UNIVERSITY OF SOUTHAMPTON

Faculty of Social Sciences
Department of Applied Mathematics

Black Holes in String Theory: Beyond Supergravity.

by

Davide Bufalini

ORCiD: 0000-0001-8700-4394

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Abstract

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The aim of this thesis is to demonstrate the effectiveness of worldsheet string theory in the context of black hole physics, going beyond the supergravity approximation. For a certain class of microstates, which in a particular limit admit an α'-exact description in terms of gauged Wess-Zumino-Witten (gWZW) models, we will derive consistency conditions, spectrum of massless excitations, and compute an extensive set of correlation functions. In particular, we will show how the consistency of the worldsheet conformal field theory (CFT) is in one-to-one correspondence with the absence of closed timelike curves, singularities, and horizon of the background. These conditions give rise to a set of quantisation constraints on the gauging parameters that will allow us to prove that the family of microstates we consider is the most general solution of our gWZW model. In addition, we will construct the physical states in the full gauged model and in its three-dimensional anti-de Sitter (AdS_3) limit. We will also derive a formula that expresses heavy-light correlators involving an arbitrary number of massless fields in the AdS_3 limit of the coset model in terms of $AdS_3 \times \mathbb{S}^3$ vacuum correlator. Finally, we prove a conjecture on three-point functions in the $SL(2, \mathbb{R})$ WZW model involving spectrally flowed vertex operators. These results constitute an important contribution for the understanding of black hole microstates beyond the usual supergravity approximation, and for AdS_3/CFT_2 holography at finite 't Hooft coupling.

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Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as: [1, 2, 3, 4]

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"The fact that we humans, who are ourselves mere collections of fundamental particles of nature, have been able to come this close to an understanding of the laws governing us in our universe, is a great triumph.

I want to share my excitement and enthusiasm about this quest.

So, remember to look at the stars and not down at your feet.
Try to make sense of what you see and wonder about what makes the universe exists.

Be curious.

And however difficult life may seem, there is always something you can do, and succeed at. It matters that you don't just give up. There are no limits to the human spirit: I believe what makes us unique is trascending our limits.

We are here together and we need to live together with tolerance and respect.
We must become global citizens.

I have been enormously privileged through my work to be able to contribute to our understanding of the universe, but it would be an empty universe indeed if it were not for the people I love and who love me.

We are all time travelers, journeying together into the future but let us work together to make that future a place we want to visit.

Be brave, be determined, overcome the odds!

It can be done."

Stephen William Hawking.
08/01/1942 - 14/03/2018

Chapter 1

Introduction

The hypothesis of the existence of a 'theory of everything' is undoubtedly one of the most fascinating concepts in science: a unique and coherent theoretical framework capable of explaining all the measurable phenomena in our universe. Despite its status of hypothesis, expectations of its existence based on past experimental and theoretical achievements are supported by Maxwell's discovery of the electromagnetic theory and that of the electroweak model made possible by the combined efforts of many theoretical physicists, including the 1979 Nobel prize winners S. Glashow, A. Salam and S. Weinberg.

Once the strong nuclear force is taken into account, our best model for the description of non-gravitational interactions is nowadays the Standard Model of Particle Physics. Despite its incredible successes, it is well known and accepted in the scientific community that a better model should be formulated: in addition to not considering gravity, the Standard Model does not explain neutrino masses, does not take into account the matter-antimatter asymmetry, and does not provide any information about the nature of dark matter and dark energy.

On the other hand, gravitational phenomena are incredibly well described by Einstein's theory of general relativity. Indeed, LIGO's first direct observation of gravitational waves [5] followed a few years later by the publication of the first image of a black hole by the Event Horizon Telescope Collaboration (EHT) [6] represent incredible milestones for the validation of Einstein's theory of gravity. Thanks to LIGO and the EHT collaborations, we have experimental evidence of the theoretical prediction of general relativity, within its regime of validity. However, also in this case, there are reasons to believe that this theory should not be the ultimate description of gravitational interactions when short distance physics or quantum mechanics is taken into account. Notorious problems in such instances include the puzzling description of cosmological and black holes' curvature singularities, and the paradoxes associated to black holes.

The current understanding of our universe is unsatisfactory and humbling. It would thus be desirable to formulate a framework in which all the known fundamental forces can be unified in a consistent fashion, while resolving the aforementioned problems. String theory is so far the only candidate for such a remarkable endeavor. String theory is a consistent theoretical framework in which gravity and (non-)abelian gauge theories naturally emerge from the quantisation of the theory on the worldsheet of closed and open strings, respectively. In addition to fundamental strings, D-branes and NS5-branes are essential constituents: black holes in string theory are low-energy descriptions of bound states of (possibly excited) strings and branes at strong coupling. It is precisely because of this large set of degrees of freedom, together with the extended nature of strings, that the paradoxes and puzzles arising in quantum descriptions of black holes can be tackled and, possibly, solved.

1.1 Black holes and paradoxes

Classically, the 'No Hair Theorem'[1] holds [8]: every four-dimensional, asymptotically flat, black hole is described *uniquely* in terms of three parameters, namely the mass M, the angular momentum J and the charge[2] Q. Once we specify these three numbers, the black hole is completely determined in a unique way. Using this fact we can immediately note that the entropy associated to such a system is $S = k_B \ln(1) = 0$, because for an observer at asymptotic infinity the number of accessible microstates Ω possessing those numbers is exactly 1. In addition, classical black holes have no entropy $S = 0$ and their temperature is zero[3] $T = 0$. In 1973, J. Bekenstein noticed [9] that the laws of thermodynamics are completely analogous to the laws that govern black holes: as an example, the area of the horizon is never decreasing precisely as the entropy of a thermodynamical system. This means that we can *formally* associate, even in the classical case, an entropy and a temperature to a black hole. Quoting Bekenstein:

> *"It is then natural to introduce the concept of black-hole entropy as the measure of the inaccessibility of information (to an exterior observer) as to which particular internal configuration of the black hole is actually realized [...]."*

Unfortunately, classically we can not have a non-trivial entropy and temperature for a black hole, hence it seemed that this analogy was simply formal. However, taking into account quantum effects one can solve this puzzle: in 1974 Hawking [10, 11] was able to show that a black hole can emit a thermal radiation[4], thus implying that we

[1] More precisely, the "hairs" are fields associated to a stationary black hole and, today, we should speak about the "No Scalar-Hair Conjecture" [7].

[2] More generally, multiple electric and magnetic charges are allowed.

[3] This is often stated as "classical black holes are really black", they do not emit radiation.

[4] The radiation is thermal but not Planckian since it is modified by the absorption cross section.

can associate a temperature to a black hole. In addition, the entropy is proportional to the area of the black hole horizon, in Planck units. Hawking's original calculation [11] showed that the average number of outgoing particles, in a fixed frequency mode ω, is distributed in accordance with a thermal spectrum

$$\langle N \rangle \sim \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \,. \tag{1.1}$$

In addition, later it was shown [12, 13] that the full probability distribution (and not just the average number) is that of a thermal radiation.

The particle emission from the black hole is in agreement with the 'Generalised Second Law' of thermodynamics, which states that:

$$\frac{d}{dt}\left(S_{BH} + S_{rad}\right) \geq 0 \,, \tag{1.2}$$

where S_{rad} is the entropy of the emitted radiation and S_{BH} is the **B**ekenstein-**H**awking entropy of a **B**lack **H**ole

$$S_{BH} = \frac{k_B c^3}{\hbar G_N}\frac{A_{\mathcal{H}}}{4} = \frac{A_{\mathcal{H}}}{4} \,, \tag{1.3}$$

where $A_{\mathcal{H}}$ is the surface area of the horizon, k_B is the Boltzmann constant, c is the speed of light, $\hbar$ is the reduced Planck constant and G_N is the Newton constant. In the second equality we have set $\hbar = c = k_B = G_N = 1$, a convention we will adopt for the rest of the thesis unless otherwise stated. The fact that the entropy is proportional to the area of the black hole, and not proportional to its volume, is remarkable. This led to the idea of the Holographic Principle [14, 15], and one explicit realization of holography is the celebrated AdS/CFT duality [16, 17, 18]. Note that the above discussion implies that the dimension of the Hilbert space associated to the microstates of the black hole (within the full Quantum Gravity theory) is

$$\dim(\mathcal{H}_{\text{micro}}) \sim e^S \,, \tag{1.4}$$

with possible subleading corrections omitted. This is the expected number of microstates for a black hole with entropy S. As an example, the Bekenstein-Hawking entropy of a Schwarzschild black hole of mass M is

$$S_{\text{Schw}} \sim 10^{76}\left(\frac{M}{M_{\text{sun}}}\right) , \tag{1.5}$$

which is very large. The discrepancy between the classical and the quantum prediction is enormous. The Hawking temperature of a Schwarzschild black hole is

$$T_H = \frac{\hbar c^3}{k_B}\frac{1}{8\pi G_N M} \sim 6 \times 10^{-8}\left(\frac{M_{\text{sun}}}{M}\right) K \,. \tag{1.6}$$

The bigger the black hole, the lower its temperature. More generally the temperature is given in terms of the surface gravity κ as

$$T_H = \frac{\hbar c^3}{G_N k_B} \frac{\kappa}{2\pi} = \frac{\kappa}{2\pi}, \tag{1.7}$$

where the surface gravity κ is defined by

$$\kappa\, k^\mu = k^\nu \nabla_\nu k^\mu \bigg|_{\mathcal{H}}, \tag{1.8}$$

where k^μ is the Killing vector associated with the Killing horizon. As an example, the surface gravity for a four-dimensional Kerr-Newmann black hole is

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)} = \frac{\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}}, \tag{1.9}$$

where

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}, \qquad a = \frac{J}{M}. \tag{1.10}$$

1.1.1 Black hole thermodynamics

Black hole mechanics is governed by the following four laws, similar to those of thermodynamics:

0. The 'Zeroth Law' states that the surface gravity κ is constant over the horizon. This is analogous to thermal equilibrium.

1. The 'First Law' states energy conservation: the change in mass M of the black hole is related to the change of its horizon area $A_{\mathcal{H}}$, angular momentum J and charge Q by
$$\delta M = \frac{\kappa}{8\pi}\delta A_{\mathcal{H}} + \Omega \delta J + \Phi \delta Q, \tag{1.11}$$
where Ω is the angular velocity and Φ is the electrostatic potential.

2. The 'Second Law' states that the total entropy of a system consisting of a black hole and matter never decreases
$$dS_{\text{Tot}} = dS_{BH} + dS_{\text{matter}} \geq 0. \tag{1.12}$$

3. The 'Third Law' states that it is impossible to reduce the surface gravity κ to zero by a finite sequence of operations. It is analogous to Nernst's Law.

1.1.2 The Entropy Puzzle

Imagine the following *Gedankenexperiment*: consider a box with some gas (thus with a non-trivial entropy) and throw it into an asymptotically flat black hole. After a sufficiently long time, an asymptotic observer is not able to resolve the presence of the highly red-shifted matter next to the black hole horizon. Since the gas, and hence his entropy, is not accessible anymore we have decreased the entropy of the observable universe, naïvely violating the second law of thermodynamics. The solution of this problem is to associate an entropy to a black hole (1.3), following the intuition of Bekenstein and Hawking. In turn, this give rise to another profound question: what are the *microstates* of the black hole? This is the so-called Entropy Puzzle.

In a seminal paper, Strominger and Vafa [19] provided, within the string theory framework, the first microphysical calculation of a black hole entropy in terms of strings. They gave the first microscopical explanation for the Bekenstein-Hawking formula, and they showed that the entropy of a black hole is precisely a quarter of the area of the event horizon (in Planck units), as expected. This computation is perceived as one of the major successes of string theory. For an introductory review and for a conceptual analysis of the Strominger-Vafa results see also [20].

1.1.3 The Information Paradox

We have learned that black holes have an entropy, a temperature, and they radiate following a thermal distribution, but what is the physical origin of the Hawking radiation? An explanation, often given to respond to the above answer in a more intuitive fashion, is the following: quantum fluctuations of the vacuum create particle-antiparticle pairs and, in the presence of the gravitational field sourced by the black hole, one of the two particles (just inside the horizon) falls towards the singularity, while the other particle (just outside the horizon) can escape towards a far away observer. The infalling particle has negative energy[5] and subtracts mass to the black hole, which thus evaporates. These escaping particles constitutes the Hawking radiation. The radiation which emerges is *not* in a pure quantum state: indeed, the emitted quanta are in a mixed state. Before discussing the information paradox, let us recall some basic definitions.

Definition 1.1. A state ρ of a quantum system is a self-adjoint operator that is

1. Trace-class[6]

$$\mathrm{Tr}(\rho) = \sum_{e_n} \langle e_n, \rho\, e_n \rangle < \infty \,, \tag{1.13}$$

   where $\{e_n\}_n$ are elements of *any* orthonormal basis for a Hilbert space $\mathcal{H}$.

[5] With respect to the observer, i.e. with respect to the timelike Killing vector defined at infinity.

[6] Here we are remarking the fact that not every self-adjoint operator on an infinite-dimensional vector space has a trace defined. If this happens, it is called trace class.

2. Unit trace

$$\mathrm{Tr}(\rho) = 1\,, \tag{1.14}$$

3. Non-negative

$$\langle \varphi, \rho\, \varphi \rangle \geq 0\,, \quad \forall \varphi \in \mathcal{H}\,. \tag{1.15}$$

Remark 1.2. The second and the first points in the above definition above can be combined by noting that, when well defined, the trace is independent of the orthonormal basis chosen.

Definition 1.3. Often, the above ρ is referred to as density matrix. We can thus equivalently say that a density matrix is a non-negative unit-trace self-adjoint operator.

Definition 1.4. Given a Hilbert space $\mathcal{H}$, given two states $\rho_1, \rho_2 \in \mathcal{H}$ and $c_1, c_2 \geq 0$ such that $c_1 + c_2 = 1$, a state $\Psi \in \mathcal{H}$ is called convex combination of ρ_1, ρ_2 if it is written as

$$\Psi = c_1\rho_1 + c_2\rho_2 = c_1\rho_1 + (1 - c_1)\rho_2\,. \tag{1.16}$$

Definition 1.5. A state Ψ is called pure if it can not be written as a convex combination of $\rho_1, \rho_2 \neq \Psi$, unless trivially $c_1 = 1, c_2 = 0$ or $c_1 = 0, c_2 = 1$.

Remark 1.6. Pictorially, pure states are the extreme points of a convex set of states.

Definition 1.7. An equivalent formulation of a pure state can be given in terms of the density matrix ρ, by looking at its properties. Given[7] $|\psi\rangle \in \mathcal{H}$ a state is called a pure state if the density matrix can be written as

$$\rho = \frac{\langle \psi, \,\cdot\, \rangle\, \psi}{\langle \psi, \psi \rangle} = \frac{|\psi\rangle \otimes \langle \psi|}{||\psi||^2}\,, \tag{1.17}$$

Equivalently, $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$.

Definition 1.8. All the other states, for which such $|\psi\rangle \in \mathcal{H}$ does not exist, are called mixed states. The density matrix is written, as opposed to the one above, as

$$\rho = \sum_{\ell \geq 1} c_\ell\, |\psi_\ell\rangle \otimes \langle \psi_\ell|\;, \qquad c_\ell \geq 0\,, \quad \sum_{\ell \geq 1} c_\ell = 1\,. \tag{1.18}$$

Remark 1.9. Note that for a pure state we have $c_1 = 1,\; c_\ell = 0, \ell > 1$ and the above decomposition is trivial. Equivalently, a state is mixed if $\mathrm{Tr}(\rho^2) < 1$.

Remark 1.10. Let us denote the density matrix associated to ψ_ℓ as ρ_{ψ_ℓ}. One should note that, given $c \in (0, 1)$,

$$c\rho_{\psi_\ell} + (1 - c)\rho_{\psi_k} \neq \rho_{c\psi_\ell + (1-c)\psi_k}\,, \tag{1.19}$$

since the left hand side describes a mixed state (convex combination of density matrices) while the right hand side is a pure state. Indeed, using (1.17), one can easily check that the right hand side satisfies $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$.

[7] Note that we are considering also the case when $|\psi\rangle$ is not normalized.

Definition 1.11. Consider two systems A, B with associated Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and basis $\{|\psi_i\rangle_A\}_i, \{|\psi_j\rangle_B\}_j$ respectively. The most general state Ψ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is given by

$$|\Psi\rangle = \sum_{i,j} c_{ij} |\psi_i\rangle_A \otimes |\psi_j\rangle_B \ . \tag{1.20}$$

The state $|\Psi\rangle$ is called **separable** if $c_{ij} = c_i^A c_j^B$, so that

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \ , \qquad |\psi_{A,B}\rangle = \sum_k c_k^{A,B} |\psi_k\rangle_{A,B} \ . \tag{1.21}$$

Otherwise, the state is called **entangled** (or inseparable).

Mixed states inevitably arise from pure states when, for a composite quantum system defined on $\mathcal{H}_1 \otimes \mathcal{H}_2$ with an entangled state constructed with states living in their own Hilbert space, the part $\mathcal{H}_2$ is inaccessible to an observer. The state is thus expressed with a partial trace over $\mathcal{H}_2$. It is important to note that, if we have a pure state and then we perform a partial trace, we lose information about the system we are tracing over. This does not imply that the original system was not in a pure state, but rather that we have an effective description of the system, and we are left with a mixed state. Nevertheless, in principle, we should be able to recover the fine-grained description and hence have access to the properties of the original state.

The information paradox does not solely rely on the thermal nature of the emitted radiation. It is a consequence of the combined properties of the radiation *and* entanglement. Let's explain this carefully, following [21]. According to Hawking's arguments, one can show that the state of the emitted radiation is of the form

$$\begin{aligned} |\psi\rangle_{\text{emit}} &= C\left(|0\rangle_b \otimes |0\rangle_c + \gamma\, \hat{b}^\dagger |0\rangle_b \otimes \hat{c}^\dagger |0\rangle_c + \frac{1}{2}\gamma^2\, \hat{b}^\dagger \hat{b}^\dagger |0\rangle_b \otimes \hat{c}^\dagger \hat{c}^\dagger |0\rangle_c + \dots\right) \\ &= C\left(|0\rangle_b \otimes |0\rangle_c + \gamma\, |1\rangle_b \otimes |1\rangle_c + \gamma^2\, |2\rangle_b \otimes |2\rangle_c + \dots\right) \ , \end{aligned} \tag{1.22}$$

where $|0\rangle_b, |0\rangle_c$ are two different vacua of two different locations in the curved spacetime, a crucial feature of quantum field theory on a curved background. Without entering in details about the origin of this state, we can immediately note that the above state is not factorised. This implies that the subsystems are entangled. Suppose now that the quanta created by the operator $\hat{c}^\dagger$ fall towards the singularity while the quanta created by the operator $\hat{b}^\dagger$ propagate to infinity. The two are entangled. If the black hole evaporates, the outgoing radiation is entangled with nothing. We know that the emitted quanta cannot be described by a pure state, but now we cannot write a mixed state either since we have nothing to mix the radiation with! The only way to describe the system B is by using a reduced density matrix ρ_B. This description is statistical rather than quantum-mechanical. It is thus impossible to reconstruct the whole system (i.e. obtain a pure state) and the information about the original pure system is lost, violating unitarity.

Proposition 1.12 (Information Paradox)**.** *The information paradox states the inability to reconstruct the original pure state from the thermal radiation collected by an observer at infinity due to the disappearance of entangled quanta during the black hole evaporation process. Hence, for an observer at infinity, the original pure state has undergone a non-unitary transformation, in contrast to quantum mechanics' unitarity postulate.*

For this reason, a modification in the formulation of quantum mechanics was originally deemed necessary by Hawking. However, as we will discuss below, using string theory's description of black holes (instead of quantum field theory on curved spacetime) it is possible to preserve the postulates of quantum mechanics *and* potentially solve the information paradox.

How can we solve this paradox? One possibility would be to consider *small*, *local* and subtle correlations between the emitted quanta. In other words, one could imagine that perturbative corrections to Hawking's computation could lead to a successful unitary picture of black hole evaporation. Under certain physically-reasonable assumptions, one can show however that this is not possible. Mathur [22] proved the following "Small correction" no-go theorem that we recall without proof:

Theorem 1.13 (Mathur's small correction no-go theorem)**.** *Consider a black hole solution of mass M and Schwarzschild radius R_S. Define ℓ_P, m_P to be the Planck length and mass respectively. Consider the following hypothesis (not all independent):*

1. *Small curvatures: the quantum state is defined on a spacelike slice whose intrinsic and extrinsic (when embedded in a 4D spacetime) curvatures are small: ${}^{(3)}R \ll \frac{1}{\ell_P}$, $K \ll \frac{1}{\ell_P}$. Furthermore, the spacetime curvature in a neighbourhood of the spacelike slice is small as well: ${}^{(4)}R \ll \frac{1}{\ell_P}$. We assume this to be true at sufficiently large times, as long as the mass of the evaporating black hole is $M \gg m_P$.*

2. *Energy conditions: we assume that the matter fields obey the usual energy conditions (say, the dominant energy condition).*

3. *Local Hamiltonian Evolution. The quantum state of the matter evolves according to a local Hamiltonian.*

4. *Information-free horizon (traditional black hole): we assume that at any point in the neighbourhood of the event horizon, the evolution of field modes with wavelength $\ell_P \ll \lambda \lesssim R_S$ is given by the semi-classical evolution of quantum fields on empty curved spacetime, up to terms suppressed as m_P/M. In other words, the No-Hair theorem holds.*

Under the above assumptions, the formation and evaporation of the black hole will lead to mixed states or remnants.

Roughly put, the theorem is telling us that, under physically-reasonable assumptions, small and local corrections are not enough to restore unitarity: we would be left with a mixed state or remnants. To evade this no-go theorem, at least one of the assumptions must be violated.

