A Refined Convergence Analysis of Popov's Algorithm for Pseudo-monotone Variational Inequalities

Le Thi Thanh Hai [*]   Thanh Quoc Trinh[‡]   Phan Tu Vuong [§]

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Abstract. In this work, we present a refined convergence analysis of the Popov's projection algorithm for solving pseudo-monotone variational inequalities in Hilbert spaces. Our analysis results in a larger range of stepsize, which is achieved by using a new Lyapunov function. Furthermore, when the operator is strongly pseudo-monotone and Lipschitz continuous, we establish the linear convergence of the sequence generated by the Popov's algorithm. As a by-product of our analysis, we extend the range of stepsize in the projected reflected gradient algorithm for solving unconstrained pseudo-monotone variational inequalities.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and a generated norm $\| \cdot \|$. Let K be a nonempty closed convex subset of H and let F be an operator from H to H. We are interested in the classical variational inequality problem, denoted by $\mathrm{VI}(K, F)$: Find $u^* \in K$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0 \quad \forall u \in K. \tag{1}$$

We denote the solution set of $\mathrm{VI}(K, F)$ is $\mathrm{Sol}(K, F)$, which is assumed to be nonempty. When $K = H$, we have an unconstrained VI, which is finding $x^* \in H$ such that $F(x^*) = 0$.

[*]Faculty of Applied Sciences, HCMC University of Technology and Education, Ho Chi Minh City, Vietnam. E-mail: hailtt@hcmute.edu.vn

[†]Corresponding author

[‡]Faculty of Fundamental Sciences, Van Lang University, Ho Chi Minh City, Vietnam. Email: thanh.tq@vlu.edu.vn

[§]School of Mathematical Sciences, University of Southampton, SO17 1BJ, Southampton, UK. Email: t.v.phan@soton.ac.uk

The variational inequality (VI) model is a versatile tool that unifies many important models in applied mathematics. It encompasses various types of problems, such as optimization problems, complementary problems, saddle-point (min-max) problems, Nash equilibrium problems, and fixed point problems. [6]. To solve a VI, it is often necessary to assume that the given operator satisfies certain (generalized) monotonicity conditions (see [11]). These conditions ensure that the solution of the VI is well-posed and provide important properties for designing numerical methods that can efficiently compute the solution. There have been numerous solution algorithms proposed for solving generalized monotone variational inequalities (VIs). Among these algorithms, extragradient-type methods have been shown to be effective for solving VIs that belong to the pseudo-monotone class. These algorithms can be used to solve a wide range of optimization problems and other problems that can be formulated as VIs (see[1, 4, 5, 10, 16, 21, 23, 24, 25]). For a recent overview development of VIs and its general models, the readers are referred to [19].

Popov's algorithm [20] is an improved version of extragradient algorithm [16] with an advantage that it requires only one operator value instead of two as in the extragradient algorithm. This advantage is particularly important for solving VIs in high-dimensional spaces or VIs arising in optimal control problems where operator evaluation is expensive. However, the stepsize is also a crucial factor in practice, as a larger stepsize leads to faster termination of the algorithm. The original convergence analysis of Popov's algorithm showed that the upper bound of the stepsize to guarantee convergence of the iterative sequence is $\frac{1}{3L}$, which is considerable smaller than $\frac{1}{L}$ as in the extragradient algorithm [16, 13], here $L > 0$ is the Lipschitz continuity constant of the corresponding operator. Therefore, a natural question is whether it is possible to enlarge the range of the stepsize. A careful examination of Popov's original proof has shown that the upper bound can be extended to $\frac{\sqrt{2}-1}{L}$ (see e.g. [7, 26]). In addition, it was obtained in [8] that this stepsize's upper bound can be enlarged further to $\frac{1}{2L}$ but only for the *convergence of the ergodic average sequence*[1] provided that the operator is monotone. It is also stated in [8] that *"because the role of averaging is essential in this argument, the convergence of the algorithm's last iterate requires significantly different techniques"*. One of the aims of this paper is to prove the convergence of the last iterate with the stepsize's upper bound $\frac{1}{2L}$. The main idea of the proof is to use a new Lyapunov function with tight estimations.

In addition to the enlargement of the stepsize, we adapt the technique recently developed in [13, 25] for the extragradient algorithm to make the Popov's algorithm works for pseudo-monotone VIs in infinite dimensional Hilbert spaces. We observe that all the nice properties of the extragradient algorithm established in [13, 25] still hold for the Popov's algorithm. A modified version of Popov's algorithm [18] is also investigated. This algorithm is one of the few methods that only requires one operator value and one projection per iteration for solving monotone VIs. We establish the convergence results of the modified Popov's algorithm for solving pseudo-monotone VIs with larger stepsizes.

Another aim of the paper is to investigate the convergence rate of the Popov's algorithm when

[1] a weighted average of the sequence of points generated by the algorithm

the operator is strongly pseudo-monotone. It is known that the simple projection algorithm [12] and extragradient-type algorithms [13] generated iterative sequences converging linearly to the unique solution of a strongly pseudo-monotone and Lipschitz continuous VIs. So, it is an interesting question to see whether *the linear convergence rate is still guaranteed for the Popov's algorithm.* We provide a positive answer for this question when the setpsize is smaller that $\frac{1}{2L}$, which is also new to the best of our knowledge.

As a final note, it is worth mentioning that in the unconstrained case, the Popov's algorithm is equivalent to the projected reflected gradient method studied in [17] for solving monotone variational inequalities. The upper bound for the stepsize derived in [17] is given by $\frac{\sqrt{2}-1}{L}$. Therefore, by using the extended upper bound for stepsize obtained in this paper for Popov's algorithm, we can obtain the (linear) convergence rate for the projected reflected gradient method for solving unconstrained (strongly) pseudo-monotone VIs with a larger range of stepsize.

