrightarrow & |\mathcal{H}^p| \times |\mathcal{H}^{k-p}| & \xrightarrow{\otimes} & |\mathcal{H}^k| \\ \downarrow {\scriptstyle |\Xi^p| \times |\Xi^{k-p}|} & & & & \downarrow {\scriptstyle |\Xi^k|} \\ \{|\Xi^p|(M)\} \times |G^{k-p}| & \hookrightarrow & |G^p| \times |G^{k-p}| & \xrightarrow{\otimes} & |G^k| \end{array}$$

Proof. Let $\mathrm{pr}^*\mathrm{Sub}_{k-p}\mathcal{H}^{k-p}$ be the k-simplicial set given by

$$\mathrm{pr}^*\mathrm{Sub}_{k-p}\mathcal{H}^{k-p}(A_1, \dots, A_k) := \mathcal{H}^{k-p}(A_{p+1} \cdot \ldots \cdot A_k)$$

($A_1, \dots, A_k \in \mathrm{Ord}$). We define

$$\alpha : \mathrm{Sub}_k\mathcal{H}^{k-p} \to \mathrm{pr}^*\mathrm{Sub}_{k-p}\mathcal{H}^{k-p}$$

to be the k-simplicial map given by

$$\begin{array}{ccc} \mathcal{H}^{k-p}(A_1 \cdot \ldots \cdot A_k) & \to & \mathcal{H}^{k-p}(A_{p+1} \cdot \ldots \cdot A_k) \\ M & \mapsto & i^*(M) \end{array}$$

($A_1, \dots, A_k \in \mathrm{Ord}$) where $i : A_{p+1} \cdot \ldots \cdot A_k \to A_1 \cdot \ldots \cdot A_k$ denotes the canonical inclusion. By lemma 6 its realization is homotopic to the composition of identifications

$$|\mathrm{Sub}_k\mathcal{H}^{k-p}| \xrightarrow{\sim} |\mathcal{H}^{k-p}| \xrightarrow{\sim} |\mathrm{pr}^*\mathrm{Sub}_{k-p}\mathcal{H}^{k-p}|$$

(compose with α_k, for instance) and in particular it is a homotopy equivalence. Now it is enough to show that the following diagram of k-simplicial maps commutes:

$$\begin{array}{ccc} \mathrm{Sub}_k\mathcal{H}^{k-p} & \xrightarrow{M\otimes -} & \mathrm{Sub}_k\mathcal{H}^k \\ \downarrow {\scriptstyle \alpha} & & \\ \mathrm{pr}^*\mathrm{Sub}_{k-p}\mathcal{H}^{k-p} & & \downarrow {\scriptstyle \Xi^k} \\ \downarrow {\scriptstyle \Xi^{k-p}} & & \\ \mathrm{pr}^*G^{k-p} & \xrightarrow{\Xi^p(M)\otimes -} & G^k \end{array}$$

We fix objects $A_1, \dots, A_k \in \mathrm{Ord}$, a functor $N \in \mathcal{H}^{k-p}(A_1 \cdot \ldots \cdot A_k)$ and an element $(\frac{i_1}{j_1}, \dots, \frac{i_k}{j_k}) \in \Gamma(A_1) \times \ldots \times \Gamma(A_k)$ and put

$$\alpha := (\tfrac{i_1}{l_1}, *_2, \dots, *_k, \tfrac{i_k}{l_k}) := \Xi^k(\tfrac{i_1}{j_1}, \dots, \tfrac{i_k}{j_k})$$
$$\alpha' := (\tfrac{i_{p+1}}{l'_{p+1}}, *_{p+1}, \dots, *_k, \tfrac{i_k}{l'_k}) := \Xi^{k-p}(\tfrac{i_{p+1}}{j_{p+1}}, \dots, \tfrac{i_k}{j_k}).$$

If $j_1 = \ldots = j_p = L$, we define s by $j_1 = \ldots = j_{p+s} = L$ and $j_{p+s+1} \neq L$. Then by definition of Ξ we have $l_{p+s+1} \neq L, \dots, l_k \neq L$ and $*_2 = \ldots = *_{p+s} = \wedge$. In particular we have $l'_{p+1} = l_{p+1}, \dots, l'_k = l_k$. Now the following calculations show the commutativity of the above diagram:

$$\begin{aligned}
&\Big(\Xi^p(M) \otimes \Xi^{k-p}(N)\Big)(\frac{i_1}{j_1}, \dots, \frac{i_k}{j_k}) \\
&\quad = \Xi^p(M)(\frac{i_1}{j_1}, \dots, \frac{i_p}{j_p}) \otimes \Xi^{k-p}(N)(\frac{i_{p+1}}{j_{p+1}}, \dots, \frac{i_k}{j_k}) \\
&\quad = M(\frac{i_1}{l_1}, *_2, \dots, *_p, \frac{i_p}{l_p}) \otimes N(\frac{i_{p+1}}{l'_{p+1}}, *_{p+2}, \dots, *_k, \frac{i_k}{l'_k}) \\
&\quad = \begin{cases} M(\frac{0}{L}, \wedge, \dots, \wedge, \frac{0}{L}) \otimes N(\frac{i_{p+1}}{l_{p+1}}, *_{p+2}, \dots, *_k, \frac{i_k}{l_k}) & \text{, if } j_1 = \ldots = j_p = L \\ 0 & \text{, else (by assumption on } M) \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\Xi^k \circ \mathrm{Sym}^{p,k-p}(M \otimes N)(\frac{i_1}{j_1}, \dots, \frac{i_k}{j_k}) \\
&\quad = \mathrm{Sym}^{p,k-p}(M \otimes N)(\frac{i_1}{l_1}, *_2, \dots, *_k, \frac{i_k}{l_k}) \\
&\quad = \bigoplus_{\mathcal{R} \in \mathcal{P}(k,p)/\sim_\alpha} \sum_{R \in \mathcal{R}} M(\alpha^R) \otimes N(\alpha^{\bar{R}}) \\
&\quad = \begin{cases} \sum\limits_{R \subseteq \{1, \dots, p+s\}, |R| = p} M(\alpha^R) \otimes N(\alpha^{\bar{R}}) & \text{, if } j_1 = \ldots = j_p = L \\ 0 & \text{, else (by assumption on } M) \end{cases} \\
&\quad = \begin{cases} M(\frac{0}{L}, \wedge, \dots, \wedge, \frac{0}{L}) \otimes N(\frac{i_{p+1}}{l_{p+1}}, *_{p+2}, \dots, *_k, \frac{i_k}{l_k}) & \text{, if } j_1 = \ldots = j_p = L \\ 0 & \text{, else.} \end{cases}
\end{aligned}$$

Corollary. *If $\mathcal{M}$ has an identity element I (i. e. the functor $I \otimes -$ is isomorphic to the identical functor), the continuous map*

$$|G^k| \overset{\mathrm{Sym}^{1,\dots,1}}{\longrightarrow} |\mathcal{H}^k| \overset{\Xi^k}{\longrightarrow} |G^k|$$

is homotopic to the identity.

