Adams Operations
for Projective Modules over Group Rings

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Abstract

Let R be a commutative ring, Γ a finite group acting on R, and let $k \in \mathbb{N}$ be invertible in R. Generalizing a definition of Kervaire, we construct an Adams operation ψ^k on the Grothendieck group and on the higher K-theory of projective modules over the twisted group ring $R\#\Gamma$. For this, we generalize Atiyah's cyclic power operations and use shuffle products in higher K-theory. For the Grothendieck group, we show that ψ^k is multiplicative and that it commutes with base change, with the Cartan homomorphism, and with ψ^l for any other l which is invertible in R.

Introduction

On the classical ring of characters $K_0(\bar{\mathbb{Q}}\Gamma)$ of a finite group Γ, the k-th Adams operation ψ^k is defined by

$$\psi^k(\chi)(\gamma) := \chi(\gamma^k)$$

(for any character $\chi \in K_0(\bar{\mathbb{Q}}\Gamma)$ and any $\gamma \in \Gamma$) or, equivalently, as the k-th Newton polynomial in the exterior power operations $\lambda_1, \ldots, \lambda_k$.

For the Grothendieck group $K_0(\mathbb{Z}\Gamma)$ of projective modules over the integral group ring $\mathbb{Z}\Gamma$, Fröhlich has given the following description, the so-called Hom-description:

$$\frac{K_0(\mathbb{Z}\Gamma)}{\langle \text{free ones} \rangle} \cong \frac{\operatorname{Hom}_G(K_0(\bar{\mathbb{Q}}\Gamma), \mathcal{I}(\bar{\mathbb{Q}}))}{\operatorname{Hom}_G(K_0(\bar{\mathbb{Q}}\Gamma), \bar{\mathbb{Q}}^{\times}) \cdot \operatorname{Det}(\mathcal{U}(\mathbb{Z}\Gamma))}$$

Here G denotes the absolute Galois group $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $\mathcal{I}(\bar{\mathbb{Q}})$ the idele group of $\bar{\mathbb{Q}}$, Det the generalized determinant map, and $\mathcal{U}(\mathbb{Z}\Gamma)$ the group of unit ideles of $\mathbb{Z}\Gamma$ (see [F] for precise definitions and proofs). In [CNT], Cassou-Noguès and

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Taylor have shown that (for odd k) the homomorphism induced by ψ^k on the numerator leaves the denominator of this formula invariant, i. e. ψ^k induces a homomorphism ψ_k on $K_0(\mathbb{Z}\Gamma)/\langle\text{free ones}\rangle$. This paper grew out of the desire to describe this homomorphism ψ_k in algebraic terms, i. e. to express ψ_k in terms of power operations. Investigating the proofs of Fröhlich and Cassou-Noguès/Taylor, one is led to the conjecture that there exist Adams operations on the Whitehead group $K_1(\mathbb{Z}_p\Gamma)$ of the group ring of Γ over the p-adic integers $\mathbb{Z}_p$. The aim of this paper is to prove this conjecture in the following much more general setting.

Let R be a commutative Γ-ring and let $k \in \mathbb{N}$ be invertible in R. Then, for each $q \geq 0$, there exists an algebraically defined, additive Adams operation ψ^k on Quillen's q-th K-group $K_q(R\#\Gamma)$ associated with the exact category of finitely generated, projective modules over the twisted group ring $R\#\Gamma$.

To construct ψ^k, we generalize Kervaire's construction of ψ^k on $K_0(K\Gamma)$ for a (finite) field K (cf. [Ke]). In particular, our definition is based on generalizations of Atiyah's cyclic power operations (cf. formula (2.7) of [At]). It uses shuffle products in higher K-theory (cf. [Ko2]) and Grayson's construction of power operations on higher K-theory (cf. [Gr1]).

We show that the k-th Adams operation ψ^k on the Grothendieck group $K_0(R\#\Gamma)$ is multiplicative and that it commutes with the Cartan homomorphism

$$c : K_0(R\#\Gamma) \rightarrow K_0(\Gamma, R),$$

with base change, and with the Adams operation ψ^l for any other l which is invertible in R. Furthermore, we formulate a conjectural relationship between cyclic powers and exterior powers which one should be able to prove in representation theory and which would suffice to prove that ψ^k commutes with the Cartan homomorphism $c : K_q(R\#\Gamma) \rightarrow K_q(\Gamma, R)$ in higher K-theory. Here $K_q(\Gamma, R)$ denotes Quillen's q-th K-group associated with finitely generated, R-projective $R\#\Gamma$-modules. Finally, if the characteristic of R is a prime number, we define an Adams operation ψ^k on $K_q(R\#\Gamma)$ for an arbitrary $k \in \mathbb{N}$ by combining the above construction with the base change homomorphism induced by the Frobenius endomorphism of R.

For k coprime to the order $|\Gamma|$ of Γ, let k' be the inverse of k modulo $|\Gamma|$. Then $\psi^{k'}$ on $K_0(\bar{\mathbb{Q}}\Gamma)$ is the adjoint of ψ^k with respect to the classical character pairing

$$K_0(\bar{\mathbb{Q}}\Gamma) \times K_0(\bar{\mathbb{Q}}\Gamma) \rightarrow \mathbb{Z}$$

(cf. the proof of Théorème 7-5 of [CNT]). I believe that the Adams operations defined here are adjoints (in a sense which has still to be made precise) of those defined by Cassou-Noguès and Taylor. Having established such an interpretation, the Adams-Riemann-Roch theorem of [Ko1] and [Ko3] (for K_1) will presumably lead to an algebro-geometric explanation of the formula of Burns and Chinburg in [BC] which computes the Adams operations of Cassou-Noguès and Taylor for

certain ambiguous ideals in a tame Galois extension of a number field. These topics are the objects of further investigations (see [Ko4] and [Ko5]).

1. The Descent Construction

Let l be a prime number, C a (multiplicative) cyclic group of order l, and $c \in C$ a fixed generator. The group $G := (\mathbb{Z}/l\mathbb{Z})^{\times}$ acts on C by group automorphisms in the usual way, and we denote the corresponding semidirect product by $C \rtimes G$ (defined by $(c^a, \sigma) \cdot (c^b, \tau) := (c^{a+\sigma b}, \sigma\tau)$ for all a, $b \in \mathbb{Z}/l\mathbb{Z}$, σ, $\tau \in G$).

If in a commutative ring R the prime number l is invertible and if R contains a primitive l-th root of unity ζ, then, as is well-known, any module V over the group ring RC decomposes into the direct sum $V = \oplus_{a \in \mathbb{Z}/l\mathbb{Z}} V_a$ of "eigenspaces" $V_a := \{v \in V : c(v) = \zeta^a v\}$. The object of this section is to construct an analogous decomposition for V in the case when the assumption "R contains a primitive l-th root of unity" is replaced by the following assumption: There is an action of G on V which together with the given action of C on V induces an action of $C \rtimes G$ on V. This construction generalizes the descent argument of Kervaire given in §5 of [Ke] and will be needed in the following sections in the even more general situation when V in addition carries a (semilinear) action of a further group Γ.

So let Γ be an (arbitrary, abstract) group, R a commutative ring such that l is invertible in R, $\Gamma \times R \to R$ an action of Γ on R by ring automorphisms, and let $R\#\Gamma$ be the corresponding twisted group ring. Let $\Phi_l := 1 + \ldots + X^{l-1} \in R[X]$ be the l-th cyclotomic polynomial and $S := R[X]/(\Phi_l)$. Then the residue class ζ of X is a primitive l-th root of unity in S and the group G acts on S by R-algebra automorphisms given by $\sigma(\zeta) := \zeta^{\sigma}$ for $\sigma \in G$.

Now let V be a $(C \rtimes G)$-module over $R\#\Gamma$. The generator c of C induces an S-linear and $R\#\Gamma$-linear automorphism of $S \otimes_R V$ which will be denoted by c again. For any $a \in \mathbb{Z}/l\mathbb{Z}$, the eigenspace

$$(S \otimes_R V)_a := \{x \in S \otimes_R V : c(x) = \zeta^a x\}$$

is an S- and $R\#\Gamma$-submodule of $S \otimes_R V$. Now we define an S-*semi*linear and $R\#\Gamma$-linear action of G on $S \otimes_R V$ by $\sigma(\alpha \otimes v) := \sigma(\alpha) \otimes \sigma(v)$ for $\sigma \in G$, $\alpha \in S$ and $v \in V$.

Lemma 1.1. *For each $a \in \mathbb{Z}/l\mathbb{Z}$, the eigenspace $(S \otimes_R V)_a$ is stable under this G-action.*

Proof. For each $\sigma \in G$, we have $c \circ \sigma = \sigma \circ c^{(\sigma^{-1})}$ in the semidirect product $C \times_{\rfloor} G$. Hence, for any $x \in (S \otimes_R V)_a$, we have $c(\sigma(x)) = \sigma(c^{(\sigma^{-1})}(x)) = \sigma(\zeta^{a\sigma^{-1}} x) = \zeta^a \sigma(x)$ in $S \otimes_R V$, and $\sigma(x)$ is contained in $(S \otimes_R V)_a$ again.

