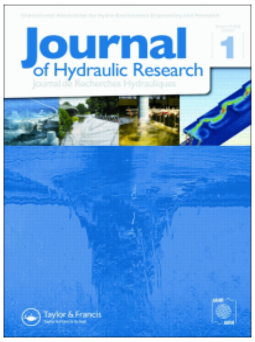

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Educational paper

Two simple theoretical models for teaching wave mechanics in coastal engineering

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ABSTRACT
Waves are an integral component of teaching in coastal engineering. Some aspects of wave theory are however complex and outside the scope of e.g. introductory courses, so that only the results of the theory are used. For other wave effects such as overtopping, no theory exists, and purely empirical formulas are employed. This limits the students' understanding of the problems. At Southampton University, we developed simple models for wave effects to improve the teaching. The models rely on basic hydraulic engineering principles such as continuity, conservation of energy and momentum, with the condition that the results are reasonably close to those from more complex theories or from experiments. In this article, two such models for the propagation speed of a solitary wave, and for the shallow water breaking criterion, will be presented. The results from both models are surprisingly close to the textbook formulas or values.

Keywords: Coastal engineering; coastal hydraulics; hydraulic education; solitary waves; wave breaking

1 Introduction

In the teaching of coastal engineering, waves and wave related effects are of course of prime importance. However, in many cases, such as solitary waves or wave breaking, the theory is complex and outside the scope of e.g. general or introductory courses. In addition, there are several important areas where no theory exists and where we rely on empirical equations, e.g. in wave overtopping. This makes the student's understanding of the basic phenomena, and of the influence of the parameters involved, more difficult. To address this problem, the author has developed simple theoretical models which describe these effects. These models rely on basic hydraulic engineering theory such as continuity, conservation of energy and momentum with which the students are familiar. The basic conditions for the models are that they:

(1) are coherent and conform with the principles of fluid mechanics; and
(2) have results which show a reasonable to good agreement with theoretical models or, in case of empirical relationships, with experimental results (Müller, 2022).

In the following, two such models will be described.

2 Speed of propagation of a solitary wave

Solitary waves are waves which consist of a crest only. They are used in coastal engineering to analyse shallow water effects such as wave breaking, or the propagation of tsunami waves. Their theory is however rather complex (see the overview in Daily & Stephan, 1952). For engineering purposes, the speed of propagation v_0 of such a wave is of course of great interest since it determines the effective loadings generated by such a wave. Currently, we simply use a formula given in the literature where the velocity v_0 is a function of the water depth d and the wave height H. This widely used formula is the result of a fairly complex mathematical model based on potential flow theory (Boussinesq, 1872):

$$v_0 = \sqrt{g(d + H)} \tag{1}$$

This speed can however also be determined by analysing the energy flux. Figure 1 shows a wave with a control volume/section at the crest. Here, the wave contains potential and kinetic energy. In the illustrative explanation for students (Fig. 1a), the potential energy induces the collapse of the wave whilst the kinetic energy balances this tendency. Both must be equal for the wave to be stable. Since the wave is moving, we need to consider the power. Figure 1b shows the potential and

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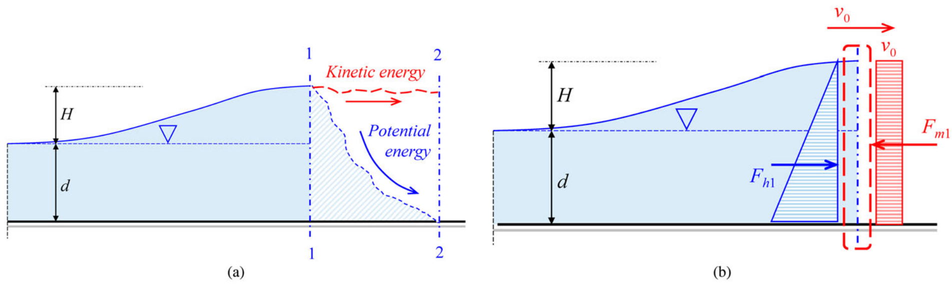