In particular the conjecture' holds in the case $k = 2$, $p = 1$.

Proof. In the case $k = 1$ there is nothing to prove. For the general case we proceed by induction on k: Obviously the k-simplicial map

$$\mathrm{pr}_k^* G^1 \stackrel{I\otimes\ldots\otimes I\otimes -}{\longrightarrow} G^k$$

is equal to the map j_k introduced in section 1. Applying the theorem to $M := \mathrm{Sym}^{1,\ldots,1}(I \otimes \ldots \otimes I)$ ($k-1$ factors) we obtain the commutativity of the following diagram (up to homotopy):

$$\begin{array}{ccccc} |G^1| & \stackrel{|j_k|}{\to} & |G^k| & \stackrel{|\mathrm{Sym}^{1,\ldots,1}|}{\to} & |\mathcal{H}^k| \\ \| & & & & \downarrow |\Xi^k| \\ |G^1| & & \stackrel{|\Xi^{k-1}\circ \mathrm{Sym}^{1,\ldots,1}(I\otimes\ldots\otimes I)\otimes -|}{\longrightarrow} & & |G^k| \end{array}$$

Because j_k is a homotopy equivalence and the point $\Xi^{k-1} \circ \mathrm{Sym}^{1,\ldots,1}(I \otimes \ldots \otimes I)$ lies in the same component as $I \otimes \ldots \otimes I$ by the inductive hypothesis, this proves the corollary.

Remark. Recently A. Nenashev told me that he is able to prove the conjecture' in the case $p = 1$ and k arbitrary.

5. The Rule $\lambda^k(x + y) = \sum_{p=0}^{k}(\lambda^p x) \cdot (\lambda^{k-p} y)$

Let $\mathcal{M}$ be an exact category as in section 2, equipped with power operations. In addition to the axioms (E1) to (E5) we assume:

(Add1) The map (E1) is an epimorphism (in general not admissible).

(Add2) The tensor product is commutative in the usual sense and for any U in $\mathcal{M}$, $k \in \mathbb{N}$, and $\sigma \in \Sigma_k$ the following diagram commutes:

$$\begin{array}{ccc} \otimes^k U & \stackrel{\sigma}{\longrightarrow} & \otimes^k U \\ \downarrow \text{(E1)} & & \downarrow \text{(E1)} \\ \Lambda^k U & \stackrel{\mathrm{sgn}(\sigma)}{\longrightarrow} & \Lambda^k U \end{array}$$

We will call an exact category with power operations which satisfy (Add1) and (Add2), an *exact category with exterior power operations.* Let $k \geq 1$ be a fixed integer. The aim of this section is to prove

Theorem 2 (The rule $\lambda^k(x+y) = \sum_{p=0}^k (\lambda^p x)\cdot(\lambda^{k-p}y)$ in the homotopy category). *The diagram*

$$\begin{array}{ccccc} |G\mathcal{M}| \times |G\mathcal{M}| & \stackrel{(\Lambda^p\otimes\Lambda^{k-p})_{p=0}^k}{\longrightarrow} & \prod_{p=0}^k |\mathcal{H}^k\mathcal{M}| & \stackrel{\prod_{p=0}^k |\Xi^k|}{\longrightarrow} & \prod_{p=0}^k |G^k\mathcal{M}| \\ \downarrow \oplus & & & & \downarrow \oplus \\ |G\mathcal{M}| & & \stackrel{\lambda^k}{\longrightarrow} & & |G^k\mathcal{M}| \end{array}$$

of continuous maps commutes up to homotopy.

Remark. On higher K-groups $K_q(\mathcal{M})$, $q \geq 1$, the λ-operation λ^k is a homomorphism. Defining the product of two homogeneous elements $x, y \in K(\mathcal{M}) := \oplus_{q\geq 0} K_q(\mathcal{M})$ of positive degree to be zero, this immediately yields the rule $\lambda^k(x+y) = \sum_{p=0}^k (\lambda^p x)\cdot(\lambda^{k-p}y)$ for $x,\ y$. While this reasoning is based on the cogroup structure of the sphere S^q and hence can only be applied to the K-groups of $\mathcal{M}$, the above theorem shows, that the rule $\lambda^k(x+y) = \sum_{p=0}^k (\lambda^p x)\cdot(\lambda^{k-p}y)$ already holds for $|G\mathcal{M}|$, i. e. in the category of topological spaces up to homotopy.

Proof. Obviously it is enough to show that the diagram

$$\begin{array}{ccc} G\mathcal{M} \times G\mathcal{M} & \stackrel{(\Lambda^p\otimes\Lambda^{k-p})_{p=0}^k}{\longrightarrow} & \prod_{p=0}^k \mathcal{H}^k\mathcal{M} \\ \downarrow \oplus & & \downarrow \oplus \\ G\mathcal{M} & \stackrel{\Lambda^k}{\longrightarrow} & \mathcal{H}^k\mathcal{M} \end{array}$$

of 1-simplicial maps commutes up to homotopy. This immediately follows from the following proposition (To construct the homotopy precisely, compare also section 7):

Proposition 2. *Let $A \in Ord$. Then for any $M, N \in Exact(\Gamma(A), \mathcal{M})$ there is a natural isomorphism*

$$\bigoplus_{p=0}^k (\Lambda^p M \otimes \Lambda^{k-p} N) \stackrel{\sim}{\longrightarrow} \Lambda^k(M \oplus N)$$

of functors from $\Gamma^k(A)$ to $\mathcal{M}$.

