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RESEARCH ARTICLE

Homotopy types of $\mathrm{Spin}^c(n)$-gauge groups over S^4

Simon Rea[1]

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Abstract

The gauge group of a principal G-bundle P over a space X is the group of G-equivariant homeomorphisms of P that cover the identity on X. We consider the gauge groups of bundles over S^4 with $\mathrm{Spin}^c(n)$, the complex spin group, as structure group and show how the study of their homotopy types reduces to that of $\mathrm{Spin}(n)$-gauge groups over S^4. We then advance on what is known by providing a partial classification for $\mathrm{Spin}(7)$- and $\mathrm{Spin}(8)$-gauge groups over S^4.

Keywords Gauge groups · Homotopy types · Spin groups

Mathematics Subject Classification 55P15 · 55Q05

1 Introduction

Let G be a topological group and X a space. The *gauge group* $\mathcal{G}(P)$ of a principal G-bundle P over X is defined as the group of G-equivariant bundle automorphisms of P which cover the identity on X. A detailed introduction to the topology of gauge groups of bundles can be found in [24, 42]. The study of gauge groups is important for the classification of principal bundles, as well as understanding moduli spaces of connections on principal bundles [7, 50, 52]. As is well known, Donaldson [12] discovered a deep link between the gauge groups of certain $\mathrm{SU}(2)$-bundles and the differential topology of 4-manifolds.

Key properties of gauge groups are invariant under continuous deformation and so studying their homotopy theory is important. Having fixed a topological group G and

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✉ Simon Rea
s.rea.hw@gmail.com

1 Department of Mathematical Sciences, University of Southampton, Highfield Campus, Southampton SO17 1BJ, UK

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a space X, an interesting problem is that of classifying the possible homotopy types of the gauge groups $\mathcal{G}(P)$ of principal G-bundles P over X.

Crabb and Sutherland showed [8, Theorem 1.1] that if G is a compact, connected, Lie group and X is a connected, finite CW-complex, then the number of distinct homotopy types of $\mathcal{G}(P)$, as $P \to X$ ranges over all principal G-bundles over X, is finite. In fact, since isomorphic G-bundles give rise to homeomorphic gauge groups, it will suffice to the let $P \to X$ range over the set of isomorphism classes of principal G-bundles over X.

Explicit classification results have been obtained, especially for the case of gauge groups of bundles with low rank, compact, Lie groups as structure groups and $X = S^4$ as base space. In particular, the first such result was obtained by Kono [31] in 1991. Using the fact that isomorphism classes of principal $\mathrm{SU}(2)$-bundles over S^4 are classified by $k \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(2))$ and denoting by $\mathcal{G}_k$ the gauge group of the principal $\mathrm{SU}(2)$-bundle $P_k \to S^4$ corresponding to the integer k, Kono showed that there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_l$ if and only if $(12, k) = (12, l)$, where (m, n) denotes the greatest common divisor of m and n. Since 12 has six divisors, it follows that there are precisely six homotopy types of $\mathrm{SU}(2)$-gauge groups over S^4.

Results formally similar to that of Kono have been obtained for principal bundles over S^4 with different structure groups, among others, by: Hamanaka and Kono [17] for $\mathrm{SU}(3)$-gauge groups; Theriault [53, 54] for $\mathrm{SU}(n)$-gauge groups, as well as [48] for $\mathrm{Sp}(2)$-gauge groups; Cutler [9, 10] for $\mathrm{Sp}(3)$-gauge groups and $\mathrm{U}(n)$-gauge groups; Kishimoto and Kono [27] for $\mathrm{Sp}(n)$-gauge groups; Kishimoto, Theriault and Tsutaya [30] for G_2-gauge groups; Kamiyama, Kishimoto, Kono and Tsukuda [25] for $\mathrm{SO}(3)$-gauge groups; Kishimoto, Membrillo-Solis and Theriault [29] for $\mathrm{SO}(4)$-gauge groups; Hasui, Kishimoto, Kono and Sato [20] for $\mathrm{PU}(3)$- and $\mathrm{PSp}(2)$-gauge groups; and Hasui, Kishimoto, So and Theriault [21] for bundles with exceptional Lie groups as structure groups.

There are also several classification results for gauge groups of bundles with base spaces other than S^4 [2, 16, 18, 20, 22, 23, 28, 32, 33, 35, 36, 39–41, 43, 45, 46, 51, 55, 57].

The complex spin group $\mathrm{Spin}^c(n)$ was first introduced in 1964 in a paper of Atiyah, Bott and Shapiro [4]. There has been an increasing interest in the $\mathrm{Spin}^c(n)$ groups ever since the publication of the Seiberg–Witten equations for 4-manifolds [58], whose formulation requires the existence of $\mathrm{Spin}^c(n)$-structures, and more recently for the role they play in string theory [6, 13, 44].

In this paper we examine $\mathrm{Spin}^c(n)$-gauge groups over S^4. We begin by recalling some basic properties of the complex spin group $\mathrm{Spin}^c(n)$ and showing that it can be expressed as a product of a circle and the real spin group $\mathrm{Spin}(n)$. For $n \geqslant 6$, we show that this decomposition is reflected in the corresponding gauge groups.

Theorem 1.1 *For $n \geqslant 6$ and any $k \in \mathbb{Z}$, we have*

$$\mathcal{G}_k(\mathrm{Spin}^c(n)) \simeq S^1 \times \mathcal{G}_k(\mathrm{Spin}(n)).$$

The homotopy theory of $\mathrm{Spin}^c(n)$-gauge groups over S^4 therefore reduces to that of the corresponding $\mathrm{Spin}(n)$-gauge groups. We advance on what is known on $\mathrm{Spin}(n)$-

gauge groups by providing a partial classification for Spin(7)- and Spin(8)-gauge groups over S^4.