Figure 1 Breaking wave: (a) propagating wave; (b) energy balance with moving reference frame

the kinetic power in a reference frame which moves with the wave speed. For the following analysis, a strip with a width of 1 m is assumed. With a propagation speed v_0, the kinetic power P_{kin} is given as:

$$P_{kin} = \rho(d+H)\frac{v_0^3}{2} \tag{2}$$

The hydrostatic force F_{h1} acts in the positive direction since it has to counteract the momentum force F_{m1}. Power equals force times velocity, so the potential power P_{pot} becomes:

$$P_{pot} = v_0\rho g\frac{(d+H)^2}{2} \tag{3}$$

The power balance can be written as:

$$\rho(d+H)\frac{v_0^3}{2} = \rho g\frac{(d+H)^2}{2}v_0 \tag{4}$$

This leads to:

$$(d+H)v_0^2 = g(d+H)^2 \tag{5}$$

And:

$$v_0 = \sqrt{g(d+H)} \tag{6}$$

Equation (6) is identical to the solution for the speed of propagation of a solitary wave given by Boussinesq (1872), i.e. Eq. (1), and solutions given by Raleigh, Stakes and Stoker (Daily & Stephan, 1952). Equation (6) slightly overpredicts experimental results reported by Daily and Stephan by 2.7% for $H/d = 0.65$, and the solution can therefore be regarded as reasonably accurate.

3 Wave breaking in shallow water

3.1 *Overview*

Wave breaking is an important phenomenon in the coastal zone; it limits the wave height and dissipates energy. Depth induced wave breaking involves wave steepening, where the particle motion becomes predominantly horizontal, and the subsequent overturning of the crest. When hitting a seawall, breaking waves can create so-called impulsive overtopping and extremely high impact pressures. There are several theories for the mechanism of wave breaking reported in the literature, the most common ones being the stability limit of a solitary wave, and the assumption that the particle velocity at the wave crest reaches and exceeds the wave speed itself.

3.2 *Textbook models*

Solitary wave theory breaking criterion

The textbook breaker criterion states that waves break if the ratio of wave height and water depth $H\ /\ d$ reaches 0.78. This is based on McCowan (1894), who developed a potential flow theory for solitary waves. For increasing values of $H\ /\ d$, the solitary wave forms a discontinuity – a cusp – at the crest. When $H\ /\ d = 0.78$, the curvature radius r at the cusp becomes infinite, indicating a point of inflexion, so that a further increase in height is not possible (Fig. 2). This is considered as the breaking limit. In a recent review of wave breaking, new theoretical work was described which led to a more accurate value of $H\ /\ d = 0.82$ (Robertson et al., 2013).

Particle velocity assumption

The most common assumption made in textbooks for wave breaking is that the onset of breaking occurs at the point where the maximum horizontal particle velocity reaches the speed of the wave (e.g. Wood & Fleming, 1981; Sorensen, 2006). This occurs under the following preconditions:

(1) As the water depth reduces, the celerity of the wave becomes a function of the water depth only.

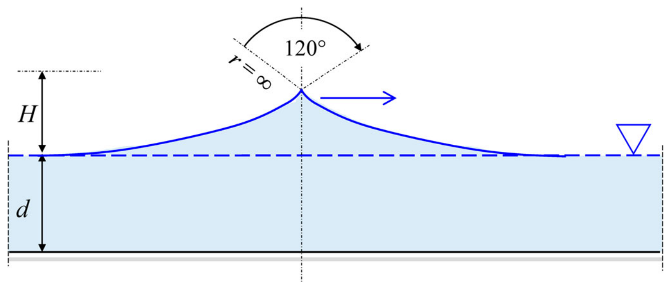

Figure 2 Breaking point of a solitary wave, $H/d = 0.78$ (McCowan, 1894)

(2) As the water depth reduces, the wave height increases due to shoaling. The particle velocity is a function of the wave height and therefore also increases.
(3) Wave breaking starts when the particle velocity of the crest reaches the speed of propagation of the wave, the wave crest then starts to overtake the wave.

