Inertial projection and contraction methods for solving variational inequalities with applications to image restoration problems

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Abstract

In this paper, we introduce two inertial self-adaptive projection and contraction methods for solving the pseudomonotone variational inequality problem with a Lipschitz-continuous mapping in real Hilbert spaces. The adaptive stepsizes provided by the algorithms are simple to update and their computations are more efficient and flexible. Also we prove some weak and strong convergence theorems without prior knowledge of the Lipschitz constant of the mapping. Finally, we present some numerical experiments to demonstrate the effectiveness of the proposed algorithms

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by comparisons with related methods and some applications of the proposed algorithms to the image deblurring problem.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H and $A : H \to H$ be a continuous mapping.

The *variational inequality problem* (shortly, VIP) is defined as follows:

$$\text{Find} \quad z \in C \quad \text{such that} \quad \langle Az, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the solution set of the VIP (1.1) by $VI(C,A)$. Several important applications of the VIP (1.1) have been discussed in, for instance, [2, 4, 12, 22–24, 28]. It is well known that a point z is a solution of the VIP (1.1) if and only if z solves the fixed point equation:

$$z = P_C(z - \lambda Az), \quad \forall \lambda > 0,$$

where P_C is the projection operator from H onto C. One of the earliest projection methods for solving VIP is the *extragradient method* (EGM) introduced independently by Antipin [3] and Korpelevich [25] as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

where $A : H \to H$ is monotone and L-Lipschitz continuous and suitable stepsize $\lambda \in \left(0, \frac{1}{L}\right)$. It was proved that the EGM converges weakly to a solution of VIP in finite dimensional spaces. However, the EGM requires two projections onto the feasible set C which can be computationally costly if A is not simple.

A question of interest in projection-type algorithms is how to reduce the number of projections in the algorithm. This has led to many modifications and improvements of the EGM by many authors.

In particular, Censor et al. [7] introduced the *subgradient extragradient method* (SEGM) for solving the VIP with a monotone and L-Lipschitz continuous mapping. The algorithm is described as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{1.2}$$

where

$$T_n := \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}.$$

The authors proved that SEM converges weakly to a solution of the VIP provided the stepsize $\lambda \in \left(0, \frac{1}{L}\right)$. Note that the T_n in (1.2) is a half-space and P_{T_n} can be easily calculated using the closed form formula.

Also, Tseng [38] introduced the following single projection method for solving the VIP. This method is known as the *Tseng extragradient method* (TEGM), which is described as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda (A y_n - A x_n). \end{cases} \tag{1.3}$$

It was proved that TEGM converges weakly to a solution of the VIP if the stepsize satisfies $\lambda \in \left(0, \frac{1}{L}\right)$. Note that the TEGM is more efficient than the EGM and its modifications due to its single projection onto the feasible set C per each iteration.

Another method based on the single projection onto C for solving the monotone VIP is the *projection and contraction method* (PCM) introduced by He [14] (see also Sun [33]). The algorithm is stated as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = x_n - \gamma \eta_n d_n, \end{cases} \tag{1.4}$$

where $\gamma \in (0,2)$, $\lambda_n \in \left(0, \frac{1}{L}\right)$ and

$$\eta_n := \frac{\langle x_n - y_n, d_n \rangle}{\|d_n\|^2}, \quad d_n := x_n - y_n - \lambda_n (A x_n - A y_n). \tag{1.5}$$

They proved that PCM converges weakly to a solution of the VIP under appropriate assumptions. Recently, PCM for solving VIP has received great attention from many authors, who improved it in various ways (see, for example, [9, 11, 15, 16]).

However, the stepsizes of the methods SEGM, TEGM and PCM depend on the prior estimate of the Lipschitz constant L of the cost operator which is very difficult to estimate in practice.

In order to modified the method which stepsize does not require prior estimate of the Lipschitz constant and extend to more general class of the monotone VIPs, Thong and Vuong [36] proposed a modification of the TEGM with a linesearch procedure for solving the VIP with a pseudomonotone and Lipschitz continuous mapping in Hilbert spaces. To be more precise, they proposed the following algorithm:

Algorithm A. [The TEGM for the pseudomonotone VIP]

Step 0: Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in (0,1)$. Let $x_1 \in H$ be arbitrary.

Step 1: Calculate

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|.$$

Step 2: Calculate

$$x_{n+1} = y_n - \lambda_n (Ay_n - Ax_n).$$

Update $n := n+1$ go to **Step 1**.

They proved that, if we assume that $A : H \to H$ is sequentially weakly continuous, then the sequence $\{x_n\}$ generated by Algorithm A converges weakly to a point of $VI(C,A)$.

Very recently, Khanh et al. [21] also proposed the following SEGM for solving the VIP with a pseudomonotone and Lipschitz continuous mapping in Hilbert spaces:

Algorithm B. [The SEGM for the pseudomonotone VIP]

Step 0: Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in (0,1)$. Let $x_1 \in H$ be arbitrary.

Step 1: Calculate

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|.$$

Step 2: Construct the half-space

$$T_n := \{x \in H : \langle x_n - \lambda_n Ax_n - y_n, x - y_n \rangle \leq 0\}$$

and calculate

$$x_{n+1} = P_{T_n}(x_n - \lambda_n Ay_n).$$

Update $n := n+1$ go to **Step 1**.

The weak convergence of the sequence $\{x_n\}$ generated by Algorithm B was established under assuming the weak sequential continuity of A, which often assumed in many recent works related to the pseudomonotone VIP (see, for example, [5,8,17,21,26,36,41]). In most cases, the strong convergence is also preferable to the weak convergence in many problems that arise in infinite-dimensional spaces because the weak convergence of algorithms does not allow to enable efficient.

On the other hand, the inertial method has been a technique of interest and has received a lot of attention from many researchers. Recently, the inertial technique is often used to accelerated the convergence rate of algorithms to solves many kinds of optimization (see, for example, [1, 8, 9, 16, 18, 27, 30, 35, 39, 41]).

Motivated and inspired by the above work, in this paper, we propose two modified inertial projection and contraction methods with self adaptive stepsize rules to the solve pseudomonotone variational inequality problem in real Hilbert spaces. This adaptive stepsize rules are more efficient and flexible in computations without any linesearch procedure which can be time-consuming and expensive. Also we prove some weak and strong convergence theorems for the proposed methods without any prior knowledge of the Lipschitz constant of the mapping and without assuming the weak sequential continuity of the mapping.

The rest of the paper is divided as follows: In Sect. 2, we provide some preliminary results which are need for our work. In Sect. 3, we prove some weak and strong convergence theorems for the proposed methods. Finally, in Sect. 4, we give some numerical experiments including comparisons with other algorithms and the applications of the proposed algorithms in the image debluring problem.

2 Preliminaries

Let H be a real Hilbert space. For a sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup z$ to indicate that the sequence $\{x_n\}$ converges weakly to a point $x \in H$ and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to a point $x \in H$. A point $x \in H$ is called a *weak cluster point* of a sequence $\{x_n\}$ in H if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point $x \in H$.

For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we know the following inequalities:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \tag{2.1}$$

and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.2}$$

Let $A : H \to H$ be a mapping. Then A is said to be:

(1) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \le L\|x-y\|, \quad \forall x, y \in H,$$

and, if $L \in [0,1)$, then A is called *contraction*;

(2) *monotone* if

$$\langle Ax - Ay, x-y\rangle \ge 0, \quad \forall x, y \in H;$$

(3) *pseudomonotone* if

$$\langle Ax, y-x\rangle \ge 0 \implies \langle Ay, y-x\rangle \ge 0, \quad \forall x, y \in H;$$

(4) *sequentially weakly continuous* if, for each sequence $\{x_n\} \in H$, $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

Remark 2.1. It is observe that every monotone mapping is a pseudomonotone mapping. Indeed, let $A : H \to H$ be a monotone mapping such that $\langle Ax, y-x\rangle \ge 0$ for all $x, y \in H$. It follows that

$$\langle Ay, y-x\rangle = \underbrace{\langle Ay - Ax, y-x\rangle}_{\ge 0} + \underbrace{\langle Ax, y-x\rangle}_{\ge 0} \ge 0$$

for all $x, y \in H$. Hence A is a pseudomonotone mapping, but the converse implication is not true. Several examples of a pseudomonotone mapping which is not necessarily monotone can be found in [5, 20, 32].

