Amplitudes at strong coupling as hyperkähler scalars

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Abstract

Alday & Maldacena conjectured an equivalence between string amplitudes in $\mathrm{AdS}_5 \times S^5$ fixed by null polygonal boundaries in Minkowski-space with both amplitudes and Wilson loops in planar $\mathcal{N} = 4$ super-Yang-Mills (SYM). At strong coupling this leads to an identification of SYM amplitudes with areas of minimal surfaces in AdS. Together with Gaiotto, Sever & Vieira, they introduced a 'Y-system' for computing this area. We first establish a correspondence between Y-systems and twistor spaces that will apply more generally, and which, in the cases considered here determine a geometry on the space of kinematic data. In the case of minimal surfaces in AdS_3 with boundaries on null polygons with $4k+2$ edges, we show that the geometry in question is a split signature pseudo-hyperkähler structures and that the remainder function for the amplitude is a Plebanski scalar that generates the geometry. This geometry leads to explicit overdetermined completely integrable systems of differential equations for the area, and we also give its Lax system.

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1 Introduction

In $\mathcal{N}=4$ super Yang-Mills, the geometry of the spaces of kinematic data, $\mathcal{K}$, has been a fertile ground for advancing the understanding of scattering amplitudes. Most of this work has focused on the cluster structures and positive geometries that arise in perturbation theory [1–3]. This paper initiates a study of the geometry that arises at strong coupling. We focus here on restricted kinematics. We find that the amplitude defines local differential geometric structures on the space of kinematics, $\mathcal{K}$. Our results are distinct in flavour from the more combinatorial ideas that arise naturally at weak coupling.

Alday-Maldacena conjectured a correspondence between three distinct objects: planar amplitudes, $\mathcal{A}$; planar null-polygonal Wilson-loops, $\langle \mathcal{W}_\gamma \rangle$, both for $\mathcal{N}=4$ super-Yang-Mills; and type IIB string amplitudes in $\mathrm{AdS}_5 \times S^5$. The correspondences are summarised by the equations:

$$\mathcal{A} = \langle \mathcal{W}_\gamma \rangle = \int_{\partial\Sigma=\gamma} \mathcal{D}[\Sigma \subset AdS_5 \times S^5]\, \mathrm{e}^{-\frac{1}{\alpha'} S_{\mathrm{string}}}, \tag{1.1}$$

where γ is a null polygon made up from the null momenta in the amplitude. The α' string parameter in this correspondence is related to the 't Hooft coupling, λ, by $R^2_{AdS}/\alpha' = \sqrt{\lambda}$. The first equality has been proved[3] in perturbation theory using MHV diagrams [4] and also by recursion arguments [5]. The second equality is a conjecture arising from the AdS/CFT correspondence. It has only been systematically investigated at strong coupling as $\lambda \to \infty$ (and $\alpha' \to 0$), where the equality becomes, using the semi-classical approximation for the string,

$$\langle \mathcal{W}_\gamma \rangle \sim \mathrm{e}^{-\mathrm{Area}(\Sigma)/\alpha'}, \tag{1.2}$$

where $\mathrm{Area}(\Sigma)$ is the area of the minimal surface, Σ, bounded by γ. Like the Wilson-loop, $\langle \mathcal{W}_\gamma \rangle$, the area of Σ is divergent. The minimal surface has cusps at infinity where it

[3] for the 4d all-loop integrand; the tree-level MHV amplitude is removed in the definition of $\mathcal{A}$.

meets the boundary. The divergences in the infinite areas of these cusps correspond to the infrared divergences of the amplitude. These divergences can be removed in such a way that agrees with our expectations for both the amplitude and Wilson loop. The result is a regularized area or remainder function, $R(\gamma)$, which is our main object of study.

Alday-Maldacena relate minimal surfaces in AdS to a Hitchin system and express the area as the Hamiltonian for a circle action that acts on a certain subspace of the Hitchin moduli space that has coordinates given by the kinematic data [6]. Hitchin moduli spaces are often hyperkähler [7, 8]. But to relate the Hitchin systems to minimal surfaces, discrete symmetries are imposed that restrict us to a subspace of the full Hitchin moduli space. This means that standard results (from, e.g., [9, 10]) do not directly apply, and this subspace is not generally expected to be hyperkähler, see for example those in [11]. However, we will show that these smaller moduli spaces are often *pseudo-hyperkähler*, i.e., the analogue of hyperkähler appropriate to metrics of split signature.