Theorem 1.2 (a) *If* $(168, k) = (168, l)$*, there is a homotopy equivalence*

$$\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$$

after localising rationally or at any prime.

(b) *If* $\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$ *then* $(84, k) = (84, l)$.

We note that the discrepancy by a factor of 2 between parts (a) and (b) is due to the same discrepancy for G_2-gauge groups.

Theorem 1.3 (a) *If* $(168, k) = (168, l)$*, there is a homotopy equivalence*

$$\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$$

after localising rationally or at any prime.

(b) *If* $\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$ *then* $(28, k) = (28, l)$.

For the Spin(8) case, in addition to the same 2-primary indeterminacy appearing in the Spin(7) case, there are also known [26, 49] difficulties at the prime 3 due to the non-vanishing of $\pi_{10}(\mathrm{Spin}(8))_{(3)}$.

2 $\mathrm{Spin}^c(n)$ groups

For $n \geqslant 1$, the complex spin group $\mathrm{Spin}^c(n)$ is defined as the quotient

$$\frac{\mathrm{Spin}(n) \times \mathrm{U}(1)}{\mathbb{Z}/2\mathbb{Z}}$$

where $\mathbb{Z}/2\mathbb{Z} \cong \{(1, 1), (-1, -1)\} \subseteq \mathrm{Spin}(n) \times \mathrm{U}(1)$ denotes the central subgroup of order 2. The group $\mathrm{Spin}^c(n)$ is a special case of the more general notion of $\mathrm{Spin}^k(n)$ group introduced in [1].

The first low rank $\mathrm{Spin}^c(n)$ groups can be identified as follows:

- $\mathrm{Spin}^c(1) \cong \mathrm{U}(1) \simeq S^1$;
- $\mathrm{Spin}^c(2) \cong \mathrm{U}(1) \times \mathrm{U}(1) \simeq S^1 \times S^1$;
- $\mathrm{Spin}^c(3) \cong \mathrm{U}(2) \simeq S^1 \times S^3$;
- $\mathrm{Spin}^c(4) \cong \{(A, B) \in \mathrm{U}(2) \times \mathrm{U}(2) \mid \det A = \det B\}$.

The group $\mathrm{Spin}^c(n)$ fits into a commutative diagram

$$\begin{tikzcd}
\{\pm 1\} \arrow[d, hook] & \{(1,1),(-1,-1)\} \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook] & \{\pm 1\} \arrow[d, hook] \\
\mathrm{Spin}(n) \arrow[d, "\lambda"] & \mathrm{Spin}(n) \times S^1 \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "q"] & S^1 \arrow[d, "2"] \\
\mathrm{SO}(n) & \mathrm{Spin}^c(n) \arrow[l] \arrow[r] & S^1,
\end{tikzcd}$$

where q is the quotient map, $\lambda\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ denotes the double covering map of the group $\mathrm{SO}(n)$ by $\mathrm{Spin}(n)$ and $2\colon S^1 \to S^1$ denotes the degree 2 map. Furthermore, we observe that the map

$$\lambda \times 2\colon \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times S^1$$

is a double covering of $\mathrm{SO}(n) \times S^1$ by $\mathrm{Spin}^c(n)$.

3 Method of classification

A principal bundle isomorphism determines a homeomorphism of gauge groups induced by conjugation [42]. We therefore begin by considering isomorphism classes of principal $\mathrm{Spin}^c(n)$-bundles over S^4. These are classified by the free homotopy classes of maps $S^4 \to \mathrm{BSpin}^c(n)$. Since $\mathrm{Spin}^c(n)$ is connected, $\mathrm{BSpin}^c(n)$ is simply-connected and hence there are isomorphisms

$$[S^4, \mathrm{BSpin}^c(n)]_{\mathrm{free}} \cong \pi_3(\mathrm{Spin}^c(n)) \cong \pi_3(\mathrm{SO}(n)) \cong \begin{cases} 0 & n = 1, 2, \\ \mathbb{Z}^2 & n = 4, \\ \mathbb{Z} & n = 3, n \geqslant 5. \end{cases}$$

Remark 3.1 Note that for $n = 3$ we have $\mathrm{Spin}^c(3) \cong \mathrm{U}(2)$, and the homotopy types of $\mathrm{U}(2)$-gauge groups over S^4 have been studied by Cutler in [10].

For $n \geqslant 5$, let $\mathcal{G}_k$ denote the gauge group of the $\mathrm{Spin}^c(n)$-bundle $P_k \to S^4$ classified by $k \in \mathbb{Z}$. By [3, 15], there is a homotopy equivalence

$$\mathrm{B}\mathcal{G}_k \simeq \mathrm{Map}_k(S^4, \mathrm{BSpin}^c(n)),$$

the latter space being the k-th component of $\mathrm{Map}(S^4, \mathrm{BSpin}^c(n))$, meaning the connected component containing the map classifying $P_k \to S^4$.

