On Adams Operations on the Higher K-Theory of Group Rings

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Abstract. Using Quillen's universal transformation we verify some (standard) properties of Adams operations on the higher K-theory of projective modules over group rings. Furthermore, we rather explicitly describe Adams operations on the Whitehead group $K_1(C\Gamma)$ associated with the group ring $C\Gamma$ of a finite group Γ over an algebraically closed field of characteristic 0.

Introduction

Let Γ be a finite group, p a prime number, and L a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_L\Gamma$ denote the group ring associated with Γ and the ring of integers $\mathcal{O}_L$ in L and let $K_0T(\mathcal{O}_L\Gamma)$ denote the Grothendieck group of finite $\mathcal{O}_L\Gamma$-modules possessing a $\mathcal{O}_L\Gamma$-free resolution of length 1. This Grothendieck group is a fundamental object in the Galois module theory à la Fröhlich. In particular, it enters the picture in studying the projective class group of the group ring of Γ over the ring of integers in a number field.

The best way to describe $K_0T(\mathcal{O}_L\Gamma)$ in a little bit more familiar terms is the exact sequence

$$K_1(\mathcal{O}_L\Gamma) \to K_1(L\Gamma) \to K_0T(\mathcal{O}_L\Gamma) \to 0$$

which comes from a localization sequence in K-theory (e. g. see sequence (1.3) of [T] on p. 2). It leads to the so-called Hom-description

$$K_0T(\mathcal{O}_L\Gamma) \cong \frac{\mathrm{Hom}_G(K_0(\bar{L}\Gamma), \bar{L}^\times)}{\mathrm{Det}((\mathcal{O}_L\Gamma)^\times)}$$

of $K_0T(\mathcal{O}_L\Gamma)$; here $G = \mathrm{Gal}(\bar{L}/L)$ denotes the absolute Galois group, $K_0(\bar{L}\Gamma)$ is the classical ring of virtual characters of Γ, and Det is the determinant map (e. g. see §2 of Chapter 1 in [T] for a precise definition of Det and Theorem 3.2 of [T] on p. 10 for a proof of the Hom-description). Using Taylor's group logarithm techniques Cassou-Noguès and Taylor have shown that the k-th Adams operation ψ^k on $K_0(\bar{L}\Gamma)$ induces an operation on $K_0T(\mathcal{O}_L\Gamma)$, if $L/\mathbb{Q}_p$ is non-ramified (e. g. see Theorem 1.2 of [T] on p. 98). The aim of this paper is to give a more or less satisfactory explanation of this operation in K-theoretical terms using power operations on modules.

For this we remark that we have constructed an Adams operation ψ^k on $K_1(L\Gamma)$ using exterior power operations (see §3 of [Ko1]) and that, if p does not divide k, we have constructed an Adams operation ψ^k on $K_1(\mathcal{O}_L\Gamma)$ using generalizations of Atiyah's cyclic power operations and shuffle products in higher K-theory (see section 3 of [Ko2]). In this paper we show that the homomorphism $K_1(\mathcal{O}_L\Gamma) \to K_1(L\Gamma)$ in the above sequence commutes with ψ^k (see Corollary c) of Proposition 1). Hence ψ^k induces an operation ψ^k on $K_0T(\mathcal{O}_L\Gamma)$, if p does not divide k. The question whether a p-th Adams operation ψ^p on $K_1(\mathcal{O}_L\Gamma)$ can be canonically defined remains open.

Furthermore we show that, via the well-known Hom-description

$$K_1(\bar{L}\Gamma) \cong \mathrm{Hom}(K_0(\bar{L}\Gamma), \bar{L}^\times)$$

of $K_1(\bar{L}\Gamma)$, the k-th exterior power operation λ^k on $K_1(\bar{L}\Gamma)$ (defined in §3 of [Ko1]) corresponds to the homomorphism on the right hand side induced by $(-1)^{k-1}\hat{\psi}^k$ where $\hat{\psi}^k$ is the adjoint of ψ^k on $K_0(\bar{L}\Gamma)$ with respect to the classical character pairing (see Theorem 1). One easily deduces from this that the k-th Adams operation ψ^k on $K_1(\bar{L}\Gamma)$ corresponds to the homomorphism on the right hand side induced by $k \cdot \hat{\psi}^k$ (see Corollary 1 of Theorem 1).

If k is coprime to the order of Γ, the adjoint homomorphism $\hat{\psi}^k$ equals $\psi^{k'}$ where k' is a natural number which is an inverse of k modulo the order of Γ (see the proof of formula (1.7) of [T] on p. 101). This suggests that, up to a sign, the operation on $K_0T(\mathcal{O}_L\Gamma)$ defined by Cassou-Noguès and Taylor should be called an exterior power operation rather than an Adams operation. Since the canonical base change homomorphism $K_1(L\Gamma) \to K_1(\bar{L}\Gamma)$ is injective (see Proposition 2.8 of [Que] on p. 247) the results explained above prove the more precise fact that, via the above Hom-description, the operation ψ^k on $K_0T(\mathcal{O}_L\Gamma)$ corresponds to the homomorphism on the right hand side induced by $k \cdot \psi^{k'}$, if k is coprime to p and to the order of Γ. Thus it differs from the operation defined by Cassou-Noguès and Taylor by passing from k to k' and by the factor k.

Finally we rather explicitly describe the $K_0(\bar{L}\Gamma)$-module structure (see Theorem 2) and the Grothendieck filtration on $K_1(\bar{L}\Gamma)$ (see Proposition 6).

As a byproduct of the explicit description (of the $K_0(\bar{L}\Gamma)$-module structure and) of the Adams operation on $K_1(\bar{L}\Gamma)$ we strengthen the induction formula (6.2) of [Ko3] in the situation considered in this paper (see Theorem 3).

In the appendix we give a new proof of Queyrut's Hom-description (see section 3 of [Que]) of the Grothendieck group $K_0(\Gamma, l)$ associated with finitely generated $l\Gamma$-modules where l is a field of positive characteristic.

