Conditional Interchangeability of Nash Equilibria

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Abstract

The notion of interchangeability was introduced by Nash in one of his original papers on equilibria in strategic games. It has been recently shown that propositional theory of this relation is the same as propositional theories of the nondeducibility relation in the information flow theory, the independence relation in probability theory, and the noninterference relation in concurrency theory.

Propositional theories of conditional nondeducibility and conditional independence have been studied before. This article introduces a notion of conditional interchangeability and gives complete axiomatization of this relation with conditioning by a single player.

1 Introduction

In this article we study properties of conditional interchangeability of Nash equilibria in strategic games. The notion of (non-conditional) equilibria interchangeability first appeared in one of Nash's original papers [16] on equilibria in strategic games. Interchangeability is easiest to define in a two-player game: equilibria in such a game are interchangeable if for any two equilibria $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$, strategy profiles $\langle a_1, b_2\rangle$ and $\langle a_2, b_1\rangle$ are also equilibria. In a multiplayer game setting, we say that a set of players A is interchangeable with a disjoint set of players B if for any two Nash equilibria e_1 and e_2 of the game there is equilibrium e of the same game such that equilibria e and e_1 agree on strategies of players in set A and equilibria e and e_2 agree on strategies of players in set B. We denote this by $A \parallel B$.

As shown by Naumov and Nicholls [17], propositional theory of interchangeability relation can be completely axiomatized by the following axioms:

1. Empty Set: $A \parallel \varnothing$,
2. Symmetry: $A \parallel B \to B \parallel A$,
3. Monotonicity: $A \parallel B, C \to A \parallel B$,

4. Exchange: $A, B \parallel C \to (A \parallel B \to A \parallel B, C)$,

where here and everywhere below A, B stand for the union of the sets A and B. The above axioms 1.-4. were first introduced by Geiger, Paz, and Pearl [4] to describe properties of independence in the probability theory. They have shown that this axiomatic system is complete with respect to the probabilistic semantics.

A property similar to interchangeability, but between two different pieces of *information*, was introduced by Sutherland [24]. In the information flow theory this property became known as *nondeducibility*. Cohen [2] presented a related property called strong dependence. More recently, Halpern and O'Neill [7] introduced a variation of nondeducibility, that they called f-secrecy, to reason about multiparty protocols. Miner More and Naumov [14] generalized nondeducibility to a relation between two *sets* of pieces of information and have shown that it can be completely described by the same axioms 1.-4. The axioms 1.-4. also give a complete axiomatization of a non-interference relation in concurrency theory [15].

The properties of interchangeability are different in the case of *zero-sum* games. It is well-known [16] that the set of all equilibria in any two-player zero-sum game is interchangeable. Naumov and Simonelli [20] described interchangeability properties in multi-player zero-sum games.

Not only is independence a well-studied relation in probability theory, but so also is *conditional* independence. We write $A \parallel_B C$ if sets of random variables A and C are independent conditionally upon B. Attempts to axiomatize conditional independence relation have been made [22]. Studený [23] has shown that conditional independence has no finite complete characterization. Similarly, one can define conditional *nondeducibility* between sets of pieces of information. Unlike their non-conditional counterparts, the propositional properties of conditional independence and conditional nondeducibility are different. Conditional nondeducibility has been studied in database theory, where it became known as *embedded multivalued dependency*. Parker and Parsaye-Ghomi [21] have shown that this relation can not be described by a finite system of inference rules. Herrmann [10, 11] proved the undecidability of the propositional theory of this relation. Lang, Liberatore, and Marquis [13] studied the complexity of conditional nondeducibility between sets of propositional variables. More recently, Grädel and Väänänen discussed incomplete logical systems describing properties of the conditional nondeducibility in the propositional and the first order languages [6] and suggested model checking game semantics for these systems [5]. Naumov and Nicholls [19] gave a complete recursively enumerable axiomatization of conditional nondeducibility.

So far, we have been assuming that sets A and B in relation $A \parallel B$ are disjoint. If this assumption is removed, then an additional axiom

5. Determinicity: $A \parallel B \to (C \parallel C \to A \parallel B, C)$

should be added to the Geiger, Paz, and Pearl system to make it complete with respect to probabilistic, information flow, game theory, and concurrency semantics. The propositional theory in all four of these cases still remains the same.

The situation becomes very different, however, in the case of conditional relations, where, for example, Naumov and Nicholls [19] axiomatization can not be easily generalized to the case when sets A and C are not disjoint in the conditional nondeducibility relation $A \parallel_B C$. The reason for this is that conditional nondeducibility statement $A \parallel_B A$ is equivalent to the statement that A is *functionally dependent* on B. Functional dependency of A on B, that we also denote by $B \rhd A$, is another well-known relation. This relation was shown by Armstrong [1] to be completely described by the following axioms:

1. *Reflexivity*: $A \rhd B$, if $A \supseteq B$,
2. *Augmentation*: $A \rhd B \to A, C \rhd B, C$,
3. *Transitivity*: $A \rhd B \to (B \rhd C \to A \rhd C)$.

The above axioms are known in database literature as Armstrong axioms [3, p. 81]. Thus, any axiomatization of relation $A \parallel_B C$, where sets A and C are not necessarily disjoint, would have to capture properties of both nondeducibility and functional dependency relations, as well as the properties that connect these two relations. The only result known to us in this direction is Kelvey, More, Naumov, and Sapp [12] axiomatic system combining relations $a \parallel b$ and $a \rhd b$ under information flow and probabilistic semantics.

In this article we introduce conditional *interchangeability* relation $A \parallel_B C$ and give a complete axiomatization of this relation when sets A and C are not necessarily disjoint. We restrict our consideration, however, to the case when set B contains exactly one element. We discuss the more general case of an arbitrary finite set B in the conclusion.

There are at least two different ways to define conditional interchangeability. One way is to closely follow the information flow definition and say that $A \parallel^1_B C$ means that for each two Nash equilibria e_1 and e_2 that agree on players in set B, there is a Nash equilibrium e of the same game such that e agrees with e_1 on players in sets A and B and e agrees with e_2 on players in sets B and C. The second way is to say that $A \parallel^2_B C$ means that in any restricted game, where all players in set B publicly commit to some strategies, sets A and C are interchangeable in the unconditional sense.

To illustrate the difference between these two definitions, consider a game between players a, b, and c in which each of the players chooses an integer number. If all three numbers have the same parity, then each of the players is paid one dollar, otherwise nobody is paid. The Nash equilibria of this game are all triples (x, y, z) such that either all three numbers are odd or all three numbers are even. Thus, if (x_1, y, z_1) and (x_2, y, z_2) are two Nash equilibria that agree on strategy of player b, then strategy profile (x_1, y, z_2) is also a Nash equilibria. Hence, $a \parallel^1_b c$.

On the other hand, let player b publicly commit to a strategy $\hat{y}$. Then the game is essentially reduced to a two-player game between players a and c. In this new game, Nash equilibria are all pairs (x, z) such that x and z have the same parity[1]. Thus, no matter what the value of $\hat{y}$ is, strategy profiles $(1, 1)$ and $(0, 0)$ are Nash equilibria and strategy profile $(1, 0)$ is not. Therefore, $a \parallel_b^2 c$ is false.