Figure 3a shows the deep-water situation, Fig. 3b the increase in maximum particle velocities when entering shallow water. These assumptions can be tested quite easily using linear wave theory. The maximum horizontal particle velocity $u_{\max}$ is given as:

$$u_{max} = \frac{H}{2}\sqrt{\frac{g}{d}} \tag{7}$$

The local wave celerity at the crest v_{cr} can be estimated as:

$$v_{cr} = \sqrt{g(d + H/2)} \tag{8}$$

This takes the effect of the wave crest in shallow water into account. Both must be equal:

$$u_{max} = v_{cr};\ \frac{H}{2}\sqrt{\frac{g}{d}} = \sqrt{g(d + H/2)} \tag{9}$$

This gives a quadratic equation for H for a constant value d:

$$\frac{H^2}{4d} - \frac{H}{2} - d = 0 \tag{10}$$

The equation has two roots, $H_1 = 3.236d$ and $H_2 = -1.235d$. The first solution is clearly not applicable, whilst the second implies that it is not the particle in the crest which overtakes the wave, but the particle in the trough which comes to a standstill so that the crest must overtake it. A change in sign for the $H\ /\ 2$ term in the r.h.s. of Eq. (9) gives a positive result, verifying this statement. The $H\ /\ d$ ratio from this assumption is however unrealistically high. An increase in trough depth ratio would result in lower ratios of $H\ /\ d$. However, such a wave would contradict the observed characteristics of shallow water waves where the crest becomes shorter and higher whilst the trough becomes longer and shallower when the wave travels into shallow water.

3.3 *Momentum model*

Momentum balance and moving reference frame

Both assumptions for wave breaking as presented in the previous sections rely on very different criteria. The results do not agree well, and the solitary wave analysis is mathematically very complex. Therefore, a momentum analysis which assumes wave breaking to constitute a stability problem was developed to determine a critical $H\ /\ d$ ratio.

In a breaking wave approaching a vertical wall, the orbital motion of the particles is replaced by a predominantly horizontal motion (e.g. Hull & Müller, 2002). For the analysis, we therefore assume the breaking wave to consist of a body of water of height $d + H$, which moves horizontally with a velocity v_0, in a water depth d (Fig. 4a). The forward velocity can be determined with Eq. (6). For an analysis of the stability of the wave, we consider the momentum balance with a moving reference frame as shown in Fig. 4b.

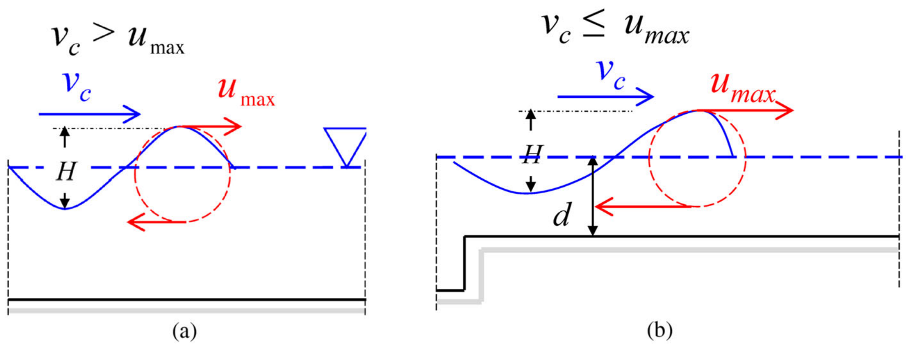

Figure 3 Wave and maximum horizontal particle velocities: (a) deep water, (b) shallow water

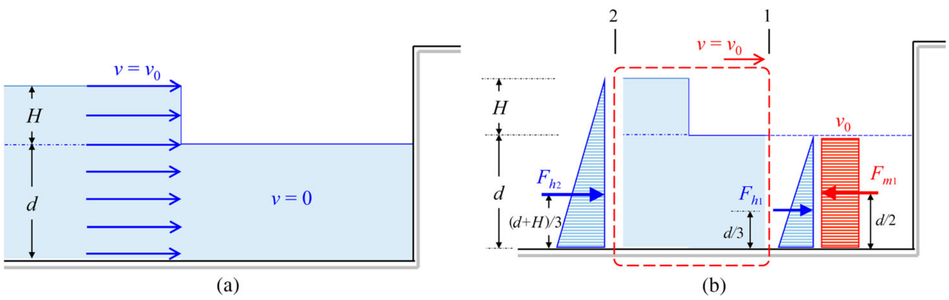

Figure 4 Breaking wave: (a) simplified velocity distribution; (b) momentum balance with moving reference frame

Stability of a rigid block

A closer look at Fig. 4b shows that the lines of action (l.o.a.) of the different forces / the distances between the l.o.a. and the bed are not the same. This implies that the effect of the different forces on the stability or the overturning of the wave will also be different. We can now borrow a concept from the mechanics of solid bodies, the stability of a rigid block subjected to a horizontal load against overturning (Fig. 5) to analyse the problem. A block with a base width b and a weight W is subjected to a horizontal force F_H, which acts at a vertical distance e above ground level. The block is assumed to rotate over point "1".