Let C be a nonempty closed and convex subset of H. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

Such a mapping P_C is called the *metric projection* of H onto C. The following is well known:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0. \tag{2.3}$$

Let A be a mapping of C into H. Then we know the following property [34]:

$$z \in VI(C,A) \iff z = P_C(z - \lambda Az), \quad \forall \lambda > 0. \tag{2.4}$$

The following are explicit formulas of the metric projection on various feasible sets [6]:

(1) A half-space in H has the form $H_{(a,\beta)} := \{x \in H : \langle a, x\rangle \leq \beta\}$, where $a \in H$, $a \neq 0$ and $\beta \in \mathbb{R}$. Then the projection of x onto $H_{(a,\beta)}$ is given by

$$P_{H_{(a,\beta)}}(x) = \begin{cases} x - \max\left\{\frac{\langle a,x\rangle - \beta}{\|a\|^2}, 0\right\} a & \text{if } \langle a, x\rangle > \beta, \\ x & \text{if } \langle a, x\rangle \leq \beta. \end{cases}$$

(2) A ball $B[p,r] := \{x \in H : \|x - p\| \leq r\}$, where $r > 0$. Then the projection of x onto $B[p,r]$ is given by

$$P_{B[p,r]}(x) = \begin{cases} p + \frac{r}{\max\{\|x-p\|, r\}}(x - p) & \text{if } \|x - p\| > r, \\ x & \text{if } \|x - p\| \leq r. \end{cases}$$

(3) A box constraints in $\mathbb{R}^n$ have the form $\text{Box}[a,b] := \{x \in \mathbb{R}^n : a \leq x \leq b\}$, where $a, b \in \mathbb{R}^n$ and $a \leq b$. Then the projection of x onto $\text{Box}[a,b]$ is given by

$$P_{\text{Box}[a,b]}(x)_i = \min\{b_i, \max\{x, a_i\}\}.$$

We need the following lemmas and facts, which will play an important role in proving our main results.

Lemma 2.2. [29] *Let C be a nonempty set of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each $x \in C$;*

(ii) *every weak cluster point of $\{x_n\}$ is in C.*

Then $\{x_n\}$ converges weakly to a point in C.

Let $\{a_n\}$ be a real sequence. Then we have

$$\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty}(a_n).$$

In particular, if $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then we obtain the following:

(1) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$;

(2) $\liminf_{n\to\infty}(a_n + b_n) \ge \liminf_{n\to\infty}(a_n) + \liminf_{n\to\infty}(b_n)$.

Lemma 2.3. [1] *Let $\{\varphi_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be three nonnegative real sequences such that*

$$\varphi_{n+1} \le \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \beta_n, \quad \forall n \ge 1,$$

with $\sum_{n=1}^{\infty} \beta_n < \infty$ and there exists a real number α such that $0 \le \alpha_n \le \alpha < 1$ for all $n \in \mathbb{N}$. Then the following results hold:

(1) $\sum_{n=1}^{\infty}[\varphi_n - \varphi_{n-1}]_+ < \infty$, *where* $[t]_+ := \max\{t, 0\}$.

(2) *There exists $\varphi^* \in [0,\infty)$ such that* $\lim_{n\to\infty} \varphi_n = \varphi^*$.

Lemma 2.4. [27] *Let $\{a_n\}$ and $\{c_n\}$ be two nonnegative real sequences such that*

$$a_{n+1} \le (1-\alpha_n)a_n + b_n + c_n, \quad \forall n \ge 1.$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(1) *If* $b_n \leq \alpha_n M$ *for some* $M \geq 0$*, then* $\{a_n\}$ *is a bounded sequence.*

(2) *If* $\sum_{n1}^{\infty} \alpha_n = \infty$ *and* $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [31] *Let* $\{a_n\}$ *be a nonnegative real sequence,* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{b_n\}$ *be a real sequence. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we introduce two new modified inertial projection and contraction algorithms with adaptive stepsize rule for solving the psuedomonotone VIP. In order to prove the convergence results of the proposed algorithms, we need the following conditions:

Condition 1: The feasible set C is a closed and convex subset of a real Hilbert space H.

Condition 2: The mapping $A : H \to H$ is L-Lipschitz continuous and pseudomonotone on H.

Condition 3: The mapping $A : H \to H$ satisfies the following condition:

$$\text{whenever } \{q_n\} \subset C,\ q_n \rightharpoonup q \text{ one has } \|Aq\| \leq \liminf_{n\to\infty} \|Aq_n\|.$$

Condition 4: The solution set of VIP is nonempty, that is, $VI(C,A) \neq \emptyset$.

Remark 3.1. (1) If H is a finite-dimensional space, then it suffices to assume that the mapping A is continuous pseudomonotone and the Condition 3 is not necessary to assume.

(2) The Condition 3 is weaker than the sequential weak continuity of the mapping A. Indeed, let $A : \ell_2 \to \ell_2$ be a mapping defined by $Ax = x\|x\|$ for all $x \in \ell_2$. Let $\{q_n\} \subset \ell_2$ such that $q_n \rightharpoonup q$. By the weak lower semicontinuity of the norm, we have $\|q\| \leq \liminf_{n\to\infty} \|q_n\|$ and so

$$\|Aq\| = \|q\|^2 \leq (\liminf_{n\to\infty} \|q_n\|)^2 \leq \liminf_{n\to\infty} \|q_n\|^2 = \liminf_{n\to\infty} \|Aq_n\|.$$

To show that A is not sequentially weakly continuous, choose $q_n = e_n + e_1$, where $\{e_n\}$ is a standard basis of ℓ_2, that is, $e_n = (0,0,\cdots,1,\cdots)$ with 1 at the n-th position. It is clear that $q_n \rightharpoonup$

e_1 and $Aq_n = A(e_n + e_1) = (e_n + e_1)\|e_n + e_1\| \rightharpoonup \sqrt{2}e_1$ but $Ae_1 = e_1\|e_1\| = e_1$. However, if A is monotone, then the Condition 3 is not necessary to assume.

3.1 The weak convergence

In this subsection, we propose a modified inertial projection and contraction algorithm for solving the psuedomonotone VIP.

Algorithm 1:

Initialization: Given $\lambda_1 > 0$, $\mu \in (0,1)$ and $\gamma \in \left(1, \frac{2}{\sigma}\right)$, where $\sigma \in (1,2)$. Choose $\{\theta_n\} \subset [0,1)$.

Iterative Steps: Let $x_0, x_1 \in H$ be arbitrary and calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Compute

$$u_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2. Compute

$$y_n = P_C(u_n - \lambda_n Au_n).$$

If $u_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of VIP. Otherwise, go to **Step 3**.

Step 3. Compute

$$x_{n+1} = u_n - \gamma\eta_n d_n,$$

where η_n and d_n are defined as follows:

$$\eta_n := (1-\mu)\frac{\|u_n - y_n\|^2}{\|d_n\|^2}, \quad d_n := u_n - y_n - \lambda_n(Au_n - Ay_n), \tag{3.1}$$

and update stepsize by

$$\lambda_{n+1} = \min\left\{\frac{\mu\|u_n - y_n\|}{\|Au_n - Ay_n\|}, \lambda_n\right\}. \tag{3.2}$$

Set $n := n+1$ and return to **Step 1**.

Lemma 3.2. [42] *The sequence $\{\lambda_n\}$ generated by (3.2) is nonincreasing and* $\lim_{n\to\infty}\lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}$.