There is an evaluation fibration

$$\mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)) \longrightarrow \mathrm{Map}_k(S^4, \mathrm{BSpin}^c(n)) \xrightarrow{\mathrm{ev}} \mathrm{BSpin}^c(n),$$

where ev evaluates a map at the basepoint of S^4 and the fibre is the k-th component of the pointed mapping space $\mathrm{Map}^*(S^4, \mathrm{BSpin}^c(n))$. This fibration extends to a homotopy fibration sequence

$$\mathcal{G}_k \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)) \longrightarrow \mathrm{B}\mathcal{G}_k \longrightarrow \mathrm{BSpin}^c(n).$$

Furthermore, by [47] there is, for each $k \in \mathbb{Z}$, a homotopy equivalence

$$\mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)) \simeq \mathrm{Map}_0^*(S^4, \mathrm{BSpin}^c(n)).$$

The space on the right-hand side is homotopy equivalent to $\mathrm{Map}_0^*(S^3, \mathrm{Spin}^c(n))$ by the pointed exponential law, and is more commonly denoted as $\Omega_0^3\mathrm{Spin}^c(n)$. We therefore have the following homotopy fibration sequence:

$$\mathcal{G}_k \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\partial_k} \Omega_0^3\mathrm{Spin}^c(n) \longrightarrow B\mathcal{G}_k \longrightarrow B\mathrm{Spin}^c(n),$$

which exhibits the gauge group $\mathcal{G}_k$ as the homotopy fibre of the map ∂_k. This is a key observation, as it implies that the homotopy theory of the gauge groups $\mathcal{G}_k$ depends on the maps ∂_k.

Lemma 3.2 (Lang [34, Theorem 2.6]) *The adjoint of* $\partial_k \colon \mathrm{Spin}^c(n) \to \Omega_0^3\mathrm{Spin}^c(n)$ *is homotopic to the Samelson product* $\langle k\epsilon, 1\rangle \colon S^3 \wedge \mathrm{Spin}^c(n) \to \mathrm{Spin}^c(n)$*, where* $\epsilon \in \pi_3(\mathrm{Spin}^c(n))$ *is a generator and 1 denotes the identity map on* $\mathrm{Spin}^c(n)$.

As the Samelson product is bilinear, we have $\langle k\epsilon, 1\rangle \simeq k\langle \epsilon, 1\rangle$, and hence, taking adjoints once more, $\partial_k \simeq k\partial_1$.

Lemma 3.3 (Theriault [48, Lemma 3.1]) *Let X be a connected CW-complex and let Y be an H-space with a homotopy inverse. Suppose that $f \in [X, Y]$ has finite order and let $m \in \mathbb{N}$ be such that $mf \simeq *$. Then, for any integers $k, l \in \mathbb{Z}$ such that $(m, k) = (m, l)$, the homotopy fibres of kf and lf are homotopy equivalent when localised rationally or at any prime.*

Remark 3.4 The lemma of Theriault is the local analogue of a lemma used by Hamanaka and Kono in their study [17] of $SU(3)$-gauge groups over S^4.

Part (a) of Theorems 1.2 and 1.3 will follow as applications of Lemma 3.3, whereas for part (b) we will need to determine suitable homotopy invariants of the gauge groups.

4 $\mathrm{Spin}^c(n)$-gauge groups

We begin with a decomposition of $\mathrm{Spin}^c(n)$ as a product of spaces which will be reflected in an analogous decomposition of $\mathrm{Spin}^c(n)$-gauge groups.

From the definition of $\mathrm{Spin}^c(n)$, we can construct the commutative diagram

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, hook] \arrow[d, equal] & \mathrm{Spin}(n) \times S^1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "q"] & S^1 \arrow[d, "2"] \\ \mathrm{Spin}(n) \arrow[r] & \mathrm{Spin}^c(n) \arrow[r] & S^1. \end{tikzcd}$$

There is, therefore, an exact sequence

$$1 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{Spin}^c(n) \longrightarrow S^1 \longrightarrow 1,$$

and hence a fibration

$$\mathrm{Spin}(n) \longrightarrow \mathrm{Spin}^c(n) \longrightarrow S^1. \qquad (\star)$$

A section for (⋆) can be obtained as follows:

$$\begin{tikzcd} S^1 \arrow[r, "\Delta"] \arrow[d, "2"] & S^1 \times S^1 \arrow[r, hook] & \mathrm{Spin}(n) \times S^1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "q"] & S^1 \arrow[d, "2"] \\ S^1 \arrow[rr] & & \mathrm{Spin}^c(n) \arrow[r] & S^1. \end{tikzcd}$$

Hence (⋆) splits, and we have a homeomorphism

$$\mathrm{Spin}^c(n) \cong \mathrm{Spin}(n) \times S^1.$$

We are now ready to show that the decomposition

$$\mathcal{G}_k(\mathrm{Spin}^c(n)) \simeq S^1 \times \mathcal{G}_k(\mathrm{Spin}(n))$$

for $n \geqslant 6$ holds as stated in Theorem 1.1.

Proof of Theorem 1.1 Let ϱ and g denote the maps in the fibration

$$\mathrm{Spin}(n) \xrightarrow{\varrho} \mathrm{Spin}^c(n) \xrightarrow{g} S^1,$$

and let $s\colon S^1 \to \mathrm{Spin}^c(n)$ denote a section of g.

As $\pi_4(\mathrm{Spin}^c(n)) \cong 0$ for $n \geqslant 6$, there is a lift in the diagram

$$\begin{tikzcd} & S^1 \arrow[dl, dashed, "a"'] \arrow[d, "s"] \arrow[dr, "*"] & \\ \mathcal{G}_k(\mathrm{Spin}^c(n)) \arrow[r] & \mathrm{Spin}^c(n) \arrow[r, "\partial_k"] & \Omega^3_0\mathrm{Spin}^c(n). \end{tikzcd}$$

Define the map b to be the composite

$$\mathcal{G}_k(\mathrm{Spin}^c(n)) \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{g} S^1.$$

Since, in particular, s is a right homotopy inverse for g, the map a is a right homotopy inverse for b. Therefore we have $\mathcal{G}_k(\mathrm{Spin}^c(n)) \simeq S^1 \times F_b$, where F_b denotes the homotopy fibre of b.

