University of Southampton

Faculty of Social Sciences
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Loop Space Decompositions of Highly Connected Poincaré Duality Complexes

by

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A thesis for the degree of
Doctor of Philosophy

July 2023

University of Southampton

Abstract

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One of the main goals of homotopy theory is to determine the homotopy types of topological spaces. Furthermore, given a space, one may hope there is a way to decompose this space into simpler spaces whose homotopy theory is well understood. In particular, it can be useful to study product decompositions of the based loop space of the object we wish to consider – doing so provides useful data about the object's homotopy groups, amongst other things. Furthermore, by considering manifolds through the lens of homotopy theory, it is natural to broaden one's scope to Poincaré Duality complexes: such complexes are a topological generalisation of manifolds and have an underlying structure that is readily exploitable. Thus, by studying the loop spaces of Poincaré Duality complexes, we may answer questions about the homotopy theory of manifolds.

In particular, given two manifolds of the same dimension, a natural object to consider is their connected sum. This situation is often flipped: one asks the question of whether a given manifold is decomposable as a connected sum of simpler manifolds. In areas such as surgery theory and differential topology this problem is of fundamental importance, despite the fact that in dimensions higher than two the problem is often inaccessible. This thesis studies this highly geometric problem from a new topological viewpoint, using elements of classical homotopy theory together with recent results.

In expanding upon these methods, we find that the loop space decompositions of several classes of highly connected manifolds coincide with those of the loop spaces of certain connected sums, and thus we have a homotopy theoretic perspective on the above question. Indeed, we apply these results to comment on the Vigué-Poirrier Conjecture, a particular long standing question from rational homotopy theory. We also prove a higher dimensional homotopy theoretic analogue to a theorem of C.T.C. Wall – a fundamental calculation from differential topology that shows one may decompose a simply connected 6-manifold as a connected sum of two simpler manifolds – for $(n-2)$-connected $2n$-dimensional Poincaré Duality complexes.

Key to these discussions is a consideration of inert maps, a concept brought across from rational homotopy theory. By combining other results from this area, we provide an answer to another problem: under what circumstances does the total space of a pullback fibration over a connected sum have the rational homotopy type of a connected sum? We conclude with a reformatted version of a recent paper of the author, which gives a condition on rational cohomology to yield an affirmative answer, but only after taking based loop spaces. This takes inspiration from recent work of Jeffrey and Selick, in which they study pullback bundles of this type, but under stronger hypotheses compared to our result.

Contents

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

[1] S. Chenery, *The rational homotopy type of homotopy fibrations over connected sums*, Proceedings of the Edinburgh Mathematical Society, 66(1), 133-142, doi:10.1017/S001309152300007X, 2023.

[2] S. Chenery, *A homotopy theoretic analogue to a theorem of Wall*, preprint, available at arXiv:2210.04548 [math.AT], 2022.

Signed:... Date:..................

Acknowledgements

The journey that took me to producing this thesis has been an arduous one, and the full cast of characters is too numerous to count. I have never been one to shrink from such a task, however, so I shall at least attempt a list.

First and foremost, I give my deepest thanks to my family for their undying love and support. Following swiftly, I would like to extend my most sincere gratitude and humble respect for my supervisor, Stephen Theriault, for the multitude of valuable lessons he has taught me throughout these past few years. I also wish to thank the faculty at Southampton for all their help and support over the course of my doctoral studies. More internationally, I must also thank Paul Selick and Lisa Jeffrey for their generosity during the preparation of what became my first paper.

To all the graduate students at Southampton, especially my fellow topologists, thank you for enabling me to retain something resembling sanity. In particular (though in no particular order) I especially wish to thank Lawk Mineh, Briony Eldridge, Jane Turner, Lewis Stanton, George Davenport, George Simmons, Matthew Staniforth, Will Warhurst, Harry Iveson, Rhys Counsell, Holly Paveling and Simon Rea. I have to also thank Guy Boyde, for being a fantastic and inspiring academic older brother. Moreover, to my partners in crime: Geraint Evans, Dawid Drelinkiewicz and Phillip Wells - thank you for being the best housemates a young man could wish for.

Musically, to my many friends at Southampton University Brass Band and beyond, especially Andy Wareham, Edward Guy, Jacob Povall, Peter Forbes, Daisy Woodley, Kieran Potter, Kevin Ponte and Sophie Blundell: thank you all for the good times, and for helping me through the bad.

Those that know me well will be aware that without Southampton University Archery Club, I would not be standing today. I owe a debt of gratitude to SUAC's head coach, Gary Carr, for teaching me so much beyond just sport. Also to Niambh Jones, whose enduring personal strength never ceases to amaze, no matter how dark the situation. My thanks also go to Bethany Logan, Callum Newlands, Yanna Fidai, Rachel Coombs, Sophie Brown and Alex Brentnall. Furthermore, my everlasting respect to the three idiots and a fool that are Callum Anderson, Kennie Hayward, Michael Sessions and Stuart MacFarquhar. I salute you!

During the final stages of writing of this thesis, I was in medical isolation for a clinical trial. The hardship of being forced to live on instant coffee aside, I give my thanks to the National Institute for Health and Care Research for providing 17 days of the perfect working environment.

This thesis is dedicated to the memory of

Matthew Wickes

15th November 2000 – 30th June 2022

"It is not worth an intelligent man's time to be in the majority. By definition, there are already enough people to do that." – G.H. Hardy

"Build a man a fire, and he'll be warm for a day. Set a man on fire, and he'll be warm for the rest of his life." – Terry Pratchett

"Time is an illusion. Lunchtime doubly so." – Douglas Adams

Notation

$\mathbb{1}_X$	The identity map on a space X
i_j	The inclusion map $X_j \to \bigvee_{i=1}^n X_i$ of the j^{th} wedge summand
p_j	The projection map $\bigvee_{i=1}^n X_i \to X_j$ to the j^{th} wedge summand
$X \simeq Y$	The spaces X and Y are homotopy equivalent
$X \cong Y$	The spaces X and Y are homeomorphic
ΩX	The based loop space of a space X
$\mathcal{L}X$	The free loop space of a space X
ΣX	The reduced suspension $S^1 \wedge X$ of a space X
$X^{\wedge k}$	The smash product of k copies of the space X
S^n	The n-dimensional sphere
$P^n(\ell)$	The mod-ℓ Moore space, with $\widetilde{H}^m(P^n(\ell);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/\ell & m = n, \\ 0 & \text{otherwise.} \end{cases}$

Chapter 1

Introduction

Given a topological space X, the gold standard of unstable homotopy theory is to determine its homotopy groups, denoted $\pi_*(X)$. Doing so in general is a notoriously difficult problem, even for relatively simple spaces like spheres (see for example [Tod16]). Alternatively, one may hope to express X as a product of simpler spaces whose homotopy groups may be easier to understand, taking advantage of the fact that $\pi_*(A \times B) \cong \pi_*(A) \times \pi_*(B)$. Clearly, however, the vast majority of spaces are not products.

Another space to study is the *based loop space* of X, denoted ΩX, which is the space of basepoint preserving maps $S^1 \to X$. Understanding ΩX opens a door to finding the homotopy groups of X, as there is an isomorphism $\pi_n(\Omega X) \cong \pi_{n+1}(X)$. So, combining this with our first fact, finding product decompositions of ΩX is a valuable technique for seeking to understand $\pi_*(X)$.

A natural class of spaces to study is the class of manifolds: these are highly geometric objects, which carry deep structures. Indeed, the algebraic topology of manifolds has a rich and varied history, and when studying their homotopy theory one can start by considering those that are highly connected – we say a space X is n-connected if the group $\pi_i(X)$ is trivial for all $i \leq n$. Poincaré Duality complexes (of which smooth, closed, oriented manifolds are a subclass; see Section 2.4 for a full definition) provide us with a useful topological analogue to manifolds. By understanding their loop spaces, we may answer questions about the homotopy theory of manifolds. This is the titular task that will occupy our efforts for the following pages: to find loop space decompositions of Poincaré Duality complexes, often those that are highly connected, and thereby develop our homotopy theoretic understanding of manifolds.

The past decade has seen much activity studying highly connected manifolds in this way, notably by Beben and Theriault [BT14, BT22], in which they consider the based loop spaces of $(n-1)$-connected $2n$-manifolds. More recently, work of Huang

[Hua22a] incorporated a study of torsion free $(n-2)$-connected $2n$-manifolds with vanishing cohomology in dimension n. More broadly, extensive study of the wider homotopy theoretic situation was made in [The20], which has lead to further recent work, such as [Hua21, Hua22b, HT22, Che22, Che23].

In particular, given a manifold one may ask whether it is decomposable as a connected sum of other manifolds, analogous to decomposing an integer as a product of primes. Using our loop space decompositions, we will approach this highly geometric problem from a novel topological viewpoint, using elements of classical homotopy theory together with recent results to show that several classes of Poincaré Duality complexes have the loop space homotopy type of connected sums. The techniques for this are developed in Chapter 4, and they underpin much of what follows. Chiefly, three main applications of this approach are demonstrated in this thesis.

The Vigué-Poirrier Conjecture

The first comes in Section 4.4, as a direct result of the preceding material in Chapter 4, and is related to a particular long standing question of rational homotopy theory: the Vigué-Poirrier Conjecture. Letting X be a simply connected space, we denote its *free loop space* by $\mathcal{L}X$. That is to say, $\mathcal{L}X := Map(S^1, X)$, the space of continuous (not necessarily based) maps from S^1 to X. A simply connected space X is called *rationally elliptic* if $dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$, and called *rationally hyperbolic* otherwise [FHT01]. Vigué-Poirrier made the following conjecture in [VP84].

Conjecture (Vigué-Poirrier)**.** *If X is rationally hyperbolic, then $H_*(\mathcal{L}X; \mathbb{Q})$ grows exponentially.*

In the same paper, Vigué-Poirrier herself proved that the Conjecture holds for finite wedges of spheres. Furthermore, the Vigué-Poirrier Conjecture may be viewed as a development of another conjecture (due to Gromov [Gro78]) that when X is a closed manifold then $H_*(\mathcal{L}X; \mathbb{Q})$ 'almost always' grows exponentially. This has profound implications in Riemannian geometry, in which one may give a lower bound for the number of geometrically distinct closed geodesics on a simply connected closed Riemannian manifold M using the rate of growth in the dimension of $H_*(\mathcal{L}M; \mathbb{Q})$.

The Vigué-Poirrier Conjecture has also been shown to hold for several other classes of spaces, notably for non-trivial connected sums of closed manifolds which are not monogenic in cohomology [Lam01]. This brings us to our application, which broadens the class of spaces for which the Conjecture holds to Poincaré Duality complexes with the loop space homotopy type of certain connected sums; in particular, such a homotopy equivalence need not hold before taking loop spaces. The statement of the

Theorem A below uses the notions of *inert* maps and *good exponential growth* – these are defined precisely in Definition 4.1 and Definition 4.14, respectively.

Theorem A. *Let $n > 3$ and let M be a n-dimensional Poincaré Duality complex, and suppose that there exist n-dimensional Poincaré Duality complexes N and P such that $\Omega M \simeq \Omega(N\#P)$. If P is rationally elliptic and the attaching map of the top-cell of P is inert, and if $\mathcal{L}(N\#P)$ has good exponential growth, then M satisfies the Vigué-Poirrier Conjecture.* □

In this context, $\mathcal{L}(N\#P)$ having good exponential growth implies that it satisfies the Vigué-Poirrier Conjecture – that is to say, $N\#P$ is an example of a rationally hyperbolic space such that $H_*(\mathcal{L}(N\#P);\mathbb{Q})$ grows exponentially. As said above, the Conjecture is known to hold for connected sums of certain types of manifolds, so such examples arise in nature. Theorem A therefore shows that the Conjecture holds for spaces with the loop space homotopy type of connected sums of manifolds which satisfy the simple cohomological condition of [Lam01].

A Theorem of Wall

In Chapter 5 we return explicitly to the highly connected case. The study of highly connected manifolds has a long pedigree of intense research: Milnor recounts in [Mil00] that much of his early work in the 1950s concerned $(n-1)$-connected $2n$-manifolds, Ishimoto classified π-manifolds of this type in [Ish69] for $n \geq 3$, and the study of simply connected 4-manifolds produced two Fields Medalists in Freedman and Donaldson [Fre82, Don83]. Another prolific author in this area (and many more besides) was C.T.C. Wall, who in [Wal62] classified $(n-1)$-connected $2n$-manifolds which have a boundary component diffeomorphic to a sphere, and later also worked on attempts to classify simply connected 4-manifolds up to diffeomorphism [Wal64].

Among the questions that developed from this study was whether these methods of differential topology and surgery theory were applicable to more complicated families of manifolds. For example, one could consider simply connected 6-manifolds, and indeed, in [Wal66a] Wall formulated the following.

Theorem (Wall)**.** *Let M be a closed, smooth, simply connected 6-manifold. Then there is a diffeomorphism*

$$M \cong M_1 \# M_2$$

where M_1 is a connected sum of finitely many copies of $S^3 \times S^3$ and $H_3(M_2)$ is finite.

This is a highly influential theorem, often referenced as a fundamental result of manifold theory. Generalising the theorem to higher dimensions leads one to consider decomposing $(n-2)$-connected $2n$-manifolds into constituent parts via the operation of connected sums, and indeed, such generalisations have been the subject of active research for several decades. Tamura gave decomposition results for closed, oriented, torsion free $(n-2)$-connected differentiable $2n$-manifolds (for certain congruence classes of n modulo 8) with vanishing n^{th} homology group [Tam68]. Later, in the 70s, Ishimoto was able to expand on this, giving a partial analogue to Wall's Theorem for $(n-2)$-connected $2n$-manifolds with torsion free homology, using results about parallelisability [Ish73]. Indeed, [Ish73, Theorem 4] shows that a unique connected sum decomposition (up to reordering the summands) always exists for these manifolds, and a partial answer to the consequent classification problem is subsequently developed - see for example [Ish73, Theorem 7]. In analogue to Wall's Theorem, both [Tam68] and [Ish73] detect copies of $(n-1)$-connected $2n$-dimensional manifolds as summands in their connected sum decompositions. They also work hard to gain more control over the diffeomorphism type of the space analogous to Wall's M_2. Further work from the last century, for example [Fan96], continued this trend of using geometric and differential methods developed from those of Wall in order to provide higher dimensional analogues.

We, however, will take a different approach. Drawing on recent work in homotopy theory and making use of known results for based loop spaces of certain complexes (including but not limited to [BT14, BT22, The20, Hua22a]), we give the following homotopy theoretic analogue to Wall's Theorem, which we prove in Theorem 5.8.

Theorem B. *Let $n > 3$ be an integer such that $n \notin \{4, 8\}$, and let M be a $(n-2)$-connected $2n$-dimensional Poincaré Duality complex with $\mathrm{rank}(H_n(M)) = d > 1$. Then there exists a homotopy equivalence*

$$\Omega M \simeq \Omega(M_1 \# M_2 \# M_3)$$

where

(i) M_1 is an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex, with $\mathrm{rank}(H_n(M_1)) = d$;

(ii) M_2 is a connected sum of finitely many copies of $S^{n-1} \times S^{n+1}$ and;

(iii) M_3 is a CW-complex with a single top-cell and $H_n(M_3)$ finite.

This result has several implications, notably that the homotopy groups of M are determined by those of the connected sum $M_1 \# M_2 \# M_3$. More deeply, it also implies that in almost all cases, the homotopy groups of a $(n-2)$-connected $2n$-manifold are rationally hyperbolic (see Corollary 5.10). Note however that we specifically exclude the case of a simply connected and 6-dimensional Poincaré Duality complex – the techniques required for decomposing such complexes are very different to those discussed in Chapter 5 (see for example [CS22, Hua21]). Furthermore, it also bears mentioning that the complexes M_1 and M_3 in Theorem B may in some cases have the homotopy type of a manifold: when the integer d is even we may take M_1 to be a connected sum of $\frac{d}{2}$-many copies of $S^n \times S^n$, and for M_3 it depends on the total surgery obstruction of Ranicki [Ran92].

As we discussed, Theorem B is by no means the first higher dimensional analogue to Wall's Theorem, and the methods used to prove Theorem B in Chapter 5 differ greatly from those of the past authors mentioned above. Indeed, we focus on decomposing Poincaré Duality complexes (again: smooth, closed, oriented manifolds are a subclass), and our restrictions are far milder. We do not need to make assumptions about parallelisability or restrict to the case when homology is torsion free. Though the price we pay is to sacrifice geometric precision by passing to based loop spaces, we still recover useful homotopy theoretic information, and demonstrate the value in considering such decomposition problems from this point of view.

Pullback Fibrations over Connected Sums

Taking inspiration from [JS21], the final chapter of this thesis begins with a homotopy fibration $F \to L \xrightarrow{f} C$ in which all spaces have the homotopy type of Poincaré Duality complexes. Writing $dim(C) = n$ and $dim(L) = m$, let B be another n-dimensional Poincaré Duality complex. Form the connected sum $B\#C$, and take the natural collapsing map $p : B\#C \to C$. Defining the m-dimensional complex M as the pullback of f across p, we have a homotopy fibration diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & B\#C \arrow[d, "p"] \\ F \arrow[r] & L \arrow[r, "f"] & C. \end{tikzcd} \tag{1.1}$$

A natural question follows: to what extent does M behave like a connected sum?

Jeffrey and Selick give a partial answer in [JS21]; they consider the question when each space is a closed, oriented, smooth, simply connected manifold, but in the stricter setting of fibre bundles, and construct a space X' with the property that there is an isomorphism of homology groups

$$H_k(M;\mathbb{Z}) \cong H_k(X';\mathbb{Z}) \oplus H_k(L;\mathbb{Z})$$

for $0 < k < m$ [JS21, Theorem 3.3]. This suggests that in certain circumstances we might expect there to be an m-dimensional manifold X, such that $M \simeq X\#L$. Jeffrey and Selick show that there are contexts in which such an X exists, and others where it cannot exist.

Similar questions to the above have been asked recently. Duan [Dua22] approaches the topic from a much more geometric, surgery theoretic viewpoint. In this work, the principal objects of concern are manifolds which exhibit a regular circle action; namely, a free circle action on an n-dimensional closed, oriented, smooth, simply connected manifold whose quotient space is an $(n-1)$-dimensional closed, oriented, smooth, simply connected manifold. Translated into the context of [JS21], Duan studies the situation when $F \simeq S^1$. If L is of dimension at least 5, it is shown in [Dua22] that the total space of the pullback fibration is indeed always diffeomorphic to a connected sum. Although the thrust of [Dua22] is mainly concerned with constructing smooth manifolds that admit regular circle actions, it is interesting to remark that its strategy yields a specific class of examples for the situation as in Diagram (6.1). Other recent work includes that of Huang and Theriault [HT22], in which they consider the loop space homotopy type of manifolds after stabilisation by connected sum with a projective space. They do so by combining the results of [Dua22] with a homotopy theoretic analysis of special cases of Diagram (1.1).

In Chapter 6 we give a special circumstance, recorded in Proposition 6.4, in which the based loop space of M is homotopy equivalent to the based loops of a connected sum. This takes its most dramatic form in the context of rational homotopy theory, which we state in the Theorem below. Let $\overline{C}$ and $\overline{L}$ denote the $(n-1)$- and $(m-1)$-skeleta of C and L, respectively.

Theorem C. *Given spaces and maps as in Diagram (1.1), if*

(i) the map $\alpha : F \to P$ is (rationally) null homotopic, and,

(ii) both $H^*(\overline{C};\mathbb{Q})$ and $H^*(\overline{L};\mathbb{Q})$ are generated by more than one element,

there is a rational homotopy equivalence $\Omega M \simeq \Omega(X\#L)$ where X is an appropriate CW-complex, which we construct in Section 6.2.

Thus we are able to give an affirmative answer in this situation, but after looping and up to rational homotopy equivalence. Examples of homotopy fibrations that fulfil the criteria of Theorem C include certain sphere bundles. Furthermore, note that a consequence of Theorem C is that there is an isomorphism of rational homotopy groups: $\pi_*(M) \otimes \mathbb{Q} \simeq \pi_*(X\#L) \otimes \mathbb{Q}$.

Outline of this Thesis

Chapter 2 introduces the necessary background in homotopy theory for the material that follows in this thesis, with a few prerequisites: we will make use of elementary concepts such as homotopy fibrations and cofibrations, and trust that the reader has knowledge of these. This ground has been well covered by many authors of a capability far higher than this one, so the uninitiated reader is encouraged to consult, for example, [Hat02, Ark11].

Throughout this thesis, all topological spaces have the homotopy type of pointed finite type CW-complexes unless otherwise stated, and products are taken with the compactly generated topology. Furthermore, when discussing homology and cohomology, we follow the convention that an omission of coefficients denotes coefficients in $\mathbb{Z}$.

Chapter 3 continues our exploration by discussing the context in which this thesis sits: we cover much of the literature on loop space decompositions, and provide some developments of that theory in Sections 3.3 and 3.4. Chapters 4, 5 & 6 are reformatted versions of papers of the author, together with Sections 4.3 and 4.4, which are more recent work.

Chapter 2

Preliminary Homotopy Theory

This chapter lays the foundations of what is to come in this thesis. Throughout, we will assume the reader has some prior knowledge of elementary homotopy theory and homological algebra, though where necessary we will restate key results in order to make this chapter somewhat self-contained. In particular however, we take our definitions of homotopy pushout and homotopy pullback to be as in [Ark11, Definition 6.2.1] and [Ark11, Definition 6.2.13], respectively.

Section 2.1 gives several elementary constructions and an exposition of *generalised connected sums*, which forms one of our principal ideas throughout this work. Next, Section 2.2 discusses the background of [The20] as well as some results from that paper. Chief amongst these results is Theorem 2.14, which we will regularly use in the chapters that follow. We then have Section 2.3, which provides two lemmas regarding homotopy cofibrations involving connected sums.

Our foremost concern is the utility of these techniques is in describing the homotopy theory of highly connected, closed, smooth manifolds. Much of this work will be done using Poincaré Duality complexes, which define and discuss in Section 2.4.

2.1 Generalised Connected Sums

As mentioned above, we begin with some elementary constructions that form the basis of much of this work. Manipulating homotopy cofibrations, and indeed producing new homotopy cofibrations from given data, will enable us to prove the loop space decompositions that are the central focus of this thesis. Given two homotopy cofibrations

$$A \xrightarrow{f} B \xrightarrow{h} C \text{ and } X \xrightarrow{g} B \xrightarrow{j} Y$$

one may form the following commutative diagram, in which the bottom-right square is a homotopy pushout

$$\begin{tikzcd}
& X \arrow[r, equal] \arrow[d, "g"] & X \arrow[d, "h\circ g"] \\
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "h"] \arrow[d, "j"] & C \arrow[d] \\
A \arrow[r, "j\circ f"] & Y \arrow[r] & Q.
\end{tikzcd} \tag{2.1}$$

The following Lemma is an elementary fact of homotopy theory.