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1. Quillen's Universal Transformation

The purpose of this section is to introduce some notations used throughout this paper and to recall Quillen's universal transformation introduced in Hiller's paper [Hi].

For any (abstract) group G and any (not necessarily commutative) G-ring A let $K_q(G, A)$ denote Quillen's q-th K-group associated with the exact category of finitely generated, projective A-modules with semilinear G-action. (If no action of G on A is given, the trivial action is meant and the action of G on modules is then assumed to be linear.) If G is the trivial group, we put $K_q(A) := K_q(G, A)$. Furthermore let

$$\tilde{K}_0(G, A) := \ker(K_0(G, A) \stackrel{\text{can}}{\longrightarrow} K_0(A))$$

be the reduced Grothendieck group. We obviously have $K_0(G, A) = K_0(A) \oplus \tilde{K}_0(G, A)$, if G acts trivially on A.

For any ring A let $BGL(A)^+$ be the plus construction associated with the classifying space $BGL(A)$ of the general linear group $GL(A) = \cup_{n \geq 0} GL_n(A)$ (e. g. see [L]). For any CW-complex X let $[X, BGL(A)^+]$ denote the set of free homotopy classes of free continuous maps from X to $BGL(A)^+$. This is the same as the set of pointed homotopy classes of pointed continuous maps from X to $BGL(A)^+$ since $BGL(A)^+$ is a connected H-space (see Theorem 9 of [S] on p. 384). Now Quillen's natural transformation

$$q(-) : \tilde{K}_0(\pi_1(-), A) \to [-, BGL(A)^+]$$

of functors from the category of connected finite CW-complexes to the category of groups (or more generally to the category of pointed sets) is defined as follows (see section 1 of [Hi]): Let X be a connected finite CW-complex and let $\rho : \pi_1(X) \to \mathrm{Aut}(P)$ be a representation of the fundamental group $\pi_1(X)$ of X on a finitely generated, projective A-module P. We choose a projective A-module P' such that $P \oplus P'$ is free over A, say of rank n. Then $q(X)(\rho)$ is defined to be the homotopy class of the composition of the maps

$$X \longrightarrow B\pi_1(X) \stackrel{B(\rho \oplus 1)}{\longrightarrow} BGL_n(A) \stackrel{\text{can}}{\longrightarrow} BGL(A) \stackrel{\text{can}}{\longrightarrow} BGL(A)^+.$$

Here $X \to B\pi_1(X)$ is the canonical 2-coskeleton. It is shown in Proposition 1.1 of [Hi] on p. 243 that $q(X)(\rho)$ does not depend on the chosen module P' and that the association $\rho \mapsto q(X)(\rho)$ induces a well-defined map $q(X) : \tilde{K}_0(\pi_1(X), A) \to [X, BGL(A)^+]$.

Furthermore the natural transformation $q(-)$ is universal in the following sense (see Corollary 2.3 of [Hi] on p. 246): For any H-space Z and for any natural transformation $\tilde{K}_0(\pi_1(-), A) \to [-, Z]$ there is a unique natural transformation $[-, BGL(A)^+] \to [-, Z]$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{K}_0(\pi_1(-), A) & \xrightarrow{q(-)} & [-, BGL(A)^+] \\ & \searrow \quad \swarrow & \\ & [-, Z]. & \end{array}$$

Note that the ring A need not to be commutative neither for the construction of $q(-)$ nor for the proof of the universality of $q(-)$. In particular we may take a group ring for A.

2. On Adams Operations on $K_q(R\#\Gamma)$

Let Γ be a group, R a commutative (noetherian) Γ-ring, and k a natural number which is invertible in R. Let $A := R\#\Gamma$ denote the associated twisted group ring. Using generalizations of Atiyah's cyclic power operations, shuffle products in higher K-theory, and Grayson's construction of power operations on higher K-theory we have constructed an Adams operation ψ^k on $K_q(R\#\Gamma)$, $q \geq 0$, in [Ko2]. The object of this section is to carry over some properties of ψ^k proved in section 2 of [Ko2] for K_0 to higher K-theory using Quillen's universal transformation recalled in section 1.

Let l be a prime which is invertible in R. For any $a \in \mathbb{Z}/l\mathbb{Z}$ and for any R-module V let $V[a]_l$ be the cyclic l-th power of V with eigenvalue ζ_l^a. It is defined as follows (see [Ko2]): Let $S = R[\zeta_l] = R[T]/(1 + \ldots + T^{l-1})$ be the l-th cyclotomic extension of R, $c : V^{\otimes l} \to V^{\otimes l}$, $v_0 \otimes \ldots \otimes v_{l-1} \mapsto v_{l-1} \otimes v_0 \otimes \ldots \otimes v_{l-2}$, the cyclic permutation on $V^{\otimes l}$, and $G := (\mathbb{Z}/l\mathbb{Z})^\times$ the group of invertible elements in $\mathbb{Z}/l\mathbb{Z}$ acting on S as Galois group as usual and on $V^{\otimes l}$ via

$$\sigma(v_0 \otimes \ldots \otimes v_{l-1}) := v_{\sigma^{-1}\cdot 0} \otimes \ldots \otimes v_{\sigma^{-1}\cdot(l-1)} \quad (\text{for } \sigma \in G, v_0, \ldots, v_{l-1} \in V).$$

Then $V[a]_l$ is defined to be the G-fixed module associated with the ζ_l^a-eigenspace of the endomorphism $1 \otimes c$ of $S \otimes_R V^{\otimes l}$ (see section 2 of [Ko2]). If V carries a (semilinear) Γ-action, then $V[a]_l$ obviously does as well. If V is furthermore R-projective or $R\#\Gamma$-projective, the same holds for $V[a]_l$ (see Corollary b) of Proposition 1 in [Ko2]).