In restricted kinematics, we take the momenta and Wilson loop to lie in 1+1 dimensions and the spanning minimal surface lives in AdS_3. For restricted kinematics, we prove that the regularized area is a Kahler scalar for a pseudo-hyperkähler structure on the kinematic space $\mathcal{K}$, when $\mathcal{K}$ has $4k$ dimensions. A key step in the proof is to use the Y-system [12] to define a twistor space for $\mathcal{K}$, analogous to the twistor spaces for full Hitchin moduli spaces; we expect this novel connection between Y-systems and twistor constructions to be of much wider applicability. Having proved this, we derive a system of integrable equations satisfied by the regularized area, which can be used to solve for the area. In §2 we introduce the kinematic space $\mathcal{K}$ and both its cluster and associated Poisson or symplectic structure. In §3 we recall the Y-system [12] and explain how it defines a twistor space for $\mathcal{K}$. In §4 we find the hyperkähler structure explicitly and show the regularized area is a split signature analogue of a Kähler scalar that satisfies an integrable system of generalized Plebanski equations. Finally in §5 we mention a number of checks and further developments. We also consider how these results might relate to amplitudes at finite coupling.

2 The spaces of kinematic data in $1+1$-dimensions

In restricted kinematics, the kinematic space $\mathcal{K}_n$ is the moduli space of null polygonal Wilson loops in $1+1$-dimensions with $2n$ sides. Such a Wilson loop is given by a set of ordered null momenta (the 'edges' of the loop) that sum to zero (so that the loop closes). Take null coordinates (X^+, X^-) on Minkowski space with metric

$$\mathrm{d}s^2 = 2\mathrm{d}X^+\mathrm{d}X^-. \tag{2.1}$$

The null edges of a polygonal Wilson-loop alternate between lines of constant X^+ and lines of constant X^-. The kinematic data for a $2n$-sided Wilson loop in AdS_3 is therefore given by two cyclically ordered sets of real numbers $\{X_i^+\}, \{X_i^-\}$, with $i = 0, \ldots, n-1$. Vertices of the polygon are given by the points (X_i^+, X_{i-1}^-), (X_i^+, X_i^-) then (X_{i+1}^+, X_i^-) and so on. Conformal invariance means that functions of these parameters should be invariant under Möbius transformations on the X_i^+ and X_i^- separately. Thus the space of kinematic data

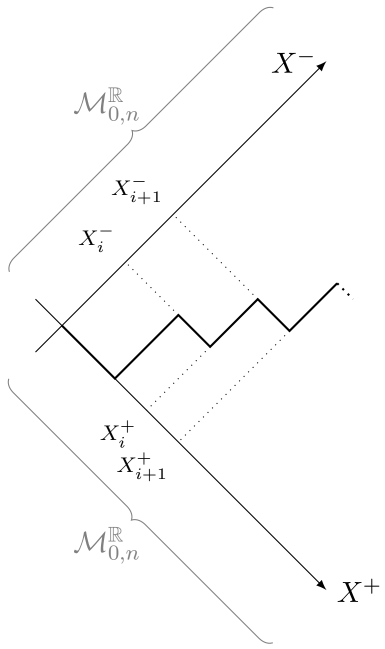

Figure 1. *Null coordinates on* $\mathbb{M}^{1,1}$.

$\mathcal{K}_n$ is

$$\mathcal{K}_n = \mathcal{M}_{0,n}^{\mathbb{R}} \times \mathcal{M}_{0,n}^{\mathbb{R}} \,. \tag{2.2}$$

where

$$\mathcal{M}_{0,n}^{\mathbb{R}} = \{X_i^{\pm}, i = 1, \ldots, n\}/PSL_2 \,. \tag{2.3}$$

is the moduli space of n points on $\mathbb{RP}^1$ modulo Mobius transformations.

Möbius invariant functions on $\mathcal{K}$ are given by cross-ratios

$$(ij|kl)^{\pm} = \frac{(X_i^{\pm} - X_j^{\pm})(X_k^{\pm} - X_l^{\pm})}{(X_i^{\pm} - X_l^{\pm})(X_j^{\pm} - X_k^{\pm})}. \tag{2.4}$$

There are special sets of cross-ratios that can be grouped into so-called *clusters*, which give local systems of coordinates for $\mathcal{K}$.