The remaining sections of this paper are structured as follows: Section 2 introduces some necessary preliminaries. Section 3 presents the Popov's algorithm with an extended upper bound of stepsize. In Section 4, we establish the linear convergence of the iterative sequence when the mapping is both strongly pseudo-monotone and Lipschitz continuous. Finally, Section 5 concludes the paper and discusses some open questions for future research.

2 Preliminaries

In the sequel, we will need some of the following definitions: The operator F is called

- Lipschitz continuous on K if there exists $L > 0$ such that $\|F(u) - F(v)\| \leq L\|u - v\|$ for all $u, v \in K$;
- sequentially weakly continuous if it maps a weakly convergent sequence to a weakly convergent sequence;
- monotone on K if for all $u, v \in K$ it holds $\langle F(u) - F(v), u - v\rangle \geq 0$;
- pseudo-monotone on K if for any $u, v \in K$ such that $\langle F(v), u-v\rangle \geq 0$ then $\langle F(u), u-v\rangle \geq 0$;
- γ-strongly pseudo-monotone on K if there exists $\gamma > 0$ and for any $u, v \in K$ such that $\langle F(v), u - v\rangle \geq 0$ then $\langle F(u), u - v\rangle \geq \gamma\|u - v\|^2$.

Remark 2.1 *Obviously, if the operator F is strongly (pseudo-) monotone, then it is (pseudo-) monotone. Besides, in this case, if VI(K, F) cannot have more than one solution. Indeed, if F is strongly pseudo-monotone mapping and $u^*, v^* \in$ Sol (K, F), then $\langle F(v^*), u^* - v^*\rangle \geq 0$ and $\langle F(v^*), v^* - u^*\rangle \geq \gamma\|v^* - u^*\|^2$. Adding these inequalities yields $0 \geq \gamma\|v^* - u^*\|^2$, we can infer $u^* = v^*$.*

Remark 2.2 *A very important class of pseudo-monotone VIs is the optimization problems of pseudo-convex objective. It is well known that a differentiable function is pseudo-convex if and only* *if its gradient is pseudo-monotone [11]. An important subclass of the one of pseudo-convex functions are ratios of convex and concave functions. Indeed, if $C \subseteq \mathbb{R}^n$ is a convex set, $g : C \to [0, +\infty)$ is a convex function, $h : C \to (0, +\infty)$ is a concave function, and both g and h are differentiable on C, then the function*

$$f : C \to [0, +\infty), \quad f(x) := \frac{g(x)}{h(x)},$$

is pseudo-convex on C (see e.g. [2]).

Let K be a nonempty, closed and convex subset of a real Hilbert space H. The metric projection of an element u in H on K, denoted by $P_K(u)$, is a unique element of K such that $\|u - P_K(u)\| \leq \|u - v\|$ for all $v \in K$. The metric projection is non-expansive (i.e. Lipschitz continuous with modules $L = 1$) and has two important properties as follows [9].

Theorem 2.1 *For any $u \in H$ and $v \in K$ we have*

(a) $\langle u - P_K(u), v - P_K(u) \rangle \leq 0$;

(b) $\|P_K(u) - v\|^2 \leq \|u - v\|^2 - \|u - P_K(u)\|^2$.

From (1) and above properties, we can deduce that

Remark 2.3 $u^* \in \mathrm{Sol}(K, F)$ *if and only if* $u^* = P_K(u^* - \lambda F(u^*))$ *for every* $\lambda > 0$.

We will need Minty lemma [15, Lemma 1.5 on p. 85] in the later convergence analysis.

Proposition 2.1 *Consider the problem* $\mathrm{VI}(K, F)$ *with K being a nonempty, closed, convex subset of a real Hilbert space H and $F : K \to H$ being pseudo-monotone and continuous. Then,* $\mathrm{Sol}(K, F)$ *is closed convex and u^* is a solution of* $\mathrm{VI}(K, F)$ *if and only if*

$$\langle F(u), u - u^* \rangle \geq 0 \quad \forall u \in K.$$

3 Convergence Analysis

In this section, we consider the problem $\mathrm{VI}(K, F)$ with K being nonempty, closed, convex and F being pseudo-monotone and Lipschitz continuous with modulus $L > 0$ on K.

Popov's Algorithm

Data: $u^0, v^0 \in K$ and $\lambda \in]0, \frac{1}{2L}[$.

Step 0: Set $k = 0$.

Step 1: If $u^k = P_K(u^k - \lambda F(v^k)) = v^k$ then Stop. Otherwise go to **Step 2**.

Step 2: Set $u^{k+1} = P_K(u^k - \lambda F(v^k))$ and $v^{k+1} = P_K(u^{k+1} - \lambda F(v^k))$ and replace k by $k+1$; go to **Step 1**.

If the computation terminates at a step k, then one puts $u^{k'} = u^k$ for all $k' \geq k + 1$. Thus, for a given stepsize $\lambda \in]0, \frac{1}{2L}[$, Popov's Algorithm produces for each initial point $u^0, v^0 \in K$ a unique iterative sequence $\{u^k\}$ and $\{v^k\}$.

Remark 3.1 *If at some iteration we have $F(v^k) = 0$, then v^k is a solution and the Popov's Algorithm terminates at step k. From now on, we assume that $F(v^k) \neq 0$ for all k and the Popov's Algorithm generates an infinite sequence.*

Remark 3.2 *In the unconstrained case, i.e. when $K = H$ the Popov's Algorithm coincides with the projected reflected gradient method for solving monotone VIs studied in [17], which generated the sequences as*

$$u^{k+1} = P_K(u^k - \lambda F(v^k)), \quad v^{k+1} = 2u^{k+1} - u^k.$$

The convergence analysis in [17] requires $\lambda < \frac{\sqrt{2}-1}{L}$. This upper bound will be enlarged to $\frac{1}{2L}$ in our work.