These two different definitions of conditional interchangeability lead to two different notions of functional dependency. We denote relation $A \parallel_B^1 A$ by $B \rhd^1 A$ and relation $A \parallel_B^2 A$ by $B \rhd^2 A$. The first of these relations satisfies Armstrong axioms. Harjes and Naumov [9] have not only shown the completeness of Armstrong axioms with respect to this semantics, but have also described an extension of Armstrong system for games with a fixed dependency graph for pay-off functions. They also studied the same relation in what they called cellular games [8]. The second dependency relation has been called *rationally functional dependency* by Naumov and Nicholls [18], who gave the following complete axiomatization of this relation:

1. *Reflexivity*: $A \rhd^2 A$,
2. *Right Monotonicity*: $A \rhd^2 B, C \to A \rhd^2 B$,
3. *Union*: $A \rhd^2 B \to (A \rhd^2 C \to A \rhd^2 B, C)$,
4. *Weak Transitivity*: $A \rhd^2 B \to (A, B \rhd^2 C \to A \rhd^2 C)$.

As we have seen from the example above, $A \parallel_B^1 C$ and $A \parallel_B^2 C$ are two different relations. Propositional properties of these relations are different as well. For example, the following three principles are valid for relation $A \parallel_B^1 C$:

$$A \parallel_C^1 B \wedge A \parallel_{B,C}^1 D \to A \parallel_C^1 B, D,$$

$$A, B \parallel_C^1 D \to A \parallel_{B,C}^1 D,$$

$$B \parallel_A^1 C \wedge E \parallel_B^1 D \wedge D \parallel_C^1 F \wedge E \parallel_D^1 F \wedge A \parallel_E^1 F \to E \parallel_A^1 F.$$

The same principles are also valid for conditional nondeducibility, but none of them is valid for relation $A \parallel_B^2 C$. We think, although we did not prove this, that complete axiomatization of relation $A \parallel_B^1 C$ could be given by the same axioms as complete axiomatization of conditional nondeducibility [19].

In relation $A \parallel_B^1 C$ we essentially restrict the set of all equilibria of the original game to those that have specific strategies of players in set B, without any intuition as to why this restriction should be considered. In the relation $A \parallel_B^2 C$, on the other hand, the restriction comes from the public commitment of players in set B. For this reason, we think that $A \parallel_B^2 C$ is a more meaningful relation to consider. Thus, in this article we study only the relation $A \parallel_B^2 C$, which, from now on, will be denoted simply by $A \parallel_B C$ and called conditional interchangeability. Our main result is a complete axiomatization of this relation when B is a single-element set. In the conclusion we discuss some of the principles that we have found for the case when set B is an arbitrary set of players.

[1]Later in the article, it will be technically more convenient to refer to the Nash equilibria of the restricted game as $(x, \hat{y}, z)$.

2 Syntax and Semantics

In this section we review basic notations and definitions from game theory, specify the propositional language that we study, and formally define conditional interchangeability.

Definition 1 *A strategic game is a triple* $G = (P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$*, where*

1. *P is a non-empty finite set of "players".*
2. *S_p is a non-empty set of "strategies" of a player $p \in P$. Elements of the cartesian product $\prod_{p\in P} S_p$ are called "strategy profiles".*
3. *u_p is a "pay-off" function from strategy profiles into the set of real numbers.*

As is common in the game theory literature, for any tuple $a = \langle a_i \rangle_{i\in I}$, any $i_0 \in I$, and any value b, by (a_{-i_0}, b) we mean the tuple a in which i_0-th component is changed from a_{i_0} to b.

Definition 2 *Nash equilibrium of a strategic game* $G = (P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$*, is a strategy profile s such that $u_{p_0}(s_{-p_0}, s_0) \leq u_{p_0}(s)$ for each $p_0 \in P$ and each $s_0 \in S_{p_0}$.*

The set of all Nash equilibria of a game G is denoted by $NE(G)$.

Definition 3 *Let* $G = (P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$ *be any strategic game, $b \in P$ be any player, and $\hat{b}$ be any strategy from set S_b. By restricted game $G[b \mapsto \hat{b}]$ we mean game $(P, \{S'_p\}_{p\in P}, \{u_p\}_{p\in P})$, where*

$$S'_p = \begin{cases} \{\hat{b}_p\} & \textit{if } p = b, \\ S_p & \textit{otherwise.} \end{cases}$$

Definition 4 *For any finite set of players P, the set of formulas $\Phi(P)$ is the minimal set of formulas such that:*

1. *$\perp \in \Phi(P)$,*
2. *$A \parallel_b C \in \Phi(P)$, where A and C are two subsets of P and $b \in P$.*
3. *$\varphi \to \psi \in \Phi(P)$, if $\varphi, \psi \in \Phi(P)$.*

If $x = \langle x_i \rangle_{i\in I}$ and $y = \langle y_i \rangle_{i\in I}$ are two tuples such that $x_a = y_a$ for each $a \in A$, then we write $x \equiv_A y$. Next, we define the truth relation $G \vDash \varphi$ between a game G and a formula φ:

Definition 5 *For any game* $G = (P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$ *and any formula $\varphi \in \Phi(P)$, binary relation $G \vDash \varphi$ is defined as follows:*

1. *$G \nvDash \perp$,*

2. *$G \vDash \varphi \to \psi$ if and only if $G \nvDash \varphi$ or $G \vDash \psi$,*

3. *$G \vDash A \parallel_b C$ if for each $\hat{b} \in S_b$ and each $e_1, e_2 \in NE(G[b \mapsto \hat{b}])$ there is $e \in NE(G[b \mapsto \hat{b}])$ such that $e \equiv_A e_1$ and $e \equiv_C e_2$.*

The third part of the above definition is the key definition of this article. It formally specifies conditional interchangeability relation.

3 Axioms

For any set of players P, our logical system consists of propositional tautologies in language $\Phi(P)$, Modus Ponens inference rule, and the following axioms:

1. Reflexivity: $A \parallel_b b$,
2. Symmetry: $A \parallel_b C \to C \parallel_b A$,
3. Monotonicity: $A \parallel_b C, D \to A \parallel_b C$,
4. Exchange: $A \parallel_b C \to (A, C \parallel_b D \to A \parallel_b C, D)$,
5. Determinicity: $A \parallel_b C \to (D \parallel_b D \to A \parallel_b C, D)$.

The first four of these axioms are a natural adaptation of the Geiger, Paz, and Pearl axioms mentioned in the introduction. The non-conditional version of Determinicity axiom has also been mentioned in the introduction. We write $X \vdash \varphi$ if formula φ is provable in our system using an additional set of axioms X. We write $\vdash \varphi$ instead of $\varnothing \vdash \varphi$.

4 Soundness

In this section we prove soundness of axioms 1.-5.

Theorem 1 *For any finite set of parties P and any $\varphi \in \Phi(P)$, if $\vdash \varphi$, then $G \vDash \varphi$ for each game $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$.*

Proof. It will be sufficient to verify that $G \vDash \varphi$ for each axiom φ of our logical system. Soundness of propositional tautologies and the Modus Ponens rule is trivial.

Reflexivity Axiom. Let $\hat{b} \in S_b$. Consider any two Nash equilibria $e' = \langle e'_p \rangle_{p \in P} \in NE(G[b \mapsto \hat{b}])$ and $e'' = \langle e''_p \rangle_{p \in P} \in NE(G[b \mapsto \hat{b}])$. We need to show that there is $e = \langle e_p \rangle_{p \in P} \in NE(G[b \mapsto \hat{b}])$ such that $e' \equiv_A e \equiv_b e''$. Indeed, let e be equilibrium e'. Then, $e_b = e'_b = \hat{b} = e''_b$ and $e \equiv_A e'$. Therefore, $e' \equiv_A e \equiv_b e''$.

Symmetry Axiom. Assume $G \vDash A \parallel_b C$. Let $\hat{b} \in S_b$. Consider any two Nash equilibria $e' \in NE(G[b \mapsto \hat{b}])$ and $e'' \in NE(G[b \mapsto \hat{b}])$. We need to show that there is $e \in NE(G[b \mapsto \hat{b}])$ such that $e' \equiv_C e \equiv_A e''$. Indeed, by the assumption, there exists $e \in NE(G[b \mapsto \hat{b}])$ such that $e \equiv_A e''$ and $e \equiv_C e'$. Therefore, $e' \equiv_C e \equiv_A e''$.

