Data-driven simulation of continuous-time linear time-invariant systems: the autonomous case

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Abstract: We use orthogonal bases for continuous-time function spaces to state a version of Willems' lemma (Willems et al., 2005) for autonomous systems. We assume that an infinite number of representation coefficients of a "sufficiently rich" trajectory is given, and we illustrate how to compute solutions to initial and boundary conditions problems from such data.

Keywords: orthogonal basis, data-driven simulation, autonomous system, function approximation, fundamental lemma.

1. INTRODUCTION

The "fundamental lemma" (Theorem 1 p. 327 of (Willems et al., 2005)) is extensively used in discrete-time data-driven control, see e.g. (Berberich and Allgöwer, 2019; Berberich et al., 2020; Coulson et al., 2019; Persis and Tesi, 2020; van Waarde et al., 2022). More recently, also continuous-time problems have been investigated, see e.g. Bisoffi et al. (2022); Strässer et al. (2021); Berberich et al. (2021); Dai and Sznaier (2021). These approaches assume at least that state and the input variables are measured, and that either the state derivative is measurable, or that a numerical approximation of it is computed.

A continuous-time version of the fundamental lemma has been stated in Theorem 2 of (Lopez and Müller, 2022), where a parametrization of admissible continuous-time system trajectories is obtained solving a system of linear time-varying differential equations determined from input-state data. In this paper we adopt a different approach based on orthogonal bases for spaces of continuous-time functions square integrable on a finite interval. We focus on linear, time-invariant *autonomous* systems. We give a parametrization of *all* system trajectories based on matrices with a finite number of rows and an infinite number of columns, derived from the coefficients of the orthogonal basis representation of *one* "sufficiently rich" trajectory. Using such parametrization we solve boundary condition problems, a special case of the more general data-driven simulation problem considered in (Markovsky and Rapisarda, 2008) for discrete-time systems.

Our approach offers some advantages over the results of (Lopez and Müller, 2022). We do not need to measure also the state trajectory: we use only the *external* variables. Admittedly, in the present contribution we only consider the special case of systems without inputs; however, recent results (see (Rapisarda et al., 2023)) show that the MIMO case can be dealt with in a similar way. Moreover, the computation of system trajectories from data occurs via linear algebraic operations, not by the more involved method of numerically solving time-varying linear differential equations.

Orthogonal basis concepts (polynomial chaos expansions) have been used only in a couple of recent publications concerned with data-driven control problems for stochastic systems (see (Mühlpfordt et al., 2018; Pan et al., 2022)). With this contribution we hope to offer a glimpse of the potential that such concepts have also for the study of *deterministic* continuous-time systems.

Notation

We denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ respectively the set of natural, real and complex numbers, and by $\mathbb{R}[s]$ the ring of polynomials with real coefficients. $\mathbb{R}^n$, respectively $\mathbb{C}^n$, denote the space of n-dimensional vectors with real, respectively complex, entries. $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices with real entries; $\mathbb{R}^{n\times\infty}$ the set of real matrices with n rows and an infinite number of columns; and $\mathbb{R}^{\infty\times\infty}$ the set of real matrices with an infinite number of rows and columns. The transpose of a matrix M is denoted by $M^\top$, its complex conjugate transpose by M^*, and its pseudoinverse by $M^\dagger$.

Define $\mathbb{I} := (t_0, t_1)$. We denote by $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ the space of square-integrable real-valued functions defined on $\mathbb{I}$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$. The inner product on $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ defined by a *weight-function* w is denoted by $\langle f, g\rangle_w := \int_{\mathbb{I}} f(t)g(t)w(t)dt$. Notation and definitions extend to vector-valued functions.

Given an orthogonal basis $\{b_k\}_{k\in\mathbb{N}}$ for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$, we define $\mathfrak{b} := \begin{bmatrix} b_0 & b_1 & \dots \end{bmatrix}^\top$. Given a complete orthogonal basis $\{b_k\}_{k\in\mathbb{N}}$ and $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$, the *orthogonal basis representation* of f is $f = \sum_{k=0}^{\infty} \widetilde{f}_k b_k$, where $\widetilde{f}_k \in \mathbb{R}$, $k \in \mathbb{N}$. We call $\widetilde{f}_k$ the *k-th coefficient* of f (in the orthogonal basis representation). If $f = \sum_{k=0}^{\infty} \widetilde{f}_k b_k$, we define $\widetilde{f} := \begin{bmatrix} \widetilde{f}_0 & \widetilde{f}_1 & \dots \end{bmatrix} \in \mathbb{R}^{1\times\infty}$. The map associating $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ with $\widetilde{f}$ is denoted by Π, i.e. $\widetilde{f} = \Pi(f)$. If $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R}^n)$ and f_i is the i-th component of f, $i = 1, \dots, n$, then we denote by $\widetilde{f} \in \mathbb{R}^{n\times\infty}$

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the matrix whose i-th row is the orthogonal basis representation of f_i, $i = 1, \dots, n$. We call $\sum_{k=0}^{N} \widetilde{f}_k b_k$ the *truncation* or *projection to degree* N of the orthogonal representation of f. If $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ has a square-integrable derivative, the vector of coefficients of the orthogonal basis representation of $\frac{d}{dt} f$ is denoted by $\widetilde{f}^{(1)}$.