Before presenting the convergence analysis, let us consider a simple example to illustrate the advantage of the larger stepsize. This is typical numerical example for monotone VI with F is a rotation operator in $\mathbb{R}^2$, where the forward- backward method fails.

Example 3.1 *Let $K = H = \mathbb{R}^2$ and $F(x_1, x_2) = (-x_2, x_1)$ which is monotone and Lipschitz continuous with $L = 1$. In the figure below, we plot the iterates of Popov's Algorithm with $\lambda = 0.4999 < \frac{1}{2L}$ (this work), $\lambda = 0.4142 < \frac{\sqrt{2}-1}{L}$ (obtained in [7, 26]) and $\lambda = 0.3 < \frac{1}{3L}$ obtained by Popov [20]. It is clear that the Popov's Algorithm with $\lambda = 0.4999 < \frac{1}{2L}$ converges to the unique solution $(0, 0)$ after 23 iterations, while the iterates generated with other smaller stepsizes are still far away from the solution.*

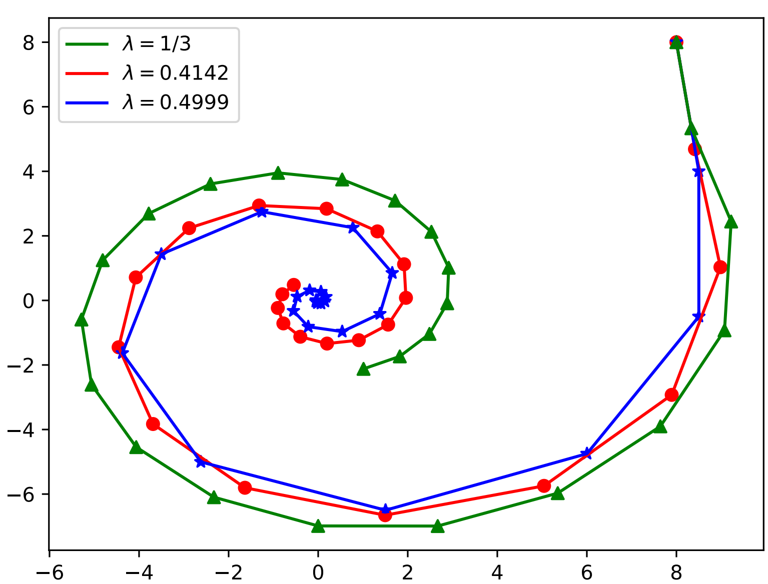

Figure 1: Performance of sequences generated by Popov's Algorithm with different values of stepsize.

We establish some properties of the iterative sequences $\{u^k\}$ and $\{v^k\}$ generated by Popov's Algorithm.

Proposition 3.1 *Assume that F is pseudo-monotone and L-Lipschitz continuous on K. Then for any solution u^* of $\mathrm{VI}(K,F)$ and for every $k \geq 2$ we have*

$$\|u^{k+1}-u^*\|^2 + \frac{8\alpha^2}{1+4\alpha}\|v^k - v^{k-1}\|^2 \leq \|u^k - u^*\|^2 + \frac{8\alpha^2}{1+4\alpha}\|v^{k-1}-v^{k-2}\|^2 - \frac{1-4\alpha}{1+4\alpha}\|u^k - v^k\|^2 - \frac{1-4\alpha}{2}\|u^{k+1}-v^k\|^2 - \frac{2\alpha(1-4\alpha)}{1+4\alpha}\|v^k - v^{k-1}\|^2, \tag{2}$$

where $\alpha = \lambda^2 L^2 < 1/4$.

Proof. Let u^* be a solution of $\mathrm{VI}(K,F)$, since $v^k \in K$ and F is pseudo-monotone on K

$$\langle F(v^k), v^k - u^*\rangle \geq 0,$$

which implies

$$\langle F(v^k), v^k - u^{k+1}\rangle \geq -\langle F(v^k), u^{k+1} - u^*\rangle. \tag{3}$$

Since $u^{k+1} = P_K(u^k - \lambda F(v^k))$, by Theorem 2.1(b) we have

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &\leq \|u^k - \lambda F(v^k) - u^*\|^2 - \|u^k - \lambda F(v^k) - u^{k+1}\|^2\\ &= \|u^k - u^*\|^2 - \|u^k - u^{k+1}\|^2 - 2\lambda\langle F(v^k), u^{k+1}-u^*\rangle\\ &= \|u^k - u^*\|^2 - \|u^k - v^k\|^2 - \|v^k - u^{k+1}\|^2 - 2\langle u^k - v^k, v^k - u^{k+1}\rangle\\ &\quad - 2\lambda\langle F(v^k), u^{k+1}-u^*\rangle.\end{aligned}$$

Combining the above inequality with (3) we get

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &\leq \|u^k - u^*\|^2 - \|u^k - v^k\|^2 - \|v^k - u^{k+1}\|^2 - 2\langle u^k - \lambda F(v^k) - v^k, v^k - u^{k+1}\rangle\\ &= \|u^k - u^*\|^2 - \|u^k - v^k\|^2 - \|v^k - u^{k+1}\|^2 + 2\lambda\langle F(v^k) - F(v^{k-1}), v^k - u^{k+1}\rangle\\ &\quad + 2\langle u^k - \lambda F(v^{k-1}) - v^k, u^{k+1} - v^k\rangle.\end{aligned} \tag{4}$$