Definition 1.2. For any $a \in \mathbb{Z}/l\mathbb{Z}$, let

$$F_a(V) := (S \otimes_R V)_a^G$$

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be the $R\#\Gamma$-submodule of $(S \otimes_R V)_a$ consisting of the G-invariant elements in $(S \otimes_R V)_a$.

Obviously, the association $V \mapsto F_a(V)$ defines a functor

$$F_a : ((C \rtimes G)\text{-modules over } R\#\Gamma) \to (R\#\Gamma\text{-modules}).$$

Proposition 1.3. *The direct sum $\oplus_a F_a$ of the functors F_a, $a \in \mathbb{Z}/l\mathbb{Z}$, is isomorphic to the forgetful functor.*

Proof. Let V be a $(C \rtimes G)$-module over $R\#\Gamma$. On the $R\#\Gamma$-module $\mathrm{Maps}(G, V)$, the group G acts via left translation on G and via the given action on V. We will construct natural $R\#\Gamma$-isomorphisms

$$\oplus_a F_a(V) \cong (S \otimes_R V)^G \cong \mathrm{Maps}_G(G, V) \cong V,$$

and these prove Proposition 1.3. The first isomorphism comes from Lemma 1.1 and the fact that the elements $e_a := \frac{1}{l} \sum_{i=0}^{l-1} \zeta^{-ai} c^i$, $a \in \mathbb{Z}/l\mathbb{Z}$, form a complete system of orthogonal idempotents in the group ring SC with $e_a(S \otimes_R V) = (S \otimes_R V)_a$ for all $a \in \mathbb{Z}/l\mathbb{Z}$. The second isomorphism is the restriction of the G-equivariant map $\mathrm{Maps}(G, V) \to S \otimes_R V$, $f \mapsto \sum_{\sigma \in G} \sigma(\zeta) \otimes f(\sigma)$ which is bijective because ζ is a normal basis element for the Galois extension S/R. The third isomorphism is the map $V \to \mathrm{Maps}_G(G, V)$, $v \mapsto (\sigma \mapsto \sigma(v))$, which obviously is bijective.

Corollary 1.4.
a) The functors F_a, $a \in \mathbb{Z}/l\mathbb{Z}$, are exact.
b) For any $(C \rtimes G)$-module V over $R\#\Gamma$ which is projective over $R\#\Gamma$ (respectively over R), the modules $F_a(V)$, $a \in \mathbb{Z}/l\mathbb{Z}$, are projective over $R\#\Gamma$ (respectively over R) again.
c) (Base change) The functors F_a, $a \in \mathbb{Z}/l\mathbb{Z}$, commute with base change with respect to any homomorphism $R \to R'$ of Γ-rings.

Proof. This is clear.

Examples 1.5.
a) For any $(C \rtimes G)$-module V over $R\#\Gamma$, the module $F_0(V)$ is isomorphic to the fixed module $V^c := \{v \in V : c(v) = v\}$.
b) For any a, $b \in \mathbb{Z}/l\mathbb{Z}\backslash\{0\}$, the functors F_a and F_b are isomorphic.
c) Let W be a G-module over $R\#\Gamma$. The canonical maps

$$C \rtimes G \to \mathrm{Permutations}(C) \quad \text{and} \quad C \rtimes G \to G$$

define an action of $C \rtimes G$ on the $R\#\Gamma$-module $\mathrm{Maps}(C, W)$. Then, for each $a \in \mathbb{Z}/l\mathbb{Z}$, we have $F_a(\mathrm{Maps}(C, W)) \cong W$.

Proof. It easy to see that the isomorphism $V \to (S \otimes_R V)^G$, $v \mapsto \sum_{\sigma \in G} \zeta^\sigma \otimes \sigma(v)$, in the proof of Proposition 1.3 maps V^c onto $F_0(V)$. This shows claim a). For

claim b), we choose $\sigma \in G$ with $a = \sigma b$. Then $(S \otimes_R V)_a \to (S \otimes_R V)_b$, $\alpha \otimes v \mapsto \alpha \otimes \sigma(v)$, yields the desired isomorphism. For each $a \in \mathbb{Z}/l\mathbb{Z}$, the map

$$S \otimes_R W \to (\mathrm{Maps}(C, S \otimes_R W))_a, \quad x \mapsto (c^i \mapsto \zeta^{-ia}x),$$

is a well-defined isomorphism of G-modules. Restricting to G-invariants yields the desired isomorphism in c).

The final assertions of this section will be based on the following Gauss sum argument.

Lemma 1.6. *Let S be a commutative ring such that l is invertible in S and such that S contains a primitive l-th root of unity ζ. Then, the element $\sum_{\sigma \in G} \zeta^\sigma[\sigma]$ of the group ring SG is invertible.*

Proof. The following calculation shows that the element $\frac{1}{l} \sum_{\tau \in G}(\zeta^{-\tau} - 1)[\tau^{-1}]$ is the inverse of $\sum_{\sigma \in G} \zeta^\sigma[\sigma]$:

$$\left(\sum_{\sigma \in G} \zeta^\sigma[\sigma]\right) \cdot \left(\sum_{\tau \in G}(\zeta^{-\tau} - 1)[\tau^{-1}]\right)$$
$$= \sum_{\sigma,\tau \in G} \zeta^\sigma(\zeta^{-\tau} - 1)[\sigma\tau^{-1}] = \sum_{\eta \in G}\left(\sum_{\tau \in G} \zeta^{\eta\tau}(\zeta^{-\tau} - 1)\right)[\eta] = l[1].$$

Here, the last equality follows from

$$\sum_{\tau \in G} \zeta^{(\eta-1)\tau} - \sum_{\tau \in G} \zeta^{\eta\tau} = \begin{cases} (l-1) + 1 = l & \text{for } \eta = 1 \\ 0 & \text{for } \eta \in G\backslash\{1\} \end{cases}.$$

Proposition 1.7. *If already R contains a primitive l-th root of unity $\tilde{\zeta}$, then, for each $a \in \mathbb{Z}/l\mathbb{Z}$ and for each $(C \rtimes G)$-module V over $R\#\Gamma$, the canonical $R\#\Gamma$-linear map from $F_a(V)$ to the eigenspace $V_a := \{v \in V : c(v) = \tilde{\zeta}^a v\}$ given by $\zeta \otimes v \mapsto \tilde{\zeta}v$ is an isomorphism.*

Proof. The composition of the isomorphism $V \to (S \otimes_R V)^G$, $v \mapsto \sum_{\sigma \in G} \zeta^\sigma \otimes \sigma(v)$, used in the proof of Proposition 1.3 with the map $(S \otimes_R V)^G \to V$, $\zeta \otimes v \mapsto \tilde{\zeta}v$, is bijective by Lemma 1.6. Hence, the canonical map $(S \otimes_R V)^G \to V$ is an isomorphism, and restricting to the eigenspaces yields the assertion.

Corollary 1.8. *For any $(C \rtimes G)$-module V over $R\#\Gamma$ and for any $a \in \mathbb{Z}/l\mathbb{Z}$, the canonical map $S \otimes_R F_a(V) \to (S \otimes_R V)_a$, $\alpha \otimes x \mapsto \alpha x$, is an isomorphism.*

Proof. Obviously, this map is the composition of the base change isomorphism $S \otimes_R F_a(V) \cong F_a(S \otimes_R V)$ with the isomorphism $F_a(S \otimes_R V) \cong (S \otimes_R V)_a$ of Proposition 1.7.

The next example shows that our construction agrees with Kervaire's construction given in §5 of [Ke].

Example 1.9. Let $R = K$ be a field with $(\mathrm{char}(K), l) = 1$, L the splitting field of the polynomial $X^l - 1$ over K, and $\tilde{\zeta} \in L$ a primitive l-th root of unity. Let H

be the Galois group of L/K considered as a subgroup of G as usual. As above, for any $(C \rtimes G)$-module V over $K\#\Gamma$, we have a semilinear H-action on $L \otimes_K V$ and we let $\tilde{F}_a(V) := \{\sum_{\sigma \in H} \sigma(x) : x \in (L \otimes_K V)_a\}$ be the image of the eigenspace $(L \otimes_K V)_a$ under the trace map. Then the canonical map $F_a(V) \to \tilde{F}_a(V)$ given by $\zeta \otimes v \mapsto \tilde{\zeta} \otimes v$ is an isomorphism.

Proof. By the usual descent argument of Galois theory (e. g. see the appendix of [Dr]), we have $L \otimes_K (L \otimes_K V)_a^H \cong (L \otimes_K V)_a$ and hence $\tilde{F}_a(V) = (L \otimes_K V)_a^H$ because the trace map from L to K is surjective. By Proposition 1.7 and base change, we furthermore have $(L \otimes_K V)_a \cong F_a(L \otimes_K V) \cong L \otimes_K F_a(V)$. Now it easy to see that the composed isomorphism $L \otimes_K F_a(V) \cong L \otimes_K (L \otimes_K V)_a^H$ is the base change of the canonical map $F_a(V) = (S \otimes_R V)_a^G \to (L \otimes_K V)_a^H$ which therefore must be an isomorphism.