At first we formulate and prove the underlying assertion for exterior powers. For this let $U_1 \hookrightarrow \ldots \hookrightarrow U_k =: U$ and $V_1 \hookrightarrow \ldots \hookrightarrow V_k =: V$ be two chains of admissible monomorphims in $\mathcal{M}$. Then for each subset $R \subseteq \{1, \ldots, k\}$ the following commutative diagram defines a natural map

$$i_R : (\mathop{\Lambda}_{r\in R} U_r) \otimes (\mathop{\Lambda}_{r\in \bar{R}} V_r) \to \mathop{\Lambda}_{r=1}^{k} (U_r \oplus V_r).$$

(The commutativity of the outer square shows that the map from the upper left corner to the right term factorizes through the left middle term):

$$
\begin{array}{ccccccc}
\bigotimes\limits_{r\in R} U_r \otimes \bigotimes\limits_{r\in \bar{R}} V_r & \hookrightarrow & \bigotimes\limits_{r\in R} (U_r\oplus V_r) \otimes \bigotimes\limits_{r\in \bar{R}} (U_r\oplus V_r) & \xrightarrow{\mathrm{sgn}(\sigma)\cdot\sigma} & \bigotimes\limits_{r=1}^{k} (U_r\oplus V_r) & & \\
\downarrow \text{(E1)} & & \downarrow & & \downarrow & \searrow \text{(E1)} & \\
(\mathop{\Lambda}\limits_{r\in R} U_r) \otimes (\mathop{\Lambda}\limits_{r\in \bar{R}} V_r) & & \otimes^p(U\oplus V)\otimes\otimes^{k-p}(U\oplus V) & \xrightarrow{\mathrm{sgn}(\sigma)\cdot\sigma} & \otimes^k(U\oplus V) & & \mathop{\Lambda}\limits_{r=1}^{k}(U_r\oplus V_r) \\
\downarrow & & \downarrow \text{(E1)} & \searrow \text{(E1)} & \downarrow \text{(E1)} & \swarrow & \\
\Lambda^p U\otimes \Lambda^{k-p} V & \hookrightarrow & \Lambda^p(U\oplus V)\otimes \Lambda^{k-p}(U\oplus V) & \xrightarrow{\text{(E1)}} & \Lambda^k(U\oplus V) & &
\end{array}
$$

(The permutation σ corresponds to R.)

Lemma 7. *The above morphisms i_R, $R\subseteq\{1,\ldots,k\}$, induce an isomorphism*

$$\bigoplus_{p=0}^{k} \sum_{R\in\mathcal{P}(k,p)} (\mathop{\Lambda}_{r\in R} U_r)\otimes(\mathop{\Lambda}_{r\in\bar{R}} V_r) \tilde{\to} \mathop{\Lambda}_{r=1}^{k}(U_r\oplus V_r).$$

Proof of lemma 7. If $U_r = U$ and $V_r = V$ for all $r = 1,\ldots,k$ we have the following (well-known) exact sequences (by (E5)):

$$0\to \underbrace{V\wedge\ldots\wedge V}_{k} \to \underbrace{V\wedge\ldots\wedge V}_{k-1}\wedge(V\oplus W)\to (\underbrace{V\wedge\ldots\wedge V}_{k-1})\otimes W\to 0$$

$$0\to \underbrace{V\wedge\ldots\wedge V}_{k-1}\wedge(V\oplus W)\to \underbrace{V\wedge\ldots\wedge V}_{k-2}\wedge(V\oplus W)\wedge(V\oplus W)\to (\underbrace{V\wedge\ldots\wedge V}_{k-2})\otimes(W\wedge W)\to 0$$

$$\vdots$$

$$0\to V\wedge\underbrace{(V\oplus W)\wedge\ldots\wedge(V\oplus W)}_{k-1}\to \underbrace{(V\oplus W)\wedge\ldots\wedge(V\oplus W)}_{k}\to \underbrace{W\wedge\ldots\wedge W}_{k}\to 0$$

This sequences split, because by (E3) for each p the composition

$$\Lambda^p V\otimes\Lambda^{k-p}W \overset{\text{(E1)}}{\to} V\wedge\ldots\wedge V\wedge(V\oplus W)\wedge\ldots\wedge(V\oplus W) \overset{\text{(E2)}}{\to} \Lambda^p V\otimes \Lambda^{k-p}W$$

is the identity. This shows the lemma in this classical case. In the general case the injectivity follows from this and from the following commutative diagram:

$$
\begin{array}{ccc}
\bigoplus\limits_{p=0}^{k} \sum\limits_{R\in\mathcal{P}(k,p)} (\mathop{\Lambda}\limits_{r\in R} U_r)\otimes(\mathop{\Lambda}\limits_{r\in\bar{R}} V_r) & \to & \mathop{\Lambda}\limits_{r=1}^{k}(U_r\oplus V_r) \\
\downarrow & & \downarrow \\
\bigoplus\limits_{p=0}^{k} \Lambda^p U\otimes\Lambda^{k-p}V & \tilde{\to} & \Lambda^k(U\oplus V)
\end{array}
$$

To prove the surjectivity, we will show by descending induction on i that the lemma holds, if $V_1 = \ldots = V_i = 0$. For the case $i = k$ there is nothing to prove. For the induction step $i \to i-1$ we consider the following diagram with the obvious maps:

$$U_1 \wedge \ldots \wedge U_{i-1} \wedge U_i \wedge (U_{i+1} \oplus V_{i+1}) \wedge \ldots \wedge (U_k \oplus V_k)$$

$$\downarrow$$

$$\bigoplus_{p=0}^{k} \sum_{\substack{R \subseteq \{i+1,\ldots,k\} \\ |R|=p}} (U_1 \wedge \ldots \wedge U_{i-1} \wedge \mathop{\Lambda}_{r\in R} U_r) \otimes (\mathop{\Lambda}_{r \in \bar{R} \cup \{i\}} V_r)$$

$$\searrow$$

$$U_1 \wedge \ldots \wedge U_{i-1} \wedge (U_i \oplus V_i) \wedge (U_{i+1} \oplus V_{i+1}) \wedge \ldots \wedge (U_k \oplus V_k)$$

$$\downarrow \text{(E2)}$$

$$\downarrow$$

$$\bigoplus_{p=0}^{k} \sum_{\substack{R \subseteq \{i+1,\ldots,k\} \\ |R|=p}} (U_1 \wedge \ldots \wedge U_{i-1} \otimes \mathop{\Lambda}_{r\in R} \tfrac{U_r}{U_i}) \otimes (\mathop{\Lambda}_{r \in \bar{R} \cup \{i\}} V_r)$$

$$\searrow$$

$$U_1 \wedge \ldots \wedge U_{i-1} \otimes V_i \wedge (\tfrac{U_{i+1}}{U_i} \oplus V_{i+1}) \wedge \ldots \wedge (\tfrac{U_k}{U_i} \oplus V_k)$$

By axiom (E5) the right column is a short exact sequence. By axiom (E5) the left vertical map is an epimorphism. Applying the inductive hypothesis to $\frac{U_{i+1}}{U_i} \hookrightarrow \ldots \hookrightarrow \frac{U_k}{U_i}$ and $V_i \hookrightarrow \ldots \hookrightarrow V_k$ (with k replaced by $k-i+1$) we see that the lower map is an isomorphism. From this it follows that $U_1 \wedge \ldots \wedge U_{i-1} \wedge (U_i \oplus V_i) \wedge \ldots \wedge (U_k \oplus V_k)$ is the sum of the two subobjects which are written in the picture. Applying the inductive hypothesis to the upper subobject we obtain the lemma.