Lemma 3.3. *Let $\{u_n\}$, $\{y_n\}$ and $\{d_n\}$ be the sequences generated by Algorithm 1. If there exists $n \geq n_0 \in \mathbb{N}$ such that $u_n = y_n$ or $d_n = 0$, then $y_n \in VI(C,A)$.*

Proof. By the definition of d_n, we have

$$\begin{aligned}\|d_n\| &= \|u_n - y_n - \lambda_n(Au_n - Ay_n), u_n - y_n\| \\ &\geq \|u_n - y_n\| - \lambda_n\|Au_n - Ay_n\| \\ &\geq \|u_n - y_n\| - \mu\frac{\lambda_n}{\lambda_{n+1}}\|u_n - y_n\| \\ &= \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|u_n - y_n\|.\end{aligned}$$

Since

$$\lim_{n\to\infty}\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > \frac{1-\mu}{\sigma} > 0,$$

there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0, \quad \forall n \geq n_0$$

and so

$$\|d_n\| \geq \frac{1-\mu}{\sigma}\|u_n - y_n\|, \quad \forall n \geq n_0.$$

It is observe that $\|d_n\| > 0$ for all $n \geq n_0$. Indeed, if there exists $n \geq n_0$ such that $\|d_n\| = 0$ or, equivalently, $d_n = 0$, then $u_n = y_n$. Therefore, y_n is a solution of the VIP. This completes the proof. □

Lemma 3.4. *Suppose that Conditions 1-4 hold. Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then, for each $p \in VI(C,A)$ and $n \geq n_0$, we have*

$$\|x_{n+1} - p\|^2 \leq \|u_n - p\|^2 - \frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - u_n\|^2.$$

Proof. For each $p \in VI(C,A)$, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|u_n - \gamma\eta_n d_n - p\|^2 \\ &= \|u_n - p\|^2 - 2\gamma\eta_n\langle u_n - p, d_n\rangle + \gamma^2\eta_n^2\|d_n\|^2. \qquad (3.3)\end{aligned}$$

By the definition of d_n, it follows that

$$\begin{aligned}\langle u_n - p, d_n\rangle &= \|u_n - y_n\|^2 - \lambda_n\langle u_n - y_n, Au_n - Ay_n\rangle + \langle y_n - p, d_n\rangle \\ &\geq \|u_n - y_n\|^2 - \lambda_n\|u_n - y_n\|\|Au_n - Ay_n\| + \langle y_n - p, d_n\rangle \\ &\geq \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|u_n - y_n\|^2 + \langle y_n - p, d_n\rangle.\end{aligned}$$

Since

$$\lim_{n\to\infty}\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > \frac{1-\mu}{\sigma} > 0,$$

there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$1 - \mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0.$$

Thus we have

$$\langle u_n - p, d_n\rangle \geq \frac{1-\mu}{\sigma}\|u_n - y_n\|^2 + \langle d_n, y_n - p\rangle, \quad \forall n \geq n_0. \tag{3.4}$$

Since $y_n = P_C(u_n - \lambda_n Au_n)$, it follows from (2.3) that

$$\langle u_n - \lambda_n Au_n - y_n, y_n - p\rangle \geq 0. \tag{3.5}$$

Using the fact that $\langle Ap, y_n - p\rangle \geq 0$ and the pseudomonotonicity of A, we have

$$\langle Ay_n, y_n - p\rangle \geq 0. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}\langle d_n, y_n - p\rangle &= \langle u_n - y_n - \lambda_n(Au_n - Ay_n), y_n - p\rangle \\ &= \langle u_n - \lambda_n Au_n - y_n, y_n - p\rangle + \lambda_n\langle Ay_n, y_n - p\rangle \\ &\geq 0. \end{aligned}\tag{3.7}$$

Combining (3.4) and (3.7), we obtain

$$\langle u_n - p, d_n\rangle \geq \frac{1-\mu}{\sigma}\|u_n - y_n\|^2, \quad \forall n \geq n_0.$$

By the definition of η_n, we have

$$\langle u_n - p, d_n\rangle \geq \frac{1}{\sigma}\eta_n\|d_n\|^2, \quad \forall n \geq n_0. \tag{3.8}$$

Combining (3.3) and (3.9), we get

$$\|x_{n+1}-p\|^2 \leq \|u_n-p\|^2 - \gamma\Big(\frac{2}{\sigma}-\gamma\Big)\eta_n^2\|d_n\|^2, \quad \forall n \geq n_0.$$

Since $x_{n+1} = u_n - \gamma\eta_n d_n$, we have

$$\eta_n^2\|d_n\|^2 = \frac{1}{\gamma^2}\|x_{n+1}-u_n\|^2.$$

Therefore, it follows that

$$\|x_{n+1}-p\|^2 \leq \|u_n-p\|^2 - \frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|x_{n+1}-u_n\|^2, \quad \forall n \geq n_0.$$

This completes the proof. □

Lemma 3.5. *Suppose that Conditions 1-4 hold. Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then we have*

$$\|u_n-y_n\|^2 \leq \Big(\frac{1+\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\Big)^2\|x_{n+1}-u_n\|^2.$$

Proof. By the definition of η_n, we have

$$\begin{aligned}\|u_n-y_n\|^2 &= \frac{1}{1-\mu}\cdot\eta_n\|d_n\|^2 \\ &= \frac{1}{1-\mu}\cdot\frac{1}{\gamma^2\eta_n}(\gamma^2\eta_n^2\|d_n\|^2) \\ &= \frac{1}{1-\mu}\cdot\frac{1}{\gamma^2\eta_n}\|x_{n+1}-u_n\|^2. \end{aligned} \tag{3.9}$$

Since $\|d_n\|^2 \leq (1+\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|u_n-y_n\|^2$, it follows that

$$\frac{1}{\|d_n\|^2} \geq \frac{1}{(1+\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|u_n-y_n\|^2}.$$

Hence we have

$$\eta_n = (1-\mu)\frac{\|u_n-y_n\|^2}{\|d_n\|^2} \geq \frac{1-\mu}{(1+\mu\frac{\lambda_n}{\lambda_{n+1}})^2}. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\|u_n - y_n\|^2 \le \left(\frac{1+\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\right)^2 \|x_{n+1} - u_n\|^2.$$

This completes the proof. □

Lemma 3.6. *Suppose that Conditions 1-4 hold. Let $\{x_n\}$ be the sequence generated by Algorithm 1. If $\{\theta_n\}$ is a nondecreasing sequence, then the following estimate holds: for each $p \in VI(C,A)$ and $n \ge n_0$,*

$$\Gamma_{n+1} \le \Gamma_n - \left(\left(\frac{2}{\sigma} - \gamma\right)\left(\frac{1-\theta_n}{\gamma}\right) - \xi_{n+1}\right)\|x_{n+1} - x_n\|^2,$$

where

$$\Gamma_n := \|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 + \xi_n\|x_n - x_{n-1}\|^2$$

and

$$\xi_n := \theta_n\left(1 + \theta_n + \left(\frac{2}{\sigma} - \gamma\right)\left(\frac{1-\theta_n}{\gamma}\right)\right).$$

Proof. Let $p \in VI(C,A)$. From (2.2), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|(1+\theta_n)(x_n - p) - \theta_n(x_{n-1} - p)\|^2 \\ &= (1+\theta_n)\|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.11}$$

It follows from Lemma 3.4 and (3.11) that, for all $n \ge n_0$,

$$\begin{aligned} \|x_{n+1} - p\|^2 \le\ & (1+\theta_n)\|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \\ & -\frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - u_n\|^2. \end{aligned} \tag{3.12}$$