Monotonicity Axiom. Assume $G \vDash A \parallel_b C, D$. Let $\hat{b} \in S_b$. Consider any two Nash equilibria $e' \in NE(G[b \mapsto \hat{b}])$ and $e'' \in NE(G[b \mapsto \hat{b}])$. We need to show that there is $e \in NE(G[b \mapsto \hat{b}])$ such that $e' \equiv_A e \equiv_C e''$. Indeed, by the assumption, there must exist $e \in NE(G[b \mapsto \hat{b}])$ such that $e' \equiv_A e \equiv_{C,D} e''$. Therefore, $e' \equiv_A e \equiv_C e''$.

Exchange Axiom. Assume $G \vDash A \parallel_b C$ and $G \vDash A, C \parallel_b D$. Let $\hat{b} \in S_b$. Consider any two Nash equilibria $e' \in NE(G[b \mapsto \hat{b}])$ and $e'' \in NE(G[b \mapsto \hat{b}])$. We need to show that there is $e \in NE(G[b \mapsto \hat{b}])$ such that $e' \equiv_A e \equiv_{C,D} e''$. By the assumption $G \vDash A \parallel_b C$, there is a Nash equilibrium $e''' \in NE(G[b \mapsto \hat{b}])$ such that $e''' \equiv_A e'$ and $e''' \equiv_C e''$. Since $G \vDash A, C \parallel_b D$, there is a Nash equilibrium $e \in NE(G[b \mapsto \hat{b}])$ such that $e \equiv_{A,C} e'''$ and $e \equiv_D e''$. Thus, $e \equiv_A e''' \equiv_A e'$ and $e \equiv_C e''' \equiv_C e''$ and $e \equiv_D e''$. Therefore, $e' \equiv_A e \equiv_{C,D} e''$.

Determinicity Axiom. Assume $G \vDash A \parallel_b C$ and $G \vDash D \parallel_b D$. Let $\hat{b} \in S_b$. Consider any two Nash equilibria $e' \in NE(G[b \mapsto \hat{b}])$ and $e'' \in NE(G[b \mapsto \hat{b}])$. We need to show that there is $e \in NE(G[b \mapsto \hat{b}])$ such that $e' \equiv_A e \equiv_{C,D} e''$. By the assumption $G \vDash A \parallel_b C$, there exists $e \in NE(G[b \mapsto \hat{b}])$ such that $e \equiv_A e'$ and $e \equiv_C e''$. By the assumption $G \vDash D \parallel_b D$, there exists $e''' \in NE(G[b \mapsto \hat{b}])$ such that $e''' \equiv_D e$ and $e''' \equiv_D e''$. Thus, $e \equiv_D e''' \equiv_D e''$. Therefore, $e' \equiv_A e \equiv_{C,D} e''$. ⊠

5 Completeness

We state and prove completeness of our logical system later in the article as Theorem 2. The proof of the completeness theorem will construct a counter-example game for each formula not provable in our system. This game will be defined as a composition of multiple "mini" games played concurrently. We start first by defining the mini games and proving their basic properties to be used later in the proof of completeness.

5.1 Game $G_1(P, a, b)$

In the first type of mini game, called G_1, there are two special players a and b. Player a is rewarded for choosing strategy 1. Player b is also rewarded to choose 1, but only if player a chooses 1 as well. All other players are rewarded to match the choice of player a.

Definition 6 *For any set P and any two distinct $a, b \in P$, by $G_1(P, a, b)$ we mean triple $(P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ such that:*

1. *$S_p = \{0, 1\}$, for each $p \in P$,*

2. *For each $p \in P$ and each strategy profile $\langle s_q \rangle_{q \in P} \in \prod_{q \in P} S_q$,*

   (a) *if $p = a$, then $u_p(\langle s_q \rangle_{q \in P}) = s_p$,*

   (b) *if $p = b$, then*

$$u_p(\langle s_q \rangle_{q \in P}) = \begin{cases} s_p & \text{if } s_a = 1, \\ 0 & \text{otherwise,} \end{cases}$$

   (c) *if $p \notin \{a, b\}$, then*

$$u_p(\langle s_q \rangle_{q \in P}) = \begin{cases} 1 & \text{if } s_p = s_a, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 *For any $y \in P$ and any $\hat{y} \in \{0, 1\}$, game $G_1(P, a, b)[y \mapsto \hat{y}]$ has at least one Nash equilibrium.*

Proof. If $y = a$, then strategy profile $\langle e_p \rangle_{p \in P}$, where $e_p = \hat{y}$, for each $p \in P$, is a Nash equilibrium of the game $G_1(P, a, b)[y \mapsto \hat{y}]$. If $y \neq a$, then strategy profile $\langle e_p \rangle_{p \in P}$ such that, for each $p \in P$,

$$e_p = \begin{cases} \hat{y} & \text{if } p = y, \\ 1 & \text{otherwise.} \end{cases}$$

is a Nash equilibrium of the same game $G_1(P, a, b)[y \mapsto \hat{y}]$. ⊠

The next key lemma describes which atomic conditional interchangeability formulas are true in game G_1.

Lemma 2 *$G_1(P, a, b) \nvDash X \parallel_y Z$ if and only if $y = a$ and $b \in X \cap Z$, for all subsets $X, Z \subseteq P$, and all $y \in P$.*

Proof. $(\Rightarrow)$: We need to show that if either $y \neq a$ or $b \notin X \cap Z$, then $G_1(P, a, b) \vDash X \parallel_y Z$.

Let us assume first that $y \neq a$. Consider any $\hat{y} \in S_y$. By Definition 6, $s_a = 1$ for each Nash equilibrium $\langle s_p \rangle_{p \in P}$ of the game $G_1(P, a, b)[y \mapsto \hat{y}]$. Hence, again by Definition 6, $s_p = 1$ for each $p \in P$ and each Nash equilibrium $\langle s_p \rangle_{p \in P}$ of the game $G_1(P, a, b)[y \mapsto \hat{y}]$. Thus, game $G_1(P, a, b)[y \mapsto \hat{y}]$ has at most one Nash equilibrium. Then, for each two equilibria s' and s'' of this game, there is equilibrium s of the same game such that $s' \equiv_X s \equiv_Z s''$. Therefore, $G_1(P, a, b) \vDash X \parallel_y Z$.

Let us now assume that $y = a$ and $b \notin X \cap Z$. By Definition 6, $s_p = \hat{a}$ for each $p \in P \setminus \{b\}$, each $\hat{a} \in S_a$, and each Nash equilibrium $\langle s_p \rangle_{p \in P}$ of the game $G_1(P, a, b)[a \mapsto \hat{a}]$. Then,

$$s' \equiv_{P \setminus \{b\}} s'' \tag{1}$$

for each two equilibria s' and s'' of the game $G_1(P, a, b)[a \mapsto \hat{a}]$ and for each $\hat{a} \in S_a$.