A cluster for $\mathcal{M}_{0,n}^{\mathbb{R}}$ is specified by choosing a triangulation of the n-gon. A triangulation defines an antisymmetric matrix, $\epsilon_{ss'}$ where $s, s' = 1, \ldots, n-3$ index the chords of the triangulation. For two such chords s and s', write $\epsilon_{ss'} = 0$ if the two chords do not share a triangle. If they share a triangle, write $\epsilon_{ss'} = 1$ if s' is clockwise of s, or write $\epsilon_{s's} = 1$ is s' is counter-clockwise of s.[4] The matrix $\epsilon_{ss'}$ in turn defines a quiver as its incidence matrix with nodes placed on the edges of the n-gon and the chords of its triangulation. Fix a triangulation. A chord s in the triangulation will be the diagonal, $i-k$, or some quadrilateral (i, j, k, l) of the triangulation. To this chord we associate the coordinate

$$\chi_s^{\pm} = (il|kj)^{\pm}. \tag{2.5}$$

The set of these cross ratios, $\{\chi_s^{\pm}\}$ define a cluster of coordinates on $\mathcal{K}_n$.

[4] This matrix is related to the so-called b-matrix of the A-type cluster algebra: $\operatorname{sgn} b_{ij} = \operatorname{sgn} \epsilon_{ij} = \epsilon_{ij}$.

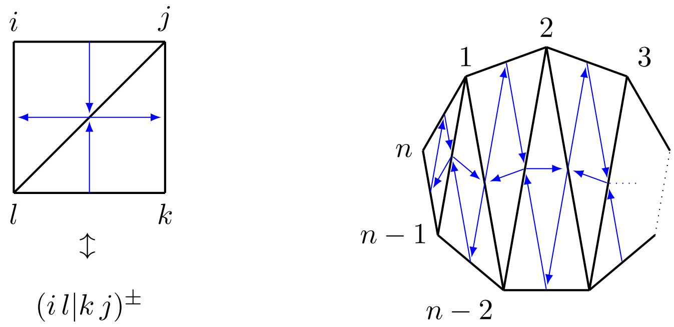

Figure 2. Left: The correspondence between a chord of a triangulation and the cross ratios. Right: The "zig-zag" triangulation of the polygon and corresponding chart on kinematic space.

Different choices of clusters of coordinates are related by *mutation relations*. Flipping a chord s, inside a quadrilateral that it is a diagonal of, gives a new chord, s', and the new cross-ratios are related to the old ones by

$$\mu(\chi_s) = \chi_s^{-1}, \qquad \mu(\chi_t) = \chi_t(1 + \chi_s^{\epsilon_{st}})^{\epsilon_{st}}. \tag{2.6}$$

The only cross-ratios that change are those sharing a triangle with s in the triangulation.

Finally, there is a natural 2-form on $\mathcal{K}_n$ that is symplectic when $\mathcal{K}_n$ is even dimensional, i.e., for n odd. Fixing a triangulation of the n-gon as above, on each copy of $\mathcal{M}_{0,n}^{\mathbb{R}}$ define

$$\omega^\pm = \sum_{i,j} \epsilon_{ij}\, \mathrm{d}\log \chi_i^\pm \wedge \mathrm{d}\log \chi_j^\pm\,. \tag{2.7}$$

It follows directly from the mutation rule (2.6) that Mutating, or flipping, a chord of the triangulation preserves the 2-form, i.e., after one mutation

$$\omega^\pm = \mu(\omega^\pm) := \sum_{i,j} \tilde{\epsilon}_{ij} \mathrm{d}\log \mu(\chi_i) \wedge \mathrm{d}\log \mu(\chi_j), \tag{2.8}$$

where $\mu(\chi_i)$ are the new cross-ratios, and $\tilde{\epsilon}_{ij}$ is the matrix of the new triangulation. Thus, a series of mutations leaves $\omega^\pm$ invariant and so we write $\omega^\pm$ independently of the choice of cluster.

Example. A useful example is the 'zig-zag' triangulation given by Figure 2. The associated cluster has cross-ratios given by $(s = 1, ..., n-3)$

$$\chi_s^\pm = \begin{cases} (s-1, s|-s-1, -s)^\pm & s \text{ odd} \\ (s-1, s|-s, -s+1)^\pm & s \text{ even}, \end{cases} \tag{2.9}$$

where we specify vertices of the polygon mod n. The matrix $\epsilon_{ss'}$ is given by $\epsilon_{s,s+1} = 1$ (for s odd) and $\epsilon_{s,s+1} = -1$ (for s even). For this triangulation, as in [12], we have

$$\omega = \sum \mathrm{d}\log \chi_{2i} \wedge (\mathrm{d}\log \chi_{2i-1} - \mathrm{d}\log \chi_{2i+1}). \tag{2.10}$$