On the other hand, from the equality $\|a-b\|^2 = \|a\|^2 - 2\langle a,b\rangle + \|b\|^2 \ge 0$, we have $2\langle a,b\rangle \le \|a\|^2 + \|b\|^2$. Hence we have

$$\begin{aligned} \|x_{n+1} - u_n\|^2 &= \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\ge \|x_{n+1} - x_n\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - \theta_n(\|x_{n+1} - x_n\|^2 + \|x_n - x_{n-1}\|^2) \\ &= (1-\theta_n)\|x_{n+1} - x_n\|^2 + (\theta_n^2 - \theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\|x_{n+1}-p\|^2 \leq (1+\theta_n)\|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2-\left(\frac{2}{\sigma}-\gamma\right)\left(\frac{1-\theta_n}{\gamma}\right)\|x_{n+1}-x_n\|^2 \\ +\theta_n\left(1+\theta_n+\left(\frac{2}{\sigma}-\gamma\right)\left(\frac{1-\theta_n}{\gamma}\right)\right)\|x_n-x_{n-1}\|^2 \tag{3.14}$$

for all $n \geq n_0$. We put

$$\xi_n := \theta_n\left(1+\theta_n+\left(\frac{2}{\sigma}-\gamma\right)\left(\frac{1-\theta_n}{\gamma}\right)\right)$$

for all $n \geq n_0$. Since the sequence $\{\theta_n\}$ is nondecreasing, it follows from (3.14) that

$$\begin{aligned} & \|x_{n+1}-p\|^2-\theta_{n+1}\|x_n-p\|^2+\xi_{n+1}\|x_{n+1}-x_n\|^2 \\ \leq\ & \|x_n-p\|^2-\theta_n\|x_{n-1}-p\|^2+\xi_n\|x_n-x_{n-1}\|^2 \\ & +\left(\xi_{n+1}-\left(\frac{2}{\sigma}-\gamma\right)\left(\frac{1-\theta_n}{\gamma}\right)\right)\|x_{n+1}-x_n\|^2. \end{aligned} \tag{3.15}$$

By the definition of Γ_n, we can write (3.15) as

$$\Gamma_{n+1} \leq \Gamma_n-\left(\left(\frac{2}{\sigma}-\gamma\right)\left(\frac{1-\theta_n}{\gamma}\right)-\xi_{n+1}\right)\|x_{n+1}-x_n\|^2, \quad \forall n \geq n_0.$$

This completes the proof. □

Lemma 3.7. [37] *Suppose that Conditions 1-4 hold. Let $\{u_n\}$and $\{y_n\}$ be the sequences generated by Algorithm 1. If there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $v \in H$ and $\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0$, then $v \in VI(C,A)$.*

Now, we prove the weak convergence theorem of Algorithm 1.

Theorem 3.8. *Suppose that Conditions 1-4 hold. Let $\beta := \frac{1}{\gamma}\left(\frac{2}{\sigma}-\gamma\right)$. Suppose, in addition, that $\{\theta_n\}$ is a nondecreasing sequence such that $0 \leq \theta_n \leq \theta_{n+1} \leq \theta$ for all $n \geq n_0$, where $\theta < \frac{\sqrt{1+8\beta}-2\beta-1}{2(1-\beta)}$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $VI(C,A)$.*

Proof. Since $0 \le \theta_n \le \theta_{n+1} \le \theta$ with $\theta < \frac{\sqrt{1+8\beta}-2\beta-1}{2(1-\beta)}$, it follows that

$$\begin{aligned}
(1-\theta_n)\beta - \xi_{n+1} &= (1-\theta_n)\beta - \theta_{n+1}(1+\theta_{n+1}+(1-\theta_{n+1})\beta)\\
&\ge (1-\theta_{n+1})\beta - \theta_{n+1}(1+\theta_{n+1}+(1-\theta_{n+1})\beta)\\
&\ge (1-\theta)\beta - \theta(1+\theta+(1-\theta)\beta)\\
&= -(1-\beta)\theta^2 - (2\beta+1)\theta + \beta.
\end{aligned}$$

Let $\delta := -(1-\beta)\theta^2 - (2\beta+1)\theta + \beta$. It is easy to see that $\delta > 0$. Then it follows from Lemma 3.6 that

$$\Gamma_{n+1} - \Gamma_n \le -\delta \|x_{n+1} - x_n\|^2, \quad \forall n \ge n_0. \tag{3.16}$$

This implies that $\Gamma_{n+1} - \Gamma_n \le 0$ for all $n \ge n_0$ and so $\{\Gamma_n\}$ is nonincreasing. Thus, by the definition of Γ_n, it follows that, for all $n \ge n_0$,

$$\begin{aligned}
\|x_n - p\|^2 &= \Gamma_n + \theta_n\|x_{n-1}-p\|^2 - \xi_n\|x_n - x_{n-1}\|^2\\
&\le \Gamma_n + \theta_n\|x_{n-1}-p\|^2\\
&\le \Gamma_{n_0} + \theta\|x_{n-1}-p\|^2\\
&\cdots\\
&\le \theta^{n-n_0}\|x_{n_0}-p\|^2 + \Gamma_{n_0}(1+\theta+\theta^2+\ldots+\theta^{n-n_0-1})\\
&\le \theta^{n-n_0}\|x_{n_0}-p\|^2 + \frac{\Gamma_{n_0}}{1-\theta}.
\end{aligned} \tag{3.17}$$

We also observe that

$$\begin{aligned}
\Gamma_{n+1} &= \|x_{n+1}-p\|^2 - \theta_{n+1}\|x_n - p\|^2 + \xi_{n+1}\|x_{n+1}-x_n\|^2\\
&\ge -\theta_{n+1}\|x_n - p\|^2\\
&\ge -\theta\|x_n - p\|^2.
\end{aligned} \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we have

$$\begin{aligned}
\delta\sum_{n=n_0}^{k}\|x_{n+1}-x_n\|^2 &\le \delta\sum_{n=n_0}^{k}(\Gamma_n - \Gamma_{n+1})\\
&= \Gamma_{n_0} - \Gamma_{k+1}\\
&\le \Gamma_{n_0} + \theta\|x_k - p\|^2\\
&\le \Gamma_{n_0} + \theta^{k-n_0+1}\|x_{n_0}-p\|^2 + \frac{\theta\Gamma_{n_0}}{1-\theta}.
\end{aligned}$$

Thus we have

$$\delta \sum_{n=n_0}^{\infty} \|x_{n+1}-x_n\|^2 = \lim_{k\to\infty} \left(\delta \sum_{n=n_0}^{k} \|x_{n+1}-x_n\|^2\right) < \infty. \tag{3.19}$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0.$$

Consequently, we have

$$\|x_n-u_n\| = \theta_n\|x_n-x_{n-1}\| \le \theta\|x_n-x_{n-1}\| \to 0 \text{ as } n\to\infty. \tag{3.20}$$

Now, we see that

$$\begin{aligned}\|x_{n+1}-u_n\| &= \|x_{n+1}-x_n-\theta_n(x_n-x_{n-1})\| \\ &\le \|x_{n+1}-x_n\|+\theta_n\|x_n-x_{n-1}\| \\ &\le \|x_{n+1}-x_n\|+\theta\|x_n-x_{n-1}\|\end{aligned}$$

and so

$$\lim_{n\to\infty} \|x_{n+1}-u_n\| = 0.$$

Also, by Lemma 3.5, we obtain

$$\lim_{n\to\infty} \|u_n-y_n\| = 0. \tag{3.21}$$

Since $\{x_n\}$ is bounded, without loss of generality, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$ for some $v \in H$. From (3.20), we also get $u_{n_k} \rightharpoonup v$. This together with (3.20) and Lemma 3.7 concludes that $v \in VI(C,A)$.

On the other hand, from (3.12), it follows that, for all $n \ge n_0$,

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2+\theta_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2)+\theta(1+\theta)\|x_n-x_{n-1}\|^2. \tag{3.22}$$

From (3.19) and Lemma 2.3, we can show that $\lim_{n\to\infty}\|x_n-p\|^2$ exists. In summary, we have shown that

- $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p \in VI(C,A)$;

- every weak cluster point of $\{x_n\}$ is in $VI(C,A)$.