2. ORTHOGONAL BASES FOR FUNCTION SPACES

2.1 Basics

Let $\mathbb{I} = (t_0, t_1)$, with $t_0, t_1 \mathbb{R}$; an orthogonal basis for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ is defined by: a set of *basis elements* $b_k \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$, $k \in \mathbb{N}$; a *weight function* $w : \mathbb{I} \to \mathbb{R}$; an *inner product* on $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ defined by $\langle f, g \rangle_w := \int_{\mathbb{I}} f(t) g(t) w(t) dt$, such that $\langle b_j, b_k \rangle_w = \gamma_{jk} \delta_{j,k}$, $j, k \in \mathbb{N}$, where $\delta_{\cdot,\cdot}$ denotes the Kronecker delta, and $\gamma_{jk} > 0$, $j, k = 0, \dots$.

A basis $\{b_k\}_{k \in \mathbb{N}}$ is *complete* if its linear span is dense in $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$. For proofs of the following statements, see section 6 of (Sansone, 1959).

Theorem 1. *The following statements are equivalent:*

(1) $\{b_k\}_{k \in \mathbb{N}}$ *is complete;*
(2) If $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ *and* $\langle f, b_k \rangle_w = 0 \ \forall \ k \in \mathbb{N}$, *then* $f = 0$*;*
(3) If $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$, *there exist unique* $\widetilde{f}_k \in \mathbb{R}$, $k \in \mathbb{N}$, *such that the sequence* $\left\{ \sum_{k=0}^{N} \widetilde{f}_k b_k \right\}_{N \in \mathbb{N}}$ *converges in the mean to* f*; moreover,* $\widetilde{f}_k = \langle f, b_k \rangle_w$.

We denote by $\mathfrak{b}$ the infinite vector defined by

$$\mathfrak{b}(\cdot) := [b_0(\cdot) \ b_1(\cdot) \ \dots]^\top \ ; \tag{1}$$

given $f = \sum_{k=0}^{\infty} \widetilde{f}_k b_k \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$, we denote by $\widetilde{f}$ the infinite vector defined by

$$\widetilde{f} := [\widetilde{f}_0 \ \widetilde{f}_1 \ \dots] \ . \tag{2}$$

The equality $f = \sum_{k=0}^{\infty} \widetilde{f}_k b_k$ is equivalent with

$$f(t) = \widetilde{f} \mathfrak{b}(t) \ , \ t \in \mathbb{I} \ . \tag{3}$$

When dealing with vector functions, we denote by f_i, $i = 1, \dots, n$ the i-th component of $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R}^n)$. Let $f_i = \sum_{k=0}^{\infty} \widetilde{f}_{i,k} b_k$; we write

$$f = \underbrace{\begin{bmatrix} \widetilde{f}_{1,0} & \widetilde{f}_{1,1} & \dots \\ \vdots & \vdots & \dots \\ \widetilde{f}_{n,0} & \widetilde{f}_{n,1} & \dots \end{bmatrix}}_{=:\widetilde{f}} \mathfrak{b} \ . \tag{4}$$

Define the map $\Pi_{\mathfrak{b}} : \mathcal{L}_2(\mathbb{I}, \mathbb{R}) \to (\mathbb{R})^{\mathbb{N}}$ by

$$\Pi_{\mathfrak{b}}(f) := \left\{ \widetilde{f}_k \right\}_{k=0,\dots} \ ; \tag{5}$$

it follows from Theorem 23 in section 6 of (Sansone, 1959) that $\Pi_{\mathfrak{b}}$ is a bijective isometry between $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ and $\ell_2(\mathbb{N}, \mathbb{R})$. The definition of Π generalizes in a straightforward way to the vector case using (4).

2.2 Differentiation

Let $f = \sum_{k=0}^{\infty} \widetilde{f}_k b_k$; assume that f is differentiable and that $\frac{d}{dt} f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$. Because of completeness, the following equality holds $\forall \ t \in \mathbb{I}$:

$$\frac{d}{dt} f(t) = \sum_{k=0}^{\infty} \widetilde{f}_k \ \frac{d}{dt} b_k(t) \ . \tag{6}$$

In many cases, e.g. when using *polynomial* orthogonal bases such as Chebyshev or Legendre polynomials, $\frac{d}{dt} b_k$ can be written as linear combination of the basis elements: there exist $d_{k,j} \in \mathbb{R}$, $k, j \in \mathbb{N}$, such that

$$\frac{d}{dt} b_k(t) = \sum_{j=0}^{\infty} d_{k,j} b_j(t) \ . \tag{7}$$

A *matrix representation of differentiation* follows from (7). Let $t \in \mathbb{I}$; define

$$\frac{d}{dt} \mathfrak{b}(t)^\top := \left[\frac{d}{dt} b_0(t) \ \frac{d}{dt} b_1(t) \ \dots \right]^\top \ , \tag{8}$$

and define from (7) the infinite matrix

$$\mathcal{D}_{\mathfrak{b}} := [d_{k,j}]_{k,j \in \mathbb{N}} \ . \tag{9}$$

With these positions, (7) can be written as $\frac{d}{dt} \mathfrak{b}(t) = \mathcal{D}_{\mathfrak{b}} \mathfrak{b}(t)$, and (6) is equivalent with

$$\frac{d}{dt} f(t) = \widetilde{f} \frac{d}{dt} \mathfrak{b}(t) = \underbrace{\widetilde{f} \mathcal{D}_{\mathfrak{b}}}_{=:\widetilde{f}^{(1)}} \mathfrak{b}(t) \ . \tag{10}$$

The differentiation operator $\frac{d}{dt}$ on $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ induces an operator $\mathcal{D}_{\mathfrak{b}}$ defined by:

$$\begin{aligned} \mathcal{D}_{\mathfrak{b}} : \ell_2(\mathbb{N}, \mathbb{R}) &\to \ell_2(\mathbb{N}, \mathbb{R}) \\ \widetilde{f} &\to \widetilde{f} \mathcal{D}_{\mathfrak{b}} \ , \end{aligned} \tag{11}$$

i.e. the (orthogonal basis representation of the) derivative of a function is *directly* computed from the (orthogonal basis representation of the) function itself.