3 From the Y-system to the twistor space

We now show how the 'Y-system' can be used to construct a twistor space, which will we use in Section 4 to prove new results about the remainder function. In the first instance, fix the 'zig-zag' choice of cluster coordinates (χ_s^+, χ_s^-) on $\mathcal{K}_n$ as given in (2.9). The Y-system of [12] is the family of functions ($s = 1, \dots, n-3$)

$$\mathcal{Y}_s = \mathcal{Y}_s(\chi_r^+, \chi_r^-, \zeta) : \mathcal{K}_n \times \mathbb{CP}^1 \longrightarrow \mathbb{C}, \tag{3.1}$$

that are complex analytic in the *spectral parameter* $\zeta \in \mathbb{CP}^1$. We define a Y-system associated to this cluster by the following conditions. First, the $\mathcal{Y}_s$ are fixed to yield the cluster coordinates (χ_s^+, χ_s^-) at $\zeta = 1$ and i:

$$\mathcal{Y}_s(1) = \chi_s^+ , \qquad \mathcal{Y}_s(i) = \chi_s^- . \tag{3.2}$$

Second, we impose that the $\mathcal{Y}_s$, are holomorphic except for branching singularities at $\zeta = 0$ and $\zeta = \infty$, taking the branch cut to be $\mathbb{R}^-$. We further require exponential asymptotics, so that $\log \mathcal{Y}_s$ has residues at 0 and ∞.[5] Finally, to fully determine the $\mathcal{Y}_s$ we define their analytic continuation across the $\mathbb{R}^-$ branch cut. Write $\mathcal{Y}_s^{++}(\zeta) = \mathcal{Y}(\mathrm{e}^{i\pi}\zeta)$. In the case of the zigzag triangulation, the analytic continuation is given by the relations

$$\begin{aligned} \mathcal{Y}_{2k+1}^{++}\mathcal{Y}_{2k+1} &= (1 + \mathcal{Y}_{2k+2})(1 + \mathcal{Y}_{2k}) , \\ \mathcal{Y}_{2k}^{++}\mathcal{Y}_{2k} &= (1 + \mathcal{Y}_{2k+1}^{++})(1 + \mathcal{Y}_{2k-1}^{++}) . \end{aligned} \tag{3.3}$$

These relations are a consequence of the mutation relations, (2.6), applied to the $\mathcal{Y}_s$. In general, the $\mathcal{Y}_s^{++}$ are given by performing a series of $2(n-3)$ mutations that implement a 'rotation' of the cluster triangulation by $2\pi/n$.

The $\mathcal{Y}_s$ are expected to be uniquely determined by these conditions, and solutions can be obtained from iterating integral equations known in the form of a Thermodynamic Bethe Ansätze described in [12, 13], which we do not review here.

As a smooth manifold, define the twistor space to be $\mathcal{T}_n = \mathcal{K}_n \times \mathbb{CP}^1$. The $\mathcal{Y}_s$-functions defined above turn $\mathcal{T}_n$ into a complex manifold defining holomorphic coordinates on $\mathcal{T}_n$. The following proposition summarizes the key properties of $\mathcal{T}_n$ that we will need.

Proposition 3.1 *$\mathcal{T}_n$ is a complex $n-2$-manifold with local holomorphic coordinates $(\mathcal{Y}_s, \zeta)$, with a holomorphic projection: $p : \mathcal{T}_n \to \mathbb{CP}^1$. There is a family of symplectic 2-forms $\Sigma(\zeta)$ on the fibres of p. For odd n, $\Sigma(\zeta)$ is non-degenerate. Moreover, $\Sigma(\zeta)$ is invariant under the holomorphic circle action*

$$(\mathcal{Y}_s, \zeta) \longrightarrow (\mathcal{Y}_s, \mathrm{e}^{i\theta}\zeta).$$

Finally, there is an anti-holomorphic involution on $\mathcal{T}_n$ given by

$$(\mathcal{Y}_s, \zeta) \longrightarrow (\overline{\mathcal{Y}}_s, 1/\bar{\zeta}),$$

so that the $\mathcal{Y}_s$ are real on the unit circle $|\zeta| = 1$.

[5] These can be written as $\log \mathcal{Y}_s = Z_s\zeta^{-1} + \dots$ as $\zeta \to 0$ and $\log \mathcal{Y}_s = \bar{Z}_s\zeta + \dots$ as $\zeta \to \infty$ for some $Z_s(\chi_r^+, \chi_r^-)$, but we will not use this in what follows.

Proof: The twistor space can be constructed by gluing together holomorphic coordinate patches. Take $U = \{-\pi < \arg\zeta < \pi\}$ and $U^{++} = \{0 < \arg\zeta < 2\pi\}$. The two patches are related by $\zeta \mapsto e^{i\pi}\zeta$. The $\mathcal{Y}_s$ functions are holomorphic on U. The $\mathcal{Y}_s^{++}$ functions are holomorphic on U^{++}. These two holomorphic patches can be glued together on the overlap, $U \cap U^{++}$, by the Y-system equations, (3.3).