Lemma 2.1. *The bottom row and right-hand column of (2.1) are homotopy cofibrations.*

Proof. Representing the initial two homotopy cofibrations as homotopy pushouts, and combining them with the homotopy pushout in the bottom-right square of (2.1), we have the following homotopy commutative diagram of homotopy pushout squares

$$\begin{tikzcd}
& A \arrow[r] \arrow[d, "f"] & * \arrow[d] \\
X \arrow[r, "g"] \arrow[d] & B \arrow[r, "h"] \arrow[d, "j"] & C \arrow[d] \\
* \arrow[r] & Y \arrow[r] & Q.
\end{tikzcd}$$

Consider the bottom two squares and their outer rectangle. The two bottom squares are homotopy pushouts, so recalling [Ark11, Theorem 6.3.3], the outer rectangle is therefore a homotopy pushout as well. This is equivalent to $X \xrightarrow{h\circ g} C \to Q$ being a homotopy cofibration, and arguing by symmetry, we have that $A \xrightarrow{j\circ f} Y \to Q$ is also a homotopy cofibration. □

Another way of stating Lemma 2.1 is that Diagram (2.1) is a *homotopy cofibration diagram*, i.e. that the diagram is homotopy commutative and every complete row and column is a homotopy cofibration. A *homotopy fibration diagram* is similarly defined, namely that such a diagram is homotopy commutative and every complete row and column is a homotopy *fibration*.

Remark 2.2. Lemma 2.1 is particularly useful if we have prior knowledge of the homotopy cofibre of $j \circ f$ (for example), as this must then also be the homotopy cofibre of $h \circ g$. One special instance of this is in the situation of generalised connected sums, which we explain after first giving the definition below.

Definition 2.3. Given a co-H-space A with co-multiplication σ, spaces X and Y and maps $f : A \to X$ and $g : A \to Y$, define $f \check{+} g$ to be the composite

$$f \check{+} g : A \xrightarrow{\sigma} A \vee A \xrightarrow{f \vee g} X \vee Y.$$

Note that the choice of co-multiplication need not be unique for a given co-H-space, and (letting p_1 denote a pinch map, as per page *xiii* of this thesis) that $p_1 \circ (f \check{+} g) \simeq f$. By symmetry we also have that $p_2 \circ (f \check{+} g) \simeq g$.

Suppose now that we have two homotopy cofibrations of simply connected spaces

$$A \xrightarrow{f} B \xrightarrow{j} C \text{ and } A \xrightarrow{g} D \xrightarrow{l} E$$

where A is a co-H-space with comultiplication σ.

Definition 2.4. With the set-up as above, the homotopy cofibre of $f \check{+} g$ is called the *generalised connected sum* of C and E over A, written $C\#_A E$.

When the co-H-space A is clear, we will often omit the subscript. We wish to use the construction of Diagram (2.1) to deduce some facts about generalised connected sums.

Proposition 2.5. *There is a homotopy cofibration diagram*

$$\begin{tikzcd} & D \arrow[r, equal] \arrow[d, "i_2"] & D \arrow[d] \\ A \arrow[r, "f \check{+} g"] \arrow[d, equal] & B \vee D \arrow[r] \arrow[d, "p_1"] & C\#_A E \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r, "j"] & C \end{tikzcd} \tag{2.2}$$

Proof. The middle row of (2.2) is a homotopy cofibration, by definition, and the middle column is evidently also a homotopy cofibration. By Lemma 2.1, these intersecting cofibrations yield a homotopy cofibration diagram

$$\begin{tikzcd} & D \arrow[r, equal] \arrow[d, "i_2"] & D \arrow[d, "q \circ i_2"] \\ A \arrow[r, "f \check{+} g"] \arrow[d, equal] & B \vee D \arrow[r, "q"] \arrow[d, "p_1"] & C\#_A E \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r] & Q \end{tikzcd}$$

where the bottom-right square is a homotopy pushout, defining the space Q. We have, however, that the homotopy cofibre of f is C, so therefore $Q \simeq C$ and we have Diagram (2.2) as claimed. □

Example 2.6. A particular consequence of Proposition 2.5 is that there is a homotopy cofibration $D \to C\#_A E \to C$. Indeed, suppose we had two path connected n-dimensional CW-complexes M and N, both of which have a single n-cell. Let M_{n-1} and N_{n-1} denote their $(n-1)$-skeleta, respectively. Then there are homotopy cofibrations

$$S^{n-1} \to M_{n-1} \to M \text{ and } S^{n-1} \to N_{n-1} \to N$$

which by Proposition 2.5 yields a homotopy cofibration $N_{n-1} \to M\#_{S^{n-1}}N \to M$. If M and N are smooth, closed, oriented manifolds manifolds then we choose $M\#_{S^{n-1}}N$ such that it coincides (up to homotopy) with the usual orientation preserving connected sum of M and N.

2.2 Homotopy Fibrations with a Section after Looping

We give a homotopy theoretic construction that will form the basis of our method for decomposing the loop spaces of certain highly connected complexes. The key statements are those of Theorem 2.13 and Theorem 2.14. Much of what is given here follows [The20, Chapters 1-2] but with some details added. To get started, we state the Cube Lemma (due originally to Mather [Mat76]).

Theorem 2.7 (Cube Lemma)**.** *Let A, B, C, D, E, F, G and H be topological spaces, with maps between them as in the cube diagram below, in which all faces are homotopy commutative squares.*

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr] \arrow[dd] & & F \arrow[dr] \arrow[dd] & \\
& G \arrow[rr] \arrow[dd] & & H \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

If the bottom square $ABCD$ is a homotopy pushout and the four vertical squares $EGAC$, $GHCD$, $FHBD$ and $EFAB$ are homotopy pullbacks, then the top square $EFGH$ is a homotopy pushout. □

Now, let us introduce the situation we wish to study. In what follows, we will assume all spaces to be path connected. Consider a map $f : A \to B$, whose homotopy cofibre is denoted by C. Suppose also that there is another map $h : B \to Z$ that extends to a map $h' : C \to Z$. Further, let us denote the homotopy fibres of the maps h and h' by E and E', respectively. We can arrange this data into the following homotopy commutative diagram

$$
\begin{tikzcd}
 & E \arrow[r] \arrow[d] & E' \arrow[d] \\
A \arrow[r, "f"] & B \arrow[r] \arrow[d, "h"] & C \arrow[d, "h'"] \\
 & Z \arrow[r, equal] & Z
\end{tikzcd}
\tag{2.3}
$$

where the middle and right columns are homotopy fibrations and the middle row is a homotopy cofibration.

Lemma 2.8. *With the setup as in (2.3) above, there is a homotopy pushout square*

$$
\begin{tikzcd}
\Omega Z \times A \arrow[r, "a"] \arrow[d, "\pi_1"] & E \arrow[d] \\
\Omega Z \arrow[r] & E'
\end{tikzcd}
$$

where the map a is to be determined, and the map π_1 denotes projection in the first factor.

Proof. Taking (2.3), we can augment it to give the following diagram.

$$
\begin{tikzcd}[row sep=small, column sep=small]
\Omega Z \times A \arrow[rr, "a"] \arrow[dr] \arrow[dd] & & E \arrow[dr] \arrow[dd] & \\
 & \Omega Z \arrow[rr, crossing over] & & E' \arrow[dd] \\
A \arrow[rr, "f" near start] \arrow[dr] & & B \arrow[dr] \arrow[dd, "h"] & \\
 & * \arrow[rr] \arrow[from=uu, crossing over] & & C \arrow[dd, "h'"] \\
 & & Z \arrow[dr, equal] & \\
 & & & Z
\end{tikzcd}
\tag{2.4}
$$

The bottom face of the cube is a homotopy pushout because $A \xrightarrow{f} B \longrightarrow C$ is a homotopy cofibration. All four vertical faces are forced to be homotopy pullbacks because all the vertical homotopy fibrations are over a common base space. More specifically, the right-hand face of the cube must be a homotopy pullback because both the maps $E \to B$ and $E' \to C$ have the same homotopy fibre, namely ΩZ. The front face of the cube is made by taking a homotopy pullback in the canonical way, and so is the left-hand face. Thus we are in precisely the situation of the Cube Lemma, so the top face is a homotopy pushout.

We now show that the map $\Omega Z \times A \to \Omega Z$ may be chosen to be a projection. Indeed, in general for spaces X and Y there exists a homotopy pullback

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

where the maps π_1 and π_2 are projections. Up to homotopy, this is exactly the left face of the cube in (2.4), so we have the lemma as claimed. □

We now work towards identifying the map a from Lemma 2.8. In general, suppose there exists a homotopy fibration sequence

$$\Omega Z \xrightarrow{\delta} X \to Y \to Z.$$

There exists a homotopy action $\theta : \Omega Z \times X \to X$ that extends the map $\Omega Z \vee X \xrightarrow{\delta \vee 1} X$. This action is canonical, and letting μ denote the usual loop multiplication in the space ΩZ, it satisfies the two homotopy pullback squares:

$$\begin{tikzcd} \Omega Z \times \Omega Z \arrow[r, "\mu"] \arrow[d, "1\times\delta"] & \Omega Z \arrow[d, "\delta"] \\ \Omega Z \times X \arrow[r, "\theta"] & X \end{tikzcd} \qquad \begin{tikzcd} \Omega Z \times X \arrow[r, "\theta"] \arrow[d, "\pi_2"] & X \arrow[d] \\ X \arrow[r] & Y. \end{tikzcd} \tag{2.5}$$

Lemma 2.9. *The map $a : \Omega Z \times A \to E$ from Lemma 2.8 is homotopic to the composite*

$$\varphi : \Omega Z \times A \xrightarrow{1\times g} \Omega Z \times E \xrightarrow{\theta} E$$

where g is a lift of the map f that factors through the homotopy fibre of the map $E \to E'$.

Proof. The condition on the lift g is equivalent to requiring that the composite of g with the map $E \to E'$ be null homotopic. Thus, by the commutativity of (2.4), up to homotopy we may choose g to be the composite

$$A \xrightarrow{\iota_2} \Omega Z \times A \xrightarrow{a} E$$

where ι_2 denotes the inclusion of the second factor. Now, consider the diagram below

$$\begin{tikzcd}
\Omega Z \times A \arrow[r, "1\times g"] \arrow[d, "\pi_2"] & \Omega Z \times E \arrow[r, "\theta"] \arrow[d, "\pi_2"] & E \arrow[d] \\
A \arrow[r, "g"] & E \arrow[r] & B
\end{tikzcd}$$

where the right-hand square is one of the homotopy pullbacks given in (2.5). The left square is also homotopy pullback, because of the projections, and therefore the whole outer rectangle is a homotopy pullback.

The composite along the bottom recovers the map f. Thus we have constructed the rear face of (2.4), and hence we have that $a \simeq \theta \circ (1 \times g)$. □

Before we continue, we must introduce an operation on topological spaces.

Definition 2.10. Let X and Y be pointed path connected spaces. The *(left) half-smash* of X and Y is the quotient space

$$X \ltimes Y = (X \times Y)/(X \times y_0)$$

where y_0 denotes the basepoint of Y.

Observe that the half-smash fits into the cofibration sequence

$$X \xrightarrow{\iota_1} X \times Y \to X \ltimes Y \xrightarrow{\partial} \Sigma X$$

where the map ι_1 is the inclusion of the first factor. Moreover, we have the following well known homotopy equivalence.

Lemma 2.11. *Let X and Y be pointed path connected spaces. If Y is a co-H-space, then $X \ltimes Y \simeq (X \wedge Y) \vee Y$.*

Proof. First, observe that the spaces $X \ltimes (Y \vee Y)$ and $(X \ltimes Y) \vee (X \ltimes Y)$ are homeomorphic. Thus, the co-H structure σ on Y induces one on $X \ltimes Y$, via the map

$$X \ltimes Y \xrightarrow{1\ltimes\sigma} X \ltimes (Y \vee Y) \cong (X \ltimes Y) \vee (X \ltimes Y).$$

Now, consider the homotopy cofibration $Y \xrightarrow{i} X \ltimes Y \xrightarrow{q} X \wedge Y$ where i is the inclusion into the second coordinate. The map i has a left inverse, denoted by α, which is given by factoring the projection map $\pi_2 : X \times Y \to Y$ through $X \ltimes Y$ as in the commutative diagram below.

$$\begin{tikzcd}
X \times Y \arrow[rr, "\pi_2"] \arrow[dr] & & Y \\
& X \ltimes Y \arrow[ur, dashed, "\alpha"] &
\end{tikzcd}$$

Thus we have that Y retracts off of $X \ltimes Y$. The co-H structure on $X \ltimes Y$ then yields a composite

$$\epsilon : X \ltimes Y \xrightarrow{1\ltimes\sigma} (X \ltimes Y) \vee (X \ltimes Y) \xrightarrow{q\vee\alpha} (X \wedge Y) \vee Y$$

which is a homotopy equivalence, by virtue of [Har02, Section 3.3]. □

Observe that the Lemma holds in the particular case when the space Y is a suspension. Now we return to the situation we have been analysing, and move to considering the consequences of the map h in (2.3) having a right homotopy inverse after looping. Note that this implies that ΩZ retracts off ΩB.

Lemma 2.12. *If the map Ωh has a right homotopy inverse, there exists a homotopy cofibration*

$$\Omega Z \ltimes A \xrightarrow{\overline{\varphi}} E \to E'$$

for some map $\overline{\varphi}$ such that the composite $\Omega Z \times A \to \Omega Z \ltimes A \xrightarrow{\overline{\varphi}} E$ is homotopic to φ.

Proof. Let us briefly consider a more general situation. As alluded to previously, the left half-smash fits into the homotopy cofibration sequence

$$X \xrightarrow{\iota_1} X \times Y \to X \ltimes Y \xrightarrow{\partial} \Sigma X.$$

The inclusion ι_1 has a left inverse, and therefore the map ∂ is null homotopic. Therefore, for any space Z, the induced map $[X \ltimes Y, Z] \to [X \times Y, Z]$ is an injection. Moreover, if there is a map $f : X \times Y \to Z$ such that the composite $f \circ \iota_1$ is null homotopic, then there exists a map $\overline{f} : X \ltimes Y \to Z$ whose homotopy class is determined by f, illustrated in the commutative diagram below.

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] & X \times Y \arrow[r] \arrow[d, "f"] & X \ltimes Y \arrow[dl, dashed, "\overline{f}"] \\
& Z &
\end{tikzcd}$$

Returning now to our specific case, if Ωh has a right homotopy inverse, this forces the connecting map $\delta : \Omega Z \to E$ to be null homotopic. Moreover, by definition, the restriction of the map φ to ΩZ is δ. By the homotopy commutativity of the pushout square in Lemma 2.8, we see that this causes the map $\Omega Z \to E'$ to be null homotopic as well. This allows us to pinch out a copy of ΩZ from the square in Lemma 2.8, and thus obtain the homotopy pushout square

$$\begin{tikzcd}
\Omega Z \ltimes A \arrow[r, "\overline{\varphi}"] \arrow[d] & E \arrow[d] \\
* \arrow[r] & E'
\end{tikzcd}$$

which is equivalent to having a homotopy cofibration $\Omega Z \ltimes A \xrightarrow{\overline{\varphi}} E \to E'$. □

Let us record the previous results together in following theorem, which the reader may find as [The20, Theorem 2.2(a)].

Theorem 2.13 (Theriault)**.** *Suppose there exists a homotopy commutative diagram*

$$\begin{tikzcd}
 & E \arrow[r, "\alpha"] \arrow[d] & E' \arrow[d] \\
\Sigma A \arrow[r, "f"] & B \arrow[r] \arrow[d, "h"] & C \arrow[d, "h'"] \\
 & Z \arrow[r, equal] & Z
\end{tikzcd}$$

where the middle and right columns are homotopy fibrations, the map α is an induced map of fibres and the middle row is a homotopy cofibration. If Ωh has a right homotopy inverse, then there exists a homotopy cofibration $\Omega Z \ltimes \Sigma A \xrightarrow{\overline{\varphi}} E \to E'$. □

In the full statement of [The20, Theorem 2.2], much more detail over the homotopy class of the map $\overline{\varphi}$ is given. We will not need this level of precision for the study we wish undertake, so we neglect to state it - all we require is the existence of the stated homotopy cofibration.

A fundamental special case of Theorem 2.13 is when we have $Z = C$. Indeed, it underpins much of the work that will follow in this thesis. We record it the theorem below, a version of which the reader will also find in [BT22, Proposition 3.5], where it first appeared. Note that the need for the suspension ΣA is dropped.

Theorem 2.14 (Beben-Theriault)**.** *Suppose we have a homotopy cofibration $A \xrightarrow{f} B \xrightarrow{h} C$ such that the map Ωh has a right homotopy inverse. Then there exists a homotopy fibration*

$$\Omega C \ltimes A \to B \xrightarrow{h} C$$

which splits after looping. Thus, there exists a homotopy equivalence

$$\Omega B \simeq \Omega C \times \Omega(\Omega C \ltimes A).$$

Proof. To identify the fibre of the map h, let us consider the diagram

$$\begin{tikzcd}
 & E \arrow[r] \arrow[d] & E' \arrow[d] \\
A \arrow[r, "f"] & B \arrow[r, "h"] \arrow[d, "h"] & C \arrow[d, equal] \\
 & C \arrow[r, equal] & C
\end{tikzcd}$$

where E and E' denote the appropriate homotopy fibres. Observe that this is a special case of (2.3). One sees immediately that $E' \simeq *$, since it is the fibre of an identity map, so by Theorem 2.13 we obtain a homotopy cofibration

$$\Omega C \ltimes A \xrightarrow{\overline{\varphi}} E \to *$$

which forces the map $\overline{\varphi}$ to be a homotopy equivalence, proving the first part of the corollary. Since Ωh has a right homotopy inverse, the homotopy fibration

$$\Omega(\Omega C \ltimes A) \to \Omega B \xrightarrow{\Omega h} \Omega C$$

splits, and therefore we have the homotopy equivalence as claimed. □

Example 2.15. For two pointed, path connected spaces X and Y, let us use the above Corollary to identify the homotopy fibre of the pinch map $X \vee Y \xrightarrow{p_1} X$. The map p_1 has a clear right inverse, namely the inclusion of the first wedge summand. Thus, Ωp also has a right homotopy inverse. The map p_1 fits into the homotopy cofibration

$$Y \xrightarrow{i_2} X \vee Y \xrightarrow{p_1} X$$

and thus Theorem 2.14 gives us that the homotopy fibre of p_1 is the space $\Omega X \ltimes Y$. Moreover, we have that

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega(\Omega X \ltimes Y).$$

Note that if X and Y were suspensions, we could repeatedly apply Lemma 2.11 and Theorem 2.14 and recover the Hilton-Milnor Theorem, which we state below so that we may refer to it later (the second equivalence is from [Sel97, Theorem 7.9.4]).

Hilton-Milnor Theorem. *Let X and Y be pointed, path connected spaces. Then there is a homotopy equivalence*

$$\begin{aligned}\Omega(\Sigma X \vee \Sigma Y) &\simeq \Omega\Sigma X \times \Omega\Sigma Y \times \Omega\Sigma\left(\bigvee_{i=1}^{\infty} X \wedge Y^{\wedge i}\right) \\ &\simeq \prod_{w\in W} \Omega\Sigma\left(X^{\wedge(w_x)} \wedge Y^{\wedge(w_y)}\right)\end{aligned}$$

where $W = W\langle x, y\rangle$ is a (vector space) basis for the free Lie algebra on generators x and y. □

The Hilton-Milnor Theorem also functions in its own right as an example of a loop space decomposition. Indeed, the results we will give later in this thesis on the homotopy type of certain loop spaces will have a similar form.

Remark 2.16. We sketch how the homotopy equivalence in the above statement of the Hilton-Milnor Theorem can be constructed, for the benefit of a later example. Taking adjoints of inclusions maps i_1 and i_2 into the wedge, we have maps

$$x : X \to \Omega(\Sigma X \vee \Sigma Y) \;\text{ and }\; y : Y \to \Omega(\Sigma X \vee \Sigma Y)$$

with which we can take a Samelson product $s = \langle x, y\rangle : X \wedge Y \to \Omega(\Sigma X \vee \Sigma Y)$. Next, we use our knowledge of the James Construction (see [Sel97, Section 7.9]), thus giving an extension as in the dashed arrow in diagram below

$$\begin{tikzcd} X \wedge Y \arrow[r, "s"] \arrow[d] & \Omega(\Sigma X \vee \Sigma Y) \\ \Omega\Sigma(X \wedge Y). \arrow[ur, dashed] & \end{tikzcd} \tag{2.6}$$

It can be shown that the dashed arrow above is in fact $\Omega[i_1, i_2]$ the loop map on the Whitehead product of the inclusions we began with. Taking advantage of the suspension comultiplications on ΣX and ΣY, one can take multiples of the inclusion maps i_1 and i_2 before taking adjoints and Samelson products. In this way one forms a free Lie algebra $W\langle x, y\rangle$ of maps, each with an extension analogous to the dashed arrow in (2.6) with the homotopy class of a looped Whitehead product. Multiplying all these looped Whitehead products together, via the usual loop space multiplication, gives the homotopy equivalence in the Theorem. Furthermore it implies that for such a Whitehead product $w : \Sigma(X^{\wedge(w_x)} \wedge Y^{\wedge(w_y)}) \to \Sigma X \vee \Sigma Y$, letting π_w be the projection from the product to the relevant factor in the decomposition, the composite

$$\Omega\Sigma(X^{\wedge(w_x)} \wedge Y^{\wedge(w_y)}) \xrightarrow{\Omega w} \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\pi_w} \Omega\Sigma(X^{\wedge(w_x)} \wedge Y^{\wedge(w_y)})$$

is a homotopy equivalence.

2.3 Homotopy Cofibrations with Wedge Sums

We give two elementary constructions of homotopy cofibrations that involve wedge sums. These will be of particular use when we consider homology decompositions of certain CW-complexes in Chapter 5. Recall that for a space X to be a *homotopy retract* of another space Y, we mean that there exists a sequence of maps $X \to Y \to X$ whose composite is homotopic to $\mathbb{1}_X$.

Lemma 2.17. *Suppose that we have a homotopy cofibration*

$$A \xrightarrow{f} B \vee C \xrightarrow{q} D$$

such that the composition of f with the pinch map $p_1 : B \vee C \to B$ is null homotopic. Then B is a homotopy retract of D.

Proof. Consider the homotopy commutative diagram below.

$$\begin{tikzcd}
 & B \arrow[d, "i_1"] & \\
A \arrow[r, "f"] & B \vee C \arrow[r, "q"] \arrow[d, "p_1"] & D \arrow[dl, dashed, "\overline{p}"] \\
 & B &
\end{tikzcd}$$

The composition $p_1 \circ f$ is null homotopic, which gives rise to the extension $\overline{p}$. The composition of the inclusion map i_1 with p is homotopic to the identity on B, and by the homotopy commutativity of the diagram, so is $\overline{p} \circ q \circ i_1$. Thus we have that B is a homotopy retract of D. □

Lemma 2.18. *Suppose that we have a homotopy cofibration*

$$A \vee B \xrightarrow{h} C \to D$$

such that restriction of the map h to A is null homotopic. Then ΣA is a homotopy retract of D.