The main result of section 3 in [Ko2] is that the association $V \mapsto V[a]_l$ yields an operation $[a]_l$ on $K_q(R\#\Gamma)$ and on $K_q(\Gamma, R)$ for any $q \geq 0$. The l-th Adams operation on $K_q(R\#\Gamma)$ is then defined to be $\psi^l := [0]_l - [1]_l$. Now let X be a connected finite CW-complex. Applying this construction to the group $\pi_1(X) \times \Gamma$ in place of Γ we in particular obtain an operation $[a]_l$ on $K_0(A[\pi_1(X)]) = K_0(R\#(\pi_1(X) \times \Gamma))$ and on $K_0(\pi_1(X) \times \Gamma, R)$. Similarly one can define an operation $[a]_l$ on $\tilde{K}_0(\pi_1(X), A)$. By the universality of the natural transformation $q(-)$ this operation induces a natural transformation $[a]_l$ on $[-, BGL(A)^+]$. In particular, we have once more defined an operation $[a]_l$ on $K_q(R\#\Gamma) = K_q(A) = [S^q, BGL(A)^+]$ for any $q \geq 1$.

Proposition 1. *The operation $[a]_l$ on $K_q(R\#\Gamma)$, $q \geq 1$, defined in this way agrees with the operation $[a]_l$ defined in section 3 of [Ko2].*

Proof. This can be proved similarly to section 9 of [Gr].

Corollary. *Let $q \geq 0$.*
a) For all $x \in K_0(R\#\Gamma)$ and $y \in K_q(R\#\Gamma)$ we have:

$$\psi^l(x \cdot y) = \psi^l(x) \cdot \psi^l(y) \quad \text{in} \quad K_q(R\#\Gamma).$$

b) Let l' be another prime which is invertible in R. Then we have for all $x \in K_q(R\#\Gamma)$:

$$\psi^l(\psi^{l'}(x)) = \psi^{l'}(\psi^l(x)) \quad \text{in} \quad K_q(R\#\Gamma).$$

c) If Γ is finite and the group order $\mathrm{ord}(\Gamma)$ is invertible in R, then ψ^l commutes with the Cartan homomorphism $c : K_q(R\#\Gamma) \to K_q(\Gamma, R)$.

Proof. These assertions have already been proved in section 2 of [Ko2], if $q = 0$. So let $q \geq 1$.
a) Tensoring with a projective A-module over R transforms projective A-modules into projective A-modules (see Lemma 3 of [Ko2]). Hence by the usual techniques (see [Q] or [Hi]) we obtain a $K_0(A)$-module structure on $K_q(A)$. For any $x \in K_0(A)$ the map

$$K_q(A) \to K_q(A), \quad y \mapsto \psi^l(x) \cdot \psi^l(y),$$

is the S^q-level of a natural transformation $[-, BGL(A)^+] \to [-, BGL(A)^+]$ which makes the following diagram commutative:

$$\begin{array}{ccccc} z & & \tilde{K}_0(\pi_1(-), A) & \stackrel{q(-)}{\longrightarrow} & [-, BGL(A)^+] \\ \downarrow & & \downarrow & & \downarrow \\ \psi^l(x) \cdot \psi^l(z) & & \tilde{K}_0(\pi_1(-), A) & \stackrel{q(-)}{\longrightarrow} & [-, BGL(A)^+]. \end{array}$$

Since $\psi^l(x) \cdot \psi^l(z) = \psi^l(x \cdot z)$ for all $z \in \tilde{K}_0(\pi_1(X), A)$ (cf. Proposition 5 of [Ko2]) the natural transformation $y \mapsto \psi^l(x \cdot y)$ makes the above diagram

commutative as well. Now the universality of $q(-)$ proves assertion a).
b) Similarly to a) this follows from the fact that $\psi^l \circ \psi^{l'} = \psi^{l'} \circ \psi^l$ on $\tilde{K}_0(\pi_1(X), A)$ (cf. Proposition 6 in [Ko2]).
c) If Γ is finite and $\mathrm{ord}(\Gamma)$ is invertible in R, an $R\#\Gamma$-module is finitely generated and projective over R, if and only if it is finitely generated and projective over $R\#\Gamma$. (This is an easy generalization of Maschke's theorem.) Hence the Cartan homomorphism is an isomorphism and we only have to show that the Adams operation ψ^l on $K_q(\Gamma, R)$ defined via cyclic power operations agrees with the Adams operation ψ^l defined via the l-th Newton polynomial in the exterior power operations $\lambda_1, \ldots, \lambda_l$. Similarly to a) this follows from the corresponding fact for $\tilde{K}_0(\pi_1(X), A) = \tilde{K}_0(\pi_1(X) \times \Gamma, R)$ (cf. Proposition 4 of [Ko2]). (The same proof even shows that $[a]_l = C_{l,a}(\lambda^1, \ldots, \lambda^l)$ on $K_q(R\#\Gamma) = K_q(\Gamma, R)$ where $C_{l,a}$ is the polynomial defined in Lemma 6 of [Ko2]).

Remark.
(i) Conjecturally assertion c) is true even when the group order is not invertible in R. At the end of the paper [Ko2] some speculations are presented how one should be able to prove this in the general case.
(ii) Let $k \in \mathbb{N}$ be invertible in R. Using a factorization of k into prime factors we may define an Adams operation ψ^k on $K_q(R\#\Gamma)$ for any $q \geq 1$. By assertion b) this does not depend on the ordering of the prime factors.
(iii) If $p := \mathrm{char}(R)$ is a prime, we may define an Adams operation ψ^k even for *all* $k \in \mathbb{N}$ by defining ψ^p to be the base change homomorphism associated with the Frobenius endomorphism (see last Remark in [Ko2]). If Γ is finite and $\mathrm{ord}(\Gamma)$ is invertible in R, one can deduce from Proposition 7 of [Ko2] similarly to the proof of assertion a) that ψ^p agrees with the usual Adams operation on $K_q(R\#\Gamma) = K_q(\Gamma, R)$.