For fixed ζ, define the closed 2-form

$$\Sigma(\zeta) := \sum \epsilon_{ij} \mathrm{d}\log\mathcal{Y}_i \wedge d\log\mathcal{Y}_j. \tag{3.4}$$

By the same argument given below (2.7), $\Sigma(\zeta)$ is preserved by mutations. In particular, $\Sigma^{++}(\zeta) \equiv \Sigma(e^{i\pi}\zeta) = \Sigma(\zeta)$. So $\Sigma(\zeta)$ is defined for all ζ, except for $\zeta = 0$ and $\zeta = \infty$. Moreover, $\mathcal{Y}_s$ are invariant under the circle symmetry, so Σ is likewise circle invariant.

Finally, the functions $\overline{\mathcal{Y}}_s(1/\bar{\zeta})$ have the same analytic properties and special values as the functions $\mathcal{Y}_s(\zeta)$, and satisfy the same Y-system equations as $\mathcal{Y}_s(\zeta)$. But the $\mathcal{Y}_s$ functions are unique, so we must have that $\overline{\mathcal{Y}}_s(1/\bar{\zeta}) = \mathcal{Y}_s(\zeta)$. □

4 Integrable system for the remainder function

The nontrivial part of the regularised area of the minimal surface in AdS_3 is called the *remainder function*, $R(\chi_r^+, \chi_s^-)$. The remainder function can be presented in integral form using the Thermodynamic Bethe Ansatze (as in appendix E of [14]). Here we take a different approach. Following §3 of [6], we identify R with the Hamiltonian for the circle-action on $\mathcal{T}_n$, we find a completely integrable system of differential equations for R. This system generalizes the Plebanski equations for four-dimensional self-dual metrics:

Proposition 4.1 *For n odd, the remainder function satisfies the equations*

$$R^{pq}R^{rs}\epsilon_{pr} = \epsilon^{qs}, \tag{4.1}$$

together with circle invariance, e.g. (4.21)*. Here $\epsilon_{pq}\epsilon^{qr} = \delta_p^r$ and R^{pq} is the Hessian matrix:*

$$R^{rs} = \frac{\partial^2 R}{\partial x_r^+ \partial x_s^-},$$

with $x_p^\pm := \log\chi_p^\pm$. The system (4.1) *and* (4.21) *follow from the consistency of the Lax system $\{\mathcal{L}_r, \tilde{V}\}$ where*

$$\mathcal{L}_r := (\zeta^2-1)\frac{\partial}{\partial x_r^+} + (\zeta^2+1)iR^{rs}\frac{\partial}{\partial x_s^-}, \qquad \widetilde{V} := \epsilon_{rs}\left(\frac{\partial R}{\partial x_s^+}\frac{\partial}{\partial x_r^+} + \frac{\partial R}{\partial x_s^-}\frac{\partial}{\partial x_r^-}\right) + i\zeta\frac{\partial}{\partial\zeta}. \tag{4.2}$$

A *pseudo-hyperkähler* structure is the analogue of a hyperkahler structure appropriate to a metric of split signature, in which two of the three Kähler structures become transverse maximal null foliations. As the proof of Proposition 4.1 will show, our integrable system implies that $\mathcal{K}_n$ is pseudo-hyperkähler:

Proposition 4.2 *For n odd, $\mathcal{K}_n$ is pseudo-hyperkähler, with split-signature metric*

$$ds^2 := R^{rs}\mathrm{d}x_r^+\mathrm{d}x_s^-, \tag{4.3}$$

and the three symplectic 2-forms

$$\omega^\pm = \epsilon^{rs}\mathrm{d}x_r^+ \wedge \mathrm{d}x_s^+ \pm \epsilon^{rs}\mathrm{d}x_r^+ \wedge \mathrm{d}x_s^+\,, \qquad \Omega = R^{rs}\mathrm{d}x_r^+ \wedge \mathrm{d}x_s^-\,. \tag{4.4}$$

Proof (of Proposition 4.1). The 2-form, $\Sigma(\zeta)$, defined in (3.4), is globally defined on the ζ-plane, with poles at $\zeta = 0, \infty$. Consider a Laurent series expansion of $\Sigma(\zeta)$ in ζ. It follows from (3.2) that $\Sigma(\zeta)$ has the special values

$$\Sigma(1) = \sum \epsilon_{ij}\mathrm{d}x_i^+ \wedge \mathrm{d}x_j^+, \qquad \Sigma(i) = \sum \epsilon_{ij}\mathrm{d}x_i^- \wedge \mathrm{d}x_j^-\,. \tag{4.5}$$