Remarks 1.10.
a) Like in Example 1.9, one may use the l-th cyclotomic extension of R (e. g. see page 34 of [Gre]) in place of $S = R[X]/(\Phi_l)$ for the construction of $F_a(V)$. However, one striking advantage of using $S = R[X]/(\Phi_l)$ is that we have the more or less canonical normal basis element ζ in S and thus, for instance, base change becomes a very easy matter.
b) The quintessence of the above Gauss sum argument is the following assertion: For each $a \in \mathbb{Z}/l\mathbb{Z}$, the association $V \mapsto (\frac{1}{l} \sum_{\tau \in G} (\zeta^{-\tau} - 1)[\tau^{-1}])V_a$ is already defined over $\mathbb{Z}[\frac{1}{l}]$, i. e. for any $(C \rtimes G)$-module V over $R\#\Gamma$ there exists a unique submodule $\tilde{V}_a$ of V such that $S \otimes_R \tilde{V}_a = \frac{1}{l}(\sum_{\tau \in G}(\zeta^{-\tau} - 1)[\tau^{-1}])((S \otimes_R V)_a)$ as submodules of $S \otimes_R V$.

2. Adams Operations on $K_0(R\#\Gamma)$

As in the previous section, let Γ be a group, R a (noetherian) commutative ring, and $\Gamma \times R \to R$ an action of Γ on R. In this section, for each $k \in \mathbb{N}$ which is invertible in R, we will construct an Adams operation ψ^k on the Grothendieck group $K_0(R\#\Gamma)$ of projective modules over the twisted group ring $R\#\Gamma$ which are finitely generated over R. Our construction generalizes the definition given by Kervaire in §5 of [Ke] which in turn is based on formula (2.7) of Atiyah's paper [At]. Furthermore, we will show that these Adams operations are multiplicative and that they commute with each other, with base change, and with the Cartan homomorphism from $K_0(R\#\Gamma)$ to the Grothendieck group $K_0(\Gamma, R)$ of finitely generated, projective R-modules with (semilinear) Γ-action. Finally, if the characteristic of R is a prime number, we will construct an Adams operation ψ^k on $K_0(R\#\Gamma)$ for an arbitrary $k \in \mathbb{N}$.

In general, there do not exist exterior power operations on $K_0(R\#\Gamma)$ which are compatible with the Cartan homomorphism (see Remarque on page 19 of [Ke]). But we still have tensor products in $K_0(R\#\Gamma)$ by the following lemma.

Lemma 2.1. *Let W be a projective $R\#\Gamma$-module and V an $R\#\Gamma$-module such that there is an $R\#\Gamma$-module V' with $V \oplus V'$ free over R. (For instance, this assumption is satisfied if V is even free over R, or if V is $R\#\Gamma$-projective, or if V is projective over R and the action of Γ on R is trivial.) Then, the $R\#\Gamma$-module $V \otimes_R W$ (with diagonal Γ-action) is projective over $R\#\Gamma$.*

Proof. We may assume that $W = R\#\Gamma$ and that V is R-free, say with basis x_i, $i \in I$. Then $x_i \otimes 1$, $i \in I$, is a basis of $V \otimes W$ over $R\#\Gamma$: The i-th dual basis element $(x_i \otimes 1)^*$ is given by $x \otimes [\gamma] \mapsto x_i^*([\gamma^{-1}] \cdot x)[\gamma]$ for $x \in V$ and $\gamma \in \Gamma$.

Now let l be a prime number which is invertible in R and V an $R\#\Gamma$-module. The symmetric group $\Sigma_l := \mathrm{Permutations}(\mathbb{Z}/l\mathbb{Z})$ acts on the $R\#\Gamma$-module $V^{\otimes l} := \otimes_{i \in \mathbb{Z}/l\mathbb{Z}} V$ by $\sigma(\otimes_i v_i) := \otimes_i v_{\sigma^{-1}(i)}$ for $\sigma \in \Sigma_l$, $v_i \in V$. We consider the semidirect product $C \rtimes G$ defined in section 1 as a subgroup of Σ_l by mapping the generator c of C to the cycle $\mathbb{Z}/l\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z}$, $i \mapsto i+1$, and by embedding G in Σ_l via $G = (\mathbb{Z}/l\mathbb{Z})^\times = \mathrm{Aut}(\mathbb{Z}/l\mathbb{Z}) \subseteq \mathrm{Permutations}(\mathbb{Z}/l\mathbb{Z}) = \Sigma_l$. So $V^{\otimes l}$ becomes a $(C \rtimes G)$-module over $R\#\Gamma$.

Definition 2.2.
a) For any $a \in \mathbb{Z}/l\mathbb{Z}$, the $R\#\Gamma$-module
$$V[a] := V[a]_l := F_a(V^{\otimes l})$$
is called the *cyclic l-th power of V with eigenvalue ζ_l^a*.
b) For any projective $R\#\Gamma$-module V which is finitely generated over R, we let
$$\psi^l(V) := V[0] - V[1] \in K_0(R\#\Gamma).$$
c) For any finitely generated, projective R-module V with (semilinear) Γ-action, we let
$$\hat{\psi}^l(V) := V[0] - V[1] \in K_0(\Gamma, R).$$

Remark 2.3. In [At], Atiyah gives the name "cyclic l-th power" to a different but related object. The expression "with eigenvalue ζ_l^a" in the above definition is only a name and actually has not a real meaning. Definitions 2.2b) and c) make sense because of Lemma 2.1 and Corollary 1.4b). Note that in b), if Γ is infinite, we deal with the category of modules which are finitely generated not only over $R\#\Gamma$ but even over R in contrast to the usual definition of $K_0(R\#\Gamma)$. Otherwise tensor products (over R) and cyclic powers could not be defined within this category.

Example 2.4. For any $R\#\Gamma$-module V which is projective of rank 1 over R, we have $\hat{\psi}^l(V) = V^{\otimes l}$ in $K_0(\Gamma, R)$.

Proof. The symmetric group and in particular C act on $V^{\otimes l}$ trivially. Hence, we have $V[0] = F_0(V^{\otimes l}) \cong V^{\otimes l}$ by Example 1.5a) and $V[1] = 0$ by Proposition 1.3.

Proposition 2.5. *Let $0 \to V_0 \to V_1 \to V_2 \to 0$ be an exact sequence of projective $R\#\Gamma$-modules which are finitely generated over R. Then $\psi^l(V_1) = \psi^l(V_0) + \psi^l(V_2)$ in $K_0(R\#\Gamma)$. The analogous assertion holds for $\hat{\psi}^l$ on $K_0(\Gamma, R)$.*

Proof. We recall the essential steps of Kervaire's proof (see §5 of [Ke]) for the reader's convenience since this proof motivates the construction of Adams operations on higher K-groups given in the next section and since some part of this proof will be referred to in the proof of the Proposition 2.11 and Lemma 3.4. We consider the following filtration on $V_1^{\otimes l}$. For $\varepsilon \in \{0,1\}^{\mathbb{Z}/l\mathbb{Z}}$, let $|\varepsilon| := \sum_{i \in \mathbb{Z}/l\mathbb{Z}} \varepsilon_i$, $V_\varepsilon := \otimes_i V_{\varepsilon_i}$ and $W_\varepsilon := \otimes_i V_{\delta_i}$ with $\delta_i = 0$ if $\varepsilon_i = 0$ and $\delta_i = 2$ if $\varepsilon_i = 1$. Now, for $\lambda = 0, \ldots, l$, let $Q_\lambda := \sum_{|\varepsilon| = \lambda} V_\varepsilon$. Then, obviously,

$$0 \subset Q_0 = V_0^{\otimes l} \subset Q_1 \subset \ldots \subset Q_l = V_1^{\otimes l}$$

is a filtration of $V_1^{\otimes l}$ by $(C \rtimes G)$-submodules over $R\#\Gamma$. For each $\lambda = 0, \ldots, l$, the canonical surjections $V_\varepsilon \to W_\varepsilon$ induce a $(C \rtimes G)$-isomorphism of the successive quotient $Q_\lambda/Q_{\lambda-1}$ with $\oplus_{|\varepsilon|=\lambda} W_\varepsilon$. (This fact has been implicitly proved in sections 2 and 3 of [Ko2] for the construction of shuffle products on higher K-theory. It should be regarded as an analogue of the fact that the successive quotients of the (Koszul) filtration of the exterior power $\Lambda^l(V_1)$ by the submodules $V_0 \wedge \ldots \wedge V_0 \wedge V_1 \wedge \ldots \wedge V_1$ are isomorphic to $\Lambda^{l-\lambda} V_0 \otimes \Lambda^\lambda V_2$, $\lambda = 0, \ldots, l$.) Hence, the successive quotients are projective $R\#\Gamma$-modules by Lemma 2.1, and for each $a \in \mathbb{Z}/l\mathbb{Z}$, we have