As the map $\varrho\colon \mathrm{Spin}(n) \to \mathrm{Spin}^c(n)$ is a group homomorphism, it classifies to a map

$$\mathrm{B}\varrho\colon \mathrm{BSpin}(n) \to \mathrm{BSpin}^c(n).$$

Since ϱ induces an isomorphism in π_3, it respects path-components in $\mathrm{Map}_k(S^4, -)$ and $\mathrm{Map}_k^*(S^4, -)$ for any $k \in \mathbb{Z}$. We therefore have a diagram of fibration sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Map}_k^*(S^4, \mathrm{BSpin}(n)) \arrow[r] \arrow[d, "(\mathrm{B}\varrho)_*"] & \mathrm{Map}_k(S^4, \mathrm{BSpin}(n)) \arrow[r] \arrow[d, "(\mathrm{B}\varrho)_*"] & \mathrm{BSpin}(n) \arrow[d, "\mathrm{B}\varrho"] \\
\cdots \arrow[r] & \mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)) \arrow[r] & \mathrm{Map}_k(S^4, \mathrm{BSpin}^c(n)) \arrow[r] & \mathrm{BSpin}^c(n).
\end{tikzcd} \tag{1}$$

Furthermore, observe that for all $k \in \mathbb{Z}$ we have

$$\pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}(n))) \cong \pi_m(\Omega_0^3 \mathrm{Spin}(n)) \cong \pi_{m+3}(\mathrm{Spin}(n))$$

and, similarly, $\pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n))) \cong \pi_{m+3}(\mathrm{Spin}^c(n))$. Since ϱ induces isomorphisms on π_m for $m \geqslant 2$, it follows that $(\mathrm{B}\varrho)_*$ induces isomorphisms

$$\pi_m((\mathrm{B}\varrho)_*) \colon \pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}(n))) \xrightarrow{\cong} \pi_m(\mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)))$$

for all m and is therefore a homotopy equivalence by Whitehead's theorem.

We can extend the fibration diagram (1) to the left as

$$\begin{tikzcd}
\mathcal{G}_k(\mathrm{Spin}(n)) \arrow[r] \arrow[d, "\mathcal{G}_k(\varrho)"] & \mathrm{Spin}(n) \arrow[r, "\partial_k'"] \arrow[d, "\varrho"] & \mathrm{Map}_k^*(S^4, \mathrm{BSpin}(n)) \arrow[r] \arrow[d, "(\mathrm{B}\varrho)_*", "\simeq"'] & \cdots \\
\mathcal{G}_k(\mathrm{Spin}^c(n)) \arrow[r] & \mathrm{Spin}^c(n) \arrow[r, "\partial_k"] & \mathrm{Map}_k^*(S^4, \mathrm{BSpin}^c(n)) \arrow[r] & \cdots
\end{tikzcd}$$

where ∂_k' denotes the boundary map associated to $\mathrm{Spin}(n)$-gauge groups over S^4.

Since $(\mathrm{B}\varrho)_*$ is a homotopy equivalence, the leftmost square is a homotopy pullback. Since we know that there is a fibration

$$\mathrm{Spin}(n) \xrightarrow{\varrho} \mathrm{Spin}^c(n) \xrightarrow{g} S^1,$$

it follows that we also have a fibration

$$\mathcal{G}_k(\mathrm{Spin}(n)) \xrightarrow{\mathcal{G}_k(\varrho)} \mathcal{G}_k(\mathrm{Spin}^c(n)) \xrightarrow{b} S^1.$$

In particular, the space $\mathcal{G}_k(\mathrm{Spin}(n))$ is seen to be the homotopy fibre F_b of the map $b \colon \mathcal{G}_k(\mathrm{Spin}^c(n)) \to S^1$ and hence we have

$$\mathcal{G}_k(\mathrm{Spin}^c(n)) \simeq S^1 \times \mathcal{G}_k(\mathrm{Spin}(n)). \qquad \square$$

Remark 4.1 Alternatively, the referee suggested the following approach to a proof of Lemma 1.1. Since the map $\mathrm{B}\varrho$ induces an isomorphism $[S^4, \mathrm{BSpin}(n)] \cong [S^4, \mathrm{BSpin}^c(n)]$, to any principal $\mathrm{Spin}^c(n)$-bundle P over S^4 we can associate a principal Spin(n)-bundle P' over S^4 such that $P \cong P' \times_{\mathrm{Spin}(n)} \mathrm{Spin}^c(n)$. There exists then a fibrewise exact sequence of adjoint bundles

$$\begin{aligned} P' \times_{\mathrm{Ad}} \mathrm{Spin}(n) \longrightarrow P' \times_{\mathrm{Ad}} \mathrm{Spin}^c(n) \cong P \times_{\mathrm{Ad}} \mathrm{Spin}^c(n) \\ \longrightarrow P' \times_{\mathrm{Ad}} S^1 = S^4 \times S^1. \end{aligned}$$

Recalling [3, Section 2] that gauge groups can be defined as spaces of sections of adjoint bundles, we obtain a diagram of fibration sequences

$$\begin{tikzcd} \mathcal{G}(P') \arrow[r] \arrow[d, "\mathrm{ev}"] & \mathcal{G}(P) \arrow[r] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(S^4, S^1) \arrow[d, "\mathrm{ev}", "\simeq"'] \\ \mathrm{Spin}(n) \arrow[r] & \mathrm{Spin}^c(n) \arrow[r] & S^1. \end{tikzcd}$$

Showing, as we have done, that $\mathcal{G}(P) \to S^1$ admits a homotopy section then leads to the statement of Lemma 1.1.

In light of Theorem 1.1, the homotopy theory of Spin$^c(n)$-gauge groups over S^4 for $n \geqslant 6$ is completely determined by that of Spin(n)-gauge groups over S^4.