Therefore, by Lemma 2.2, we conclude that $\{x_n\}$ converges weakly to a point in $VI(C,A)$. This completes the proof. □

3.2 The strong convergence

In this subsection, we propose another inertial algorithm which combines the viscosity approximation method and the projection and contraction algorithm with adaptive stepsize rule for solving the psuedomonotone VIP.

In order to obtain the strong convergence, we assume that $f : H \to H$ is a contraction mapping with constant $\alpha \in [0,1)$. Suppose, in addition, that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0,$$

where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The algorithm is formulated as follows:

Algorithm 2:

Initialization: Given $\lambda_1 > 0$, $\mu \in (0,1)$ and $\gamma \in \left(0, \frac{2}{\sigma}\right)$, where $\sigma \in (1,2)$. Choose $\{\theta_n\} \subset [0,\theta)$ for some $\theta > 0$.

Iterative Steps: Let $x_0, x_1 \in H$ be arbitrary and calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Compute

$$u_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2. Compute

$$y_n = P_C(u_n - \lambda_n A u_n).$$

If $u_n = y_n$ or $Ay_n = 0$, then stop and y_n is a solution of VIP. Otherwise, go to **Step 3**.

Step 3. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(u_n - \gamma\eta_n d_n),$$

where η_n and d_n are defined in (3.1), and update the stepsize by (3.2).

Set $n := n+1$ and return to **Step 1**.

Theorem 3.9. *Suppose that Conditions 1-4 hold. Then the sequence $\{x_n\}$ generated by Algorithm 2* *converges strongly to* $z = P_{VI(C,A)}f(z)$.

Proof. For each $n \geq 1$, let $z_n := u_n - \gamma\eta_n d_n$ and $p \in VI(C,A)$. Following the similar argument in Lemma 3.4, it follows that, for each $n \geq n_0$,

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \frac{1}{\gamma}\Big(\frac{2}{\sigma} - \gamma\Big)\|z_n - u_n\|^2. \tag{3.23}$$

This gives

$$\|z_n - p\| \leq \|u_n - p\|. \tag{3.24}$$

Moreover, we have

$$\begin{aligned} \|u_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(z_n - p)\| \\ &\leq \alpha_n\|f(x_n) - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\theta_n\|x_n - x_{n-1}\| \\ &= (1-(1-\alpha)\alpha_n)\|x_n - p\| + (1-\alpha)\alpha_n\Big[\frac{\|f(p) - p\|}{1-\alpha} + \frac{(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|}{(1-\alpha)\alpha_n}\Big]. \end{aligned}$$

Put

$$\mu_n := \frac{\|f(p) - p\|}{1-\alpha} + \frac{(1-\alpha_n)\theta_n\|x_n - x_{n-1}\|}{(1-\alpha)\alpha_n}.$$

It is easy to see that $\lim_{n\to\infty}\mu_n$ exists. So there exists $M > 0$ such that $\mu_n \leq M$ for all $n \in \mathbb{N}$. By Lemma 2.4, we know that $\{\|x_n - p\|\}$ is bounded. Moreover, we see that $\|x_n\| \leq \|x_n - p\| + \|p\|$. This implies that $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{y_n\}$ and $\{d_n\}$.

Now, let $z = P_{VI(C,A)}f(z)$. From (2.1), we have

$$\begin{aligned} \|u_n - z\|^2 &= \|x_n - z + \theta_n(x_n - x_{n-1})\|^2 \\ &\leq \|x_n - z\|^2 + 2\theta_n\langle x_n - x_{n-1}, u_n - z\rangle \\ &\leq \|x_n - z\|^2 + 2\theta_n\|x_n - x_{n-1}\|K, \end{aligned} \tag{3.26}$$

where $K=\sup_{n\geq 1}\{\|u_n-z\|\}$. By the convexity of $\|\cdot\|^2$ and (3.23), it follows that, for all $n\geq n_0$,

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n(f(x_n)-z)+(1-\alpha_n)(z_n-p)\|^2\\ &\leq \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\|z_n-z\|^2\\ &\leq \alpha_n\|f(x_n)-z\|^2+\|z_n-z\|^2\\ &\leq \alpha_n\|f(x_n)-z\|^2+\|u_n-z\|^2-\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|z_n-u_n\|^2. \end{aligned}\tag{3.27}$$

Combining (3.26) and (3.27), it follows that, for all $n\geq n_0$,

$$\|x_{n+1}-z\|^2\leq \alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2+2\theta_n\|x_n-x_{n-1}\|K-\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|z_n-u_n\|^2,$$

which implies that

$$\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|z_n-u_n\|^2\leq \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\|f(x_n)-z\|^2+2\theta_n\|x_n-x_{n-1}\|K.\tag{3.28}$$

On the other hand, from (2.1), (3.23) and (3.26), it follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n(f(x_n)-z)+(1-\alpha_n)(z_n-p)\|^2\\ &= \|\alpha_n(f(x_n)-f(z))+(1-\alpha_n)(z_n-z)+\alpha_n(f(z)-z)\|^2\\ &\leq \|\alpha_n(f(x_n)-f(z))+(1-\alpha_n)(z_n-z)\|^2+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &\leq \alpha_n\|f(x_n)-f(z)\|^2+(1-\alpha_n)\|z_n-z\|^2+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &\leq \alpha_n\alpha\|x_n-z\|^2+(1-\alpha_n)\big[\|x_n-z\|^2+2\theta_n\|x_n-x_{n-1}\|K\big]\\ &\quad +2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &= (1-(1-\alpha)\alpha_n)\|x_n-z\|^2+2(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|K\\ &\quad +2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle. \end{aligned}\tag{3.29}$$

Now, we show that the sequence $\{\|x_n-z\|^2\}$ converges to zero. In order to do this, using Lemma 2.5, it is sufficient to show that

$$\limsup_{k\to\infty}\langle f(z)-z,x_{n_k+1}-z\rangle\leq 0$$

for every subsequence $\{\|x_{n_k}-z\|\}$ of $\{\|x_n-z\|\}$ satisfying

$$\liminf_{k\to\infty}\big[\|x_{n_k+1}-z\|-\|x_{n_k}-z\|\big]\geq 0.$$

Let $\{\|x_{n_k}-z\|\}$ be a subsequence of $\{\|x_n-z\|\}$ such that

$$\liminf_{k\to\infty}\left[\|x_{n_k+1}-z\|-\|x_{n_k}-z\|\right]\geq 0.$$

Thus we have

$$\begin{aligned}\liminf_{k\to\infty}\left[\|x_{n_k+1}-z\|^2-\|x_{n_k}-z\|^2\right] &= \liminf_{k\to\infty}\left[(\|x_{n_k+1}-z\|-\|x_{n_k}-z\|)(\|x_{n_k+1}-z\|+\|x_{n_k}-z\|)\right]\\ &\geq 0.\end{aligned}$$

From (3.28), it follows that

$$\begin{aligned}\limsup_{k\to\infty}\frac{1}{\gamma}\Big(\frac{2}{\sigma}-\gamma\Big)\|z_{n_k}-u_{n_k}\|^2 &\leq \limsup_{k\to\infty}\left[\|x_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2+\alpha_{n_k}\|f(x_{n_k})-z\|^2\right.\\ &\quad \left.+2\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|K\right]\\ &\leq \limsup_{k\to\infty}\left[\|x_{n_k}-z\|^2-\|x_{n_k+1}-z\|^2\right]+\limsup_{k\to\infty}\alpha_{n_k}\|f(x_{n_k})-z\|^2\\ &\quad +\limsup_{k\to\infty}2\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|K\\ &= -\liminf_{k\to\infty}\left[\|x_{n_k+1}-z\|^2-\|x_{n_k}-z\|^2\right]\\ &\leq 0.\end{aligned}$$