$$F_a(V_1^{\otimes l}) = F_a(V_0^{\otimes l}) + F_a(V_2^{\otimes l}) + \sum_{\lambda=1}^{l-1} F_a(Q_\lambda/Q_{\lambda-1}) \quad \text{in} \quad K_0(R\#\Gamma)$$

by Corollary 1.4a). Now, in order to prove Proposition 2.5, it suffices to show that for each $\lambda \neq 0, l$, the module $F_a(Q_\lambda/Q_{\lambda-1})$ is independent of $a \in \mathbb{Z}/l\mathbb{Z}$. By the subsequent lemma there is a G-stable system of representatives R_λ in $M_\lambda := \{\varepsilon \in \{0,1\}^{\mathbb{Z}/l\mathbb{Z}} : |\varepsilon| = \lambda\}$ for the natural action of C on M_λ. Obviously, for $\lambda \neq 0, l$ there are no fixed points for this action of C on M_λ. Hence, $\oplus_{|\varepsilon|=\lambda} W_\varepsilon$ may be identified with the $(C \rtimes G)$-module $\text{Maps}(\mathbb{Z}/l\mathbb{Z}, \oplus_{\varepsilon \in R_\lambda} W_\lambda)$ where $C \rtimes G$ acts on $\mathbb{Z}/l\mathbb{Z}$ via the embedding $C \rtimes G \subset \Sigma_l$ and on $\oplus_{\varepsilon \in R_\lambda} W_\varepsilon$ via the canonical surjection $C \rtimes G \to G$. By Example 1.5c), the $R\#\Gamma$-module $F_a(Q_\lambda/Q_{\lambda-1})$ is isomorphic to $\oplus_{\varepsilon \in R_\lambda} W_\varepsilon$ for each $a \in \mathbb{Z}/l\mathbb{Z}$ and, in particular, it is independent of $a \in \mathbb{Z}/l\mathbb{Z}$. This ends the proof of Proposition 2.5 for $K_0(R\#\Gamma)$. The same proof works for $K_0(\Gamma, R)$.

Lemma 2.6. *For each $(C \rtimes G)$-set M, there exists a G-stable system of representatives in M for the set of C-orbits $N := M/C$.*

Proof. We may assume that the induced action of G on N is transitive. Let $I \subseteq G$ be the isotropy group of some $n \in N$. It suffices to show that there is a preimage of n in M which is fixed by I. Let $m \in M$ be any preimage of n and σ a generator of I. There is some $a \in \mathbb{Z}/l\mathbb{Z}$ such that $\sigma m = c^a m$. Then $c^b m$ with $b = 0$ if $\sigma = 1$ and $b = (1-\sigma)^{-1} a$ if $\sigma \neq 1$ is an I-fixed preimage of n.

By Proposition 2.5, the association $V \mapsto \psi^l(V)$ extends to a well-defined homomorphism

$$\psi^l : K_0(R\#\Gamma) \to K_0(R\#\Gamma).$$

Definition 2.7. The homomorphism ψ^l is called the *l-th Adams operation on* $K_0(R\#\Gamma)$.

Remarks 2.8.

a) In the next section, we will show that for any $a \in \mathbb{Z}/l\mathbb{Z}$ already the association $V \mapsto V[a]_l$ extends to a well-defined *map* on $K_0(R\#\Gamma)$.

b) Let $R = K$ be a field with $(\text{char}(K), l) = 1$, let Γ be a finite group, and let Γ act on R trivially. Then, Example 1.9 shows that the definition of the l-th Adams operation given here agrees with Kervaire's definition in §5 of [Ke].

c) Let $f : R \to R'$ be an homomorphism of Γ-rings. Then, Corollary 1.4c) shows that the Adams operation ψ^l commutes with the base change homomorphism $f^* : K_0(R\#\Gamma) \to K_0(R'\#\Gamma)$ given by $V \mapsto R' \otimes_R V$.

d) Obviously, ψ^l commutes with the restricting homomorphism $K_0(R\#\Gamma) \to K_0(R\#\Gamma')$ with respect to any homomorphism of groups $\Gamma' \to \Gamma$.

By Proposition 2.5, the association $V \mapsto \hat{\psi}^l(V)$ induces a well-defined homomorphism

$$\hat{\psi}^l : K_0(\Gamma, R) \to K_0(\Gamma, R).$$

By section (2.5) of [Ko1], the exterior power operations give $K_0(\Gamma, R)$ a (special) λ-ring structure. Thus, we can define an Adams operation ψ^l on $K_0(\Gamma, R)$ via the l-th Newton polynomial (e. g. see chapter I, §6 of [FL]).

Proposition 2.9. *The l-th Adams operation ψ^l agrees with the homomorphism $\hat{\psi}^l$ on $K_0(\Gamma, R)$. In particular, the Cartan homomorphism*

$$c : K_0(R\#\Gamma) \to K_0(\Gamma, R)$$

commutes with the l-th Adams operation.

Proof. Let V be a finitely generated, projective R-module, say of rank r, with a (semilinear) Γ-action. We have to show that $\psi^l(V) = \hat{\psi}^l(V)$ in $K_0(\Gamma, R)$. By the geometric splitting principle (see section (2.5) of [Ko1]), there is a Γ-scheme $f : X \to \text{Spec}(R)$ such that the base change homomorphism $f^* : K_0(\Gamma, R) \to K_0(\Gamma, X)$ is injective and such that we have $f^*(V) = \mathcal{L}_1 + \ldots + \mathcal{L}_r$ with Γ-modules $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on X of rank 1. (Even if the action of Γ on R is trivial, the action of Γ on X usually is not trivial. This is the essential reason for considering not only usual group rings but twisted group rings in this paper.) Similarly to the affine case, for any locally free Γ-module $\mathcal{V}$ on X and for any $a \in \mathbb{Z}/l\mathbb{Z}$, we let $\mathcal{V}[a] := p_*((p^*\mathcal{V}^{\otimes l})_a^G)$, where $p : S \otimes_R X \to X$ is the canonical projection, and we let $\hat{\psi}^l(\mathcal{V}) := \mathcal{V}[0] - \mathcal{V}[1] \in K_0(\Gamma, X)$. The same proofs as above show that the association $\mathcal{V} \mapsto \hat{\psi}^l(\mathcal{V})$ induces a well-defined homomorphism $\hat{\psi}^l$ on $K_0(\Gamma, X)$ and that $\hat{\psi}^l$ commutes with the base change homomorphism $f^* : K_0(\Gamma, R) \to K_0(\Gamma, X)$. Now, an analogue of Example 2.4 for X shows Proposition 2.9.

By Lemma 2.1, we have a ring structure on $K_0(R\#\Gamma)$ (without 1 in general). If the action of Γ on R is trivial, again by Lemma 2.1, we have a $K_0(\Gamma, R)$-module structure on $K_0(R\#\Gamma)$.

Proposition 2.10. *The l-th Adams operation $\psi^l : K_0(R\#\Gamma) \to K_0(R\#\Gamma)$ is multiplicative. If the action of Γ on R is trivial, ψ^l is semilinear with respect to the l-th Adams operation ψ^l on $K_0(\Gamma, R)$.*

Proof. By Corollary 1.8, for any $(C \rtimes G)$-module V over $R\#\Gamma$, the canonical map $S\otimes_R(S\otimes_R V)^G \to S\otimes_R V$ is an isomorphism. Hence, for any $(C \rtimes G)$-modules V, W over $R\#\Gamma$, the canonical map

$$(S \otimes_R V)^G \otimes_R (S \otimes_R W)^G \to (S \otimes_R V \otimes_R W)^G, \quad (\alpha \otimes v) \otimes (\beta \otimes w) \mapsto \alpha\beta \otimes v \otimes w$$

becomes an isomorphism after tensoring with S over R. Thus, this map itself is an isomorphism. Restricting to the eigenspaces yields an isomorphism $\oplus_{a+b=c} F_a(V) \otimes_R F_b(W) \cong F_c(V \otimes_R W)$ for each $c \in \mathbb{Z}/l\mathbb{Z}$. Hence, using Example 1.5b), we get the following equality in $K_0(R\#\Gamma)$ for all projective $R\#\Gamma$-modules V, W which are finitely generated over R:

$$
\begin{aligned}
\psi^l(V \otimes W) &= \sum_{a \in \mathbb{Z}/l\mathbb{Z}} V[a] \otimes W[-a] - \sum_{a \in \mathbb{Z}/l\mathbb{Z}} V[a] \otimes W[1-a] \\
&= (V[0] \otimes W[0] + (l-1)V[1] \otimes W[1]) \\
&\qquad -(V[0] \otimes W[1] + V[1] \otimes W[0] + (l-2)V[1] \otimes W[1]) \\
&= (V[0] - V[1]) \cdot (W[0] - W[1]) = \psi^l(V) \cdot \psi^l(W).
\end{aligned}
$$

This shows the first assertion of Proposition 2.10. The same proof shows the second assertion.