Recall that remnants [23] are long-lived objects with bounded mass and size but unbounded entanglement with a distant external system. Their formation is conjectured to have place when quantum gravity stabilising effects take over during the final stages of the evaporation process of a black hole, whose size approaches the Planck scale. The system comprised of a remnant and Hawking radiation would be in a pure state. However, their existence is nowadays deemed unlikely due to their conflict with Bekenstein's entropy bound and the generalised second law of thermodynamics [24, 25]. For other arguments against their existence see also [26, 27]. From now on we assume that the unique final stage of the evaporation process under the assumptions of Mathur's theorem is a thermal radiation.

A possible way to evade the above claim is to consider some sort of non-local interactions between the interior of the black hole and the asymptotic observer, thus relaxing the assumption (3) of the theorem. This idea is part of the so-called 'island paradigm' [28, 29, 30, 31], for a review see [32]. In this approach, most successfully applied to 2D gravity, the formula for the entanglement entropy of the radiation is computed via a gravitational path integral. The authors assume the validity in flat space-time of the so-called Quantum Extremal Surface proposal [33], which states that the fine-grained (i.e. Von Neumann) entropy of the radiation is given by an extremisation problem over appropriate spacetime regions. The solution to this extremisation problem is called (quantum) extremal surface. Their work suggests that, in order to account for the entanglement entropy of the Hawking radiation, there must exist a disconnected spacetime region called "islands" located in the interior of the black hole. By considering Hawking modes in this interior region together with the outgoing radiation one is able to recover the Page curve, nowadays considered a necessary condition for solving the information paradox. However, the description of the final state of the evaporation process is still problematic: the radiation at infinity collected by an observer is still mixed and is unclear how to recover the full pure state from it, especially in the context of four-dimensional gravity. Furthermore, one assumes some non-local relation between the interior of the black hole spacetime and the collected modes that seem not justified *a priori*.

More criticisms have been moved towards this set of semi-classical ideas. In particular, the authors of [34] highlight the contrasts with the so-called 'Fuzzball proposal' [22], which instead is based on the knowledge of the microscopic constituents of black hole and utilises fundamental ingredients from string theory. For an enlightening review see [35]. The main claim of this proposal can be summarised as follows:

Proposition 1.14 (The Fuzzball Proposal)**.** *Consider a low-energy effective theory of string theory, with a black hole solution. The Fuzzball proposal claims that quantum gravity effects are non-negligible on the scale of the would-be horizon.*

The point of the above claim is that quantum gravity effects cease to be negligible already at scales of the order of the *would-be* black hole horizon [36], violating the assumption (4) of Mathur's small correction theorem. The fuzzballs in string theory are the microstates of the black hole, the latter being a low-curvature solution of the low-energy effective theory of string theory. Hence, the black hole represents a *coarse-grained* description of the underlying stringy system. In other words, such effective description of the stringy bound state has the characteristic features of the black hole like, for instance, its highly absorptive nature. It is widely expected that only a subset of all the possible Fuzzballs will be well-described by solutions of the low-energy theory (for instance, supergravity) and thus we expect the existence of some "stringy" fuzzballs, not well-described in the supergravity approximation. Out of these stringy solutions, one can recognize coherent-like states that well approximate classical behaviors: for black hole microstates, we expect them to be described by smooth and horizonless solutions of the supergravity equations of motion.

1.2 Outline of the thesis

The aim of this thesis is to present research results published in [1, 2, 3, 4], which constitute a set of original contributions for the understanding of the role of string theory in the resolution of black hole paradoxes and their internal structure. Most of the thesis is devoted to the study of a family of non-supersymmetric microstates called JMaRT (after the authors of [37]) that, in a special regime, admit an α'-exact worldsheet description in terms of gauged Wess-Zumino-Witten (gWZW) models. The thesis is organised as follows.

In Chapter 2 we provide the necessary technical ingredients for the understanding of the subsequent chapters. In particular, we consider compactifications of Type IIB supergravity and the D1-D5-P black hole. After reviewing the essential properties of this well-studied system, we show how its weak coupling description provides a natural and elegant explanation for the existence of the dual D1-D5 conformal field theory (CFT). We review the marvelous relation between the attracted moduli space of the D1-D5-P system, the dual CFT moduli space and the instanton moduli space. We then proceed in setting up the notation for the D1-D5 CFT system at the orbifold point. From there, and for the rest of the thesis, we focus on worldsheet techniques. We review the Ramond-Neveu-Schwarz (R-NS) formalism for superstrings in $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ with pure NS-NS fluxes and, finally, we present the null-gauged Wess-Zumino-Witten models that encode the dynamics of a closed string in the NS5-decoupling limit of the JMaRT background.

In Chapter 3 we present the results contained in [1]. We show that the consistency of the spectrum of the worldsheet CFT implies a set of quantisation conditions and parity restrictions on the gauging parameters. We also derive these constraints from an independent geometrical analysis of smoothness, absence of horizons and absence of closed timelike curves. This allows us to prove that the complete set of consistent backgrounds in this class of models is precisely the general family of (NS5-decoupled) JMaRT solutions, together with their various (BPS and non-BPS) limits. We clarify several aspects of these backgrounds by expressing their six-dimensional solutions explicitly in terms of five non-negative integers and a single length-scale. Finally we study non-trivial two-charge limits, and exhibit a novel set of non-BPS supergravity solutions describing bound states of NS5 branes carrying momentum charge.

Chapter 4 is an exposition of the PRL publication [2] and its extension [3]. Here we compute a large collection of string worldsheet correlators describing light probes interacting with heavy black hole microstates. We construct physical vertex operators in these cosets, including all massless fluctuations. We compute a large class of novel heavy-light-light-heavy correlators in the AdS_3 limit, where the light operators include those dual to chiral primaries of the holographically dual CFT. We compare a subset of these correlators to the holographic CFT at the symmetric product orbifold point, and find precise agreement in all cases, including for light operators in twisted sectors of the orbifold CFT. The agreement is highly non-trivial, and includes amplitudes that describe the analogue of Hawking radiation for these microstates. We further derive a formula for worldsheet correlators consisting of n light insertions on these backgrounds, and discuss which subset of these correlators are likely to be protected. As a test, we compute a heavy-light five-point function, obtaining precisely the same result both from the worldsheet and the symmetric orbifold CFT.

In Chapter 5 we present the results contained in [4]. AdS_3 correlators are essential building blocks for the correlators in the JMaRT microstate backgrounds. However, correlation functions of the $SL(2,\mathbb{R})$-WZW model involving spectrally flowed vertex operators are notoriously difficult to compute. An explicit integral expression for the corresponding three-point functions was recently conjectured in [38]. In this chapter, we provide a proof for this conjecture. For this, we extend the methods of [39] based on the so-called $SL(2,\mathbb{R})$ series identifications, which relate vertex operators belonging to different spectral flow sectors. We also highlight the role of holomorphic covering maps in this context. These results constitute an important milestone for proving this instance of the AdS_3/CFT_2 holographic duality at finite 't Hooft coupling.

Chapter 2

Supergravity, D1-D5 CFT, and worldsheet string theory

In this chapter we provide the technical tools necessary to understand the original results published in [1, 2, 3, 4] and presented in the remaining chapters of the thesis. We start by reviewing the relevant supergravity compactifications and some important solutions in six-dimensions. We then focus on the D1-D5(-P) system and recall the derivation of the D1-D5 CFT in both the strong coupling and weak coupling regimes, and review the description of the dual CFT at the orbifold point. In doing so, we review the incredibly fascinating, albeit subtle, relation between the attracted supergravity moduli space, the dual CFT moduli space and the instanton moduli space in the context of the D1-D5-P system. We follow with an extensive review of the R-NS formulation of superstring theory in $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ with pure NS-NS fluxes and in a particular family of coset models which play a crucial role for the understanding of string theory in a specific family of black hole microstates.

2.1 Supergravity and compactifications

Here we recall basic facts about Type IIB supergravity and some compactifications that will play a prominent role for the rest of the thesis.

2.1.1 Type IIB Supergravity

We start by considering Type IIB Supergravity, the massless truncation of Type IIB Superstring theory. The fields are given by the following representations of $\mathfrak{so}(8)$:

- NS-NS: Dilaton, B-field, Graviton $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v = \{\Phi, B^{(2)}, G\}$

- NS-R: Dilatino, Gravitino $\mathbf{8}_c \oplus \mathbf{56}_c = \{\lambda_\alpha^{(1)}, \chi_\alpha^{(1),\mu}\}$
- R-NS: Dilatino, Gravitino $\mathbf{8}_c \oplus \mathbf{56}_c = \{\lambda_\alpha^{(2)}, \chi_\alpha^{(2),\mu}\}$
- R-R: 0-form (Axion), 2-form, 4-form $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_s^{(+)} = \{C^{(0)}, C^{(2)}, C^{(4),SD}\}$

where v, s, c label the three different representations of $\mathfrak{so}(8)$ related by triality, and the (SD) in the RR 4-form the denotes the self duality of its field-strength. The two gravitini have the same chirality. The theory has 128_B bosonic and 128_F fermionic (on-shell) degrees of freedom, as supersymmetry requires. Indeed, the IIB supergravity theory is a chiral $\mathcal{N} = (2,0)$ theory in 10 dimensions with 32 supercharges[8]. The IIB theory is also obtained from superstring theory, by choosing the Ramond vacua to have the same eigenvalue under the worldsheet fermion number $(-1)^F = (-1)^{\bar{F}}$ when performing the GSO projection.

The string-frame action for the bosonic fields of Type IIB Supergravity is given by

$$S_{IIB}^{(Str)} = \frac{1}{2\kappa^2}\Bigg\{ \int d^{10}x\sqrt{-G}\, e^{-2\Phi}\, R - \frac{1}{2}\int \Big[e^{-2\Phi}\left(-8\, d\Phi \wedge \star d\Phi + H^{(3)} \wedge \star H^{(3)}\right) + F^{(1)} \wedge \star F^{(1)} + \tilde{F}^{(3)} \wedge \star \tilde{F}^{(3)} + \frac{1}{2}\tilde{F}^{(5)} \wedge \star \tilde{F}^{(5)} + C^{(4)} \wedge H^{(3)} \wedge F^{(3)} \Big] \Bigg\}, \tag{2.1}$$

where $2\kappa^2 = 16\pi G_N^{10D} = (2\pi)^7 g_s^2 \ell_s^8$ and

$$G_N^d = \frac{G_N^{10D}}{(2\pi)^{10-d} V_{10-d}}, \qquad G_N^{10D} = 8\pi^6 g_s^2 \ell_s^8 . \tag{2.2}$$

We have used $H^{(3)} = dB^{(2)}$ and

$$\tilde{F}^{(3)} = F^{(3)} - C^{(0)} \wedge H^{(3)}, \qquad \tilde{F}^{(5)} = F^{(5)} - \frac{1}{2}C^{(2)} \wedge H^{(3)} + \frac{1}{2}B^{(2)} \wedge F^{(3)}, \tag{2.3}$$

with $F^{(n)} = dC^{(n-1)}$, and the self-duality of $\tilde{F}^{(5)}$ must be imposed on-shell. In Einstein frame[9] the same action becomes

$$S_{IIB}^{(Ein)} = \frac{1}{2\kappa^2}\Bigg\{ \int d^{10}x\sqrt{-G}\, R - \frac{1}{2}\int \Big[d\Phi \wedge \star d\Phi + e^{-\Phi} H^{(3)} \wedge \star H^{(3)} + e^{2\Phi} F^{(1)} \wedge \star F^{(1)} + e^{\Phi}\tilde{F}^{(3)} \wedge \star \tilde{F}^{(3)} + \frac{1}{2}\tilde{F}^{(5)} \wedge \star \tilde{F}^{(5)} - C^{(4)} \wedge H^{(3)} \wedge F^{(3)} \Big] \Bigg\}. \tag{2.4}$$

The action in Einstein frame can be rewritten in a manifestly $SL(2,\mathbb{R})$-invariant fashion, hence the symmetry group is classically $SL(2,\mathbb{R})$. However, in the full string theory it is broken to a discrete $SL(2,\mathbb{Z})$ subgroup which includes T- and S-duality transformations.

[8]The spacetime supercharges are the integrals of the *worldsheet* gravitini's vertex operators.

[9]The string and Einstein frame metric are related by $G_{MN,Ein} = e^{-\Phi/2}\, G_{MN,Str}$, where Φ is the Dilaton.

The RR-forms $C^{(p+1)}$ couple to Dp-branes. The string frame action of a stack of N coincident Dp-brane is the sum of the Dirac-Born-Infeld (DBI) and a Wess-Zumino (WZ) term, namely

$$\begin{aligned} S^{(Str)}_{Dp} =& - T_{Dp} \int_{\mathcal{M}_{p+1}} d^{p+1}x\, e^{-\Phi}\, \mathrm{Tr}\left(\sqrt{-\det\left(G_{\alpha\beta} + 2\pi\alpha'\,\mathcal{F}_{\alpha\beta}\right)}\right) \\ &+ \mu_p \int_{\mathcal{M}_{p+1}} \sum_q C^{(q)} \wedge \mathrm{ch}\,(\mathcal{F}) \wedge \sqrt{\frac{\widehat{A}(R_{T\mathcal{M}_{p+1}})}{\widehat{A}(R_{N\mathcal{M}_{p+1}})}}\,, \end{aligned} \tag{2.5}$$

where the brane tension is given by

$$T_{Dp} = \frac{1}{g_s (2\pi)^p\, (\alpha')^{\frac{p+1}{2}}} = \frac{1}{g_s \ell_s (2\pi\ell_s)^p}\,, \tag{2.6}$$

and $\mu_p = \pm T_{Dp}$ is the RR charge for the (anti-)branes. In the DBI term, the pull-back on the brane worldvolume $\mathcal{M}_{p+1}$ of the metric tensor and of the gauge invariant field strength is denoted by $G_{\alpha\beta}$ and $2\pi\alpha'\,\mathcal{F}_{\alpha\beta} = 2\pi\alpha'\,F^{(2)}_{\alpha\beta} + B^{(2)}_{\alpha\beta}\mathbb{1}$, $\alpha, \beta = 0, \ldots, p$ respectively. Here $F^{(2)}$ (not to be confused with the above RR field strengths) is the curvature of the connection $A^{(1)}$ of the $U(N)$ bundle over $\mathcal{M}_{p+1}$, while the Kalb-Ramond field $B^{(2)}$ is a singlet under the $U(N)$ group (and only couples to $\mathrm{Tr}(F)$). In the WZ term, the formal sum is over the pull-back RR-forms $C^{(q)}$ and is understood that in the wedge products one has to consider only $(p+1)$-forms. The Chern class is $\mathrm{ch}\,(\mathcal{F}) = \mathrm{Tr}\left(e^{2\pi\alpha'\mathcal{F}}\right)$ and $\widehat{A}(R)$ is the Dirac (A-roof) polynomial of the tangent or normal bundle, which can be written in terms of Pontryagin classes as $\hat{A}(R) = p_0 - \frac{1}{24}p_1 + \frac{1}{5670}(7p_1^2 - 4p_2) + \ldots$, where $p_0 = 1, p_1 = -\frac{1}{2}\frac{1}{(2\pi)^2}\,\mathrm{Tr}(R^2), p_2 = \frac{1}{8}\frac{1}{(2\pi)^4}\left[(\mathrm{Tr}\,R^2)^2 - 2\,\mathrm{Tr}\,R^4\right]$ where R is the curvature 2-form of the bundle and $R^n = \wedge^n R$. The CS term does not require a metric and is independent on the choice of the connection of the bundle. For future reference, note that

$$\begin{aligned} S_{CS} = \mu_p \int_{\mathcal{M}_{p+1}} &\Big\{ N\, C^{(p+1)} + 2\pi\alpha'\, C^{(p-1)} \wedge \mathrm{Tr}\ F^{(2)} \\ &+ \frac{(2\pi\alpha')^2}{2} C^{(p-3)} \wedge \left[\mathrm{Tr}(F^{(2)} \wedge F^{(2)}) + \frac{N}{48}\left(\mathrm{Tr}(R^{(2)}_T \wedge R^{(2)}_T) - \mathrm{Tr}(R^{(2)}_N \wedge R^{(2)}_N)\right)\right] \Big\}\,, \end{aligned} \tag{2.7}$$

with N the number of branes, showing the appearance of lower-dimensional branes inside the worldvolume of the Dp-brane [40].

2.1.2 Type IIB on $\mathbb{T}^5$: maximal 5D $\mathcal{N} = 8$ Supergravity

Consider Type IIB Supergravity compactified on $\mathbb{T}^5$. Toroidal compactifications preserve the original amount of supersymmetries, hence the number of supercharges in five dimensions is still 32. The resulting theory is the maximally supersymmetric five-dimensional $\mathcal{N} = 8$ supergravity. By compactifying Type IIB supergravity to five dimensions on circles, the U-duality group[10] is given by $E_{6,(6)}$. In the full string theory, the duality group is broken by quantum effects to a discrete subgroup $E_{6,(6)}(\mathbb{Z})$. This group contains both the T-duality group $O(5,5;\mathbb{Z})$ and the S-duality group $SL(2,\mathbb{Z})$ [43]. When compactifying on the $\mathbb{T}^5$, the massless fields of Type IIB Supergravity give rise to 27 $U(1)$ gauge fields and 42 scalars. The 27 vectors, transforming in the fundamental representation **27** of $E_{6,(6)}$, are given by:

$$\{(5)\, G_{\mu i},\ (5)\, B_{\mu i},\ (5)\, C_{\mu i},\ (10)\, C_{\mu ijk},\ (1)\, A^1_\mu,\ (1)\, A^2_\mu\}\,, \qquad \mu = 0, \ldots 4,\ i = 5, \ldots 9. \tag{2.8}$$

The two (magnetic) vectors $A^{1,2}_\mu$ are obtained as follows. Consider the NS B-field $B^{(2)}_{\mu\nu}$ and the RR 2-form $C^{(2)}_{\mu\nu}$. Their exterior derivate give rise to two field strength 3-forms, namely $H_{\mu\nu\rho}$ and $F_{\mu\nu\rho}$. Using Hodge duality in five dimensions, these can be dualized into two (magnetic) field strengths $F^1_{\mu\nu}$ and $F^2_{\mu\nu}$ respectively. We have $F^{1,2} = dA^{1,2}$. The above vectors define 27 abelian charges, namely 5 KK momenta, 5 F1 winding modes, 5 D1 winding modes, 10 D3 winding modes, 1 NS5 winding mode for wrapping the $\mathbb{T}^5$, 1 D5 winding mode for wrapping the $\mathbb{T}^5$. Note that the spectrum of charges is compatible with S-duality.

The 42 scalars parametrise the moduli space of the supergravity theory $E_{6,(6)}/Sp(4)$, where $Sp(4) = USp(8)$, which is $9 \times 8/2 = 36$ dimensional, is the maximally compact subgroup of $E_{6,(6)}$ [43, 44]. The scalars are given by

$$\{(15)\, G_{ij},\ (10)\, B_{ij},\ (10)\, C_{ij},\ (5)\, C_{ijkl},\ (1)\, \Phi,\ (1)\, C^0\}\,, \qquad i = 5, \ldots 9\,. \tag{2.9}$$

Taking into account discrete identifications, the moduli space is given by

$$E_{6(6)}(\mathbb{Z})\backslash E_{6(6)}/USp(8)\,. \tag{2.10}$$

2.1.3 Type IIB on $\mathbb{T}^4$: maximal 6D $\mathcal{N} = (2,2)$ Supergravity

In this subsection we briefly review the structure of the moduli space of vacua for the Type IIB Supergravity theory on $\mathbb{T}^4$.

Consider the ten-dimensional spacetime to be of the form $\mathbb{R}^{1,5} \times \mathbb{T}^4$, and compactify the $\mathbb{T}^4$. One obtains the maximal $\mathcal{N} = (2,2)$ Supergravity in six dimensions. This

[10]The maximally non-compact (split) real form of a Lie group of rank n has precisely n more non-compact generators than compact ones, see for example [41]. For instance, the number of non-compact generators of $\mathfrak{e}_{8,(8)}$,$\mathfrak{e}_{7,(7)}$ and $\mathfrak{e}_{6,(6)}$ are respectively 128, 70, 42. Recall that the maximally non-compact exceptional (classical) global symmetry groups $E_{n,(n)}$ arise after $\mathbb{T}^n$ compactifications of M-theory or $\mathbb{T}^{n-1}$ compactifications of Type II string theory. Note that one can continue the exceptional sequence as $\mathfrak{e}_{5,(5)} \equiv \mathfrak{so}(5,5)$, $\mathfrak{e}_{4,(4)} \equiv \mathrm{sl}(5,\mathbb{R})$, $\mathfrak{e}_{3,(3)} \equiv \mathrm{sl}(3,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$, $\mathfrak{e}_{2,(2)} \equiv \mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$, $\mathfrak{e}_{1,(1)} \equiv \mathrm{sl}(2,\mathbb{R})$, see page 314 in [42].

is equivalent to the $\mathbb{T}^5$ reduction of 11D Supergravity, yielding an $E_{5,(5)} = SO(5,5)$ global symmetry in 6-dimensions, which can be seen as the unbroken maximally non-compact subgroup of the exceptional group $E_{6,(6)}$, after one considers an $\mathbb{S}^1 \subset \mathbb{T}^5$ to be much bigger than the other compact one-cycles. Similarly, the U-duality group of the full string theory $E_{6(6)}(\mathbb{Z})$ gets broken to the subgroup $E_{5(5)}(\mathbb{Z}) = SO(5,5;\mathbb{Z})$. Consequently, the **27** of $E_{6,(6)}$ gets broken to the **10** (vector), **16** (spinor), **1** (singlet) representation of $SO(5,5)$.