Moreover, the logarithms, $y_s := \log \mathcal{Y}_s \sim 1/\zeta$ at 0 and ζ at ∞. So $\Sigma(\zeta)$, which is now single valued, has *double poles* at $\zeta = 0$ and $\zeta = \infty$. By Proposition 3.1,

$$\Sigma(-\zeta) = \Sigma(\zeta), \tag{4.6}$$

so that the Laurent series of $\Sigma(\zeta)$ does not have terms linear in ζ, ζ^{-1}. Equations (4.5) and (4.6) together imply that the Laurent expansion takes the form

$$\Sigma(\zeta) = \frac{(\zeta^2+1)^2}{4\zeta^2}\Sigma(1) - \frac{(\zeta^2-1)^2}{4\zeta^2}\Sigma(i) + \frac{(\zeta^4-1)}{4\zeta^2}i\Omega\,. \tag{4.7}$$

for some ζ-independent closed 2-form Ω. Grouping terms gives

$$\Sigma(\zeta) = \frac{1}{\zeta^2}A^{(-2)} + A^{(0)} + \zeta^2 A^{(2)}, \tag{4.8}$$

where

$$A^{(\pm 2)} = \frac{1}{4}\Sigma(1) - \frac{1}{4}\Sigma(i) \mp \frac{i}{4}\Omega, \qquad A^{(0)} = \frac{1}{2}\Sigma(1) + \frac{1}{2}\Sigma(i). \tag{4.9}$$

Since $\Sigma(\zeta)$ is non-degenerate with rank $n-3$,

$$(\Sigma(\zeta))^{(n-1)/2} = 0\,. \tag{4.10}$$

Consider the Laurent expansion of (4.10) in ζ. The coefficients of the leading terms (multiplying ζ^{n-3} and $\zeta^{-(n-3)}$) are

$$(\Sigma(1))^{(n-3)/2} \wedge \Omega = 0\,, \qquad (\Sigma(i))^{(n-3)/2} \wedge \Omega = 0\,. \tag{4.11}$$

It follows that Ω is linear in both the $\mathrm{d}x_r^+$ and the $\mathrm{d}x_s^-$, so that

$$\Omega = \frac{1}{4}J^{rs}\mathrm{d}x_r^+ \wedge \mathrm{d}x_s^- \tag{4.12}$$

for some functions $J^{rs}(x^+, x^-)$. But Ω is closed, so

$$J^{rs} = \frac{\partial^2 J}{\partial x_r^+ \partial x_s^-}, \tag{4.13}$$

for some potential function, $J(x^+, x^-)$. The vanishing of the sub-leading terms in the expansion of (4.10) imply the following Plebanski-like system of differential equations for J:

$$\epsilon_{rr'} J^{rs} J^{r's'} = \epsilon^{ss'} \,. \tag{4.14}$$

We briefly note that we can give a Lax formulation for (4.14). The rank of $\Sigma(\zeta)$ implies that we can find $n-3$ operators $\mathcal{L}_r$ satisfying $\mathcal{L}_r \lrcorner \Sigma(\zeta) = 0$. By (4.7) and (4.12), we can take $\mathcal{L}_r$ to be

$$\mathcal{L}_r = (\zeta^2 - 1)\frac{\partial}{\partial x_r^+} + (\zeta^2 + 1) i J^{rs} \frac{\partial}{\partial x_s^-} \,. \tag{4.15}$$

Then it follows that $\mathcal{L}_r y_s(\zeta) = 0$. Moreover, by the closure of $\Sigma(\zeta)$, $[\mathcal{L}_r, \mathcal{L}_s] = 0$, and this is equivalent to (4.14). The Lax system can be used to solve for the $\mathcal{Y}_r$ functions. Near $\zeta = 1, i$, we find

$$y_r(\zeta) = x_r^{\pm} + (\zeta^2 \mp 1)\epsilon_{rs} i \frac{\partial J}{\partial x_s^{\pm}} + O((\zeta^2 \mp 1)^2) \,, \tag{4.16}$$

and the Lax system further determines all higher order terms.