Recall that $b \notin X \cap Z$. Thus, $b \notin X$ or $b \notin Z$. Without loss of generality, assume that $b \notin X$. To show that $G_1(P, a, b) \vDash X \parallel_a Z$, consider any two Nash

equilibria s' and s'' of the game $G_1(P,a,b)[a \mapsto \hat{a}]$ and an arbitrary $\hat{a} \in S_a$. Note that $s' \equiv_X s''$ due to equality (1) and the assumption $b \notin X$. Hence, $s' \equiv_X s'' \equiv_Z s''$.
$(\Leftarrow)$: Assume that $y = a$ and $b \in X \cap Z$ and, at the same time, $G_1(P,a,b) \vDash X \parallel_y Z$. Consider strategy profiles $s' = \langle s'_p \rangle_{p \in P}$ and $s'' = \langle s''_p \rangle_{p \in P}$ such that $s'_p = 0$ for each $p \in P$ and

$$s''_p = \begin{cases} 1 & \text{if } p = b, \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 6, $s', s'' \in NE(G_1(P,a,b)[a \mapsto 0])$. Thus, due to the assumption $G_1(P,a,b) \vDash X \parallel_y Z$, there must exist $s \in NE(G_1(P,a,b)[a \mapsto 0])$ such that $s' \equiv_X s \equiv_Z s''$. Hence, $s' \equiv_{X \cap Z} s''$. Thus, $s'_b = s''_b$ due to the assumption $b \in X \cap Z$. Therefore, $0 = s'_b = s''_b = 1$, which is a contradiction. ⊠

5.2 Game $G_2(P,a,B)$

We now introduce the second type of mini game used in the proof of completeness, called G_2. This game has a special player a and a set of special players B. Player a is always rewarded to choose strategy 1. Each player in set B is also rewarded to choose strategy 1, but only if player a chooses strategy 1 as well; otherwise all players in set B are rewarded if the sum of their strategies is even. All other players are rewarded to choose the same strategy as player a.

Definition 7 *For any set P, any $a \in P$, and any $B \subseteq P$, such that $a \notin B$ and $|B| \geq 2$, by game $G_2(P,a,B)$ we mean triple $(P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ such that:*

1. *$S_p = \{0,1\}$, for each $p \in P$,*
2. *For each $p \in P$ and each strategy profile $\langle s_q \rangle_{q \in P} \in \prod_{q \in P} S_q$,*
   (a) *if $p = a$, then $u_p(\langle s_q \rangle_{q \in P}) = s_p$,*
   (b) *if $p \in B$, then*

$$u_p(\langle s_q \rangle_{q \in P}) = \begin{cases} s_p & \textit{if } s_a = 1, \\ 1 & \textit{if } s_a = 0 \textit{ and } \sum_{b \in B} s_b \equiv 0 \pmod 2, \\ 0 & \textit{otherwise,} \end{cases}$$

   (c) *if $p \notin \{a\} \cup B$, then*

$$u_p(\langle s_q \rangle_{q \in P}) = \begin{cases} 1 & \textit{if } s_p = s_a, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 3 *For any $y \in P$ and any $\hat{y} \in \{0,1\}$, game $G_2(P,a,B)[y \mapsto \hat{y}]$ has at least one Nash equilibrium.*

Proof. If $y = a$, then strategy profile $\langle e_p \rangle_{p\in P}$, where $e_p = \hat{y}$, for each $p \in P$, is a Nash equilibrium of the game $G_2(P, a, b)[y \mapsto \hat{y}]$. If $y \neq a$, then strategy profile $\langle e_p \rangle_{p\in P}$ such that, for each $p \in P$,

$$e_p = \begin{cases} \hat{y} & \text{if } p = y, \\ 1 & \text{otherwise.} \end{cases}$$

is a Nash equilibrium of the same game $G_2(P, a, b)[y \mapsto \hat{y}]$. ⊠

The next lemma is the key lemma about game G_2. It describes which atomic conditional interchangeability properties are true in this game.

Lemma 4 *$G_2(P, a, B) \nvDash X \parallel_y Z$ if and only if $y = a$, $X \cap B \neq \varnothing$, $Z \cap B \neq \varnothing$, and at least one of the following two conditions is satisfied:*

1. *$B \subseteq X \cup Z$,*
2. *$X \cap Z \cap B \neq \varnothing$,*

for all subsets $X, Z \subseteq P$, and all $y \in P$.

Proof. $(\Rightarrow)$: We consider four cases separately.
Case 1: Assume that $y \neq a$. Consider any $\hat{y} \in S_y$. By Definition 7, $s_a = 1$ for each Nash equilibrium $\langle s_p \rangle_{p\in P}$ of the game $G_2(P, a, B)[y \mapsto \hat{y}]$. Hence, again by Definition 7, $s_p = 1$ for each $p \in P$ and each Nash equilibrium $\langle s_p \rangle_{p\in P}$ of the game $G_2(P, a, B)[y \mapsto \hat{y}]$. Thus, game $G_2(P, a, B)[y \mapsto \hat{y}]$ has at most one Nash equilibrium. Then, for each two equilibria s' and s'' of this game, there is equilibrium s of the same game such that $s' \equiv_X s \equiv_Z s''$. Therefore, $G_2(P, a, B) \vDash X \parallel_y Z$.
Case 2: Let $X \cap B = \varnothing$ and $y = a$. Let $\hat{a}$ be any element of $\{0, 1\}$. We need to show that for any two Nash equilibria s' and s'' of the game $G_2(P, a, B)[a \mapsto \hat{a}]$, there is an equilibrium s of the same game such that $s' \equiv_X s \equiv_Z s''$.

Due to Definition 7, $s_p = s_a$ for any $p \in P \setminus B$ and for any Nash equilibrium $\langle s_p \rangle_{p\in P}$ of the game $G_2(P, a, B)[a \mapsto \hat{a}]$. Thus, $s' \equiv_{P\setminus B} s''$. Therefore, $s' \equiv_X s'' \equiv_Z s''$ due to the assumption $X \cap B = \varnothing$.
Case 3: Let $Z \cap B = \varnothing$ and $y = a$. This case is similar to Case 2.
Case 4: Assume that $y = a$, that there is $b_0 \in B$ such that $b_0 \notin X \cup Z$, and that $X \cap Z \cap B = \varnothing$. We will show that $G_2(P, a, B) \vDash X \parallel_a Z$. Consider an arbitrary $\hat{a} \in \{0, 1\}$. We need to prove that for any Nash equilibria s' and s'' of the game $G_2(P, a, B)[a \mapsto \hat{a}]$ there is equilibrium s of the same game such that $s' \equiv_X s \equiv_Z s''$.

If $\hat{a} = 1$, then by Definition 7, $s_p = 1$ for each $p \in P$ and for each Nash equilibrium $\langle s_p \rangle_{p\in P}$ of the game $G_2(P, a, B)[a \mapsto 1]$. Thus, game $G_2(P, a, B)[a \mapsto 1]$ has at most one Nash equilibrium. Then, for each two equilibria s' and s'' of this game, there is equilibrium s of the same game such that $s' \equiv_X s \equiv_Z s''$. Therefore, $G_2(P, a, B) \vDash X \parallel_a Z$.

Let now $\hat{a} = 0$. Consider any two equilibria $s' = \langle s'_p \rangle_{p\in P}$ and $s'' = \langle s''_p \rangle_{p\in P}$ of the game $G_2(P, a, B)[a \mapsto 0]$. Note that by Definition 7, $s'_p = s''_p = 0$ for each $p \in P \setminus B$. Recall the assumption $X \cap Z \cap B = \varnothing$. Let strategy profile $s = \langle s_p \rangle_{p \in P}$ of the game $G_2(P, a, B)[a \mapsto 0]$ be defined as follows

$$s_p = \begin{cases} s'_p & \text{if } p \in X \cap B, \\ s''_p & \text{if } p \in Z \cap B, \\ u & \text{if } p = b_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $u \in \{0, 1\}$ is such that

$$u \equiv \sum_{b \in X \cap B} s'_b + \sum_{b \in Z \cap B} s''_b \pmod 2.$$

Thus, due to the assumption $X \cap Z \cap B = \varnothing$,

$$\sum_{b \in B} s_b = \quad s_{b_0} + \textstyle\sum_{b \in X \cap B} s_b + \sum_{b \in Z \cap B} s_b + \sum_{b \in B \setminus (X \cup Z)} s_b =$$
$$u + \textstyle\sum_{b \in X \cap B} s_b + \sum_{b \in Z \cap B} s_b + 0 \equiv 2u \equiv 0 \pmod 2.$$

Therefore, by Definition 7, strategy profile s is a Nash equilibrium of the game $G_2(P, a, B)[a \mapsto 0]$.