The bosonic content is given by 1 graviton, 5 tensor gauge fields, 16 vectors, 25 scalars. The five 2-forms should be thought in terms of their self-dual and anti-self-dual parts

$$\text{five 2-forms } = \left(B^{+I}_{\mu\nu}, B^{-I}_{\mu\nu}\right) , \qquad I = 1, \dots, 5 , \tag{2.11}$$

which transform in the **10** of $SO(5,5)$. The vectors, on the other hand, transform in the spinor **16** of $SO(5,5)$. The R-Symmetry group[11] is $USp(4) \times USp(4) \simeq SO(5) \times SO(5)$. The 25 scalar parametrize the Teichmuller space[12]

$$\mathcal{K} = \frac{SO(5,5)}{SO(5) \times SO(5)} . \tag{2.12}$$

To obtain the fermionic content of the 6D theory we need to reduce the two dilatini and the two gravitini from 10 to 6 dimensions. We also recall that in 10 dimension the minimal spinor representation is that of a Majorana-Weyl spinor of dimension 16, while in 6 dimensions the minimal spinor representation is that of a symplectic-Weyl (or symplectic-Majorana) spinor and has dimension 8. Making use results present in App. B of [45] Eq. (B.1.43) and (B.1.44), the fermionic content of the maximal $\mathcal{N} = (2,2)$ Supergravity theory in 6D is given by $(4+4)$ gravitini and $(20+20)$ spin $1/2$ fermions. To summarise, the matter content of the maximal $\mathcal{N} = (2,2)$ Supergravity theory in 6D is given by 1 graviton, 5 tensor gauge fields, 16 vectors, 25 scalars (4+4) gravitini, and (20+20) spin $1/2$ fermions.

Following the presentation in [46], we organise the charges and the gauge fields according to their transformation properties and masses:

[11] Sometimes one finds the notation $Sp(n_L) \times Sp(n_R)$, where the total number of chiral (anti-chiral) Weyl spinors in 6D is denoted by $2n_L$ ($2n_R$).

[12] We refer to the Teichmuller space as the covering space of the moduli space. The latter is obtained from the former by discrete (global) identifications. Note that $\mathcal{K}$ is also a Grassmannian.

The 27 $U(1)$ charges in IIB Sugra on $\mathbb{T}^4$				
$SO(1,5)$ rep	$SO(5,5)$ rep	charges	gauge field	mass
tensor	vector **10**	$\{n_5, d_5, f_1, d_1, D^{5ij}\}$	$\{B^{(m)}_{\mu 5}, C^{(m)}_{\mu 5}, B_{\mu 5}, C_{\mu 5}, C_{\mu 5 ij}\}$	ℓ_s^{-2}
vector	spinor **16**	$\{w^i_{F1}, w^i_{D1}, D_{ijk}, p^i\}$	$\{B_{\mu i}, C_{\mu i}, C_{\mu ijk}, G_{\mu i}\}$	ℓ_s^{-1}
scalar	scalar **1**	n_p	$G_{\mu 5}$	ℓ_s^0

TABLE 2.1: The re-arrangement of the 27 $U(1)$ charges in Type IIB Sugra on $\mathbb{T}^4$ is shown. In the first column, we report the transformation properties of the fields under Lorentz transformations for an observer in the non-compact directions. In the fourth column, the letter m in $B^{(m)}_{\mu 5}, C^{(m)}_{\mu 5}$ stands for "magnetic". In the last column, the mass scaling with ℓ_s is shown (in the decoupling limit). The non-compact and circle indices run over $\mu = 0, \ldots, 5$. The fifth direction is that of the distinguished $\mathbb{S}^1_y$. The torus directions are denoted by $i = 6, \ldots, 9$.

By looking at the first row of the above Table 2.1, one can see that the masses of the NS5, D5, F1, the D1 and the 6 D3 branes (all winding the $\mathbb{S}^1_y$ circle) scale as ℓ_s^{-2}. In the decoupling limit $R_y^4 \gg V_{\mathbb{T}^4}$, they are infinitely massive and will constitute the background for the other fields [44]. The 16 branes that do not wrap the circle have masses scaling as ℓ_s^{-1}.

For applications present in this thesis, we will not turn on all the fields listed above. This will have important consequences for the properties of the moduli space, as we will see in due course. Preliminarily, note that the coset in Eq. (2.12) is a particular case of that obtained in the pure supergravity multiplet coupled to n_t tensor multiplets [47, 48]. The more general coset is the Grassmannian

$$\frac{SO(5, n_t)}{SO(5) \times SO(n_t)} \,. \tag{2.13}$$

One finds five self-dual tensor gauge fields and n_t anti-self-dual tensor gauge fields transforming in the vector of $SO(5, n_t)$. The moduli space of vacua is the given by a further quotient with the discrete duality group $SO(5, n_t; \mathbb{Z})$. For compactifications on K3, anomaly cancellation is readily achieved since in this case $n_t = 21$, which follows from the Hodge diamond structure of K3.

A technical detour on the D1-D5 system

Consider now the D1-D5 system, where the D5 wraps $\mathbb{T}^4$ or $K3$. Following [49] and the presentation in [50], one can describe the brane bound state by a primitive[13] (Mukai) charge vector $v \in \Gamma^{5,n_t}$, where Γ^{5,n_t} is an even unimodular lattice. For the D1-D5 system,

[13]We consider the charge vector to be not a multiple of another vector. If v is not primitive the brane system is singular: the system breaks into sub-systems with no cost in energy for *all* values of the moduli.

the lattice decomposes as $\Gamma^{5,n_t} = \Gamma^{1,1} \oplus \Gamma^{4,n_t-1}$, where the additional $\Gamma^{1,1}$ parametrizes F1 and NS5 charges, while Γ^{4,n_t-1} parametrizes the RR charges. The two $\Gamma^{1,1}$ lattices in $\Gamma^{1,1} \oplus \Gamma^{4,n_t-1}$ are exchanged by S-duality that acts as a $\mathbb{Z}_2$ group. In one of them, the D1 and D5 branes are parametrized by multiples of the vectors $(1,0)$ and $(0,1)$ respectively. Hence, for the 'pure' D1-D5 system we consider a Mukai charge vector (d_1, d_5), where d_1 and d_5 are taken to be coprime in order to ensure that the vector is primitive. In the pure D1-D5 system, the charge vector can be shown to have norm $v^2 = 2N = 2d_1d_5$ and to saturate the BPS bound. Because of this, the system has no binding energy and the branes can separate with no cost in energy. This is reflected on the structure of the moduli space of the theory, as discussed below.

For our black hole applications, we will focus on the D1-D5-P system or, most of the time, on the S-dual F1-NS5-P system. In the latter frame, a specific class of microstates of the NS5-F1-P black hole can be described with an exact null-gauged WZW model [51, 52, 53, 54, 1] since only NSNS fluxes are turned on. With this worldsheet model, one can probe the physics of these microstates (perturbatively) to all orders in α' and is the main focus of this thesis.

2.1.4 Attracted supergravity moduli space

We have seen above that the 25 scalars parametrise a 25-dimensional moduli space. However, only a 20-dimensional subspace survives the near horizon limit [49]. In other words, 5 scalars get fixed near the horizon and perturbations around their VEV are massive. This is a consequence of the attractor mechanism [55]. As an example, consider n_1 D1-branes wrapping the distinguished $\mathbb{S}^1_y$ circle of radius R_y and n_5 D5-branes wrapping the circle and the four-torus $V_{\mathbb{T}^4}$. This defines a five-dimensional black hole, when a reduction on the $\mathbb{S}^1_y$ is performed. In six-dimensions, the system is a black string with $\mathrm{AdS}_3 \times \mathbb{S}^3$ near horizon geometry. In this case, the fixed scalars are the torus volume (proportional to G_{ii}), a linear combination of $C^{(0)}$ and $C^{(4)}_{6789}$, and three anti-self-dual components of the B-field are set to zero (forcing the B-field to be self-dual on the torus) [44]. Thus, in the D1-D5 frame, the remaining 20 moduli are given as follows: 9 from the traceless G_{ij}, 6 from C_{ij}, 1 from $C^{(0)}$, 3 from B^{SD}_{ij} and 1 from the dilaton Φ. It is important to note that for the NS5-F1 system (the S-dual of the D1-D5) the 20 moduli are given by: 9 from traceless G_{ij}, 6 from B_{ij}, 1 from $C^{(0)}$, 3 from C^{SD}_{ij} and 1 from the dilaton Φ. Clearly, the role of $B^{(2)}$ and $C^{(2)}$ has been exchanged.

These near-horizon moduli parametrize the 'attracted' coset (Grassmannian) [49]

$$\mathcal{K}^* = \frac{SO(4,5)}{SO(4) \times SO(5)} \subset \mathcal{K} = \frac{SO(5,5)}{SO(5) \times SO(5)} \,. \tag{2.14}$$

Global identifications are implemented by those elements in $\mathcal{H}_v \subseteq SO(5,5;\mathbb{Z})$ that preserve the background charge vector (d_1, d_5), namely its stabiliser. The elements contained in the U-duality group $SO(5,5;\mathbb{Z})$ but not in $\mathcal{H}_v$, do not preserve the conditions that fix the moduli at the horizon [44]. One has to quotient the above coset by $\mathcal{H}_v$ to obtain the fundamental domain $\mathcal{F}$, made out of copies of the form[14]

$$\mathcal{M}^*_{\text{Sugra}} = \mathcal{H}_v \backslash SO(4,5)/SO(4) \times SO(5)\,. \tag{2.15}$$

The charge vector is nevertheless covariant under the action of $\mathcal{H}_v$, and a pair of charge (d_1, d_5) can be mapped to the so-called canonical vector $(1, d_1 d_5)$. As an example, this is achieved by an $SO(2,2;\mathbb{Z}) = SL(2,\mathbb{Z})_L \times SL(2,\mathbb{Z})_R$ subgroup, which contains the $SL(2,\mathbb{Z})$ symmetry of Type IIB string theory in ten dimensions which acts via S- and T-duality on the moduli. In particular, $SL(2,\mathbb{Z})_R$ acts on the axio-dilaton $\tau = C^{(0)} + i/g_s$ while $SL(2,\mathbb{Z})_L$ acts on the complexified modulus $\tilde{\tau} = C^{(4)}_{6789} + iv/g_s$, with v the (fixed) volume of $\mathbb{T}^4$.

It is useful to consider a basis for this 20-dimensional moduli space. The obvious choice is given by the field component that remain massless, up to a minor modification: instead of $C^{(0)}$ alone, we consider a linear combination of $\Xi = C^{(0)} + aC^{(4)}_{6789}$, for an appropriate value of a that we do not specify here. Note that the latter is not the same combination that gets attracted at the horizon and is indeed still massless. These fields have specific transformation properties under the $SO(4)_E \simeq SU(2)_L \times SU(2)_R$ isometry group of $\mathbb{S}^3$ and the $SO(4) = SU(2)_1 \times SU(2)_2$ symmetry group of the tangent space of $\mathbb{T}^4$. In particular, the moduli are all singlets under the former and some of them transform non-trivially under the latter, as reported in the following table for the case of the D1-D5 system [46]:

The 20 near-horizon moduli			
Field	$SO(4)_E \simeq SU(2)_L \times SU(2)_R$	$SO(4) = SU(2)_1 \times SU(2)_2$	DOF
$G_{ij} - \frac{1}{4}\delta_{ij}G_{kk}$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{3})$	9
B^{SD}_{ij}	$(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1})$	3
$C^{(2)}_{ij}$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$	6
$\Xi = C^{(0)} + a\,C^{(4)}_{6789}$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	1
Φ	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	1

TABLE 2.2: The 20 moduli parametrising the attracted moduli space of $\mathbb{T}^4$ compactifications of IIB supergravity. Note that all the moduli fields are singlet under $SU(2)_{L,R}$.

[14] For an arbitrary number of tensor multiplets, the attracted moduli space is made of copies of the form $\mathcal{M}^*_{\text{Sugra}} = \mathcal{H}_v \backslash SO(4,n_t)/SO(4) \times SO(n_t)$.

A co-dimension 4 subspace of the 20-dimensional moduli space is actually singular [56, 57, 50]. In the D1-D5 frame, this subspace is defined by $\Xi = B_{ij}^{SD} = 0$. Note that, as a consequence, the full B-field on the torus $B_{ij}^{(2)}$ is set to zero. In this region, the D1-D5 system becomes unstable under fragmentation and the brane can nucleate at no cost in energy and is exactly the point where the charge vector v has norm $v^2 = 2d_1d_5$, as discussed above. Indeed, one can show that when $B_{ij}^{SD} \neq 0$, the branes have a non-zero binding energy, see also [58]. In the NS5-F1 frame all works analogously, with the B-field dualised to the RR two-form and the singular subspace is defined by $\Xi = C_{ij}^{(2),SD} = 0$. Note that a pure NS worldsheet description necessarily lies in the singular locus of the moduli space.

2.1.5 Half-maximal and quarter-maximal 6D Sugra

There are two half-maximal Supergravity theories in six dimensions: the non-chiral $\mathcal{N} = (1,1)$ theory and the chiral $\mathcal{N} = (2,0)$ theory. The latter is of interest for us, since we will focus on the near-horizon geometry of D1-D5 (or F1-NS5) branes present in the Type IIB theory. There are two ways of obtaining this theory: the first is to compactify IIB Supergravity on K3, being a half-BPS Calabi-Yau space. This implies the holonomy constraint on the IIB Killing spinors which, upon the condition $\Gamma^{0\ldots9}\epsilon = \epsilon$, now must satisfy

$$\Gamma^{6789}\epsilon = \epsilon \quad \Rightarrow \quad \Gamma^{012345}\epsilon = \epsilon \,. \tag{2.16}$$

The second way is to take the *chiral* compactification of IIB Supergravity which consists in imposing the same condition above on the $\mathcal{N} = (2,2)$ theory. We thus remain with 16 supersymmetries and the R-symmetry reduces to $USp(4)$ only. Note that the near horizon of the D1-D5 system is $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ and this space preserves 16 supercharges, exhibiting an enhancement of supersymmetries.

Finally, we conclude this section by briefly recalling the field content of the quarter-maximal 6D $\mathcal{N} = (1,0)$ Supergravity theory, which will be useful for later applications. We now list the $(1,0)$ multiplets:

- $(1,0)$ Graviton multiplet "**G**": 1 graviton, 1 self-dual tensor gauge field, two left-handed gravitini
- $(1,0)$ Gravitino (L) multiplet "**g1**": 4 self-dual tensor gauge field, two left-handed gravitini
- $(1,0)$ Gravitino (R) multiplet "**g2**": 4 vector fields, two right-handed gravitini, 2 left-handed spinors
- $(1,0)$ Tensor multiplet "**T**": 1 anti-self-dual tensor gauge field, 2 right-handed spinors (tensorini), 1 scalar

- $(1,0)$ Vector multiplet "**V**": 1 vector field, 2 left-handed spinors
- $(1,0)$ Hypermultiplet "**H**": 4 scalars, 2 right-handed spinors

Remark 2.1. We can reconstruct the $(2,2)$ and $(2,0)$ multiplets as follows: $(2,2)$ Graviton multiplet = **G** + **g1** + 5× (**T+H**) + 2× (**g2** + 4× **V**); $(2,0)$ Graviton multiplet = **G** + **g1**; $(2,0)$ Tensor multiplet = **T** + **H**.

In most of the present thesis, we will study the properties a non-supersymmetric solitonic solution of the 6D $\mathcal{N} = (1,0)$ Sugra theory with $n_t = 1$ tensor multiplets, using worldsheet techniques. Thus, the multiplets of our interests will be the $(1,0)$ Graviton multiplet and one $(1,0)$ tensor multiplet. The particular Supergravity fields we are interested in define what is known as JMaRT solution [37]. The latter is found as a particular case of the Cvetic-Youm family [59, 60], which we now discuss.

2.2 Cvetic-Youm solution and its limits

We consider the most general rotating three-charge D1-D5-P solutions of maximal 6D supergravity presented in [59, 60]. We will focus on the interpretation of the five-dimensional black hole solution in terms of the rotating six-dimensional black string. The reasons for this are two: the solution (although still involved) is more simple if written as a six-dimensional black string and, furthermore, it will be useful when discussing the 6D JMaRT family in the rest of this thesis. We partly follow the notation and discussion of [61].

The string frame metric and dilaton of the so-called Cvetic-Youm solution [60] are

$$
\begin{aligned}
ds^2_{6D,S} = \frac{1}{\sqrt{H_1 H_5}} \Bigg[& -\left(1 - \frac{2mf_D}{r^2}\right) d\tilde{t}^2 + d\tilde{y}^2 + H_1 H_5 f_D^{-1} \frac{r^4}{(r^2+l_1^2)(r^2+l_2^2) - 2mr^2} dr^2 \\
& - \frac{4mf_D}{r^2} \cosh\alpha_1 \cosh\alpha_5 (l_2 \cos^2\theta d\psi + l_1 \sin^2\theta d\phi) d\tilde{t} \\
& - \frac{4mf_D}{r^2} \sinh\alpha_1 \sinh\alpha_5 (l_1 \cos^2\theta d\psi + l_2 \sin^2\theta d\phi) d\tilde{y} \\
& + \left(\left(1 + \frac{l_2^2}{r^2}\right) H_1 H_5 r^2 + (l_1^2 - l_2^2)\cos^2\theta \left(\frac{2mf_D}{r^2}\right)^2 \sinh^2\alpha_1 \sinh^2\alpha_5 \right) \cos^2\theta d\psi^2 \\
& + \left(\left(1 + \frac{l_1^2}{r^2}\right) H_1 H_5 r^2 + (l_2^2 - l_1^2)\sin^2\theta \left(\frac{2mf_D}{r^2}\right)^2 \sinh^2\alpha_1 \sinh^2\alpha_5 \right) \sin^2\theta d\phi^2 \\
& + \frac{2mf_D}{r^2} \left(l_2 \cos^2\theta d\psi + l_1 \sin^2\theta d\phi\right)^2 + H_1 H_5 f_D^{-1} r^2 d\theta^2 \Bigg],
\end{aligned}
\tag{2.17}
$$

$$
e^{2\Phi} = g_s^2 \frac{H_1}{H_5}, \tag{2.18}
$$

where

$$d\tilde{t} = \cosh\alpha_p dt - \sinh\alpha_p dy\,, \qquad d\tilde{y} = -\sinh\alpha_p dt + \cosh\alpha_p dy\,, \tag{2.19}$$

$$H_{1,5} = 1 + \frac{2mf_D}{r^2}\sinh\alpha_{1,5}\,, \qquad f_D^{-1} = 1 + \frac{l_1^2\cos^2\theta + l_2^2\sin^2\theta}{r^2}\,, \tag{2.20}$$

and $y \sim y + 2\pi R_y$. The parameters $m, l_1, l_2, \alpha_{1,5,p}$ are related to mass, charges angular momenta as

$$M = m\sum_{i=1,5,p}\cosh 2\alpha_i\,, \qquad Q_i = m\sinh 2\alpha_i\,, \qquad i = 1,5,p\,, \tag{2.21a}$$

$$J_{L/R} = \frac{1}{2}(J_\phi \mp J_\psi) = \frac{\pi}{4G_N^{5D}}\,m(l_1 \mp l_2)\left(\prod_{i=1,5,p}\cosh\alpha_i \pm \prod_{i=1,5,p}\sinh\alpha_i\right)\,, \tag{2.21b}$$

and the 5D Newton constant is related to the six and ten dimensional one as $(2\pi)^4 V(2\pi R_y)G_N^{5D} = (2\pi)^4 V G_N^{6D} = G_N^{10D} = 8\pi^6 g_s^2\alpha'^4$. The brane charges are quantised and are given by

$$Q_1 = c_1 n_1\,, \qquad c_1 = \frac{gs\alpha'^3}{V}\,, \qquad n_1 = \frac{mV}{g_s\alpha'^3}\sinh 2\alpha_1\,, \tag{2.22a}$$

$$Q_5 = c_5 n_5\,, \qquad c_5 = g_s\alpha'\,, \qquad n_5 = \frac{m}{g_s\alpha'}\sinh 2\alpha_5\,, \tag{2.22b}$$

$$Q_p = c_p n_p\,, \qquad c_p = \frac{gs^2\alpha'^4}{R_y^2 V}\,, \qquad n_p = \frac{mR_y^2 V}{g_s^2\alpha'^4}\sinh 2\alpha_p\,, \tag{2.22c}$$

where $n_{1,5,p} \in \mathbb{Z}$. The angular momenta are quantised as well:

$$J_\phi = J_L + J_R = \frac{n_\phi}{2}\,, \qquad J_\phi = -J_L + J_R = \frac{n_\psi}{2}\,, \tag{2.23}$$

with $n_{\phi,\psi} \in \mathbb{Z}$. It is very important to note that the parameter space of the above solution contains black holes, smooth solitons, conical defects or naked singularities. We now temporarily focus on the black hole region in this parameter space.

2.2.1 Cvetic-Youm black hole

The Cvetic-Youm black hole is defined as the dimensional reduction on $\mathbb{S}_y^1$ of supergravity fields in Eq. (2.17) with the additional important constraint

$$m \geq \frac{(|l_1| + |l_2|)^2}{2}\,. \tag{2.24}$$

The Cvetic-Youm black hole is also known as the non-extremal five-dimensional rotating D1-D5-P black hole. As particular cases, it includes the BMPV black hole and the celebrated non-rotating extremal Strominger-Vafa black hole, as we will discuss below.