Hence we have

$$\lim_{k\to\infty}\|z_{n_k}-u_{n_k}\|=0. \tag{3.30}$$

As in the proof lines of Lemma 3.5, we can deduce that

$$\|u_{n_k}-y_{n_k}\|^2\leq\Big(\frac{1+\mu\frac{\lambda_{n_k}}{\lambda_{n_k+1}}}{\gamma(1-\mu)}\Big)^2\|z_{n_k}-u_{n_k}\|^2.$$

Thus it follows from (3.30) that

$$\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0. \tag{3.31}$$

Moreover, we see that

$$\|u_{n_k}-x_{n_k}\|=\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|=\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0. \tag{3.32}$$

It follows from (3.30) and (3.31) that

$$\|z_{n_k} - x_{n_k}\| \le \|z_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \tag{3.33}$$

and so

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \|\alpha_{n_k} f(x_{n_k}) + (1-\alpha_{n_k}) z_{n_k} - x_{n_k}\| \\ &\le \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\|z_{n_k} - x_{n_k}\| \\ &\to 0. \end{aligned}\tag{3.34}$$

Since $\{x_{n_k}\}$ is bounded, without loss of generality, we assume that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup v$ for some $v \in H$ and

$$\limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle = \lim_{j\to\infty}\langle f(z) - z, x_{n_{k_j}} - z\rangle.$$

From (3.32), we also get $u_{n_{k_j}} \rightharpoonup v$. This together with (3.31) and Lemma 3.7 concludes that $v \in VI(C,A)$. Hence we have

$$\limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle = \langle f(z) - z, v - z\rangle \le 0. \tag{3.35}$$

Combining (3.34) and (3.35), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle &\le \limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle \\ &\le 0. \end{aligned}\tag{3.36}$$

From (3.29), we can write it as

$$\begin{aligned}\|x_{n_k+1} - z\|^2 &\le (1-(1-\alpha)\alpha_{n_k})\|x_{n_k} - z\|^2 \\ &\quad + (1-\alpha)\alpha_{n_k}\Big[\frac{2(1-\alpha_{n_k})\theta_{n_k}\|x_{n_k} - x_{n_k-1}\|K}{(1-\alpha)\alpha_{n_k}} + \frac{2}{1-\alpha}\langle f(z) - z, x_{n_k+1} - z\rangle\Big].\end{aligned}$$

This together with (3.36) and Lemma 2.5 yields that $\lim_{k\to\infty}\|x_n - z\|^2 = 0$. Therefore, $x_n \to z$. This completes the proof. □

4 Numerical experiments

In this section, we give some numerical experiments in two parts. In the first part, we provide a comparison the numerical behaviour of the proposed algorithms and their algorithms with non-inertial terms to illustrate the efficiency and advantages of the proposed algorithms and also compare them with the following:

- Algorithm A: The TEGM for the pseudomonotone VIP [36];
- Algorithm B: The SEGM for the pseudomonotone VIP [21].

In the second part, we apply the proposed algorithms to solve the image restoration problem and compare the computational results with Algorithm A and Algorithm B. In the following, we denote "iter." and "time" by the number of iteration and the running time in seconds, respectively.

4.1 Numerical results

Problem 4.1. The variational inequality problem in infinite-dimensional spaces

Consider a Hilbert space $H := \ell_2 = \{x = (x_1, x_2, x_3, \cdots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with the norm $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$ and the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ for all $x = (x_1, x_2, x_3, \cdots), y = (y_1, y_2, y_3, \cdots) \in \ell_2$. Let $A : \ell_2 \to \ell_2$ be a mapping defined by

$$A(x_1, x_2, x_3, \cdots) = (x_1 e^{-x_1^2}, 0, 0, \cdots).$$

It was shown in [5, Example 2.1] that A is pseudomonotone, Lipschitz continuous and sequentially weakly continuous (hence A satisfies the Condition 3), but not monotone on ℓ_2. The feasible set is $C = \{x = (x_1, x_2, x_3, \cdots) \in \ell_2 : \|x\| \leq 1\}$ and then the projection onto C is easily calculated by the following formula:

$$P_C(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } \|x\| > 1, \\ x & \text{otherwise.} \end{cases}$$

In this experiment, for Algorithm 1, we take $\lambda_1 = 0.36, \mu = 0.54$, $\sigma = \frac{3}{2}$, $\gamma = \frac{7}{6}$, $\theta_n = \frac{9}{10}$ and, in addition, for Algorithm 2, we take $f(x) = \frac{x}{4}$, $\alpha_n = \frac{1}{n+1}$, $\theta_n = \alpha_n^2$. More so, for Algorithm A and Algorithm B, we take $\gamma = 3$, $l = 0.68$, $\mu = 0.34$. We perform numerical test of all algorithms with three different cases of the starting point as follows:

Case A: $x_0 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ and $x_1 = (1, 2, 3, \cdots)$;

Case B: $x_0 = (5, 5, 5, \cdots)$ and $x_1 = (2, 1, 2, \cdots)$;

Case C: $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $x_1 = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \cdots)$.

We use $TOL_n = \|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criteria in each algorithm. The numerical results are shown in Table 1 and Figure 1.

Table 1: Numerical results for Problem 4.1.

x_0, x_1	Alg 1		Alg 1 ($\theta_n = 0$)		Alg 2		Alg 2 ($\theta_n = 0$)		Alg A		Alg B	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
Case A	14	0.0065	28	0.0122	10	0.0024	14	0.0031	55	0.0193	87	0.0256
Case B	14	0.0057	33	0.0119	10	0.0041	20	0.0076	56	0.0129	64	0.0229
Case C	13	0.0048	37	0.0097	10	0.0022	20	0.0088	84	0.0209	76	0.0241

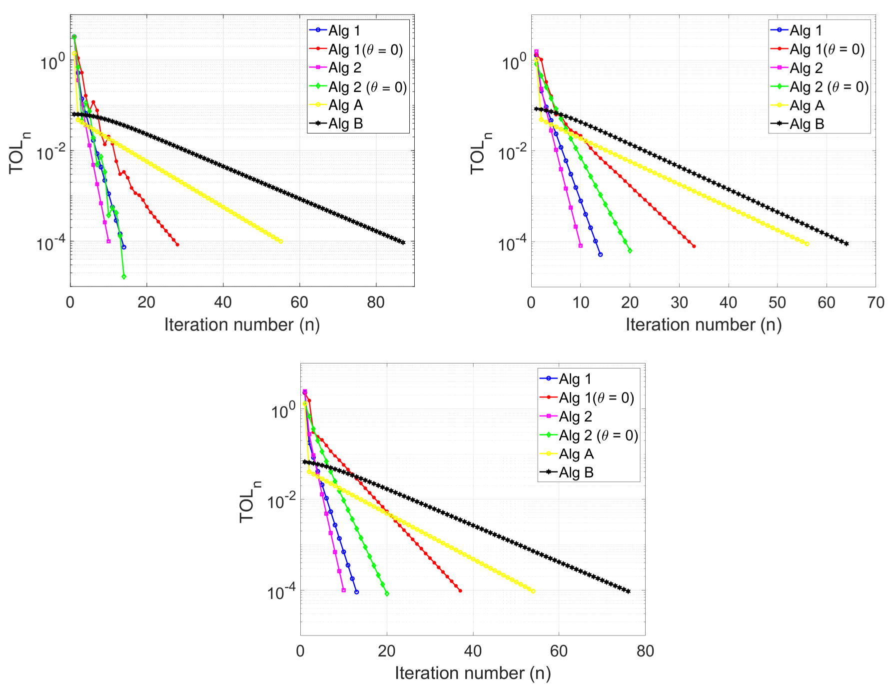

Figure 1: Example 4.1, Top Left: Case A; Top Right: Case B; Bottom: Case C.