Remark 4.2 By a result of Cutler [10], there is a decomposition

$$\mathcal{G}_k(\mathrm{U}(2)) \simeq S^1 \times \mathcal{G}_k(\mathrm{SU}(2))$$

of U(2)-gauge groups over S^4 whenever k is even. Given that $\mathrm{Spin}^c(3) \cong \mathrm{U}(2)$ and $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, the statement of Theorem 1.1 still holds true when $n = 2$ provided that k is even. Cutler also shows that $\mathcal{G}_k(\mathrm{U}(2)) \simeq S^1 \times \mathcal{G}_k(\mathrm{PU}(2))$ for odd k, so Theorem 1.1 does not hold for $n = 2$.

5 Spin(n)-gauge groups

We now shift our focus to principal Spin(n)-bundles over S^4 and the classification of their gauge groups. In the interest of completeness, we recall that, for $n \leqslant 6$, the following exceptional isomorphisms hold.

The cases $n = 1, 2$ are trivial. Indeed, as $\pi_3(\mathrm{O}(1)) \cong \pi_3(\mathrm{U}(1)) \cong 0$, there is only one isomorphism class of O(1)- and U(1)-bundles over S^4 (namely, that of the trivial bundle), and hence there is only one possible homotopy type for the corresponding gauge groups. The case $n = 3$ was studied by Kono in [31]. The case $n = 4$ can be reduced to the $n = 3$ case by [5, Theorem 5]. The case $n = 5$ was studied by Theriault in [48]. Finally, the case $n = 6$ was studied by Cutler and Theriault in [11].

We shall now explore the $n = 7$ case. Recall that we have a fibration sequence

$$\mathcal{G}_k(\mathrm{Spin}(7)) \longrightarrow \mathrm{Spin}(7) \xrightarrow{k\partial_1} \Omega_0^3\mathrm{Spin}(7).$$

Table 1 The exceptional isomorphisms

n	$\mathrm{Spin}(n)$
1	$O(1)$
2	$U(1)$
3	$SU(2)$
4	$SU(2) \times SU(2)$
5	$Sp(2)$
6	$SU(4)$

Lemma 5.1 *Localised away from the prime 2, the boundary map*

$$\mathrm{Spin}(7) \xrightarrow{\partial_1} \Omega_0^3\mathrm{Spin}(7)$$

has order 21.

Proof Harris [19] showed that $\mathrm{Spin}(2m+1) \simeq_{(p)} \mathrm{Sp}(m)$ for odd primes p. This result was later improved by Friedlander [14] to a p-local homotopy equivalence of the corresponding classifying spaces. Then, in particular, localising at an odd prime p, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spin}(7) \arrow[r, "\partial_1"] \arrow[d, "\simeq"] & \Omega_0^3\mathrm{Spin}(7) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_1(S^4, B\mathrm{Spin}(7)) \arrow[r] \arrow[d, "\simeq"] & B\mathrm{Spin}(7) \arrow[d, "\simeq"] \\
\mathrm{Sp}(3) \arrow[r, "\partial_1'"] & \Omega_0^3\mathrm{Sp}(3) \arrow[r] & \mathrm{Map}_1(S^4, B\mathrm{Sp}(3)) \arrow[r] & B\mathrm{Sp}(3)
\end{tikzcd}$$

where $\partial_1' \colon \mathrm{Sp}(3) \to \Omega_0^3\mathrm{Sp}(3)$ denotes the boundary map associated to $\mathrm{Sp}(3)$-gauge groups over S^4 studied in [9]. Hence the result follows from the calculation in [9, Theorem 1.2] where it is shown that ∂_1' has order 21 after localising away from the prime 2. $\square$

Lemma 5.2 *Let $F \to X \to Y$ be a homotopy fibration, where F is an H-space, and let $\partial \colon \Omega Y \to F$ be the homotopy fibration connecting map. Let $\alpha \colon A \to \Omega Y$ and $\beta \colon B \to \Omega Y$ be maps such that*

(1) *$\mu \circ (\alpha \times \beta) \colon A \times B \to \Omega Y$ is a homotopy equivalence, where μ is the loop multiplication on ΩY;*
(2) *$\partial \circ \beta \colon B \to F$ is nullhomotopic.*

Then the orders of ∂ and $\partial \circ \alpha$ coincide.

Proof Let $\theta\colon \Omega Y \times F \to F$ denote the canonical homotopy action of the loopspace ΩY onto the homotopy fibre F, and let $e = \mu \circ (\alpha \times \beta)$. Consider the diagram

$$\begin{tikzcd}
 & A \times B \arrow[dl, "\mathrm{pr}_1"'] \arrow[d, "\alpha\times\beta"] \arrow[dr, "e"] & \\
A \arrow[d, "\alpha"] & \Omega Y \times \Omega Y \arrow[r, "\mu"] \arrow[d, "\mathrm{id}\times\partial"] & \Omega Y \arrow[d, "\partial"] \\
\Omega Y \arrow[r, hook] \arrow[rr, bend right, "\partial"'] & \Omega Y \times F \arrow[r, "\theta"] & F.
\end{tikzcd}$$

The left portion of the diagram commutes by the assumption that $\partial \circ \beta \simeq *$, while the right and bottom portions commute by properties of the canonical action θ. Therefore

$$\partial \simeq \partial \circ \alpha \circ \mathrm{pr}_1 \circ e^{-1},$$

and hence the orders of ∂ and $\partial \circ \alpha$ coincide. □

Lemma 5.3 *Localised at the prime* 2*, the order of the boundary map*

$$\mathrm{Spin}(7) \xrightarrow{\partial_1} \Omega_0^3 \mathrm{Spin}(7)$$

is at most 8.