$(\Leftarrow)$: We consider two cases separately.

Case 1: Assume that $y = a$, that there is $b_0 \in X \cap Z \cap B$, and that $G_2(P, a, B) \vDash X \parallel_a Z$. Due to the condition $|B| \geq 2$ of Definition 7, there must exist $b_1 \in B$ such that $b_1 \neq b_0$. Consider strategy profiles $s' = \langle s'_p \rangle_{p \in P}$ and $s'' = \langle s''_p \rangle_{p \in P}$ such that $s'_p = 0$ for each $p \in P$ and that

$$s''_p = \begin{cases} 1 & \text{if } p \in \{b_0, b_1\}, \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 7, strategy profiles s' and s'' are Nash equilibria of the game $G_2(P, a, B)[a \mapsto 0]$. Thus, by the assumption $G_2(P, a, B) \vDash X \parallel_a Z$, there exists Nash equilibrium $s = \langle s_p \rangle_{p \in P}$ of the game $G_2(P, a, B)[a \mapsto 0]$ such that $s' \equiv_X s \equiv_Z s''$. Therefore, $0 = s'_{b_0} = s_{b_0} = s''_{b_0} = 1$, which is a contradiction.

Case 2: Suppose now that $X \cap Z \cap B = \varnothing$, that $B \subseteq X \cup Z$, that $G_2(P, a, B) \vDash X \parallel_a Z$, and that there are $b_X \in X \cap B$ and $b_Z \in Z \cap B$. Note that $b_X \neq b_Z$ due to the assumption $X \cap Z \cap B = \varnothing$. Consider strategy profiles $s' = \langle s'_p \rangle_{p \in P}$ and $s'' = \langle s''_p \rangle_{p \in P}$ such that $s'_p = 0$ for each $p \in P$ and that

$$s''_p = \begin{cases} 1 & \text{if } p \in \{b_X, b_Z\}, \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 7, strategy profiles s' and s'' are Nash equilibria of the game $G_2(P, a, B)[a \mapsto 0]$. Thus, by the assumption $G_2(P, a, B) \vDash X \parallel_a Z$, there exists Nash equilibrium $s = \langle s_p \rangle_{p \in P}$ of the game $G_2(P, a, B)[a \mapsto 0]$ such that $s' \equiv_X s \equiv_Z s''$. Hence, due to the assumptions $X \cap Z \cap B = \varnothing$ and $B \subseteq X \cup Z$,

$$\sum_{b \in B} s_b = \sum_{b \in X \cap B} s_b + \sum_{b \in Z \cap B} s_b = 0 + s_{b_Z} = 1,$$

which is a contradiction to Definition 7 and the assumption that s is Nash equilibrium of the game $G_2(P, a, B)[a \mapsto 0]$. $\boxtimes$

5.3 Critical Sets

Any proof of completeness could be viewed as a bridge connecting provability with semantics. In the previous section we prepared for this bridge construction on the semantics site by introducing mini games G_1 and G_2. Let us now turn to provability site and define the notion of d-critical set. In the proof of completeness, d-critical set will be used as set of special players B for mini game G_2. We start with a sequence of lemmas in which we assume a fixed finite set of parties P and a fixed set of formulas $X \subseteq \Phi(P)$.

Definition 8 *For any $d \in P$, a set $C \subseteq P$ is called d-critical if there is a disjoint partition $C = C_1 \cup C_2$, called a "d-critical partition", such that*

1. $X \nvdash C_1 \parallel_d C_2$,
2. $X \vdash C_1 \cap E \parallel_d C_2 \cap E$, *for any* $E \subsetneq C$.

Lemma 5 *Any d-critical partition is a non-trivial partition.*

Proof. It is sufficient to prove that for any set A, we have $X \vdash A \parallel_d \varnothing$ and $X \vdash \varnothing \parallel_d A$. Indeed, by the Reflexivity axiom, $\vdash A \parallel_d d$. Hence, by the Monotonicity axiom, $X \vdash A \parallel_d \varnothing$. Therefore, by the Symmetry axiom, $X \vdash \varnothing \parallel_d A$. $\boxtimes$

Lemma 6 *$X \nvdash A \parallel_d B$, for any non-trivial (but not necessarily d-critical) disjoint partition $C = A \cup B$ of a d-critical set C.*

Proof. Suppose $X \vdash A \parallel_d B$ and let $C = C_1 \cup C_2$ be a d-critical partition of C. By the Monotonicity and Symmetry axioms, $X \vdash A \cap C \parallel_d B \cap C$. Thus,

$$X \vdash A \cap C_1, A \cap C_2 \parallel_d B \cap C_1, B \cap C_2. \tag{2}$$

Since $A \cup B$ is a non-trivial partition of C, sets A and B are both non-empty. Thus, $A \subsetneq C$ and $B \subsetneq C$. Hence, by the definition of a d-critical set, $X \vdash A \cap C_1 \parallel_d A \cap C_2$ and $X \vdash B \cap C_1 \parallel_d B \cap C_2$.

Note that $A \cap C$ is not empty since $A \cup B$ is a non-trivial partition of C. Thus, either $A \cap C_1$ or $A \cap C_2$ is not empty. Without loss of generality, assume that $A \cap C_1 \neq \varnothing$. From (2) and our earlier observation that $X \vdash A \cap C_1 \parallel_d A \cap C_2$, the Exchange axiom yields

$$X \vdash A \cap C_1 \parallel_d A \cap C_2, B \cap C_1, B \cap C_2.$$

By the Symmetry axiom,

$$X \vdash A \cap C_2, B \cap C_1, B \cap C_2 \parallel_d A \cap C_1. \tag{3}$$

The assumption $A \cap C_1 \neq \varnothing$ implies that $(A \cap C_2) \cup (B \cap C_1) \cup (B \cap C_2) \subsetneq C$. Hence, by the definition of a critical set,

$$X \vdash B \cap C_1 \parallel_d A \cap C_2, B \cap C_2.$$

By Symmetry axiom,

$$X \vdash A \cap C_2, B \cap C_2 \parallel_d B \cap C_1.$$

From (3) and the above statement, using the Exchange axiom,

$$X \vdash A \cap C_2, B \cap C_2 \parallel_d A \cap C_1, B \cap C_1.$$

Since $A \cup B$ is a partition of C, we can conclude that $X \vdash C_2 \parallel_d C_1$. By the Symmetry axiom, $X \vdash C_1 \parallel_d C_2$, which contradicts the assumption that $C_1 \cup C_2$ is a d-critical partition. ⊠

Lemma 7 *For any two disjoint subsets $A, B \subseteq P$, if $X \nvdash A \parallel_d B$, then there is a d-critical partition $C_1 \cup C_2$, such that $C_1 \subseteq A$ and $C_2 \subseteq B$.*

Proof. Consider the partial order $\preceq$ on set $2^A \times 2^B$ such that $(E_1, E_2) \preceq (F_1, F_2)$ if and only if $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$. Define

$$\mathcal{E} = \{(E_1, E_2) \in 2^A \times 2^B \mid X \nvdash E_1 \parallel_d E_2\}.$$

$X \nvdash A \parallel_d B$ implies that $(A, B) \in \mathcal{E}$. Thus, $\mathcal{E}$ is a non-empty finite set. Take (C_1, C_2) to be a minimal element of set $\mathcal{E}$ with respect to partial order $\preceq$. ⊠

Lemma 8 *If $X \vdash A \parallel_d B$ and if C is a d-critical set, then $A \cap B \cap C = \varnothing$.*

Proof. Suppose that $c \in A \cap B \cap C$. Assumption $X \vdash A \parallel_d B$, by Monotonicity and Symmetry axioms, implies that $X \vdash c \parallel_d c$. Let $C_1 \cup C_2$ be a d-critical partition such that $C = C_1 \cup C_2$. Without loss of generality, assume that $c \in C_2$. By the definition of a d-critical partition, $X \vdash C_1 \parallel_d C_2 \setminus \{c\}$. From $X \vdash c \parallel_d c$ and $X \vdash C_1 \parallel_d C_2 \setminus \{c\}$, by Determinicity axiom, we can conclude that $X \vdash C_1 \parallel_d C_2$, which is a contradiction to the definition of a d-critical partition. ⊠

5.4 Combining games G_1 and G_2

The following lemma, essentially, combines properties of games G_1 and G_2, earlier expressed in Lemma 2 and Lemma 4.