Proof. By the dual statement of [Sel97, Theorem 7.6.2], there exists a homotopy commutative diagram of homotopy cofibrations (i.e. every row and column is a homotopy cofibration sequence)

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i_1"] & * \arrow[r] \arrow[d] & \Sigma A \arrow[r, equal] \arrow[d, "f"] & \Sigma A \arrow[d, "i_1"] \\
A \vee B \arrow[r, "h"] \arrow[d] & C \arrow[r] \arrow[d, equal] & D \arrow[r, "\partial"] \arrow[d, "q"] & \Sigma A \vee \Sigma B \\
B \arrow[r] & C \arrow[r] & E &
\end{tikzcd} \tag{2.7}$$

where f is the induced map of homotopy cofibres and the space E is its homotopy cofibre. Furthermore, taking the righthand square and composing with the pinch map $p_1 : \Sigma A \vee \Sigma B \to \Sigma A$ along the bottom row, we get a second homotopy commutative diagram

$$\begin{tikzcd}
\Sigma A \arrow[r, equal] \arrow[d, "f"] & \Sigma A \arrow[d, "i_1"] \arrow[dr, equal] & \\
D \arrow[r, "\partial"] & \Sigma A \vee \Sigma B \arrow[r, "p_1"] & \Sigma A.
\end{tikzcd}$$

which gives $p_1 \circ \partial \circ f \simeq \mathbb{1}_{\Sigma A}$, and thus the desired homotopy retraction. □

2.4 Poincaré Duality Complexes

When Henri Poincaré originally defined his duality in the late 19^{th} century, it was in terms of Betti numbers: the quite simple statement that, given an integer $n > 0$ and a closed, orientable n-manifold, the k^{th} and $(n-k)^{th}$ Betti numbers are equal. The subsequent modern formulation expresses Poincaré Duality in terms of cup and cap products (which we will assume the reader has knowledge of - see for example [Hat02, Chapter 3]) and enables one to concretely ask which spaces satisfy Poincaré Duality, other than closed, orientable manifolds.

Asking which CW-complexes have Poincaré Duality open us to the broader class of spaces that are *Poincaré Duality complexes*. In brief language, for us such a complex will be a finite, simply connected CW-complex whose (co)homology satisfies Poicaré Duality for every coefficient ring. This works particularly well for our purposes: as we are using homotopy theory to conduct our study, it makes sense to use spaces with an underlying structure that we can exploit.

Our first concrete definition is due to Wall, which we state for posterity, who defines what he called *Poincaré complexes* abstractly in [Wal67]. In stating it, we shall incorporate notation from [Wal66b]: for a given CW-complex X let us write $\pi = \pi_1(X)$ and let $\Lambda = \mathbb{Z}[\pi]$ be its associated integral group ring. Recall that a CW-complex is called *finitely dominated* if there exists a finite CW-complex of which it is a homotopy retract. A connected CW-complex X is called a *connected Poincaré complex* if

(i) X is finitely dominated;

(ii) there exists a homomorphism $w : \pi \to \{\pm 1\}$ that defines a Λ-module structure, denoted $\mathbb{Z}^t$, on $\mathbb{Z}$;

(iii) these exists an integer n and a class $[X] \in H_n(X; \mathbb{Z}^t)$ such that for all integers r, cap product with $[X]$ induces an isomorphism

$$[X] \cap - : H^r(X; \Lambda) \to H_{n-r}(X; \Lambda \otimes \mathbb{Z}^t).$$

We will be making some simplifications to this abstract definition for our work in this thesis. First, we will insist that our CW-complexes are finite, so we have finite domination trivially. Second, we demand that our complexes are simply connected, so that π is the trivial group. In practice, therefore, throughout this thesis we will make use of the following simpler definition.

Definition 2.19. Let $n > 0$ be an integer, and let X be a space with the homotopy type of a finite CW-complex of dimension n. The space X is a *Poincaré Duality complex* if for every coefficient ring R there exists a class $\mu_X \in H_n(X; R)$, called the *fundamental class*, such that the homomorphism induced by the cap product

$$\mu_X \cap - : H^k(X; R) \to H_{n-k}(X; R)$$

is an isomorphism for every integer k.

Remark 2.20. Note that we require something stricter than just an isomorphism between $H^k(X; R)$ and $H_{n-k}(X; R)$: we need such an isomorphism to be induced by some fundamental class.

Poincaré Duality complexes are indeed a large class of spaces. Examples include all smooth, closed, oriented manifolds. In later sections, we will make use of the symmetry inherent in Poincaré Duality complexes to aid us in unravelling their loop spaces. An example of this symmetry is detailed in the Lemma below, a key observation which the reader will also find proved in [BT14, Lemma 3.3].

Lemma 2.21. *Let X be an n-dimensional Poincaré Duality complex such that $H_*(X)$ is torsion-free, and let μ_X^* denote the cohomology dual of the fundamental class. Then for any positive integer $i \leq n$ and basis element $x^* \in H^i(X)$, there exists a choice of basis for $H^{n-i}(X)$ such that $x^* y^* = \mu_X^*$ for some y^* in this basis.*

Proof. Let $x \in H_i(X)$ be the homology dual of the class x^*. Consider a new element $y = \mu_X \cap x$. Since X is a Poincaré Duality complex, the cap product with the class μ_X induces an isomorphism, and therefore y is in a basis of $H_{n-i}(X)$.

Since $H_*(X)$ is torsion-free, the cup product is dual to the cap product. In particular, the homomorphism $(- \cap x^*)$ sends the class μ_X to y, and therefore its dual $(- \cap x^*)^*$ sends y^* to μ_X^*. But this dual homomorphism is nothing but the cup product with x^*, and therefore we have shown that $x^* y^* = \mu_X^*$. Since y is an element of a basis for $H_{n-i}(X)$, the class y^* is therefore an element in the dual basis for $H^{n-i}(X)$, so we have the Lemma as claimed. □

Next we record an observation about even-dimensional Poincaré Duality complexes.

Proposition 2.22. *Let X be an $2n$-dimensional Poincaré Duality complex. If n is odd then the rank of the middle (co)homology group is even.*

Proof. Let $d = rank(H_n(X))$. By Poincaré Duality, $H_n(X) \cong H^n(X)$. If n is odd, then taking coefficients in $\mathbb{R}$, the cup product induces a non-degenerate skew-symmetric bilinear form

$$H^n(X, \mathbb{R}) \times H^n(X, \mathbb{R}) \to H^{2n}(X, \mathbb{R})$$

which we may represent by a $(d \times d)$-matrix A. By some elementary linear algebra we therefore have $det(A) = (-1)^d det(A)$, so d must be even, otherwise we obtain a contradiction. □

Remark 2.23. The reader may well (rightly) ask if there are examples of Poincaré Duality complexes that do not have the homotopy type of a manifold. We are not especially concerned with this question, but it does deserve a remark. Indeed, the answer is yes. In pursuit of his classification of simply connected 5-manifolds, Stöcker studied simply connected 5-dimensional Poincaré Duality complexes in [Stö82]. There he showed that there are indeed such complexes that are not homotopy equivalent to manifolds, via a classification of invariants - most notably an exotic order. Stöcker showed that this exotic order, taking the value either zero or one, indicates whether there exists an orthogonal sphere bundle structure on the Spivak fibration. It then follows, from some very deep theory due to Browder and Novikov, that a simply connected 5-dimensional Poincaré Duality complex has the homotopy type of a manifold if and only if the exotic order is zero. Remarkably, these complexes are not pathological examples but arise in well known natural ways, such as the total space of an S^2-fibration over S^3 [GS65].

Chapter 3

Loop Space Decompositions

This chapter functions both as a development of the preliminary homotopy theory of the previous chapter, and as a detailed discussion of the existing literature on loop space decompositions of Poincaré Duality complexes. Many of the methods used fit into a more general framework of the author, which we present in Chapter 4.

Much of what we write in Sections 3.1 and 3.2 of this chapter may be found in [BT14] and [BT22], in which $(n-1)$-connected Poincaré Duality complexes of dimension $2n$ are discussed, where $n > 1$. In later chapters we will draw inspiration from this work, enabling us to widen the scope of these arguments to Poincaré Duality complexes of lower connectivity. We will use homotopy theoretic methods to provide product decompositions of the loop spaces of highly connected complexes. In pursuit of the goal of decomposing the loop spaces of highly connected Poincaré Duality complexes, our main tool will be Theorem 2.14.

Sections 3.3 and 3.4 contain discussions of two developments that may be made from the theory detailed in this chapter. Section 3.3 widens the scope of our earlier exposition to a larger class of Poincaré Duality complexes of lower connectivity than those covered in Sections 3.1 and 3.2. Indeed, the methods of Section 3.3 are also developed later in this thesis in Chapter 5, which uses similar ideas to give a homotopy theoretic analogue to well-known theorem of C.T.C. Wall. Section 3.4 focuses on making a brief comment on some aspects of the homotopy theory of this larger class of Poincaré Duality complexes in Section 3.4.

3.1 $(n-1)$-Connected $2n$-Dimensional Poincaré Duality Complexes

Throughout this section, let $n > 1$ be an integer and let M denote a $2n$-dimensional Poincaré Duality complex that is $(n-1)$-connected. Our goal is to find the homotopy type of the loop spaces of such complexes. From the initial hypotheses on M, we can immediately deduce that its cohomology has the form

$$H^m(M) \cong \begin{cases} \mathbb{Z} \text{ if } m = 0 \text{ or } m = 2n \\ \mathbb{Z}^d \text{ if } m = n \\ 0 \text{ otherwise} \end{cases}$$

for some integer d. Applying the Universal Coefficient Theorem, we also have

$$H_m(M) \cong \begin{cases} \mathbb{Z} \text{ if } m = 0 \text{ or } m = 2n \\ \mathbb{Z}^d \text{ if } m = n \\ 0 \text{ otherwise.} \end{cases}$$

Let M_n denote the n-skeleton of M. Since M in $(n-1)$-connected and has the homotopy type of a CW-complex, $M_n \simeq \bigvee_{i=1}^d S^n$, and there is a homotopy cofibration

$$S^{2n-1} \xrightarrow{f} M_n \simeq \bigvee_{i=1}^{d} S^n \longrightarrow M \tag{3.1}$$

for some attaching map f. To find the homotopy type of ΩM, we must divide this into the following three cases, depending on the value of the integer d. Note that by Proposition 2.22, if n odd then the integer d is even.

Case 1: $d = 0$

In this scenario, M is in fact $(2n-1)$-connected. By the Hurewicz theorem there is a map $\psi : S^{2n} \to M$ which induces an isomorphism on H_{2n}. Indeed, this map ψ induces an isomorphism in all degrees of homology, and is therefore a homotopy equivalence, by Whitehead's Theorem. Thus we have $\Omega M \simeq \Omega S^{2n}$.

Case 2: $d = 1$

The case when $d = 1$ is closely related to the Hopf invariant one problem. To begin, let $x^* \in H^n(M) \cong \mathbb{Z}$ be a generator. Since M is at least path-connected, the class μ_M^* generates $H^{2n}(M) \cong \mathbb{Z}$. The fact that $H^*(M)$ is torsion-free allows us to use Lemma 2.21, which yields the relation $(x^*)^2 = \pm\mu_M^*$, and so the map f has a Hopf invariant of 1 or -1. Adams showed in [Ada60] that this can only happen for $n \in \{2, 4, 8\}$. In these cases, we obtain the following homotopy equivalences:

$$M \simeq \begin{cases} \mathbb{C}P^2 \text{ if } n = 2 \\ \mathbb{H}P^2 \text{ if } n = 4 \\ \mathbb{O}P^2 \text{ if } n = 8. \end{cases}$$

By the splitting of the complex and quaternionic Hopf fibrations, we have that

$$\Omega\mathbb{C}P^2 \simeq S^1 \times \Omega S^5 \text{ and } \Omega\mathbb{H}P^2 \simeq S^3 \times \Omega S^{11}$$

so we have a suitable decomposition of ΩM when $n = 2$ and $n = 4$. There is no corresponding decomposition for the octonionic case, when $n = 8$, due in part to the fact that S^7 in not homotopy associative and therefore does not have a classifying space (see for example [Sel97, Theorem 14.6.4]). This is as far as we are currently able to pursue this situation.

Case 3: $d > 1$

In this much more general case, we aim to make use of some of the homotopy theoretic results laid out in Section 2.2. For the moment, we shall assume that $n \notin \{2, 4, 8\}$ so that we may avoid Hopf invariant one cases; we postpone this discussion until later in this chapter.

Let $\{x_1^*, \ldots, x_d^*\}$ be a basis for $H^n(M)$. By Lemma 2.21, there exists integers k and l such that $1 \leq k < l \leq n$ and $x_k^* x_l^* = \mu_M^*$. Note that $x_k^* \neq \pm x_l^*$, as otherwise we would be in a Hopf invariant one case, which we have explicitly excluded. Dualizing, we also have a homology basis $\{x_1, \ldots, x_d\}$ for $H_n(M)$. Our first step in this case will be to make a precise correspondence between our choice basis and the wedge of spheres M_n.

Our Poincaré Duality complex M is $(n-1)$-connected, so by the Hurewicz Theorem there is a group isomorphism $\pi_n(M) \cong H_n(M)$. Thus, for each $i \in \{1, \ldots, d\}$ there exist maps $s_i : S^n \to M$ with Hurewicz image $(s_i)_*(\iota_n) = x_i$, where ι_n denotes the generator of $\pi_n(S^n)$. Let $\sigma : \bigvee_{i=1}^d S^n \to M$ be the wedge sum of the s_i. Note that for dimensional reasons, this map σ must factor through M_n, so there is a factorisation as below.

$$\begin{tikzcd} \bigvee_{i=1}^d S^n \arrow[rr, "\sigma"] \arrow[dr, "\sigma'"'] & & M \\ & M_n \arrow[ur, dashed] & \end{tikzcd}$$

The induced map $(\sigma')_*$ is an isomorphism on homology, and therefore homotopy equivalence by Whitehead's Theorem. Thus we may adjust (3.1), the initial homotopy cofibration, by a self-equivalence of $\bigvee_{i=1}^{d} S^n$, giving us a new homotopy cofibration

$$S^{2n-1} \xrightarrow{f} \bigvee_{i=1}^{d} S^n \xrightarrow{j} M$$

where the map j is such that the restriction to the i^{th} wedge summand has Hurewicz image x_i. Reordering the wedge summands if necessary, we may also assume without loss of generality that the spheres corresponding to our distinguished basis elements x_k and x_l are in the $(d-1)^{st}$ and d^{th} places in the wedge sum, respectively. Furthermore, there is a cofibration

$$\bigvee_{i=1}^{d-2} S^n \xrightarrow{I} \bigvee_{i=1}^{d} S^n \xrightarrow{p} S^n \vee S^n$$

where I is the inclusion of the first $(d-2)$ wedge summands, and p is the pinch map to the $(d-1)^{st}$ and d^{th} wedge summands.

For the next step, we combine the two above cofibrations, in the diagram illustrated below. This defines a new space Q and the maps j_Q and h. In this diagram, every complete row and column is a homotopy cofibration.

$$\begin{tikzcd}
 & \bigvee_{i=1}^{d-2} S^n \arrow[r, equal] \arrow[d, "I"] & \bigvee_{i=1}^{d-2} S^n \arrow[d, "j\circ I"] \\
S^{2n-1} \arrow[r, "f"] \arrow[d, equal] & \bigvee_{i=1}^{d} S^n \arrow[r, "j"] \arrow[d, "p"] & M \arrow[d, "h"] \\
S^{2n-1} \arrow[r, "p\circ f"] & S^n \vee S^n \arrow[r, "j_Q"] & Q
\end{tikzcd} \tag{3.2}$$

Lemma 3.1. *There is a homotopy equivalence $\Omega Q \simeq \Omega S^n \times \Omega S^n$.*

Proof. We first show that there is a ring isomorphism $H^*(Q) \cong H^*(S^n \times S^n)$. By the above construction, j^* sends our distinguished classes x_k^* and x_l^* to the cohomology generators of the $(d-1)^{st}$ and d^{th} wedge summands. The map p pinches to these two wedge summands, and it is clear that j_Q^* induces an isomorphism on H^n, so there is a basis $\{u^*, v^*\}$ of $H^n(Q)$ such that $h^*(u^*) = x_k^*$ and $h^*(v^*) = x_l^*$. Naturality of the cup product implies that

$$h^*(u^*v^*) = h^*(u^*)h^*(v^*) = x_k^* x_l^* = \mu_M^*.$$

Moreover, Q and M both have their top-dimensional cells in dimension $2n$, so h^* is an isomorphism on H^{2n} as well. Therefore if w^* is a generator of $H^{2n}(Q)$ such that $h^*(w^*) = \mu_M^*$, we must have $u^*v^* = w^*$. Hence we have the desired ring isomorphism.

Dualising this ring isomorphism gives a co-algebra isomorphism in homology, namely $H_*(Q) \cong H_*(S^n \times S^n)$. Moreover, letting $i_1 : S^n \to S^n \vee S^m$ and $i_2 : S^m \to S^n \vee S^m$ denote the inclusions of respective wedge summands, the composites $j_Q \circ i_1$ and $j_Q \circ i_2$ induce the inclusions of the submodules $H_n(S^n \times *)$ and $H_n(* \times S^n)$ into $H_n(S^n \times S^n)$, respectively.

A Serre spectral sequence calculation gives that $H_*(\Omega Q) \cong H_*(\Omega S^n \times \Omega S^n)$ (see [BT14, Lemma 2.2]), and similarly to before, the composites $\Omega j_Q \circ \Omega i_1$ and $\Omega j_Q \circ \Omega i_2$ induce the inclusions of the submodules $H_n(\Omega S^n \times *)$ and $H_n(* \times \Omega S^n)$ into $H_n(\Omega S^n \times \Omega S^n)$, respectively. Now, consider another composite, given by

$$\alpha : \Omega S^n \times \Omega S^n \xrightarrow{\Omega i_1 \times \Omega i_2} \Omega(S^n \vee S^n) \times \Omega(S^n \vee S^n) \xrightarrow{\mu} \Omega(S^n \vee S^n) \xrightarrow{\Omega j_Q} \Omega Q$$

where μ denotes the usual loop multiplication. The above description of the maps $\Omega j_Q \circ \Omega i_1$ and $\Omega j_Q \circ \Omega i_2$, together with the deduced co-algebra isomorphism, implies that α_* is a homology isomorphism, and therefore α is a homotopy equivalence by Whitehead's Theorem. □

A consequence of Lemma 3.1 is that it implies the existence of a right homotopy inverse for the map Ωj_Q, which we will denote by s. Alternatively, given the homotopy equivalence of the Lemma, one may infer the existence of such an inverse from the Hilton-Milnor Theorem. Note that the pinch map p in (3.2) also has a right homotopy inverse, given by the looping of the inclusion map of the $(d-1)^{st}$ and d^{th} summands into the wedge sum $\bigvee_{i=1}^{d} S^n$. The existence of these two homotopy inverses yields the following Lemma.

Lemma 3.2. *The map $\Omega h : \Omega M \to \Omega Q$ has a right homotopy inverse. Moreover, we may choose this inverse to be the composite*

$$\Omega Q \xrightarrow{s} \Omega(S^n \vee S^n) \xrightarrow{\Omega J} \Omega\left(\bigvee_{i=1}^{d} S^n\right) \xrightarrow{\Omega j} \Omega M$$

where J denotes the inclusion map of the $(d-1)^{st}$ and d^{th} wedge summands.

Proof. Consider the lower-right square of (3.2). The composite $\Omega J \circ s$ is a right homotopy inverse for $\Omega j_Q \circ \Omega p$, so by the homotopy commutativity of the diagram we have that $\Omega j \circ \Omega J \circ s$ is a right homotopy inverse for h. □

So, we have deduced that the homotopy cofibration $\bigvee_{i=1}^{d-2} S^n \to M \xrightarrow{h} Q$ in the right-hand column of (3.2) is such that the map h has a right homotopy inverse after looping. We may now apply results from the previous chapter.

Theorem 3.3. *Let M be an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex with $n > 1$, $n \notin \{2,4,8\}$ and $H^n(M)$ of rank $d > 1$. Then there is a homotopy fibration*

$$(\Omega S^n \times \Omega S^n) \ltimes \left(\bigvee_{i=1}^{d-2} S^n\right) \longrightarrow M \longrightarrow Q$$

that splits after looping. Furthermore, there is a homotopy equivalence

$$\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega\left(\left((\Omega S^n \times \Omega S^n) \wedge \bigvee_{i=1}^{d-2} S^n\right) \vee \bigvee_{i=1}^{d-2} S^n\right).$$

Proof. Applying Theorem 2.14 to the right-hand column of (3.2), we obtain the following homotopy fibration, which splits after looping:

$$\Omega Q \ltimes \left(\bigvee_{i=1}^{d-2} S^n\right) \longrightarrow M \longrightarrow Q.$$

Applying the homotopy equivalence of Lemma 3.1 yields the desired homotopy fibration. The splitting yields a homotopy equivalence

$$\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega\left((\Omega S^n \times \Omega S^n) \ltimes \left(\bigvee_{i=1}^{d-2} S^n\right)\right).$$

Since $\bigvee_{i=1}^{d-2} S^n \simeq \Sigma\left(\bigvee_{i=1}^{d-2} S^{n-1}\right)$, we may apply Lemma 2.11, which gives the homotopy equivalence claimed in the statement of the Theorem. $\square$

Remark 3.4. Note that if $d = 2$, since the left half-smash with a point is homotopy equivalent to a point, the homotopy fibration in Theorem 3.3 becomes $* \to M \to Q$. We then have that M has the same homotopy type as Q, and so $\Omega M \simeq \Omega S^n \times \Omega S^n$. Furthermore, for $d > 2$ we may use the homotopy equivalence in Theorem 3.3 in conjunction with the James splitting, $\Sigma\Omega S^n \simeq \bigvee_{k=0}^{\infty} S^{k(n-1)+1}$, to show that the space

$$\left((\Omega S^n \times \Omega S^n) \wedge \bigvee_{i=1}^{d-2} S^n\right) \vee \bigvee_{i=1}^{d-2} S^n$$

is homotopy equivalent to an infinite wedge of spheres, which we will call W. Explicitly, we have

$$W \simeq \left(\bigvee_{i=1}^{d-2} S^n\right) \vee \left(\bigvee_{l,m=0}^{\infty}\left(\bigvee_{i=1}^{d-2} S^{(l+m)(n-1)+n}\right)\right) \vee \left(\bigvee_{j,k=0}^{\infty}\left(\bigvee_{i=1}^{d-2} S^{j(n-1)+n} \vee S^{k(n-1)+n}\right)\right).$$

We may write that $\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega W$. The Hilton-Milnor Theorem then implies that ΩM is homotopy equivalent to a product of loops on spheres, so the homotopy groups of M are calculable to the same extent as the homotopy groups of spheres.