Problem 4.2. The quadratic fractional programming problem

Consider the following quadratic fractional programming problem:

$$\min_{x\in C} f(x),$$

where $f(x)=\frac{x^TQx+a^Tx+c}{b^Tx+d}$ and $C=\{x\in\mathbb{R}^4: 1\le x_i\le 10,\ i=1,2,3,4\}$. Let

$$Q=\begin{pmatrix}5&-1&2&0\\-1&5&-1&3\\2&-1&3&0\\0&3&0&5\end{pmatrix},\ a=\begin{pmatrix}1\\-2\\-2\\1\end{pmatrix},\ b=\begin{pmatrix}2\\1\\1\\0\end{pmatrix},\ c=-2,\ d=4.$$

It is easy to see that Q is symmetric and positive definite on $\mathbb{R}^4$ and so f is pseudoconvex on $\mathbb{R}^4$. It is known that this problem is equivalent to the VIP [13, 23] with

$$Ax:=\nabla f(x)=\frac{(b^Tx+d)(2Qx+a)-b(x^TQx+a^Tx+c)}{(b^Tx+d)^2}.$$

It was shown in [19] that ∇f is continuous pseudomonotone. The VIP has unique solution is $z=(1,1,1,1)^T\in C$.

In this experiment, for Algorithm 1, we take $\lambda_1=0.28$, $\mu=0.45$, $\sigma=\frac{4}{3}$, $\gamma=\frac{5}{4}$, $\theta_n=\frac{3}{5}$ and, in addition, we take $f(x)=\frac{x}{8}$, $\alpha_n=\frac{1}{\sqrt{n+1}}$, $\theta_n=\frac{1}{n+1}$ for Algorithm 2. Also, for Algorithms A and B, we choose $\gamma=0.33$, $l=0.66$ and $\mu=0.64$.. We perform numerical test of all algorithms with three different cases of the starting point as follows:

Case A: $x_0=(2,2,2,2)^T$ and $x_1=(4,4,4,4)^T$;

Case B: $x_0=(3,3,3,3)^T$ and $x_1=(5,5,5,5)^T$;

Case C: $x_0=(2,0,0,4)^T$ and $x_1=(3,1,3,1)^T$.

Since we know the solution of the problem, we use $TOL_n=\|x_n-z\|<10^{-4}$ as the stopping criteria in each algorithm. The numerical results are shown in Table 2 and Figure 2.

Remark 4.3. From the above experimental results, we can summarize in the following points:

(1) Algorithm 1 and Algorithm 2 have less iteration numbers and computation times than algorithms without the inertial. It is remarkable that by adding the inertial $\theta_n(x_n-x_{n-1})$ makes convergence faster. This is main advantage of adding inertial term to algorithms in solving the problem.

(2) Algorithm 1 and Algorithm 2 with adaptive stepsizes have a better performance in terms

Table 2: Numerical results for Problem 4.2.

x_0, x_1	Alg 1		Alg 1 ($\theta_n = 0$)		Alg 2		Alg 2 ($\theta_n = 0$)		Alg A		Alg B	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
Case A	7	0.0038	13	0.0046	8	0.0044	12	0.0059	28	0.0138	37	0.0317
Case B	8	0.0036	14	0.0049	8	0.0038	12	0.0049	25	0.0145	37	0.0220
Case C	8	0.0036	16	0.0083	8	0.0030	12	0.0043	27	0.0132	37	0.0155

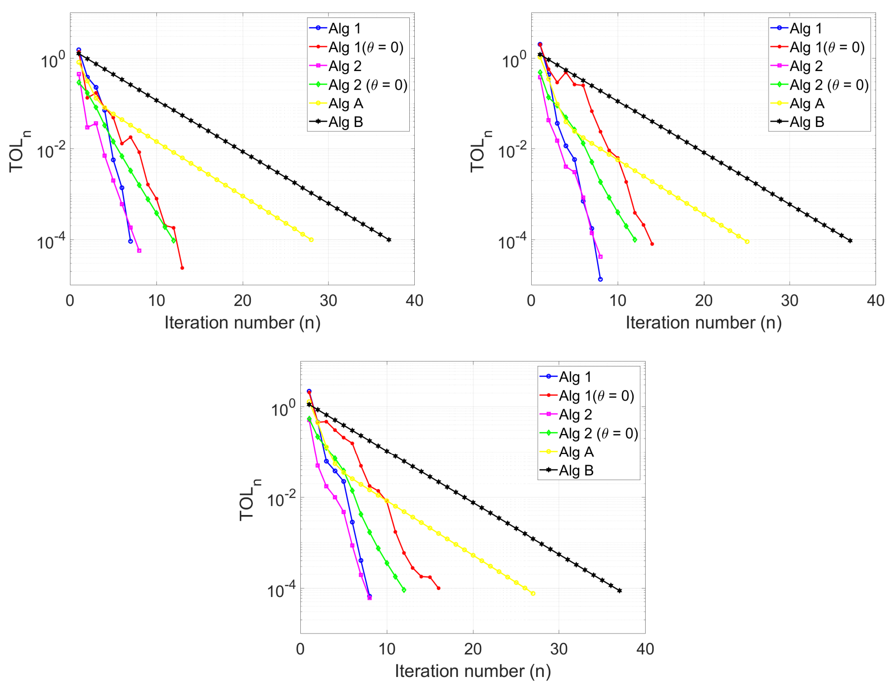

Figure 2: Example 4.2, Top Left: Case A; Top Right: Case B; Bottom: Case C.

of convergence speed than Algorithm A and Algorithm B with Armijo linesearch procedures. This shows that the proposed algorithms have higher superiority and efficiency than Algorithm A and Algorithm B in solving the pseudomonotone VIP. It is due to the fact that Armijo linesearch procedures use an inner-loop until some stopping criterion is reached. This may takes time-consuming in evaluations of the projections on the feasible set in each iteration.

4.2 Applications to the image restoration problem

The image restoration problem is one of the interest topics in image processing and computer vision. This problem has been extensively studied by many authors because of its applications in almost every field such as film restoration, image and video coding, medical and astronomical imaging (see, for example, [10, 40]). Image restoration is a process of recovering images from blurring and noise observation which is to improve the quality of the image. Recall that the image restoration problem can be formulated as the following linear inverse problem:

$$b = Bx + v, \tag{4.1}$$

where $x \in \mathbb{R}^N$ is the original image, $b \in \mathbb{R}^M$ is the degraded image, $B \in \mathbb{R}^{M\times N}$ is the blurring matrix and v is an additive noise. An efficient method for recovering the original image is the ℓ_1-norm regularized least square method given by

$$\min_{x\in\mathbb{R}^N} \left\{ \frac{1}{2}\|Bx-b\|_2^2 + \lambda\|x\|_1 \right\}. \tag{4.2}$$

Our main task is to restore the original image x given the data of the blurred image b. The least square problem (4.2) can be expressed as a variational inequality problem by setting $A := B^T(Bx-b)$. It is known that the operator A in this case is monotone (hence it is pseudomonotone) and Lipschitz continuous with $L = \|B^T B\|$. We consider the grey scale image of M pixels wide and N pixel height, each value is known to be in the range $[0,255]$. The quality of the restored image is measured by the signal to noise ratio (SNR) which is defined by

$$SNR = 20\log_{10}\left(\frac{\|x\|_2}{\|x-x^*\|_2}\right),$$

where x is the original image and x^* is the restored image. Note that the larger the value of SNR, the better the quality of the restored image. In our experiments, we use the grey test image Tire (291×240) and Cameraman (256×256), each test image is degraded by Gaussian 7×7 blur kernel with standard deviation 4 and the maximum iteration is set to be 1000. We choose $\lambda_1 = 0.5$, $\mu = 0.8$, $\sigma = 1.5$, $\gamma = 1$, $\theta_n = 0.99$, $\alpha_n = \frac{1}{100(n+1)}$, $f(x) = \frac{x}{4}$, $l = 0.3$, $\mu = 0.6$, $x_0 = \mathbf{0} \in \mathbb{R}^{\mathbf{D}}$ and $x_1 = \mathbf{1} \in \mathbb{R}^{\mathbf{D}}$, where $\mathbf{D} = M \times N$.