Example 1 (Chebyshev polynomials)**.** Let $\mathbb{I} = (-1, 1)$. The Chebyshev polynomials[1] are defined by (see Gil et al. (2007)): $C_0(t) := 1$, $C_1(t) := t$, and $C_{n+1}(t) = 2tC_n(t) - C_{n-1}(t)$, $n \geqslant 1$. They are orthogonal with respect to the inner product defined by $w(t) = \frac{1}{\sqrt{1-t^2}}$, and form a complete basis for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$. Define $\mathfrak{C}(t) := [C_0(t) \ C_1(t) \ \dots]^\top$. Using formula (2.4.22) p. 87 of (Canuto et al., 2006), it can be proved that

$$\mathcal{D}_{\mathfrak{C}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 4 & 0 & 0 & 0 & \dots \\ 3 & 0 & 6 & 0 & 0 & \dots \\ 0 & 8 & 0 & 8 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \ . \tag{12}$$

□

Remark 1 (Computation of coefficients)**.** In principle, computing $\widetilde{f}_k = \int_{\mathbb{I}} f(t) b_k(t) w(t) dt$ requires numerical integration. However, for some bases (most notably, the Chebyshev and Legendre ones), the orthogonal basis representation can be computed by *interpolating* f on an appropriate sampling grid (see (3.4) p. 14 of Trefethen (2019), and section 2.3 of Canuto et al. (2006), respectively)[2].

[1] Chebyshev polynomials can also be defined on (t_0, t_1), with some scaling involved.

[2] A description of the process used to compute the coefficients of the Chebyshev expansion up to machine precision is given in pp. 18-20 of Trefethen (2019).

For the Chebyshev basis, the coefficients can be efficiently computed using the FFT (see section 3.3 of Trefethen (2019)). Consequently, the Chebyshev and Legendre basis representations of trajectories can be be computed *directly* from *sampled* data. □

Remark 2 (Convergence of the series representation)**.** A general principle in orthogonal bases is that the smoother a function is, the faster the *approximation error* $f - \sum_{k=0}^{N} \tilde{f}_k b_k$ decays with N. In this contribution we consider autonomous systems only, whose trajectories are infinitely differentiable. Consequently, fast convergence rates can be expected. For the Chebyshev basis, for example, if $f \in \mathcal{C}^\infty$ then the approximation error for a truncation to N coefficients goes to zero faster than $O(N^{-k})$ for every finite k ("exponential convergence"), see p. 47 of (Canuto et al., 2006). It follows that the trajectories of the systems considered in this paper can be represented *up to machine precision* by truncated series involving a *relatively small* number of coefficients. □

3. SUFFICIENTLY INFORMATIVE TRAJECTORIES AND THEIR CHARACTERIZATION

Some of the concepts used in this section assume familiarity with the behavioral approach; we refer the interested reader to (Polderman and Willems, 1997; Rapisarda and Willems, 1997) as suitable introductions. In the following the symbol $\mathcal{N}(\mathfrak{B})$ denotes the set of annihilators of a linear differential behavior $\mathfrak{B} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, considered as a module on $\mathbb{R}^{1\times q}[s]$ or $\mathbb{R}^{1\times q}[\frac{d}{dt}]$, depending on the context. The notation $\ell(\mathfrak{B})$ denotes the *lag* of $\mathfrak{B}$; this is the highest order of differentiation in a shortest lag description of $\mathfrak{B}$, see (Willems, 1986). The symbol $n(\mathfrak{B})$ denotes the McMillan degree of $\mathfrak{B}$, i.e. the minimal dimension of a state-space description of $\mathfrak{B}$; for autonomous systems, $n(\mathfrak{B}) = \dim\ (\mathfrak{B})$. Given $p_i \in \mathbb{R}^{1\times q}[s]$, $i = 1, \ldots, N$, the notation $\langle p_1, \ldots, p_N\rangle$ denotes the module generated by the p_i, $i = 1, \ldots, N$.

Definition 1. *Let $\mathfrak{B}$ be an autonomous behavior with q-dimensional trajectories, and let $\mathbb{I} = (t_0, t_1)$, $t_0, t_1 \in \mathbb{R}$. A trajectory $w \in \mathfrak{B}$ is* informative for identification of $\mathfrak{B}$ *if for every $L \in \mathbb{N}$, $L \geqslant \ell(\mathfrak{B})$ it holds that*

$$\mathcal{N}(\mathfrak{B}) = \left\langle \left\{ \sum_{i=0}^{L} \eta_i s^i \in \mathbb{R}^{1\times q}[s] \mid \sum_{i=0}^{L} \eta_i \frac{d^i}{dt^i} w = 0 \text{ on } \mathbb{I} \right\} \right\rangle . \tag{13}$$

Note that in (13) the inclusion $\supseteq$ holds for every $w \in \mathfrak{B}$, since annihilators of every trajectory in $\mathfrak{B}$ also annihilate the particular trajectory $w \in \mathfrak{B}$. The opposite inclusion holds only if the trajectory w contains enough information to deduce a basis for the module of *all* annihilators of $\mathfrak{B}$.