Using the projection characterization in Theorem 2.1, since $v^k = P_K(u^k - \lambda F(v^{k-1}))$ and $u^{k+1} \in K$, it holds

$$\langle u^k - \lambda F(v^{k-1}) - v^k, u^{k+1} - v^k\rangle \leq 0. \tag{5}$$

Let us estimate the term $2\lambda\langle F(v^k) - F(v^{k-1}), v^k - u^{k+1}\rangle$ in the inequality (4) as follows:

$$\begin{aligned}2\lambda\langle F(v^k) - F(v^{k-1}), v^k - u^{k+1}\rangle &\leq 2\lambda\|F(v^k) - F(v^{k-1})\|\|v^k - u^{k+1}\|\\ &\leq 2\lambda L\|v^k - v^{k-1}\|\|v^k - u^{k+1}\|\\ &\leq \frac{2\alpha}{1+4\alpha}\|v^k - v^{k-1}\|^2 + \frac{1+4\alpha}{2}\|v^k - u^{k+1}\|^2,\end{aligned} \tag{6}$$

where we have used the Lipschitz continuity of F in the second inequality and the Cauchy-Schwartz inequality in the last inequality.
Substituting the estimations in (6) and (5) into the right side of inequality (4), we obtain

$$\|u^{k+1}-u^*\|^2 \leq \|u^k-u^*\|^2 - \|u^k-v^k\|^2 - \frac{1-4\alpha}{2}\|v^k-u^{k+1}\|^2 + \frac{2\alpha}{1+4\alpha}\|v^k-v^{k-1}\|^2. \qquad (7)$$

On the other hand, by the non-expansiveness of P_K and the Lipschitz continuity of F we have

$$\begin{aligned}\|u^k-v^{k-1}\| &= \|P_K(u^{k-1}-\lambda F(v^{k-1})) - P_K(u^{k-1}-\lambda F(v^{k-2}))\| \\ &\leq \lambda\|F(v^{k-1})-F(v^{k-2})\| \\ &\leq \lambda L\|v^{k-1}-v^{k-2}\|. \qquad (8)\end{aligned}$$

Hence

$$\|v^k-v^{k-1}\|^2 \leq 2\|v^k-u^k\|^2 + 2\|u^k-v^{k-1}\|^2 \leq 2\|u^k-v^k\|^2 + 2\alpha\|v^{k-1}-v^{k-2}\|^2,$$

which implies

$$\begin{aligned}\|v^k-v^{k-1}\|^2 &= 2\|v^k-v^{k-1}\|^2 - \|v^k-v^{k-1}\|^2 \\ &\leq 4\|u^k-v^k\|^2 + 4\alpha\|v^{k-1}-v^{k-2}\|^2 - \|v^k-v^{k-1}\|^2.\end{aligned}$$

Therefore,

$$\frac{2\alpha}{1+4\alpha}\|v^k-v^{k-1}\|^2 \leq \frac{8\alpha}{1+4\alpha}\|u^k-v^k\|^2 + \frac{8\alpha^2}{1+4\alpha}\|v^{k-1}-v^{k-2}\|^2 - \frac{2\alpha}{1+4\alpha}\|v^k-v^{k-1}\|^2. \qquad (9)$$

Combining (7) and (9) we obtain

$$\begin{aligned}\|u^{k+1}-u^*\|^2 \leq \|u^k-u^*\|^2 - \frac{1-4\alpha}{1+4\alpha}\|u^k-v^k\|^2 - \frac{1-4\alpha}{2}\|v^k-u^{k+1}\|^2 \\ + \frac{8\alpha^2}{1+4\alpha}\|v^{k-1}-v^{k-2}\|^2 - \frac{2\alpha}{1+4\alpha}\|v^k-v^{k-1}\|^2,\end{aligned}$$

which implies (2). ■

We are now in the position to present the main result of this section.

Theorem 3.1 *Assume that F is pseudo-monotone on H, L-Lipschitz continuous and sequentially weakly continuous on K. Assume also that* Sol(K,F) *is nonempty. Then, the sequences $\{u^k\}$ and $\{v^k\}$ generated by Popov's Algorithm converge weakly to a solution of* VI(K,F).

Proof. Let u^* be a solution of VI(K,F) and let

$$A_k := \|u^k-u^*\|^2 + \frac{8\alpha^2}{1+4\alpha}\|v^{k-1}-v^{k-2}\|^2 \geq 0,$$

for all $k \geq 3$. It follows from (2) that

$$A_{k+1} \leq A_k - \frac{1-4\alpha}{1+4\alpha}\|u^k-v^k\|^2 - \frac{1-4\alpha}{2}\|u^{k+1}-v^k\|^2 - \frac{2\alpha(1-4\alpha)}{1+4\alpha}\|v^k-v^{k-1}\|^2. \qquad (10)$$

Since $\lambda \in]0; \frac{1}{2L}[$ we have $4\alpha = 4\lambda^2 L^2 < 1$, hence the sequence $\{A_k\}$ is decreasing and bounded from below by 0, therefore it converges. Hence the sequence $\{u^k\}$ is bounded. Moreover, from (10) we obtain

$$\lim_{k\to\infty} \|u^k - v^k\|^2 = \lim_{k\to\infty} \|u^{k+1} - v^k\|^2 = \lim_{k\to\infty} \|v^k - v^{k-1}\|^2 = 0. \tag{11}$$

As $\{u^k\}$ is a bounded sequence in a Hilbert space, there exists a subsequence $\{u^{k_i}\}$ of $\{u^k\}$ converging weakly to $\overline{u} \in K$. Follows from (11), the sequence $\{v^{k_i}\}$ also converges weakly to $\overline{u}$. We will prove that $\overline{u} \in \text{Sol }(K, F)$ by employing the technique developed in [25]. Indeed, use the projection characterization in Theorem 2.1, we have

$$\langle u^{k_i+1} - \lambda F(v^{k_i}) - v^{k_i+1}, w - v^{k_i+1}\rangle \leq 0 \quad \forall w \in K.$$

or equivalently

$$\begin{aligned} 0 &\leq \langle v^{k_i+1} - u^{k_i+1} + \lambda F(v^{k_i}), w - v^{k_i+1}\rangle \\ &= \langle v^{k_i+1} - u^{k_i+1}, w - v^{k_i+1}\rangle + \lambda\langle F(v^{k_i}), w - v^{k_i}\rangle + \lambda\langle F(v^{k_i}), v^{k_i} - v^{k_i+1}\rangle \quad \forall w \in K. \end{aligned} \tag{12}$$