Lemma 9 *For any set of formulas X in the language $\Phi(P)$, any subsets $R, T \subseteq P$, and any $s \in P$, if $X \nvdash R \parallel_s T$, then there exists a game G with the set of players P such that*

1. *game $G[y \mapsto \hat{y}]$ has at least one Nash equilibrium for any player $y \in P$ and any strategy $\hat{y}$ of the player y in the game G,*

2. *$G \nvDash R \parallel_s T$,*

3. *$G \vDash E \parallel_d F$ for any $E, F \subseteq P$ and any $d \in P$ such that $X \vdash E \parallel_d F$.*

Proof. By Lemma 7, there exists s-critical partition $R' \cup T'$ such that $R' \subseteq R$ and $T' \subseteq T$. We consider two separate cases.
Case 1: $R' \cap T'$ is not empty. Let $q \in R' \cap T'$. Then $X \vdash R' \setminus \{q\} \parallel_s T' \setminus \{q\}$ because $R' \cup T'$ is an s-critical partition. By Determinicity axiom, $X \vdash q \parallel_s q \to R' \setminus \{q\} \parallel_s T'$. By Symmetry axiom, $X \vdash q \parallel_s q \to T' \parallel_s R' \setminus \{q\}$. By Determinicity axiom, $X \vdash q \parallel_s q \to T' \parallel_s R'$. By Symmetry axiom, $X \vdash q \parallel_s q \to R' \parallel_s T'$. By the definition of an s-critical set, $X \nvdash R' \parallel_s T'$. Thus,

$$X \nvdash q \parallel_s q. \tag{4}$$

Let G be game $G_1(P, s, q)$. By Lemma 1, game $G_1(P, s, q)[y \mapsto \hat{y}]$ has at least one Nash equilibrium for any $y \in P$ and any strategy $\hat{y}$ of the player y in the game. Note also that $q \in R' \cap T' \subseteq R \cap T$. Therefore, by Lemma 2, $G_1(P, s, q) \nvDash R \parallel_s T$. We are now left to show that $G_1(P, s, q) \vDash E \parallel_d F$ for any $E, F \subseteq P$ and any $d \in P$ such that $X \vdash E \parallel_d F$. Assume $G_1(P, s, q) \nvDash E \parallel_d F$ for some $E, F \subseteq P$ and some $d \in P$ such that $X \vdash E \parallel_d F$. By Lemma 2, $d = s$ and $q \in E \cap F$. By the assumption that $X \vdash E \parallel_d F$ and the Monotonicity axiom, $X \vdash E \parallel_s q$. By the Symmetry and Monotonicity axioms, $X \vdash q \parallel_s q$, which is a contradiction to statement (4).
Case 2: $R' \cap T'$ is empty. By Lemma 5, R' and T' each contain at least one element. Therefore, $|R' \cup T'| \geq 2$ due to assumption that $R' \cap T'$ is empty. Let game G be game $G_2(P, s, R' \cup T')$. By Lemma 3, game $G_2(P, s, R' \cup T')[y \mapsto \hat{y}]$ has at least one Nash equilibrium for any $y \in P$ and any strategy $\hat{y}$ of the player y in the game. Note that $R \cap (R' \cup T')$ is not empty because $R \cap (R' \cup T') \supseteq R'$ and $T \cap (R' \cup T')$ is not empty because $T \cap (R' \cup T') \supseteq T'$. Also, $R' \cup T' \subseteq R \cup T$. Hence, $G_2(P, s, R' \cup T') \nvDash R \parallel_s T$, by Lemma 4.

We are now again left to show that $G_2(P, s, R' \cup T') \vDash E \parallel_d F$ for any $E, F \subseteq P$ and any $d \in P$ such that $X \vdash E \parallel_d F$. Assume $G_2(P, s, R' \cup T') \nvDash E \parallel_d F$ for some $E, F \subseteq P$ and some $d \in P$ such that $X \vdash E \parallel_d F$. By Lemma 4, $d = s$, $E \cap (R' \cup T') \neq \varnothing$, $F \cap (R' \cup T') \neq \varnothing$, and one of the following two conditions is true:

1. $R' \cup T' \subseteq E \cup F$,

2. $E \cap F \cap (R' \cup T') \neq \varnothing$.

Recall that $X \vdash E \parallel_d F$ and that $R' \cup T'$ is an s-critical set. Thus, the second of the two conditions above does not hold due to Lemma 8. Hence, we can assume that $R' \cup T' \subseteq E \cup F$ and that $(R' \cup T') \cap E \cap F = \varnothing$. The latter implies that sets $E \cap (R' \cup T')$ and $F \cap (R' \cup T')$ are disjoint. These sets form a partition of $R' \cup T'$ because $R' \cup T' \subseteq E \cup F$. This disjoint partition is nontrivial because $E \cap (R' \cup T') \neq \varnothing$ and $F \cap (R' \cup T') \neq \varnothing$. Then,

$X \nvDash E \cap (R' \cup T') \parallel_d F \cap (R' \cup T')$, by Lemma 6. Therefore, due to Symmetry and Monotonicity axioms, $X \nvDash E \parallel_d F$, which is a contradiction. $\boxtimes$

5.5 Game Composition

Informally, by a composition of several games we mean a game in which each of the composed games is played independently. Pay-off of any player is defined as the sum of the pay-offs in the individual games.

Definition 9 *Let $\{G^i\}_{i \in I} = \{(P, \{S^i_p\}_{p \in P}, \{u^i_p\}_{p \in P})\}_{i \in I}$ be a finite family of strategic games with the same set of players P. By composition game $\bigotimes_i G^i$ we mean game $(P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ such that*

1. $S_p = \prod_i S^i_p$,
2. $u_p(\langle\langle s^i_p \rangle_{i \in I}\rangle_{p \in P}) = \sum_i u^i_p(\langle s^i_p \rangle_{p \in P})$.