Proposition 2.11. *Let $k \in \mathbb{N}$ be another prime number which is invertible in R. Then, the Adams operations ψ^k and ψ^l on $K_0(R\#\Gamma)$ commute.*

Proof. Here, let C be the cyclic group of order kl, c a generator of C, $G := (\mathbb{Z}/kl\mathbb{Z})^\times$ acting on C, Φ_{kl} the kl-th cyclotomic polynomial, $S := R[X]/(\Phi_{kl})$ and $\zeta := \bar{X} \in S$. We will denote the corresponding objects for k respectively l with a subscript k respectively l. We have canonical isomorphisms

$$\mathbb{Z}/kl\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z} \quad \text{and} \quad S \xrightarrow{\sim} S_k \otimes_R S_l, \ \zeta \mapsto \zeta_k \otimes \zeta_l,$$

and for any $a \in \mathbb{Z}/kl\mathbb{Z}$, we will denote both the element $a + k\mathbb{Z}$ in $\mathbb{Z}/k\mathbb{Z}$ and the preimage of $(a + k\mathbb{Z}, 0)$ in $\mathbb{Z}/kl\mathbb{Z}$ by a_k.

Now, let V be a projective $R\#\Gamma$-module which is finitely generated over R. As above, for any $a \in \mathbb{Z}/kl\mathbb{Z}$, let

$$V[a] := (S \otimes_R V^{\otimes kl})_a^G.$$

We claim that

$$V[0] - V[1_k] \ (\stackrel{\text{def}}{=} V[(0,0)] - V[(0,1)]) = V[0]_k[0]_l - V[0]_k[1]_l \quad \text{in} \quad K_0(R\#\Gamma),$$

$$V[1_l] - V[1] \ (\stackrel{\text{def}}{=} V[(1,0)] - V[(1,1)]) = V[1]_k[0]_l - V[1]_k[1]_l \quad \text{in} \quad K_0(R\#\Gamma).$$

Then we have

$$\psi^l \circ \psi^k(V) = V[0] - V[1_l] - V[1_k] + V[1] \quad \text{in} \quad K_0(R\#\Gamma),$$

and this expression is symmetric in k and l. Thus, these claims show Proposition 2.11.

To prove the above claims, we proceed like in the proof of Proposition 2.10. Using Lemma 1.6 for k and l, one shows like in the proof of Corollary 1.8 that the canonical map

$$S \otimes_R \left(S \otimes_R V^{\otimes kl} \right)^G \to S \otimes_R V^{\otimes kl}, \quad \alpha \otimes x \mapsto \alpha x,$$

is an isomorphism. Hence, after tensoring with $S = S_k \otimes S_l$, the map

$$\Phi : \left(S_l \otimes_R ((S_k \otimes_R V^{\otimes k})^{G_k})^{\otimes l} \right)^{G_l} \to \left(S \otimes_R V^{\otimes kl} \right)^G$$

induced by the canonical isomorphism $(V^{\otimes k})^{\otimes l} \cong V^{\otimes kl}$ becomes an isomorphism. Thus, Φ itself is an isomorphism.

Now we consider the eigenspace decompositions on both sides of Φ. By Lemma 2.6, there exists a G-stable system of representatives R in $(\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}/l\mathbb{Z}}$ for the set of C-orbits under the natural action of C on $(\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}/l\mathbb{Z}} \setminus \{(a,\dots,a) : a \in \mathbb{Z}/k\mathbb{Z}\}$. Like in the proof of Proposition 2.5, this action has no fixed points, and we have

$$
\begin{aligned}
((S_k \otimes V^{\otimes k})^{G_k})^{\otimes l} &\cong (V[0]_k \oplus \dots \oplus V[k-1]_k)^{\otimes l} \\
&\cong \bigoplus_{\varepsilon \in (\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}/l\mathbb{Z}}} V[\varepsilon_0]_k \otimes \dots \otimes V[\varepsilon_{l-1}]_k \\
&\cong \bigoplus_{a=0}^{k-1} V[a]_k^{\otimes l} \oplus \bigoplus_{\varepsilon \in R} \mathrm{Maps}(\mathbb{Z}/l\mathbb{Z}, V[\varepsilon_0]_k \otimes \dots \otimes V[\varepsilon_{l-1}]_k).
\end{aligned}
$$

Thus, for each $b \in \mathbb{Z}/l\mathbb{Z}$, we have

$$((S_k \otimes V^{\otimes k})^{G_k})[b]_l \cong \bigoplus_{a=0}^{k-1} V[a]_k[b]_l \oplus \bigoplus_{\varepsilon \in R} V[\varepsilon_0]_k \otimes \dots \otimes V[\varepsilon_{l-1}]_k$$

by Example 1.5c). For any $\varepsilon \in (\mathbb{Z}/k\mathbb{Z})^{\mathbb{Z}/l\mathbb{Z}}$, let $|\varepsilon| := \sum_{i=0}^{l-1} \varepsilon_i \in \mathbb{Z}/k\mathbb{Z}$. Then, for each $(a,b) \in \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z} = \mathbb{Z}/kl\mathbb{Z}$, we have

$$\Phi^{-1}(V[(a,b)]) \cong V[l^{-1}a]_k[b]_l \oplus \bigoplus_{\varepsilon \in R : |\varepsilon| = a} V[\varepsilon_0]_k \otimes \dots \otimes V[\varepsilon_{l-1}]_k.$$

Because of Example 1.5b), these isomorphisms show the above claims and Proposition 2.11 is proved.

Remarks 2.12.

a) Let $k \in \mathbb{N}$ be invertible in R and let $k = l_1 \cdot \dots \cdot l_r$ be its prime factor decomposition. Then, by Proposition 2.11, the composition $\psi^k := \psi^{l_1} \circ \dots \circ \psi^{l_r}$ does not depend on the order of the prime numbers $l_1, \dots, l_r$. We call this composition

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the k-th Adams operation on $K_0(R\#\Gamma)$. Obviously, we have $\psi^k \circ \psi^l = \psi^{kl}$ for all $k, l \in \mathbb{N}$ which are invertible in R. Like in the proof of Proposition 2.11, one can prove the following formula for $\psi^k(V)$, V a finitely generated $R\#\Gamma$-module:

$$\psi^k(V) = \sum_{d|k} \mu(d) F_d(V^{\otimes k}) \quad \text{in} \quad K_0(R\#\Gamma)$$

(cf. Proposition 2 of [Be]). Here, μ is the Möbius function of multiplicative number theory, and $F_d(V^{\otimes k})$ is the analogously defined cyclic k-th power of V with eigenvalue $\zeta_k^{k/d}$.

b) In the situation considered by Kervaire in [Ke] (cf. Remark 2.8b)), the Cartan homomorphism c is injective, and the assertions of Propositions 2.10 and 2.11 already follow from the corresponding assertions for $K_0(\Gamma, R)$.

In the final part of this section, we assume in addition that the characteristic of R is a prime number p. Let

$$F : R \to R, \quad r \mapsto r^p,$$

be the Frobenius endomorphism. Obviously, F is a homomorphism of Γ-rings. Thus, we have the base change homomorphism

$$F^* : K_0(R\#\Gamma) \to K_0(R\#\Gamma), \quad V \mapsto R \otimes_R V,$$

where the tensor product $R \otimes_R V$ is taken with respect to F and the R-module structure of $R \otimes_R V$ is given by $r(s \otimes v) := rs \otimes v$ for $r, s \in R$, $v \in V$ as usual.

Definition 2.13. The base change homomorphism F^* is called *the p-th Adams operation on $K_0(R\#\Gamma)$* and will be denoted by ψ^p.

Remark 2.14. By Remark 2.8c), $F^* = \psi^p$ commutes with ψ^l for each prime number l. Thus, here we can define an Adams operation ψ^k for an arbitrary $k \in \mathbb{N}$.

Proposition 2.15. *The base change homomorphism F^* on $K_0(\Gamma, R)$ agrees with the p-th Adams operation ψ^p on $K_0(\Gamma, R)$. In particular, ψ^p commutes with the Cartan homomorphism $c : K_0(R\#\Gamma) \to K_0(\Gamma, R)$.*

Proof (see also Theorem 5.1 in [Hi] and Proposition 5.4 in [Kr]). Let V be a finitely generated, projective R-module with a (semilinear) Γ-action. We have to show that $\psi^p(V) = F^*(V)$ in $K_0(\Gamma, R)$. Like in the proof of Proposition 2.9, by the geometric splitting principle, we may assume that V is of rank 1. Then,

$$F^*(V) = R \otimes_R V \to V^{\otimes p}, \quad r \otimes v \mapsto r \cdot v^{\otimes p},$$

is a well-defined isomorphism of $R\#\Gamma$-modules. This shows Proposition 2.15.