If $\mathfrak{B}$ is a linear differential behavior, no finite-time escape to infinity occurs, and all its trajectories are square-integrable on $\mathbb{I}$. Consequently every $w \in \mathfrak{B}$ has a basis representation $\tilde{w}$. Define

$$\Pi_{\mathfrak{b}}(\mathfrak{B}) := \{\tilde{w} : \mathbb{N} \to \mathbb{R}^q \mid \exists\, w \in \mathfrak{B} \ \text{ such that } \tilde{w} = \Pi_{\mathfrak{b}}(w)\} ,$$

where $\Pi_{\mathfrak{b}}$ is the projection defined in (5). Since $\Pi_{\mathfrak{b}}$ is a bijective isometry, $\Pi_{\mathfrak{b}}(\mathfrak{B})$ is a subspace of $\ell_2(\mathbb{N}, \mathbb{R})$. If $\mathfrak{B}$ is autonomous, then its dimension is finite and it equals $n(\mathfrak{B})$, the minimal dimension of the state in any state-representation of $\mathfrak{B}$. It follows that $\Pi_{\mathfrak{b}}(\mathfrak{B})$ is also finite-dimensional, and that $\dim\ \Pi_{\mathfrak{b}}(\mathfrak{B}) = n(\mathfrak{B})$.

For simplicity of exposition, in the rest of the paper we consider the case of *scalar* ($q = 1$) trajectories only. The extension to the case $q > 1$ requires a more cumbersome notation and some adaptation to some of the arguments, but it is relatively straightforward.

In statement 4 of the next result we characterize $\Pi_{\mathfrak{b}}(\mathfrak{B})$ in terms of the row space of the following matrix constructed from the coefficients of an informative trajectory:

$$\widetilde{\mathcal{W}}_L := \begin{bmatrix} \tilde{w} \\ \tilde{w}\mathcal{D} \\ \vdots \\ \tilde{w}\mathcal{D}^L \end{bmatrix} \in \mathbb{R}^{(L+1)\times\infty} . \tag{14}$$

Theorem 2. *Let $\mathfrak{B}$ be an autonomous behavior with scalar trajectories, and let $w \in \mathfrak{B}$. Let $\mathfrak{b}$ be a complete orthogonal basis for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$, with differentiation matrix* (9) *denoted by $\mathcal{D}_{\mathfrak{b}}$. Define $\widetilde{\mathcal{W}}$ by* (14)*. Let $L \in \mathbb{N}$, $L \geqslant \ell(\mathfrak{B})$.*

The following statements are equivalent:

(1) w *is informative for identification;*
(2) $\mathcal{N}(\mathfrak{B})$ *equals* $\left\langle \left\{ \sum_{i=0}^{L} \eta_i s^i \in \mathbb{R}[s] \mid [\eta_0 \ \ldots \ \eta_L] \in \text{left ker } \widetilde{\mathcal{W}} \right\} \right\rangle$*;*
(3) rank $\widetilde{\mathcal{W}}_L = n(\mathfrak{B})$*;*
(4) row space $\widetilde{\mathcal{W}}_L = \Pi_{\mathfrak{b}}(\mathfrak{B})$.

Proof. Denote by left ker $\widetilde{\mathcal{W}}_L$ the set of left-annihilators of $\widetilde{\mathcal{W}}_L$ defined by (14); the equality

$$\text{left ker } \widetilde{\mathcal{W}}_L = \left\{ [\eta_0 \ \ldots \ \eta_L] \mid \sum_{i=0}^{L} \eta_i \frac{d^i}{dt^i} w = 0 \text{ on } \mathbb{I} \right\} , \tag{15}$$

follows from completeness of $\mathfrak{b}$ and from $w = \sum_{i=0}^{\infty} \tilde{w}_i b_i$. The equivalence of statements (1) and (2) follows from (15), from $L \geqslant \ell(\mathfrak{B})$, and the definition of informative trajectory, see (13).

To prove (2) $\Longrightarrow$ (3), let η be a lowest-degree annihilator of $\mathfrak{B}$; its degree equals $\ell(\mathfrak{B}) = n(\mathfrak{B})$, the dimension of $\mathfrak{B}$.