Proof of proposition 2. Let $\alpha \in \Gamma^k(A)$. For any $r = 1, \ldots, k$ we put $M_r := M(\frac{i_r(\alpha)}{l_r(\alpha)})$ and $N_r := N(\frac{i_r(\alpha)}{l_r(\alpha)})$. Then by lemma 7 and by the exactness of the tensor product there is a unique isomorphism

$$\bigoplus_{p=0}^{k} (\Lambda^p M \otimes \Lambda^{k-p} N)(\alpha) \to \Lambda^k (M \oplus N)(\alpha)$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\bigoplus\limits_{R\subseteq\{1,\ldots,k\}} (\bigotimes\limits_{r\in R} M_r)\otimes(\bigotimes\limits_{r\in\bar{R}} N_r) & \overset{\sum_{R\subseteq\{1,\ldots,k\}}\mathrm{sgn}(\sigma(R))\cdot\sigma(R)}{\longrightarrow} & \bigotimes\limits_{r=1}^{k}(M_r\oplus N_r) \\
\downarrow \text{(E1)} & & \\
\bigoplus\limits_{p=0}^{k}\ \bigoplus\limits_{R\in\mathcal{P}(k,p)/\sim_\alpha}\ \sum\limits_{R\in\mathcal{R}} (\Lambda^p M)(\alpha^R)\otimes(\Lambda^{k-p}N)(\alpha^{\bar{R}}) & & \downarrow \text{(E1)} \\
\| & & \\
\bigoplus\limits_{p=0}^{k}(\Lambda^p M\otimes\Lambda^{k-p}N)(\alpha) & \to & \Lambda^k(M\oplus N)(\alpha)
\end{array}$$

By construction this isomorphism is natural in M and N and compatible with the maps (E1) and (E2) and hence it induces the desired functor isomorphism.

Remark. Replacing the map $\mathrm{sgn}(\sigma)$ in axiom (Add2) by the identity we obtain the axiomatic definition of symmetric power operations and theorem 2 also holds for symmetric powers.

6. The Rule $\sum_{p=0}^{k}(-1)^p(\lambda^p x)\cdot(s^{k-p}x) = 0$

In this section we will desist from describing things axiomatically, because in the proof of proposition 3 we will construct maps by defining them on basis elements. So in this section $\mathcal{M}$ is the category of finitely generated projective modules over a fixed ring or, more generally, the category of locally free modules of finite presentation on a fixed locally ringed space (Because the ring or space will not occur explicitly, we don't introduce a symbol to denote it). Then $\mathcal{M}$ is an exact category with exterior power operations λ^k and with symmetric power operations s^k, $k \in \mathbb{N}$, in the usual sense. We fix an integer $k \geq 1$.

Theorem 3. *The continuous map*

$$\sum_{p=0}^{k}(-1)^p|\lambda^p\cdot s^{k-p}| : |G\mathcal{M}| \to |G^k\mathcal{M}|$$

is homotopic to the zero map.

Remark. While the rule $\lambda^k(x+y) = \sum_{p=0}^{k}(\lambda^p x)\cdot(\lambda^{k-p}y)$ considered in the last section for homogeneous elements $x,\ y \in K(\mathcal{M}) = \oplus_{q\geq 0}K_q(\mathcal{M})$ of positive degree is already an immediate consequence of the linearity of the map λ^k (see remark after theorem 2), I don't know such a reasoning for the rule $\sum_{p=0}^{k}(-1)^p(\lambda^p x)\cdot(s^{k-p}x) = 0$. So for the rule considered in this section the shuffle products are essential not only to clearify the situation on the topological space $|G\mathcal{M}|$ but also to prove the rule for K-groups.

Similar to the last section we will at first formulate and prove the underlying proposition for functors $M \in \mathrm{Exact}(\Gamma(A), \mathcal{M})$. In order to get a quick survey of the succeeding constructions, the reader should take a look at the following exact sequences in the case $k = 2$ and $k = 3$.

$$
\begin{array}{cccc}
\boxed{V \hookrightarrow W} & \boxed{V, W} & & \\
0 & 0 & & \\
\downarrow & \downarrow & & \\
V \wedge W & V \otimes W & vw & \\
\downarrow & \downarrow & \bar{\downarrow} & \\
V \otimes W & V \otimes W & vw & v_1 w_1 \\
+ & \oplus & & \\
W \otimes V & W \otimes V & -wv & +w_2 v_2 \\
\downarrow & \downarrow & & \bar{\downarrow} \\
V \cdot W & V \otimes W & & v_1 w_1 + v_2 w_2 \\
\downarrow & \downarrow & & \\
0 & 0 & &
\end{array}
$$

$\boxed{V \hookrightarrow W \hookrightarrow X}$	$\boxed{V \hookrightarrow W, X}$	$\boxed{V, W \hookrightarrow X}$	$\boxed{V, W, X}$		
0	0	0	0		
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
$V \wedge W \wedge X$	$V \wedge W \otimes X$	$V \otimes W \wedge X$	$V \otimes W \otimes X$	vwx	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	↧	
$(W \wedge X) \otimes V$	$(W \otimes X) \otimes V$	$(W \wedge X) \otimes V$	$(W \otimes X) \otimes V$	wxv	$w_1x_1v_1$
$+$	$+$	$\oplus$	$\oplus$		
$(V \wedge X) \otimes W$	$(V \otimes X) \otimes W$	$(V \otimes X) \otimes W$	$(V \otimes X) \otimes W$	$-vxw$	$+v_2x_2w_2$
$+$	$\oplus$	$+$	$\oplus$		
$(V \wedge W) \otimes X$	$(V \wedge W) \otimes X$	$(V \otimes W) \otimes X$	$(V \otimes W) \otimes X$	$+vwx$	$+v_3w_3x_3$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	———	↧
$V \otimes (W \cdot X)$	$V \otimes (W \otimes X)$	$V \otimes (W \cdot X)$	$V \otimes (W \otimes X)$	$v_1w_1x_1$	$-v_2w_2x_2$ $-v_3w_3x_3$
$+$	$+$	$\oplus$	$\oplus$		
$W \otimes (V \cdot X)$	$W \otimes (V \otimes X)$	$W \otimes (V \otimes X)$	$W \otimes (V \otimes X)$	$+w_2v_2x_2$	$-w_1v_1x_1$ $+w_3v_3x_3$
$+$	$\oplus$	$+$	$\oplus$		
$X \otimes (V \cdot W)$	$X \otimes (V \cdot W)$	$X \otimes (V \otimes W)$	$X \otimes (V \otimes W)$	$+x_3v_3w_3$	$+x_1v_1w_1$ $+x_2v_2w_2$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	↧	
$V \cdot W \cdot X$	$V \cdot W \otimes X$	$V \otimes W \cdot X$	$V \otimes W \otimes X$	$v_1w_1x_1$ $+v_2w_2x_2$ $+v_3w_3x_3$	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		
0	0	0	0		

Here for each sequence the assumed situation is fixed in the framed box. The essential point is that the formal description of the maps on the right hand side works for all sequences simultaneously. This essentially means that the map d_p defined later on is a functor morphism.