The horizons of the five-dimensional Cvetic-Youm black hole are located at

$$r_\pm^2 = m - \frac{l_1^2 + l_2^2}{2} \pm \sqrt{\left(m - \frac{l_1^2 + l_2^2}{2}\right)^2 - l_1^2 l_2^2}\,, \tag{2.25}$$

and the Bekenstein-Hawking entropy is given by

$$\begin{aligned} S_{BH} = \frac{\pi^2 m}{2G_5}\Bigg[& \left(\prod_{i=1,5,p} \cosh 2\alpha_i + \prod_{i=1,5,p} \sinh 2\alpha_i\right)\sqrt{2m - (l_1 - l_2)^2} \\ & + \left(\prod_{i=1,5,p} \cosh 2\alpha_i - \prod_{i=1,5,p} \sinh 2\alpha_i\right)\sqrt{2m - (l_1 + l_2)^2}\Bigg]\,. \end{aligned} \tag{2.26}$$

Understanding the microscopic origin of the above entropy from a bulk perspective is a major fundamental and open question. We can consider this to be one of the most important (and difficult) aims of string theorists working in the field of black hole microstates.

2.2.2 The extremal rotating D1-D5-P black hole

We have seen that Eq. (2.24) is the necessary and sufficient condition for having a black hole. In the particular case when[15]

$$m = \frac{(l_1 + l_2)^2}{2}\,, \tag{2.27}$$

then the outer and inner horizon coincide, and we recover the three-charge extremal rotating D1-D5-P black hole solution. The horizon is at $r_H = \sqrt{l_1 l_2}$. The Bekenstein-Hawking entropy reduces to

$$S_{BH} = 2\pi\sqrt{\left(\frac{\pi}{4G_N^{5D}}\right)^2 Q_1 Q_5 Q_p - J_L^2 + J_R^2} = 2\pi\sqrt{n_1 n_5 n_p + \frac{n_\phi n_\psi}{4}}\,. \tag{2.28}$$

In this formula, the black hole attractor mechanism is manifest in a simple manner.

2.2.3 The BMPV black hole

Note that the above solution is extremal but not supersymmetric. If we further require that the black hole is BPS saturated, we obtain the three-charge BMPV black hole [62]. The BPS bound reads

$$M \geq Q_1 + Q_5 + Q_p\,. \tag{2.29}$$

[15] We are assuming $l_{1,2}$ positive for simplicity.

The BPS limit must be taken carefully, by appropriately rescaling certain quantities. Let's define

$$e^{\alpha_i} = \frac{\eta_i}{\sqrt{m}}\,, \qquad l_{1,2} = \sqrt{m}\, j_{1,2}\,, \tag{2.30}$$

then the BPS limit is defined by sending $m \to 0, \alpha_i \to \infty$ while keeping $\eta_i, j_{1,2}, G_N^{10D}, V, R_y$ fixed. In this way, we saturate the BPS bound Eq. (2.29) and the line element becomes simpler:

$$\begin{aligned} ds^2_{6D,S} = \frac{1}{\sqrt{H_1 H_5}} \Bigg[& -dt^2 + dy^2 + H_p(dt-dy)^2 + H_1 H_5 (dr^2 + r^2 d\Omega^2_{S^3}) \\ & - \frac{8 G_N^{5D} J_L}{\pi r^2} (\sin^2\theta d\phi - \cos^2\theta d\psi)(dt - dy) \Bigg]\,, \end{aligned} \tag{2.31}$$

where this time the "1" has been dropped from H_p, namely $H_p = Q_p/r^2$, while $H_i = 1 + Q_i/r^2$ for $i = 1,5$. Note that the right angular momentum is zero $J_R = 0$ due to the minus sign in the last parenthesis in the definition of J_R in Eq. (2.21b).

2.3 The D1-D5(-P) black hole

By further setting $l_1 = l_2$ (namely $j_1 = j_2$) in the BMPV black hole we have $J_L = 0$ as well. In this case we obtain the non-rotating three-charge D1-D5-P black hole, also known as[16] the Strominger-Vafa black hole. The latter has proven to be crucial for the first microscopic counting of microstates in string theory [19], and the first instance of what would have become the AdS/CFT correspondence [16, 17, 18]. Because of its importance, we review its brane construction and features.

Before diving into the details of the three-charge black hole, we first explain the construction and properties for the two-charge D1-D5 black hole. The reason for this are two-fold. First, its simplicity allows us to emphasize the important details shared also by the D1-D5-P black hole. Second, the bulk microstates reproducing the entropy of the D1-D5 black hole have been found in [64, 65, 66, 67]. These microstates admit a supergravity description, and an appropriate quantisation of the moduli space shows that they account for the entropy of the two-charge D1-D5 black hole [68].

In the following, we will denote by d_1, d_5 the number of D1 and D5-branes, while we will use n_1, n_5 to the denote the number of fundamental strings and NS5-branes. This echoes the conventions of [44].

[16] Actually, Strominger and Vafa used more general configurations in Type II and Heterotic string theory. The D1-D5-P solution was explicitly considered in [63].

2.3.1 The naive D1-D5 small black hole

Let's consider Type IIB String Theory compactified on $\mathbb{S}^1_y \times \mathbb{T}^4$, namely on a distinguished circle $\mathbb{S}^1_y$ of radius R_y and a 4-torus $\mathbb{T}^4$. The D-brane setup that leads to the (naive) two-charge extremal D1-D5 black hole is the following:

IIB	0	1	2	3	4	5	6	7	8	9
d_1 D1	$\times$					$\times$	$\sim$	$\sim$	$\sim$	$\sim$
d_5 D5	$\times$					$\times$	$\times$	$\times$	$\times$	$\times$

where $\sim$ means that the D1-branes have been smeared along the given direction. The coordinates $x_1, \ldots, x_4$ parametrise the non-compact $\mathbb{R}^4$ directions not wrapped by the brane system. The $\mathbb{S}^1_y$ circle, parametrised by y, is the fifth coordinate and is shared by both types of branes. The last four coordinates parametrise the torus. The ten-dimensional Lorentz group is broken to $SO(1,1) \times SO(4)_E \times SO(4)_I$, where "E" stands for external and "I" for internal[17]. The Killing spinor equations lead to

$$\Gamma^{056789}\epsilon_L = \epsilon_R \,, \qquad \Gamma^{05}\epsilon_L = \epsilon_R \,, \tag{2.32}$$

which shows that the brane configuration is 1/4-BPS[18], hence the total number of supercharges for the asymptotically flat solution is 8. The supergravity fields sourced by this brane configuration are given by

$$\begin{aligned} d\tilde{s}^2 &= \frac{1}{(H_1 H_5)^{\frac{1}{2}}}(-dt^2 + dy^2) + (H_1 H_5)^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega^2_{\mathbb{S}^3}\right) + \left(\frac{H_1}{H_5}\right)^{\frac{1}{2}} ds^2_{\mathbb{T}^4} \,, \\ F^{(3)} &= -\frac{1}{\tilde{g}_s}\, dH_1^{-1} \wedge dt \wedge dy + 2\alpha' d_5\, \Omega_{\mathbb{S}^3} \,, \\ e^{2\tilde{\Phi}} &= \tilde{g}_s^2 \frac{H_1}{H_5} \,, \qquad H_{1,5} = 1 + \frac{Q_{1,5}}{r^2} \,, \end{aligned} \tag{2.33}$$

where we have denoted by a tilde the quantities in the D1-D5 frame, to distinguish them from the NS5-F1 one that we will use throughout this thesis. The metric is given in string frame, and $\tilde{g}_s$ is the coupling constant at infinity[19]. Note that the r dependence of the harmonic function H_1 takes into account the smearing of the D1 branes. The coordinate r is the transverse radial distance from the brane sources in the non-compact directions. The charges Q_1, Q_5 are quantised and, when parametrising the volume of the (square) $\mathbb{T}^4$ in terms of its asymptotic volume v as measured from infinity, $V_{\mathbb{T}^4} =$

[17] Properly the $SO(4)_I$ group is broken to the $\mathbb{T}^4$ to $U(1)^4$. We still consider the $SO(4)_I$ acting on the tangent bundle of the torus, which will be useful for organising the matter fields.
[18] Recall that each spinor $\epsilon_{L,R}$ has 16 real independent components.
[19] It can be seen as a boundary condition, since the dilaton has a profile that changes with r.

$(2\pi\ell_s)^4\tilde{v}$, their expressions are given by

$$Q_1 = \frac{\tilde{g}_s d_1 \alpha'}{\tilde{v}}\,, \qquad Q_5 = \tilde{g}_s d_5 \alpha'\,. \tag{2.34}$$

In what follows, we will sometimes consider the $\mathbb{T}^4$ to be very small. In practice, this means we consider $\tilde{v} \sim \mathcal{O}(1)$, so that $V_{\mathbb{T}^4} \sim (2\pi\ell_s)^4$. As a consequence, the charges are now very similar $Q_i = \tilde{g}_s d_i \ell_s^2$, clearly exhibiting a symmetry under the exchange $d_1 \leftrightarrow d_5$. Note that the $\mathbb{S}^1_y$ circle is macroscopically large compared to the torus, since $R_y^4 \gg V_{\mathbb{T}^4}$. This means that the solution in Eq. (2.33) describes a *black string* in six dimensions. To obtain a black hole solution, one must dimensionally reduce on the circle.

We can consider the near horizon limit of (2.33) for the metric and dilaton. Roughly, this means neglecting the "1" in the harmonic functions[20]. We obtain

$$\begin{aligned}
d\tilde{s}_S^2 &= \left(\frac{Q_1}{Q_5}\right)^{\frac{1}{4}} d\tilde{s}_E^2 = \frac{r^2}{\sqrt{Q_1Q_5}}(-dt^2+dy^2) + \sqrt{Q_1Q_5}\frac{dr^2}{r^2} + \sqrt{Q_1Q_5}\,d\Omega^2_{\mathbb{S}^3} + \sqrt{\frac{Q_1}{Q_5}}\,ds^2_{\mathbb{T}^4}\,,\\
e^{2\tilde{\Phi}_{\rm hor}} &= \tilde{g}^2_{s,\rm hor} = \tilde{g}_s^2\frac{Q_1}{Q_5} = \frac{\tilde{g}_s^2}{\tilde{v}}\frac{d_1}{d_5}.
\end{aligned} \tag{2.35}$$

where "S" and "E" stands for string and Einstein frame respectively. The near-horizon ten-dimensional string and gravitational coupling constants are

$$\tilde{g}^2_{s,\rm hor} = \frac{\tilde{g}_s^2}{\tilde{v}}\frac{d_1}{d_5}\,, \qquad 2\kappa^2_{\rm hor} = (2\pi)^7\alpha'^4\,\tilde{g}^2_{s,\rm hor}\,. \tag{2.36}$$

Notably, the near horizon geometry is $\mathrm{AdS}_3\times\mathbb{S}^3\times\mathbb{T}^4$, as can be clearly seen by defining $z = \sqrt[2]{Q_1Q_5}/r$. The isometries are thus $(SL(2,\mathbb{R})\times SU(2))_L \times (SL(2,\mathbb{R})\times SU(2))_R \times U(1)^4$. More precisely, in presence of supersymmetry we should consider the isometry supergroup $\mathrm{PSU}(1,1|2)_L\times\mathrm{PSU}(1,1|2)_R$. The AdS_3 and $\mathbb{S}^3$ part have the same size fixed in terms of the charges of the brane system as $L_{\rm AdS} = R_{\mathbb{S}^3} = \sqrt[4]{Q_1Q_5}$. In the near-horizon region, the number of supersymmetries gets enhanced from 8 to 16.

The scalar curvatures in this regime are given by

$$R = 0\,, \qquad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = R_{\mu\nu}R^{\mu\nu} = \frac{24}{Q_1Q_5} = \frac{\tilde{v}}{\tilde{g}_s^2 d_1 d_5}\frac{24}{\alpha'^2}\,. \tag{2.37}$$

In order to have a reliable Supergravity description, we require the above quantities to be small or, more precisely, $L_{\rm AdS} \gg \ell_s$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = R_{\mu\nu}R^{\mu\nu} = \frac{24}{Q_1Q_5} \ll \frac{1}{\ell_s^4}$. This means that stringy α' corrections are negligible and that the background is gently curved. In addition, in order to suppress string loops, we require the near-horizon string coupling to be small $e^{2\tilde{\Phi}_{\rm hor}} = \tilde{g}_s^2\frac{Q_1}{Q_5} \ll 1$. Keeping $\tilde{v}$ generic for this analysis, we

[20]The Maldacena decoupling limit is properly performing by considering $\alpha' \to 0$ while keeping $U = r/\alpha'$ and $\tilde{v}$ fixed. Note that $\tilde{v}$ is a fixed scalar in the D1-D5 frame.

need to assume

$$L_{\mathrm{AdS}}^4 = Q_1 Q_5 \gg \alpha'^2 \,, \qquad \tilde{g}_s^2 \frac{Q_1}{Q_5} \ll 1 \,. \tag{2.38}$$

From (2.34) we learn that

$$Q_1 Q_5 = \frac{\tilde{g}_s^2 \alpha'^2}{\tilde{v}} d_1 d_5 \,, \qquad \frac{Q_1}{Q_5} = \frac{d_1}{d_5 \, \tilde{v}} \,, \tag{2.39}$$

hence the above requirements become

$$\frac{\tilde{g}_s^2}{\tilde{v}} \gg \frac{1}{d_1 d_5} \,, \qquad \frac{\tilde{g}_s^2}{\tilde{v}} \ll \frac{d_5}{d_1} \,, \tag{2.40}$$

where the first condition is to suppress stringy corrections while the second is to suppress loops. We thus need

$$\frac{1}{d_1 d_5} \ll \frac{\tilde{g}_s^2}{\tilde{v}} \ll \frac{d_5}{d_1} \,. \tag{2.41}$$

Note that the first inequality is consistent with what is expected for the supergravity regime to hold. At this point, it is natural and convenient to define the six-dimensional coupling constant

$$\tilde{g}_6^2 = \frac{\tilde{g}_s^2}{\tilde{v}} \,, \tag{2.42}$$

which will appear in later discussions. Note that, $\tilde{g}_6$ is a modulus in the D1-D5 frame since it explicitly depends on the asymptotic value $\tilde{g}_s$ of the dilaton. On the other hand, the attractor mechanism [55] fixes $\tilde{v}$ via the "fixed volume condition" in [44] to $\tilde{v} = d_1/d_5$. This can be seen by requiring that the volume of the torus in the near horizon regime is unit normalised $\sqrt{\frac{Q_1}{Q_5}} = 1$. Note that the above inequalities can also be recast as

$$\frac{d_1}{d_5} \ll \tilde{g}_6^{-2} \ll d_1 d_5 \,, \tag{2.43}$$

where the first inequality suppresses loops and the second suppresses stringy corrections.

The horizon of the black string solution is topologically $\mathbb{S}^1_y \times \mathbb{S}^3$. We now want to calculate its horizon area and the macroscopic entropy. To achieve this, we must write the metric in the Einstein frame, which reads

$$ds^2_{6D,E} = H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} \left(-dt^2 + dy^2\right) + H_1^{\frac{1}{4}} H_5^{\frac{3}{4}} \left(dr^2 + r^2 d\Omega^2_{\mathbb{S}^3}\right) \,. \tag{2.44}$$

The near horizon geometry is

$$ds^{2, r\to 0}_{6D,E} = \frac{r^2}{Q_1^{3/4} Q_5^{1/4}} \left(-dt^2 + dy^2\right) + \frac{Q_1^{1/4} Q_5^{3/4}}{r^2} dr^2 + Q_1^{1/4} Q_5^{3/4} d\Omega_{\mathbb{S}^3} \,. \tag{2.45}$$

The area of the horizon is thus (naively)

$$A_{\mathcal{H}} = \int_{S^1\times S^3} \sqrt{g_{\mathcal{H}}}\bigg|_{r=0} = 2\pi^2(2\pi)\sqrt{r^2\, Q_1^{-3/4} Q_5^{-1/4}\left(Q_1^{1/4}Q_5^{3/4}\right)^3}\,\Bigg|_{r=0} = 0\,. \qquad (2.46)$$

Since the horizon area vanishes, the entropy is trivial. We thus conclude that the *naive* two-charge black hole solution has vanishing entropy. In addition, the Ricci scalar behaves near $r=0$ as $R\sim r^{-2/3}$, and we find a naked singularity. Some observations are in order:

Remark 2.2. The reason why the horizon of six-dimensional black string solution has zero area is associated to the fact that the S^1_y shrinks to zero size. This is due to the tension of the branes wrapping the circle.

Remark 2.3. The fact that, naively, the two-charge six-dimensional black string solution has zero entropy implies that the two-charge five-dimensional black hole has zero entropy as well. This can be seen after performing a dimensional reduction along the S^1_y, that leads to the following five-dimensional metric in Einstein frame:

$$ds^2_{5D,Ein} = -(H_1H_5)^{-2/3}dt^2 + (H_1H_5)^{1/3}(dr^2 + r^2 d\Omega^2_{S^3})\,. \qquad (2.47)$$

Repeating the above steps for the computation of the horizon area, we obtain

$$A_{\mathcal{H}} = \int_{\mathcal{H}=S^3}\sqrt{g_{\mathcal{H}}} = 2\pi^2\left[\sqrt[6]{\frac{Q_1}{r^2}\frac{Q_5}{r^2}}\; r\right]^3\Bigg|_{r=0} = 2\pi^2\sqrt{Q_1Q_5}\; r\bigg|_{r=0} = 0\,, \qquad (2.48)$$

as claimed.

There is an important subtlety about the entropy of the two-charge naive solution. By looking at the above results, we would be tempted to conclude that the small black hole has zero macroscopic entropy. However, by a sequence of T and S dualities, one can map the D1-D5 system to a fundamental string with momentum, whose microscopic entropy is given by

$$S_{F1-P} = 2\pi\sqrt{2n_1n_p}\,, \qquad (2.49)$$

which clearly differs from our macroscopic expectation. Sen [69] argued that in the D1-D5 frame, the entropy should be obtained by counting solutions with no horizons, while in the U-dual NS1-P frame the entropy should be obtained in a different manner, using Wald's entropy formula for geometries with horizons. This could lead to the idea that the computation of the entropy somehow frame (gauge) dependent. However, in [70] it is argued that this is *not* the case: the microstate are smooth and horizonless (or, more generally, fuzzballs) in any frame.

2.3.2 The NS5-F1 system

The D1-D5 system can be mapped, via S-duality, to the NS5-F1 system. The advantage of the latter is the presence of only NS-NS fluxes. Indeed, a closed string propagating in those backgrounds can be described via a class of α'-exact cosets involving $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ Wess-Zumino-Witten models. For this reason, it is important to study the NS5-F1 system in great detail and understand its differences with respect to its D1-D5 cousin.

The Supergravity fields sourced by the NS5-F1 system are

$$\begin{aligned} ds^2 &= \frac{1}{H_1}(-dt^2+dy^2) + H_5\left(dr^2+r^2 d\Omega^2_{\mathbb{S}^3}\right) + ds^2_{\mathbb{T}^4}\,, \\ H^{(3)} &= -\frac{1}{g_s}\, dH_1^{-1}\wedge dt\wedge dy + 2\alpha' n_5\, \Omega_{\mathbb{S}^3}\,, \\ e^{2\Phi} &= g_s^2\frac{H_5}{H_1}\,, \qquad\qquad H_{1,5} = 1+\frac{Q_{1,5}}{r^2}\,, \end{aligned} \tag{2.50}$$

with

$$Q_1 = \frac{g_s^2 n_1 \alpha'}{v}\,, \qquad\qquad Q_5 = n_5\alpha'\,, \tag{2.51}$$

where v enters the torus volume as $V_{\mathbb{T}^4} = (2\pi\ell_s)^4 v$. Note that when $g_s \to 0$, the above solution coincides with the CHS background [71] since the NS5-branes become much heavier than the background fundamental strings.

The near-horizon fields are

$$\begin{aligned} ds^2 &= \frac{r^2}{Q_1}(-dt^2+dy^2) + Q_5\frac{dr^2}{r^2} + Q_5 d\Omega^2_{\mathbb{S}^3} + ds^2_{\mathbb{T}^4}\,, \\ H^{(3)} &= 2\alpha' n_5\left(\mathrm{vol}_{\mathrm{AdS}_3} + \mathrm{vol}_{\mathbb{S}^3}\right)\,, \\ e^{2\Phi_{\mathrm{hor}}} &= g^2_{s,\mathrm{hor}} = g_s^2\frac{Q_5}{Q_1} = v\,\frac{n_5}{n_1}\,. \end{aligned} \tag{2.52}$$

The near-horizon geometry is again $\mathrm{AdS}_3\times\mathbb{S}^3\times\mathbb{T}^4$ as one can see with the same change of coordinates $z = \sqrt[2]{Q_1Q_5}/r$. However, this time the curvature scale is set by Q_5 only. Indeed, in this regime, the curvature invariants read

$$R = 0\,, \qquad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = R_{\mu\nu}R^{\mu\nu} = \frac{24}{Q_5^2} = \frac{24}{n_5^2\alpha'^2}\,. \tag{2.53}$$

To suppress curvature and loop effects respectively, one demands

$$1 \ll n_5 \ll \frac{n_1}{v}\,, \tag{2.54}$$

and can equivalently recast the above bounds as

$$\frac{v}{n_5} \ll v \ll \frac{n_1}{n_5}\,, \tag{2.55}$$

where v is a modulus for the NS5-F1 system. Indeed, in the NS5-F1 frame the string coupling constant at the horizon is a fixed scalar independent of the asymptotic value g_s of the dilaton. As a consequence, the six-dimensional coupling constant is attracted to the value [44]

$$g_6^2 = \frac{g_{s,\text{hor}}^2}{v} = \frac{n_5}{n_1}\,, \tag{2.56}$$

and is important to note that is independent of the modulus v. Note that T-duality transformations can exchange $v \mapsto \frac{1}{v}$, hence we can restrict ourselves to $v \geq 1$. When considering the α'-exact WZW models, the condition ensuring absence of stringy corrections can be relaxed. In other words, when dealing with worldsheet string theory, it is not necessary to consider $n_5 \gg 1$, which is only required for the validity of the supergravity solution. However, we will continue to suppress string loops and thus we will always work in a regime where $n_1 \gg n_5$.