The stabilizing moment M_s is:

$$M_s = W\frac{b}{2} \tag{11}$$

And the overturning moment M_{OT} becomes:

$$M_{OT} = F_H e \tag{12}$$

As long as $M_s > M_{OT}$, the block remains stable. Once $M_s = M_{OT}$, the block is in an unstable equilibrium; if $M_s < M_{OT}$, then the block overturns.

Wave stability and breaking

The velocity of the wave increases with increasing wave height (Eq. 6). The forces in Fig. 4b must balance. However, the lines of action of these forces have different distance to the bed, which is considered as the reference line. Subsequently, we can consider this as a stability problem similar to Fig. 5 and determine a wave height for which the sum of the moments equals zero. This can then be considered as the stability limit. Applying this concept to the momentum forces in Fig. 4b, we take the moments over the seabed for the critical condition. Here, the momentum force F_{m1} causes the stabilizing, the hydrostatic forces F_{h1} and F_{h2} the overturning moment. Eq. (10) gives the moment balance:

$$\rho d v_0^2 \frac{d}{2} - \rho g \frac{d^2}{2}\frac{d}{3} - \rho g \frac{(d+H)^2}{2}\left(\frac{d+H}{3}\right) = 0 \tag{13}$$

Simplifying, and substituting with Eq. (6):

$$d^2(d+H) - \frac{(d+H)^3}{3} - \frac{d^3}{3} = 0 \tag{14}$$

This cubic equation can be solved numerically; it has a solution for $H = 0.534\, d$. For larger values of H, the hydrostatic moment created by F_{h2}, which is the overturning factor, becomes dominant and the wave loses its stability.

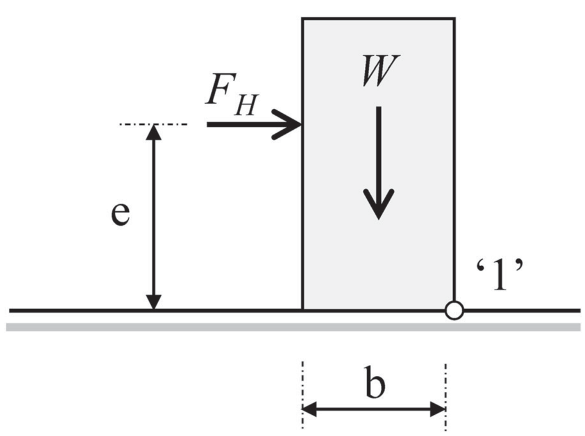

Figure 5 Stability of a block against overturning

3.4 *Analysis and interpretation*

The common assumptions of particle velocities reaching and exceeding the wave speed give results which are too high to be realistic. The solitary wave criterion is based on a mathematical limit for the wave height and does not include the steepening of the wave front. Experiments reported by Hull and Müller (2002) showed that breakers occurred from a value of $H / d = 0.58$ to 0.74, and results from experiments mentioned by Sorensen (2006) give breaking ratios of $H / d = 0.62$ to 0.74, well below the ratio of $H / d = 0.78$ which is derived from solitary wave theory. The stability analysis described in Section 3.3 interprets wave breaking as a stability problem, which is nearly independent of the local particle velocity. It introduces a new concept, the moment balance, in addition to the equilibrium of forces. The analysis results in a breaking wave height to depth ratio of $H / d = 0.534$. This is below the limit given in Sorensen (2006) but still a value which can be regarded as approximately realistic.

3.5 *Extended theory including particle velocity*

The next question to the students that comes after this result is whether or not the momentum analysis is absolutely correct. The answer is of course "no" – the velocity formula does not take the effect of the particle velocity in the solitary wave into account. The velocity determined from Eq. (6) therefore slightly overpredicts the wave speed since the particle motion is not taken into account (Daily & Stephan, 1952). This velocity is small compared with the propagation velocity, but near the breaking point it does have an effect, slowing the wave down. It can be quantified from the force analysis of the momentum balance shown in Fig 4b: the forces do not balance completely, and an additional momentum force has to be introduced to create force equilibrium.