To conclude this section, we note that the decomposition of Theorem 3.3 does not depend on the attaching map f, but only on the rank of $H_n(M)$. From this observation, we obtain the following Corollary.

Corollary 3.5. *Let M and N be two $(n-1)$-connected $2n$-dimensional Poincaré Duality complexes, with $n \notin \{2,4,8\}$. If the rank of $H_n(M)$ equals that of $H_n(N)$, then there is a homotopy equivalence $\Omega M \simeq \Omega N$.* □

3.2 Hopf Invariant One Cases

In the preceding discussions, we omitted the cases when $n \in \{2,4,8\}$. To see why, take such an n and consider an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex M. Let $\{x_1^*, \dots, x_d^*\}$ be a basis for $H^n(M)$, as before. Then Lemma 2.21 implies that for each basis element x_i^* there exists an x_j^* such that $x_i^* x_j^* = \mu_M^*$, again as before. However, in this case we cannot guarantee that x_i^* and x_j^* are distinct. Indeed, consider the composite map

$$S^{2n-1} \xrightarrow{f} \bigvee_{i=1}^{d} S^n \xrightarrow{p} S^n$$

where p is the pinch map to the i^{th} wedge summand. Because of the possible values of n we have chosen, $p \circ f$ could yield a homotopy class in $\pi_{2n-1}(S^n)$ of Hopf invariant one. If it does, then assuming the generator x_i^* corresponds to the i^{th} wedge summand, we must have $(x_i^*)^2 = \mu_X^*$. Thus we may not be able to manipulate the homotopy cofibration (3.1) in order to have two distinguished spheres retracting off the n-skeleton of M, and our method from the previous section is lost.

We can however introduce another method to describe how to decompose the loop spaces of simply-connected 4-dimensional Poincaré Duality complexes, from [BT14, Chapter 4]. This approach also informs how one might find the homotopy type of the loop space of a 3-connected 8-dimensional Poincaré Duality complex.

Our first step is to introduce a technique due to Duan and Liang, from [DL05]. Throughout this section, let M be a simply-connected 4-dimensional Poincaré Duality complex, with $H_2(M)$ of rank d. From the last section of this report, recall that when $d = 0$ we have $M \simeq S^4$, and $M \simeq \mathbb{C}P^2$ when $d = 1$. Thus we may assume that $d \geq 2$.

In general, for a space X and a coefficient ring R, any cohomology class in $H^m(X;R)$ may be represented by a map from X to the Eilenberg-MacLane space $K(R,m)$. In our case, we will be concerned with maps $M \to K(\mathbb{Z},2)$ representing classes in $H^2(M)$. Note that $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$ and $\Omega\mathbb{C}P^\infty \simeq S^1$.

Given a basis $\{x_1^*, \ldots, x_d^*\}$ of $H^2(M)$, take the particular element x_d^*. By Lemma 2.21 we may assume that the basis has been chosen so that there exists an k such that we have $x_d^* x_k^* = \mu_X^*$. Note that this k may be equal to d. Moreover, we could have $x_k^* = -x_d^*$, but is fixed by taking a different orientation. Represent the class x_d^* by the map $q : M \to \mathbb{C}P^\infty$, and define the space Z by the homotopy fibration sequence

$$\Omega\mathbb{C}P^\infty \simeq S^1 \to Z \to M \xrightarrow{q} \mathbb{C}P^\infty.$$

It is an observation due to Quinn [Qui72, Remark 1.6] that in a fibration of spaces with the homotopy type of finite CW-complexes, the total space is a Poincaré Duality complex if and only if both the base and fibre spaces are Poincaré Duality complexes. This also holds for homotopy fibrations, so since both S^1 and M are Poincaré Duality complexes, so is the space Z.

Now, by the Universal Coefficient Theorem, $H^2(M) \cong H_2(M)$. Since M is simply connected, the Hurewicz Theorem implies that there is a map $S^2 \to M$ whose Hurewicz image is the homology class dual to x_d^*. Therefore the composite

$$S^2 \to M \xrightarrow{q} \mathbb{C}P^\infty$$

is homotopic to the inclusion of the bottom cell. By looping and precomposing with the suspension map E, we obtain another composite

$$S^1 \xrightarrow{E} \Omega S^2 \to \Omega M \xrightarrow{\Omega q} \Omega\mathbb{C}P^\infty \simeq S^1$$

which is a degree 1 map, and therefore a homotopy equivalence. Thus the homotopy fibration $Z \to M \to \mathbb{C}P^\infty$ splits after looping, and we have the following lemma.

Lemma 3.6. *With the set-up as above, $\Omega M \simeq S^1 \times \Omega Z$.*

So our new objective is to discern the homotopy type of the loop space ΩZ. As a first step, we have the following result. The proof uses a Serre spectral sequence argument, which the reader may find in [BT14, Lemma 4.1].

Lemma 3.7. *The Poincaré Duality complex Z satisfies the following:*

(i) there is a homotopy cofibration

$$S^4 \xrightarrow{\gamma} \bigvee_{i=1}^{d-1} (S^2 \vee S^3) \to Z$$

for some map γ;

(ii) *$H^*(Z)$ is torsion-free.* □

That is, Z is a torsion-free, simply-connected 5-dimensional Poincaré Duality complex. In this section we have so far not given the necessary general approach required to identify ΩZ, though we shall briefly do so now; we proceed similarly to the earlier case of $(n-1)$-connected $2n$-dimensional Poincaré Duality complexes.

Let $n > 2$ be an integer, and take another positive integer m such that $m < n$. Let P be an $(m-1)$-connected n-dimensional Poincaré Duality complex. Moreover, assume that $H_*(P)$ is torsion free, and that there is a homotopy equivalence

$$P_{n-m} \simeq S^m \vee S^{n-m} \vee J$$

where P_{n-m} denotes the $(n-m)$-skeleton of P, and J is some CW-complex. Further, if x_m^* and x_{n-m}^* denote the cohomology generators associated with the spheres in the above homotopy equivalence, we require that $x_m^* x_{n-m}^* = \mu_P^*$. By pinching out to these two distinguished spheres and following a similar program to Section 3.1, we may produce a space Q' and a map $h' : P \to Q'$ such that:

(a) there is a ring isomorphism $H^*(Q') \cong H^*(S^m \times S^{n-m})$;

(b) $\Omega Q' \simeq \Omega S^m \times \Omega S^{n-m}$;

(c) the map $\Omega h'$ has a right homotopy inverse;

(d) there is a homotopy fibration $\Omega Q' \ltimes J \to P \xrightarrow{h'} Q'$, that splits after looping.

This yields the following theorem, which may also be found in [BT14, Theorem 2.6].

Theorem 3.8. *Let P be an $(m-1)$-connected n-dimensional Poincaré Duality complex, under the assumptions detailed above, and let $j' : P_{n-m} \to P$ be the skeletal inclusion of the $(n-m)$-skeleton of P. Then the following hold:*

(i) *there is a homotopy equivalence*

$$\Omega P \simeq \Omega S^m \times \Omega S^{n-m} \times \Omega\left((\Omega S^m \times \Omega S^{n-m}) \ltimes J\right)$$

which, if J is a suspension, refines to

$$\Omega P \simeq \Omega S^m \times \Omega S^{n-m} \times \Omega\left(((\Omega S^m \times \Omega S^{n-m}) \wedge J) \vee J\right);$$

(ii) *the map $\Omega j'$ has a right homotopy inverse.* □

Now, we return to our specific case for the complex Z. Since we have assumed that $d \geq 2$, by Lemma 2.21 we have the exact case of Theorem 3.8. We are thus able to refine the homotopy equivalence of Lemma 3.6.

Theorem 3.9. *Let M be a simply-connected 4-dimensional Poincaé Duality complex, with $H_2(M)$ of rank $d \geq 2$. If $d = 2$ there is a homotopy equivalence*

$$\Omega M \simeq S^1 \times \Omega S^2 \times \Omega S^3$$

and if $d > 2$ there is a homotopy equivalence

$$\Omega M \simeq S^1 \times \Omega S^2 \times \Omega S^3 \times \Omega \left(\Omega \left(\Omega S^2 \times \Omega S^3\right) \ltimes \left(\bigvee_{i=1}^{d-2} \left(S^2 \vee S^3\right) \right) \right).$$

Proof. In the context of Theorem 3.8, $n = 5$, $m = 2$ and $J = \bigvee_{i=1}^{d-2} \left(S^2 \vee S^3\right)$. When $d = 2$, $\Omega Z \simeq \Omega S^2 \times \Omega S^3$, and when $d > 2$ there is a homotopy equivalance

$$\Omega Z \simeq \Omega S^2 \times \Omega S^3 \times \Omega \left(\Omega \left(\Omega S^2 \times \Omega S^3\right) \ltimes \left(\bigvee_{i=1}^{d-2} \left(S^2 \vee S^3\right) \right) \right).$$

We then apply Lemma 3.6 to complete the proof of the Theorem. □

Remark 3.10. Earlier in this section, we indicated that there in an analogous approach for 3-connected 8-dimensional Poincaré Duality complexes. Let N be such a complex. We may apply similar techniques to those above if there is a map $N \to \mathbb{H}P^2$ that induces a surjection in cohomology. When we have such a map, by composing with the inclusion $\mathbb{H}P^2 \to \mathbb{H}P^\infty$ and using that $\Omega \mathbb{H}P^\infty \simeq S^3$, one obtains a homotopy fibration

$$S^3 \to Z' \to N$$

where Z' can be shown to be a 3-connected 11-dimensional Poincaré duality complex. Indeed, the 7-skeleton of Z' is homotopy equivalent to a wedge of 4-spheres and 7-spheres. The case of a 7-connected 16-dimensional complex remains unclear; this, again, is complicated by the fact that S^7 does not have a classifying space.

3.3 An Improvement

This section details a preliminary observation of the author, which widens the scope of our earlier exposition to a larger class of Poincaré Duality complexes. The concepts laid out here will be developed and expanded in Chapter 5.

Fix an integer $n > 3$, and consider an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex N. Further, assume that that $H_*(N)$ is torsion-free. Similar to the case of Section 3.1, this assumption together with Poincaré Duality allows us to deduce that

$$H^m(N) \cong \begin{cases} \mathbb{Z} \text{ if } m = 0 \text{ or } m = 2n \\ \mathbb{Z}^l \text{ if } m = n-1 \text{ or } m = n+1 \\ \mathbb{Z}^d \text{ if } m = n \\ 0 \text{ otherwise} \end{cases}$$

for some non-negative integers d and l. Likewise, by the Universal Coefficient Theorem, we have the following for the homology of N:

$$H_m(N) \cong \begin{cases} \mathbb{Z} \text{ if } m = 0 \text{ or } m = 2n \\ \mathbb{Z}^l \text{ if } m = n-1 \text{ or } m = n+1 \\ \mathbb{Z}^d \text{ if } m = n \\ 0 \text{ otherwise.} \end{cases}$$

The case where $l = 0$ reduces us to N being $(n-1)$-connected, which we have already discussed. Henceforth, let us assume that $l > 0$.

Now, consider the skeletal structure of N. For an integer $k < 2n$, let N_k denote the k-skeleton of N. As N is $(n-1)$-connected, there is a homotopy equivalence $N_{n-1} \simeq \bigvee_{i=1}^{l} S^{n-1}$. Similar to our earlier discussion, we have varying outcomes depending on the integer d; we will briefly consider the situation when $d = 0$, and then move to a more complete picture for when $d > 0$.

Case 1: $d = 0$

For the case when $d = 0$, we shall draw upon a result of Huang from [Hua22a]. Having $d = 0$ is equivalent to demanding that $H_n(N) \cong 0$, and therefore there are no cells of consecutive dimensions in the CW-structure of N. Thus there is a homotopy cofibration

$$\bigvee_{i=1}^{l} S^n \xrightarrow{\varphi} \bigvee_{i=1}^{l} S^{n-1} \to N_{n+1}$$

that defines the $(n+1)$-skeleton of N. Further, letting i_r denote the inclusion of the r^{th} wedge summand, for each $r \in \{1, \ldots, l\}$ the composite

$$\varphi_r : S^n \xrightarrow{i_r} \bigvee_{i=1}^{l} S^n \xrightarrow{\varphi} \bigvee_{i=1}^{l} S^{n-1}$$

defines a homotopy class in the group $\pi_n(\bigvee_{i=1}^{l} S^{n-1})$. Since $n > 3$, this group is isomorphic to $\bigoplus_{i=1}^{l} \mathbb{Z}/2\mathbb{Z}$ [Tod16]. Thus we may from an $(l \times l)$-matrix C with entries in $\mathbb{Z}/2\mathbb{Z}$, where the r^{th} column is the image of the homotopy class of φ_r under this group isomorphism.

Huang shows that this matrix may be manipulated by row and column operations. This yields the following result, whose proof we neglect to state here, but may be found in [Hua22a, Lemma 6.1].

Lemma 3.11. *With the set-up as above, there is a homotopy equivalence*

$$N_{n+1} \simeq \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2\right) \vee \left(\bigvee_{i=1}^{l-c} (S^{n-1} \vee S^{n+1})\right)$$

where $c = rank(C)$. □

To proceed with finding a decompositon of ΩN when $d = 0$, we restrict to the scenario when we have $c < l$, and make use the more general results summarised in Theorem 3.8. A direct application of this Theorem gives us the following.

Proposition 3.12. *Let $n > 3$ and let N be an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex, with $H_*(N)$ torsion-free, $H_{n-1}(N)$ of rank $l > 0$ and $H_n(N) = 0$. Further, assume that the matrix C detailed above has rank $c < l$. Then there is a homotopy equivalence*

$$\Omega N \simeq \Omega S^{n-1} \times \Omega S^{n+1} \times \Omega\left(\left(\Omega S^{n-1} \times \Omega S^{n+1}\right) \ltimes J\right)$$

where $J = \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2\right) \vee \left(\bigvee_{i=1}^{l-c-1} (S^{n-1} \vee S^{n+1})\right)$. □

Note that in the above Proposition, the complex J is a suspension, so we may refine the homotopy equivalence of ΩN further, as in Theorem 3.8 (i).

Remark 3.13. Note that if $l = 1$, we must have that $c = 0$. In that case, the Proposition above simplifies to the statement that $\Omega N \simeq \Omega S^{n-1} \times \Omega S^{n+1}$.

Case 2: $d > 1$

In this situation, our assumption that $H_*(N)$ is torsion-free plays a major role. Note also that having $d = 1$ gives rise to restrictions due to the Hopf invariant one problem; we regrettably omit a discussion of these cases. Our first Proposition is more general, however, and holds for $d > 0$.

Proposition 3.14. *Let $n > 3$ and let N be an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex, with $H_*(N)$ torsion-free, $H_{n-1}(N)$ of rank $l > 0$ and $H_n(N)$ of rank $d > 0$. Then there exists a homotopy equivalence*

$$N_{n+1} \simeq \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2\right) \vee \left(\bigvee_{i=1}^{l-c} (S^{n-1} \vee S^{n+1})\right) \vee \left(\bigvee_{i=1}^{d} S^n\right)$$

for some non-negative integer $c \leq l$.

Proof. First, restrictions of attaching maps to cells of consecutive dimensions must be null homotopic, otherwise we would contradict our assumption that $H_*(N)$ is torsion-free. Therefore we have a homotopy equivalence

$$N_n \simeq \left(\bigvee_{i=1}^{l} S^{n-1}\right) \vee \left(\bigvee_{i=1}^{d} S^n\right)$$

and a homotopy cofibration

$$\bigvee_{i=1}^{l} S^n \longrightarrow \left(\bigvee_{i=1}^{l} S^{n-1}\right) \vee \left(\bigvee_{i=1}^{d} S^n\right) \longrightarrow N_{n+1}$$

which defines N_{n+1}. Again, since maps between cells of consecutive dimensions must be null homotopic, we must have $N_{n+1} \simeq \left(\bigvee_{i=1}^{d} S^n\right) \vee J'$, where J' is a CW-complex consisting of l many $(n-1)$-cells and l many $(n+1)$-cells.

To complete the proof, we shall deduce the homotopy type of the complex J'. Let $f : S^{2n-1} \to N_{n+1}$ be the attaching map of the top-dimensional cell of N, and consider the homotopy commutative diagram (3.3) below, which defines the complex P.

$$\begin{tikzcd}
 & \bigvee_{i=1}^{d} S^n \arrow[r, equal] \arrow[d, "I"] & \bigvee_{i=1}^{d} S^n \arrow[d] \\
S^{2n-1} \arrow[r, "f"] \arrow[d, equal] & \left(\bigvee_{i=1}^{d} S^n\right) \vee J' \arrow[r] \arrow[d] & N \arrow[d] \\
S^{2n-1} \arrow[r] & J' \arrow[r] & P
\end{tikzcd} \tag{3.3}$$

Each complete row and column in (3.3) is a homotopy cofibration. Moreover, the middle row is the homotopy cofibration by which we attach the top cell of N, and the middle column pinches out the wedge of spheres in N_{n+1}.

Now, observe that the bottom row of (3.3) implies that

$$H^m(P) \cong \begin{cases} \mathbb{Z} \text{ if } m = 0 \text{ or } m = 2n \\ \mathbb{Z}^l \text{ if } m = n-1 \text{ or } m = n+1 \\ 0 \text{ otherwise} \end{cases}$$

and that $H^*(P)$ also inherits the cup and cap product structures from $H^*(N)$ in all dimensions except for n. Therefore, P is an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex, with $H^n(P) \cong 0$. Hence Lemma 3.11 applies, and there exists an integer c such that

$$J' \simeq \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2 \right) \vee \left(\bigvee_{i=1}^{l-c} (S^{n-1} \vee S^{n+1}) \right)$$

which completes the proof. $\square$

Now, let us assume that $d > 1$ and $n \notin \{4, 8\}$, in order to avoid cases where maps of Hopf invariant one may arise. Let $I' : J' \to N_{n+1}$ be the inclusion of the complex J' into the $(n+1)$-skeleton of N. There is a diagram similar to (3.3), where we change the inclusion I for I', which defines the complex M and the maps j_M and h.

$$\begin{tikzcd}
 & J' \arrow[r, equal] \arrow[d, "I'"] & J' \arrow[d] \\
S^{2n-1} \arrow[r, "f"] \arrow[d, equal] & \left(\bigvee_{i=1}^{d} S^n\right) \vee J' \arrow[r] \arrow[d, "q"] & N \arrow[d, "h"] \\
S^{2n-1} \arrow[r, "q \circ f"] & \bigvee_{i=1}^{d} S^n \arrow[r, "j_M"] & M
\end{tikzcd} \tag{3.4}$$

Similar to the proof of Proposition 3.14, we may infer that the complex M is an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex. As such, we may invoke [BT14, Theorem 2.6 (ii)], giving the existence of a right homotopy inverse for Ωj_M. Further, there is also a clear right homotopy inverse for the pinch map q, and therefore one for Ωq.

The existence of these inverses imply the existence of a right homotopy inverse for Ωh, similar to the proof of Lemma 3.2. Hence we may immediately apply Theorem 2.14 and Theorem 3.3, obtaining the following result.

Proposition 3.15. *Let $n > 3$ be an integer such that $n \notin \{4, 8\}$, and let N be an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex, with $H_*(N)$ torsion-free, $H_{n-1}(N)$ of rank $l > 0$ and $H_n(N)$ of rank $d > 1$. Then there exists a homotopy equivalence*

$$\Omega N \simeq \Omega M \times \Omega(\Omega M \ltimes J')$$

where

$$\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega \left(\left((\Omega S^n \times \Omega S^n) \wedge \bigvee_{i=1}^{d-2} S^n \right) \vee \bigvee_{i=1}^{d-2} S^n \right)$$

and

$$J' \simeq \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2 \right) \vee \left(\bigvee_{i=1}^{l-c} (S^{n-1} \vee S^{n+1}) \right)$$

for some non-negative integer $c \leq l$. $\square$

3.4 p-Hyperbolicity of Poincaré Duality complexes

In this section, we will apply the previous results on the loop spaces of highly connected Poincaré Duality complexes to comment on their homotopy groups. More specifically, we shall discuss the p-hyperbolicity of these complexes, using some results due to Boyde in [Boy21]. The definitions in this section are from the introduction of this paper.

In rational homotopy theory, a finite simply-connected CW-complex X is called *rationally elliptic* if $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, and *rationally hyperbolic* otherwise. We may also define similar terminology for discussing the growth of torsion summands with respect to a chosen prime, the details of which were first introduced by [HW20]. For any prime p, by a *p-torsion summand* in an abelian group A, we mean a direct summand isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$ for some $r \geq 1$.

Definition 3.16. Let X be a finite simply-connected CW-complex, and let p be a prime. We say that X is *p-hyperbolic* if the number of p-torsion summands in $\pi_*(X)$ grows exponentially, in the sense that

$$\liminf_m \frac{\ln(T_m)}{m} > 0$$

where T_m is the number of p-torsion summands in $\bigoplus_{i \leq m} \pi_i(X)$.

This definition counts $\mathbb{Z}/p^r\mathbb{Z}$-summands in $\bigoplus_{i \leq m} \pi_i(X)$ for all values of r, as m increases. It is also possible to consider a single r, which gives rise to the next definition.

Definition 3.17. Let X be a finite simply-connected CW-complex, let p be a prime and fix $r \in \mathbb{N}$. We say that X is *$\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic* if the number of $\mathbb{Z}/p^r\mathbb{Z}$-summands in $\pi_*(X)$ grows exponentially, in the sense that

$$\liminf_m \frac{\ln(t_m)}{m} > 0$$

where t_m is the number of $\mathbb{Z}/p^r\mathbb{Z}$-summands in $\bigoplus_{i \leq m} \pi_i(X)$.

Note that $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolicty for any r implies p-hyperbolicty, and so is a stronger result. Many examples of spaces that exhibit these kinds of hyperbolicity are given in [Boy21]; we record two of direct interest to us in the Lemmas below.

Lemma 3.18. *Let $n_1, n_2 > 0$ be integers. Then $S^{n_1+1} \vee S^{n_2+1}$ is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$.* □

Lemma 3.19. *Let $n > 0$ be an integer. Then $\Sigma^n \mathbb{C}P^2$ is p-hyperbolic for all primes $p \neq 2$.* □

We will use these to comment on the hyperbolicity of the highly connected Poincaré Duality complexes we studied earlier in this chapter. We first look at $(n-1)$-connected $2n$-dimensional complexes.