2.3.3 Comparing the D1-D5 and the NS5-F1 system and their moduli

We now remark similarities and differences of the two S-dual systems. Recall that the tilde quantities refer to the D1-D5 system.

Remark 2.4. In the D1-D5 system, the six-dimensional coupling constant $\tilde{g}_6^2$ is a modulus while the torus volume parametrised by $\tilde{v}$ is a fixed scalar. In the NS5-F1 system, v is a modulus while the six-dimensional coupling constant g_6^2 is a fixed scalar.

We now show how S-duality relates the two modulus and the two fixed scalars. We start with the moduli. Recall that

$$\tilde{g}_{s,\text{hor}}^2 = \frac{\tilde{g}_s^2}{\tilde{v}}\frac{d_1}{d_5}\,, \qquad g_{s,\text{hor}}^2 = v\frac{n_5}{n_1}\,. \tag{2.57}$$

However, because of S-duality $\tilde{g}_{s,\text{hor}}^2 = g_{s,\text{hor}}^{-2}$, hence we get

$$\frac{\tilde{g}_s^2}{\tilde{v}}\frac{d_1}{d_5} = \frac{n_1}{vn_5}\,, \tag{2.58}$$

and since under S-duality the number of branes do not change $d_i = n_i$, we finally get

$$\frac{\tilde{g}_s^2}{\tilde{v}} = \tilde{g}_6^2 = \frac{1}{v}\,. \tag{2.59}$$

This shows that in the NS5-F1 frame v behaves like the six-dimensional coupling constant. It is very easy to show how to relate the fixed moduli. Recall their fixed value:

$$g_6^{-2} = \frac{n_1}{n_5}\,, \qquad \tilde{v} = \frac{d_1}{d_5}\,, \tag{2.60}$$

but since under S-duality $n_i = d_i$, it immediately follows that

$$g_6^{-2} = \tilde{v}\,. \tag{2.61}$$

Remark 2.5. Recall that the unit normalisation of the torus volume in the D1-D5 system required $\tilde{v} = \frac{d_1}{d_5}$, which gives the fixed scalar condition for $\tilde{v}$. At the same time, from the previous remark we have

$$\tilde{g}_{s,\text{hor}}^2 = \frac{\tilde{v}}{v} = \frac{n_1}{v\, n_5}\,. \tag{2.62}$$

where we have used the fact that $d_i = n_i$.

Remark 2.6. The two pictures are related via the non-perturbative S-duality and hence we do not expect both tree-level supergravity to be valid at the same time. Allowing general stringy corrections, let's consider the various strong coupling regimes. Recall that

$$\tilde{g}_{s,\text{hor}}^2 = \frac{\tilde{g}_s^2}{\tilde{v}}\frac{d_1}{d_5} = \frac{n_1}{vn_5} = g_{s,\text{hor}}^{-2}\,. \tag{2.63}$$

Requiring that the D-brane picture is weakly coupled demands

$$\frac{\tilde{g}_s^2}{\tilde{v}}\frac{d_1}{d_5} = \frac{n_1}{vn_5} \ll 1 \tag{2.64}$$

which is clearly a *strong coupling* requirement for the NS5-F1 picture. We thus conclude that both pictures cannot be valid at the same time if we demand the coupling to be small.

2.3.4 Bulk microstates of the 2-charge system

We now recall, in two different U-dual frames, the supergravity solutions corresponding to microstates of the two-charge black hole. As explained in [67], in order to properly account for the full entropy of the two-charge system [68], one needs to consider microstates with both bosonic and fermionic [67] excitations. In [65], the bosonic excitations were purely external along the transverse $\mathbb{R}^4$, while in [64, 72] excitations along the internal manifold were considered. For simplicity, we focus on those with external bosonic excitations, thus a subset of microstates collectively known as Lunin-Mathur solutions [64, 65].

F1-P frame

Consider the same compactification manifolds discussed above for the two-charge black hole. Let's parametrise the distinguished circle with $0 \leq y < L$. Let's consider a fundamental string (F1) with n_w units of winding and n_p units of momentum, both on the circle $\mathbb{S}_y^1$. The strings wounds n_w times the physical circle, but in the covering space of the $\mathbb{S}^1_y$ it looks single wounded with length $L_T = n_w L$. Let's parametrise this covering space with the coordinate $0 \le \hat{y} < L_T$, and let's define $\hat{v} = t - \hat{y}$. In the classical supergravity limit of large $n_w n_p$, coherent (micro-)states are well described by a continuous[21] periodic vibration profile function $\mathbf{F}(\hat{v}) = \mathbf{F}(\hat{v} + L_T)$ which we take to be a function in $\mathbb{R}^4$, ignoring the four torus for simplicity.

The F1-P supergravity solutions are given by

$$ds^2 = \frac{1}{H}\Big(-du dv + K\, dv^2 + 2A_i dx^i dv\Big) + ds^2_{\mathbb{R}^4} + ds^2_{\mathbb{T}^4} \tag{2.65a}$$

$$B_{uv} = -\frac{1}{2}(H^{-1} - 1)\,, \qquad B_{vi} = H^{-1} A_i \tag{2.65b}$$

$$e^{2\Phi} = H^{-1}\,, \tag{2.65c}$$

where

$$H = 1 + \frac{Q_1}{L_T}\int_0^{L_T} \frac{d\hat{v}}{|\mathbf{x} - \mathbf{F}(\hat{v})|^2}\,, \qquad A_i = -\frac{Q_1}{L_T}\int_0^{L_T} \frac{\dot{\mathbf{F}}(\hat{v}) d\hat{v}}{|\mathbf{x} - \mathbf{F}(\hat{v})|^2}\,,$$
$$K = \frac{Q_1}{L_T}\int_0^{L_T} \frac{|\dot{\mathbf{F}}(\hat{v})|^2 d\hat{v}}{|\mathbf{x} - \mathbf{F}(\hat{v})|^2}\,. \tag{2.66}$$

These solutions are horizonless, as microstates are expected to be. They are also smooth, up to a physical (thus allowed) singularity associated to the presence of the string source. The same properties continue to hold for solutions with fermionic and internal excitations.

D1-D5 frame

After a sequence of T and S dualities, we can map Eq. (2.65) to the D1-D5 frame. The sequence of dualities that leads to the D1-D5 system is given by $S\, T_{\mathbb{T}^4}\, S\, T_{\mathbb{S}^1_y}\, S$.

The D1-D5 supergravity solutions, in the more compact notation of [68], are then given by

$$ds^2_{\mathbb{S}} = \frac{1}{f_1 f_5}\left[-\Big(dt + A_i dx^i\Big)^2 + \Big(dy + B_i dx^i\Big)^2\right] + \sqrt{f_1 f_5}\, ds^2_{\mathbb{R}^4} + \sqrt{\frac{f_1}{f_5}}\, ds^2_{\mathbb{T}^4}\,,$$
$$C^{(2)} = \frac{1}{f_1}(dt + A) \wedge (dy + B) + \mathcal{C}\,,$$
$$e^{2\Phi} = \frac{f_1}{f_5}\,, \tag{2.67}$$

[21] In other words, we are smearing over the strands of the string in the base space, assuming a continuous distribution.

where the harmonic functions and forms are

$$f_1 = 1 + \frac{Q_5}{L}\int_0^L \frac{|\mathbf{F}'(s)|^2}{|\mathbf{x}-\mathbf{F}(s)|^2}ds\,, \qquad f_5 = 1 + \frac{Q_5}{L}\int_0^L \frac{1}{|\mathbf{x}-\mathbf{F}(s)|^2}ds$$
$$A^{(1)} = \frac{Q_5}{L}\int_0^L \frac{F_i(s)}{|\mathbf{x}-\mathbf{F}(s)|^2}ds\,dx^i\,, \qquad dB^{(1)} = \star_{\mathbb{R}^4} dA^{(1)}\,, \qquad dC^{(2)} = -\star_{\mathbb{R}^4} df_5\,. \tag{2.68}$$

The charges are related as [73]

$$Q_1 = \frac{Q_5}{L_T}\int_0^L |\mathbf{F}'(s)|^2 ds\,. \tag{2.69}$$

Again, these solutions are parametrized by the closed curve

$$x_i = F_i(s)\,, \qquad s \in (0, L)\,, \forall i = 1, \ldots, 4\,, \tag{2.70}$$

and in order to ensure regularity of the fields, it is important that the curve is smooth and non-selfintersecting. These solutions are horizonless and, as opposed to the previous case, completely smooth.

One can immediately notice that if we take

$$f_{1,5} \to 1 + \frac{Q_{1,5}}{|x|^2} = H_{1,5}\,, \qquad A_i, B_i \to 0\,, \tag{2.71}$$

one recovers the *naive* D1-D5 metric Eq. (2.33). This shows how at large distances the above microstates resemble the naive black hole, even though they start to differ at small distances from $r = 0$.

It has been shown in [68] that an appropriate quantisation of the moduli space of the Lunin-Mathur solution, together with the more general solutions of [67], gives the correct counting of the entropy for the D1-D5 black hole. This has a remarkable conceptual consequence: for the small black hole, all the microstates are described within supergravity and the metric field describes smooth and horizonless geometries that asymptotically look like the naive solution. This is a striking confirmation, although so far limited to the two charge case, of the Fuzzball Proposal. It should be noted, however, that in the case of three charge black holes it is expected that stringy physics may actually be important in recovering the full entropy of a macroscopic black hole.

2.3.5 The D1-D5-P large black hole

We now turn to the extremal non-rotating D1-D5-P black hole [63]. The setup is the same as that of the D1-D5 solution, but now n_p units of momentum wave are added along the S^1_y circle, and smeared along the $\mathbb{T}^4$. We add this wave holomorphically, in order to preserve some supersymmetries. In this case, the asymptotically flat solution preserves 1/8 of the original 32 supercharges.

The line element in Einstein frame of the five-dimensional black hole reads

$$ds^2_{5D,E} = -\lambda^{-2/3}dt^2 + \lambda^{1/3}\left(dr^2 + r^2 d\Omega^2_{S^3}\right) , \tag{2.72a}$$

$$\lambda = \prod_{i=1,5,p}\left(1+\frac{Q_i}{r^2}\right) , \tag{2.72b}$$

$$Q_1 = \frac{g_s n_1 \ell_s^6}{V} , \qquad Q_5 = g_s n_5 \ell_s^2 , \qquad Q_P = \frac{g_s^2 n_p \alpha'^2}{R_y^2 V} . \tag{2.72c}$$

This time the area of the horizon is finite, giving

$$A_{\mathcal{H}} = 2\pi^2\sqrt{Q_1 Q_5 Q_P} = 2\pi^2 \frac{g_s^2 \alpha'^4}{R_y V}\sqrt{n_1 n_5 n_p} , \tag{2.73}$$

and thus the Bekenstein-Hawking entropy is given by

$$S_{BH} = \frac{A_{\mathcal{H}}}{4G_N} = 2\pi\sqrt{n_1 n_5 n_p} , \tag{2.74}$$

where the expression of the five-dimensional Newton constant has been used.

As expected, the moduli do not appear in the expression of the entropy and the U-duality symmetry is manifest. More generally, the five-dimensional expression for the entropy is given in terms of the Surd cubic invariant $\triangle$ as $S = 2\pi\sqrt{\triangle}$. This is a consequence of the non-compact $E_{6,(6)}$ exceptional symmetry that characterises M or string theory toroidal compactifications.

The microscopic counting of the above entropy was performed for the first time in [19] by Strominger and Vafa. This was achieved by counting the degeneracy of massless open string states stretched between the D1 and the D5 branes. The result, obtained at weak coupling, was then successfully extrapolated at strong coupling, being the elliptic genus protected by supersymmetry. Strominger and Vafa showed how String Theory was able to account for the microscopic degrees of freedom of the D1-D5-P black hole. Their work was then generalised, in various dimensions, to different non-extremal and rotating black holes. See for instance the non-exhaustive list [74, 75, 62, 76]

2.4 From the brane system to the D1-D5 CFT

In this section we review the definition of the D1-D5 CFT starting from the brane setup in string theory. We will see that the D1-D5 CFT naturally arises from the low-energy dynamics of the sigma model on the moduli space of n_1 instantons of a $U(n_5)$ gauge theory on $\mathbb{T}^4$.

2.4.1 The instanton description

Consider the pure D1-D5 system, where both branes are wrapped on the circle and the n_5 D5-branes also wrap the torus. The theory on the D5-branes is a $5+1$-dimensional theory, whose support is $\mathbb{R}_t \times \mathbb{S}^1_y \times \mathbb{T}^4$. The n_1 D-strings are extended along $\mathbb{R}_t \times \mathbb{S}^1_y$ and are thus localised with respect to the torus directions. From the DBI action, one notices that the low-energy theory living on the worldvolume of the D5-branes is the six-dimensional SYM theory. Considering the normal and tangent bundles to be flat and $B^{(2)} = 0$, expanding the Wess-Zumino part of the D5-brane action as in Eq. (2.7) one obtaines a term of the form

$$\int_{\mathcal{M}_6} C^{(2)} \wedge \mathrm{Tr}\left(F^{(2)} \wedge F^{(2)}\right) \propto \int_{\mathcal{M}_6} C^{(2)} \wedge \mathrm{ch}_2(F)\,. \tag{2.75}$$

We note that Yang-Mills instantons (contributing to the second Chern class) carry a RR two-form charge. In particular, the D1-branes are instantons for the Euclidean four-dimensional theory on $\mathbb{T}^4$ inside the D5-branes. To see this, one has to consider gauge field configurations which are independent of $\mathbb{R}_t \times \mathbb{S}^1_y$ and whose field strength is self-dual with respect to the $\mathbb{T}^4$. Noting that, when integrating the (t, y)-independent connections on $\mathbb{T}^4$, we find $\mathrm{ch}_2(F) = n_1$. This is the instanton number of the gauge group, and hence we learn that in the n_1 instanton sector of the gauge theory we have n_1 D1-brane charge units[22] [77]. Note that for the pure D1-D5 system, the first Chern-class vanishes as indicated by the absence of a coupling of the form $C^{(4)} \wedge \mathrm{Tr}(F)$.

As explained in [49], in the limit where the $\mathbb{T}^4$ is small, the theory on $\mathbb{R}_t \times \mathbb{S}^1_y \times \mathbb{T}^4$ effectively reduces to a 2D sigma model with base space $\mathbb{R}_t \times \mathbb{S}^1_y$ with target space the moduli space of $U(n_5)$ instantons on $\mathbb{T}^4$. The moduli space of n_1 instantons of a $U(n_5)$ gauge theory on $\mathbb{T}^4$ is the the smooth manifold $\mathcal{M}_{\mathrm{Inst}}$ which is a resolution of the symmetric product orbifold $\mathrm{Sym}_{n_1 n_5}(\tilde{\mathbb{T}}^4) = (\tilde{\mathbb{T}}^4)^{\otimes n_1 n_5} / S_{n_1 n_5}$, namely

$$\mathcal{M}_{\mathrm{Inst}} \xrightarrow{\ \pi\ } \mathrm{Sym}_{n_1 n_5}(\tilde{\mathbb{T}}^4)\,, \tag{2.76}$$

where S_N is the permutation group of N elements, $\tilde{\mathbb{T}}^4$ is in general different from the compactification torus $\mathbb{T}^4$ [78] and π is a suitable projection map. The space $\mathcal{M}_{\mathrm{Inst}}$ is a simply-connected Hyperkähler space[23] and has the same cohomology as $\mathrm{Sym}_{n_1 n_5}(\tilde{\mathbb{T}}^4)$. If the charge vector v is taken to be primitive, then this space is also smooth [49].

[22] The case of K3 has a correction due to the non-vanishing first Pontryagin class [77].

[23] A manifold (M, I, J, K, g) is Hyperkähler if $I, J, K \in \mathrm{End}(TM)$ satisfy Hamilton's quaternionic relations, are integrable complex structures (the Nijenhuis tensor vanishes by Newlander–Nirenberg theorem) and the Riemannian metric g is Kähler with respect to I, J, K. Assuming a Levi-Civita connection ∇, one can show that I, J, K are parallel $\nabla I = \nabla J = \nabla K = 0$ and hence the holonomy is contained in the group $Sp(n)$. The three Kähler forms are $\omega_I(\cdot,\cdot) = g(\cdot, I\,\cdot)$ and similarly for J, K.

As discussed in previous sections, our 'pure' brane system is marginally unstable and suffers from fragmentation at no cost in energy. How can we see this feature in the instanton description? The intuitive explanation is that singularities of the moduli space are associated to a D1-brane instanton that can become small and thus separate from the D1-D5 system. These are the so-called 'long strings' [50], whose signature will be present also in the pure NS worldsheet description in terms of the so-called 'missing chiral primaries'. To dive into the details of this process, we first turn to the gauge theory description of the D1-D5 system.

2.4.2 The gauge theory description

Given that the number of supercharges preserved by the system is 8, from the discussion above we conclude that the low-energy dynamics of the pure D1-D5 brane system is described by a 2D small[24] $\mathcal{N} = (4,4)$ sigma model from $\mathbb{R}_t \times \mathbb{S}^1_y$ to $\mathcal{M}_{\text{Inst}}$. Performing an S-duality, one gets analogous conclusions for the NS5-F1 system. As we will discuss below, this sigma model flows in the IR to a 2D $\mathcal{N} = (4,4)$ Superconformal field theory known as the D1-D5 CFT. However, in this case the number of fermionic symmetries is enhanced from 8 to 16, where the new 8 symmetries are given by fermionic generators of the superconformal algebra (usually denoted by S). This is a similar phenomenon that is found in the case of the 4d $\mathcal{N} = 4$ Super Yang-Mills theory, where the fermionic symmetries are 32: 16 supercharges and 16 fermionic generators of the superconformal algebra.

As we will discuss in due course, the above sigma model has a moduli space which coincides with Eq. (2.15) [78, 49]. One thus concludes that the D1-D5 system is on the moduli space of this sigma model [50], which legitimates the study of the dynamics of low-energy open string excitations.

The gauge theory description, less general then the instanton description, is used when $1 \ll n_1, n_5 \ll 1/g_s$, since the backreaction of the branes is negligible and thus gravity effects can be ignored. The branes are boundary conditions for the open strings stretched between them. Let's denote the $\mathbb{R}_t$ and $\mathbb{S}^1_y$ directions as x_0, x_5 respectively. The non-compact direction are $x_1, \ldots, x_4$ while the $\mathbb{T}^4$ is parametrised by $x_6, \ldots, x_9$. The various massless strings are $(1,1), (1,5), (5,1), (5,5)$ strings. They give rise to the following matter content:

- **(1,1) Strings**
  These strings have NN boundary conditions along x_0, x_5 and DD along the others. Along the first two directions, the massless excitations give two components of

[24] There are two different $\mathcal{N} = 4$ superconformal algebras in two dimensions. The case of the D1-D5 CFT is referred as the small $\mathcal{N} = 4$ superalgebra, where only one $SO(4)_R$ is realised as R-symmetry instead of the usual $SO(\mathcal{N}_+) \times SO(\mathcal{N}_-)$ R-symmetry.

a vector, A_{x_0}, A_{x_5}. The vector transforms in the adjoint of $U(n_1)$ and is a $n_1 \times n_1$ matrix. Excitations transverse to the D1-branes are scalars. We have 8 transverse directions and thus 8 adjoint scalars. Fermions are obtained imposing supersymmetry. Arranging the fields in multiplets of $\mathcal{N} = 2$ in 4D, we can note that the fields give rise to one Hypermultiplet (it consists of 4 real scalars + 2 Weyl fermions) and a Vector multiplet (one vector and two reals scalars + 2 Weyl fermions).

How can we decide whether four of the eight scalars should be part of the Hyper or part of the Vector multiplet? The answer is R-Symmetry! We know that inside the[25] $SO(4)_{\text{Int}} \simeq SO(3) \times SO(3) \simeq SU(2) \times SU(2)$ one of the $SU(2)_R$ acts as an R-Symmetry group, rotating the Hypers. The Hypermultiplet is thus formed by those scalars that are charged under R-Symmetry and hence they correspond to the internal coordinates $x_6, \ldots, x_9$. In conclusion, we have (at fixed n_1, n_5)

– One Hypermultiplet whose four scalars are denoted by

$$Y^{(1,1)}_{x_6} \qquad Y^{(1,1)}_{x_7} \qquad Y^{(1,1)}_{x_8} \qquad Y^{(1,1)}_{x_9} \tag{2.77}$$

These real scalars can be recast in a complex doublet transforming in the **2** of $SU(2)_R$. The total number of scalars in the hypermultiplets is $4n_1^2$.

– One Vector multiplet with one vector and two scalars:

$$A^{(1,1)}_{x_0} \qquad A^{(1,1)}_{x_5} \qquad Y^{(1,1)}_{x_1} \qquad Y^{(1,1)}_{x_2} \qquad Y^{(1,1)}_{x_3} \qquad Y^{(1,1)}_{x_4} \tag{2.78}$$

All these fields transforms in the adjoint of $U(n_1)$ since the strings have endpoints on the same stack of D1-branes.