3. On Adams Operations on $K_1(\mathbb{C}\Gamma)$

Let Γ be a finite group. The aim of this section is to describe the Adams operations and the $K_0(\mathbb{C}\Gamma)$-module structure on $K_1(\mathbb{C}\Gamma)$ (defined in [Ko1] or [Ko2]) in explicit terms. This will enable us to strengthen the induction formula (6.2) of [Ko3] in this situation. Computing the Grothendieck filtration on $K_1(\mathbb{C}\Gamma)$ explicitly we will furthermore show that, for any subgroup Γ' of Γ, the induction map $K_1(\mathbb{C}\Gamma') \to K_1(\mathbb{C}\Gamma)$ is continuous with respect to the Grothendieck filtrations as conjectured in (5.6) of [Ko3].

More generally, let C be an algebraically closed field such that the group order of Γ is invertible in C and let $C\Gamma$ denote the group ring associated with Γ and C. Then $K_0(C\Gamma)$ is the classical ring of virtual characters of Γ. It is a free abelian group with basis the set $\mathcal{S}$ of isomorphism classes of simple finitely generated $C\Gamma$-modules. Furthermore $K_0(\mathbb{Z}, C\Gamma)$ can be identified with the Grothendieck group associated with the category of pairs (M, α) consisting of a finitely generated $C\Gamma$-module M and a $C\Gamma$-automorphism α of M. Let $\mathbb{Z}[\mathcal{S} \times C^\times]$ denote the free abelian group with basis $\mathcal{S} \times C^\times$.

Proposition 2. *The group homomorphism*

$$\Phi : \mathbb{Z}[\mathcal{S} \times C^\times] \to K_0(\mathbb{Z}, C\Gamma) \quad \textit{given by} \quad (S, \beta) \mapsto (S, \beta)$$

is bijective.

Proof. We define an inverse map as follows: Let (M, α) be a pair as above. Then there are natural numbers n_S, $S \in \mathcal{S}$, such that $M \cong \oplus_{S \in \mathcal{S}} S^{n_S}$ and there are matrices $A_S \in GL_{n_S}(C)$, $S \in \mathcal{S}$, such that α corresponds to $\oplus_{S \in \mathcal{S}} A_S$ under $M \cong \oplus S^{n_S}$. For $A \in GL_n(C)$ and $\beta \in C$ let $m_\beta(A) := \dim_C \ker((A - \beta \cdot \mathrm{id})^\infty)$. Then the association

$$(M, \alpha) \mapsto \sum_{(S,\beta) \in \mathcal{S} \times C^\times} m_\beta(A_S) \cdot (S, \beta)$$

obviously defines a group homomorphism

$$\Psi : K_0(\mathbb{Z}, C\Gamma) \to \mathbb{Z}[\mathcal{S} \times C^\times]$$

such that $\Psi \circ \Phi = \mathrm{id}$.
Furthermore for all matrices $A \in GL_n(C)$ there is an upper triangular matrix $B \in GL_n(C)$ which is equivalent to A. In other words, the $\mathbb{Z}$-representation (S^n, A) is isomorphic to the $\mathbb{Z}$-representation (S^n, B) for any $S \in \mathcal{S}$. Now

$$0 \subset (S \times 0 \times \ldots \times 0, B) \subset (S \times S \times 0 \times \ldots \times 0, B) \subset \ldots \subset (S^n, B)$$

is a filtration of (S^n, B) by $\mathbb{Z}$-representations whose successive quotients are isomorphic to (S, β) for some $\beta \in C^\times$ and the the pair (S, β) occurs precisely $m_\beta(B) = m_\beta(A)$ times. This shows that $(S^n, A) = \sum_{\beta \in C^\times} m_\beta(A) \cdot (S, \beta)$ in $K_0(\mathbb{Z}, C\Gamma)$ and hence that $\Phi \circ \Psi = \mathrm{id}$.

According to Proposition 2 we have $K_0(\mathbb{Z}, C\Gamma) \cong K_0(C\Gamma)[C^\times]$. Henceforth we will write elements of $K_0(\mathbb{Z}, C\Gamma)$ also in the form $\sum_{\beta \in C^\times} z_\beta[\beta]$ with $z_\beta \in K_0(C\Gamma)$ for all $\beta \in C^\times$. Let

$$\begin{array}{rccl} \langle\ ,\ \rangle : & K_0(C\Gamma) \times K_0(C\Gamma) & \to & K_0(C) \;\tilde{\to}\; \mathbb{Z} \\ & (P, M) & \mapsto & \mathrm{Hom}_{C\Gamma}(P, M) \mathrel{\hat{=}} \dim_C \mathrm{Hom}_{C\Gamma}(P, M) \end{array}$$

be the classical character pairing. It is a perfect symmetric pairing. Let $\langle\ ,\ \rangle$ also denote the following pairing:

$$\begin{array}{rccl} \langle\ ,\ \rangle : & K_0(C\Gamma) \times K_0(\mathbb{Z}, C\Gamma) & \to & K_0(\mathbb{Z}, C) \\ & (P, (M, \alpha)) & \mapsto & (\mathrm{Hom}_{C\Gamma}(P, M), \mathrm{Hom}_{C\Gamma}(P, \alpha)) \end{array}.$$

We will furthermore write $\langle\ ,\ \rangle$ for the composition of $\langle\ ,\ \rangle$ with the natural map

$$\begin{array}{ccccc} K_0(\mathbb{Z}, C) & \xrightarrow{\det} & \mathrm{Pic}(\mathbb{Z}, C) & \tilde{\to} & C^\times \\ (N, \gamma) & \mapsto & (\Lambda_C^{\mathrm{top}} N, \Lambda_C^{\mathrm{top}} \gamma) & \mapsto & \det(\gamma|_N). \end{array}$$

Then for all x, $z \in K_0(C\Gamma)$ and $\beta \in C^\times$ we obviously have $\langle x, z[\beta]\rangle = \langle x, z\rangle[\beta]$ in $K_0(\mathbb{Z}, C)$ respectively $\langle x, z[\beta]\rangle = \beta^{\langle x,z\rangle}$ in $C^\times$.