Proposition 3.20. *Let $n > 1$ be an integer such that $n \notin \{4, 8\}$, and let M be an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex with $H_n(M)$ of rank $d > 2$. Then M is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$.*

Proof. We start with the case where $n = 2$, and make use of Theorem 3.9. When $d > 2$, we had the decomposition

$$\Omega M \simeq S^1 \times \Omega S^2 \times \Omega S^3 \times \Omega \left(\Omega \left(\Omega S^2 \times \Omega S^3 \right) \ltimes \left(\bigvee_{i=1}^{d-2} \left(S^2 \vee S^3 \right) \right) \right).$$

Let us write $A = \Omega \left(\Omega S^2 \times \Omega S^3 \right) \ltimes \left(\bigvee_{i=1}^{d-2} (S^2 \vee S^3) \right)$. The wedge of spheres in A is evidently a suspension, so applying Lemma 2.11, we see that

$$A \simeq \left(\Omega \left(\Omega S^2 \times \Omega S^3 \right) \wedge \left(\bigvee_{i=1}^{d-2} \left(S^2 \vee S^3 \right) \right) \right) \vee \left(\bigvee_{i=1}^{d-2} \left(S^2 \vee S^3 \right) \right).$$

Since we demanded that $d > 2$, the above homotopy equivalence shows that there is at least one copy of $S^2 \vee S^3$ that retracts off A. Therefore $\Omega(S^2 \vee S^3)$ retracts off ΩA. Consequently, since $\Omega M \simeq S^1 \times \Omega S^2 \times \Omega S^3 \times \Omega A$, we deduce that $\pi_*(S^2 \vee S^3)$ retracts off $\pi_*(M)$ as a product, and therefore M is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$, by Lemma 3.18.

For $n > 2$, recall that in Remark 3.4 we showed that an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex M, with $n \notin \{2, 4, 8\}$ and $H_n(M)$ of rank greater than 2, has the property that

$$\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega W$$

where W is a wedge of infinitely many spheres (all of which are at least simply-connected, since $n > 2$). Similar to the previous case, we deduce that there exist $p, q > 1$ such that $\Omega(S^p \vee S^q)$ retracts off ΩM. We therefore have that $\pi_*(S^p \vee S^q)$ retracts off $\pi_*(M)$ as a product, and again by Lemma 3.18, M is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$. □

Next, we consider torsion-free $(n-2)$-connected $2n$-dimensional Poincaré Duality complexes. We first deal with the simplest case, when H_n is non-trivial.

Proposition 3.21. *Let $n > 3$ be an integer such that $n \notin \{4, 8\}$, and let N be an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex with $H_{n-1}(N)$ of rank $l > 0$ and $H_n(N)$ of rank $d > 1$. Then N is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$.*

Proof. Recall that Proposition 3.15 gave us a decomposition of the form

$$\Omega N \simeq \Omega M \times \Omega(\Omega M \ltimes J')$$

where M is an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex. By Proposition 3.20, the complex M is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$. Therefore, because ΩN contains ΩM as a product, N exhibits this hyperbolicity as well. □

Next, we move to when $H_n(N)$ is trivial. Recall that the decomposition we obtained in Proposition 3.12 depended on the rank of a matrix C. For what follows, we require that $H_{n-1}(N)$ be of rank $l > 1$. Further, let $c = rank(C)$, and assume that $c < l$. Before we continue with the hyperbolicity of Poincaré Duality complexes, we have a preparatory lemma. As in Proposition 3.12, we define the complex J by

$$J = \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2\right) \vee \left(\bigvee_{i=1}^{l-c-1} (S^{n-1} \vee S^{n+1})\right).$$

Lemma 3.22. *If $c = l-1$ then J is p-hyperbolic for all primes $p \neq 2$. On the other hand, if $c < l-1$, then J is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$.*

Proof. When $c = l-1$, the wedge sum of spheres in J vanishes. Clearly, a copy of $\Sigma^{n-3}\mathbb{C}P^2$ retracts off J, and therefore $\pi_*(\Sigma^{n-3}\mathbb{C}P^2)$ retracts off $\pi_*(J)$, so J is p-hyperbolic for all odd primes p. Applying Lemma 3.19 completes this part of the proof.

If $c < l-1$, then we instead have at least one $S^{n-1} \vee S^{n+1}$ retracting off J. Hence $\pi_*(S^{n-1} \vee S^{n+1})$ retracts off $\pi_*(J)$, and we apply Lemma 3.18. □

With the hyperbolicity of the auxiliary complex J now described, we are able to consider such characteristics for torsion-free $(n-2)$-connected $2n$-dimensional Poincaré Duality complexes that have trivial homology in dimension n.

Proposition 3.23. *Let $n > 3$ be an integer such that $n \notin \{4, 8\}$, and let N be an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex with $H_{n-1}(N)$ of rank $l > 1$ and $H_n(N) \cong 0$. Further, let $c = rank(C)$, and assume that $c < l$. Then we have the following:*

(i) if $c = l-1$, then N is p-hyperbolic for all primes $p \neq 2$;

(ii) if $c < l-1$ then N is $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolic for all primes p and all $r \in \mathbb{N}$.

Proof. We will make use of Proposition 3.12, which gave the homotopy equivalence

$$\Omega N \simeq \Omega S^{n-1} \times \Omega S^{n+1} \times \Omega\left(\left(\Omega S^{n-1} \times \Omega S^{n+1}\right) \ltimes J\right)$$

with J as above. The complex J is evidently a suspension, so let us write $J \simeq \Sigma\bar{J}$. By applying Lemma 2.11, we then have that

$$\Omega\left(\left(\Omega S^{n-1} \times \Omega S^{n+1}\right) \ltimes J\right) \simeq \Omega\left(\Sigma\left(\left(\Omega S^{n-1} \times \Omega S^{n+1}\right) \wedge \bar{J}\right) \vee \Sigma\bar{J}\right)$$

which by the Hilton-Milnor Theorem, we may write as $\Omega\Sigma\bar{J} \times X$, where X is some infinite product of loop spaces. Therefore we have

$$\Omega N \simeq \Omega S^{n-1} \times \Omega S^{n+1} \times \Omega J \times X.$$

Applying Lemma 3.22 then proves the Proposition. □

Remark 3.24. Proposition 3.23 (i) may not be the optimal statement of hyperbolicity for a complex such as N when $c = l - 1$. If $c > 1$ then $\Omega(\Sigma^{n-3}\mathbb{C}P^2 \vee \Sigma^{n-3}\mathbb{C}P^2)$ retracts off ΩJ. It may be the case that this wedge sum exhibits some stronger property like $\mathbb{Z}/p^r\mathbb{Z}$-hyperbolicity for some odd primes p and some r, but further investigation is needed here.

Chapter 4

Loops on Connected Sums and Applications to the Vigué-Poirrier Conjecture

Having discussed much established work in the literature in the previous chapter, we are now able to progress to developing the existing theory and prove new results. Our first step in this chapter is to make some adjustments to an existing construction, so that we have the precision required to introduce the concept of *inert map* and discuss it in detail. Indeed, it must be emphasised that inert maps are a cornerstone of much of what follows. We use this to give new insights into loop space decompositions of connected sums of Poincaré Duality complexes, culminating in Theorem 4.8.

We will use this result to remark on a long standing conjecture of rational homotopy theory, due to Vigué-Poirrier, in Section 4.4. Let X be a simply connected space, and as per page *xiii*, denote its free loop space by $\mathcal{L}X$. Such a space X is called *rationally elliptic* if $dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$, and called *rationally hyperbolic* otherwise [FHT01]. As stated in the Introduction, Vigué-Poirrier made the following conjecture in [VP84].

Conjecture (Vigué-Poirrier)**.** *If X is rationally hyperbolic, then $H_*(\mathcal{L}X;\mathbb{Q})$ grows exponentially.*

Theorem 4.17 shows that (under certain conditions) the Vigué-Poirrier Conjecture holds for Poincaré Duality complexes with the loop space homotopy type of a connected sum when one of these summands in rationally elliptic, which expands the cases for which it is known to hold.

The structure of this chapter is as follows. Sections 4.1 and 4.2 are reformatted versions of [Che22, Sections 2 & 3] (a paper of the author) together with new material in Sections 4.3 and 4.4. Indeed, Section 4.1 establishes some modifications to a construction of Theriault [The20, Section 8] in order to prove Theorem 4.2, which underpins much of what follows and provides a basis for how we will use inert maps. In particular this is employed in Section 4.2 to give an important fact in Lemma 4.5, and then to give further results in the context of the homotopy theory of Poincaré Duality complexes, including a new proof of [The20, Theorem 9.1(b)-(c)].

As alluded to earlier, the key theorem of this chapter is Theorem 4.8, which gives a general framework for our analysis. We follow this by giving a family of examples of Poincaré Duality complexes which have the homotopy type of a connected sum after looping, but not before. We then conclude the chapter with Section 4.4 by using Theorem 4.8 to expand class of spaces for which the Vigué-Poirrier Conjecture is known to hold, which we give in Corollary 4.16 and Theorem 4.17.

4.1 Adapting a Construction of Theriault

In this section, we shall make some small adjustments to a construction of Theriault from [The20, Section 8] in order to prove a slightly more general result, which forms the basis of what follows in this chapter. Unless otherwise stated, all spaces are assumed to be simply connected. We also give the following definition.

Definition 4.1. For a homotopy cofibration $A \xrightarrow{f} B \xrightarrow{j} C$ the map f is called *inert* if Ωj has a right homotopy inverse.

This an integral version of a notion used in rational homotopy theory, namely *rational inertness*, which we will see in Section 6.3.

Second, recall our standard notation for pinch maps and inclusions regarding wedges of spaces (see page *xiii*) . Note that every projection map p_j has a right inverse, given by the inclusion i_j. Our focus in this section will be on analysing homotopy cofibrations of the form

$$\Sigma A \xrightarrow{f} X \vee Y \xrightarrow{q} C$$

and the key to our considerations will be the following homotopy commutative diagram of homotopy cofibrations

$$\begin{tikzcd}
 & Y \arrow[r, equal] \arrow[d, "i_2"] & Y \arrow[d] \\
\Sigma A \arrow[r, "f"] \arrow[d, equal] & X \vee Y \arrow[r, "q"] \arrow[d, "p_1"] & C \arrow[d, "\varphi"] \\
\Sigma A \arrow[r, "p_1 \circ f"] & X \arrow[r, "j"] & M
\end{tikzcd} \tag{4.1}$$

where the bottom-right square is a homotopy pushout. Our goal is to prove the following theorem.

Theorem 4.2. *Consider Diagram (4.1). If the map Ωj has a right homotopy inverse, then so do $\Omega\varphi$ and Ωq. In particular, if the composite $p_1 \circ f$ is inert, then so is f.*

We will be following the method set out by Theriault in [The20, Section 8] very closely, though with some alterations. We include much of the argument here for the sake of transparency, though we will not include sections where we argue identically to Theriault; we will instead make this clear and give precise references for where the arguments can be found.

First, recall that for two path connected and based spaces X and Y, the *(left) half-smash* of X and Y is the quotient space

$$X \ltimes Y = (X \times Y)/(X \times y_0)$$

where y_0 denotes the basepoint of Y. We begin by recalling some of the main results from Section 2.2, starting with Theorem 2.13.

Theorem 2.13 (Theriault)**.** *Suppose there exists a homotopy commutative diagram*

$$\begin{tikzcd}
 & E \arrow[r, "\alpha"] \arrow[d] & E' \arrow[d] \\
\Sigma A \arrow[r, "f"] & B \arrow[r] \arrow[d, "h"] & C \arrow[d, "h'"] \\
 & Z \arrow[r, equal] & Z
\end{tikzcd}$$

where the middle and right columns are homotopy fibrations, the map α is an induced map of fibres and the middle row is a homotopy cofibration. If Ωh has a right homotopy inverse, then there exists a homotopy cofibration

$$\Omega Z \ltimes \Sigma A \xrightarrow{\theta} E \to E'$$

for some map θ. □

Further, recall the special case in which $C = Z$ and h' is the identiy map, which implies that E' is contractible and therefore that θ is a homotopy equivalence. This gave Theorem 2.14, which we restate here for the benefit of the reader.

Theorem 2.14 (Beben-Theriault)**.** *Suppose there is a homotopy cofibration*

$$A \xrightarrow{f} B \xrightarrow{h} C$$

such that the map Ωh has a right homotopy inverse. Then there exists a homotopy fibration

$$\Omega C \ltimes A \to B \xrightarrow{h} C.$$

Moreover, this homotopy fibration splits after looping, so there is a homotopy equivalence $\Omega B \simeq \Omega C \times \Omega(\Omega C \ltimes A)$. $\square$

Returning to the situation of Diagram (4.1), since $p_1 \circ f$ is inert, the map Ωj has a right homotopy inverse. Let E'' be the homotopy fibre of j. Theorem 2.14 applies to the bottom row of (4.1), which implies that $E'' \simeq \Omega M \ltimes \Sigma A$. Equivalently, there is a homotopy cofibration

$$\Omega M \ltimes \Sigma A \xrightarrow{\theta} E'' \to *.$$

Now, let s denote a right homotopy inverse of Ωj, and t that of Ωp_1. Then the composite $\Omega q \circ t \circ s$ is a right homotopy inverse for $\Omega\varphi$. Let $h = j \circ p_1$ and let E and E' denote the homotopy fibres of h and φ, respectively. We have the following homotopy commutative diagram

$$\begin{tikzcd}
 & E \arrow[r, "\alpha"] \arrow[d] & E' \arrow[d] \\
\Sigma A \arrow[r, "f"] & X \vee Y \arrow[r, "q"] \arrow[d, "h"] & C \arrow[d, "\varphi"] \\
 & M \arrow[r, equal] & M
\end{tikzcd} \tag{4.2}$$

where the middle and right columns are homotopy fibrations, the map α is an induced map of fibres and the middle row is a homotopy cofibration. Therefore, by Theorem 2.13, there exists a homotopy cofibration

$$\Omega M \ltimes \Sigma A \xrightarrow{\theta_f} E \xrightarrow{\alpha} E'. \tag{4.3}$$

Moreover, the right homotopy inverse for $\Omega\varphi$ enables us to apply Theorem 2.14 to the right-most column of (4.1), so we have homotopy equivalences

$$E' \simeq \Omega M \ltimes Y \text{ and } \Omega C \simeq \Omega M \times \Omega(\Omega M \ltimes Y).$$

The proof strategy for Theorem 4.2 will be as follows: we wish to gain more control over the homotopy class of the first equivalence, and use this knowledge to deduce further facts about the second. The next step is to consider the homotopy fibration diagram

$$\begin{tikzcd} E \arrow[r, dashed, "l"] \arrow[d] & E'' \arrow[d] \\ X \vee Y \arrow[r, "p_1"] \arrow[d, "h"] & X \arrow[d, "j"] \\ M \arrow[r, equal] & M. \end{tikzcd} \tag{4.4}$$

The bottom square of (4.4) is commutative by definition of the map h, so the induced map of fibres l exists. A naturality condition for Theorem 2.13 is given in [The20, Remark 2.7], which is satisfied in by virtue of (4.4). Thus there is a homotopy cofibration diagram

$$\begin{tikzcd} \Omega M \ltimes \Sigma A \arrow[r, "\theta_f"] \arrow[d, equal] & E \arrow[r, "\alpha"] \arrow[d, "l"] & E' \arrow[d] \\ \Omega M \ltimes \Sigma A \arrow[r, "\theta"] & E'' \arrow[r] & *. \end{tikzcd} \tag{4.5}$$

Note that since θ is a homotopy equivalence, (4.5) implies that the map θ_f always has a left homotopy inverse. Moreover, observe also that in constructing the above we only considered with the behaviour of f when restricted to X. We record this in the lemma below, for ease of reference.

Lemma 4.3. *With the set-up of Diagram (4.5) above, the map θ_f has a left homotopy inverse whose homotopy class depends only on the homotopy class of the composite f when restricted to X.* □

This enables us to switch focus for the time being, and consider the special case in which $C \simeq M \vee Y$. Diagram (4.1) now becomes the homotopy cofibration diagram

$$\begin{tikzcd} & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] \\ \Sigma A \arrow[r, "f"] \arrow[d, equal] & X \vee Y \arrow[r, "j \vee 1"] \arrow[d, "p_1"] & M \vee Y \arrow[d, "p_1"] \\ \Sigma A \arrow[r, "p_1 \circ f"] & X \arrow[r, "j"] & M \end{tikzcd} \tag{4.6}$$

where we have $q = j \vee 1$ and $\varphi = p_1$. Since the map Ωp_1 has a right homotopy inverse, we may apply Theorem 2.14 to the homotopy cofibration in the right-most column of (4.6), again giving a homotopy equivalence $E' \simeq \Omega M \ltimes Y$. Thus Diagram

(4.2) becomes

$$\begin{tikzcd}
 & E \arrow[r, "\alpha'"] \arrow[d] & \Omega M \ltimes Y \arrow[d] \\
\Sigma A \arrow[r, "f"] & X \vee Y \arrow[r, "j \vee 1"] \arrow[d, "h"] & M \vee Y \arrow[d, "p_1"] \\
 & M \arrow[r, equal] & M
\end{tikzcd} \tag{4.7}$$

and, analogously to (4.3), there is a homotopy cofibration

$$\Omega M \ltimes \Sigma A \xrightarrow{\theta'_f} E \xrightarrow{\alpha'} \Omega M \ltimes Y.$$

Noting that the upper square of (4.7) is a homotopy pullback and arguing exactly as in the proof of [The20, Lemma 8.3] gives the following.

Lemma 4.4. *The map α' has a right homotopy inverse $r : \Omega M \ltimes Y \to E$ such that the composite $l \circ r$ is null homotopic.* □

Combining this with the general situation, we have homotopy cofibrations

$$\Omega M \ltimes \Sigma A \xrightarrow{\theta_f} E \xrightarrow{\alpha} E' \text{ and } \Omega M \ltimes \Sigma A \xrightarrow{\theta'_f} E \xrightarrow{\alpha'} \Omega M \ltimes Y.$$

By Lemma 4.3 there is a left homotopy inverse k for both θ_f and θ'_f. Lemma 4.4 gives a right homotopy inverse r for α', and since $l \circ r \simeq *$, Diagram (4.5) implies that we have $k \circ r \simeq *$. By [The20, Lemma 8.5], this implies that the composite

$$\Omega M \ltimes Y \xrightarrow{r} E \xrightarrow{\alpha} E' \tag{4.8}$$

is a homotopy equivalence. This achieves our first goal of gaining more control over the homotopy equivalence $E' \simeq \Omega M \ltimes Y$; we will use the fact that it factors through E to prove Theorem 4.2. Recall that applying Theorem 2.14 to the right-most column of Diagram (4.2) yields a homotopy equivalence

$$\Omega C \simeq \Omega M \times \Omega(\Omega M \ltimes Y).$$

Proof of Theorem 4.2. We have already shown that the map $\Omega\varphi$ has a right homotopy inverse given by $\Omega q \circ s \circ t$, due to homotopy commutativity of (4.1), thus all that remains to prove is that the map Ωq also has a right homotopy inverse. We shall do this by showing that the above homotopy equivalence for ΩC factors through Ωq, from which the existence of a right homotopy inverse for Ωq follows immediately. Let $\lambda = s \circ t$, which is a right homotopy inverse for the map Ωh. Taking loops on the middle and right columns of (4.2) gives

$$\begin{tikzcd}
\Omega E \arrow[r, "\Omega\alpha"] \arrow[d] & \Omega E' \arrow[d] \\
\Omega(X \vee Y) \arrow[r, "\Omega q"] \arrow[d, "\Omega h"] & \Omega C \arrow[d, "\Omega\varphi"] \\
\Omega M \arrow[r, equal] & \Omega M.
\end{tikzcd} \tag{4.9}$$

Letting r' denote $\Omega M \ltimes Y \xrightarrow{r} E \to X \vee Y$, we have the composite

$$e : \Omega M \times \Omega(\Omega M \ltimes Y) \xrightarrow{\lambda \times \Omega r'} \Omega(X \vee Y) \times \Omega(X \vee Y) \xrightarrow{\mu} \Omega(X \vee Y) \xrightarrow{\Omega q} \Omega C$$

where μ is the loop multiplication. The homotopy commutativity of (4.9) together with the fact that Ωq is an H-map implies that that e is a homotopy equivalence, so the proof is complete. □

4.2 Inert Maps and Decompositions of Connected Sums

We wish to apply Theorem 4.2 to the situation of connected sums. Recall from Section 2.1 that if we have two homotopy cofibrations of simply connected spaces

$$\Sigma A \xrightarrow{f} B \xrightarrow{j} C \quad \text{and} \quad \Sigma A \xrightarrow{g} D \xrightarrow{l} E$$

we may form the generalised connected sum of C and E over ΣA, as per Definition2.4, written $C\#_{\Sigma A}E$.

Recall also that $p_1 \circ (f \check{+} g) \simeq f$. By Proposition 2.5, we have the diagram below, in which each complete row and column is a homotopy cofibration and the bottom-right square is a homotopy pushout. We label the induced map $C\#_{\Sigma A}E \to C$ by h.

$$\begin{tikzcd}
 & D \arrow[r, equal] \arrow[d] & D \arrow[d] \\
\Sigma A \arrow[r, "f \check{+} g"] \arrow[d, equal] & B \vee D \arrow[r, "q"] \arrow[d, "p_1"] & C\#_{\Sigma A}E \arrow[d, "h"] \\
\Sigma A \arrow[r, "f"] & B \arrow[r, "j"] & C.
\end{tikzcd} \tag{4.10}$$

It follows immediately from Theorem 4.2 and by applying Theorem 2.14 to the rightmost column of Diagram (4.10).

Lemma 4.5. *Take the setup of Diagram (4.10). If the map f is inert, then so is $f \check{+} g$. Moreover, there is a homotopy equivalence $\Omega(C\#_{\Sigma A}E) \simeq \Omega C \times \Omega(\Omega C \ltimes D)$.* □

Remark 4.6. Lemma 4.5 shows that whatever the homotopy class of the map g, the map $f \check{+} g$ inherits inertness from f regardless.

We may also consider connected sums of Poincaré Duality complexes. Recall that a Poincaré Duality complex is a finite, simply connected CW-complex whose cohomology ring exhibits Poincaré Duality. For such a complex, there exists a cell structure that has a single top-dimensional cell, and we may define the connected sum operation similarly to that of manifolds. Namely, for two n-dimensional Poincaré Duality complexes M and N, the space $M\#N$ is formed by removing an n-dimensional open disc from the interior of the top-cells of M and N and joining the resulting complexes along their boundaries. Up to homotopy, $M\#N$ coincides with the generalised connected sum $M\#_{S^{n-1}}N$.

In pursuit of our homotopy theoretic analysis, we seek a framework whereby it may be shown that Poincaré Duality complex has the homotopy type of a connected sum, after looping. To begin to give this we have the following Proposition, which is a restatement of [The20, Theorem 9.1 (b)-(c)], though we provide a new proof.

Proposition 4.7 (Theriault)**.** *Let M and N be two Poincaré Duality complexes of dimension n, where $n > 3$, such that the attaching map of the top-cell of M is inert. Then there is a homotopy equivalence*

$$\Omega(M\#N) \simeq \Omega M \times \Omega(\Omega M \ltimes N_{n-1})$$

where N_{n-1} denotes the $(n-1)$-skeleton of N. Furthermore, the attaching map of the top-cell of $M\#N$ is inert.