Question 2.16. Let p be a prime number which is neither invertible nor zero in R. Is there an Adams operation ψ^p on $K_0(R\#\Gamma)$ which commutes with both

the base change homomorphism $K_0(R\#\Gamma) \to K_0(R/(p)\#\Gamma)$ and the base change homomorphism $K_0(R\#\Gamma) \to K_0(R[\frac{1}{p}]\#\Gamma)$?

3. Cyclic Powers and Adams Operations on $K_q(R\#\Gamma)$

Again, let Γ be a group, R a commutative Γ-ring and l a prime number which is invertible in R. For any $q \geq 0$, let $K_q(R\#\Gamma)$ denote Quillen's q-th K-group associated with the exact category $\mathcal{M}$ of projective $R\#\Gamma$-modules which are finitely generated over R. In this section, we will construct cyclic power operations $[a]_l$, $a \in \mathbb{Z}/l\mathbb{Z}$, and an Adams operation ψ^l on $K_q(R\#\Gamma)$. This construction uses shuffle products in higher K-theory (cf. [Ko2]) and Grayson's techniques of constructing exterior power operations on higher K-theory (cf. [Gr1]). We will show that, for $q = 0$, this construction agrees with the construction given in section 2. Furthermore, we will give some speculations how one should be able to prove that ψ^l commutes with the Cartan homomorphism $c : K_q(R\#\Gamma) \to K_q(\Gamma, R)$.

At first, we will recall some definitions of [Gr1] and [Ko2]; in particular, we will give a definition of Quillen's q-th K-group $K_q(R\#\Gamma)$. Let Ord be the category of finite, non-empty, (partially) ordered sets. For any $A \in \text{Ord}$, let the disjoint union $\{L, R\} \cup A$ be ordered in such a way that A is an ordered subset, that $L < a$ and $R < a$ for all $a \in A$, and that L and R are not comparable. Let $\Gamma(A) := \{\frac{i}{j} : i \in A, j \in \{L, R\} \cup A, j \leq i\}$ and, more generally, let $\Gamma^l(A) := \{(\frac{i_1}{j_1}, *_2, \ldots, *_l, \frac{i_l}{j_l}) :$ for all $r = 1, \ldots, k : i_r \in A, j_r \in \{L, R\} \cup A, j_r \leq i_r$; for all $r = 2, \ldots, k :$ $*_r \in \{\wedge, \otimes\}$ and, if $*_r = \wedge$, then $j_{r-1} = j_r$ and $i_{r-1} \leq i_r\}$. In section 5 of [Gr1], Grayson defines a certain (partial) order on $\Gamma(A)$ and $\Gamma^l(A)$ and calls certain sequences $\alpha' \to \alpha \to \alpha''$ in $\Gamma(A)$ and $\Gamma^l(A)$ (short) exact sequences. Then, the so-called G-*construction* is the simplicial set

$$G\mathcal{M} : \text{Ord}^{\text{op}} \to \text{Sets}, \quad A \mapsto \text{Exact}(\Gamma(A), \mathcal{M}),$$

where $\text{Exact}(\Gamma(A), \mathcal{M})$ denotes the set of all functors of $\Gamma(A)$ to $\mathcal{M}$ which transform short exact sequences in $\Gamma(A)$ to short exact sequences of $\mathcal{M}$. By [GG], the q-th homotopy group $\pi_q(|G\mathcal{M}|)$ of the geometric realization $|G\mathcal{M}|$ of $G\mathcal{M}$ is Quillen's q-th K-group $K_q(R\#\Gamma)$. Similarly, let $\mathcal{H}^l\mathcal{M}$ be the simplicial set

$$\mathcal{H}^l\mathcal{M} : \text{Ord}^{\text{op}} \to \text{Sets}, \quad A \mapsto \text{Exact}(\Gamma^l(A), \mathcal{M}).$$

By Lemma 2.1, we have an associative, commutative, and bi-exact tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. Associated with this functor, we have a simplicial map

$$\otimes : \underbrace{G\mathcal{M} \times \ldots \times G\mathcal{M}}_{l} \to \mathcal{H}^l\mathcal{M},$$

the so-called $(1, \ldots, 1)$-shuffle product (cf. [Ko2]). It is defined as follows: For any $A \in \text{Ord}$ and any $\alpha = (\frac{i_1}{j_1}, *_2, \ldots, *_l, \frac{i_l}{j_l}) \in \Gamma^l(A)$, let $\sim_\alpha$ be the equivalence

relation on $\Sigma_l := \text{Permutations}\{1,\ldots,l\}$ defined by $\sigma \sim_\alpha \tau$, iff, for all $r = 1,\ldots,l$, there are only $\wedge$'s in α between the places $\sigma(r)$ and $\tau(r)$. Then, on the level A, the simplical map $\otimes$ is given by

$$GM(A) \times \ldots \times GM(A) \quad \to \quad \mathcal{H}^l\mathcal{M}(A) = \text{Exact}(\Gamma^l(A), \mathcal{M})$$

$$(M_1,\ldots,M_l) \quad \mapsto \quad \left(\alpha \mapsto \bigoplus_{\tilde\sigma \in \Sigma_l/\sim_\alpha} \sum_{\sigma \in \tilde\sigma} M_1\left(\frac{i_{\sigma(1)}}{j_{\sigma(1)}}\right) \otimes \ldots \otimes M_l\left(\frac{i_{\sigma(l)}}{j_{\sigma(l)}}\right)\right)$$

(cf. Remark at the end of section 3 of [Ko2]). We call the composition

$$\otimes^l : GM \overset{\Delta}{\to} GM \times \ldots \times GM \overset{\otimes}{\to} \mathcal{H}^l\mathcal{M}$$

of the diagonal Δ with the $(1,\ldots,1)$-shuffle product $\otimes$ the l-*th shuffle power.*

Next, for any $A \in \text{Ord}$ and any $M \in GM(A)$, we will define a natural action of Σ_l on $M^{\otimes l} \in \text{Exact}(\Gamma^l(A), \mathcal{M}) = \mathcal{H}^l\mathcal{M}(A)$. Let $\alpha = \left(\frac{i_1}{j_1}, *_2,\ldots,*_l, \frac{i_l}{j_l}\right) \in \Gamma^l(A)$. Then, interchanging the direct summands yields an action of Σ_l on $\oplus_{\sigma \in \Sigma_l} M\left(\frac{i_{\sigma(1)}}{j_{\sigma(1)}}\right) \otimes \ldots \otimes M\left(\frac{i_{\sigma(l)}}{j_{\sigma(l)}}\right)$. Obviously, for any $\sigma_1, \sigma_2 \in \Sigma_l$ with $\sigma_1 \sim_\alpha \sigma_2$, we have $\sigma_1 \circ \tau \sim_\alpha \sigma_2 \circ \tau$ for all $\tau \in \Sigma_l$. Hence, by Lemma 2.1a) of [Ko2], this action of Σ_l induces an action of Σ_l on $M^{\otimes l}(\alpha) = \oplus_{\tilde\sigma \in \Sigma_l/\sim_\alpha} \sum_{\sigma \in \tilde\sigma} M\left(\frac{i_{\sigma(1)}}{j_{\sigma(1)}}\right) \otimes \ldots \otimes M\left(\frac{i_{\sigma(l)}}{j_{\sigma(l)}}\right)$. Finally, for any α, $\beta \in \Gamma^l(A)$ with $\alpha \to \beta$, the corresponding map $M^{\otimes l}(\alpha) \to M^{\otimes l}(\beta)$ is compatible with this Σ_l-action. Thus, $M^{\otimes l}$ may be viewed as an element of $\text{Exact}(\Gamma^l(A), \mathcal{M}^{\Sigma_l})$; here, $\mathcal{M}^{\Sigma_l}$ denotes the category of Σ_l-modules over $R\#\Gamma$ which are finitely generated over R and projective over $R\#\Gamma$.

Remark 3.1. Let $\Gamma = \{1\}$ be the trivial group. In the last example of [Gr1], Grayson considers the operations

$$\mathcal{M}^{\Sigma_n} \times \mathcal{M}^{\Sigma_p} \to \mathcal{M}^{\Sigma_{n+p}}, \quad (V,W) \mapsto \text{Ind}_{\Sigma_n \times \Sigma_p}^{\Sigma_{n+p}} V \otimes_R W$$

and

$$F_k(\mathcal{M}^{\Sigma_n}) \to \mathcal{M}^{\Sigma_{nk}}, \quad (V_1 \hookrightarrow \ldots \hookrightarrow V_k) \mapsto \text{Ind}_{\Sigma_n \int \Sigma_k}^{\Sigma_{nk}} \sum_{\sigma \in \Sigma_k} V_{\sigma(1)} \otimes_R \ldots \otimes_R V_{\sigma(k)}$$

for k, n, $p \in \mathbb{N}$. He claims that these operations satisfy the axioms of power operations given in section 7 of [Gr1] (for checking this see also [GS]) and, similarly to the above construction, he obtains simplicial maps

$$\Lambda^k : GM^{\Sigma_n} \to \mathcal{H}^k\mathcal{M}^{\Sigma_{nk}}, \quad k,n \in \mathbb{N}.$$

It is easy to see that for $n = 1$ and $k = l$ this map Λ^k agrees with the above l-th shuffle power $\otimes^l : GM \to \mathcal{H}^l\mathcal{M}^{\Sigma_l}$.