To construct this sequences for general k we at first introduce the following sign: Let $I = \{i_1 < \ldots < i_p\}$ and $J = \{j_1 < \ldots < j_q\}$ be subsets of $\{1, \ldots, k\}$ with $I \cap J = \emptyset$ and

$I \cup J = \{l_1 < \ldots < l_{p+q}\}$. Then:

$$\operatorname{sgn}(I,J) := \operatorname{sgn}\begin{pmatrix} i_1 \ldots i_p & j_1 \ldots j_q \\ l_1 \ldots & \ldots l_{p+q} \end{pmatrix}.$$

Now we fix an object $A \in \mathrm{Ord}$, a functor $M \in \mathrm{Exact}(\Gamma(A), \mathcal{M})$ and an integer $1 \leq p \leq k$. We are going to construct a morphism

$$d_p : \Lambda^p M \otimes S^{k-p} M \to \Lambda^{p-1} M \otimes S^{k-p+1} M$$

of functors from $\Gamma^k(A)$ to $\mathcal{M}$.

At first let $\beta \in \Gamma^k(A)$ with $*_r(\beta) = \otimes$ for all $r = 2, \ldots, k$ and put $M_r := M(\frac{i_r(\beta)}{l_r(\beta)})$ for $r = 1, \ldots, k$. Then the map

$$d_p(\beta) : \bigoplus_{R \in \mathcal{P}(k,p)} (\bigotimes_{r \in R} M_r) \otimes (\bigotimes_{\bar{r} \in \bar{R}} M_{\bar{r}}) \to \bigoplus_{R \in \mathcal{P}(k,p-1)} (\bigotimes_{r \in R} M_r) \otimes (\bigotimes_{\bar{r} \in \bar{R}} M_{\bar{r}})$$

is defined by tensoring the map

$$\begin{array}{cccc} \tilde{d}_p : & \bigoplus_{R \in \mathcal{P}(k,p)} \mathbb{Z} e_R & \to & \bigoplus_{R \in \mathcal{P}(k,p-1)} \mathbb{Z} e_R \\ & e_R & \mapsto & \sum_{s \in R} \operatorname{sgn}(s, R \backslash \{s\}) e_{R \backslash \{s\}} \end{array}$$

with $\bigotimes_{r=1}^{k} M_r$ and composing with the canonical isomorphisms $\bigotimes_{r=1}^{k} M_r \cong (\bigotimes_{r \in R} M_r) \otimes (\bigotimes_{\bar{r} \in \bar{R}} M_{\bar{r}})$ (for each $R \in \mathcal{P}(k,p)$ respectively $R \in \mathcal{P}(k,p-1)$).

Given a general $\alpha \in \Gamma^k(A)$ we put $\beta := (\frac{i_1(\alpha)}{l_1(\alpha)}, \otimes, \ldots, \otimes, \frac{i_k(\alpha)}{l_k(\alpha)}) \in \Gamma^k(A)$ and as before we put $M_r := M(\frac{i_r(\beta)}{l_r(\beta)})$ for $r = 1, \ldots, k$. We claim that there is a unique map $d_p(\alpha) : (\Lambda^p M \otimes S^{k-p} M)(\alpha) \to (\Lambda^{p-1} M \otimes S^{k-p+1} M)(\alpha)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\Lambda^p M \otimes S^{k-p} M)(\beta) & \stackrel{d_p(\beta)}{\longrightarrow} & (\Lambda^{p-1} M \otimes S^{k-p+1} M)(\beta) \\ \downarrow \beta \longrightarrow \alpha & & \downarrow \beta \longrightarrow \alpha \\ (\Lambda^p M \otimes S^{k-p} M)(\alpha) & \stackrel{d_p(\alpha)}{\longrightarrow} & (\Lambda^{p-1} M \otimes S^{k-p+1} M)(\alpha) \end{array}$$

To prove this we will first show that the map $(\beta \to \alpha) \circ d_p(\beta)$ factorizes through

$$(\Lambda^p M \otimes S^{k-p} M)(\beta) \to \bigoplus_{R \in \mathcal{P}(k,p)} (\Lambda^p M)(\alpha^R) \otimes (S^{k-p} M)(\alpha^{\bar{R}})$$

and then that it factorizes as claimed. For the first step we fix $R = \{r_1 < \ldots < r_p\} \in \mathcal{P}(k,p)$. Let $x = (x_1 \otimes \ldots \otimes x_p) \otimes (y_1 \otimes \ldots \otimes y_{k-p})$ be an element of $(\bigotimes_{r \in R} M_r) \otimes (\bigotimes_{\bar{r} \in \bar{R}} M_{\bar{r}})$ such that there are two indices $i' < i''$ in $\{1, \ldots, p\}$ such that $r_{i'} \sim_\alpha r_{i''}$ and such that $x_{i'} \mapsto x_{i''}$ under $M_{r_{i'}} \hookrightarrow M_{r_{i''}}$. Then we have (as usual)

$$\begin{aligned}
&(\beta \to \alpha) \circ d_p(\beta)(x) = \\
&\quad = \sum_{i=1}^{p} (-1)^{i-1} (x_1 \cdots x_{i-1} x_{i+1} \cdots x_p) \otimes (y_1 \cdots x_i \cdots y_{k-p}) \\
&\quad = \sum_{i \in \{i', i''\}} (-1)^{i-1} (x_1 \cdots x_{i-1} x_{i+1} \cdots x_p) \otimes (y_1 \cdots x_i \cdots y_{k-p}) = 0.
\end{aligned}$$

This proves the first step. For the second step note that for any $R,\ R' \in \mathcal{P}(k,p)$ with $R \sim_\alpha R'$ we have also $R \backslash \{r_i\} \sim_\alpha R \backslash \{r'_i\}$ for $r = 1, \ldots, p$. Now proposition 1 and lemma 1 a) give the second step in the usual way.