Fixing $w \in K$, letting $i \to \infty$, remembering the limits in (11) and $\lambda \in]0, \dfrac{1}{2L}[$, we obtain

$$\lim_{i\to\infty} \inf \langle F(v^{k_i}), w - v^{k_i}\rangle \geq 0. \tag{13}$$

Now we choose a sequence $\{\epsilon_i\}_i$ of positive number decreasing and tending to 0. For each ϵ_i, we denote by n_i the smallest integer such that

$$\langle F(v^{k_j}), w - v^{k_j}\rangle + \epsilon_i \geq 0 \quad \forall j \geq n_i, \tag{14}$$

where the existence of n_i follows from (13). Since $\{\epsilon_i\}_i$ is decreasing, we can see that the sequence $\{n_i\}$ is increasing. Moreover, for each i, $F(v^{k_i}) \neq 0$. Setting

$$w^{k_{ni}} = \frac{F(v^{k_{ni}})}{\|F(v^{k_{ni}})\|^2}$$

we have $\langle F(v^{k_{ni}}), w^{k_{ni}}\rangle = 1$ for each i. Now we can deduce from (14) that for each i

$$\langle F(v^{k_{ni}}), w + \epsilon_i w^{k_{ni}} - v^{k_{ni}}\rangle \geq 0$$

and since F is pseudo-monotone, we obtain

$$\langle F(w + \epsilon_i w^{k_{ni}}), w + \epsilon_i w^{k_{ni}} - v^{k_{ni}}\rangle \geq 0. \tag{15}$$

On the other hand, we have that $\{v^{k_i}\}$ converges weakly to $\overline{u}$ when $i \to \infty$. Since F is sequentially weakly continuous on K, $\{F(v^{k_i})\}$ converges weakly to $F(\overline{u})$. We can assume that $F(\overline{u}) \neq 0$ (if not, $\overline{u}$ is a solution). Since the norm mapping is sequentially lower semicontinuous, we have

$$\|F(\overline{u})\| \leq \lim_{i\to\infty} \inf \|F(v^{k_i})\|. \tag{16}$$

Since $\{v^{k_{n_i}}\} \subset \{v^{k_i}\}$ and $\epsilon_i \to 0$ as $i \to \infty$, we get

$$0 \le \lim_{i\to\infty} \|\epsilon_i w^{k_{n_i}}\| = \lim_{i\to\infty} \frac{\epsilon_i}{\|F(v^{k_{n_i}})\|} \le \frac{0}{\|F(\overline{u})\|} = 0.$$

Thus, passing to the limit as $i \to \infty$ in (15), we obtain

$$\langle F(w), w - \overline{u}\rangle \ge 0.$$

It follows from Proposition 2.1 that $\overline{u} \in$ Sol(K, F). Finally, we need to show the sequence $\{u^k\}$ converges weakly to $\overline{u}$ (then $\lim_{k\to\infty} \|u^k - v^k\| = 0$ implies the sequence $\{v^k\}$ also converges weakly to $\overline{u}$). To do this, we assume that there is another sub-sequence of $\{u^k\}$, called $\{u^{k_j}\}$, converging weakly to $\widehat{u}$. We will prove that $\overline{u} = \widehat{u}$. Repeat the proof above with $\widehat{u}$ instead of $\overline{u}$, we get that $\widehat{u} \in$ Sol (K, F). From (10), (11) and $\lambda \in]0; \frac{1}{2L}[$, the sequence $\{\|u^k - \overline{u}\|^2\}$ converges. Similarly, the sequence $\{\|u^k - \widehat{u}\|^2\}$ also converges. Since for all $k \in \mathbb{N}$,

$$2\langle u^k, \widehat{u} - \overline{u}\rangle = \|u^k - \overline{u}\|^2 - \|u^k - \widehat{u}\|^2 + \|\widehat{u}\|^2 - \|\overline{u}\|^2,$$

we infer that the sequence $\{\langle u^k, \widehat{u} - \overline{u}\rangle\}$ also converges. Setting

$$t = \lim_{k\to\infty} \langle u^k, \widehat{u} - \overline{u}\rangle,$$

and passing to the limits along $\{u^{k_i}\}$ and $\{u^{k_j}\}$ yields, respectively,

$$t = \langle \overline{u}, \widehat{u} - \overline{u}\rangle = \langle \widehat{u}, \widehat{u} - \overline{u}\rangle.$$

This implies that $\|\overline{u} - \widehat{u}\|^2 = 0$ and therefore $\overline{u} = \widehat{u}$. ■

Remark 3.3 (Modified Popov's Algorithm:) *In [18], employing the idea of the subgradient extragradient algorithm [4], the authors replaced the first projection onto K by the the projection onto half-space*

$$H_k = \{t \in H : \langle u^k - \lambda F(u^{k-1}) - v^k, t - v^k\rangle \le 0\}$$

and compute

$$\begin{aligned} u^{k+1} &= P_{H_k}(u^k - \lambda F(v^k)) \\ v^{k+1} &= P_K(u^{k+1} - \lambda F(v^k)). \end{aligned} \tag{17}$$

The convergence analysis also requires $\lambda \in]0, \frac{1}{3L}[$ to guarantee the convergence of the iterative sequences $\{u^k\}$ and $\{v^k\}$. Similarly, we can use above estimations to widen the upper bound to $\frac{1}{2L}$ without breaking the convergence results.