The effect of the particle velocity u_0 can be incorporated by substituting the speed of propagation v_0 from Eq. (6) into the momentum balance of the breaking, Eq. (13), and adding a momentum force F_{m2} (Fig. 6). F_{m2}, is the force generated by the particle velocity u_0. For simplicity, the particle velocity is

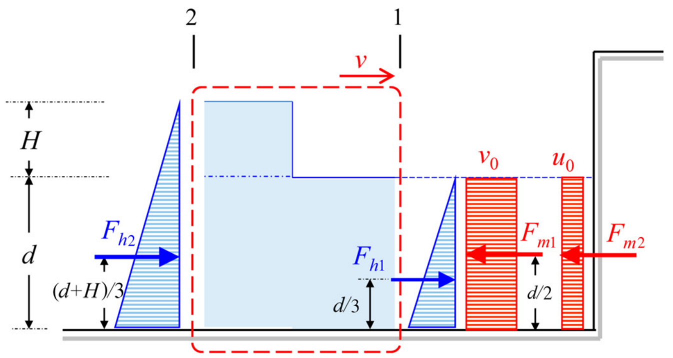

Figure 6 Momentum balance with momentum force from particle velocity

here assumed to be constant over the depth. The control volume is now moving with the actual velocity $v = v_0 - u_0$, not the velocity determined from the simplified approach in Eq. (4):

$$\rho d v_0^2 - \rho g \frac{d^2}{2} - \rho g \frac{(d+H)^2}{2} = 0? \tag{15}$$

Subsequently, a force imbalance exists which needs to be addressed. Substituting with Eq. (6):

$$\rho dg(d+H) - \rho g \frac{d^2}{2} - \rho g \frac{(d+H)^2}{2} = -\rho g \frac{H^2}{2} \neq 0 \, for \, H > 0 \tag{16}$$

The particle velocity u_0 therefore creates an additional momentum force F_{m2}:

$$F_{m2} = -\rho g \frac{H^2}{2} \tag{17}$$

The moment over the bed can then be written as:

$$\rho dg(d+H)\frac{d}{2} - \rho g \frac{d^2}{2}\frac{d}{3} - \rho g \frac{(d+H)^2}{2}\left(\frac{d+H}{3}\right) + \rho g \frac{H^2}{2}\left(\frac{d}{2}\right) = 0 \tag{18}$$

The additional momentum forces F_{m2} is positive here since it acts in the opposite direction to F_{m1}. Equation (18) reduces to:

$$\frac{d^3 + Hd^2}{2} - \frac{d^3}{6} - \frac{(d+H)^3}{6} + \frac{H^2 d}{4} = 0 \tag{19}$$

which has a solution for $H \, / \, d = 0.678$. This corresponds surprisingly well with the experimental results mentioned in Hull and Müller (2002) and Sorensen (2006). The actual velocity of the wave v can then be determined by summing up the two contributions:

$$\rho d v^2 = \rho dg(d+H) - \rho g \frac{d^2}{2} \rightarrow v = \sqrt{d + H - H^2/2d} \tag{20}$$

Equation (20) slightly underpredicts the experimental results given in Daily and Stephan (1952) by 4.2% for $H/d = 0.678$. Interestingly, it is very close to the velocities given by McCowan (1894). The discrepancy between the experimental results and the velocity from the extended model is probably caused by the fact that the particle velocity distribution in a solitary wave is not constant as assumed in the model, but variable with a maximum at the crest and a minimum near the bed.

4 Discussion and conclusions

The models described previously show how some complex fluid mechanics problems can be approximated using basic principles. Simple linear wave theory allowed the assumption to be tested that wave breaking begins when the particle velocity of the wave reaches the wave speed; this led to a different interpretation and a quantification of that assumption.

The comparison of the different models for wave breaking also introduces a more critical view of commonly accepted concepts. The subsequent analysis of the moments in a momentum analysis introduced the concept of stability against overturning, borrowed from structural analysis. Two models for wave breaking were developed. Both relied on a new concept, namely stability of a wave. The first simpler model gave a wave height to depth ratio of $H/d = 0.534$ for the onset of breaking, The second more complex model, which incorporates the effect of the particle velocities in a solitary wave, gave a ratio of $H/d = 0.678$. Theoretical work suggested ratios of $H/d = 0.78$ to 0.82 whilst experiments gave ratios of 0.58 to 0.74. The results from in particular the second model are therefore realistic.