By construction the maps $d_p(\alpha)$, $\alpha \in \Gamma^k(A)$, are functorial in M and compatible with the maps (E1) and (E2). So they define a morphism

$$d_p : \Lambda^p M \otimes S^{k-p} M \to \Lambda^{p-1} M \otimes S^{k-p+1} M$$

of functors from $\Gamma^k(A)$ to $\mathcal{M}$.

Proposition 3 (Generalized Koszul complex). *The sequence*

$$0 \to \Lambda^k M \overset{d_k}{\to} \Lambda^{k-1} M \otimes M \overset{d_{k-1}}{\to} \ldots \overset{d_2}{\to} M \otimes S^{k-1} M \overset{d_1}{\to} S^k M \to 0$$

of functors from $\Gamma^k(A)$ to $\mathcal{M}$ is exact (i. e. pointwise).

Proof. For each subset $R \subseteq \{1, \ldots, k\}$ and for any $r,\ s \in R$ with $r \neq s$ we have

$$\operatorname{sgn}(r, R \backslash \{r\}) \cdot \operatorname{sgn}(s, R \backslash \{r, s\}) + \operatorname{sgn}(s, R \backslash \{s\}) \cdot \operatorname{sgn}(r, R \backslash \{r, s\}) = 0.$$

This shows that for each $1 \leq p \leq k$ we have $\tilde{d}_{p-1} \circ \tilde{d}_p = 0$ and hence $d_{p-1} \circ d_p = 0$. There remains to prove that for each $\alpha \in \Gamma^k(A)$ the complex

$$0 \to \Lambda^k M(\alpha) \to \ldots \to S^k M(\alpha) \to 0$$

is exact. For this we may assume that $M_r := M(\frac{i_r(\alpha)}{l_r(\alpha)})$ is free for all r (by localization). We define a homotopy

$$h_p : (\Lambda^p M \otimes S^{k-p} M)(\alpha) \to (\Lambda^{p+1} M \otimes S^{k-p-1} M)(\alpha)$$

between the identity and the zero map as follows: For each $r \in \{1, \ldots, k\}$ we choose a basis $\{x_j^r : j = 1, \ldots, m_r\}$ of M_r such that for all $r < r'$ with $r \sim r'$ and for all $j < m_r$ we have

$x_j^r \mapsto x_j^{r'}$ under $M_r \hookrightarrow M_{r'}$. Now let $R = \{r_1 < \ldots < r_p\} \in \mathcal{P}(k,p)$, $\bar{R} = \{\bar{r}_1 < \ldots < \bar{r}_{k-p}\}$. Then, if $J = (j_1, \ldots, j_p)$ runs through all elements of $\{1, \ldots, m_{r_1}\} \times \ldots \times \{1, \ldots, m_{r_p}\}$ with $j_i < j_{i'}$, if $i < i'$ and $r_i \sim r_{i'}$, and if $\bar{J} = (\bar{j}_1, \ldots, \bar{j}_{k-p})$ runs through all elements of $\{1, \ldots, m_{\bar{r}_1}\} \times \ldots \times \{1, \ldots, m_{\bar{r}_{k-p}}\}$ with $\bar{j}_i \leq \bar{j}_{i'}$, if $i < i'$ and $\bar{r}_i \sim \bar{r}_{i'}$, the elements

$$x_J \otimes x_{\bar{J}} := (x_{j_1}^{r_1} \cdots x_{j_p}^{r_p}) \otimes (x_{\bar{j}_1}^{\bar{r}_1} \cdots x_{\bar{j}_{k-p}}^{\bar{r}_{k-p}})$$

form a basis of $(\Lambda^p M)(\alpha^R) \otimes (S^{k-p}M)(\alpha^{\bar{R}})$. Now we define a homomorphism

$$(\Lambda^p M)(\alpha^R) \otimes (S^{k-p}M)(\alpha^{\bar{R}}) \to (\Lambda^{p+1}M)(\alpha^{R \cup \{\bar{r}_1\}}) \otimes (S^{k-p-1}M)(\alpha^{\bar{R} \setminus \{\bar{r}_1\}})$$

by the following assignment for these basis elements:

$$x_J \otimes x_{\bar{J}} \mapsto \begin{cases} x_{(\bar{j}_1, J)} \otimes x_{(\bar{j}_2, \ldots, \bar{j}_{k-p})} & \text{, if } \bar{r}_1 \sim 1 \text{ and if } \bar{j}_1 < j_1 \text{ , if } \bar{r}_1 \sim r_1 \\ 0 & \text{, else.} \end{cases}$$

Note that in the upper case the element $x_{(\bar{j}_1, J)} \otimes x_{(\bar{j}_2, \ldots, \bar{j}_{k-p})}$ is an element of $(\Lambda^{p+1}M)(\alpha^{R \cup \{\bar{r}_1\}}) \otimes (S^{k-p-1}M)(\alpha^{\bar{R} \setminus \{\bar{r}_1\}})$ also, if $\bar{r}_1 > r_1$. For any $R,\ R' \in \mathcal{P}(k,p)$ with $R \sim R'$ we have $(\bar{r}_1 \sim 1) \Leftrightarrow (\bar{r}'_1 \sim 1)$, $(\bar{r}_1 \sim r_1) \Leftrightarrow (\bar{r}'_1 \sim r'_1)$, and $R \cup \{r_1\} \sim R' \cup \{\bar{r}'_1\}$. So the above maps for R and R' are equal on the intersection of their ranges of definition and hence by proposition 1 and lemma 1 a) these maps induce a homomorphism

$$h_p : (\Lambda^p M \otimes S^{k-p}M)(\alpha) \to (\Lambda^{p+1}M \otimes S^{k-p-1}M)(\alpha).$$