Write $\eta(s) = \sum_{j=0}^{\ell(\mathfrak{B})} \eta_j s^j$; then $\tilde{\eta} := \begin{bmatrix}\eta_0 \ \ldots \ \eta_{\ell(\mathfrak{B})} \ 0 \ \ldots \ 0\end{bmatrix} \in$ left ker $\widetilde{\mathcal{W}}_L$. Since $\frac{d^k}{dt^k}\eta\left(\frac{d}{dt}\right)$, $k \in \mathbb{N}$ also annihilates $\mathfrak{B}$, it follows that

$$\begin{aligned} \sigma\tilde{\eta} :=& \quad \begin{bmatrix}0 \ \eta_0 \ \eta_1 \ \ldots \ \eta_{\ell(\mathfrak{B})} \ 0 \ \ldots \ 0 \ 0\end{bmatrix} \\ \sigma^2\tilde{\eta} :=& \quad \begin{bmatrix}0 \ 0 \ \eta_0 \ \eta_1 \ \ldots \ \eta_{\ell(\mathfrak{B})} \ 0 \ \ldots \ 0\end{bmatrix} \\ \vdots& \\ \sigma^{L-\ell(\mathfrak{B})}\tilde{\eta} :=& \quad \begin{bmatrix}0 \ 0 \ 0 \ 0 \ \ldots \ 0 \ \eta_0 \ \ldots \ \eta_{\ell(\mathfrak{B})}\end{bmatrix} , \end{aligned} \tag{16}$$

also belong to the left kernel of $\widetilde{\mathcal{W}}_L$. The vectors $\sigma^j\tilde{\eta}$, $j = 0, \ldots, L - \ell(\mathfrak{B})$ are linearly independent. Standard behavioral theory can be invoked to prove the equality $\langle\eta(s)\rangle = \mathcal{N}(\mathfrak{B})$; it follows that $\tilde{\eta}$ and (16) generate left ker $\widetilde{\mathcal{W}}$. Consequently rank $\widetilde{\mathcal{W}}_L = (L+1) - (L - \ell(\mathfrak{B}) + 1) = \ell(\mathfrak{B}) = n(\mathfrak{B})$. The implication (2) $\Longrightarrow$ (3) is proved.

To prove the implication (3) $\Longrightarrow$ (2), let $\eta(s)$ be a minimal degree annihilator of $\mathfrak{B}$. Denote the coefficient vector

$$\widetilde{\eta} := \left[\eta_0 \ \dots \ \eta_{\ell(\mathfrak{B})} \ 0 \ \dots \ 0\right] \in \mathbb{R}^{1\times(L+1)} \ .$$

Evidently $\widetilde{\eta} \in$ left ker $\widetilde{\mathcal{W}}_L$. Now define the set of vectors (16), and use the assumption (3) to conclude that (16) is a basis of left ker $\widetilde{\mathcal{W}}_L$. Since $\mathcal{N}(\mathfrak{B}) = \langle \eta(s) \rangle$, the implication (3) $\Longrightarrow$ (2) is proved. We show the implication (3) $\Longrightarrow$ (4). Because of linearity of $\mathfrak{B}$, its closeness under differentiation, and because the basis $\mathfrak{b}$ is complete, any linear combination of the rows of $\widetilde{\mathcal{W}}$ is the image under $\Pi_{\mathfrak{b}}$ of a unique trajectory in $\mathfrak{B}$. Consequently row space $\widetilde{\mathcal{W}} \subseteq \Pi_{\mathfrak{b}}(\mathfrak{B})$. From (3) it follows that dim row space $\widetilde{\mathcal{W}} = n(\mathfrak{B})$. Since $\Pi_{\mathfrak{b}}$ is a bijection, $\dim(\mathfrak{B}) = \dim\ \Pi(\mathfrak{B})$; consequently $\dim\ \Pi(\mathfrak{B}) = n(\mathfrak{B})$, and the claim follows from row space $\widetilde{\mathcal{W}} \subseteq \Pi_{\mathfrak{b}}(\mathfrak{B})$. Finally, the implication (4) $\Longrightarrow$ (3) follows from the fact that $\Pi_{\mathfrak{b}}$ is a bijection, and $n(\mathfrak{B}) = \dim(\mathfrak{B}) = \dim\ \Pi(\mathfrak{B})$. □

In the next sections we show that using the characterization of $\Pi_{\mathfrak{b}}(\mathfrak{B})$ in Theorem 2 we can extend the fundamental lemma to continuous-time autonomous systems, giving data-driven solutions to initial conditions and boundary conditions problems.

4. THE DATA-DRIVEN SIMULATION PROBLEM WITH INITIAL CONDITIONS

In this section we solve the following problem.

Autonomous data-driven simulation problem with initial conditions

Let $\mathfrak{B}$ be an autonomous scalar linear differential system. Let $w \in \mathfrak{B}$ be informative for identification. Given $d_i \in \mathbb{R}$, $i = 0, \dots, d_{\ell(B)-1}$, compute a trajectory $w' \in \mathfrak{B}$ such that

$$\left(\frac{d^i}{dt^i} w'\right)(0) = d_i \ , \ i = 0, \dots, \ell(\mathfrak{B}) - 1 \ .$$

In order to state our solution to the initial condition problem, we first state how "initial conditions" are related to the basis representation of a trajectory.

Proposition 1. *Let $\mathfrak{b}$ be a complete orthogonal basis for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ with differentiation matrix (9). Let $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$, with basis representation $\widetilde{f}$. For every $i \in \mathbb{N}$ it holds that $\left(\frac{d^i}{dt^i} f\right)(0) = \left(\widetilde{f}\mathcal{D}^i\right)\mathfrak{b}(0)$.*

Proof. Follows from $\left(\frac{d^i}{dt^i} f\right)(t) = \left(\widetilde{f}\mathcal{D}^i\right)\mathfrak{b}(t)\ \forall\ t \in \mathbb{I}$. □

Statement 3 of Theorem 2 shows that an upper bound on the lag of an autonomous system can be computed from an informative trajectory; from the following result a procedure can be straightforwardly devised to solve the data-driven simulation problem with initial conditions.