For any $k \geq 1$ let λ^k and ψ^k denote the k-th exterior power operation and k-th Adams operation, respectively, on $K_0(\mathbb{Z}, C\Gamma)$ or $K_0(C\Gamma)$ or $K_0(\mathbb{Z}, C)$ or $K_1(C\Gamma)$ (see [Ko1] and section 2). Then for all $z \in K_0(C\Gamma)$ and $\beta \in C^\times$ we obviously have

$$\lambda^k(z[\beta]) = \lambda^k(z)[\beta^k] \quad \text{and} \quad \psi^k(z[\beta]) = \psi^k(z)[\beta^k] \quad \text{in} \quad K_0(\mathbb{Z}, C\Gamma).$$

Since the classical character pairing is perfect there is an adjoint homomorphism $\hat{\psi}^k$ on $K_0(C\Gamma)$ associated with ψ^k. If k is coprime to $\mathrm{ord}(\Gamma)$, we have $\hat{\psi}^k = \psi^{k'}$ where k' is a natural number such that $k \cdot k' \equiv 1 \mod \mathrm{ord}(\Gamma)$ (see the proof of formula (1.7) of [T] on p. 101). If G is abelian, one can easily show that $\hat{\psi}^k$ is induced by the association $M \mapsto C\Gamma \otimes_{C\Gamma} M$ where $C\Gamma$ is considered as an $C\Gamma$-algebra via the k-multiplication on Γ.

Proposition 3. *For all $x \in K_0(C\Gamma)$ and $y \in \tilde{K}_0(\mathbb{Z}, C\Gamma)$ we have*

$$\langle x, \lambda^k(y)\rangle = \langle (-1)^{k-1}\hat{\psi}^k(x), y\rangle \quad \textit{in} \quad C^\times.$$

Proof. Since for all z_1, $z_2 \in K_0(C\Gamma)$ and β_1, $\beta_2 \in C^\times$ we have

$$\langle x, (z_1[\beta_1] - z_1[1]) \cdot (z_2[\beta_2] - z_2[1])\rangle = (\beta_1\beta_2)^{\langle x, z_1 \cdot z_2\rangle} \cdot \beta_1^{-\langle x, z_1\cdot z_2\rangle} \cdot \beta_2^{-\langle x, z_1 \cdot z_2\rangle} = 1$$

both sides of the above formula are linear in y. Hence we may assume that $y = z[\beta] - z[1]$ with some $z \in K_0(C\Gamma)$ and $\beta \in C^\times$. Using the equation $\lambda_t'(z) \cdot \lambda_t(-z) = \frac{\mathrm{d}}{\mathrm{d}t} \log \lambda_t(z) = \sum_{j=1}^\infty (-1)^{k-1}\psi^k(z)t^{k-1}$ of power series (see p. 23 of [FL]) we then obtain that

$$\begin{aligned} \langle x, \lambda^k(y)\rangle &= \langle x, \lambda^k(z)[\beta^k] + \lambda^{k-1}(z)\lambda^1(-z)[\beta^{k-1}] + \ldots + \lambda^k(-z)[1]\rangle \\ &= \beta^{k\langle x, \lambda^k(z)\rangle} \cdot \beta^{(k-1)\langle x, \lambda^{k-1}(z)\lambda^1(-z)\rangle} \cdot \ldots \cdot \beta^{\langle x, \lambda^1(z)\lambda^{k-1}(-z)\rangle} \\ &= \beta^{\langle x, k\lambda^k(z) + (k-1)\lambda^{k-1}(z)\lambda^1(-z) + \ldots + \lambda^1(z)\lambda^{k-1}(-z)\rangle} \\ &= \beta^{\langle x, (-1)^{k-1}\psi^k(z)\rangle} \\ &= \beta^{\langle (-1)^{k-1}\hat{\psi}^k(x), z\rangle} \\ &= \langle (-1)^{k-1}\hat{\psi}^k(x), y\rangle \end{aligned}$$

as was to be shown.

Now we will use the following description of the Whitehead group $K_1(C\Gamma) = [S^1, BGL(C\Gamma)^+]$: It is the factor group of the free abelian group with basis the isomorphism classes of pairs (M,α) as above modulo the relations defined e. g. on p. 348 of [Ba]. Furthermore, by Proposition 2.2 of [Que] on p. 244, the map

$$\begin{array}{rcl} K_1(C\Gamma) & \to & \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma), C^\times) \\ (M,\alpha) & \mapsto & (P \mapsto \det(\mathrm{Hom}_{C\Gamma}(P,\alpha)|_{\mathrm{Hom}_{C\Gamma}(P,M)})) \end{array}$$

is a well-defined group isomorphism.

Theorem 1. *The following diagram commutes:*

$$\begin{array}{ccc} K_1(C\Gamma) & \tilde{\to} & \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma), C^\times) \\ \lambda^k \downarrow & & \downarrow \mathrm{Hom}((-1)^{k-1}\hat{\psi}^k, C^\times) \\ K_1(C\Gamma) & \tilde{\to} & \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma), C^\times). \end{array}$$

Proof. We consider the following diagram:

$$\begin{array}{cccc} y & \mapsto & (x \mapsto \langle x, y\rangle) & \\ \tilde{K}_0(\mathbb{Z}, C\Gamma) & \to & \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma), C^\times) & \\ \| & & \| & \\ \tilde{K}_0(\mathbb{Z}, C\Gamma) & \overset{D}{\to} & K_1(C\Gamma) & \\ & & & \searrow \\ \| & & \| & \quad GL(C\Gamma)^{\mathrm{ab}} \\ & & & \swarrow \\ \tilde{K}_0(\pi_1(S^1), C\Gamma) & \overset{q(S^1)}{\to} & [S^1, BGL(C\Gamma)^+] & \end{array} .$$

Here the map D is defined in such a way that the upper square commutes. Then we obviously have $D(S[\beta] - S[1]) = (S,\beta)$ in $K_1(C\Gamma)$ for all $S \in \mathcal{S}$ and $\beta \in C^\times$. Let S' be a $C\Gamma$-module such that $S \oplus S'$ is free over $C\Gamma$, say of rank n, and let $\alpha \in GL_n(C\Gamma)$ be the automorphism of $(C\Gamma)^n$ corresponding to $\beta \oplus \mathrm{id}_{S'}$. Then by Theorem (1.2)(1) of [Ba] on p. 448 the isomorphism $K_1(C\Gamma) \tilde{\to} [S^1, BGL(C\Gamma)^+]$ maps the class of (S,β) to the homotopy class of the continuous map

$$S^1 = B\mathbb{Z} \overset{B\alpha}{\to} BGL_n(C\Gamma) \overset{\mathrm{can}}{\to} BGL(C\Gamma) \overset{\mathrm{can}}{\to} BGL(C\Gamma)^+.$$

Since $q(S^1)$ maps the element $S[\beta] - S[1]$ of $\tilde{K}_0(\pi_1(S^1), C\Gamma)$ to the same homotopy class (see section 1) the lower square commutes. Hence the map D commutes with λ^k by definition of λ^k on $K_1(C\Gamma)$. Now Proposition 3 proves Theorem 1 since D is surjective.