Lemma 10

$$NE\left(\bigotimes_i G^i\right) = \{\langle\langle e^i_p \rangle_{i \in I}\rangle_{p \in P} \mid \langle e^i_p \rangle_{p \in P} \in NE(G^i) \text{ for each } i \in I\}.$$

Proof. First, assume that $\langle e_p \rangle_{p \in P} = \langle\langle e^i_p \rangle_{i \in I}\rangle_{p \in P} \in NE\left(\bigotimes_i G^i\right)$. We need to show that $\langle e^i_p \rangle_{p \in P} \in NE(G^i)$ for each $i \in I$. Indeed, suppose that for some $i_0 \in I$, some $p_0 \in P$, and some $s_0 \in S_{p_0}$ we have

$$u^{i_0}_{p_0}((\langle e^{i_0}_p \rangle_{p \in P})_{-p_0}, s_0) > u^{i_0}_{p_0}(\langle e^{i_0}_p \rangle_{p \in P}). \tag{5}$$

Define strategy profile $\langle \hat{e}_p \rangle_{p \in P} = \langle\langle \hat{e}^i_p \rangle_{i \in I}\rangle_{p \in P}$ of the game $\bigotimes_i G^i$ as follows:

$$\hat{e}^i_p = \begin{cases} s_0 & \text{if } i = i_0 \text{ and } p = p_0, \\ e^i_p & \text{otherwise.} \end{cases}$$

Note that, taking into account inequality (5),

$$\begin{aligned} u_{p_0}(\langle \hat{e}_p \rangle_{p \in P}) &= \sum_{i \in I} u^i_{p_0}(\langle \hat{e}^i_p \rangle_{p \in P}) = u^{i_0}_{p_0}(\langle \hat{e}^{i_0}_p \rangle_{p \in P}) + \sum_{i \neq i_0} u^i_{p_0}(\langle \hat{e}^i_p \rangle_{p \in P}) = \\ &= u^{i_0}_{p_0}((\langle e^{i_0}_p \rangle_{p \in P})_{-p_0}, s_0) + \sum_{i \neq i_0} u^i_{p_0}(\langle e^i_p \rangle_{p \in P}) > \\ &> u^{i_0}_{p_0}(\langle e^{i_0}_p \rangle_{p \in P}) + \sum_{i \neq i_0} u^i_{p_0}(\langle e^i_p \rangle_{p \in P}) = \\ &= \sum_i u^i_{p_0}(\langle e^i_p \rangle_{p \in P}) = u_{p_0}(\langle e_p \rangle_{p \in P}), \end{aligned}$$

which is a contradiction with the assumption that $\langle e_p \rangle_{p \in P}$ is a Nash equilibrium of the game $\bigotimes_i G^i$.

Next, assume that $\{\langle e_p^i \rangle_{p\in P}\}_{i\in I}$ is such a set that for any $i \in I$,

$$\langle e_p^i \rangle_{p\in P} \in NE(G^i) \tag{6}$$

We will prove that $\langle \langle e_p^i \rangle_{i\in I} \rangle_{p\in P} \in NE\left(\bigotimes_i G^i\right)$. Indeed, consider any p_0 and any $\langle s_0^i \rangle_{i\in I} \in \prod_{i\in I} S_{p_0}^i$. By assumption (6) and Definition 2, for any $i \in I$

$$u_{p_0}^i((\langle e_p^i \rangle_{p\in P})_{-p_0}, s_0^i) \leq u_{p_0}^i(\langle e_p^i \rangle_{p\in P}).$$

Thus,

$$\begin{aligned} u_{p_0}((\langle \langle e_p^i \rangle_{i\in I} \rangle_{p\in P})_{-p_0}, \langle s_0^i \rangle_{i\in I}) &= \sum_{i\in I} u_{p_0}^i((\langle e_p^i \rangle_{p\in P})_{-p_0}, s_0^i) \leq \\ &\leq \sum_{i\in I} u_{p_0}^i(\langle e_p^i \rangle_{p\in P}) = u_{p_0}(\langle \langle e_p^i \rangle_{i\in I} \rangle_{p\in P}). \end{aligned}$$

Therefore, $\langle \langle e_p^i \rangle_{i\in I} \rangle_{p\in P} \in NE\left(\bigotimes_i G^i\right)$. ⊠

Lemma 11 *Let $\{G_i\}_{i\in I} = \{(P, \{S_p^i\}_{p\in P}, \{u_p^i\}_{p\in P})\}_{i\in I}$ be a finite set of games with the same set of players P and d be a player in P. If $\hat{d} \in S_d^i$ for each $i \in I$, then*

$$\bigotimes_i (G^i[d \mapsto \hat{d}^i]) = (\bigotimes_i G^i)[d \mapsto \langle \hat{d}^i \rangle_{i\in I}].$$

⊠

Lemma 12 *Let $\{G_i\}_{i\in I} = \{(P, \{S_p^i\}_{p\in P}, \{u_p^i\}_{p\in P})\}_{i\in I}$ be a finite set of games with the same set of players. For any subsets A and B of the set P and any $d \in P$, if for each $i \in I$ and each $\hat{d} \in S_d^i$ game $G_i[d \mapsto \hat{d}]$ has at least one Nash equilibrium, then*

$$\bigotimes_i G^i \vDash A \parallel_d B \qquad \text{iff} \qquad \forall i\, (G^i \vDash A \parallel_d B).$$

Proof. $(\Rightarrow)$: Suppose that $\bigotimes_i G^i \vDash A \parallel_d B$ and consider any $i_0 \in I$. We will prove that $G^{i_0} \vDash A \parallel_d B$. Indeed, let $d_0 \in S_d^{i_0}$ and $f = \langle f_p \rangle_{p\in P} \in NE(G^{i_0}[d \mapsto d_0])$ and $g = \langle g_p \rangle_{p\in P} \in NE(G^{i_0}[d \mapsto d_0])$. We will construct $h = \langle h_p \rangle_{p\in P} \in NE(G^{i_0}[d \mapsto d_0])$ such that $f \equiv_A h \equiv_B g$.

By Definition 1, set S_d^i is not empty for each $i \in I$. Let $\langle \hat{d}^i \rangle_{i\in I}$ be any tuple in $\prod_{i\in I} S_d^i$ such that $\hat{d}^{i_0} = d_0$. By the assumption of the lemma, for any $i \in I$ there is at least one Nash equilibrium $\langle e_p^i \rangle_{p\in P}$ of the game $G^i[d \mapsto \hat{d}^i]$.

Consider strategy profiles $\hat{f} = \langle \langle \hat{f}_p^i \rangle_{i\in I} \rangle_{p\in P}$ and $\hat{g} = \langle \langle \hat{g}_p^i \rangle_{i\in I} \rangle_{p\in P}$ for the game $\bigotimes_i (G^i[d \mapsto \hat{d}^i])$ such that

$$\hat{f}_p^i = \begin{cases} f_p & \text{if } i = i_0, \\ e_p^i & \text{otherwise,} \end{cases} \tag{7}$$

and

$$\hat{g}^i_p = \begin{cases} g_p & \text{if } i = i_0, \\ e^i_p & \text{otherwise.} \end{cases} \tag{8}$$

By Lemma 10, $\hat{f}, \hat{g} \in NE(\bigotimes_i (G^i[d \mapsto \hat{d}^i]))$. By Lemma 11,

$$\hat{f}, \hat{g} \in NE\left(\left(\bigotimes_i G^i\right)[d \mapsto \langle \hat{d}^i \rangle_{i \in I}]\right).$$

Thus, by assumption $\bigotimes_i G^i \vDash A \parallel_d B$, there must exist

$$\hat{h} \in NE\left(\left(\bigotimes_i G^i\right)[d \mapsto \langle \hat{d}^i \rangle_{i \in I}]\right) \tag{9}$$

such that

$$\hat{f} \equiv_A \hat{h} \equiv_B \hat{g}. \tag{10}$$

From (9) by Lemma 11

$$\hat{h} \in NE\left(\bigotimes_i (G^i[d \mapsto \hat{d}^i])\right).$$

Define strategy profile h for the game G^{i_0} to be $\langle h^{i_0}_p \rangle_{p \in P}$. By Lemma 10, $h \in NE(G^{i_0}[d \mapsto \hat{d}^{i_0}])$. Hence, $h \in NE(G^{i_0}[d \mapsto d_0])$. From statements (10), (7), and (8), it follows that $f \equiv_A h \equiv_B g$.