Now let $a \in \mathbb{Z}/l\mathbb{Z}$ and let $F_a : \mathcal{M}^{\Sigma_l} \overset{\text{can}}{\longrightarrow} \mathcal{M}^{C \rtimes G} \longrightarrow \mathcal{M}$ be the exact functor constructed in section 1. We call the simplicial map

$$[a]_l : GM \quad \to \quad \mathcal{H}^l\mathcal{M}$$
$$\text{Exact}(\Gamma(A), \mathcal{M}) \ni M \quad \mapsto \quad F_a \circ M^{\otimes l} \in \text{Exact}(\Gamma^l(A), \mathcal{M})$$

($A \in \mathrm{Ord}$) the *cyclic l-th shuffle power with eigenvalue ζ_l^a*.

In section 7 of [Gr1], using a certain subdivision procedure, Grayson defines a continuous map

$$\Xi_l : |\mathcal{H}^l \mathcal{M}| \to |G^l \mathcal{M}|$$

from the geometric realization of $\mathcal{H}^l \mathcal{M}$ to the geometric realization of a certain l-simplicial set $G^l \mathcal{M}$ which in turn is naturally homotopy equivalent to $G\mathcal{M}$.

Definition 3.2. The composition

$$|G\mathcal{M}| \xrightarrow{|[a]_l|} |\mathcal{H}^l \mathcal{M}| \xrightarrow{\Xi_l} |G^l \mathcal{M}|$$

of the geometric realization $|[a]_l|$ of $[a]_l$ with the continuous map Ξ_l is called the *cyclic l-th power with eigenvalue ζ_l^a* and will be denoted by $[a]_l$ again. For any $q \geq 0$, we give the same name to the map

$$[a]_l : K_q(R\#\Gamma) \to K_q(R\#\Gamma)$$

induced by $[a]_l$ on the q-th homotopy groups. The difference $\psi^l := [0]_l - [1]_l$ is called the *l-th Adams operation on $K_q(R\#\Gamma)$*.

Proposition 3.3. *The l-th Adams operation on $K_0(R\#\Gamma)$ defined here agrees with the l-th Adams operation defined in section 2.*

Proof. This immediately follows from the "minus part" of the following lemma.

Lemma 3.4. *For all finitely generated projective $R\#\Gamma$-modules V, W, we have*

$$[a]_l(V \pm W) = V[a]_l \pm W[a]_l + \frac{(V \pm W)^l - V^l \mp W^l}{l} \quad \text{in} \quad K_0(R\#\Gamma).$$

In particular, for $l \neq 2$, we have $[a]_l(-W) = -W[a]_l$ in $K_0(R\#\Gamma)$.

Proof. First, we consider the "minus part" of this formula. Let

$$M : \Gamma(\{0\}) \to \mathcal{M}, \quad \frac{0}{L} \mapsto V, \quad \frac{0}{R} \mapsto W,$$

be the 0-simplex of $G\mathcal{M}$ corresponding to $V - W$. Similarly to section 8 of [Gr1], one easily shows:

$$[a]_l(V - W) = \sum_{\substack{a+b_1+\ldots+b_u=l \\ a \geq 0,\, b_1,\ldots,b_u \geq 1}} (-1)^u F_a \left(M^{\otimes l}(\alpha(a, b_1, \ldots, b_u)) \right)$$

where

$$\alpha(a, b_1, \ldots, b_u)$$

$$:= \left(\underbrace{\frac{0}{L}, \wedge, \ldots, \wedge, \frac{0}{L}}_{a}, \otimes, \underbrace{\frac{0}{R}, \wedge, \ldots, \wedge, \frac{0}{R}}_{b_1}, \otimes, \ldots, \otimes, \underbrace{\frac{0}{R}, \wedge, \ldots, \wedge, \frac{0}{R}}_{b_u} \right) \in \Gamma^l(A).$$

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By the construction of the l-th shuffle power, we have

$$M^{\otimes l}(\alpha(a, b_1, \ldots, b_u)) = \bigoplus_{\tilde{\sigma} \in \Sigma_l/\sim_\alpha} M\left(\frac{0}{\tilde{\sigma}(1)}\right) \otimes \ldots \otimes M\left(\frac{0}{\tilde{\sigma}(l)}\right)$$

where $\tilde{\sigma}(i) = L$, if $\sigma(i) \leq a$ for one (and then for all) $\sigma \in \tilde{\sigma}$, and $\tilde{\sigma}(i) = R$ else. Like in the proof of Proposition 2.5, for all $(a, b_1, \ldots, b_u) \neq (l)$ and $\neq (0, l)$, the cycle $< 1, \ldots, l >$ acts without fixed points on $\Sigma_l/\sim_\alpha$, and the module $F_a\left(M^{\otimes l}(\alpha(a, b_1, \ldots, b_u))\right)$ is independent of $a \in \mathbb{Z}/l\mathbb{Z}$.

On the other hand, by the Corollary of Theorem 4.2 in [Ko2], the power $(V - W)^l$ may be computed via the $(1, \ldots, 1)$-shuffle product, i. e. similarly as above, we obtain

$$(V - W)^l = V^l - W^l + \sum_{(a,b_1,\ldots,b_u)\neq(l) \text{ and } \neq(0,l)} (-1)^u M^{\otimes l}(\alpha(a, b_1, \ldots, b_u))$$

$$= V^l - W^l + \sum_{(a,b_1,\ldots,b_u)\neq(l) \text{ and } \neq(0,l)} l(-1)^u F_a\left(M^{\otimes l}(\alpha(a, b_1, \ldots, b_u))\right).$$

Here, the second equality follows from Proposition 1.3 and the fact that

$$F_a\left(M^{\otimes l}(\alpha(a, b_1, \ldots, b_u))\right)$$

is independent of $a \in \mathbb{Z}/l\mathbb{Z}$. Together with the above formula for $[a]_l(V - W)$, this shows the "minus part" of the lemma. The "plus part" can be deduced in a similar fashion from the formula

$$[a]_l(V + W) = \sum_{\lambda=0}^{l} F_a\left(M^{\otimes l}\left(\underbrace{\frac{0}{L}, \wedge, \ldots, \wedge, \frac{0}{L}}_{\lambda}, \otimes, \underbrace{\frac{1}{0}, \wedge, \ldots, \wedge, \frac{1}{0}}_{l-\lambda}\right)\right)$$

(cf. section 4 of [Gr1] and the remark in parentheses in the proof of Proposition 2.5). Here, M denotes the 1-simplex of $G\mathcal{M}$ given by $M\left(\frac{0}{L}\right) = M\left(\frac{0}{R}\right) = V$, $M\left(\frac{1}{L}\right) = M\left(\frac{1}{R}\right) = V \oplus W$, and $M\left(\frac{1}{0}\right) = W$.

Corollary 3.5. *For all x, $y \in K_0(R\#\Gamma)$, we have*

$$[a]_l(x + y) = [a]_l(x) + [a]_l(y) + \frac{(x + y)^l - x^l - y^l}{l} \quad \text{in} \quad K_0(R\#\Gamma).$$

Proof. We may write $x = V - W$, $y = V' - W'$ with finitely generated, projective $R\#\Gamma$-modules V, W, V', W'. Then, using Lemma 3.4, we obtain

$$[a]_l(x + y) = [a]_l(V - W + V' - W') = [a]_l(V \oplus V' - W \oplus W')$$
$$= [a]_l(V \oplus V') - [a]_l(W \oplus W')$$

$$+\frac{(V \oplus V' - W \oplus W')^l - (V \oplus V')^l + (W \oplus W')^l}{l}$$

$$= [a]_l(V) + [a]_l(V') - [a]_l(W) - [a]_l(W')$$
$$+\frac{(V + V' - W - W')^l - V^l - V'^l + W^l + W'^l}{l}$$

$$= [a]_l(V - W) + [a]_l(V' - W')$$
$$+\frac{(V + V' - W - W')^l - (V - W)^l - (V' - W')^l}{l}$$

$$= [a]_l(x) + [a]_l(y) + \frac{(x + y)^l - x^l - y^l}{l}.$$

Remark 3.6. In order to show that exterior powers induce well-defined maps $\lambda_k : K_0(R) \to K_0(R)$, $k \geq 1$, one usually shows that the association $V \mapsto \sum_{k \geq 0}[\Lambda^k V]t^k$ extends to a well-defined *homomorphism* $K_0(R) \to 1 + tK_0(R)[[t]]$. I don't know such a simple argument for the operations $[a]_l$, $a \in \mathbb{Z}/l\mathbb{Z}$, on $K_0(R\#\Gamma)$. However, one can show directly (i. e. without using Grayson's techniques) that the map $[a]_l$ defined by

$$[a]_l(\sum_i n_i V_i) := \sum_i n_i V_i [a]_l + \frac{(\sum n_i V_i)^l - \sum n_i V_i^l}{l}$$

on the free abelian group with basis the isomorphism classes of finitely generated, projective $R\#\Gamma$-modules factorizes through the Grothendieck group $K_0(R\#\Gamma)$.