Theorem 3. *Let $\mathfrak{b}$ be a complete orthogonal basis for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ with differentiation matrix (9). Let $w \in \mathfrak{B}$ be informative for identification, and define $\widetilde{\mathcal{W}}$ by (14). There exists $w' \in \mathfrak{B}$ such that $\frac{d^i}{dt^i} w'(0) = d_i$, $i = 0, \dots, L$ if and only if there exist $\alpha_i \in \mathbb{R}$, $i = 0, \dots, L$ such that*

$$d_i = \left(\left[\alpha_0 \ \dots \ \alpha_L\right]\widetilde{\mathcal{W}}\right)\mathcal{D}^i\mathfrak{b}(0) \ , \ i = 0, \dots, L \ . \qquad (17)$$

Moreover, if (17) is solvable, then any two solutions

$$\alpha = \left[\alpha_0 \ \dots \ \alpha_L\right] \quad \textit{and} \quad \alpha' = \left[\alpha'_0 \ \dots \ \alpha'_L\right]$$

differ by a vector in left ker $\left[\widetilde{\mathcal{W}}\mathfrak{b}(0) \ \cdots \ \widetilde{\mathcal{W}}\mathcal{D}^L\mathfrak{b}(0)\right]$.

Proof. Sufficiency follows defining $w' := \sum_{i=0}^{L} \alpha_i \frac{d^i}{dt^i} w$, using 4) of Theorem 2; and applying Proposition 1.

To prove necessity, use statement 4 of Theorem 2 to conclude that the basis representation of every trajectory $w' \in \mathfrak{B}$ belongs to row space $\widetilde{\mathcal{W}}$. Equivalently, there exist $\alpha_i \in \mathbb{R}$, $i = 0, \dots, L$ such that $\widetilde{w}' = \left[\alpha_0 \ \dots \ \alpha_L\right]\widetilde{\mathcal{W}}$. Now use Proposition 1 to conclude that this linear combination of the rows of $\widetilde{\mathcal{W}}$ must satisfy (17).

To prove the second part of the Theorem, observe that (17) is a system of linear equations in the indeterminates α_i, $i = 0, \dots, L$, with coefficient matrix $\left[\widetilde{\mathcal{W}}\mathfrak{b}(0) \ \cdots \ \widetilde{\mathcal{W}}\mathcal{D}^L\mathfrak{b}(0)\right]$. Any two solutions differ by an element of the left kernel of such coefficient matrix. □

In Theorem 3 we reduce the data-driven simulation problem to that of solving the system (17) of $L \geqslant \ell(\mathfrak{B})$ linear equations in the unknowns α_i, $i = 0, \dots, L$. Such system of equations involves the infinite matrix $\widetilde{\mathcal{W}} \in \mathbb{R}^{(L+1)\times\infty}$. Since the trajectories of an autonomous system are vector-exponential, the convergence rate of the approximation is extremely fast (see Remark 2) and consequently machine-precision accuracy can be achieved with a relatively small number of coefficients computable directly from trajectory samples (see Remark 1). Consequently, for practical purposes the sequence $\{\widetilde{w}_k\}_{k\in\mathbb{N}}$ has finite support, the matrix $\widetilde{\mathcal{W}}$ has a finite number of columns, and the computation of the derivative using a truncated differentiation matrix yields an excellent approximation. The next example exemplifies some of these practical issues and our solutions.

Example 2. We use the Chebyshev basis to solve an initial condition problem for the system $\mathfrak{B} = \ker\ (\frac{d}{dt} + 1)(\frac{d}{dt} - 2)$, and the data $w(t) = e^{-2t} + e^t$. We use $N = 19$ samples of w on the Chebyshev grid $t_j := \cos\left(\frac{j\pi}{N}\right)$, $j = 0, \dots, N$. Using `chebfun` (see (Trefethen, 2019)) we compute from these samples the first 19 coefficients of the Chebyshev basis representation of w. These are sufficient to give a machine-precision approximation of w (see footnote 2 on the process of selection of N and computation of the coefficients).

We compute the 19×19 $(1,1)$-block of the differentiation matrix from (12); we denote such matrix by $\mathcal{D}'$. We denote the vector of the first 19 coefficients of w by $\widetilde{w}'$. We obtain an approximation of the vector of coefficients of the derivative of w by multiplying $\widetilde{w}'$ by the i-th power of $\mathcal{D}'$, instead of using the infinite product $\widetilde{w}\mathcal{D}^i$. Since the coefficients of the Chebyshev representation of w are numerically zero from the 20th on, the error incurred in approximating the i-th derivative of w with the truncated product $\widetilde{w}'\mathcal{D}'^i$ is negligible.

We computed an informative w ensuring that both natural frequencies of the system are excited by the initial conditions; note that this is generically true for a random choice of the initial conditions. The dimension of the state space can be computed using the normalized singular values of $\widetilde{\mathcal{W}}'_L := \begin{bmatrix} \widetilde{w}' \\ \vdots \\ \widetilde{w}'\mathcal{D}'^L \end{bmatrix} \in \mathbb{R}^{(L+1)\times 19}$ for successive values of L, as shown in Table 1. The numerical rank of $\widetilde{\mathcal{W}}'_L$ equals 2 for $L \geqslant 1$, see statement 2 of Theorem 2.