Proof. Let f_M and f_N be the attaching maps of the top-cells of M and N, respectively, onto their $(n-1)$-skeleta M_{n-1} and N_{n-1}. We have homotopy cofibrations

$$S^{n-1} \xrightarrow{f_M} M_{n-1} \to M \text{ and } S^{n-1} \xrightarrow{f_N} N_{n-1} \to N.$$

Similar to Diagram (4.10), we have the following homotopy cofibration diagram.

$$\begin{tikzcd}
 & N_{n-1} \arrow[r, equal] \arrow[d] & N_{n-1} \arrow[d] \\
S^{n-1} \arrow[r, "f_M \check{+} f_N"] \arrow[d, equal] & M_{n-1} \vee N_{n-1} \arrow[r, "q"] \arrow[d, "p"] & M\#N \arrow[d, "h"] \\
S^{n-1} \arrow[r, "f_M"] & M_{n-1} \arrow[r, "j"] & M.
\end{tikzcd}$$

The result then follows from Lemma 4.5. □

So far we have only used Theorem 4.2 to show that a sum of attaching maps is inert. The next theorem switches that focus, giving conditions for an attaching map to be inert without first supposing that it is a sum.

Theorem 4.8. *Let $n > 3$ and suppose that M, N and P are n-dimensional Poincaré Duality complexes. Let f_M and f_P denote the attaching maps of the top-cells of M and P, respectively, and suppose further that:*

(i) $M_{n-1} \simeq P_{n-1} \vee N_{n-1}$;

(ii) the composite $p_1 \circ f_M : S^{n-1} \to P_{n-1}$ *is inert;*

(iii) the homotopy cofibre of $p_1 \circ f_M$, Q, *is such that* $\Omega Q \simeq \Omega P$;

(iv) the map f_P *is inert.*

Then the attaching map f_M *is inert and* $\Omega M \simeq \Omega(N\#P)$.

Proof. From *(i)* we have the following homotopy cofibration diagram

$$\begin{tikzcd} & N_{n-1} \arrow[r, equal] \arrow[d] & N_{n-1} \arrow[d] \\ S^{n-1} \arrow[r, "f_M"] \arrow[d, equal] & P_{n-1} \vee N_{n-1} \arrow[r] \arrow[d, "p_1"] & M \arrow[d] \\ S^{n-1} \arrow[r, "p_1 \circ f_M"] & P_{n-1} \arrow[r] & Q. \end{tikzcd} \tag{4.11}$$

Condition *(ii)* places us in the situation of Theorem 4.2, therefore f_M is inert and there is a homotopy equivalence $\Omega M \simeq \Omega Q \times \Omega(\Omega Q \ltimes N_{n-1})$. By condition *(iii)*, we therefore have

$$\Omega M \simeq \Omega P \times \Omega(\Omega P \ltimes N_{n-1}). \tag{4.12}$$

Now consider the connected sum $N\#P$. There is a homotopy cofibration diagram

$$\begin{tikzcd} & N_{n-1} \arrow[r, equal] \arrow[d] & N_{n-1} \arrow[d] \\ S^{n-1} \arrow[r, "f_N \check{+} f_P"] \arrow[d, equal] & P_{n-1} \vee N_{n-1} \arrow[r] \arrow[d, "p_1"] & N\#P \arrow[d] \\ S^{n-1} \arrow[r, "f_P"] & P_{n-1} \arrow[r] & P. \end{tikzcd}$$

By *(iv)*, Lemma 4.5 gives a homotopy equivalence $\Omega(N\#P) \simeq \Omega P \times \Omega(\Omega P \ltimes N_{n-1})$ and consequently that $\Omega M \simeq \Omega(N\#P)$, due to (4.12). $\square$

4.3 A Constructive Example

In order for Theorem 4.8 to carry weight and not be vacuous, we should provide an example of a Poincaré Duality complex with the homotopy type of a connected sum after looping, but not before. Indeed, in this section we will give an explicit family of such examples. Letting $n > 2$ be an integer, we take

$$w_1 : S^{2n-1} \to S^n \vee S^n \ , \ w_2 : S^{2n-1} \to S^{n-1} \vee S^{n+1} \ \text{ and } \ w_3 : S^{2n-2} \to S^n \vee S^{n-1}$$

to be the Whitehead products attaching the top-dimensional cells of the sphere products $S^n \times S^n$, $S^{n-1} \times S^{n+1}$ and $S^n \times S^{n-1}$, respectively. Furthermore, let η denote the classical Hopf map and let $r = 2n-4$. We form three composites, namely

$$f_1 : S^{2n-1} \xrightarrow{w_1} S^n \vee S^n \xrightarrow{i_{1,2}} S^n \vee S^n \vee S^{n-1} \vee S^{n+1}$$
$$f_2 : S^{2n-1} \xrightarrow{w_2} S^{n-1} \vee S^{n+1} \xrightarrow{i_{3,4}} S^n \vee S^n \vee S^{n-1} \vee S^{n+1}$$
$$f_3 : S^{2n-1} \xrightarrow{\Sigma^r \eta} S^{2n-2} \xrightarrow{w_3} S^n \vee S^{n-1} \xrightarrow{i_{2,3}} S^n \vee S^n \vee S^{n-1} \vee S^{n+1}$$

where $i_{j,k}$ is the inclusion of the j^{th} and k^{th} sphere. Before continuing with our construction, we will verify that the composite $w_3 \circ \Sigma^r \eta$ is essential (i.e. not null homotopic). To see this, we will study adjoints. In what follows, we let E denote the usual suspension map $X \to \Omega\Sigma X$, and for a map of topological spaces $f : \Sigma X \to Y$ we let $\hat{f} : X \to \Omega Y$ denote its adjoint. Recall that such a map f is essential if and only if $\hat{f}$ is, and that $\hat{f}$ is homotopic to the composite

$$X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega f} \Omega Y.$$

Lemma 4.9. *The composite $w_3 \circ \Sigma^r \eta$ is essential.*

Proof. In what follows, let $\gamma = w_3 \circ \Sigma^r \eta$. We will describe its adjoint $\hat{\gamma}$ to show it is essential. We first consider the following diagram

$$\begin{tikzcd}
S^{2n-2} \arrow[r, "\Sigma^{r-1}\eta"] \arrow[d, "E"] & S^{2n-3} \arrow[r, "\widehat{w_3}"] \arrow[d, "E"] & \Omega(S^n \vee S^{n-1}) \arrow[d, equal] \\
\Omega S^{2n-1} \arrow[r, "\Omega\Sigma^r\eta"] \arrow[rr, bend right, "\Omega\gamma"'] & \Omega S^{2n-2} \arrow[r, "\Omega w_3"] & \Omega(S^n \vee S^{n-1})
\end{tikzcd} \tag{4.13}$$

in which right square commutes by definition of the adjoint, the left square homotopy commutes by the naturality of the suspension map E, and the bottom row is the definition of $\Omega\gamma$. Thus the whole diagram commutes up to homotopy, and in particular we have $\hat{\gamma} \simeq \hat{w} \circ \Sigma^{r-1}\eta$. Now, by the Hilton-Milnor Theorem we have a homotopy equivalence

$$\Omega(S^n \vee S^{n-1}) \simeq \prod_{k,l \in \mathbb{N}} \Omega S^{(k+l)n+(1-k-2l)}$$

in which $k = l = 1$ gives a factor of ΩS^{2n-2}. Indeed, via Remark 2.16, the composite

$$\Omega S^{2n-2} \to \prod_{k,l \in \mathbb{N}} \Omega S^{(k+l)n+(1-k-2l)} \xrightarrow{\simeq} \Omega(S^n \vee S^{n-1})$$

is homotopic to Ωw_3. Letting $p : \Omega(S^n \vee S^{n-1}) \to \Omega S^{2n-2}$ be the projection map from the product to this factor, consider the diagram

$$\begin{tikzcd}
S^{2n-2} \arrow[r, "\hat{\gamma}"] \arrow[d, "\Sigma^{r-1}\eta"'] & \Omega(S^n \vee S^{n-1}) \arrow[r, "p"] & \Omega S^{2n-2} \\
S^{2n-3} \arrow[r, "E"] \arrow[ur, "\widehat{w_3}"] & \Omega S^{2n-2}. \arrow[u, "\Omega w_3"'] \arrow[ur, equal] &
\end{tikzcd} \tag{4.14}$$

In particular, the right-hand triangle commutes, so the whole diagram commutes up to homotopy. To check that $\hat{\gamma}$ is essential, it therefore it suffices to check that $\Sigma^{r-1}\eta$ is not null homotopic, which always holds since $r > 0$. $\square$

We return now to the main content of this long-form example. Let σ be the usual comultiplication on S^{2n-1} and let ∇ denote the fold map; we write $\sigma' = (1 \vee \sigma) \circ \sigma$ and $\nabla' = \nabla \circ (1 \vee \nabla)$ for three-fold uses of these maps. Consider now the composite

$$f : S^{2n-1} \xrightarrow{\sigma'} \bigvee^{3} S^{2n-1} \xrightarrow{f_1 \vee f_2 \vee f_3} \bigvee^{3} (S^n \vee S^n \vee S^{n-1} \vee S^{n+1}) \xrightarrow{\nabla'} S^n \vee S^n \vee S^{n-1} \vee S^{n+1}$$

Letting $p_{j,k}$ be the pinch map to the j^{th} and k^{th} sphere in $S^n \vee S^n \vee S^{n-1} \vee S^{n+1}$ note that we have

$$p_{1,2} \circ f \simeq w_1 \ , \ p_{3,4} \circ f \simeq w_2 \ \text{ and } \ p_{2,3} \circ f \simeq \gamma. \tag{4.15}$$

The complex we wish to study is the homotopy cofibre of the map f, which we shall denote by M. The construction of the map f implies that M does not have the homotopy type of a connected sum before looping; if this were true, f would be representable in the form $f \simeq g \check{+} h$. This is not the case, by virtue of Lemma 4.9, otherwise we would have $M \simeq (S^n \times S^n)\#(S^{n-1} \times S^{n+1})$. After looping, however, we have the following homotopy equivalence.

Lemma 4.10. *There is a homotopy equivalence $\Omega M \simeq \Omega((S^n \times S^n)\#(S^{n-1} \times S^{n+1}))$.*

Proof. In the context of Theorem 4.8, take $N = S^n \times S^n$ and $P = S^{n-1} \times S^{n+1}$. We have the following homotopy cofibration diagram

$$\begin{tikzcd}
 & S^n \vee S^n \arrow[r, equal] \arrow[d] & S^n \vee S^n \arrow[d] \\
S^{2n-1} \arrow[r, "f"] \arrow[d, equal] & S^n \vee S^n \vee S^{n-1} \vee S^{n+1} \arrow[r] \arrow[d, "p_{3,4}"] & M \arrow[d, "h"] \\
S^{2n-1} \arrow[r, "w_2"] & S^{n-1} \vee S^{n+1} \arrow[r] & S^{n-1} \times S^{n+1}
\end{tikzcd} \tag{4.16}$$

from which, since the map w_2 is inert, Theorem 4.8 gives the homotopy equivalence. □

So, the complex M is not a connected sum before looping, but does have the loop space homotopy type of one. Our construction also carries a deeper structure, as the following lemma shows.

Lemma 4.11. *M is a Poincaré Duality complex.*

Proof. We will show that M inherits a Poincaré Duality structure via an algebra isomorphism $H^*(M) \cong H^*((S^n \times S^n)\#(S^{n-1} \times S^{n+1}))$. To begin, consider taking the pinch map $p_{1,2}$ or $p_{2,3}$ in the middle-column of Diagram (4.16), instead of $p_{3,4}$. Via (4.15) we therefore have, in addition to the map $h : M \to S^{n-1} \times S^{n+1}$, maps

$$h' : M \to S^n \times S^n \ \text{ and } \ h'' : M \to C_\gamma$$

where C_γ denotes the homotopy cofibre of γ. It has cells in dimensions 0, $n-1$, n and $2n$, so all cup products in $H^*(C_\gamma)$ are trivial. Next we must establish notation: we write

$$H^*(S^n \times S^n) = \mathbb{Z}\langle x_1, x_2, \mu_1\rangle \ \text{ and } \ H^*(S^{n-1} \times S^{n+1}) = \mathbb{Z}\langle x_3, x_4, \mu_2\rangle$$

where $|x_1| = |x_2| = n$, $|x_3| = n-1$, $|x_4| = n+1$, $|\mu_1| = |\mu_2| = 2n$, and we have cup products $x_1 \cup x_2 = \mu_1$ and $x_3 \cup x_4 = \mu_2$. We shall do similarly for M: it has cells in dimensions 0, $n-1$, n, $n+1$ and $2n$, so may write $H^*(M) = \mathbb{Z}\langle y_1, y_2, y_3, y_4, \mu_M\rangle$ where $|y_1| = |y_2| = n$, $|y_3| = n-1$, $|y_4| = n+1$, $|\mu_M| = 2n$. By construction, the induced homomorphisms h^* and $(h')^*$ on cohomology take generators to generators; explicitly, we have

$$h^*(x_1) = y_1, h^*(x_2) = y_2, (h')^*(x_3) = y_3, (h')^*(x_4) = y_4 \text{ and } h^*(\mu_1) = (h')^*(\mu_2) = \mu_M$$

which in turn induces cup products $y_1 \cup y_2 = y_3 \cup y_4 = \mu_M$, which are the only possible cup products for dimensional reasons. Thus there is a clear algebra isomorphism $H^*(M) \cong H^*((S^n \times S^n)\#(S^{n-1} \times S^{n+1}))$ by virtue of which M gains a Poincaré Duality struture as per Definition 2.19. $\square$

Lemmas 4.10 and 4.11 combine to give the following Proposition, constituting the thrust of this section.

Proposition 4.12. *For each integer $n > 2$ there exists a $2n$-dimensional Poincaré Duality complex with the homotopy type of a connected sum after looping, but not before.* $\square$

In the construction that lead to Proposition 4.12, there are two subtleties that bear noting. First, we could have used something other than the Hopf map η.

Indeed, suppose we have two wedges of spheres W_1 and W_2, in which all spheres are of dimension at least 2, and for each $i = 1, 2$ there are maps $g_i : S^{2n-1} \to W_i$ such that their homotopy cofibres M_i are $2n$-dimensional Poincaré Duality complexes (for example, our W_i could be wedges of n-spheres, and the g_i would be the attaching maps of the $2n$-cells of $(n-1)$-connected $2n$-dimensional Poincaré Duality complexes). We then take $f_i : S^{2n-1} \xrightarrow{g_i} W_i \to W_1 \vee W_2$. Suppose also that there are integers $a, b > 1$ such that $2n > a + b$ and that spheres S^a and S^b are wedge summands of W_1 and W_2, respectively.

Now, let us suppose further that there is some non-trivial class $\alpha \in \pi_{2n-1}(S^{a+b-1})$. Similar to the proof of Lemma 4.10, let γ' denote the composite

$$\gamma' : S^{2n-1} \xrightarrow{\alpha} S^{a+b-1} \xrightarrow{w} S^a \vee S^b$$

where w is again a Whitehead product. Analogous to Diagram (4.14), we have the following

$$\begin{tikzcd}
S^{2n-2} \arrow[r, "\widehat{\gamma'}"] \arrow[d, "\hat{\alpha}"] & \Omega(S^a \vee S^b) \arrow[r, "p"] & \Omega S^{a+b-1} \\
\Omega S^{a+b-1}. \arrow[ur, "\Omega w"] \arrow[urr, equal] & &
\end{tikzcd} \tag{4.17}$$

The adjoint $\hat{\alpha}$ is essential because α is, by definition, so therefore we have that $\hat{\gamma}'$ is non-trivial, and equivalently that γ' is essential. One way to guarantee this would be to have another class $\beta \in \pi_{2n-2}(S^{a+b-2})$ that suspends non-trivially and take $\alpha = \Sigma\beta$, as we did previously with $\beta = \Sigma^{r-1}\eta$.

We then define f_3 to be the composite $S^{2n-1} \xrightarrow{\gamma'} S^a \vee S^b \to W_1 \vee W_2$ and proceed as before, constructing a map f via a threefold wedge of maps. In essence, the question now is whether the homotopy cofibre M of f is a Poincaré Duality complex in this context. A constructive argument in the style of Lemma 4.11 gives an affirmative answer. Indeed, the cup product structure induced by the maps g_i is preserved, and the only way cup products could come from interactions between spheres in different

W_i would be via f_3. There are none, however, since cup products in the homotopy cofibre of γ' all vanish for dimensional reasons (because $a + b < 2n$). Thus we again build an algebra isomorphism $H^*(M) \cong H^*(M_1 \# M_2)$ from which M inherits its Poincaré Duality structure.

The second subtlety is that, in principle, one could seat this section in a more general context by considering maps of a sphere into a wedge, say $f : S^n \to A \vee B$. By Ganea's Theorem, we have $\pi_*(A \vee B) \cong \pi_*(A) \times \pi_*(B) \times \pi(\Omega A * \Omega B)$. The philosophy behind the construction of Proposition 4.12 is that we wish to construct a map whose homotopy cofibre, which we again call M,

(a) is a Poincaré Duality complex;

(b) does not have the homotopy type of a connected sum before looping;

(c) but does have the homotopy type of a connected sum after looping;

and this is acheived by constructing an appropriate map whose homotopy class is in all three factors of $\pi_*(A \vee B)$. The point here is that if M did have the homotopy type of a connected sum, f could be written as some sum of maps $g \check{+} h$, where $g : S^n \to A$ and $h : S^n \to B$. Then the homotopy class of f would have no part in $\pi_*(\Omega A * \Omega B)$. Our strategy is therefore to construct f so this is not the case, to ensure M is not a connected sum before looping. In this section we are in effect using our knowledge of the Hilton-Milnor Theorem to provide the necessary clarity when A and B are wedges of spheres.

4.4 An Application to the Vigué-Poirrier Conjecture

Let us briefly recall the framework of the Vigué-Poirrier Conjecture. We consider a simply connected space X, and call such a space rationally elliptic if $dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$, and rationally hyperbolic otherwise. The conjecture is as follows.

Conjecture (Vigué-Poirrier)**.** *If X is rationally hyperbolic, then $H_*(\mathcal{L}X; \mathbb{Q})$ grows exponentially.*

Now we establish some terminology, which we take chiefly from [HT21, FHT13]. A graded vector space $V = \{V_i\}_{i \geq 0}$ of finite type *grows exponentially* if there exist constants $1 < C_1 < C_2 < \infty$ such that for some K

$$C_1^k \leq \sum_{i \leq k} dim(V_i) \leq C_2^k \text{ for all } k \geq K.$$

The *log-index* of V is defined by

$$\text{log-index}(V) = \limsup_i \frac{\ln(dim(V_i))}{i}.$$

For a topological space X let $\text{log-index}(\pi_*(X)) = \text{log-index}(\pi_{\geq 2}(X) \otimes \mathbb{Q})$. Note that if X is rationally elliptic then $\text{log-index}(\pi_*(X)) = -\infty$ and if X is rationally hyperbolic then $\text{log-index}(\pi_*(X)) > 0$. The following two stronger definitions were formulated by Félix, Halperin and Thomas in [FHT13], after which they then give Theorem 4.15 (see [FHT13, Theorem 3]).

Definition 4.13. A graded vector space V as described above has *controlled exponential growth* if $\text{log-index}(V) \in (0, \infty)$ and for each $\lambda > 1$ there is an infinite sequence $n_1 < n_2 < \cdots$ such that $n_{i+1} < \lambda n_i$ for all $i \geq 0$ and $dim(V_{n_i}) = e^{\alpha_i n_i}$ with $\alpha_i \to \text{log-index}(V)$.

Definition 4.14. Let X be a simply connected topological space with rational homology of finite type, and such that $\text{log-index}(H_*(\Omega X; \mathbb{Q})) \in (0, \infty)$. Then $\mathcal{L}X$ has *good exponential growth* if $H_*(\mathcal{L}X; \mathbb{Q})$ has controlled exponential growth and

$$\text{log-index}(H_*(\mathcal{L}X; \mathbb{Q})) = \text{log-index}(H_*(\Omega X; \mathbb{Q})).$$

Theorem 4.15 (Félix-Halperin-Thomas). *Let $F \to Y \to Z$ be a fibration between simply connected spaces with rational homology of finite type. If*

$$\textit{log-index}(\pi_*(Z)) < \textit{log-index}(\pi_*(Y))$$

then $\mathcal{L}Y$ has good exponential growth if and only if $\mathcal{L}F$ does. □

We are now ready to provide our application using Theorem 4.8.

Corollary 4.16. *Let $n > 3$ and suppose that M, N and P are n-dimensional Poincaré Duality complexes that satisfy conditions (i)-(iv) of Theorem 4.8. If*

$$\textit{log-index}(\pi_*(P)) < \textit{log-index}(\pi_*(N\#P))$$

then $\mathcal{L}M$ has good exponential growth if and only if $\mathcal{L}(N\#P)$ does.

Proof. Recall from the proof of Theorem 4.8 that we had homotopy cofibrations

$$N_{n-1} \to M \xrightarrow{h_1} Q \text{ and } N_{n-1} \to N\#P \xrightarrow{h_2} P$$

in which both h_1 and h_2 have right homotopy inverses after looping. In particular, by Theorem 2.14, these give rise to homotopy fibrations

$$\Omega P \ltimes N_{n-1} \to M \xrightarrow{h_1} Q \text{ and } \Omega P \ltimes N_{n-1} \to N\#P \xrightarrow{h_2} P. \tag{4.18}$$

Taking the second of these homotopy fibrations, since we supposed that

$$\text{log-index}(\pi_*(P)) < \text{log-index}(\pi_*(N\#P))$$

we have by Theorem 4.15 that $\mathcal{L}(N\#P)$ has good exponential growth if and only if $\mathcal{L}(\Omega P \ltimes N_{n-1})$ does. Since we have assumption *(iii)* of Theorem 4.8, $\Omega Q \simeq \Omega P$. It is evident that

$$\text{log-index}(\pi_*(P)) = \text{log-index}(\pi_*(Q))$$

and by the result Theorem 4.8 of we have

$$\text{log-index}(\pi_*(M)) = \text{log-index}(\pi_*(N\#P)).$$

Therefore $\text{log-index}(\pi_*(Q)) < \text{log-index}(\pi_*(M))$ and we may once again apply Theorem 4.15 to the first homotopy fibration of (4.18) and deduce that the good exponential growth of $\mathcal{L}(\Omega P \ltimes N_{n-1})$ is equivalent to good exponential growth of $\mathcal{L}M$. □

Note in particular that if P is rationally elliptic, the assumption of Corollary 4.16 always holds. We record this in the Theorem below, and then close this section with an example.