Proof The strategy here will be to show that ∂_8 is nullhomotopic. This will suffice as we have $\partial_8 \simeq 8\partial_1$ by Lemma 3.2.

By a result of Mimura [37, Proposition 9.1], the fibration

$$G_2 \xrightarrow{\alpha} \mathrm{Spin}(7) \longrightarrow S^7$$

splits at the prime 2. Let $\beta\colon S^7 \to \mathrm{Spin}(7)$ denote a right homotopy inverse for $\mathrm{Spin}(7) \to S^7$. Then the composite

$$G_2 \times S^7 \xrightarrow{\alpha\times\beta} \mathrm{Spin}(7) \times \mathrm{Spin}(7) \xrightarrow{\mu} \mathrm{Spin}(7)$$

is a 2-local homotopy equivalence.

Observe that we have $\partial_8 \circ \beta \simeq *$ since $\pi_{10}(\mathrm{Spin}(7)) \cong \mathbb{Z}/8\mathbb{Z}$ and $\partial_8 \circ \beta \simeq 8\partial_1 \circ \beta$. Therefore, by Lemma 5.2, the order of ∂_8 equals the order of $\partial_8 \circ \alpha$.

As α is a group homomorphism, there is a diagram of evaluation fibrations

$$\begin{tikzcd}
G_2 \arrow[r, "\partial_8'"] \arrow[d, "\alpha"] & \Omega_0^3 G_2 \arrow[d, "\Omega^3\alpha"] \\
\mathrm{Spin}(7) \arrow[r, "\partial_8"] & \Omega_0^3 \mathrm{Spin}(7).
\end{tikzcd}$$

Since $\partial_8' \simeq 8\partial_1' \simeq *$ by [30, Theorem 1.1], we must have $\partial_8 \simeq 8\partial_1 \simeq *$. □

Proof of Theorem 1.2 (a) Lemmas 5.1 and 5.3 imply that $168\partial_1 \simeq *$, so the result follows from Lemma 3.3. □

We now move on to consider Spin(8)-gauge groups.

Lemma 5.4 *Localised at the prime* 2 (*resp.* 3), *the order of the boundary map*

$$\mathrm{Spin}(8) \xrightarrow{\partial_1} \Omega_0^3\mathrm{Spin}(8)$$

is at most 8 (*resp.* 3).

Proof There is a fibration

$$\mathrm{Spin}(7) \xrightarrow{\alpha} \mathrm{Spin}(8) \longrightarrow S^7$$

which admits a section $\beta\colon S^7 \to \mathrm{Spin}(8)$, and hence splits integrally. Therefore, we have a homeomorphism

$$\mathrm{Spin}(7) \times S^7 \xrightarrow{\alpha\times\beta} \mathrm{Spin}(8) \times \mathrm{Spin}(8) \xrightarrow{\mu} \mathrm{Spin}(8).$$

Integrally, we have

$$\pi_{10}(\mathrm{Spin}(8)) \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z},$$

(see, e.g. the table in [38]). Hence the same argument presented in the proof of Lemma 5.3 shows that $8\partial_1 \simeq *$ and $3\partial_1 \simeq *$ after localising at $p = 2$ and $p = 3$, respectively. □

Lemma 5.5 *Let $p \neq 3$ be an odd prime. Then the p-primary order of the boundary map $\partial_1\colon \mathrm{Spin}(8) \to \Omega_0^3\mathrm{Spin}(8)$ is bounded from above by that of $\partial_1\colon \mathrm{Spin}(7) \to \Omega_0^3\mathrm{Spin}(7)$.*

Proof As $\pi_{10}(\mathrm{Spin}(8)) \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, any map $S^7 \to \mathrm{Spin}(8)$ is nullhomotopic after localisation at an odd prime p different from 3. Thus, decomposing Spin(8) as $\mathrm{Spin}(7) \times S^7$ and arguing as in the proof of Lemma 5.3 yields the statement. □

Proof of Theorem 1.3 (a) Lemmas 5.4 and 5.5 imply that $168\partial_1 \simeq *$, so the result follows from Lemma 3.3. □

6 Homotopy invariants of Spin(*n*)-gauge groups

Lemma 6.1 *If* $\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$, *then* $(21, k) = (21, l)$.

Proof As in the proof of Lemma 5.1, localising at an odd prime, we have an equivalence $\mathrm{BSpin}(7) \simeq_{(p)} \mathrm{BSp}(3)$. We therefore have a diagram of homotopy fibrations

$$\begin{array}{ccccccc}
\mathrm{Spin}(7) & \xrightarrow{\partial_k} & \Omega_0^3\mathrm{Spin}(7) & \longrightarrow & \mathrm{B}\mathcal{G}_k(\mathrm{Spin}(7)) & \longrightarrow & \mathrm{BSpin}(7) \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow & & \downarrow\simeq \\
\mathrm{Sp}(3) & \xrightarrow{\partial_k'} & \Omega_0^3\mathrm{Sp}(3) & \longrightarrow & \mathrm{B}\mathcal{G}_k(\mathrm{Sp}(3)) & \longrightarrow & \mathrm{BSp}(3)
\end{array}$$

where $\partial_k' \colon \mathrm{Sp}(3) \to \Omega_0^3\mathrm{Sp}(3)$ denotes the boundary map studied in [9]. Thus, by the five lemma, we have

$$\pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Spin}(7))) \cong \pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Sp}(3))).$$

Hence the result now follows from the calculations in [9, Theorem 1.1] where it is shown that, integrally,

$$\pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Sp}(3))) \cong \mathbb{Z}/120(84, k)\mathbb{Z}. \qquad \square$$

In their study of the homotopy types of G_2-gauge groups over S^4 in [30], Kishimoto, Theriault and Tsutaya constructed a space C_k for which

$$H^*(C_k) \cong H^*(\mathcal{G}_k(G_2))$$

in mod 2 cohomology in dimensions 1 through 6. The cohomology of C_k is then shown to be as follows.