Figures 3 and 4 show the original, blurred and restored image by using Algorithm 1, Algorithm 2, Algorithm A and Algorithm B. Also, Figure 5 shows the graph of the SNR against number of iterations for each test image using the algorithms. Then we report the time for each algorithm in Table 3.

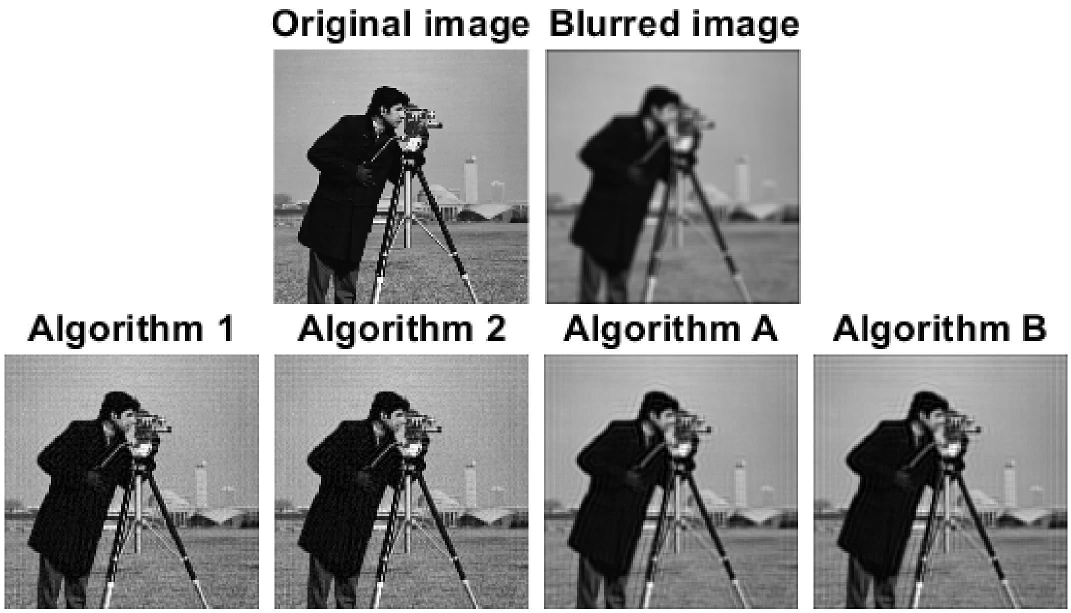

Figure 3: Top shows original image of Cameraman (left) and degraded image of Cameraman (right); Bottom shows recovered image by Algorithm 1, Algorithm 2, Algorithm A and Algorithm B.

Table 3: Computational results for Deblurring the images

Algorithms	Cameraman		Tire	
	SNR	time	SNR	time
Alg 1	34.2083	42.0504	31.4482	31.3819
Alg 2	38.7060	38.6614	30.4481	29.7643
Alg A	27.5191	91.8467	26.3005	64.4750
Alg B	29.7150	94.2898	28.9397	64.5418

Remark 4.4. From the obtained computational results, we show that both the quality of the restored images and running times of Algorithm 1 and Algorithm 2 are good as compared with Algorithm A and Algorithm B. This shows that the proposed algorithms are more efficient for restoring the degraded image than Algorithm A and Algorithm B.

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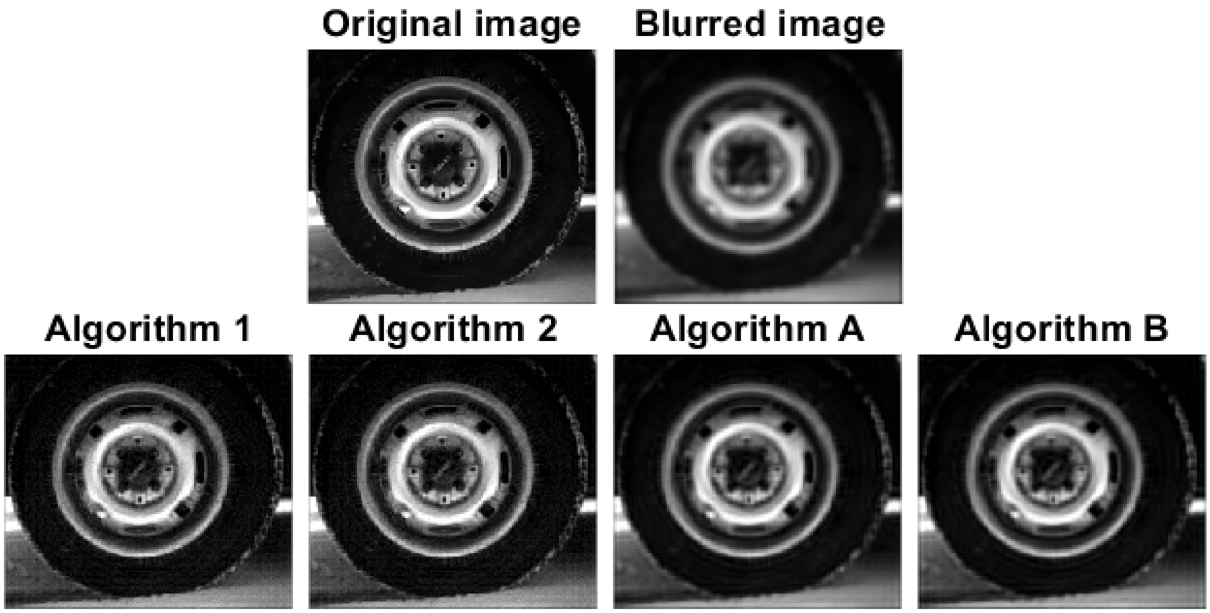

Figure 4: Top shows original image of Tire (left) and degraded image of Tire (right); Bottom shows recovered image by Algorithm 1, Algorithm 2, Algorithm A and Algorithm B.

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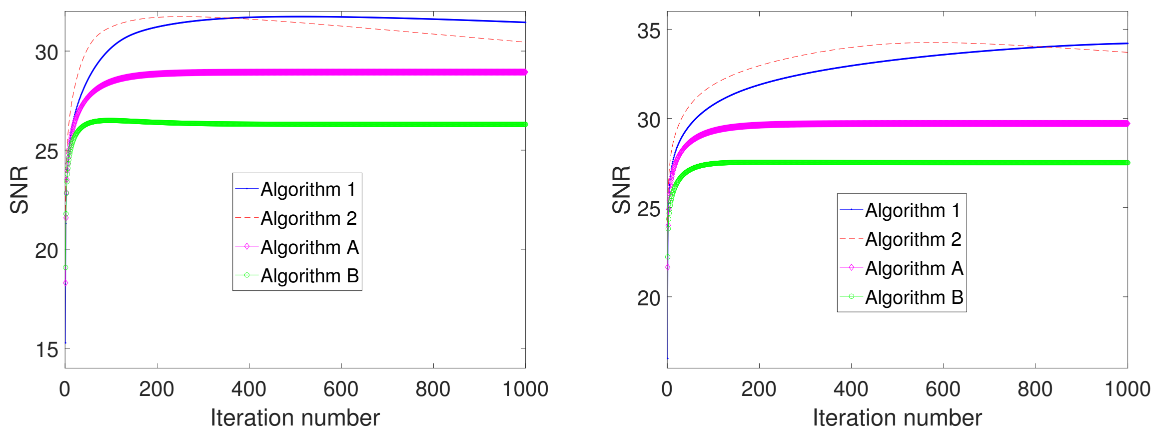

Figure 5: Graphs of the SNR values against number of iteration for Cameraman (Left) and Tire (Right).

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