Remark 3.4 *The weak continuity assumption of F is only used to obtained* (16)*. This condition can be relaxed to the following weaker assumption: the functional $f(x) := \|F(x)\|$ is weakly lower semicontinuous [22].*

When F is monotone, the weak continuity of F is not required. Indeed, if F is monotone, from (12) *we can deduce*

$$\begin{aligned} 0 &\leq \langle v^{k_i+1} - u^{k_i+1}, w - v^{k_i+1}\rangle + \lambda\langle F(v^{k_i}), v^{k_i} - v^{k_i+1}\rangle + \lambda\langle F(v^{k_i}), w - v^{k_i}\rangle \\ &\leq \langle v^{k_i+1} - u^{k_i+1}, w - v^{k_i+1}\rangle + \lambda\langle F(v^{k_i}), v^{k_i} - v^{k_i+1}\rangle + \lambda\langle F(w), w - v^{k_i}\rangle \quad \forall w \in K. \end{aligned}$$

Letting $i \to \infty$ in the last inequality, noting that $\lambda \in]0, \dfrac{1}{2L}[$ and the limits from (11)*, we get*

$$\langle F(w), w - \overline{u}\rangle \geq 0 \quad \forall w \in K,$$

which implies (from Proposition 2.1) that $\overline{u} \in$ Sol(K, F).

Remark 3.5 *In finite dimensional spaces, for solving monotone VIs, the upper bound $\frac{1}{2L}$ was obtained in [8, Theorem 1] but only for the convergence of the* **ergodic average sequence***, i.e.* $\overline{u}^n = \frac{1}{n}\sum_{k=1}^n u^k$.

The following result is similar to that of [13, Theorem 3.2] established for the extragradient algorithm.

Theorem 3.2 *Let K be a nonempty closed convex set in a Hilbert space H. Suppose that $F : K \to H$ is pseudo-monotone and Lipschitz continuous on K. Let $\{u^k\}, \{v^k\}$ be sequences generated by Popov's Algorithm. If Sol(K, F) is nonempty and there exists a subsequence $u^{k_i} \subset \{u^k\}$ converging strongly to some $u^* \in K$, then $u^* \in$ Sol(K, F) and the whole sequence $\{u^k\}$ converges strongly to u^*.*

Proof. As the subsequence $\{u^{k_i}\}$ converges in norm to $u^* \in K$, the limit $\lim\limits_{k\to\infty} \|u^k - v^k\| = 0$ in (11) implies that $\{v^{k_i}\}$ also converges in norm to u^*.

Since $v^{k_i} = P_K(u^{k_i} - \lambda F(v^{k_i-1}))$, F and $P_K(.)$ are continuous, and the limit $\lim\limits_{k\to\infty} \|u^k - v^{k-1}\| = 0$ in (11) can infer $\lim\limits_{i\to\infty} F(u^{k_i}) = \lim\limits_{i\to\infty} F(v^{k_i-1}) = F(u^*)$, we obtain

$$u^* = \lim_{i\to\infty} v^{k_i} = \lim_{i\to\infty} P_K\left(u^{k_i} - \lambda F(v^{k_i-1})\right) = P_K(u^* - \lambda F(u^*)),$$

which implies that $u^* \in \mathrm{Sol}(K, F)$.

Using (10) with $u^* \in \mathrm{Sol}(K, F)$, we can infer that the sequence $\{\|u^k - u^*\|\}$ converges. Since

$$\lim_{k\to\infty} \|u^k - u^*\| = \lim_{i\to\infty} \|u^{k_i} - u^*\| = 0,$$

$\{u^k\}$ converges strongly to u^*. ■

Noting that in two corollaries below, we do not require that $\mathrm{Sol}(K, F)$ is nonempty. Similar results were obtained in the extragradient algorithm, see [13, Theorems 3.6 and 3.7].

Corollary 3.1 *Let K be a nonempty closed convex set in a Hilbert space H. Suppose that $F : K \to H$ is pseudo-monotone and Lipschitz continuous on K. Let $\{u^k\}, \{v^k\}$ be sequences generated by Popov's Algorithm. If the sequence $\{u^k\}$ converges strongly to some u, then u is a solution of VI(K, F).*

Proof. From the Popov Algorithm and the nonexpansiveness of metric projection, for every $k \geq 1$ we have

$$\begin{aligned}\|u^k - v^k\| &= \|P_K(u^{k-1} - \lambda F(v^{k-1})) - P_K(u^k - \lambda F(v^{k-1}))\| \\ &\leq \|(u^{k-1} - \lambda F(v^{k-1})) - (u^k - \lambda F(v^{k-1}))\| \\ &= \|u^{k-1} - u^k\|. \end{aligned} \tag{18}$$

Hence, since the sequence $\{u^k\}$ converges strongly to u, the sequence $\{v^k\}$ also converges strongly to u. Use the continuity of F and $P_K(.)$ we obtain

$$u = \lim_{k\to\infty} u^k = \lim_{k\to\infty} P_K(u^{k-1} - \lambda F(v^{k-1})) = P_K(u - \lambda F(u)),$$

which implies $u \in \mathrm{Sol}(K, F)$. ■

Corollary 3.2 *Let K be a nonempty closed convex set in a Hilbert space H. Suppose that $F : K \to H$ is pseudo-monotone and Lipschitz continuous on K. Let $\{u^k\}, \{v^k\}$ be sequences generated by Popov Algorithm. If $\{u^k\}$ is bounded and it has a strongly convergent subsequence, then the whole sequence converges strongly to a solution of VI(K, F).*

Proof. From the estimation in (18) and the assumption $\{u^k\}$ is bounded, we can deduce that $\{v^k\}$ is also bounded. Therefore, $\{F(u^k)\}$ and $\{F(v^k)\}$ are also bounded by the Lipschitz continuity of F. Hence, there exists $R > 0$ such that

$$\|u^k\| \leq R, \quad \|v^k\| \leq R \quad \text{and} \quad \|F(v^k)\| \leq R \quad \forall k \in \mathbb{N}.$$