Lemma 6.2 ([30, Lemma 8.3]) *We have*

- *if* $(4, k) = 1$ *then* $C_k \simeq S^3$*, so* $H^*(C_k) \cong H^*(S^3)$*;*
- *if* $(4, k) = 2$ *or* $(4, k) = 4$ *then* $H^*(C_k) \cong H^*(S^3) \oplus H^*(P^5(2)) \oplus H^*(P^6(2))$*, where* $P^n(p)$ *denotes the nth dimensional mod p Moore space;*
- *if* $(4, k) = 2$ *then* Sq^2 *is non-trivial on the degree* 4 *generator in* $H^*(C_k)$*;*
- *if* $(4, k) = 4$ *then* Sq^2 *is trivial on the degree* 4 *generator in* $H^*(C_k)$*.*

We make use of the same spaces C_k as follows.

Lemma 6.3 *If* $\mathcal{G}_k(\mathrm{Spin}(7)) \simeq \mathcal{G}_l(\mathrm{Spin}(7))$*, then we have* $(4, k) = (4, l)$.

Proof As in the proof of Lemma 5.3, recall that we have a 2-local homotopy equivalence

$$G_2 \times S^7 \xrightarrow{\alpha\times\beta} \mathrm{Spin}(7) \times \mathrm{Spin}(7) \xrightarrow{\mu} \mathrm{Spin}(7).$$

Since the map $\alpha\colon G_2 \to \mathrm{Spin}(7)$ is a homomorphism, we have a commutative diagram

$$\begin{tikzcd} G_2 \arrow[r, "\partial_1'"] \arrow[d, "\alpha"] & \Omega_0^3 G_2 \arrow[d, "\Omega^3\alpha"] \\ \mathrm{Spin}(7) \arrow[r, "\partial_1"] & \Omega_0^3\mathrm{Spin}(7). \end{tikzcd}$$

Furthermore, as $\pi_7(\Omega_0^3 G_2) \cong \pi_{10}(G_2) \cong 0$, we have

$$\pi_7(\Omega_0^3\mathrm{Spin}(7)) \cong \pi_7(\Omega_0^3 G_2) \oplus \pi_7(\Omega^3 S^7) \cong \pi_7(\Omega^3 S^7),$$

and thus there is a commutative diagram

$$\begin{tikzcd} S^7 \arrow[r, "\gamma"] \arrow[d, "\beta"] & \Omega^3 S^7 \arrow[d, "\Omega^3\beta"] \\ \mathrm{Spin}(7) \arrow[r, "\partial_1"] & \Omega_0^3\mathrm{Spin}(7) \end{tikzcd}$$

for some γ representing a class in $\pi_7(\Omega^3 S^7) \cong \pi_{10}(S^7) \cong \mathbb{Z}/8\mathbb{Z}$.

We therefore have a commutative diagram

$$\begin{tikzcd} G_2 \vee S^7 \arrow[r, "k\partial_1' \vee k\gamma"] \arrow[d, "\alpha\vee\beta"] & \Omega_0^3 G_2 \times \Omega^3 S^7 \arrow[d, "\Omega^3\alpha\times\Omega^3\beta", "\simeq"'] \\ \mathrm{Spin}(7) \arrow[r, "k\partial_1"] & \Omega_0^3\mathrm{Spin}(7) \end{tikzcd}$$

which induces a map of fibres $\phi\colon M \to \mathcal{G}_k(\mathrm{Spin}(7))$, where M denotes the homotopy fibre of the map $k\partial_1' \vee k\gamma$.

Since the lowest dimensional cell in $G_2 \times S^7/(G_2 \vee S^7)$ appears in dimension 10, the canonical map $G_2 \vee S^7 \to G_2 \times S^7$ is a homotopy equivalence in dimensions less than 9. It thus follows that M is homotopy equivalent to the homotopy fibre of $k\partial_1' \times k\gamma$ in dimensions up to 8. Since the homotopy fibre of $k\partial_1' \times k\gamma$ is just the product $\mathcal{G}_k(G_2) \times F_k$, the zig-zag of maps

$$C_k \times F_k \longrightarrow \mathcal{G}_k(G_2) \times F_k \longleftarrow M \xrightarrow{\phi} \mathcal{G}_k(\mathrm{Spin}(7))$$

induces isomorphisms in mod-2 cohomology in dimensions 1 through 6, and therefore we have

$$H^*(\mathcal{G}_k(\mathrm{Spin}(7))) \cong H^*(C_k) \otimes H^*(F_k), \quad * \leqslant 6.$$

From the fibration sequence

$$\Omega^4 S^7 \longrightarrow F_k \longrightarrow S^7$$

we see that $H^*(F_k) \cong H^*(\Omega^4 S^7)$ in dimensions 1 through 6 for dimensional reasons, and hence we have

$$H^*(F_k) \cong \mathbb{Z}/2\mathbb{Z}[y_3, y_6], \quad * \leqslant 6,$$

where $|y_i| = i$, which, in turn, yields

$$H^*(\mathcal{G}_k(\mathrm{Spin}(7))) \cong H^*(C_k) \otimes \mathbb{Z}/2\mathbb{Z}[y_3, y_6], \quad * \leqslant 6.$$

Since $H^*(F_k)$ does not contribute any generators in degree 4 to $H^*(\mathcal{G}_k(\mathrm{Spin}(7)))$, the result now follows from Lemma 6.2. Indeed, the presence of a degree 4 generator allows us to distinguish between the $(4, k) = 1$ case and the $2 \mid k$ cases, whereas the vanishing of the Steenrod square Sq^2 on the degree 4 generator in $H^*(\mathcal{G}_k(\mathrm{Spin}(7)))$ coming from $H^*(C_k)$ can be used to distinguish between the $(4, k) = 2$ and $(4, k) = 4$ cases. □

Proof of Theorem 1.2 (b) Combine Lemmas 6.1 and 6.3. □

Lemma 6.4 *If* $\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$*, then* $(4, k) = (4, l)$.