The advantage here is, that students can relate to this interpretation of the problem and identify just why wave height is so important. Ideally, the students will learn a method to develop models for the analysis of complex and unfamiliar problems for which there are not textbook solutions available.

The analysis so far was limited to specific areas of coastal engineering. The results can however possibly be applied in other areas of hydraulic engineering as well. The energy flux analysis of the speed of propagation could possibly be used to determine the minimum velocity of the front face of a propagating mass of water on a horizontal plane, such as a flood wave. The moment analysis as described in Section 3.3 could be employed to analyse the stability of a hydraulic jump.

It appears that simple models like those presented in this article are quite suitable for teaching purposes since they employ standard approaches. They can be used in undergraduate classes and they provide some insight into the mechanism without having to use complex mathematics. The author considers the critical analysis and development of new models as important in teaching since it encourages students to question even textbook statements if they are not backed up with theory.

From the analysis described in this manuscript, the following conclusions can be drawn:

- The speed of propagation derived from a very simple model is very close to theoretical solutions for a solitary wave reported in the literature.
- Current concepts for the onset of breaking rely on very different assumptions and lead to varying results.
- The analysis of the moment equilibrium in a momentum balance of a solitary wave gave a critical wave height to water depth ratio of $H/d = 0.678$; this is reasonably close to other theoretical and experimental work.
- The solutions allow determination of these results without having to employ complex models.

The momentum models employ principles with which the students are familiar and can be employed for teaching purposes. This allows students to derive and understand solutions for complex problems with simpler means.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notation

b = width of block (m)
d = water depth (m)
e = lever arm (m)
F_h = hydrostatic force (N m^{-1})
F_H = horizontal force (N)
F_m = momentum force (N m^{-1})
g = acceleration of gravity (m s^{-2})
H = wave height (m)
M_{OT} = overturning moment (Nm)
M_s = stabilising moment (Nm)
u_0 = horizontal particle velocity of solitary wave (m s^{-1})
u_{max} = maximum horizontal particle velocity (m s^{-1})
v = solitary wave speed including particle velocity effect (m s^{-1})
v_0 = solitary wave speed (m s^{-1})
v_c = wave celerity (m s^{-1})
v_{cr} = wave crest speed (m s^{-1})
W = weight of block (N)
ρ = density of water (kg m^{-3})

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References

Boussinesq, J. (1872). Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *Journal de mathématiques pures et appliquées*, *17*, 55–108. http://www.numdam.org/item/JMPA_1872_2_17__55_0.pdf

Daily, J. W., & Stephan, S. C. Jr. (1952). The solitary wave: its celerity, profile, internal velocities and amplitude attenuation in a horizontal smooth channel. *Coastal Engineering Proceedings*, *1*, 2-2. https://doi.org/10.9753/icce.v3.2

Hull, P., & Müller, G. (2002). An investigation of breaker heights, shapes and pressures. *Ocean Engineering*, *29*(1), 59–79. https://doi.org/10.1016/S0029-8018(00)00075-5

McCowan, J. (1894). VII. On the solitary wave. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *32*(194), 45–58. doi:10.1080/14786449108621390

Müller, G. (2022). A linearized theoretical model for the assessment of wave overtopping at vertical walls. *Proceedings of the 39th IAHR World Congress*. June 2022 Granada, Spain. Paper 672. doi:10.13140/RG.2.2.17809.10082

Robertson, B., Hall, K., Zytner, R., & Nistor, I. (2013). Breaking waves: Review of characteristic relationships. *Coastal Engineering Journal*, *55*(1), 1350002-1–1350002-40. https://doi.org/10.1142/S0578563413500022

Sorensen, R. M. (2006). *Basic coastal engineering*. 3rd ed. Springer Science Business Media, Inc. doi:10.1007/0-387-23333-4_6

Wood, A. M. M., & Fleming, C. A. (1981). *Coastal hydraulics*. 2nd ed. Macmillan. https://doi.org/10.1007/978-1-349-04506-8