Table 1. Singular values of $\widetilde{\mathcal{W}}'_L$ for Example 2

L	Normalized SVDs
0	1
1	1, $2.4306 \cdot 10^{-1}$
2	1, $1.4437 \cdot 10^{-1}$, $4.0508 \cdot 10^{-14}$
3	1, $8.5684 \cdot 10^{-2}$, $1.2201 \cdot 10^{-12}$, $1.5907 \cdot 10^{-14}$
4	1, $4.8094 \cdot 10^{-2}$, $2.7146 \cdot 10^{-11}$, $6.0841 \cdot 10^{-13}$, $4.4475 \cdot 10^{-15}$

Assume that we want to compute from w a trajectory $w' \in \mathfrak{B}$ such that $w'(0) = 1$, $\frac{d}{dt}w'(0) = 3$. The solution of such initial conditions problem can be computed with pen and paper solving for α and β in the equations $\alpha e^{-2\cdot 0} + \beta e^0 = 1$, $-2\alpha e^{-2\cdot 0} + \beta e^0 = 3$, obtained from the parametrization $\alpha e^{-2t} + \beta e^t$ of trajectories in $\mathfrak{B}$. The resulting trajectory is $w'(t) = -\frac{2}{3}e^{-2t} + \frac{5}{3}e^t$. We use Theorem 3 to compute an approximation of w'; denoting $\mathfrak{b}' := \begin{bmatrix}\mathfrak{b}_0 & \ldots & \mathfrak{b}_{18}\end{bmatrix}^\top$, we first compute

$$\begin{bmatrix}\widetilde{\mathcal{W}}'_3\mathfrak{b}'(0) & \widetilde{\mathcal{W}}'_3\mathcal{D}\mathfrak{b}'(0)\end{bmatrix} = \begin{bmatrix} 2.0000 & -1.0000 \\ -1.0000 & 5.0000 \\ 5.0000 & -7.0000 \\ -7.0000 & 17.0000 \end{bmatrix},$$

and then we solve the equations in $\alpha := [\alpha_0 \ \ldots \ \alpha_3]$:

$$\alpha \begin{bmatrix} 2.0000 & -1.0000 \\ -1.0000 & 5.0000 \\ 5.0000 & -7.0000 \\ -7.0000 & 17.0000 \end{bmatrix} = [1 \ 3] \ .$$

The least-squares solution of these equations is $\alpha_0 = 0.4571$, $\alpha_1 = 0.4032$, $\alpha_2 = 0.5111$, $\alpha_3 = 0.2952$. The Chebyshev coefficients of the approximate solution of the initial conditions problem are associated with the vector $\alpha_0\widetilde{w}' + \ldots + \alpha_3\widetilde{w}'\mathcal{D}'^3$. Such coefficients and those of the exact solution $-\frac{2}{3}e^{-2t} + \frac{5}{3}e^t$ are compared in Table 2.

5. THE DATA-DRIVEN SIMULATION PROBLEM WITH BOUNDARY CONDITIONS

Using the parametrization of system trajectories in terms of their orthogonal basis expansion provided in Theorem 2, we solve the following problem.

Autonomous data-driven simulation problem with boundary conditions

Let $\mathfrak{B}$ be an autonomous scalar linear differential system. Let $w \in \mathfrak{B}$ be informative for identification.

Given $t_0, t_1 \in \mathbb{I}$ and $d_0, d_1 \in \mathbb{R}$, compute $w' \in \mathfrak{B}$ such that $w'(t_0) = d_0$, $w'(t_1) = d_1$.

Table 2. Chebyshev coefficients for Example 2

Exact solution	$\widetilde{w}'$
$5.9039 \cdot 10^{-1}$	$5.9039 \cdot 10^{-1}$
4.0047	4.0047
$-4.6611 \cdot 10^{-1}$	$-4.6611 \cdot 10^{-1}$
$3.5755 \cdot 10^{-1}$	$3.5755 \cdot 10^{-1}$
$-5.8514 \cdot 10^{-2}$	$-5.8514 \cdot 10^{-2}$
$1.4006 \cdot 10^{-2}$	$1.4006 \cdot 10^{-2}$
$-2.0586 \cdot 10^{-3}$	$-2.0586 \cdot 10^{-3}$
$3.0485 \cdot 10^{-4}$	$3.0485 \cdot 10^{-4}$
$-3.6600 \cdot 10^{-5}$	$-3.6600 \cdot 10^{-5}$
$4.0773 \cdot 10^{-6}$	$4.0773 \cdot 10^{-6}$
$-4.0134 \cdot 10^{-7}$	$-4.0134 \cdot 10^{-7}$
$3.6338 \cdot 10^{-8}$	$3.6339 \cdot 10^{-8}$
$-3.0038 \cdot 10^{-9}$	$-2.9984 \cdot 10^{-9}$
$2.3000 \cdot 10^{-10}$	$2.3143 \cdot 10^{-10}$
$-1.6344 \cdot 10^{-11}$	$-1.4037 \cdot 10^{-11}$
$1.0852 \cdot 10^{-12}$	$2.0116 \cdot 10^{-12}$
$-6.7502 \cdot 10^{-14}$	$2.1893 \cdot 10^{-13}$
$3.8488 \cdot 10^{-15}$	$3.2022 \cdot 10^{-15}$
$-2.6830 \cdot 10^{-16}$	$1.8398 \cdot 10^{-16}$

The following is a characterization of all solutions to this problem.