There remains to prove that we have

$$d_{p+1}(\alpha) \circ h_p + h_{p-1} \circ d_p(\alpha) = \mathrm{id}.$$

Let $x_J \otimes x_{\bar{J}}$ be one of the above basis elements. Then:

$$\begin{aligned} & d_{p+1}(\alpha) \circ h_p(x_J \otimes x_{\bar{J}}) \\ &\quad = \begin{cases} d_{p+1}(\alpha)(x_{(\bar{j}_1, J)} \otimes x_{(\bar{j}_2, \ldots, \bar{j}_{k-p})}) & \text{, if } \bar{r}_1 \sim 1 \text{ and if } \bar{j}_1 < j_1 \text{ , if } \bar{r}_1 \sim r_1 \\ 0 & \text{, else} \end{cases} \\ &\quad = \begin{cases} x_J \otimes x_{\bar{J}} + \sum_{i=1}^p (-1)^i x_{(\bar{j}_1, j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_p)} \otimes (x_{\bar{j}_2}^{\bar{r}_2} \cdots x_{j_i}^{r_i} \cdots x_{\bar{j}_{k-p}}^{\bar{r}_{k-p}}) & \text{, if } \ldots \\ 0 & \text{, else.} \end{cases} \end{aligned}$$

$$\begin{aligned} & h_{p-1} \circ d_p(\alpha)(x_J \otimes x_{\bar{J}}) \\ &\quad = h_{p-1}(\textstyle\sum_{i=1}^p (-1)^{i-1} x_{(j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_p)} \otimes (x_{\bar{j}_1}^{\bar{r}_1} \cdots x_{j_i}^{r_i} \cdots x_{\bar{j}_{k-p}}^{\bar{r}_{k-p}}) \\ &\quad = \begin{cases} \sum_{i=1}^p (-1)^{i-1} x_{(\bar{j}_1, j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_p)} \otimes (x_{\bar{j}_2}^{\bar{r}_2} \cdots x_{j_i}^{r_i} \cdots x_{\bar{j}_{k-p}}^{\bar{r}_{k-p}}) & \text{, if } \ldots \\ x_J \otimes x_{\bar{J}} & \text{, else.} \end{cases} \end{aligned}$$

(For the last equality note that each summand can be transformed into a distinguished basis element). This proves the proposition.

Remark.

a) In the extreme case "$*_r(\alpha) = \wedge$ for all $r = 2, \ldots, k$", the map h_p is the restriction of ($V := M(\frac{i_k(\alpha)}{l_k(\alpha)})$)

$$\begin{array}{ccc} \Lambda^p V \otimes S^{k-p} V & \to & \Lambda^{p+1} V \otimes S^{k-p-1} V \\ (x_{j_1} \wedge \ldots \wedge x_{j_p}) \otimes (x_{\bar{j}_1} \cdots x_{\bar{j}_{k-p}}) & \mapsto & \begin{cases} (x_{\bar{j}_1} \wedge x_{j_1} \wedge \ldots \wedge x_{j_p}) \otimes (x_{\bar{j}_2} \cdots x_{\bar{j}_{k-p}}) & , \text{if } \bar{j}_1 < j_1 \\ 0 & , \text{else.} \end{cases} \end{array}$$

b) In the extreme case "$*_r(\alpha) = \otimes$ for all $r = 2, \ldots, k$", the map h_p is given by

$$\begin{array}{ccc} \bigoplus_{R \in \mathcal{P}(k,p)} \mathbb{Z} e_R & \to & \bigoplus_{R \in \mathcal{P}(k,p+1)} \mathbb{Z} e_R \\ e_R & \mapsto & \begin{cases} e_{R \cup \{1\}} & , \text{if } 1 \notin R \\ 0 & , \text{else.} \end{cases} \end{array}$$

c) In a similar way one can define morphisms of functors

$$d'_p : \Lambda^{k-p} M \otimes S^p M \to \Lambda^{k-p+1} M \otimes S^{p-1} M$$

with $d'_{p-1} \circ d'_p = 0$. The corresponding d'-complex, however, is in positive characteristic in general not exact (For example, the map $d'_2 : S^2 V \to V \otimes V, xy \mapsto x \otimes y + y \otimes x$, vanishes in characteristic 2 on the elements $x \cdot x \in S^2 V$). One easily checks that $d'_{k-p+1} \circ d_p + d_{p+1} \circ d'_{k-p} = k \cdot \mathrm{id}$. If k is a unit in the ground ring, this shows that both the d'-complex and the d-complex are exact.

d) This generalized Koszul complex has also been discovered independently by D. Grayson in [5]. In the extreme case "$*_r(\alpha) = \wedge$" for all $r = 2, \ldots, k$ D. Grayson calls this complex symmetric product of the mapping cones of an admissible filtered module.

Proof of theorem 3. Let $\mathcal{E}^k$ be the exact category of exact sequences

$$0 \to V_0 \to \ldots \to V_k \to 0$$

of length $k+1$ in $\mathcal{M}$ and for any $0 \le p \le k$ let f_p be the exact functor

$$\begin{array}{ccc} \mathcal{E}^k & \to & \mathcal{M} \\ (0 \to V_0 \to \ldots \to V_k \to 0) & \mapsto & V_p. \end{array}$$

By Quillen's theorem 2 on page 105 of [11] the map $\sum_{p=0}^{k}(-1)^p|f_p|$ is homotopic to the zero map. Proposition 3 gives a simplicial map

$$\begin{array}{ccc} G\mathcal{M} & \stackrel{\Lambda^\bullet \otimes S^{k-\bullet}}{\longrightarrow} & \mathcal{H}^k\mathcal{E}^k \\ M & \mapsto & (0 \to \Lambda^k M \stackrel{d_k}{\to} \ldots \stackrel{d_1}{\to} S^k M \to 0) \end{array}$$

such that the diagram

$$\begin{array}{ccc} G\mathcal{M} & \to & \mathcal{H}^k\mathcal{E}^k \\ & {\scriptstyle (\Lambda^k, \Lambda^{k-1}\otimes S^1, \ldots, S^k)} \searrow & \downarrow {\scriptstyle (f_0, \ldots, f_p)} \\ & & \prod_{p=0}^{k} \mathcal{H}^k\mathcal{M} \end{array}$$

commutes. Now composing with the map $|\Xi^k|$ gives the theorem.

7. The Rule $\lambda^k(x \cdot y) = P_k(\lambda^1 x, \ldots, \lambda^k x, \lambda^1 y, \ldots, \lambda^k y)$

In this section $\mathcal{M}$ is the category of locally free $\mathcal{O}_X$-modules of finite rank on a fixed (noetherian) scheme X. For any $k \geq 1$ let P_k be the universal polynomial in $\mathbb{Z}[X_1, \ldots, X_k, Y_1, \ldots, Y_k]$ defined on page 5 of [2]. The aim of this section is to prove the rule

$$\lambda^k(x \cdot y) = P_k(\lambda^1 x, \ldots, \lambda^k x, \lambda^1 y, \ldots, \lambda^k y)$$

for any $x,\ y \in K(\mathcal{M}) = \oplus_{q\geq 0} K_q(\mathcal{M})$.

In contrast to the previous sections we will show this rule only for the K-groups of $\mathcal{M}$ and not for the classifying space $|G\mathcal{M}|$. The essential ingredient will be the splitting principle. The shuffle products won't be involved.