Theorem 4.17. *Let $n > 3$ and suppose that M, N and P are n-dimensional Poincaré Duality complexes that satisfy conditions (i)-(iv) of Theorem 4.8. If P is rationally elliptic, then $\mathcal{L}M$ has good exponential growth if and only if $\mathcal{L}(N\#P)$ does.* □

In the situation of Theorem 4.17, $\mathcal{L}(N\#P)$ having good exponential growth implies that it satisfies the Vigué-Poirrier Conjecture – that is to say, $N\#P$ is an example of a rationally hyperbolic space such that $H_*(\mathcal{L}(N\#P);\mathbb{Q})$ grows exponentially. From a result of [Lam01], the Conjecture is known to hold non-trivial connected sums of closed manifolds which are not monogenic in cohomology (i.e. that their cohomology rings are generated by more than one element), so such examples are known to arise in nature. Theorem 4.17 therefore shows that the Conjecture holds for spaces with the loop space homotopy type of connected sums of manifolds which satisfy this simple cohomological condition.

Example 4.18. Consider our Poincaré Duality complex M from the main part of Section 4.3. We had the homotopy equivalence

$$\Omega M \simeq \Omega((S^n \times S^n)\#(S^{n-1} \times S^{n+1})).$$

By [FHT13, Theorem 1], $\mathcal{L}((S^n \times S^n)\#(S^{n-1} \times S^{n+1}))$ has good exponential growth. Moreover, it is easily verified that both $S^n \times S^n$ and $S^{n-1} \times S^{n+1}$ are rationally elliptic. Thus Theorem 4.17 applies, and M satisfies the Vigué-Poirrier Conjecture.

Chapter 5

A Homotopy Theoretic Analogue to a Theorem of Wall

This chapter is a reformatted version of the later sections of a paper of the author, namely [Che22, Sections 4 & 5] together with some illustrative examples from [Che22, Section 3]. It provides another use of the theory we have developed in this thesis, again to do with loop space decompositions of connected sums. Our aim is to make a higher dimensional homotopy theoretic analogue to the following theorem of C.T.C. Wall, given in [Wal66a], which was proved using methods of differential topology and surgery theory.

Theorem (Wall)**.** *Let M be a closed, smooth, simply connected 6-manifold. Then there is a diffeomorphism*

$$M \cong M_1 \# M_2$$

where M_1 is a connected sum of finitely many copies of $S^3 \times S^3$ and $H_3(M_2)$ is finite. □

Generalising this theorem to higher dimensions leads one to consider decomposing $(n-2)$-connected $2n$-manifolds into constituent parts via the operation of connected sums. As we saw in Chapter 3, there has been much recent activity studying the based loop spaces of $(n-1)$-connected $2n$-manifolds, notably by Beben and Theriault [BT14, BT22]. More recently, work of Huang [Hua22a] incorporated a study of torsion free $(n-2)$-connected $2n$-manifolds with vanishing cohomology in dimension n. These papers do not explore connected sums directly, but instead give decompositions of loop spaces as products of other spaces. By drawing on this recent work, and making use earlier results covered in this thesis, we give a homotopy theoretic analogue to this Theorem of Wall in Theorem 5.8.

We begin the chapter with a corollary and some illustrative examples from [Che22, Section 3] relating to Theorem 4.8, which help to bring our focus to connected sums where we have summands that are products of pairs of spheres. In Section 5.2 we decypher the skeletal structure of $(n-2)$-connected $2n$-dimensional Poincaré Duality complexes using a homology decomposition argument; this expands on the ideas in Section 3.3 of this thesis, and we will use it to prove the Main Theorem of this chapter. The titular homotopy theoretic analogue is proved in Section 5.3, which is given by applying the methods developed throughout the preceding sections.

5.1 Some Illustrative Examples

In order to provide a more tangible link to the case where we have connected summands that are products of pairs of spheres, we have a neat corollary to Theorem 4.8, which we form by drawing on results of Beben-Theriault from [BT14]. We then follow with two examples.

Corollary 5.1. *Let $n, m \geq 2$ be integers. Suppose that M and N are two $(n+m)$-dimensional Poincaré Duality complexes such that there is a cohomology isomorphism $H^*(M) \cong H^*(N\#(S^n \times S^m))$ and there exists a homotopy equivalence*

$$M_{n+m-1} \simeq S^n \vee S^m \vee N_{n+m-1}.$$

Then there is a homotopy equivalence $\Omega M \simeq \Omega(N\#(S^n \times S^m))$ and the attaching map of the top-cell of M is inert.

Proof. Once again letting f_M denote the attaching map of the top-cell of M, there is a homotopy commutative diagram of homotopy cofibrations

$$\begin{tikzcd}
 & N_{n+m-1} \arrow[r, equal] \arrow[d, "i"] & N_{n+m-1} \arrow[d, "j\circ i"] \\
S^{n+m-1} \arrow[r, "f_M"] \arrow[d, equal] & S^n \vee S^m \vee N_{n+m-1} \arrow[r, "j"] \arrow[d, "p"] & M \arrow[d] \\
S^{n+m-1} \arrow[r, "p\circ f_M"] & S^n \vee S^m \arrow[r, "h"] & Q.
\end{tikzcd} \tag{5.1}$$

Because of the conditions we imposed in the statement of the Corollary, the complex Q precisely satisfies the situation of [BT14, Lemma 2.3], so there is a homotopy equivalence $\Omega Q \simeq \Omega S^n \times \Omega S^m$ and the map Ωh has a right homotopy inverse. Therefore the composite $p \circ f_M$ is inert (by definition), so Theorem 4.8 applies and there is a homotopy equivalence $\Omega M \simeq \Omega(N\#(S^n \times S^m))$ and f_M is inert. ◻

Example 5.2. An immediate application of Corollary 5.1 is that the attaching map of the top-cell of the r-fold connected sum

$$\mathop{\#}_{i=1}^{r}(S^n \times S^n)$$

is inert. We can generalise this further: fix an integer $k > 3$ and take an index set I and integers $n_i, m_i \geq 2$ such that $n_i + m_i = k$ for all $i \in I$. Corollary 5.1 then gives that the attaching map of the top-cell of the connected sum $\mathop{\#}_{i\in I}(S^{n_i} \times S^{m_i})$ is inert.

Example 5.3. Using Corollary 5.1 we can deduce some facts about manifolds discussed in [BT14]. Let $n > 3$ be an integer such that $n \neq 4, 8$ (to avoid cases where maps of Hopf invariant one may arise) and let M be an a smooth, closed, oriented, $(n-1)$-connected $2n$-dimensional manifold with $rank(H_n(M)) = d \geq 2$. The manifold M is Poincaré Duality complex, and we are in the situation of Corollary 5.1. Consequently, all such manifolds have their top-cells attached by inert maps. Moreover, by Proposition 2.22, if n is an odd number then Poincaré Duality implies that d must be even, and so [BT14, Theorem 1.1(b)] implies that there is a homotopy equivalence

$$\Omega M \simeq \Omega\left(\mathop{\#}_{i=1}^{d/2}(S^n \times S^n)\right)$$

and consequently an isomorphism of homotopy groups. Thus, for such a manifold M, its homotopy groups are determined entirely by the rank of its middle (co)homology group $H_n(M)$.

5.2 A Homology Decomposition

Let us fix an integer $n > 3$ such that $n \notin \{4, 8\}$, again to avoid cases where maps of Hopf invariant one may arise. Consider an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex. Take such a complex M; the Universal Coefficient Theorem together with Poincaré Duality enables us to deduce that

$$H_*(M) \cong \begin{cases} \mathbb{Z} \text{ if } * = 0 \text{ or } * = 2n \\ \mathbb{Z}^l \oplus T \text{ if } * = n-1 \\ \mathbb{Z}^d \oplus T \text{ if } * = n \\ \mathbb{Z}^l \text{ if } * = n+1 \\ 0 \text{ otherwise} \end{cases} \tag{5.2}$$

where $T \cong \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{r_i}\mathbb{Z}$ for (not necessarily distinct) primes p_i and integers $r_i \in \mathbb{N}$. In this section we shall construct an appropriate homology decomposition of M, so that the homotopy theoretic methods we have developed previously can be applied. This will provide the basis for proof of the analogue to Wall's Theorem (see Theorem 5.8).

To begin, recall from [Hat02, Section 4H] (or indeed [Ark11, Section 7.3]) that for a sequence of groups G_j, $j \geq 1$, one may inductively construct a CW-complex X via a sequence of subcomplexes $X_1 \subset X_2 \subset \dots$ with

$$H_i(X_j) \cong \begin{cases} G_i \text{ if } i \leq j \\ 0 \text{ if } i > j \end{cases}$$

such that

(i) X_1 is a Moore space $M(G_1, 1)$;

(ii) X_{j+1} is the homotopy cofibre of a cellular map $h_j : M(G_{j+1}, j) \to X_j$ that induces a trivial map in homology;

(iii) $X = \bigcup_j X_j$.

In [Hat02, Theorem 4H.3], it is noted that every simply connected CW-complex has a homology decomposition. As per our standard notation on page *xiii*, we let $P^n(p_i^{r_i})$ denote the mod-$p_i^{r_i}$ Moore space. Recall that we may one may equivalently view such a Moore space as the homotopy cofibre of the degree $p_i^{r_i}$ map $S^{n-1} \to S^{n-1}$, or use the more general notation $M(\mathbb{Z}/p_i^{r_i}\mathbb{Z}, n-1)$.

Proposition 5.4. *Let M be an $(n-2)$-connected $2n$-dimensional Poincaré Duality complex, as described in (5.2). Then there exists an integer c, with $0 \leq c \leq l$, such that there is a homotopy cofibration*

$$S^{2n-1} \to \left(\bigvee_{i=1}^{d} S^n\right) \vee \left(\bigvee_{i=1}^{l-c} S^{n-1} \vee S^{n+1}\right) \vee J \to M$$

for some appropriate CW-complex J.

To prove Proposition 5.4 we will make use of Lemmas 2.17 and 2.18 from Section 2.3, regarding homotopy cofibrations and wedge sums. We restate them here for the benefit of the reader.

Lemma 2.17. *Suppose that we have a homotopy cofibration*

$$A \xrightarrow{f} B \vee C \xrightarrow{q} D$$

such that the composition of f with the pinch map $p_1 : B \vee C \to B$ is null homotopic. Then B is a homotopy retract of D.

Lemma 2.18. *Suppose that we have a homotopy cofibration*

$$A \vee B \xrightarrow{h} C \to D$$

such that restriction of the map h to A is null homotopic. Then ΣA is a homotopy retract of D.

Proof of Proposition 5.4. To give the asserted homotopy cofibration, we will construct a homology decomposition M. We start with

$$M_1 \simeq M_2 \simeq \cdots \simeq M_{n-2} \simeq *$$

and then

$$M_{n-1} \simeq \left(\bigvee_{i=1}^{l} S^{n-1}\right) \vee \left(\bigvee_{i=1}^{k} P^n(p_i^{r_i})\right).$$

The next complex M_n is constructed via the homotopy cofibration

$$M(\mathbb{Z}^d \oplus T, n-1) \xrightarrow{h_{n-1}} M_{n-1} \to M_n$$

for which we shall take

$$M(\mathbb{Z}^d \oplus T, n-1) = \left(\bigvee_{i=1}^{d} S^{n-1}\right) \vee \left(\bigvee_{i=1}^{k} P^n(p_i^{r_i})\right).$$

Since $(h_{n-1})_* = 0$, the Hurewicz Theorem implies that the restriction of h_{n-1} to the wedge $\bigvee_{i=1}^{d} S^{n-1}$ is null homotopic. By Lemma 2.18, we therefore have that the suspension of the wedge $\bigvee_{i=1}^{d} S^{n-1}$ retracts off M_n. Hence we have a homotopy equivalence

$$M_n \simeq \bigvee_{i=1}^{d} S^n \vee E$$

where E denotes the homotopy cofibre of the inclusion $\bigvee_{i=1}^{d} S^n \to M_n$.

We now deduce some further information about the homotopy type of the complex E. Since h_{n-1} induces a trivial map in homology, we may suppose that there exists an integer c_1, with $0 \leq c_1 \leq l$, such that the composite

$$\left(\bigvee_{i=1}^{2d} S^{n-1}\right) \vee \left(\bigvee_{i=1}^{k} P^n(p_i^{r_i})\right) \xrightarrow{h_{n-1}} M_{n-1} \simeq \left(\bigvee_{i=1}^{l} S^{n-1}\right) \vee \left(\bigvee_{i=1}^{k} P^n(p_i^{r_i})\right) \xrightarrow{p} \bigvee_{i=1}^{l-c_1} S^{n-1}$$

is null homotopic, where p denotes a pinch map. Moreover, without loss of generality we may suppose that c_1 is the minimal integer with this property. Here we use Lemma 2.17, and thus we may write

$$E \simeq \left(\bigvee_{i=1}^{l-c_1} S^{n-1}\right) \vee J_1$$

for some CW-complex J_1. The next step in our construction is to form M_{n+1} via the homotopy cofibration

$$M(\mathbb{Z}^l, n) = \bigvee_{i=1}^{l} S^n \xrightarrow{h_n} M_n \to M_{n+1}.$$

First, note that the composite

$$\bigvee_{i=1}^{l} S^n \xrightarrow{h_n} M_n \simeq \left(\bigvee_{i=1}^{d} S^n\right) \vee \left(\bigvee_{i=1}^{l-c_1} S^{n-1}\right) \vee J_1 \xrightarrow{p_1} \bigvee_{i=1}^{d} S^n$$

is forced to be null homotopic, otherwise we would generate additional torsion in M_{n+1}, which is not permissible because $H_{n+1}(M) \cong \mathbb{Z}^l$. Therefore, again by Lemma 2.17, $\bigvee_{i=1}^{d} S^n$ is a homotopy retract of M_{n+1}. Furthermore, let $c_2 \leq c_1$ be the smallest integer such that the composite

$$\bigvee_{i=1}^{l} S^n \xrightarrow{h_n} M_n \xrightarrow{p} \bigvee_{i=1}^{l-c_2} S^n \subseteq \bigvee_{i=1}^{l-c_1} S^n$$

is null homotopic. Such a $c_2 \geq 0$ exists, and we may assume it is minimal without loss of generality.

Let c_3 be the least integer, $0 \leq c_3 \leq l$, such that the restiction of h_n to the sub-wedge $\bigvee_{i=1}^{l-c_3} S^n$ is null homotopic. Once again using Lemma 2.18, this gives rise to a homotopy equivalence

$$M_{n+1} \simeq \left(\bigvee_{i=1}^{d} S^n\right) \vee \left(\bigvee_{i=1}^{l-c_2} S^{n-1}\right) \vee \left(\bigvee_{i=1}^{l-c_3} S^{n+1}\right) \vee J_2$$

for some other CW-complex J_2. Rewriting this, letting $c = Max\{c_2, c_3\}$, we have

$$M_{n+1} \simeq \left(\bigvee_{i=1}^{d} S^n\right) \vee \left(\bigvee_{i=1}^{l-c} S^{n-1} \vee S^{n+1}\right) \vee J$$

where J arises from taking the wedge of J_2 with the discarded spheres. We call the CW-complex J the *auxiliary complex*. Letting f denote the attaching map of the top-cell, our Poincaré Duality complex M is then given by the homotopy cofibration

$$S^{2n-1} \xrightarrow{f} \left(\bigvee_{i=1}^{d} S^n\right) \vee \left(\bigvee_{i=1}^{l-c} S^{n-1} \vee S^{n+1}\right) \vee J \xrightarrow{j} M$$

as asserted. □

Example 5.5. In the general case, we deliberately leave the homotopy type of the auxiliary complex mysterious, but there are circumstances in which we may deduce its homotopy type. To see this, we shall revisit the construction of [Hua22a] that we used in Case 1 of Section 3.3. We demand that $H_*(M)$ is torsion free and $d = 0$. This implies $H_n(M) \cong 0$, and therefore there are no cells of consecutive dimensions in the CW-structure of M. Thus there is a homotopy cofibration

$$\bigvee_{i=1}^{l} S^n \xrightarrow{\psi} \bigvee_{i=1}^{l} S^{n-1} \to M_{n+1}$$

that defines the $(n+1)$-skeleton of M. Further, letting i_r denote the inclusion of the r^{th} wedge summand, for each $r \in \{1, \dots, l\}$ the composite

$$\psi_r : S^n \xrightarrow{i_r} \bigvee_{i=1}^{l} S^n \xrightarrow{\psi} \bigvee_{i=1}^{l} S^{n-1}$$

defines a homotopy class in the group $\pi_n(\bigvee_{i=1}^{l} S^{n-1})$. Since $n > 3$, this group is isomorphic to $\bigoplus_{i=1}^{l} \mathbb{Z}/2\mathbb{Z}$ [Tod16], where each $\mathbb{Z}/2\mathbb{Z}$ summand is generated by the homotopy class of the attaching map for the $(n+1)$-cell of $\Sigma^{n-3}\mathbb{C}P^2$. Thus we may from an $(l \times l)$-matrix C with entries in $\mathbb{Z}/2\mathbb{Z}$, where the r^{th} column is the image of the homotopy class of ψ_r under this group isomorphism. Huang shows that this matrix may be manipulated by row and column operations, and that these operations are homotopy invariant. Letting $c = rank(C)$, [Hua22a, Lemma 6.1] shows that there exists a homotopy equivalence

$$M_{n+1} \simeq \left(\bigvee_{i=1}^{l-c} (S^{n-1} \vee S^{n+1}) \right) \vee \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2 \right).$$

If $d > 0$, then as with Proposition 3.12, we have

$$M_{n+1} \simeq \left(\bigvee_{i=1}^{d} S^n \right) \vee \left(\bigvee_{i=1}^{l-c} (S^{n-1} \vee S^{n+1}) \right) \vee \left(\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2 \right)$$

so the summand $\bigvee_{i=1}^{c} \Sigma^{n-3}\mathbb{C}P^2$ is playing the role of the auxilliary complex here.

Remark 5.6. Our aim in the construction of this section was to write the skeletal structure of M in such a fashion as to have as many wedges of pairs of spheres retracting off it. This is key to our approach and to sustaining the analogy with Wall's Theorem and it's generalisations (recall our discussion of [Tam68, Ish73, Fan96] from the introduction of this thesis). One may think of Proposition 5.4 as giving a separation of the skeleton into two parts: those from which trivial and non-trivial Steenrod squares may arise. Our eventual aim is to obtain a decomposition of M which maximises the number of sphere products in the connected sum decomposition, for which the step of Proposition 5.4 is essential.

As M is a Poincaré Duality complex, it has a fundamental class, which we will denote by μ_M.

Lemma 5.7. *Assume $d > 1$ and let $x \in H^n(M)$ be a generator induced by an S^n wedge summand in M_{n+1}. Then there is a class $y \in H^n(M)$, is induced a different S^n wedge summand, such that $x \cup y = \mu_M$.*

Proof. The given class x associated with an S^n wedge summand in M_{n+1} is a basis element in the cohomology group $H^n(M)$. By [Hat02, Corollary 3.39] there exists a class $y \in H^n(M)$ that generates an infinite cyclic summand of $H^n(M)$, such that $x \cup y = \mu_M$. Therefore, up to a change of basis of $H^*(M)$ (i.e. up to a self equivalence of M_{n+1}) we have that the class y is also induced by an S^n wedge summand. Furthermore, because we excluded Hopf invariant one cases in the setup of this section, we have $\pm x \neq \pm y$, so the spheres that induce the classes x and y are distinct. □

5.3 Proving the Analogue

We now apply the methods we have developed to give the titular homotopy theoretic analogue. Recall from the previous section that for a smooth, closed, oriented, $(n-2)$-connected $2n$-dimensional manifold Poincaré Duality complex M, with $n > 3$ such that $n \notin \{4, 8\}$, we have integral homology

$$H_*(M) \cong \begin{cases} \mathbb{Z} \text{ if } * = 0 \text{ or } * = 2n \\ \mathbb{Z}^l \oplus T \text{ if } * = n-1 \\ \mathbb{Z}^d \oplus T \text{ if } * = n \\ \mathbb{Z}^l \text{ if } * = n+1 \\ 0 \text{ otherwise} \end{cases}$$

where $T \cong \bigoplus_{i=1}^k \mathbb{Z}/p_i^{r_i}\mathbb{Z}$ for primes p_i and integers $r_i \in \mathbb{N}$.

Theorem 5.8. *Let $n > 3$ be an integer such that $n \notin \{4, 8\}$, and let M be a $(n-2)$-connected $2n$-dimensional Poincaré Duality complex with $d > 1$. Then there exists a homotopy equivalence*

$$\Omega M \simeq \Omega(M_1 \# M_2 \# M_3)$$

where

(i) M_1 is an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex, with $\mathrm{rank}(H_n(M_1)) = d$;

(ii) M_2 is a connected sum of finitely many copies of $S^{n-1} \times S^{n+1}$ and;

(iii) M_3 is a CW-complex with a single top-cell and $H_n(M_3)$ finite.

Proof. We first produce a loop space decomposition for $M_1\#M_2\#M_3$. In general, we take M_1 to be as in *(i)* above, but note that if d is even (which, by Proposition 2.22, always holds when n is odd) we may simply take M_1 to be a connected sum of $\frac{d}{2}$-many copies of $S^n \times S^n$. Letting c and J be as in Proposition 5.4, we define

$$M_2 = \mathop{\#}_{i=1}^{l-c}(S^{n-1} \times S^{n+1}) \text{ and } M_3 = J \cup e^{2n}.$$

Note that we neglect to denote the map by which we attach the $2n$-cell to J, as its homotopy type is of no consequence. For brevity, let

$$W_1 = \bigvee_{i=1}^{d} S^n,\, W_2 = \bigvee_{i=1}^{l-c}(S^{n-1} \vee S^{n+1}) \;\text{ and }\; X = \left(\bigvee_{i=1}^{d-2} S^n\right) \vee W_2 \vee J$$

so by construction $(M_1)_{2n-1} \simeq W_1$ and $(M_2)_{2n-1} \simeq W_2$. By Lemma 5.7, we can always isolate two n-spheres in W_1 that are associated with classes that cup together to give the fundamental class, thus giving rise to the homotopy cofibration diagram

$$\begin{tikzcd} & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d] \\ S^{2n-1} \arrow[r] \arrow[d, equal] & W_1 \vee W_2 \vee J \arrow[r] \arrow[d, "p"] & M_1\#M_2\#M_3 \arrow[d, "h"] \\ S^{2n-1} \arrow[r] & S^n \vee S^n \arrow[r, "q"] & Q \end{tikzcd} \tag{5.3}$$

where the space Q has the property that $H^*(Q) \cong H^*(S^n \times S^n)$. By [BT14, Lemma 2.3], the map Ωq has a right homotopy inverse and $\Omega Q \simeq \Omega(S^n \times S^n)$. By homotopy commutativity of (5.3) the map Ωh therefore has a right homotopy inverse, and applying Theorem 2.14 to the right-hand column gives the loop space decomposition

$$\Omega(M_1\#M_2\#M_3) \simeq \Omega(S^n \times S^n) \times \Omega(\Omega(S^n \times S^n) \ltimes X).$$

Now consider the Poincaré Duality complex M. Similarly to above, letting f denote the attaching map of the top-cell of M, by Proposition 5.4 and Lemma 5.7 we have the following diagram of homotopy cofibrations

$$\begin{tikzcd}
 & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "j\circ i"] \\
S^{2n-1} \arrow[r, "f"] \arrow[d, equal] & W_1 \vee W_2 \vee J \arrow[r, "j"] \arrow[d, "p"] & M \arrow[d, "h"] \\
S^{2n-1} \arrow[r, "p\circ f"] & S^n \vee S^n \arrow[r, "q"] & Q'
\end{tikzcd} \tag{5.4}$$

where, again, Q' is such that $H^*(Q') \cong H^*(S^n \times S^n)$. Reasoning identically to before gives the loop space decomposition

$$\Omega M \simeq \Omega(S^n \times S^n) \times \Omega(\Omega(S^n \times S^n) \ltimes X). \tag{5.5}$$

Comparing this to the decomposition for $\Omega(M_1 \# M_2 \# M_3)$ gives us the desired homotopy equivalence. □

The proof of Theorem 5.8 shows that after we take loop spaces, the homotopy class of the attaching map of the top-cell ceases to be important. Indeed, the principal object of concern in the proof is the $(2n-1)$-skeleton of the complex M, and the connected sum $M_1 \# M_2 \# M_3$ is constructed so that its $(2n-1)$-skeleton exactly matches that of M. Observe also that Theorem 5.8 also gives inertness of the attaching map of the top-cell of M, by application of Proposition 4.7 and Example 5.3.