Remark. If more generally C is a commutative ring such that $\mathrm{ord}(\Gamma)$ is invertible in C, the above arguments essentially show that we have $\lambda^k(S,\beta) = ((-1)^{k-1}\psi^k(S), \beta^k)$ in $K_1(C\Gamma)$ for any $C\Gamma$-module S and $\beta \in C^\times$.

Corollary 1. *The following diagram commutes:*

$$\begin{array}{ccccc} K_1(C\Gamma) & \tilde{\to} & \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma), C^\times) & & f \\ \psi^k \downarrow & & \downarrow & & \downarrow \\ K_1(C\Gamma) & \tilde{\to} & \mathrm{Hom}_{\mathbb{Z}}(K_0 C\Gamma), C^\times) & & \mu^k \circ f \circ \hat{\psi}^k, \end{array}$$

where $\mu^k(\beta) = \beta^k$ *for* $\beta \in C^\times$.

Proof. Since the multiplication on $K_1(C\Gamma)$ is defined to be trivial we have $\psi^k = (-1)^{k-1} k \lambda^k$. Thus Theorem 1 proves Corollary 1. Alternatively Corollary 1 follows from the following proposition similarly to the proof of Theorem 1.

Proposition 4. *For all* $x \in K_0(C\Gamma)$ *and* $y \in K_0(\mathbb{Z}, C\Gamma)$ *we have*

$$\langle x, \psi^k(y)\rangle = \psi^k(\langle \hat{\psi}^k(x), y\rangle) \quad in \quad K_0(\mathbb{Z}, C).$$

Proof. We may assume that $y = z[\beta]$ with some $z \in K_0(C\Gamma)$ and $\beta \in C^\times$. Then we have

$$\begin{aligned} &\langle x, \psi^k(y)\rangle = \langle x, \psi^k(z)[\beta^k]\rangle \\ &\quad = \langle \hat{\psi}^k(x), z\rangle[\beta^k] = \psi^k(\langle \hat{\psi}^k(x), z\rangle[\beta]) = \psi^k(\langle \hat{\psi}^k(x), y\rangle). \end{aligned}$$

Corollary 2. *Via the isomorphism* $K_1(C\Gamma) \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma), C^\times)$ *the Grothendieck operation* γ^k *on* $K_1(C\Gamma)$ *corresponds to the (additively written) homomorphism*

$$\begin{aligned} &\sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k-1}{i} \mathrm{Hom}(\hat{\psi}^{k-i}, C^\times) \\ &\quad = (-1)^{k-1}\mathrm{Hom}(\hat{\psi}^k, C^\times) + (-1)^{k-2}(k-1)\mathrm{Hom}(\hat{\psi}^{k-1}, C^\times) + \ldots + \mathrm{id} \end{aligned}$$

on $\mathrm{Hom}(K_0(C\Gamma), C^\times)$.

Proof. By definition (see p. 47 of [FL]) we have $\gamma^k = \sum_{i=0}^{k-1} \binom{k-1}{i} \lambda^{k-i}$. Thus Theorem 1 proves Corollary 2.

Let

$$^* : K_0(C\Gamma) \to K_0(C\Gamma), \quad P \mapsto P^* := \mathrm{Hom}_C(P, C),$$

denote the dualizing map.

Proposition 5. *For all* x, $z \in K_0(C\Gamma)$ *and* $y \in K_0(\mathbb{Z}, C\Gamma)$ *we have*

$$\langle x, z \cdot y\rangle = \langle z^* \cdot x, y\rangle \quad in \quad K_0(\mathbb{Z}, C).$$

Proof. This follows from the canonical isomorphisms

$$\mathrm{Hom}_{C\Gamma}(P\otimes_C Q,R)\cong \mathrm{Hom}_{C\Gamma}(Q,\mathrm{Hom}_C(P,R))\cong \mathrm{Hom}_{C\Gamma}(Q,P^*\otimes_C R)$$

(for all finitely generated $C\Gamma$-modules P,Q,R).

Theorem 2. *For all $z\in K_0(C\Gamma)$ the following diagram commutes:*

$$\begin{array}{cccccc} y & K_1(C\Gamma) & \tilde{\rightarrow} & \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma),C^\times) & & f \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ z\cdot y & K_1(C\Gamma) & \tilde{\rightarrow} & \mathrm{Hom}_{\mathbb{Z}}(K_0(C\Gamma),C^\times) & & (x\mapsto f(z^*\cdot x)). \end{array}$$

Proof. This can be deduced from Proposition 5 similarly to the proof of Theorem 1.

The next theorem strengthens the induction formula (6.2) of [Ko3] in the situation considered in this section. For this let Γ' be a subgroup of Γ and let

$$i_*: K_1(C\Gamma')\to K_1(C\Gamma),\quad (M,\alpha)\mapsto (C\Gamma\otimes_{C\Gamma'}M, 1\otimes\alpha),$$

be the induction map (see also section 6 in [Ko3]).