- **(5,5) Strings**
Since the torus is of the order of the string scale, the winding and the momentum modes of the string goes like $\sim 1/\ell_s$. This means that if we study the system at energies $E \ll 1/\ell_s$, we are allowed to neglect these modes. The x_5 coordinate is taken to be compactified on a circle of radius R which is much bigger than the string length. In this respect, we can treat the x_5 coordinate to be non-compact $x_5 \in \mathbb{R}$ instead of $x_5 \in \mathbb{S}^1_y$. The theory on the D5-branes is seen at low energies as $(1+1)$-dimensional. The field content is thus analogous to that of the (1,1) strings (at fixed n_1, n_5):

– One Hypermultiplet

$$Y^{(5,5)}_{x_6} \qquad Y^{(5,5)}_{x_7} \qquad Y^{(5,5)}_{x_8} \qquad Y^{(5,5)}_{x_9} \tag{2.79}$$

[25]There is a subtlety: since we are wrapping the D5 branes on a $\mathbb{T}^4$ it is not completely correct to claim that the symmetry group for the internal coordinates is $SO(4)$. This is *locally* correct and the $SO(4)$ is a useful approximate symmetry.

These real scalars can also be recast in a complex doublet transforming in the **2** of $SU(2)_R$. The total number of scalars in the hypermultiplets is $4n_5^2$.

– One Vector multiplet:

$$A^{(5,5)}_{x_0} \qquad A^{(5,5)}_{x_5} \qquad Y^{(5,5)}_{x_1} \qquad Y^{(5,5)}_{x_2} \qquad Y^{(5,5)}_{x_3} \qquad Y^{(5,5)}_{x_4} \tag{2.80}$$

All the fields are in the adjoint of $U(n_5)$.

- **(1,5) and (5,1) Strings**
The quantisation of these strings is slightly more subtle and requires some care and is not often spelled out in great detail in the literature [79]. First of all, recall that worldsheet boson for the open string are integer moded when the boundary conditions are NN or DD. When the latter are of the type DN or ND, the bosons are half-integer moded. The same is true the fermions in the Ramond sector. However, fermions in the NS sector are integer moded for DN and ND boundary conditions and are half-integer moded for NN and DD boundary condition[26]. The moding affects the zero point energies "E" as follows:

$$E_{X,\mathbb{Z}} = -E_{\psi,\mathbb{Z}} = -\frac{1}{24}\,, \qquad E_{X,\frac{\mathbb{Z}}{2}} = -E_{\psi,\frac{\mathbb{Z}}{2}} = \frac{1}{48}\,, \tag{2.81}$$

where "X, ψ" stand for boson and fermion while $\mathbb{Z}$ and $\frac{\mathbb{Z}}{2}$ stands for integer and half-integer moding. Let's denote by ν the number of DN or ND directions. Taking into account all the possible boundary conditions in d space-time dimensions, the total vacuum energy for a boson is

$$E_X^{\text{tot}} = (d-2-\nu)E_{X,\mathbb{Z}} + \nu\, E_{X,\frac{\mathbb{Z}}{2}}\,. \tag{2.82}$$

Note that the vacuum energy for the Ramond sector is exactly the opposite $E_X^{\text{tot}} = -E_{\psi_R}^{\text{tot}}$ and the vacuum energies exactly cancel. On the other hand, the total vacuum energy for the fermions in the NS sector is

$$E_{\psi_{NS}}^{\text{tot}} = (d-2-\nu)E_{\psi_{NS},\frac{\mathbb{Z}}{2}} + \nu\, E_{\psi_{NS},\mathbb{Z}}\,. \tag{2.83}$$

Note that for the critical superstring, we get

$$E_X^{\text{tot}} + E_{\psi_{NS}}^{\text{tot}} = -\frac{1}{2} + \frac{\nu}{8}\,. \tag{2.84}$$

For the D1-D5 system, $\nu = 4$ and thus the bosonic and fermionic vacuum energies cancel also in the NS sector. Let's discuss now the $(1,5)$ or $(5,1)$ strings. There are four periodic ψ^i_{NS} fermions in the torus directions $i = 6,7,8,9$ and they give $2^{4/2} = 4$ degenerate ground states. Labelling their spins in the (i,j) plane with $s_{ij} = \pm\frac{1}{2}$, we get four states $|s_{67}, s_{89}\rangle$ and they behave as two $SU(2)$ doublets $\mathbf{2} \oplus$

[26] A reference for this can be [80], Eq. (7.70).

$\mathbf{2'}$ under the internal $SO(4)_I$, while being singlets under the external $SO(1,1) \times SO(4)_E$ group. We now need to impose the GSO projection[27] with the operator $\exp(i\pi F), F = \sum_a s_a$. Taking into account the extra sign coming from the ghosts, the GSO projection demands that $\exp(i\pi F) = -1$, which implies $s_{67} = s_{89} = \frac{1}{2}$. Only the state with same spins survives, and transforms as a doublet $\mathbf{2}$. This immediately implies that there are only two bosonic degrees of freedom, and not four.

One can ask what happens for the Ramond sector, which we expect would give 2 fermionic degress of freedom by supersymmetry. In the R sector, there are four transverse periodic fermions $\psi_R^m, m = 1, \ldots, 4$, along the non-compact directions external to both type of branes which thus cannot give fermionic contributions. In addition we have four internal anti-periodic fermions ψ_R^i: the R fermions along the ND and DN directions are half-integer moded and give fermionic states. The R vacuum give rise to $2^{4/2} = 4$ $SO(4)_I$-singlet ground states $|s_{12}, s_{34}\rangle$, out of which only two survive the GSO projection (which has no extra sign in the R sector of the ghosts) which requires $s_1 = -s_2$. These space-time fermions transform as a doublet $\mathbf{2'}$ under the little group $SO(4)_E$. We have thus concluded that the number of bosons coming from the (p, p') strings is only 2 and come from the NS sector. The additional 2 bosonic degrees of freedom come from the (p', p) strings in the NS sector. By supersymmetry, the R sector gives 2 fermionic degrees of freedom for each orientation.

It is straightforward to apply these results to our D1-D5 system. From the $(1,5)$ and $(5,1)$ strings we obtain $2n_1n_5 + 2n_1n_5 = 4n_1n_5$ scalars from the NS sector, which constitute another type of Hypermultiplet, different from those considered for the (1,1) and (5,5) strings. The scalars can be recast in a complex doublet $\chi = (A, B^\dagger), A, B \in \mathbb{C}$ and χ is a chiral spinor satisfying $\Gamma_{6789}\chi = \chi$. These strings are bifundamental fields: for instance, the (1,5) strings transform in the fundamental of $U(n_1)$ and in the anti-fundamental of $U(n_5)$. Note that is the presence of these strings that breaks the supersymmetry from 16 to 8, in agreement with the fact that the D1-D5 system is 1/4 BPS. From a 6D viewpoint, the theory on the D5-branes is a $\mathcal{N} = 1$ theory with R-symmetry $SU(2)_R$. Note that the fermionic superpartners of the bifundamental scalars are singlets of $SO(4)_I$ but are charged under the $SU(2)_{1,2}$ in $SO(4)_E \simeq SU(2)_1 \times SU(2)_2$.

From the above discussions, and from Section 2.4.1, we conclude that the gauge theory of the D1D5 system is a (1+1)-dimensional $\mathcal{N} = (4,4)$ sigma model with gauge group $U(n_1) \times U(n_5)$ and target space Eq. (2.76).

[27] A reference for this is [45], Eq. (10.2.24) and Eq. (13.5.19).

2.4.3 The D1-D5 superconformal field theory

The classical moduli space of vacua of the 2D sigma model is found by imposing the D-flatness equations, modulo the $U(n_1) \times U(n_5)$ gauge group. Details can be found in [58]. Depending on the VEV of the scalars in the various multiplets, these can parametrise the Coulomb or the Higgs branch[28] of the theory. The Coulomb branch is spanned by the VEVs of the scalars in the vector multiplets, while the scalars in the hypermultiplets are set to zero. In this case, the branes can separate from each other and generically the gauge group is completely broken to $U(1)^n$. The Higgs branch is defined by the VEVs of the scalars in the hypermultiplets, having set to zero the scalars in the vector multiplet. The two branches meet at the origin (where all the scalars are zero) and it has been shown in [56] that the two remain distinct also quantum mechanically. To see this, one can consider the conformal field theory limit of the 2D sigma model. In fact, the latter flows to *two* distinct CFTs in the IR, one corresponding to the Coulomb branch and one corresponding to the Higgs branch of the theory [81]. Indeed, the two conformal field theories have different R-symmetry and usually different central charges [56]: the Coulomb branch has $c = 6n_5$ while the Higgs branch has $c = 6n_1n_5$ (or $6(n_1n_5 + 1)$ for the K3 compactification). As a consequence, the moduli space of instantons has dimension $\dim(\mathcal{M}_{\text{Inst}}) = 4n_1n_5$ (or $4(n_1n_5 + 1)$ for the K3 compactification). Note that, for black hole applications, the Higgs branch is the most entropic sector of the theory[29] and the CFT limit corresponds to the near-horizon limit of the D1-D5 supergravity solution. The theory on the worldvolume of the effective 6d D-strings sees the attractor mechanism as its RG-flow. It is thus expected that the SCFT will be sensible to the 20-dimensional *attracted* moduli space $\mathcal{M}^*_{\text{Sugra}}$, so that the moduli space of the SCFT is $\mathcal{M}_{\text{SCFT}} = \mathcal{M}^*_{\text{Sugra}}$ [49]. Indeed, this is know from the works of [82, 83] that locally the moduli space of any $\mathcal{N} = (4, 4)$ SCFT is of the form $SO(4, n)/SO(4) \times SO(n)$, where $n = 5, 21$ for $\mathbb{T}^4$ and K3 respectively. This holds to be true also at first order in conformal perturbation theory [84]. Note that this is consistent with the idea found in usual AdS/CFT dualities where the radial space-time coordinate corresponds to the energy of the dual CFT theory.

We conclude that the low energy description of the D1-D5 system is a 2D $\mathcal{N} = (4, 4)$ SCFT with moduli space $\mathcal{M}_{\text{SCFT}} = \mathcal{M}^*_{\text{Sugra}}$ and with target space

$$\mathbb{R}^4 \times \left(\mathbb{T}^4 \times \mathcal{M}_{\text{Inst}}\right) , \tag{2.85}$$

where the first factor describes the Coulomb branch and the last two the Higgs branch. The $\mathbb{T}^4$ is a free theory describing the center of mass motion of the branes inside the torus.

[28] There can be mixed branches, which we ignore for simplicity of discussion.

[29] This can be seen by using the Cardy formula for the entropy, where the central charge contributes to the entropy.

The origin of the two branches is related to the small instanton singularity of the gauge theory, corresponding to the emission of long strings. Quantum mechanically, the Coulomb branch metric $ds^2_{\mathbb{R}^4} = dr^2 + r^2 d\Omega^2_{\mathbb{S}^3}$ gets deformed to a tube-like metric of the form $r^{-2}\left(dr^2 + r^2 d\Omega^2_{\mathbb{S}^3}\right)$. This shows that now the origin $r = 0$ is at infinite distance. The Higgs branch, being Hyperkähler, is protected from quantum corrections. However, using the correct variables near the singularities of the space, one can show that the physics near the origin is described in terms of a dynamical Liouville field which induces a strong coupling singularity [50]. Again, there is a tube-like behaviour at the quantum level. This means that, starting from the Higgs branch of the D1-D5 system, an instanton can travel through the infinitely long tube and 'come out' of the Coulomb branch. This emission of long strings has been shown to be a peculiarity only of the co-dimension four singular space [85]. By turning on the fields parametrising the additional four directions (for instance, $B^{SD}_{ij} \neq 0$), the singularity gets resolved and the long string emission stops.

2.4.4 Attracted moduli space and moduli space of instantons

It remains to understand what is the relation between the (attracted) moduli space $\mathcal{M}_{\text{SCFT}} = \mathcal{M}^*_{\text{Sugra}}$ and the (non-trivial, Higgs branch) target space $\mathcal{M}_{\text{Inst}}$. To understand this, we follow [49]. The instanton moduli space has a natural hyperkähler metric induced[30] by the compactification manifold $X = \mathbb{T}^4, K3$, in terms of the L^2-metric on (adjoint-valued) one-forms on X. Additionally, one can turn on a (self-dual) B-field on X, which modifies the hyperkähler moment map used in the definition of $\mathcal{M}_{\text{Inst}}$. In particular, the B-field enters in the definition of the Kähler form while the complex structure of $\mathcal{M}_{\text{Inst}}$ is determined by the complex structure of $\mathcal{M}_{\text{SCFT}} = \mathcal{M}^*_{\text{Sugra}}$. We thus conclude [49] that the 20-dimensional moduli space of the 2D $\mathcal{N} = (4,4)$ SCFT $\mathcal{M}_{\text{SCFT}} = \mathcal{M}^*_{\text{Sugra}}$ should be matched with the hyperkähler structure of the instanton moduli space $\mathcal{M}_{\text{Inst}}$, which constitute the (non-trivial) part of the Higgs branch of the theory. Importantly, in the D1-D5 frame, when the B-field is set to zero, the hyperkähler metric is of the symmetric product form and the inverse string coupling is identified with the volume of the sigma model.

We now summarise the relation between the moduli spaces with the following diagram:

[30] In general, not unique.

$$
\begin{tikzcd}
\mathcal{M}_{\text{Sugra}} \arrow[d, "\text{Attractor}"'] & & \mathcal{M}_{\text{Gauge}} \arrow[d, "\text{RG Flow}"] & \mathbb{R}_t \times \mathbb{S}^1_y \arrow[d, "\text{SCFT, Higgs branch}"] & \\
\mathcal{M}^*_{\text{Sugra}} & \equiv_{\text{loc}} & \mathcal{M}_{\text{SCFT}} \arrow[r, "\text{HK structure}"] & \mathcal{M}_{\text{Inst}} \arrow[r, "\pi"] & \text{Symm}_{n_1 n_5}(\tilde{\mathbb{T}}^4)
\end{tikzcd}
$$

FIGURE 2.1: Summary of the relation between the various moduli spaces. The moduli space $\mathcal{M}_{\text{Sugra}}$ and $\mathcal{M}_{\text{Gauge}}$ flow to the same 'attracted' moduli space $\mathcal{M}^*_{\text{Sugra}} \equiv_{\text{loc}} \mathcal{M}_{\text{SCFT}}$, where the identification is understood to be local. The latter is related to the Hyperkähler structure of the (non-trivial part of the) Higgs branch $\mathcal{M}_{\text{Inst}}$, which is seen as a compact smooth Hyperkähler resolution of the orbifold $\text{Symm}_{n_1 n_5}(\tilde{\mathbb{T}}^4)$.

Below we compare the *bulk* physics (be it supergravity or the more general string theory description) to the *boundary* physics at various points of the moduli space.

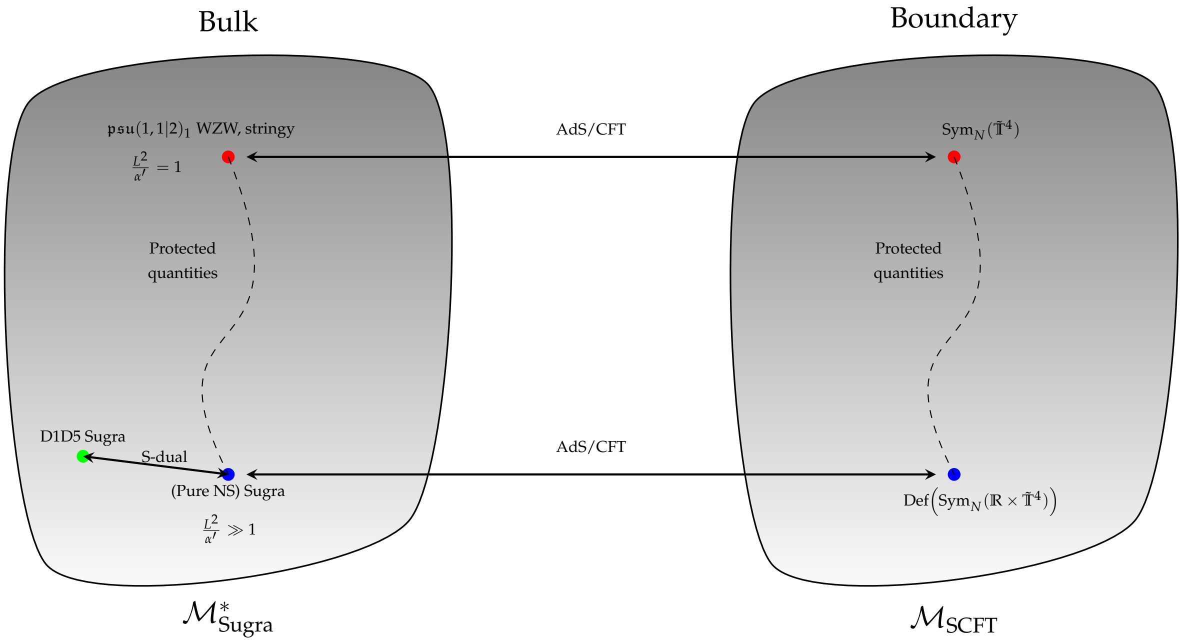

FIGURE 2.2: A very qualitative cartoon of the relation between various points in moduli space, according to the AdS/CFT duality. The figure emphasizes that, despite having $\mathcal{M}^*_{\text{Sugra}} \equiv_{\text{loc}} \mathcal{M}_{\text{SCFT}}$, quantities computed in regime of low spacetime curvature can be compared to those obtained at orbifold point only if protected, and viceversa. Note that, in a more accurate representation, the marked points may actually reside at cusps or at the boundary of the moduli space.

A more precise representation and discussion about the moduli space of the D1-D5 system can be found in [44]. In particular, parametrizing a two-dimensional subspace of the moduli space with $\tau = C^{(0)} + i g_s^{-1}$ at fixed $N = n_1 n_5$, the free symmetric orbifold point is found at $i\infty$. Deformations along the $C^{(0)}$ direction (and $C^{(3),SD}_{ij}$) would generically break the orbifold structure, and stop the long string emission [85]. On the other hand, changes along the imaginary axis would still lead to the presence of long strings in the spectrum. Notably, the bulk dual of the symmetric product orbifold theory at $i\infty$ does not have a continuous spectrum given to the shortening of the representations [86].

One may thus wonder what is the relation between the D1D5 CFT *at the orbifold point* and the supergravity description of the D1-D5 system. These two points are different and only protected quantities can be compared between the two points. In addition, the supergravity solution admits a continuum of states that are not present in the discrete spectrum of the dual D1D5 CFT at the orbifold point, as mentioned above. For the NS5-F1 system with one unit of NS flux $n_5 = 1$, at a point of the moduli space $\mathcal{M}^*_{\text{Sugra}} \simeq \mathcal{M}_{\text{SCFT}}$ one finds that the Hyperkähler structure of the Higgs branch $\mathcal{M}_{\text{Inst}}$ degenerates to that of an orbifold space and the holographic CFT becomes a particularly simple theory in terms of free bosons and fermions. Indeed, it has been recently shown [87, 88] that the bulk dual is the so-called tensionless string, described by a $\text{PSU}(1,1|2)$ WZW model whose affine algebra $\widehat{\mathfrak{psu}}(1,1|2)_1$ has level $n_5 = 1$. The level, coinciding with the number of five-branes, is identified with the size of AdS_3 in string units (which is the same as that of $\mathbb{S}^3$ for critical strings) and the tensionless regime describes the particular situation of having a string-sized target space $L^2_{\text{AdS}}/\alpha' = 1$. At the orbifold point, the relation between the string coupling and N is correct only at large values of the latter. As explained in [89], one must compare the *genus* expansion of the worldsheet theory with that of covering space of the dual orbifold theory: at a given genus g, there is an infinite series of subleading terms in powers of N.

The supergravity regime is reliable when the size of AdS is much larger to that of the string length. In case of only NS-NS fluxes (as an example, the NS5-F1-P system), some supergravity backgrounds are (partly) described by $SL(2,\mathbb{R})$ Wess-Zumino-Witten model whose affine Lie algebra $\widehat{\text{sl}(2,\mathbb{R})}_k$ has level $k \in \mathbb{Z}_{>1}$. For the case of the bosonic string in AdS_3 (a particular case of the more general model we consider in this thesis), the holographic CFT has been recently proposed [90, 91] to be a deformation of $\text{Sym}_N(\mathbb{R}_Q \times \mathbb{T}^4)$, where $\mathbb{R}_Q$ is a linear dilaton theory of slope $Q = \frac{k-3}{\sqrt{k-2}}$, where this time k is the bosonic affine level. The deformation is parametrised by the operator $\Phi = e^{-\sqrt{k-2}\phi}\sigma_2$, where ϕ is the linear dilaton field and σ_2 is a twist-2 operator of the symmetric orbifold theory.

2.5 The holographic CFT

As discussed, in the AdS_3 limit of the NS5-F1 system there is a locus in moduli space at which the holographically dual CFT is an $\mathcal{N} = (4,4)$ symmetric product orbifold

CFT with target space $\left(\mathbb{T}^4\right)^N/S_N$, where $N = n_1 n_5$. To make the presentation self-contained, we review some aspects of this theory and set the notation.

Recall that we work in Type IIB compactified on $\mathbb{S}^1 \times \mathbb{T}^4$, with n_5 NS5 branes wrapped on $\mathbb{S}^1 \times \mathbb{T}^4$, n_1 units of F1 winding on $\mathbb{S}^1$, and n_P units of momentum charge along $\mathbb{S}^1$. The moduli space is 20-dimensional and the symmetric product orbifold CFT is conjectured to lie at a particular locus of this moduli space [44], see also [92]. The configuration breaks the SO(1,9) Lorentz group to $\mathrm{SO}(1,1) \times \mathrm{SO}(4)_E \times \mathrm{U}(1)^4$, where the external R-symmetry $\mathrm{SO}(4)_E \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ corresponds to rotations in the spatial $\mathbb{R}^4$ transverse to the branes (in the IR limit, rotations of the $\mathbb{S}^3$). It is customary to introduce an approximate internal $\mathrm{SO}(4)_I \simeq \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$, which is broken to $\mathrm{U}(1)^4$ by the compactification, but which is useful for classifying states and organizing fields [58, 46].