Up to a constant, the remainder function R is the Hamiltonian that generates the circle action of $\mathcal{T}_n$. Suppose that the rotation symmetry acts on $x_r^{\pm}$ by some vector field V. Its lift to $\mathcal{T}_n$ acts on ζ by $\tilde{V}(\zeta) = i\zeta$ so that $\tilde{V} = V + i\zeta\partial_\zeta$. The circle action leaves $\Sigma(\zeta)$ invariant, so that $\tilde{V}$ annihilates $\Sigma(\zeta)$. Consider again the Laurent expansion of Σ in ζ. The coefficient of ζ^0, $A^{(0)}$, must be invariant under V. Given that R is the Hamiltonian for V, we therefore have that

$$V = \epsilon_{rs}\left(\frac{\partial R}{\partial x_s^+}\frac{\partial}{\partial x_r^+} + \frac{\partial R}{\partial x_s^-}\frac{\partial}{\partial x_r^-}\right) \,. \tag{4.17}$$

Moreover, invariance of $\Sigma(\zeta)$ under $\tilde{V}$ also implies that $A^{(\pm 2)}$ has weight ∓ 2 under V:

$$\pounds_V A^{(\pm 2)} = \mp 2i A^{(\pm 2)}, \tag{4.18}$$

where the Lie derivative is $\pounds_V A^{(-2)} = \mathrm{d}\left(i_V A^{(-2)}\right)$. Adding together both signs of (4.18) gives

$$\frac{\partial^2 R}{\partial x_r^+ \partial x_s^-} = \frac{\partial^2 J}{\partial x_r^+ \partial x_s^-} \,. \tag{4.19}$$

Since the potential J is only defined up to a sum of a function of x_r^+ and another of x_r^-, we can fix this freedom by identifying $J \equiv R$. The difference between the $\pm$ parts of (4.18) gives

$$0 = \partial_{x_r^+}(J^{ts} V_t^+) + \partial_{x_s^-}(J^{rt} V_t^-), \qquad \epsilon_{rs} = \partial_{x_s^{\pm}}(J^{rt} V_t^{\mp}), \tag{4.20}$$

With $J = R$, the first of these equations reads

$$0 = \partial_{x_r^+}\left(\frac{\partial^2 R}{\partial x_t^+ \partial x_s^-}\frac{\partial R}{\partial x_u^+}\epsilon^{tu}\right) + \partial_{x_s^-}\left(\frac{\partial^2 R}{\partial x_r^+ \partial x_t^-}\frac{\partial R}{\partial x_u^-}\epsilon^{tu}\right) . \tag{4.21}$$

Likewise, with $J = R$, the remaining two equations simplify, and are both solved by Plebanski equation, which we can now write as:

$$0 = \epsilon_{rs} R^{rr'} R^{ss'} + \epsilon^{r's'} \,. \tag{4.22}$$

Equations (4.21) and (4.22) together define an integrable system for the remainder function, R. Equivalently, in Lax form, these equations are given by the Lax system $\{\mathcal{L}_r, V + i\zeta\partial_\zeta\}$, where $V + i\zeta\partial_\zeta$ is the circle symmetry generator.

To complete the proof, we show that (4.21) is essentially a single further constraint on R over (4.22). Write $\partial^r = \partial/\partial x_r^+$ and $\partial^{r'} = \partial/\partial x_{r'}^-$ for derivatives with respect to x_r^+ and $x_{r'}^-$. Then (4.22) can be written in two equivalent ways as

$$\partial^{[r}(R^{s]s'}R^{r'}\epsilon_{r's'} + \epsilon^{qs}x_q^+) = 0, \qquad \partial^{[r'}(R^{s']s}R^r\epsilon_{rs} + \epsilon^{q's'}x_{q'}^-) = 0\,. \tag{4.23}$$

These are integrability conditions for the existence of functions S and S' satisfying

$$R^{ss'}R^{r'}\epsilon_{r's'} + \epsilon^{qs}x_q^+ = \partial^s S\,, \qquad R^{s's}R^r\epsilon_{rs} + \epsilon^{q's'}x_{q'}^- = \partial^{s'}S' \tag{4.24}$$

where S is defined up to functions of $x_{r'}^-$, and S' is defined up to functions of x_r^+. Given this, (4.21) becomes $\partial^r\partial^{r'}S' + \partial^{r'}\partial^r S = 0$, and this imposes one additional constraint on the system, namely, that $S + S' = 0$. □

Note that (4.24) together with $S+S' = 0$, provides an alternative form of the integrable system, with one fewer derivatives, at the price of introducing the additional function S. The equations have the trivial flat solution $R = \epsilon^{rs}x_r^+x_s^-$, $S = 0$ but it is clear from the expansions arising from the TBA that the true solution is much more complicated.