Theorem 3. *For all $y\in K_1(C\Gamma')$ we have*

$$\psi^k i_*(y)=i_*\psi^k(y)\quad in\quad \begin{cases} K_1(C\Gamma), & if\ (k,\mathrm{ord}(\Gamma))=1\\ \hat{K}_1(C\Gamma)[k^{-1}], & else.\end{cases}$$

Here $\hat{K}_1(C\Gamma)[k^{-1}]$ denotes the $F^1K_0(C\Gamma)[k^{-1}]$-adic completion of $K_1(C\Gamma)[k^{-1}]$.

Proof. Let $i^*: K_0(C\Gamma)\to K_0(C\Gamma')$ be the restriction map. Then by Frobenius reciprocity the diagram

$$\begin{array}{ccc} K_1(C\Gamma') & \xrightarrow{\sim} & \mathrm{Hom}(K_0(C\Gamma'),C^\times) \\ i_*\downarrow & & \downarrow \mathrm{Hom}(i^*,C^\times) \\ K_1(C\Gamma) & \xrightarrow{\sim} & \mathrm{Hom}(K_0(C\Gamma),C^\times) \end{array}$$

commutes.

If $(k,\mathrm{ord}(\Gamma))=1$ we have $\hat{\psi}^k=\psi^{k'}$ where k' is a natural number such that $k\cdot k'\equiv 1 \mod \mathrm{ord}(\Gamma)$. Hence i^* commutes with $\hat{\psi}^k$ and thus $i_*\mathrel{\hat{=}}\mathrm{Hom}(i^*,C^\times)$ commutes with $\psi^k\mathrel{\hat{=}}\mathrm{Hom}(\hat{\psi}^k,C^\times)$ as was to be shown. If $(k,\mathrm{ord}(\Gamma))\neq 1$, this is proved in Theorem (6.2) of [Ko3] using the equivariant Adams-Riemann-Roch theorem. Alternatively this can be proved as follows without using geometric arguments: Let $I:=F^1K_0(C\Gamma)$. By Example (6.8) of [Ko3] we have $\psi^k i_*(x)-i_*\psi^k(x)\in\cap_{n\geq 0}I[k^{-1}]^n$ for all

$x \in K_0(C\Gamma')$. Hence by Frobenius reciprocity we have $f \circ \hat{\psi}^k \circ i^* - f \circ i^* \circ \hat{\psi}^k \in \cap_{n\geq 0} I^n \mathrm{Hom}(K_0(C\Gamma), \mathbb{Z})[k^{-1}]$ for all $f \in \mathrm{Hom}(K_0(C\Gamma'), \mathbb{Z})$. Here the multiplication of $I^n \subseteq K_0(C\Gamma)$ on $\mathrm{Hom}(K_0(C\Gamma), \mathbb{Z})$ is defined as in Theorem 2. Writing a homomorphism $f \in \mathrm{Hom}(K_0(C\Gamma'), C^\times)$ as the image of a map $K_0(C\Gamma') \to \mathbb{Z}[\beta_1] \oplus \ldots \oplus \mathbb{Z}[\beta_r]$ we deduce from this that $f \circ \hat{\psi}^k \circ i^* - f \circ i^* \circ \hat{\psi}^k \in \cap_{n\geq 0} I^n \mathrm{Hom}(K_0(C\Gamma), C^\times)[k^{-1}]$. Now Corollary 1 of Theorem 1 and Theorem 2 prove Theorem 3.

Now we are going to describe the Grothendieck filtration on $K_1(C\Gamma)$. For this let $K(C\Gamma) := K_0(C\Gamma) \oplus K_1(C\Gamma)$ be equipped with the ring structure induced by the ring structure on $K_0(C\Gamma)$, by the $K_0(C\Gamma)$-module structure on $K_1(C\Gamma)$, and by the trivial multiplication on $K_1(C\Gamma)$. Then the exterior power operations on $K_0(C\Gamma)$ and $K_1(C\Gamma)$ make $K(C\Gamma)$ a λ-ring (see §2 of [Ko1]). Let

$$F^1K(C\Gamma) := \ker(K(C\Gamma) \overset{\mathrm{can}}{\longrightarrow} K_0(C\Gamma) \overset{\mathrm{dim}}{\longrightarrow} \mathbb{Z})$$

be the augmentation ideal and let $(F^nK(C\Gamma))_{n\geq 0}$ be the associated Grothendieck filtration. Recall that $F^nK(C\Gamma)$ is generated as abelian group by the elements

$$\gamma^{n_1}(z_1) \cdot \ldots \cdot \gamma^{n_r}(z_r), \quad z_1, \ldots, z_r \in F^1K(C\Gamma), \quad n_1 + \ldots + n_r \geq n.$$

The ideal $F^nK(C\Gamma)$ is obviously a homogeneous ideal, i. e. $F^nK(C\Gamma) = F^nK_0(C\Gamma) \oplus F^nK_1(C\Gamma)$ with a certain subgroup $F^nK_1(C\Gamma)$ of $K_1(C\Gamma)$.

Proposition 6. *We have*

$$F^nK_1(C\Gamma) = \begin{cases} K_1(C\Gamma) & \text{for } n = 0, 1 \\ \ker(K_1(C\Gamma) \overset{\mathrm{can}}{\longrightarrow} K_1(C)) & \text{for } n \geq 2. \end{cases}$$

In particular, the induction map $i_* : K_1(C\Gamma') \to K_1(C\Gamma)$ is continuous with respect to the Grothendieck filtrations as conjectured in (5.6) of [Ko3].

Proof. This is clear for $n = 0, 1$. So let $n \geq 2$. Since $F^nK_1(C)$ obviously vanishes for $n \geq 2$ we have $F^nK_1(C\Gamma) \subseteq \ker(K_1(C\Gamma) \to K_1(C)) =: K$. Using the isomorphisms $K \oplus K_1(C) \cong K_1(C\Gamma) \cong \mathrm{Hom}(K_0(C\Gamma), C^\times)$ we see that the multiplication with $\mathrm{ord}(\Gamma)$ on K is surjective. Hence we have $K = \mathrm{ord}(\Gamma) \cdot K \subseteq F^2K_1(C\Gamma)$ by Proposition (6.1) of [Ko3]. By induction on n we thus obtain the reverse inclusion

$$K = \mathrm{ord}(\Gamma) \cdot K = \mathrm{ord}(\Gamma) \cdot F^{n-1}K_1(C\Gamma) \subseteq F^nK_1(C\Gamma)$$

for all $n \geq 2$ again by Proposition (6.1) of [Ko3].
Since the diagram

$$\begin{array}{ccc} K_1(C\Gamma') & \to & K_1(C) \\ i_* \downarrow & & \downarrow [\Gamma : \Gamma'] \\ K_1(C\Gamma) & \to & K_1(C) \end{array}$$

obviously commutes we finally obtain that $i_*(F^nK_1(C\Gamma')) \subseteq F^nK_1(C\Gamma)$ for all $n \geq 0$. In particular, i_* is continuous with respect to the Grothendieck filtrations.