The following considerations are concerned with the question how to prove that the Adams operation ψ^l commutes with the Cartan homomorphism $c : K_q(R\#\Gamma) \to K_q(\Gamma, R)$ (see also Remark 3.8). Here, $K_q(\Gamma, R)$ denotes Quillen's q-th K-group associated with the exact category of finitely generated, projective R-modules with (semilinear) Γ-action, and ψ^l on $K_q(\Gamma, R)$ is defined via the l-th Newton polynomial in the exterior power operations $\lambda_1, \ldots, \lambda_l$ on $K_q(\Gamma, R)$ which in turn were constructed by Grayson in [Gr1]. As above, for any $a \in \mathbb{Z}/l\mathbb{Z}$ and any $q \geq 0$, we can define the cyclic l-th power $[a]_l$ on $K_q(\Gamma, R)$.

Lemma 3.7. *The operations $[0]_l$ and $[1]_l$ on $K_0(\Gamma, R)$ are natural transformations of λ-rings, i. e. they can be expressed as polynomials in the exterior power operations λ_k, $k \geq 1$.*

Proof. We view the symmetric polynomials

$$C_{l,1} := \frac{(t_1 + \ldots + t_l)^l - (t_1^l + \ldots + t_l^l)}{l} \in \mathbb{Z}[t_1, \ldots, t_l]$$

and

$$C_{l,0} := (t_1 + \ldots + t_l)^l - (l - 1)C_{l,1} \in \mathbb{Z}[t_1, \ldots, t_l]$$

as polynomials in the elementary symmetric polynomials $\lambda_1 := t_1 + \ldots + t_l$, ..., $\lambda_l := t_1 \cdot \ldots \cdot t_l$ and claim that

$$[1]_l = C_{l,1}(\lambda_1, \ldots, \lambda_l) \quad \text{and} \quad [0]_l = C_{l,0}(\lambda_1, \ldots, \lambda_l) \quad \text{on} \quad K_0(\Gamma, R)$$

where now $\lambda_1, \ldots, \lambda_l$ are the exterior power operations. Obviously, we have

$$C_{l,1} = \frac{\lambda_1^l - \psi^l}{l} \quad \text{and} \quad C_{l,0} = \frac{\lambda_1^l + (l-1)\psi^l}{l},$$

and Lemma 3.4 holds for $K_0(\Gamma, R)$ with $[a]_l$ replaced by $C_{l,0}$ or $C_{l,1}$. Hence, like in the proof of Proposition 2.9, by the geometric splitting principle, it suffices to show $[0]_l(V) = C_{l,0}(V)$ and $[1]_l(V) = C_{l,1}(V)$ in $K_0(\Gamma, R)$ for any $R\#\Gamma$-module V which is projective of rank 1 over R. But this follows from (the proof of) Example 2.4.

Now, in order to show that ψ^l commutes with the Cartan homomorphism $c : K_q(R\#\Gamma) \to K_q(\Gamma, R)$, it suffices to show that

$$[1]_l = C_{l,1}(\lambda_1, \ldots, \lambda_l) \quad \text{and} \quad [0]_l = C_{l,0}(\lambda_1, \ldots, \lambda_l) \quad \text{on} \quad K_q(\Gamma, R).$$

This is certainly true for $l = 2$ because $[1]_2 = \lambda_2$ and $[0]_2$ is the symmetric square σ_2 which in turn equals $\lambda_1^2 - \lambda_2$ by Theorem 6.1 of [Ko2].

For $l = 3$, one can show that, for any finitely generated, projective R-module V with (semilinear) Γ-action, the sequence

$$0 \to \Lambda^3 V \to \Lambda^2 V \otimes V \to V[1]_3 \to 0$$

is exact. Here, the first arrow comes from the Koszul complex and the second arrow is the composition of the natural embedding $\Lambda^2 V \otimes V \hookrightarrow V^{\otimes 3}$ with the projection $V^{\otimes 3} \to V[1]_3$ coming from Proposition 1. Furthermore, this sequence can be completed to a diagram as in Fig. 2 of [Ko2]. Like there, one can deduce from this that $[1]_3 = \lambda_1 \cdot \lambda_2 - \lambda_3$ already on the simplicial level. Together with the fact

$$(l-1)[1]_l + [0]_l = \lambda_1^l$$

coming from Proposition 1.3 and Example 1.5b), this shows the above claims for $l = 3$.

For $l > 3$, the essential problem is to find (universal) exact sequences which relate the cyclic power $[1]_l$ to exterior powers. The previous examples make it not unreasonable to conjecture that, for any finitely generated, projective R-module V with (semilinear) Γ-action, there is a (universal) resolution

$$0 \to D_1 V \to D_2 V \to \ldots \to D_{l-1} V \to V[1]_l \to 0$$

of $V[1]_l$ where, for each $r = 1, \ldots, l-1$, $D_r V$ is the following direct sum of tensor products of exterior powers of V:

$$D_r V = \bigoplus_{n_1 \geq \ldots \geq n_r \geq 1} (\Lambda^{n_1} V \otimes \ldots \otimes \Lambda^{n_r} V)^{m(n_1, \ldots, n_r)}.$$

Here $(-1)^{l-r+1}m_{(n_1,\ldots,n_r)}$ is the coefficient of the monomial $\lambda_1^{n_1} \cdot \ldots \cdot \lambda_r^{n_r}$ in the polynomial $C_{l,1}$. This coefficient can be rather easily computed by the formula

$$C_{l,1} = \frac{\lambda_1^l - \psi^l}{l} = \frac{\lambda_1^l - \sum_{i=1}^l (-1)^{i-1} i \lambda_i \cdot s_{l-i}}{l}$$

(cf. section 3 of [Gr2]). In particular, one can deduce from this formula that the multiplicity $m_{(n_1,\ldots,n_r)}$ is positive for all $r = 1, \ldots, l-1$ and all $n_1 \geq \ldots \geq n_r$ with $n_1 + \ldots + n_r = l$, and zero otherwise (e. g. $D_1 V = \Lambda^l V$, $D_{l-1} V = \Lambda^2 V \otimes V \otimes \ldots \otimes V$). There is a canonical choice for the map $D_{l-1} \to V[1]_l$ in the above resolution, namely the composition of the canonical embedding $\Lambda^2 V \otimes V \otimes \ldots \otimes V \hookrightarrow V^{\otimes l}$ with the projection $V^{\otimes l} \to V[1]_l$ coming from Proposition 1.3. One can show that this composition is surjective. Thus, the remaining problem is to construct (universal) homomorphisms, $D_r \to D_{r+1}$, $r = 1, \ldots, l-2$, which make the above sequence exact. For instance, for $l = 5$, one should be able to define arrows in the sequence

$$0 \to \Lambda^5 V \to \Lambda^4 V \otimes V \oplus \Lambda^3 V \otimes \Lambda^2 V \to \Lambda^3 V \otimes V \otimes V \oplus \Lambda^2 V \otimes \Lambda^2 V \otimes V$$
$$\to \Lambda^2 V \otimes V \otimes V \otimes V \to V[1]_5 \to 0$$

such that this becomes an exact sequence and one should be able to generalize this sequence like in Fig. 1 and Fig. 2 of section 6 of [Ko2].

Remark 3.8. Obviously, using standard techniques, one can prove that the Adams operation ψ^l and the cyclic l-th powers $[a]_l$, $a \in \mathbb{Z}/l\mathbb{Z}$, commute with the base change homomorphism $K_q(R\#\Gamma) \to K_q(R'\#\Gamma)$ with respect to any homomorphism $R \to R'$ of Γ-rings. If the characteristic of R is a prime number p, one can define ψ^p on $K_q(R\#\Gamma)$ to be the base change homomorphism F^* with respect to the Frobenius endomorphism F. Then, ψ^p commutes with each ψ^l. Using Quillen's universal transformation, we show in [Ko4] that (for finite Γ) the Adams operations constructed above commute with each other, that they are semilinear with respect to the corresponding Adams operations on $K_0(R\#\Gamma)$, and, finally, that they commute with the Cartan homomorphism $c : K_q(R\#\Gamma) \to K_q(\Gamma, R)$, *if the order of Γ is invertible in R.*
In particular, using a prime decomposition of k, we can define an Adams operation ψ^k on $K_q(R\#\Gamma)$, $q \geq 1$, for each $k \in \mathbb{N}$ which is invertible in R.

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