$(\Leftarrow)$: Assume that $G^i \vDash A \parallel_d B$ for all $i \in I$. Consider any $\hat{d} = \langle \hat{d}^i \rangle_{i \in I} \in \prod_{i \in I} S^i_d$. Let $f = \langle \langle f^i_p \rangle_{i \in I} \rangle_{p \in P} \in NE((\bigotimes_i G^i)[d \mapsto \hat{d}])$ and $g = \langle \langle g^i_p \rangle_{i \in I} \rangle_{p \in P} \in NE((\bigotimes_i G^i)[d \mapsto \hat{d}])$. We will show that there is $e \in NE((\bigotimes_i G^i)[d \mapsto \hat{d}])$ such that $f \equiv_A e \equiv_B g$.

Indeed, by Lemma 11, $f, g \in NE(\bigotimes_i (G^i[d \mapsto \hat{d}^i]))$. Thus, by Lemma 10, $\langle f^i_p \rangle_{p \in P} \in NE(G^i[d \mapsto \hat{d}^i])$ and $\langle g^i_p \rangle_{p \in P} \in NE(G^i[d \mapsto \hat{d}^i])$ for each $i \in I$. Hence, by the assumption, for all $i \in I$ there is $\langle e^i_p \rangle_{p \in P} \in NE(G^i[d \mapsto \hat{d}^i])$ such that $\langle f^i_p \rangle_{p \in P} \equiv_A \langle e^i_p \rangle_{p \in P} \equiv_B \langle g^i_p \rangle_{p \in P}$. Thus,

$$\langle \langle f^i_p \rangle_{i \in I} \rangle_{p \in P} \equiv_A \langle \langle e^i_p \rangle_{i \in I} \rangle_{p \in P} \equiv_B \langle \langle g^i_p \rangle_{i \in I} \rangle_{p \in P}.$$

Pick strategy profile e to be $\langle \langle e^i_p \rangle_{i \in I} \rangle_{p \in P}$ and notice that $e \in NE(\bigotimes_i (G^i[d \mapsto \hat{d}^i]))$ by Lemma 10. By Lemma 11, $e \in NE((\bigotimes_i G^i)[d \mapsto \hat{d}])$. ⊠

Theorem 2 (completeness) *For any finite P and any $\varphi \in \Phi(P)$, if $G \vDash \varphi$ for each game G with the set of players P, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Let X be any maximal consistent subset of $\Phi(P)$ such that $X \nvdash \varphi$. By Lemma 9, for all subsets $R, T \subseteq P$ and all $s \in P$, if $X \nvdash R \parallel_s T$, then there is a game $G_{R,s,T}$ with the set of players P such that

1. game $G_{R,s,T}[y \mapsto \hat{y}]$ has at least one Nash equilibrium for any $y \in P$ and any strategy $\hat{y}$ of the player y in the game $G_{R,s,T}$,

2. $G_{R,s,T} \nvDash R \parallel_s T$,

3. $G_{R,s,T} \vDash E \parallel_d F$ for any $E, F \subseteq P$ and any $d \in P$ such that $X \vdash E \parallel_d F$.

Consider game

$$\mathbb{G} = \bigotimes_{X \nvdash R \parallel_s T} G_{R,s,T},$$

where sets R and T are restricted to subsets of P and s is restricted to players from set P. Thus, game $\mathbb{G}$ is a composition of a *finite* set of games.

Lemma 13 *$X \vdash \psi$ if and only if $\mathbb{G} \vDash \psi$, for each formula $\psi \in \Phi(P)$,*

Proof. Induction on structural complexity of formula ψ. If $\psi \equiv \bot$, then $X \nvdash \psi$ due to consistency of the set X and $\mathbb{G} \nvDash \psi$ by Definition 5. Suppose now that $\psi = A \parallel_b C$.
$(\Rightarrow)$: If $X \vdash A \parallel_b C$, then $G_{R,s,T} \vDash A \parallel_b C$ for each $R, T \subseteq P$ and each $s \in P$ such that $X \nvdash R \parallel_s T$ due to the choice of the game $G_{R,s,T}$. Thus, $\mathbb{G} \vDash A \parallel_b C$ by Lemma 12.
$(\Leftarrow)$: If $X \nvdash A \parallel_b C$, then $G_{A,b,C} \nvDash A \parallel_b C$ due to the choice of the game $G_{A,b,C}$. Therefore, $\mathbb{G} \nvDash A \parallel_b C$ by Lemma 12.

The case $\psi \equiv \tau \to \sigma$ follows from the maximality and the consistency of the set X in the standard way. ⊠

To finish the proof of the theorem, notice that $\mathbb{G} \nvDash \varphi$ due to Lemma 13 and the assumption $X \nvdash \varphi$. ⊠

6 Conclusion

In this article we introduced the notion of conditional interchangeability of Nash equilibria in strategic games. Unlike the non-conditional case, the propositional theory of this relation is different from the propositional theory of conditional nondeducibility. We gave a complete axiomatization of the conditional interchangeability in the case with conditioning by a single player.

If conditioning by an arbitrary set of players is allowed, then new logical principles must be added to the system to keep it complete. Surprisingly, the new principles already appear if we allow a mix of atomic formulas conditioned by a single player and atomic formulas conditioned by an empty set of players. The latter, of course, is the same as non-conditional interchangeability. For example, the following property is true in any three-player game with players a, b, and c:

$$a \parallel c \to (b \parallel c \to (a \parallel_b c \to (b \parallel_a c \to a, b \parallel c))).$$

Indeed, consider any two equilibria $\langle a_1, b_1, c_1 \rangle$ and $\langle a_2, b_2, c_2 \rangle$ of a game G. It will be sufficient to show that $\langle a_1, b_1, c_2 \rangle$ is a Nash equilibrium of the same game. By assumption $a \parallel c$, there is a strategy x of player b such that $\langle a_1, x, c_2\rangle$ is a Nash equilibrium of the game G. Note that $\langle a_1, b_1, c_1\rangle$ and $\langle a_1, x, c_2\rangle$ are also equilibria in the restricted game $G[a \mapsto a_1]$. Thus, $\langle a_1, b_1, c_2\rangle$ is a Nash equilibrium of the same restricted game, due to the assumption that $b \parallel_a c$. Hence, by Definition 2, for any strategy q of player b and for any strategy r of player c in the game G,

$$u_b(a_1, q, c_2) \leq u_b(a_1, b_1, c_2), \tag{11}$$

$$u_c(a_1, b_1, r) \leq u_c(a_1, b_1, c_2). \tag{12}$$

Next, since $\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$ are equilibria of the game G, by assumption $b \parallel c$, there is a strategy y of player a such that $\langle y, b_1, c_2\rangle$ is a Nash equilibrium of the game G. Note that $\langle a_1, b_1, c_1\rangle$ and $\langle y, b_1, c_2\rangle$ are also equilibria in the restricted game $G[b \mapsto b_1]$. Thus, $\langle a_1, b_1, c_2\rangle$ is a Nash equilibrium of the same restricted game, by the assumption $a \parallel_b c$. Hence, by Definition 2, for any strategy t of player a in the game G,

$$u_a(t, b_1, c_2) \leq u_a(a_1, b_1, c_2). \tag{13}$$

Inequalities (11), (12), (13), by Definition 2, imply that $\langle a_1, b_1, c_2\rangle$ is a Nash equilibrium of the original game G.

Similarly, one can show that the following principle is true in any four-player game with players a, b, c, and d:

$$a \parallel d \to (b \parallel c \to (a \parallel_b c \to (a \parallel_d c \to (b \parallel_c d \to (b \parallel_a d \to (a \parallel_{b,c} d \to (b \parallel_{a,d} c \to a, b \parallel c, d))))))).$$

The complete axiomatization of propositional properties of conditional interchangeability with conditioning by an arbitrary set of players remains an open question.

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