Proof As in the proof of Lemma 5.3, the splitting of $G_2 \to \mathrm{Spin}(7) \to S^7$ at the prime 2 implies that there is a 2-local homotopy equivalence

$$\mu \circ (\alpha \times \beta)\colon G_2 \times S^7 \longrightarrow \mathrm{Spin}(7).$$

Since the fibration $\mathrm{Spin}(7) \to \mathrm{Spin}(8) \to S^7$ also splits after localising at any prime, there is a decomposition

$$\mu \circ \big((\iota \circ \alpha) \times (\iota \circ \beta) \times \gamma\big)\colon G_2 \times S^7 \times S^7 \longrightarrow \mathrm{Spin}(8),$$

where $\iota\colon \mathrm{Spin}(7) \to \mathrm{Spin}(8)$ is the inclusion homomorphism and γ is a homotopy inverse for the map $\mathrm{Spin}(8) \to S^7$.

Since the map $\iota \circ \alpha$ is a homomorphism, we have a commutative diagram

$$\begin{tikzcd} G_2 \arrow[r, "\partial_1'"] \arrow[d, "\iota\circ\alpha"] & \Omega^3_0 G_2 \arrow[d, "\Omega^3(\iota\circ\alpha)"] \\ \mathrm{Spin}(8) \arrow[r, "\partial_1"] & \Omega^3_0 \mathrm{Spin}(8). \end{tikzcd}$$

Furthermore, as $\pi_7(\Omega^3_0 G_2) \cong \pi_{10}(G_2) \cong 0$, we have

$$\pi_7(\Omega^3_0 \mathrm{Spin}(8)) \cong \pi_7(\Omega^3 S^7) \oplus \pi_7(\Omega^3 S^7),$$

and thus there are commutative diagrams

$$\begin{array}{ccc} S^7 & \xrightarrow{\delta} & \Omega^3 S^7 \times \Omega^3 S^7 \\ \downarrow{\scriptstyle \iota\circ\beta} & & \downarrow{\scriptstyle \Omega^3(\iota\circ\beta)\times\Omega^3\gamma} \\ \mathrm{Spin}(8) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(8) \end{array} \qquad \begin{array}{ccc} S^7 & \xrightarrow{\delta'} & \Omega^3 S^7 \times \Omega^3 S^7 \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Omega^3(\iota\circ\beta)\times\Omega^3\gamma} \\ \mathrm{Spin}(8) & \xrightarrow{\partial_1} & \Omega^3_0\mathrm{Spin}(8) \end{array}$$

for some δ, δ' representing classes in $\pi_7(\Omega^3 S^7 \times \Omega^3 S^7) \cong (\mathbb{Z}/8\mathbb{Z})^2$. We therefore have a commutative diagram

$$\begin{array}{ccc} G_2 \vee (S^7 \vee S^7) & \xrightarrow{k\partial_1' \vee k(\delta\vee\delta')} & \Omega^3_0 G_2 \times (\Omega^3 S^7 \times \Omega^3 S^7) \\ \downarrow{\scriptstyle \iota\alpha\vee(\iota\beta\vee\gamma)} & & \simeq\downarrow{\scriptstyle \Omega^3\iota\alpha\times(\Omega^3\iota\beta\times\Omega^3\gamma)} \\ \mathrm{Spin}(8) & \xrightarrow{k\partial_1} & \Omega^3_0\mathrm{Spin}(8). \end{array}$$

Arguing as in the proof of Lemma 6.3, we conclude that

$$H^*(\mathcal{G}_k(\mathrm{Spin}(8))) \cong H^*(C_k) \otimes H^*(\Omega^4(S^7 \times S^7)), \quad * \leqslant 6.$$

Observing that $H^*(\Omega^4(S^7 \times S^7))$ does not contribute any generators in degree 4 to $H^*(\mathcal{G}_k(\mathrm{Spin}(8)))$ and arguing as in the proof of Lemma 6.3 yields the statement. □

Lemma 6.5 *If* $\mathcal{G}_k(\mathrm{Spin}(8)) \simeq \mathcal{G}_l(\mathrm{Spin}(8))$, *then* $(7, k) = (7, l)$.

Proof Localising at $p = 7$, we have

$$\mathrm{Spin}(8) \simeq \mathrm{Spin}(7) \times S^7 \simeq G_2 \times S^7 \times S^7.$$

Applying the functor π_{11} and noting that

$$\pi_{10}(S^7) \cong \pi_{11}(S^7) \cong \pi_{14}(S^7) \cong 0$$

(see, e.g. [56]), we find that the evaluation fibration

$$\mathrm{Spin}(8) \xrightarrow{\partial_k} \Omega^3_0\mathrm{Spin}(8) \longrightarrow \mathrm{B}\mathcal{G}_k(\mathrm{Spin}(8)) \longrightarrow \mathrm{BSpin}(8)$$

reduces to the exact sequence

$$\pi_{11}(G_2) \longrightarrow \pi_{11}(\Omega^3_0 G_2) \longrightarrow \pi_{11}(\mathrm{B}\mathcal{G}_k(\mathrm{Spin}(8))) \longrightarrow 0.$$

Hence the result follows from [30]. □

Proof of Theorem 1.3 (b) Combine Lemmas 6.4 and 6.5. □

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