Therefore by the nonexpansiveness of the metric projection and by the Lipschitz continuity of F, for all $k \geq 1$, we have

$$\begin{aligned}\|P_K\left(u^k - \lambda F(v^{k-1})\right)\| &= \|P_K\left(u^k - \lambda F(v^{k-1})\right) - P_K(u^k) + P_K(u^k)\| \\ &\leq \|P_K(u^k - \lambda F(v^{k-1})) - P_K(u^k)\| + \|u^k\| \\ &\leq \|(u^k - \lambda F(v^{k-1})) - u^k\| + R \\ &\leq |\lambda| \|F(v^{k-1})\| + R \\ &\leq \frac{1}{2L} R + R =: \overline{R}. \end{aligned}$$

Similarly, we also get $\|P_K\left(u^k - \lambda F(v^k)\right)\| \leq \overline{R}$ for every $k \in \mathbb{N}$. Let B be the ball with origin center and radius $\overline{R}$ and K_B be the intersection of K with B. Then K_B is nonempty, closed, and convex in H. Since F is pseudo-monotone on K, the problem VI(K_B, F) has a solution ([28], Theorem 2.3). As $\|P_K(u^k - \lambda F(v^{k-1}))\| \le \overline{R}$, $P_K(u^k - \lambda F(v^{k-1}))$ is contained in B. Similarly, $P_K(u^k - \lambda F(v^k))$ is also contained in B. Hence, we have

$$u^{k+1} = P_K(u^k - \lambda F(v^k)) = P_{K_B}(u^k - \lambda F(v^k)),$$

$$v^k = P_K(u^k - \lambda F(v^{k-1})) = P_{K_B}(u^k - \lambda F(v^{k-1})).$$

The latter means that $\{u^k\}$ can be regarded as a sequence generated by the Popov's algorithm for solving $\mathrm{VI}(K_B, F)$. Since $\mathrm{Sol}(K_B, F)$ is nonempty, and the sequence $\{u^k\}$ has a strongly convergent subsequence by our assumption, $\{u^k\}$ converges strongly to $u^* \in K_B$ by the Theorem 3.2. Applying Corollary 3.1, we can conclude that $u^* \in \mathrm{Sol}(K, F)$. ■

Remark 3.6 *All results obtained in this section are still valid if we replace the fixed stepsize λ by variable stepsize λ_k such that $\lambda_k \in [a, b] \subset (0, \frac{1}{2L})$. Especially, when the Lipschitz constant L is not explicitly available, we can choose adaptive stepsizes*

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\mu\|v^{k-1} - v^k\|}{2\|F(v^{k-1}) - F(v^k)\|}\right\},$$

for some $\mu \in (0, 1)$. Note that in this case, the sequence $\{\lambda_k\}$ is non-increasing and bounded. Moreover $\lim_{k\to\infty} \lambda_k = \lambda \le \frac{\mu}{2L} < \frac{1}{2L}$, see e.g. [3, 27] for the similar techniques using in extragradient algorithm.

4 Linear Convergence Analysis

In this section, we assume that the mapping F is strongly pseudo-monotone with modulus γ and L-Lipschitz continuous. Then $\mathrm{VI}(K, F)$ has a unique solution [14]. We will show that sequence $\{u^k\}$ generated by Popov's algorithm converges linearly to the unique solution of $\mathrm{VI}(K, F)$.

Proposition 4.1 *Let H be a real Hilbert space and K be a nonempty closed convex set in H. Let $F : K \to H$ be a strongly pseudo-monotone on K with modulus γ and Lipschitz continuous on K with a constant L. Let $\{u^k\}, \{v^k\}$ be consequences generated by Popov's algorithm and u^* be the unique solution of* $\mathrm{VI}(K, F)$ then for every $k \in \mathbb{N}$ we have

$$\|u^{k+1} - u^*\|^2 \le \frac{(\lambda L + 1)^2 + 1}{\lambda^2\gamma^2}(\|u^k - v^k\|^2 + \|u^{k+1} - v^k\|^2). \tag{19}$$

Proof. Since $u^{k+1} = P_K(u^k - \lambda F(v^k))$ and $u^* \in K$, it follows from Theorem 2.1 (b) that

$$\langle u^k - \lambda F(v^k) - u^{k+1}, u^* - u^{k+1}\rangle \le 0 \quad \forall k \in \mathbb{N},$$

or equivalently

$$\langle u^k - \lambda F(v^k) - u^{k+1}, u^{k+1} - u^*\rangle \ge 0 \quad \forall k \in \mathbb{N}.$$

Thus,

$$\begin{aligned}\langle u^k - u^{k+1}, u^{k+1} - u^*\rangle &\geq \lambda\langle F(v^k), u^{k+1} - u^*\rangle \\ &= -\lambda\langle F(u^{k+1}) - F(v^k), u^{k+1} - u^*\rangle + \lambda\langle F(u^{k+1}), u^{k+1} - u^*\rangle.\end{aligned}$$

Since $\|u^k - u^{k+1}\|.\|u^{k+1} - u^*\| \geq \langle u^k - u^{k+1}, u^{k+1} - u^*\rangle$ we get

$$\|u^k - u^{k+1}\|.\|u^{k+1} - u^*\| \geq -\lambda\langle F(u^{k+1}) - F(v^k), u^{k+1} - u^*\rangle + \lambda\langle F(u^{k+1}), u^{k+1} - u^*\rangle. \tag{20}$$

Since F is Lipschitz continuous on K with constant L, we have

$$\begin{aligned}-\lambda\langle F(u^{k+1}) - F(v^k), u^{k+1} - u^*\rangle &\geq -\lambda\|F(u^{k+1}) - F(v^k)\|.\|u^{k+1} - u^*\| \\ &\geq -\lambda L\|u^{k+1} - v^k\|.\|u^{k+1} - u^*\|. \end{aligned} \tag{21}$$