In the symmetric product orbifold theory, for each copy of $\mathbb{T}^4$ there are four free bosons, together with their left and right-moving fermionic superpartners. Indices $\alpha, \dot{\alpha}, A, \dot{A}$ correspond respectively to $\mathrm{SU}(2)_L, \mathrm{SU}(2)_R, \mathrm{SU}(2)_1, \mathrm{SU}(2)_2$. The free fields are denoted as (we use the conventions of [93])

$$X_{A\dot{A}\,(r)}(z,\bar{z})\,, \qquad \psi^{\alpha\dot{A}}_{(r)}(z)\,, \qquad \bar{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(\bar{z})\,, \tag{2.86}$$

where the subscript (r) denotes the r-th copy of the seed $\mathbb{T}^4$ theory. Omitting this copy subscript and focusing on the holomorphic sector, the energy-momentum tensor $\mathbb{T}(z)$, the supercurrents $\mathbb{G}^{\alpha A}(z)$ and the $\mathrm{SU}(2)_L$ currents $\mathbb{J}^a$ generate the small (4,4) superconformal algebra. We denote holographic CFT conformal weights by h and R-symmetry quantum numbers by $(\mathring{\mathbb{j}}, m')$ and $(\bar{\mathring{\mathbb{j}}}, \bar{m}')$, respectively. Thanks to the quotient with the symmetric group S_N, different copies of the theory can be combined together into strands of various lengths. Thus, in the orbifold theory we can distinguish two different sectors, namely the untwisted sector, where the theory is thought as a collection of N strands of length equal to one, and the twisted sector, where multiple strands are joined together and have length greater than one. The operator that implements such transformation is the twist operator $\sigma_n := \sigma_{(12\ldots n)}$. More precisely, given a permutation $(12\ldots n), n < N$, the operator σ_n implements the following boundary conditions on the fields: $X^i_{(1)} \to X^i_{(2)} \to \cdots \to X^i_{(n)} \to X^i_{(1)}$, while the remaining strands from $n+1$ to N are left untouched. Another important feature of the orbifold theory is the presence of an automorphism of the su(2) subalgebra of the superconformal algebra, called holographic spectral flow[31]. This automorphism implements a change in the weight and R-charge of the states given by

$$h' = h + \alpha\mathring{\mathbb{j}} + \frac{c\,\alpha^2}{24}\,, \qquad m' = m + \frac{c\,\alpha}{12}\,, \qquad \alpha \in \mathbb{R}\,. \tag{2.87}$$

[31]For the rest of the thesis, the reader should be careful in distinguishing the holographic spectral denoted by α flow from the worldsheet spectral flow denoted by w, which instead represents an asymptotic winding of the strings in AdS$_3$.

The heavy states we are interested in are obtained by fractional spectral flow [94], see also [95, 96]. We start from the state dual to $\mathbb{Z}_\mathsf{k}$-orbifolded global $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$, which is a tensor product of N/k identical states of twist k, each of which is the lowest dimension state in the k-twisted sector. This state has conformal dimension

$$h = \bar{h} = \frac{c}{24}\left[1 - \frac{1}{\mathsf{k}^2}\right], \tag{2.88}$$

where the central charge $c = 6N$. The R-charges of this state are zero. Because all the component states (i.e. the “strands” mentioned above) have twist k, there is an enhancement of the usual spectral flow, such that one can perform spectral flow with fractional parameters,

$$\alpha = \frac{\mathsf{m}+\mathsf{n}}{\mathsf{k}} = \frac{2s+1}{\mathsf{k}}\,, \qquad \bar{\alpha} = \frac{\mathsf{m}-\mathsf{n}}{\mathsf{k}} = \frac{2\bar{s}+1}{\mathsf{k}}\,, \tag{2.89}$$

where $s, \bar{s} \in \mathbb{Z}$, and we note that the range $\mathsf{m} > \mathsf{n} \geq 0$ is the range $s \geq \bar{s} \geq 0$. This generates a new state with quantum numbers

$$\begin{aligned} h &= \frac{c}{24}\left[1 - \frac{1}{\mathsf{k}^2} + \alpha^2\right], \qquad & m' &= \frac{\alpha c}{12}, \\ \bar{h} &= \frac{c}{24}\left[1 - \frac{1}{\mathsf{k}^2} + \bar{\alpha}^2\right], \qquad & \bar{m}' &= \frac{\bar{\alpha} c}{12}. \end{aligned} \tag{2.90}$$

These states are “heavy” in the sense that their conformal dimensions scale linearly with the large central charge c. By constrast, the “light” perturbative string states probing these backgrounds will correspond to holographic CFT states with conformal dimensions that are independent of c.

2.5.1 Chiral primaries in the D1D5 CFT

We briefly review the construction of chiral primary operators in the symmetric orbifold CFT [97]. We focus primarily on the holomorphic sector in the following; the antiholomorphic sector is entirely analogous. In the untwisted sector, on each copy of the seed $\mathbb{T}^4$ theory, the chiral primary operators correspond to the states (suppressing the copy (r) label)

$$|0_{\mathrm{NS}}\rangle\,, \qquad \psi^{+\dot{A}}_{-\frac{1}{2}}|0_{\mathrm{NS}}\rangle\,, \qquad \mathbb{J}^{+}_{-1}|0_{\mathrm{NS}}\rangle \;=\; \psi^{+\dot{1}}_{-\frac{1}{2}}\psi^{+\dot{2}}_{-\frac{1}{2}}|0_{\mathrm{NS}}\rangle\,, \tag{2.91}$$

where $|0_{\mathrm{NS}}\rangle$ is the NS vacuum. The corresponding weights and R-charges are $h = m' = 0, \frac{1}{2}, 1$, respectively. Physical configurations in the orbifold theory are obtained by symmetrizing the states in (2.91) over the different copies of the seed theory.

By including the antiholomorphic sector we can obtain, for instance, the dimension $(\frac{1}{2}, \frac{1}{2})$ operator (see e.g. [98])

$$O^{++} = \sum_{r=1}^{N} O^{++}_{(r)} = \frac{-i}{\sqrt{2}} \sum_{r=1}^{N} \psi^{+\dot{A}}_{(r)} \epsilon_{\dot{A}\dot{B}} \bar{\psi}^{+\dot{B}}_{(r)} \,, \qquad (O^{++})^{\dagger} = O^{--} \,. \tag{2.92}$$

We will use this operator in an explicit example later in the thesis.

In order to construct more general chiral primaries, one needs to consider the twisted sectors of the theory. Consider the 'bare' twist operators σ_n, defined on the cylinder, that impose the following boundary conditions corresponding to a single-cycle permutation,

$$\begin{aligned} X_{(1)} \to X_{(2)} \to \cdots \to X_{(n)} \to X_{(1)} \,, \\ \psi_{(1)} \to \psi_{(2)} \to \cdots \to \psi_{(n)} \to -\psi_{(1)} \,, \end{aligned} \tag{2.93}$$

and likewise for the antiholomorphic fermions. The bare twist operators are defined to be the lowest-dimension twist operators that impose the above boundary conditions; they have dimension $h = \bar{h} = \frac{1}{4}\left(n - \frac{1}{n}\right)$ and zero R-charge. Chiral operators are obtained by exciting the bare twist operators operators to add R-charge. The lowest-dimension chiral operators have $h = m' = \frac{n-1}{2}$. For n odd, these operators are obtained by acting with modes of the SU(2) currents, which are bilinears in the free fermions. Due to the twist operator, the SU(2) currents are fractional-moded in units of $1/n$. The relation between these modes and those of free fermions on the n copies of the seed theory can be found in [97]. To construct the chiral operators, one acts with the currents $\mathbb{J}^{+}_{-l/n}$ for which l is odd and $l < n$,

$$n \text{ odd}: \qquad \sigma_n^{-} \;=\; \prod_{p=1}^{(n-1)/2} \mathbb{J}^{+}_{-\frac{2p-1}{n}} \sigma_n \;=\; \mathbb{J}^{+}_{-\frac{n-2}{n}} \cdots \mathbb{J}^{+}_{-\frac{3}{n}} \mathbb{J}^{+}_{-\frac{1}{n}} \sigma_n \,. \tag{2.94}$$

For n even, one first acts with a spin field S_n^{+}, which has weight $\frac{1}{4n}$ and charge $\frac{1}{2}$, putting the fermions into the Ramond sector (i.e. their boundary conditions are similar to Eq. (2.93) but with the final sign being $+\psi_{(1)}$). One then acts with the currents $\mathbb{J}^{+}_{-l/n}$ for which l is even and $l < n$,

$$n \text{ even}: \qquad \sigma_n^{-} \;=\; \prod_{p=1}^{(n-2)/2} \mathbb{J}^{+}_{-\frac{2p}{n}} S_n^{+} \sigma_n \;=\; \mathbb{J}^{+}_{-\frac{n-2}{n}} \cdots \mathbb{J}^{+}_{-\frac{4}{n}} \mathbb{J}^{+}_{-\frac{2}{n}} S_n^{+} \sigma_n \,. \tag{2.95}$$

As in the untwisted case, for both odd and even n we can act with $\psi^{+\dot{A}}_{-\frac{1}{2}} \equiv \sum_{r=1}^{n} \psi^{+\dot{A}}_{-\frac{1}{2}(r)}$ to obtain a chiral operator $\psi^{+\dot{A}}_{-\frac{1}{2}} \sigma_n^{-}$ which has $h = m' = \frac{n}{2}$. Similarly we can act with $\mathbb{J}^{+}_{-1}$ to obtain a chiral operator $\mathbb{J}^{+}_{-1} \sigma_n^{-}$ which has $h = m' = \frac{n+1}{2}$. Together with the analogous antiholomorphic operators, this exhausts the single-cycle chiral operators. Indeed, by making use of anti-commutators of the supercurrent modes $\mathbb{G}^{\pm A}_{-m/n}$ in the corresponding twisted sectors, one can show that chiral weights are bounded by [46]

$$\frac{n-1}{2} \leq h \leq \frac{n+1}{2} \,. \tag{2.96}$$

In the twisted sectors, it is often convenient to work in a basis that diagonalizes the twisted boundary conditions. We shall make use of this basis in later chapters. One defines

$$\tilde{\psi}^{\alpha\dot{A}}_{\rho} = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} e^{\alpha \frac{2\pi i r\rho}{n}} \psi^{\alpha\dot{A}}_{(r)} \,, \qquad \rho = 0, \ldots, n-1 \,, \tag{2.97}$$

where the index $\alpha = \pm$ should not be confused with the spectral flow parameter in Eq. (2.89). These are mutually orthogonal, and diagonalize the twisted boundary conditions as

$$\tilde{\psi}^{\alpha\dot{A}}_{\rho}(e^{2\pi i} z) = e^{-\alpha \frac{2\pi i \rho}{n}} \tilde{\psi}^{\alpha\dot{A}}_{\rho}(z) \,. \tag{2.98}$$

These fermions can be bosonized to construct an explicit expression for the spin fields mentioned above. Note that the fields $\tilde{\psi}^{\alpha\dot{A}}_{\rho=0}$ are invariant under the twisting. For further discussion, see [99].

We now combine the above holomorphic construction with its antiholomorphic counterpart and define the complete list of scalar left-right chiral primaries we will be interested in:

$$O^{--}_n = \sigma^{--}_n \,, \qquad O^{\dot{A}\dot{B}}_n = \tilde{\psi}^{+\dot{A}}_{\rho=0} \bar{\tilde{\psi}}^{+\dot{B}}_{\rho=0} \sigma^{--}_n \,, \qquad O^{++}_n = \tilde{\psi}^{+\dot{1}}_{\rho=0} \tilde{\psi}^{+\dot{2}}_{\rho=0} \bar{\tilde{\psi}}^{+\dot{1}}_{\rho=0} \bar{\tilde{\psi}}^{+\dot{2}}_{\rho=0} \sigma^{--}_n \,, \tag{2.99}$$

where σ^{--}_n is defined similarly to Eqs. (2.94), (2.95), with the same construction in the antiholomorphic sector. The operators in (2.99) are normalized such that they have unit two-point functions.

For later reference, we note that in each case the respective weights and twist numbers can be written in terms of $j = \frac{n+1}{2}$ as

$$h\left[O^{--}_n\right] = j - 1 \,, \qquad h\left[O^{\dot{A}\dot{B}}_n\right] = j - \frac{1}{2} \,, \qquad h\left[O^{++}_n\right] = j \,. \tag{2.100}$$

An analogous list of anti-chiral primaries (which have $h = -m'$) is obtained by acting on the bare twist fields with current and fermion modes with opposite charge, i.e. $\mathbb{J}^{-}_{-l/n}$ and $\psi^{-\dot{A}}$. As we will shortly review, and up to a shift related to worldsheet spectral flow, this j will be identified with the principal quantum number of the bosonic (global) SL(2,$\mathbb{R}$) algebra of the worldsheet theory, to which we now turn.

2.6 Superstring theory on $\mathbf{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$

We now review the basics of superstring theory on $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ using the RNS formalism with BRST quantization. We first discuss the bosonic SL(2, $\mathbb{R}$) and SU(2) WZW models and then present their supersymmetric counterparts. We present the current algebra and review the spectrum, including states arising from worldsheet spectral flow. Finally, we review the dictionary between holographic CFT operators and their counterparts in the worldsheet theory, following [99, 100].

2.6.1 Bosonic WZW model for SL(2,$\mathbb{R}$)

The SL(2,$\mathbb{R}$) WZW model was studied in detail in [101, 102, 103]. Here we will mostly follow the notation of [99, 100, 38], and normal ordering will be implicitly assumed.

The holomorphic SL(2,$\mathbb{R}$) currents will be denoted $j^a(z)$. They satisfy the OPEs

$$j^a(z)j^b(w) \;\sim\; \frac{k}{2}\frac{\eta^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c\, j^c(w)}{z-w}\,, \tag{2.101}$$

where k is the level of the affine algebra, and where

$$-2\eta^{33} = \eta^{+-} = 2\,, \qquad f^{+-}{}_3 = -2\,, \quad f^{3+}{}_+ = -f^{3-}{}_- = 1\,. \tag{2.102}$$

The holomorphic stress tensor and the central charge follow from the Sugawara construction, and are given by (likewise for the antiholomorphic sector)

$$T_{\rm sl}(z) = \frac{1}{k-2}\left[-j^3(z)j^3(z) + \frac{1}{2}\,j^+(z)j^-(z) + \frac{1}{2}\,j^-(z)j^+(z)\right]\,, \qquad c_{\rm sl} = \frac{3k}{k-2}\,. \tag{2.103}$$

We denote bosonic SL(2, $\mathbb{R}$) primary vertex operators by $V_{j,m,\bar{m}}(z,\bar{z})$. Their zero-mode wavefunctions do not factorize between holomorphic and antiholomorphic sectors, however as is often done we shall work primarily with the holomorphic sector, and suppress the $\bar{m}$ and $\bar{z}$ dependence. The relevant representations of the holomorphic zero-mode algebra are as follows. The principal series discrete representations of lowest (highest) weight are spanned by

$$\mathcal{D}_j^{\pm} = \{|j,m\rangle\;,\; m = \pm j, \pm j \pm 1, \pm j \pm 2, \cdots\}\,, \tag{2.104}$$

respectively, where $j_0^3|j,m\rangle = m|j,m\rangle$. These are unitary representations for any positive real j, and one is the charge conjugate of the other (we will restrict the range of j momentarily). There are also the principal continuous series representations, spanned by

$$\mathcal{C}_j^{\hat{\alpha}} = \{|j,\hat{\alpha},m\rangle\ ,\ 0 \leq \hat{\alpha} < 1\ ,\ j = \frac{1}{2} + is\ ,\ s \in \mathbb{R}\ ,\ m = \hat{\alpha}, \hat{\alpha} \pm 1, \hat{\alpha} \pm 2, \cdots\}\,. \tag{2.105}$$

The particular case $\hat{\alpha} = 1/2 = j$ is actually reducible. It was shown in [101] that the spectrum of the model is built out of continuous and lowest weight representations with

$$\frac{1}{2} < j < \frac{k-1}{2}\,, \tag{2.106}$$

together with their spectrally flowed images, to be introduced below. The allowed range (2.106) follows from L^2 normalization conditions, no-ghost theorems and spectral flow considerations.

Before considering worldsheet spectral flow (we refer to this as the “unflowed” sector), the action of the currents on the primary states is given by

$$j_0^3|j,m\rangle = m|j,m\rangle\,, \tag{2.107a}$$

$$j_0^{\pm}|j,m\rangle = \begin{cases} (m \mp (j-1))|j,m\pm 1\rangle & \text{if } m \neq \mp j \\ 0 & \text{if } m = \mp j\,, \end{cases} \tag{2.107b}$$

$$j_n^a|j,m\rangle = 0 \qquad \forall n > 0\,. \tag{2.107c}$$

These vertex operators can be obtained from those of the Euclidean counterpart of the model, namely the H_3^+ WZW model [104, 105, 106] (see also [107]), as follows. One introduces a set of operators depending on a complex label x, written as $V_j(x|z)$, and having conformal weight

$$\Delta = -\frac{j(j-1)}{k-2}\,. \tag{2.108}$$

The action of the currents on $V_j(x,z)$ is given by

$$j^a(z)V_j(x,w) \sim \frac{D_j^a V_j(x,w)}{(z-w)}\,, \tag{2.109}$$

where

$$D_j^+ = \partial_x\,, \qquad D_j^3 = x\partial_x + j\,, \qquad D_j^- = x^2\partial_x + 2jx\,. \tag{2.110}$$

The two-point function is given by [106]

$$\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)\rangle = \frac{1}{|z_{12}|^{4\Delta_1}}\left[\delta^2(x_1-x_2)\delta(j_1+j_2-1) + \frac{B(j_1)}{|x_{12}|^{4j_1}}\delta(j_1-j_2)\right], \tag{2.111}$$

with

$$B(j) = \frac{2j-1}{\pi}\frac{\Gamma[1-b^2(2j-1)]}{\Gamma[1+b^2(2j-1)]}\nu^{1-2j}\,, \qquad \nu = \frac{\Gamma[1-b^2]}{\Gamma[1+b^2]}\,,\ b^2 = (k-2)^{-1}\,. \tag{2.112}$$

The operators $V_{j,m}(z)$ are related to $V_j(x|z)$ by means of the following Mellin-like transform:

$$V_{j,m}(z) = \int_{\mathbb{C}} d^2x\, x^{j-m-1} \bar{x}^{j-\bar{m}-1} V_j(x,z)\,. \tag{2.113}$$

In the Euclidean H_3^+ model, j takes values $j = 1/2 + is$. To obtain the unflowed $V_{j,m}$ for Lorentzian AdS$_3$, one assumes a well-defined analytic continuation to real values of j. This procedure was discussed in [103], which identified the physical origin of the different divergences arising in correlation functions. For related work, see [108]. The two-point functions in the m-basis then follow from (2.111), (2.113). Using the shorthand $V_i \equiv V_{j_i,m_i}$, one finds

$$\langle V_1 V_2 \rangle \;=\; \frac{\delta^2(m_1+m_2)}{|z_{12}|^{4\Delta_1}} \left[\delta(j_1+j_2-1) + \delta(j_1-j_2) \frac{\pi B(j_1)}{\gamma(2j_1)} \frac{\gamma(j_1+m_1)}{\gamma(1-j_1+m_1)} \right], \tag{2.114}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-\bar{x})$, and where $\delta^2(m)$ is a Dirac delta in $m+\bar{m}$ times a Kroenecker delta in $m-\bar{m}$.

At first sight, the complex variable x may appear simply as an SL(2,$\mathbb{R}$) version of the isospin variables defined for SU(2) in [109]. However, given that the integrated zero modes of the currents realize the spacetime Virasoro modes L_0 and $L_{\pm 1}$, and by examining the expressions of the associated differential operators (2.110), one is led to interpret x as the local coordinate on the boundary theory [78]. In other words, the x variable corresponds to the insertion point of the holographic dual operator at the boundary of AdS$_3$, and computing worldsheet correlation function in the x-basis is equivalent (after an appropriate integration over the worldsheet coordinates $z, \bar{z}$) to computing correlators of the holographic CFT. Indeed, as we will see in more detail later on, according to (2.111), in the bosonic theory a z-integrated vertex operator $V_j(x)$ is identified with a local operator on the boundary theory with weight j. Conversely, the corresponding boundary modes are given by the m-basis operators. Indeed, for states in the discrete sector, the transform in Eq. (2.113) can be inverted, giving

$$V_j(x,z) \;=\; \sum_{m=j+n,\, n\in\mathbb{N}_0} x^{m-j} \bar{x}^{\bar{m}-j}\, V_{j,m\bar{m}}(z)\,. \tag{2.115}$$

The vertex $V_j(x,z)$ is realized via Eq. (2.115) as $V_{j,j}(z)$ translated from the origin to x. Poles in the integrand of (2.113) coming from the expansion around $x=0$ ($x=\infty$) are associated to states in the $\mathcal{D}_j^+$ ($\mathcal{D}_j^-$) representations [88, 38].

Spectral flow automorphisms of the current algebra (2.101) are defined as

$$j^{\pm}(z) \to \tilde{j}^{\pm}(z) = z^{\pm w} j^{\pm}(z)\,, \qquad j^3(z) \to \tilde{j}^3(z) = j^3(z) - \frac{k\,\omega}{2}\frac{1}{z}\,, \tag{2.116}$$

where the so-called spectral flow charge ω is an integer. Analogous formulas hold for the antiholomorphic sector. We work with the universal cover of SL(2,$\mathbb{R}$), which imposes that the holomorphic and antiholomorphic spectral flow parameters must be equal, $\bar{\omega} = \omega$. The action of (2.116) on the above representations defines in general inequivalent representations that must be considered in order to generate a consistent spectrum. This holds up to the so-called series identifications due to the fact that the affine modules $\hat{\mathcal{D}}_j^{+,w}$ and $\hat{\mathcal{D}}_{k/2-j}^{-,w+1}$ are isomorphic. Thus, as mentioned above, the discrete series spectrum is constructed solely upon lowest weight representations with j restricted to the range (2.106).