Remark 5.9. Proving Theorem 5.8 for Poincaré Duality complexes, as we have done, implies we have such a composition in the case when M is in fact an smooth, closed, oriented, $(n-2)$-connected $2n$-manifold. In that case, the complex M_3 may also have the homotopy type of a manifold - this depends on whether the *total surgery obstruction* of Ranicki (see [Ran92, Theorem 17.4]) is zero. It would be interesting to make further investigation here.

Theorem 5.8 enables us to make a further observation regarding rational hyperbolicity. Indeed, recall that a simply connected space Y is called *rationally elliptic* if $dim(\pi_*(Y) \otimes \mathbb{Q}) < \infty$, and called *rationally hyperbolic* otherwise [FHT01]. For example, any wedge of spheres $\bigvee_{i=1}^{r} S^{m_i}$ with $r > 1$ and all $m_i > 1$ is a rationally hyperbolic space.

Corollary 5.10. *Let $n > 3$ be an integer such that $n \notin \{4, 8\}$, and let M be a $(n-2)$-connected $2n$-dimensional Poincaré Duality complex with $d > 1$. Then M is rationally hyperbolic if and only if $d > 2$ or $H_{n-1}(M; \mathbb{Q}) \ncong 0$.*

Proof. Recall from the proof of Theorem 5.8 that we had the homotopy equivalence (5.5), namely

$$\Omega M \simeq \Omega(S^n \times S^n) \times \Omega(\Omega(S^n \times S^n) \ltimes X)$$

where $X = \left(\bigvee_{i=1}^{d-2} S^n\right) \vee W_2 \vee J$. By assuming $d > 2$ or $H_{n-1}(M;\mathbb{Q}) \ncong 0$, the construction of Proposition 5.4 guarantees that X does not have the rational homotopy type of a point. More than this, as X is an $(n-2)$-connected $(n+1)$-dimensional CW-complex, since our restrictions on n give $n \geq 5$, we are able to invoke [Gan70] and show that X in fact has the homotopy type of suspension. Let us write $X \simeq \Sigma X'$.

Thus we have $\Omega(S^n \times S^n) \ltimes \Sigma X' \simeq \Sigma(\Omega(S^n \times S^n) \wedge X') \vee X')$. Rationally, a suspension is homotopy equivalent to a wedge of spheres, so there is a rational homotopy equivalence

$$\Omega(S^n \times S^n) \ltimes \Sigma X' \simeq \bigvee_{i=1}^{r} S^{m_i}$$

for some integers $m_i > 1$ and $r > 1$. Therefore, rationally, $\Omega(\bigvee_{i=1}^{r} S^{m_i})$ retracts off ΩM, and M is consequently rationally hyperbolic.

We prove the other direction by negation: if $d = 2$ and $H_{n-1}(M;\mathbb{Q}) \cong 0$, then by the construction of Proposition 5.4 we have that X must be a torsion space, and consequently have the rational homotopy type of a point. Since left half-smash with a point is contractible (by definition), the homotopy equivalence of (5.5) then implies that there is a rational homotopy equivalence $\Omega M \simeq \Omega(S^n \times S^n)$, and so

$$\pi_*(M) \cong \pi_*(S^n) \times \pi_*(S^n).$$

The complex M is therefore rationally elliptic, and in particular, not rationally hyperbolic. □

Chapter 6

The Rational Homotopy Type of Homotopy Fibrations Over Connected Sums

This chapter, which is a reformatted version of [Che23] (a paper of the author), provides another example of results we can obtain through the methods expounded in this thesis.

We begin with a homotopy fibration $F \to L \xrightarrow{f} C$ in which all spaces have the homotopy type of Poincaré Duality complexes. Writing $dim(C) = n$ and $dim(L) = m$, let B be another n-dimensional Poincaré Duality complex. Form the connected sum $B\#C$, and take the natural collapsing map $p : B\#C \to C$. Defining the m-dimensional complex M as the pullback of f across p, we have a homotopy fibration diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & B\#C \arrow[d, "p"] \\ F \arrow[r] & L \arrow[r, "f"] & C. \end{tikzcd} \tag{6.1}$$

The central question of this chapter is as follows: to what extent does M behave like a connected sum? Using our methods, we will attempt to answer this up to the homotopy type of ΩM.

We give a special circumstance in Proposition 6.4, in which the based loop space of M is homotopy equivalent to the based loops of a connected sum. This takes its most dramatic form in the context of rational homotopy theory, in Theorem 6.7. Thus we give an affirmative answer to our question, but after looping and up to rational homotopy equivalence.

6.1 Recalling Some Preliminaries

Recall that for two path connected and based spaces X and Y, the *(left) half-smash* of X and Y is the quotient space

$$X \ltimes Y = (X \times Y)/(X \times y_0)$$

where y_0 denotes the basepoint of Y. Furthermore, we had Lemma 2.11, which gives a homotopy equivalence $X \ltimes Y \simeq (X \wedge Y) \vee Y$ when Y is a co-H-space.

Recall also from Section 4.2 that for a homotopy cofibration $A \xrightarrow{f} B \xrightarrow{j} C$, the map f is called *inert* if Ωj has a right homotopy inverse. We will see the rational homotopy theoretic definition that inspired this, namely *rational inertness*, in Section 6.3 of this chapter. We will also once again make use of the Theorem 2.14, which we restate for the benefit of the reader.

Theorem 2.14 (Beben-Theriault). *Suppose we have a homotopy cofibration $A \xrightarrow{f} B \xrightarrow{h} C$ such that the map Ωh has a right homotopy inverse. Then there exists a homotopy fibration*

$$\Omega C \ltimes A \to B \xrightarrow{h} C$$

which splits after looping. Thus, there exists a homotopy equivalence

$$\Omega B \simeq \Omega C \times \Omega(\Omega C \ltimes A).$$

Take now a different situation, in which we have two homotopy cofibrations of simply connected spaces

$$A \xrightarrow{f} B \xrightarrow{j} C \text{ and } Y \xrightarrow{i} B \xrightarrow{p} X.$$

In the diagram below, each complete row and column is a homotopy cofibration, and the bottom-right square is a homotopy pushout, defining the new space Q and the maps h and q:

$$\begin{tikzcd}
 & Y \arrow[r, equal] \arrow[d, "i"] & Y \arrow[d, "j\circ i"] \\
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "j"] \arrow[d, "p"] & C \arrow[d, "h"] \\
A \arrow[r, "p\circ f"] & X \arrow[r, "q"] & Q.
\end{tikzcd} \tag{6.2}$$

We record an elementary fact in the following lemma, for ease of reference, which is a slightly more general version of Lemma 4.5.

Lemma 6.1. *Take the setup of Diagram (6.2). If the maps Ωp and Ωq have right homotopy inverses, then so does Ωh. Moreover, there is a homotopy equivalence*

$$\Omega C \simeq \Omega Q \times \Omega(\Omega Q \ltimes Y).$$

Proof. Let us denote the right homotopy inverses of Ωq and Ωp by s and t, respectively. Then, by the homotopy commutativity of Diagram (6.2), Ωh has right homotopy inverse given by the composite $\Omega j \circ t \circ s$.

As Y and C are simply connected, the space Q is as well, and the map $j \circ i$ is by definition inert. Hence we may apply Theorem 2.14 to the right-most column of (6.2), obtaining the asserted homotopy equivalence. □

Finally, recall that the spaces considered by Jeffrey and Selick in [JS21] have the homotopy type of oriented, smooth, closed, simply connected manifolds, and are thus Poincaré Duality complexes; that is to say, they satisfy Definition 2.19. For such a complex there exists a CW structure having a single top-dimensional cell. For brevity, given a k-dimensional Poincaré Duality complex Y, let $\overline{Y}$ denote its $(k-1)$-skeleton, and note that there exists a homotopy cofibration

$$S^{k-1} \xrightarrow{f} \overline{Y} \to \overline{Y} \cup_f e^k \simeq Y$$

where f is the attaching map of the top-cell of Y. Furthermore, given two k-dimensional Poincaré Duality complexes X and Y, we may form their connected sum by means of Definition 2.4. In particular, $\overline{X\#Y} \simeq \overline{X} \vee \overline{Y}$, and there is a homotopy cofibration

$$\overline{X} \to X\#Y \to Y.$$

6.2 Pullbacks over Connected Sums

The situation we wish to study begins with a homotopy fibration $F \to L \xrightarrow{f} C$, in which each space has the homotopy type of a Poincaré Duality complex. As in the introduction, let $dim(C) = n$ and $dim(L) = m$, and let B be another n-dimensional Poincaré Duality complex. We form the connected sum $B\#C$, and take the natural collapsing map $p : B\#C \to C$. Defining the m-dimensional complex M as the pullback of f across p, we have a homotopy fibration diagram

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\pi"] & B\#C \arrow[d, "p"] \\ F \arrow[r] & L \arrow[r, "f"] & C \end{tikzcd} \tag{6.3}$$

where we denote the induced map $M \to L$ by π and the fibre map $F \to M$ by α.

Lemma 6.2. *With spaces and maps as in Diagram (6.3), there is a homotopy pushout square*

$$\begin{tikzcd}
F \times \overline{B} \arrow[r] \arrow[d, "p_1"] & M \arrow[d, "\pi"] \\
F \arrow[r] & L.
\end{tikzcd}$$

Where the map p_1 is projection to the first factor. In particular, if α is null homotopic, there is a homotopy cofibration $F \ltimes \overline{B} \to M \xrightarrow{\pi} L$.

Proof. To prove the existence of the asserted homotopy pushout, we will use Mather's Cube Lemma (see Theorem 2.7). Indeed, consider the following diagram

$$\begin{tikzcd}
F \times \overline{B} \arrow[rr, "\beta"] \arrow[dr, "p_1"] \arrow[dd, "p_2"'] & & M \arrow[dr] \arrow[dd] & \\
& F \arrow[rr] \arrow[dd] & & L \arrow[dd, "f"] \\
\overline{B} \arrow[rr] \arrow[dr] & & B\#C \arrow[dr, "p"] & \\
& * \arrow[rr] & & C.
\end{tikzcd} \tag{6.4}$$

We must show is that (6.4) commutes up to homotopy, that bottom face is a homotopy pushout and that the four vertical faces are homotopy pullbacks.

The bottom face of (6.4) arises from the homotopy cofibration $\overline{B} \to B\#C \xrightarrow{p} C$, and so is homotopy pushout. The front face is evidently a homotopy pullback, because it comes from the homotopy fibration we began with, as is the right-hand face of the cube, which is the right-hand sqaure in Diagram (6.3). Furthermore, it is an elementary fact that the left-hand face of the cube, together with the projection maps, is also homotopy pullback.

What remains to show is that the map $\beta : F \times \overline{B} \to M$ is chosen such that the diagram commutes up to homotopy and that the rear face is a homotopy pullback. Indeed, as the right-hand face is a homotopy pullback, β is induced by the existence of the composites $F \times \overline{B} \to F \to L$ and $F \times \overline{B} \to \overline{B} \to B\#C$, so the diagram does indeed homotopy commute. One then applies [Ark11, Theorem 6.3.3], which forces the rear face to be a homotopy pullback. In the special case in which the fibre map α is null homotopic, we may pinch out a copy of F in the asserted pushout, giving the square

$$\begin{tikzcd}
F \ltimes \overline{B} \arrow[r] \arrow[d] & M \arrow[d, "\pi"] \\
* \arrow[r] & L.
\end{tikzcd}$$

which is equivalent to the stated homotopy cofibration. ☐

Remark 6.3. Note that, because Diagram (6.3) is homotopy commutative, requiring α to be null homotopic forces the fibre map $F \to L$ to have also been null homotopic to begin with.

We now give the thrust of this section, providing a circumstance in which the based loop space of the Poincaré Duality complex M is homotopy equivalent to the based loop space of a connected sum. Let $X' = F \ltimes \overline{B}$ and $X = X' \cup e^m$ (the homotopy class of the attaching map $S^{m-1} \to X'$ plays no role in what is to follow, so we suppress it in the definition of X).

Proposition 6.4. *Take the situation as in Diagram (6.1), and suppose that the map Ωp has a right homotopy inverse. Then the map $\Omega\pi$ has a right homotopy inverse. Moreover, if α is null homotopic and the attaching map of the top cell of L is inert, then*

$$\Omega M \simeq \Omega(X \# L).$$

Proof. Denoting the right homotopy inverse of Ωp by $s : \Omega C \to \Omega(B\#C)$, consider the diagram

$$\begin{tikzcd}
\Omega L \arrow[r, "\Omega f"] \arrow[dr, dashed, "\lambda"] \arrow[ddr, equal] & \Omega C \arrow[dr, "s"] & \\
& \Omega M \arrow[r] \arrow[d, "\Omega\pi"] & \Omega(B\#C) \arrow[d, "\Omega p"] \\
& \Omega L \arrow[r, "\Omega f"] & \Omega C
\end{tikzcd}$$

where the map λ will be detailed momentarily. Since the right-hand square of Diagram (6.3) is a homotopy pullback, so is the square in the above. Furthermore, since $\Omega p \circ s \simeq 1_{\Omega C}$, the diagram commutes. As ΩM is the homotopy pullback of Ωf across Ωp, the map λ exists and we have that $\Omega\pi \circ \lambda \simeq 1_{\Omega L}$. In other words, the map λ is a right homotopy inverse for $\Omega\pi$.

Consequently, in the case when α is null homotopic, we apply Theorem 2.14 to the homotopy cofibration

$$X' \to M \xrightarrow{\pi} L$$

from the special case of Lemma 6.2. Indeed, since $\Omega\pi$ has a right homotopy inverse, the map $X' \to M$ is by definition inert, so Theorem 2.14 immediately gives us that

$$\Omega M \simeq \Omega L \times \Omega(\Omega L \ltimes X').$$

On the other hand, let us now consider the connected sum $X\#L$. Take two homotopy cofibrations: one is the attaching map of the top-cell of $X\#L$, and the other is from inclusion of a wedge summand

$$S^{m-1} \to X' \vee \overline{L} \to X\#L \text{ and } X' \to X' \vee \overline{L} \xrightarrow{q} \overline{L}.$$

We combine these to give a cofibration diagram, in the sense of (6.2)

$$\begin{tikzcd}
 & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] \\
S^{m-1} \arrow[r] \arrow[d, equal] & X' \vee \overline{L} \arrow[r] \arrow[d, "q"] & X\#L \arrow[d] \\
S^{m-1} \arrow[r] & \overline{L} \arrow[r, "j"] & L.
\end{tikzcd}$$

The map q pinches to the second wedge summand, and therefore has a right homotopy inverse given by inclusion; therefore Ωq also has a right homotopy inverse. Moreover, if the attaching map of the top-cell of L is inert, the map Ωj has a right homotopy inverse, by definition. Thus Lemma 6.1 applies, implying there is a homotopy equivalence

$$\Omega(X\#L) \simeq \Omega L \times \Omega(\Omega L \ltimes X').$$

Thus ΩM and $\Omega(X\#L)$ are both homotopy equivalent to $\Omega L \times \Omega(\Omega L \ltimes X')$, and are therefore homotopy equivalent to each other. □

Example 6.5. A general class of examples that satisfy the requirement that $\alpha \simeq *$ are sphere bundles, $S^r \to L \to C$, where the pullback M has trivial r^{th} homotopy group. Consider for example the classical Hopf bundle $S^3 \to S^7 \xrightarrow{\eta} S^4$. Taking products with the trivial fibration $* \to S^6 \to S^6$ yields a new homotopy fibration

$$S^3 \to S^7 \times S^6 \xrightarrow{\eta \times 1} S^4 \times S^6.$$

Applying our construction with $B = S^5 \times S^5$, we have the following pullback diagram of homotopy fibrations

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\pi"] & (S^5 \times S^5)\#(S^4 \times S^6) \arrow[d, "p"] \\
S^3 \arrow[r] & S^7 \times S^6 \arrow[r, "\eta\times 1"] & S^4 \times S^6.
\end{tikzcd}$$

Using techniques from [The20, Section 9] it can be shown that $\Omega(S^4 \times S^6)$ retracts off $\Omega((S^5 \times S^5)\#(S^4 \times S^6))$ via a right homotopy inverse for Ωp. Moreover, the attaching map for the top-cell of the product $S^7 \times S^6$ is known to be inert.

Now we show that $\pi_3(M) \cong 0$. The existence of a right homotopy inverse for the map Ωp implies that its homotopy fibre (and consequently the homotopy fibre of π) is homotopy equivalent to $(\Omega S^4 \times \Omega S^6) \ltimes (S^5 \vee S^5)$, by Theorem 2.14. It is now easy to check that the long exact sequence of homotopy groups induced by the fibration sequence $(\Omega S^4 \times \Omega S^6) \ltimes (S^5 \vee S^5) \to M \xrightarrow{\pi} S^3 \times S^4$ forces $\pi_3(M)$ to be trivial. Therefore Proposition 6.4 applies, with

$$X' \simeq S^3 \ltimes (S^5 \vee S^5) \simeq S^5 \vee S^5 \vee S^8 \vee S^8.$$

By gluing a 13-cell to X', we may take $X = (S^5 \times S^8)\#(S^5 \times S^8)$. Hence we obtain a homotopy equivalence

$$\Omega M \simeq \Omega((S^7 \times S^6)\#(S^5 \times S^8)\#(S^5 \times S^8)).$$

To conclude this example, we remark that many of the situations considered by Duan in [Dua22] also fit into this framework.

6.3 The Rational Homotopy Perspective

We wish to apply Proposition 6.4 in the context of rational homotopy theory. Let

$$S^{k-1} \xrightarrow{f} Y \xrightarrow{i} Y \cup_f e^k$$

be a homotopy cofibration, where the map f attaches a k-cell to Y and i is the inclusion. The map f is *rationally inert* if Ωi induces a surjection in rational homology. This implies that, rationally, Ωi has a right homotopy inverse. The following theorem was first proved in [HL87, Theorem 5.1], though we prefer the statement found in [FH19].

Theorem 6.6 (Halperin-Lemaire)**.** *If $Y \cup_f e^k$ is a Poincaré Duality complex and $H^*(Y;\mathbb{Q})$ is generated by more than one element (as an algebra), then the attaching map f is rationally inert.* □

This leads us to the statement and proof of the Main Theorem.

Theorem 6.7. *Given spaces and maps as in Diagram (6.3), if*

(i) the map α is (rationally) null homotopic, and,

(ii) both $H^*(\overline{C};\mathbb{Q})$ and $H^*(\overline{L};\mathbb{Q})$ are generated by more than one element (as algebras),

there is a rational homotopy equivalence $\Omega M \simeq \Omega(X\#L)$.

Proof. By Theorem 6.6, the attaching maps for the top-cells of C and L are rationally inert. We have the homotopy pullback below, which is the right-hand square of (6.3)

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\pi"] & B\#C \arrow[d, "p"] \\ L \arrow[r, "f"] & C. \end{tikzcd}$$

Rationalising spaces and maps in this pullback square, we see that Proposition 6.4 would apply if the map Ωp has a rational right homotopy inverse, as the attaching map for the top-cell of L is rationally inert. Thus we would have a rational homotopy equivalence $\Omega M \simeq \Omega(X\#L)$.

It therefore remains to show that the map Ωp has a rational right homotopy inverse. With this in mind, consider the following homotopy cofibration diagram

$$\begin{tikzcd} & \overline{B} \arrow[r, equal] \arrow[d] & \overline{B} \arrow[d] \\ S^{n-1} \arrow[r] \arrow[d, equal] & \overline{B} \vee \overline{C} \arrow[r] \arrow[d, "q"] & B\#C \arrow[d, "p"] \\ S^{n-1} \arrow[r] & \overline{C} \arrow[r, "i_C"] & C. \end{tikzcd}$$

As the pinch map q has a right homotopy inverse, so does Ωq. Furthermore, the attaching map of the top-cell of C is rationally inert, and therefore Ωi_C has a right homotopy inverse after rationalisation. Therefore the map Ωp also has a (rational) right homotopy inverse, by Lemma 6.1. □

Remark 6.8. Recall that a simply connected space Y is called *rationally elliptic* if $dim(\pi_*(Y) \otimes \mathbb{Q}) < \infty$, and called *rationally hyperbolic* otherwise [FHT01]. We remark briefly on the rational hyperbolicity of the spaces discussed above.

Indeed, suppose that the skeleton $\overline{B}$ is a suspension. Then, as $X' = F \ltimes \overline{B}$, we have a homotopy equivalence $X' \simeq (F \wedge \overline{B}) \vee \overline{B}$, which is again a suspension. Thus, rationally, X' is homotopy equivalent to a wedge of spheres. Assuming X' is rationally homotopy equivalent to a wedge containing more than one sphere of dimension greater than 1, this would imply the rational hyperbolicity of ΩX. Indeed, this is guaranteed if the ring $H^*(\overline{B};\mathbb{Q})$ has more that one generator of degree 2 or more, or if $H^*(\overline{B};\mathbb{Q})$ has one such generator and F is not rationally contractible. Since X' homotopy retracts off $\Omega L \ltimes X'$, by Theorem 6.7 we have that $\Omega X'$ retracts off ΩM. With the assumptions on X' above, this implies that ΩM is rationally hyperbolic.

As a final observation, note that a natural situation in which $\overline{B}$ has the homotopy type of a suspension would be when B is sufficiently highly connected: by [Gan70], if B is k-connected, $\overline{B}$ has the homotopy type of a suspension if $n \leq 3k+1$. For example, take B to be an $(n-1)$-connected $2n$-dimensional Poincaré Duality complex.

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