On the other hand, as $u^* \in \text{Sol }(K, F)$ and $u^{k+1} \in K$, we have $\langle F(u^*), u^{k+1} - u^*\rangle \geq 0$. Hence, by the strong pseudo-monotonicity, we get

$$\langle F(u^{k+1}), u^{k+1} - u^*\rangle \geq \gamma\|u^{k+1} - u^*\|^2. \tag{22}$$

Combining (20) with (21) and (22) give

$$\|u^k - u^{k+1}\|.\|u^{k+1} - u^*\| \geq -\lambda L\|u^{k+1} - v^k\|.\|u^{k+1} - u^*\| + \lambda\gamma\|u^{k+1} - u^*\|^2.$$

Thus,

$$\|u^{k+1} - u^*\| \leq \frac{1}{\lambda\gamma}(\|u^k - u^{k+1}\| + \lambda L\|u^{k+1} - v^k\|). \tag{23}$$

Using the triangle inequality, we obtain

$$\begin{aligned}\|u^{k+1} - u^*\| &\leq \frac{1}{\lambda\gamma}(\|u^k - v^k\| + \|v^k - u^{k+1}\| + \lambda L\|u^{k+1} - v^k\|) \\ &= \frac{1}{\lambda\gamma}(\|u^k - v^k\| + (\lambda L + 1)\|u^{k+1} - v^k\|).\end{aligned}$$

Using the Cauchy–Schwartz inequality, we get

$$\begin{aligned}\|u^{k+1} - u^*\|^2 &\leq \frac{1}{(\lambda\gamma)^2}\left[\|u^k - v^k\| + (\lambda L + 1)\|u^{k+1} - v^k\|\right]^2 \\ &\leq \left[\frac{(\lambda L + 1)^2 + 1}{\lambda^2\gamma^2}\right](\|u^k - v^k\|^2 + \|u^{k+1} - v^k\|^2).\end{aligned}$$

■

Remark 4.1 *The inequality* (23) *provides an upper error bound, which estimates the distance from* u^{k+1} *to the unique solution* u^**. This error bound is useful as a stopping criteria in the implementation of the Popov's algorithm.*

The main result of this section is established as follow.

Theorem 4.1 *Let H be a real Hilbert space and K be a nonempty closed convex set in H. Let $F : K \to H$ be a strongly pseudo-monotone on K with modulus γ and Lipschitz continuous on K with a constant L. Let $\{u^k\}, \{v^k\}$ be consequences generated by Popov's algorithm and u^* be the unique solution of* $\mathrm{VI}(K, F)$. *Then the sequence $\{u^k\}$ converges linearly to u^*.*

Proof. On the one hand, since

$$\beta := \frac{2\alpha(1-4\alpha)}{1+4\alpha} = \min\left\{\frac{1-4\alpha}{1+4\alpha}, \frac{1-4\alpha}{2}, \frac{2\alpha(1-4\alpha)}{1+4\alpha}\right\},$$

we have from (10) that

$$A_{k+1} \le A_k - \beta(\|u^k - v^k\|^2 + \|u^{k+1} - v^k\|^2 + \|v^k - v^{k-1}\|^2). \tag{24}$$

On the other hand, let $\theta := \frac{(\lambda L+1)^2+1}{\lambda^2\gamma^2}$, from Proposition 4.1 it holds

$$\|u^{k+1} - u^*\|^2 \le \theta(\|u^k - v^k\|^2 + \|u^{k+1} - v^k\|^2),$$

which implies

$$\begin{aligned} A_{k+1} &= \|u^{k+1} - u^*\|^2 + \frac{8\alpha^2}{1+4\alpha}\|v^k - v^{k-1}\|^2 \\ &\le \theta(\|u^k - v^k\|^2 + \|u^{k+1} - v^k\|^2) + \frac{8\alpha^2}{1+4\alpha}\|v^k - v^{k-1}\|^2 \\ &\le \left(\theta + \frac{8\alpha^2}{1+4\alpha}\right)(\|u^k - v^k\|^2 + \|u^{k+1} - v^k\|^2 + \|v^k - v^{k-1}\|^2). \end{aligned} \tag{25}$$

Combining (24) with (25) gives

$$A_{k+1} \le A_k - \beta\left(\theta + \frac{8\alpha^2}{1+4\alpha}\right)^{-1} A_{k+1}.$$

Setting $\delta := \beta\left(\theta + \frac{8\alpha^2}{1+4\alpha}\right)^{-1} > 0$, the last inequality gives

$$A_{k+1} \le \frac{1}{1+\delta} A_k. \tag{26}$$

which implies that the sequence $\{A_k\}$ converges linearly to 0. As a consequence, $\{u^k\}$ converges linearly the unique solution u^*. ■

Remark 4.2 *By using above proof, in case the mapping F is strongly pseudo-monotone on K we can show that the sequence $\{u^k\}$ generated by the modified Popov's algorithm* (17) *also converges linearly to the unique solution u^*.*

Remark 4.3 *In this section, the strong pseudo-monotonicity of F can be replaced by the following weaker condition*

$$\langle F(v), v - u^*\rangle \ge \gamma\|v - u^*\|^2 \quad \forall v \in K \cap \mathbf{B}(u^*, r),$$

where $\mathbf{B}(u^*, r)$ is the ball centered at u^* with radius $r > 0$.

5 Conclusion

We have extended the upper bound of stepsize in the (modified) Popov's algorithms for solving pseudo-monotone and Lipschitz continuous VIs in Hilbert spaces. In addition, we have proved that the sequences $\{u^k\}$ generated by these algorithms converge linearly to the unique solution u^* if the operator F is strongly pseudo-monotone. While it is known that the stepsize's upper bound $\frac{1}{L}$ is tight for the extragradient algorithm [13], it is not clear if the upper bound $\dfrac{1}{2L}$ obtained in this paper for the (modified) Popov's algorithms is tight or not. We leave it as an open question for future research.

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