Appendix

Let Γ be a finite group and let l be a finite field. In this appendix we will present a new proof for the Hom-description of the Grothendieck group $K_0(\Gamma, l)$ given by Queyrut in section 3 of [Que].

To recall this Hom-description, let L be a local field of characteristic 0 with residue classfield l. Let L_{nr} be the maximal non-ramified extension in the algebraic closure $\bar{L}$ of L. Then the residue classfield of L_{nr} (and $\bar{L}$) is an algebraic closure $\bar{l}$ of l. Let $\mathcal{O}_{L_{\mathrm{nr}}}$ and $\mathcal{O}_{\bar{L}}$ denote the ring of integers in L_{nr} and $\bar{L}$, respectively. Let $G_L := \mathrm{Gal}(\bar{L}/L)$ and $G_l := \mathrm{Gal}(\bar{l}/l)$ denote the corresponding absolute Galois groups. We will identify G_l with $\mathrm{Gal}(L_{\mathrm{nr}}/L)$. Our proof of the Hom-description will be based on the following two facts:

(i) In the Swan triangle

$$\begin{array}{ccccc} K_0(\bar{l}\Gamma) & & \xrightarrow{e} & & K_0(\bar{L}\Gamma) \\ & c \searrow & & \swarrow d & \\ & & K_0(\Gamma, \bar{l}) & & \end{array}$$

all homomorphisms are compatible with the obvious G_L-action (see [Se]).

(ii) The pairing

$$\begin{array}{ccc} K_0(\bar{l}\Gamma) \times K_0(\Gamma, \bar{l}) & \to & \mathbb{Z} \\ (P, M) & \mapsto & \dim_{\bar{l}} \mathrm{Hom}_{\bar{l}\Gamma}(P, M) \end{array}$$

induces an isomorphism of groups

$$K_0(\Gamma, l) \xrightarrow{\sim} \mathrm{Hom}_{G_l}(K_0(\bar{l}\Gamma), \mathbb{Z})$$

via base extension (see Théorème 2.7 of [Que] on p. 247).

Let $v : \bar{L}^\times \to \mathbb{Q}$ be the valuation normalized by $v(\pi_L) = 1$ where π_L is a prime element of L (and then of L_{nr} as well). We put

$$H(l, \Gamma) := \{f \in \mathrm{Hom}_{G_L}(K_0(\bar{L}\Gamma), \bar{L}^\times) : f(\mathrm{Image}(e)) \subseteq \mathcal{O}_{\bar{L}}^\times\}.$$

Theorem 4 (Hom-description of $K_0(\Gamma, l)$). *The homomorphism*

$$\begin{array}{ccc} \mathrm{Hom}_{G_L}(K_0(\bar{L}\Gamma), \bar{L}^\times) & \to & \mathrm{Hom}_{G_l}(K_0(\bar{l}\Gamma), \mathbb{Z}) \\ f & \mapsto & (y \mapsto v \circ f \circ e(y)) \end{array}$$

induces an isomorphism

$$\frac{\mathrm{Hom}_{G_L}(K_0(\bar{L}\Gamma), \bar{L}^\times)}{H(l, \Gamma)} \xrightarrow{\sim} K_0(\Gamma, l).$$

Proof. Note that $v \circ f \circ e(y)$ lies in $\mathbb{Z}$ since $e(y)$ and hence $f \circ e(y)$ is fixed by the inertia group.
The split exact sequence

$$0 \to \mathcal{O}_{L_{\rm nr}}^{\times} \to L_{\rm nr}^{\times} \stackrel{v}{\to} \mathbb{Z} \to 0$$

of G_l-modules induces the exact sequence

$$0 \to \mathrm{Hom}_{G_l}(K, \mathcal{O}_{L_{\rm nr}}^{\times}) \to \mathrm{Hom}_{G_l}(K, L_{\rm nr}^{\times}) \to \mathrm{Hom}_{G_l}(K, \mathbb{Z}) \to 0$$

where $K := K_0(\bar{l}\Gamma)$. Since K is fixed by the inertia group we have $\mathrm{Hom}_{G_l}(K, L_{\rm nr}^{\times}) = \mathrm{Hom}_{G_L}(K, \bar{L}^{\times})$ and $\mathrm{Hom}_{G_l}(K, \mathcal{O}_{L_{\rm nr}}^{\times}) = \mathrm{Hom}_{G_L}(K, \mathcal{O}_{\bar{L}}^{\times})$. Furthermore the map $e : K \to K_0(\bar{L}\Gamma)$ of the Swan triangle is a direct injection of G_L-modules (see [Se]). Hence the pull-back diagram

$$\begin{array}{ccccccc} 0 \to & H(l, \Gamma) & \to & \mathrm{Hom}_{G_L}(K_0(\bar{L}\Gamma), \bar{L}^{\times}) & \to & \mathrm{Hom}_{G_l}(K, \mathbb{Z}) & \to 0 \\ & \downarrow & & \downarrow e & & \| & \\ 0 \to & \mathrm{Hom}_{G_L}(K, \mathcal{O}_{\bar{L}}^{\times}) & \to & \mathrm{Hom}_{G_L}(K, \bar{L}^{\times}) & \to & \mathrm{Hom}_{G_l}(K, \mathbb{Z}) & \to 0 \end{array}$$

proves Theorem 4.

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