Theorem 4. *Let $\mathfrak{b}$ be a complete orthogonal basis for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ with differentiation matrix (9). Let $w \in \mathfrak{B}$ be informative for identification, and define $\widetilde{\mathcal{W}}$ by (14).*

There exists $w' \in \mathfrak{B}$ such that $w'(t_0) = d_0$ and $w'(t_1) = d_1$ if and only if there exist $\alpha_i \in \mathbb{R}$, $i = 0, \ldots, L$ such that

$$[d_0 \ d_1] = [\alpha_0 \ \ldots \ \alpha_L]\,\widetilde{\mathcal{W}}\,[\mathfrak{b}(t_0) \ \mathfrak{b}(t_1)] \ . \qquad (18)$$

Moreover, if (18) is solvable, then any two solutions

$$\alpha = [\alpha_0 \ \ldots \ \alpha_L] \quad \text{and} \quad \alpha' = [\alpha'_0 \ \ldots \ \alpha'_L]$$

differ by a vector in left ker $\widetilde{\mathcal{W}}$*.*

Proof. Sufficiency of the condition follows defining $w' := \sum_{i=0}^L \alpha_i \frac{d^i}{dt^i} w$, using statement 4 of Theorem 2, and the equality $f(t) = \widetilde{f}\mathfrak{b}(t)$ for all $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ and all $t \in \mathbb{I}$.

To prove necessity, use statement 4 of Theorem 2 to conclude that the basis representation of every trajectory $w' \in \mathfrak{B}$ belongs to row space $\widetilde{\mathcal{W}}$. Equivalently, there exist $\alpha_i \in \mathbb{R}$, $i = 0, \ldots, L$ such that $\widetilde{w}' = [\alpha_0 \ \ldots \ \alpha_L]\,\widetilde{\mathcal{W}}$. Now use the equality $f(t) = \widetilde{f}\mathfrak{b}(t)$ to conclude that this linear combination of the rows of $\widetilde{\mathcal{W}}$ must satisfy (17).

To prove the second part of the Theorem, observe that (18) is a system of two linear equations in the indeterminates α_i, $i = 0, \ldots, L$, with coefficient matrix $\widetilde{\mathcal{W}}\,[\mathfrak{b}(t_0) \ \mathfrak{b}(t_1)] \in \mathbb{R}^{(L+1)\times 2}$. Any two solutions differ by an element of the left kernel of such coefficient matrix. ☐

Example 3. We use the same data of Example 2 and solve the boundary condition problem of computing a trajectory $w' \in \mathfrak{B}$ such that $w'(-1) = 1$ and $w'(1) = 2$. Using the parametrization $w'(t) = \alpha e^{-2t} + \beta e^t$ of all trajectories of $\mathfrak{B}$, it can be checked that the desired trajectory is $w'(t) = \frac{e^4 - 2e^2}{e^6 - 1}e^{-2t} + \frac{2e^5 - e}{e^6 - 1}e^t$. We now use the approximate parametrization of all solutions provided by the row space of $\widetilde{\mathcal{W}}'$. We first compute $\mathfrak{b}'(-1)$ and $\mathfrak{b}'(1)$; the solution of the boundary condition problem $w'(-1) = d_{-1}$, $w'(1) = d_1$, are parametrized by the system of equations

$$[\alpha_0 \; \dots \; \alpha_3]\, \widetilde{\mathcal{W}}' \left[\mathfrak{b}'(-1) \;\; \mathfrak{b}'(1)\right]$$
$$= [\alpha_0 \; \dots \; \alpha_3] \begin{bmatrix} 7.7569 & 2.8536 \\ -14.4102 & 2.4476 \\ 29.9241 & 3.2596 \\ -58.7446 & 1.6356 \end{bmatrix} = [d_{-1} \;\; d_1] \; .$$

We solve these equations in the least-squares sense with $d_{-1} = 1$, $d_1 = 2$ and obtain

$$[\alpha_0 \; \dots \; \alpha_3] = [0.2116 \; 0.1731 \; 0.2502 \; 0.0959] \; ,$$

from which the trajectory $\sum_{i=0}^{3} \alpha_i \widetilde{w}' \mathcal{D}'^i \mathfrak{b}'(\cdot)$ is obtained.

6. CONCLUSIONS AND FURTHER WORK

Using orthogonal bases representations of square-integrable functions, in Theorem 2 we parametrized all (vectors of coefficients of) trajectories of scalar autonomous systems in terms of linear combinations of the (vector of coefficients of the) derivatives of a "sufficiently rich" one. We used such parametrization to solve initial- and boundary-values data-driven simulation problems in sections 4 and 5.

The results presented in this contribution are part of a broader research effort aimed at evaluating the potential of orthogonal basis functions in continuous-time data-driven control. The following are two of the current lines of investigation:

- Using an analogous approach to the one illustrated here, we have recently solved the problem of parametrizing restrictions of MIMO trajectories in terms of linear combinations of an informative one and its derivatives, see (Rapisarda et al., 2023);
- The parametrization obtained in (Rapisarda et al., 2023) for MIMO systems is also being used to solve data-driven optimal control problems, and it is being applied to develop a data-driven alternative to classical approaches to continuous-time Iterative Learning Control problems.

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