5 Conclusions and discussion

We have seen that the Y-system of [14] defines a twistor space for the kinematic space, $\mathcal{K}_n$, of null polygonal Wilson loops with $2n$ sides. When n is odd, the remainder function, R, satisfies a completely integrable system on $\mathcal{K}_n$. The system is a higher dimensional analogue of the first Plebanski equation for 4d self-dual gravity. It follows from the fact that $\mathcal{K}_n$ is pseudo-hyperkähler, and that the remainder function, R, is the split signature analogue of a Kahler scalar for a split signature metric on $\mathcal{K}_n$.

Given appropriate boundary conditions, our integrable system should inductively determine R at all n. Boundary conditions arise from colinear limits of the kinematics. The colinear limit, for two adjacent vertices of the Wilson loop, defines a codimension two boundary of $\mathcal{K}_n$ that can be identified with $\mathcal{K}_{n-1}$. The remainder function R on $\mathcal{K}_n$ restricts to that for $\mathcal{K}_{n-1}$ on the boundary (see [6, 15, 16] for a summary of colinear limits in restricted kinematics). Although these boundaries have co-dimension two, the equations are over determined by at least that co-dimension. For the example of $n = 5$, $\mathcal{K}_5$ is 4-dimensional, and as the 4d deterministic equations are further subject to a circle symmetry, so they should propagate data from the 2d boundary $\mathcal{K}_4$, where integral formulae for R are known [6]. It follows from the complex dimension $n-2$ of the twistor space (on which the data is holomorphic) versus that, $2n-6$ of $\mathcal{K}_n$ that the equations only become more overdetermined for larger n. Thus it should be possible to determine the remainder functions for all n inductively. However, more work needs to be done on the geometry at even n to make this systematic.

We have seen the close interaction between the cluster geometry of $\mathcal{K}_n$ and that of the twistor space and the pseudo-hyperkähler geometry via the naturally defined logarithmic 2-forms. Positive geometry also potentially plays a role. The Euclidean region in $\mathcal{K}$ is defined to be the region where non-adjacent vertices are space-like separated. On this region we expect the area and hence the remainder function R to be positive with boundaries corresponding to (multi-)collinear limits.

The pseudo-hyperkähler spaces studied here are *not* directly related to the other hyperkähler structures that arise in studies of Hitchin systems. Hitchin showed that moduli spaces of regular Hitchin systems admit hyperkähler structures, [8] and this result has been partially extended to the irregular case relevant here in [9] but this freezes the relevant data here. Gaiotto, Moore and Neitzke [10, 17] incorporate our data and more. However, these do not apply directly to amplitudes at strong coupling as $\mathcal{K}_n$ parameterises an invariant subspace of the Hitchin moduli space under an involution [6, 12, 13]. It can be checked directly that the hyperkähler structure of [10, 17] does not restrict to this subspace to yield the structures discussed here even in their simplest versions (details to appear elsewhere).

We comment on some implications of our result for further research. First, our methods here can be applied to many other cluster varieties and associated Y-systems, to find new differential equations. It will be interesting to study the generalization to other ADE Y-systems [18], and also beyond ADE to the affine and surface-type cluster algebras. These cases correspond, in physics, to other contributions in the $1/N$ expansion at strong coupling (for restricted kinematics). Beyond these cases, the Grassmannian cluster algebras appear when computing strong coupling amplitudes and form factors for full $\mathcal{N}=4$ SYM kinematics. The Y-systems associated to these cluster algebras are well known, [12] and indeed generalizations incorporating form-factors in [15]. These should all define twistor spaces, and we expect to find integrable systems for the amplitudes in these cases. However, the geometry is not as simple as in restricted kinematics. For the full strong coupling planar amplitude, $\mathcal{K}_n$ has dimension $3n-15$, and so $\mathcal{K}_n$ can not be hyperkähler unless $3n-15$ is a multiple of 4 (i.e. $n = 9, 13, 17 ...$).

Secondly, our results suggest some new approaches to going beyond the strong coupling limit. It is known from numerical studies that the perturbation series computation of the amplitude is numerically close to the strong-coupling amplitude [19]. This suggests that the differential equations that we have discovered at strong coupling might hold approximately in perturbation theory, possibly with deformations that could provide a new approach for studying the amplitude beyond strong coupling. In this direction, there are several other connections to explore. It should be possible to view our integrable system as arising from a twistor sigma model action [20, 21]; expressing the string quantization in this model might allow computations beyond the strong coupling limit. Links to the finite coupling results for the anomalous dimension spectrum problem might also arise by making contact with the quantum spectral curve of [22] via its underlying Y-system. The coupling constant should then be incorporated by incorporating the 'Joukowski correspondence' of [23–25].

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