We recall: The tensor product makes $K_q(\mathcal{M})$ into a $K_0(\mathcal{M})$-module. Defining the product of two homogeneous elements of $K(\mathcal{M})$ of positive degree to be zero $K(\mathcal{M})$ becomes a $K_0(\mathcal{M})$-algebra. Furthermore we have the exterior power operations

$$\lambda^k : K_q(\mathcal{M}) \to K_q(\mathcal{M}), \quad k \geq 1, \quad q \geq 0,$$

(see section 1). We set

$$\lambda^k(x, y) := (\lambda^k x, \sum_{p=1}^{k} (\lambda^{k-p} x) \cdot (\lambda^p y)) \text{ for } x \in K(\mathcal{M}), \quad y \in \bigoplus_{q\geq 1} K_q(\mathcal{M}).$$

Then $K(\mathcal{M})$ becomes a pre-λ-ring, i. e. in $K(\mathcal{M})$ the rule $\lambda^k(x+y) = \sum_{p=0}^{k}(\lambda^p x)\cdot(\lambda^{k-p}y)$ holds for all $k \geq 1$.

Theorem 4. *For any $k \geq 1$ and any $x,\ y \in K(\mathcal{M})$ we have*

$$\lambda^k(x\cdot y) = P_k(\lambda^1 x, \ldots, \lambda^k x, \lambda^1 y, \ldots, \lambda^k y).$$

Proof. We may assume that x, y are homogeneous of degree p respectively q with $p \leq q$. If $p \geq 1$ we have $\lambda^k(x\cdot y) = \lambda^k(0) = 0 = P_k(\lambda^1 x, \ldots, \lambda^k x, \lambda^1 y, \ldots, \lambda^k y)$, because P_k has no linear part. So we may assume that $p = 0$, i. e. we may assume that $x = [\mathcal{E}]$ with some locally free $\mathcal{O}_X$-module $\mathcal{E}$. Furthermore by the splitting principle (see page 4 and page 115 of [2]) we may assume that $\mathcal{E}$ is invertible. Because $P_k(X, 0, \ldots, 0, Y_1, \ldots, Y_k) = X^k \cdot Y_1$, there remains to prove

Proposition 4. *The diagram*

$$\begin{array}{ccc} G\mathcal{M} & \xrightarrow{\Lambda^k} & \mathcal{H}^k\mathcal{M} \\ \downarrow \mathcal{E}\otimes - & & \downarrow \mathcal{E}^{\otimes k}\otimes - \\ G\mathcal{M} & \xrightarrow{\Lambda^k} & \mathcal{H}^k\mathcal{M} \end{array}$$

of 1-simplicial maps commutes up to homotopy.

Proof. We define a homotopy

$$h : [1] \times G\mathcal{M} \to \mathcal{H}^k\mathcal{M}$$

between $\Lambda^k \circ (\mathcal{E}\otimes -)$ and $(\mathcal{E}^{\otimes k}\otimes -)\circ \Lambda^k$ as follows: Let $A \in \mathrm{Ord}$, $\varepsilon \in \mathrm{Mor}(A,[1])$, $M \in \mathrm{Exact}(\Gamma(A), \mathcal{M})$ and $\alpha \in \Gamma^k(A)$. We set

$$h(\varepsilon, M)(\alpha) := \begin{cases} \Lambda^k(\mathcal{E}\otimes M)(\alpha) & , \text{if } \varepsilon(i_1(\alpha)) = 0 \\ \mathcal{E}^{\otimes k}\otimes \Lambda^k M(\alpha) & , \text{if } \varepsilon(i_1(\alpha)) = 1 \end{cases}$$

To define $h(\varepsilon, M)(\alpha \to \beta)$ for an arrow $\alpha \to \beta$ in $\Gamma^k(A)$ we may assume $\varepsilon(i_1(\alpha)) = 0$ and $\varepsilon(i_1(\beta)) = 1$. Otherwise it is obvious. We recall that for each $\mathcal{F} \in \mathcal{M}$ we have a canonical isomorphism

$$\begin{array}{ccc} \mathcal{E}^{\otimes k}\otimes \Lambda^k\mathcal{F} & \tilde{\to} & \Lambda^k(\mathcal{E}\otimes\mathcal{F}) \\ (x_1\otimes\ldots\otimes x_k)\otimes(y_1\wedge\ldots\wedge y_k) & \mapsto & (x_1\otimes y_1)\wedge\ldots\wedge(x_k\otimes y_k). \end{array}$$

Using this isomorphism and the commutativity of the tensor product we obtain for any $\alpha \in \Gamma^k(A)$ a natural isomorphism

$$\mathcal{E}^{\otimes k} \otimes \Lambda^k M(\alpha) \xrightarrow{\sim} \Lambda^k(\mathcal{E} \otimes M)(\alpha).$$

Now $h(\varepsilon, M)(\alpha \to \beta)$ is given by the diagonal in the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}^{\otimes k} \otimes \Lambda^k M(\alpha) & \stackrel{\alpha\to\beta}{\longrightarrow} & \mathcal{E}^{\otimes k} \otimes \Lambda^k M(\beta) \\ \downarrow & & \downarrow \\ \Lambda^k(\mathcal{E} \otimes M)(\alpha) & \stackrel{\alpha\to\beta}{\longrightarrow} & \Lambda^k(\mathcal{E} \otimes M)(\beta) \end{array}$$

Then obviously $h(\varepsilon, M)$ is an exact functor and h is a homotopy between $(\mathcal{E}^{\otimes k} \otimes -) \circ \Lambda^k$ and $\Lambda^k \otimes (\mathcal{E} \otimes -)$.

Remark. In [1] K. Akin, D. A. Buchsbaum and J. Weyman construct a natural filtration on $\Lambda^k(E \otimes F)$ (E, F projective modules over a ring A) such that the associated graded object is given by Schur functors of E and F. From this the rule $\lambda^k(x \cdot y) = P_k(\lambda^1 x, \ldots, \lambda^k x, \lambda^1 y, \ldots, \lambda^k y)$ for x, $y \in K_0(A)$ can be deduced. I hope that this " intrinsic" proof of the above rule can be generalized to prove it already in the homotopy category in a similar fashion as in the previous sections.

To prove also the last axiom "$\lambda^k(\lambda^j(x)) = P_{k,j}(\lambda^1 x, \ldots, \lambda^{kj} x)$" of a λ-ring (see page 5 of [2]) in this way a sufficient fine and natural "decomposition" of $\Lambda^k \Lambda^l(E)$ would be necessary. This problem is essentially the same as the so-called "plethysm-problem" (see [9]) and is up to now unsolved.

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