At the level of vertex operators and for $\omega > 0$, the spectral flow operation introduced in (2.116) defines the so-called flowed primaries, whose OPEs with the currents take the form

$$j^+(z)V^\omega_{j,m}(w) = \frac{(m+1-j)V^\omega_{j,m+1}(w)}{(z-w)^{\omega+1}} + \sum_{n=1}^{\omega} \frac{(j^+_{n-1}V^\omega_{j,m})(w)}{(z-w)^n} + \dots, \tag{2.117a}$$

$$j^3(z)V^\omega_{j,m}(w) = \frac{\left(m+\frac{k}{2}\omega\right)V^\omega_{j,m}(w)}{(z-w)} + \dots, \tag{2.117b}$$

$$j^-(z)V^\omega_{j,m}(w) = (z-w)^{\omega-1}(m-1+j)V^\omega_{j,m-1}(w) + \dots, \tag{2.117c}$$

where the ellipses indicate higher-order terms. Similar expressions hold for $\omega < 0$, with the roles of j^+ and j^- inverted. The operators $V^\omega_{j,m}(z)$ are not affine primaries. They are, however, Virasoro primaries, with worldsheet conformal weight

$$\hat{\Delta} = -\frac{j(j-1)}{k-2} - m\omega - \frac{k}{4}\omega^2\,. \tag{2.118}$$

Note that for $\omega > 0$ ($\omega < 0$), independently of the characteristics of the original state, these correspond to lowest (highest) weight states, with SL(2,$\mathbb{R}$) spin

$$h = m + \frac{k}{2}\omega\,, \tag{2.119}$$

($h = -m - k\omega/2$, respectively). The notation h anticipates that the SL(2,$\mathbb{R}$) spin is identified with the holographic CFT conformal weight [110] (see also e.g. [99]), as we shall see in Eq. (2.121).

The flowed affine modules alluded above are built by acting with the currents on flowed primary states. In particular, the remaining states in the zero-mode algebra, which are obtained by acting with j_0^-, are *not* flowed primaries. Nevertheless, one can proceed as done for the unflowed states, and combine them into a local operator, defined initially for $\omega > 0$ as

$$V^w_{j,h}(x,z) = \sum_{n\in\mathbb{N}_0} x^n \bar{x}^{\bar{n}}\, V^w_{j,h+n,h+\bar{n}}(z)\,. \tag{2.120}$$

Moreover, by inverting $x \to 1/x$ in the expansion, one also obtains the states in the highest-weight representation with the same spin and opposite ω and m. This shows that the resulting x-basis states are actually defined in terms of the absolute value of ω, its sign being irrelevant. A direct x-basis definition for spectrally flowed vertex operators was recently derived in [39], extending the original proposal of [103] valid only for the singly flowed case.

The classical analog of the spectral flow operation (2.116) maps space-like geodesics of point-like strings into solutions in which a long string wound around the AdS$_3$ angular direction at large radius comes in to the centre of global AdS$_3$, collapses to a point, and then re-expands to large radial distance [101]. The spectral flow parameter ω is thus sometimes referred to in the literature as a "winding" number. Note that since the AdS$_3$ angular direction is contractible in the interior of global AdS$_3$, the parameter ω is not a conserved quantity. However, the m-basis two-point functions are diagonal in ω: it was shown in [103] that the m-basis two-point function of flowed primaries is as in (2.114) with an extra factor of $\delta_{\omega_1,-\omega_2}$ and the worldsheet conformal weight Δ_1 replaced by $\hat{\Delta}_1$ given in Eq. (2.118). On the other hand, in the x-basis one finds

$$\langle V^{\omega_1}_{j_1,h_1}(x_1,z_1)V^{\omega_2}_{j_2,h_2}(x_2,z_2)\rangle \;=\; \frac{1}{|x_{12}|^{4h_1}}\frac{\langle V^{\omega_1}_{j_1,m_1}V^{\omega_2}_{j_2,m_2}\rangle}{V_{\rm conf}}\,. \tag{2.121}$$

Thus, as mentioned above, the SL(2,$\mathbb{R}$) spin h is identified with the holographic CFT conformal weight [110], even though in the flowed sectors the spin is independent of the value of j of the corresponding unflowed operator. The factor $V_{\rm conf}$ stands for the divergent volume of the conformal group; it reflects the fact we are picking up the contribution from a pole, and it will cancel in the relevant computations that follow.

2.6.2 Bosonic WZW model for SU(2)

The bosonic SU(2) WZW model was studied in [111, 109]. We denote the generators of the current algebra by k^a, and for most quantities we use primes to distinguish them from their SL$(2,\mathbb{R})$ counterparts. The currents satisfy the OPEs

$$k^a(z)k^b(w) \;\sim\; \frac{k'}{2}\frac{\delta^{ab}}{(z-w)^2}+\frac{f'^{ab}_{c}k^c(w)}{z-w}\;, \tag{2.122}$$

where k' is the level of the affine Lie algebra, δ^{ab} is the Killing form, and f'^{abc} are the corresponding structure constants,

$$2\delta^{33}=\delta^{+-}=2\,, \qquad f'^{+-}_{3}=2\,,\quad f'^{3+}_{+}=-f'^{3-}_{-}=1\,. \tag{2.123}$$

The energy-momentum tensor and central charge are

$$T_{\rm su}(z)=\frac{1}{k'+2}\left[k^3(z)k^3(z)+\frac{1}{2}k^+(z)k^-(z)+\frac{1}{2}k^-(z)k^+(z)\right]\,,\qquad c_{\rm su}=\frac{3k'}{k'+2}. \tag{2.124}$$

We denote SU(2) vertex operators by $V'_{j',m',\bar{m}'}(z,\bar{z})$. Again, their zero-mode wavefunctions do not factorize into holomorphic and antiholomorphic parts, however we shall mostly work holomorphically and suppress antiholomorphic quantities ($\bar{m}'$, $\bar{z}$).

For SU(2), the unitary representations of the zero-mode algebra are labeled by

$$0 \le j' \le \frac{k'}{2}\,, \qquad j' \in \mathbb{Z}/2\,, \tag{2.125}$$

and their states are $|j',m'\rangle$ with $m' = -j', -j'+1, \ldots, j'-1, j'$. Using conventions that mimic those used above for SL$(2,\mathbb{R})$, we have

$$\begin{aligned} k_0^3|j',m'\rangle \;&=\; m'|j',m'\rangle\,, && (2.126a)\\ k_0^{\pm}|j',m'\rangle \;&=\; \begin{cases} (j'+1\pm m')|j',m'\pm 1\rangle & \text{if } \; m\neq \pm j \\ 0 & \text{if } \; m=\pm j\,, \end{cases} && (2.126b)\\ k_n^a|j',m'\rangle \;&=\; 0 \qquad \forall n>0\,, && (2.126c) \end{aligned}$$

and

$$\Delta' \;=\; \frac{j'(j'+1)}{k'+2}\,. \tag{2.127}$$

Unlike SL$(2,\mathbb{R})$, in the SU(2) WZW model spectral flow is not necessary for constructing a consistent spectrum, due to the compactness of the group manifold. Indeed, the spectral flow automorphisms merely reshuffle primary and descendant fields, and they do not introduce new inequivalent representations. Nevertheless, for superstring theory applications it is of practical use to include it in the discussion [100, 52, 1]. We will discuss this in more detail shortly.

For SU(2), spectral flow is defined as

$$k^{\pm}(z) \to \tilde{k}^{\pm}(z) = z^{\mp w'}k^{\pm}(z)\,, \qquad k^3(z) \to \tilde{k}^3(z) = k^3(z) - \frac{k'\omega'}{2}\frac{1}{z}\,. \tag{2.128}$$

In this case, however, it is possible to have $\bar{\omega}' \neq \omega'$. As before, spectrally flowed primaries $V^{\omega'}_{j',m'}(z)$ are Virasoro primaries, with weight

$$\hat{\Delta}' \;=\; \frac{j'(j'+1)}{k'-2} + m'\omega' + \frac{k'}{4}\omega'^2\,, \tag{2.129}$$

but they are not affine primaries, and for $\omega' > 0$ they are defined in terms of the OPEs

$$k^+(z)V^{\omega'}_{j',m'}(w) = (z-w)^{\omega'-1}(j'+1+m')V^{\omega'}_{j',m'+1}(w) + \dots\,, \tag{2.130a}$$

$$k^3(z)V^{\omega'}_{j',m'}(w) = \frac{\left(m' + \frac{k}{2}\omega'\right) V^{\omega'}_{j',m'}(w)}{(z-w)} + \dots\,, \tag{2.130b}$$

$$k^-(z)V^{\omega'}_{j',m'}(w) = \frac{(j'+1-m')V^{\omega'}_{j',m'-1}(w)}{(z-w)^{\omega'+1}} + \sum_{n=0}^{\omega'} \frac{(k^-_{n-1}V^{\omega'}_{j',m'})(w)}{(z-w)^n} + \dots \tag{2.130c}$$

The corresponding two-point functions are, again, the unflowed ones times $\delta_{\omega_1,-\omega_2}$, with the appropriate powers of z_{12}.

2.6.3 Superstrings in $\mathbf{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$

We now review supersymmetric generalizations of the bosonic WZW models discussed above. We introduce fermions ψ^a and χ^a which are superpartners of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ currents J^a and K^a respectively. The appropriate $\mathcal{N} = 1$ supersymmetric extensions of the affine $\mathrm{sl}(2,\mathbb{R})_k$ and $\mathrm{su}(2)_{k'}$ algebras are generated by the supercurrents $\psi^a + \theta J^a$ and $\chi^a + \theta K^a$, where θ is a Grassmann variable. The currents J^a and K^a satisfy the OPEs (2.101) and (2.122) respectively, with level n_5 in both cases, and the OPEs involving the fermions ψ^a and χ^a are

$$J^a(z)\psi^b(w) \sim \frac{f^{ab}{}_c\,\psi^c(w)}{(z-w)}\,, \qquad K^a(z)\chi^b(w) \sim \frac{f'^{ab}{}_c\,\chi^c(w)}{(z-w)}, \tag{2.131a}$$

$$\psi^a(z)\psi^b(w) \sim \frac{n_5}{2}\frac{\eta^{ab}}{(z-w)}\,, \qquad \chi^a(z)\chi^b(w) \sim \frac{n_5}{2}\frac{\delta^{ab}}{(z-w)}\,. \tag{2.131b}$$

One can split the currents into two independent contributions via

$$J^a = j^a - \frac{1}{n_5} f^a{}_{bc}\psi^b\psi^c\,, \qquad K^a = k^a - \frac{1}{n_5} f'^a{}_{bc}\chi^b\chi^c\,. \tag{2.132}$$

The "bosonic" currents j^a and k^a commute with the free fermions, and are currents of bosonic WZW models as described in Section 2.6, with levels $k = n_5 + 2$ and $k' = n_5 - 2$ respectively. In the fermionic sector, the spectral flow automorphisms act as

$$\tilde{\psi}^\pm(z) = z^{\mp\omega}\psi^\pm(z)\,, \quad \tilde{\psi}^3(z) = \psi^3(z)\,, \qquad \tilde{\chi}^\pm(z) = z^{\mp\omega}\chi^\pm(z)\,, \quad \tilde{\chi}^3(z) = \chi^3(z)\,. \tag{2.133}$$

The remaining flat compact directions are treated as usual. For the $\mathbb{T}^4$, we simply have four (canonically normalized) free bosons Y^i and their fermionic partners λ^i ($i = 6,\dots,9$), with OPEs

$$Y^i(z)Y^j(w) \sim -\delta^{ij}\log(z-w)\,, \qquad \lambda^i(z)\lambda^j(w) \sim \frac{\delta^{ij}}{(z-w)}\,. \tag{2.134}$$

We can now write down the energy-momentum tensor T and the supercurrent G of the worldsheet theory for type II superstrings in $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$. The matter contributions read

$$\begin{aligned} T &= \frac{1}{n_5}\left(j^a j_a - \psi^a \partial \psi_a + k^a k_a - \chi^a \partial \chi_a\right) + \frac{1}{2}\left(\partial Y^i \partial Y_i - \lambda^i \partial \lambda_i\right), \quad (2.135)\\ G &= \frac{2}{n_5}\left(\psi^a j_a - \frac{1}{3n_5} f_{abc}\psi^a\psi^b\psi^c + \chi^a k_a - \frac{1}{3n_5} f'_{abc}\chi^a\chi^b\chi^c\right) + i\,\lambda^j \partial Y_j\,, \quad (2.136)\end{aligned}$$

and the resulting central charge is compensated by the usual bc and $\beta\gamma$ ghost systems, leading to the BRST charge

$$\mathcal{Q} = \oint dz\left(c\left(T + T_{\beta\gamma}\right) - \gamma\, G + c(\partial c) b - \frac{1}{4} b\gamma^2\right). \quad (2.137)$$

Here $T_{\beta\gamma}$ is the energy-momentum tensor of the $\beta\gamma$ system, which is bosonized as

$$\beta = e^{-\varphi}\partial\xi\,, \qquad \gamma = \eta e^{\varphi}\,, \quad (2.138)$$

where $\varphi(z)\varphi(w) \simeq -\ln(z-w)$ has background charge 2, and $\xi(z)\eta(w) \sim (z-w)^{-1}$. The (matter) central charge is

$$c = \frac{3(n_5+2)}{n_5} + \frac{3}{2} + \frac{3(n_5-2)}{n_5} + \frac{3}{2} + 6 = 15. \quad (2.139)$$

For computational purposes it is useful to also bosonize the rest of the fermions [78, 99]. We thus define (canonically normalized) bosonic fields H_I with $I = 1, \dots 5$, and write

$$\hat{H}_I = H_I + \pi \sum_{J<I} N_J\,, \qquad N_J \equiv \oint i\partial H_J\,, \quad (2.140)$$

where the number operators N_I are introduced in order to keep track of the cocycle factors, namely

$$e^{ia\hat{H}_I} e^{ib\hat{H}_J} = e^{ib\hat{H}_J} e^{ia\hat{H}_I}\, e^{i\pi ab}\,, \qquad \text{if} \quad I > J\,. \quad (2.141)$$

We bosonize as

$$\psi^{\pm} = \sqrt{n_5}\, e^{\pm i\hat{H}_1}\,, \quad \chi^{\pm} = \sqrt{n_5}\, e^{\pm i\hat{H}_2}\,, \quad \lambda^6 \pm i\lambda^7 = e^{\pm i\hat{H}_4}\,, \quad \lambda^8 \pm i\lambda^9 = e^{\pm i\hat{H}_5}\,, \quad (2.142a)$$

$$\psi^3 = \frac{\sqrt{n_5}}{2}\left(e^{i\hat{H}_3} - e^{-i\hat{H}_3}\right), \qquad \chi^3 = \frac{\sqrt{n_5}}{2}\left(e^{i\hat{H}_3} + e^{-i\hat{H}_3}\right), \quad (2.142b)$$

where $\hat{H}_I^\dagger = \hat{H}_I$ for $I \neq 3$ and $\hat{H}_3^\dagger = -\hat{H}_3$. Then we have

$$i\partial\hat{H}_1 = \frac{1}{n_5}\,\psi^+\psi^-\,, \qquad i\partial\hat{H}_2 = \frac{1}{n_5}\,\chi^+\chi^-\,, \qquad i\partial\hat{H}_3 = \frac{2}{n_5}\,\psi^3\chi^3\,, \quad (2.143a)$$

$$i\partial\hat{H}_4 = i\lambda^6\lambda^7\,, \qquad i\partial\hat{H}_5 = i\lambda^8\lambda^9\,. \quad (2.143b)$$

The phases in (2.141) ensure that bosonized fermions anticommute, and will be important when working with states in the Ramond sector. From now on we will simply omit the hats, and explicitly include the phase factors when they are needed.

The spacetime supercharges can be written as:

$$Q_\varepsilon = \oint dz\, e^{-\varphi/2} S_\varepsilon \,, \qquad S_\varepsilon = \exp\left(\frac{i}{2}\sum_{I=1}^{5} \varepsilon_I H_I\right), \tag{2.144}$$

where S_ε are spin fields and $\varepsilon_I = \pm 1$. Imposing BRST invariance – where the relevant contributions come from the $f_{abc}\psi^a\psi^b\psi^c$ and $f'_{abc}\chi^a\chi^b\chi^c$ pieces of G in (2.136) – and mutual locality (chiral GSO) leads to the conditions

$$\prod_{I=1}^{3} \varepsilon_I = \prod_{I=1}^{5} \varepsilon_I = 1\,. \tag{2.145}$$

In the holomorphic sector this gives the expected four 'ordinary' supercharges and four 'superconformal' supercharges. The same applies in the antiholomorphic sector, giving the total 16 real supercharges of global $\mathrm{AdS}_3 \times \mathbb{S}^3$ [78].

For later use, let us also recall that the R-symmetry of the boundary theory is generated on the worldsheet by the SU(2) currents. More precisely, the zero modes of the spacetime R-currents are given by the integrated worldsheet currents [78], i.e.

$$\mathbb{J}_0^a = \oint dz\, K^a(z)\,. \tag{2.146}$$

Consequently, the holomorphic R-charge in the holographic CFT is identified with m', which from now onwards denotes the eigenvalue of K_0^3. This is why we used the notation m' in Sections 2.5 and 2.5.1. Similarly, from now onwards m denotes the eigenvalue of J_0^3.

2.6.4 Vertex operators and two-point functions

We now discuss physical vertex operators and their two-point functions, both in NS and R sectors. This section is largely review, though we also give explicit expressions for some R sector operators that to our knowledge have not appeared before in the literature.

Our main interest is in worldsheet operators that correspond to chiral primaries of the holographic CFT. We thus focus on states belonging to the discrete representations of SL(2,$\mathbb{R}$), since it has been shown that the chiral primaries of the dual CFT are in one-to-one correspondence with worldsheet states in the discrete representation [100]. Also, as discussed in the previous sections, the continuous representation states are lifted at generic points of the moduli space, and, furthermore, are not part of the spectrum of the dual CFT at the orbifold point. We also discuss the role of SL(2,$\mathbb{R}$) and SU(2) spectral flows in the string theoretical construction. These vertex operators and their two-point functions will be used as building blocks for constructing the vertex operators and two-point functions of the null-gauged models.

2.6.4.1 NS sector

The unflowed NSNS sector was considered in [112], see also [99]. We continue to suppress antiholomorphic parts of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ vertex operators $V_{j,m,\bar{m}}$ and $V'_{j',m',\bar{m}'}$. The complete NSNS vertex is obtained by including the antiholomorphic fermions and ghosts.

We work in the canonical "-1" ghost picture, and consider only states with vanishing momentum in the $\mathbb{T}^4$ directions. Then the (holomorphic part of the) BRST invariant states with up to a single fermionic excitation include the tachyon (which is projected out by GSO),

$$\mathcal{T}_{j,m,j',m'} = e^{-\varphi} V_{j,m} V'_{j',m'} \,, \tag{2.147}$$

and the following spacetime vectors. To write these, we denote the total spins corresponding to the supersymmetric currents J^a and K^a by $J = j + \varepsilon$ and $J' = j' + \varepsilon$, with $\varepsilon = \pm 1$ (recall that m and m' now denote the eigenvalues of J_0^3 and K_0^3 respectively, and that $i = 6, \ldots, 9$),

$$\begin{aligned}
\mathcal{V}^i_{J,m,J',m'} &= e^{-\varphi} \lambda^i V_{j,m} V'_{j',m'} \,, && (2.148\text{a}) \\
\mathcal{W}^\varepsilon_{J,m,J',m'} &= e^{-\varphi} \left(\psi V_j\right)_{j+\varepsilon,m} V'_{j',m'} \,, && (2.148\text{b}) \\
\mathcal{X}^\varepsilon_{J,m,J',m'} &= e^{-\varphi} V_{j,m} (\chi V'_{j'})_{j'+\varepsilon,m'} \,, && (2.148\text{c})
\end{aligned}$$

where we have introduced the linear combinations

$$(\psi V_j)_{j+\varepsilon,m} = c^r_\varepsilon \psi^r V_{j,m-r} \,, \qquad (\chi V'_{j'})_{j'+\varepsilon,m'} = d^r_\varepsilon \chi^r V'_{j',m'-r} \,, \tag{2.149}$$

in which a summation over $r = +1, -1, 0$ is implicit, "0" corresponding to the "3" direction of the respective algebras. These combine the products of bosonic primaries and free fermions into fields of total spins J and J' [112]. The Clebsh-Gordan coefficients are given in our conventions by

$$\begin{aligned}
c^r_- &= \left(\frac{1}{2}, \frac{1}{2}, -1\right), \quad d^r_+ = \left(-\frac{1}{2}, \frac{1}{2}, 1\right), \\
c^r_+ &= \left(\frac{1}{2}(j+m)(j+m-1),\ \frac{1}{2}(j-m)(j-m-1),\ (j+m)(j-m)\right), \\
d^r_- &= \left(\frac{1}{2}(j'-m')(j'-m'+1),\ -\frac{1}{2}(j'+m')(j'+m'+1),\ (j'-m')(j'+m')\right) .
\end{aligned} \tag{2.150}$$

The Virasoro condition associated to all vertex operators in Eq. (2.148) reads

$$\frac{1}{2}+\frac{1}{2}-\frac{j(j-1)}{n_5}+\frac{j'(j'+1)}{n_5}=1\,, \tag{2.151}$$

and is solved by $j=j'+1$ (or its reflection under $j\to 1-j$), thus implying that we are dealing with bosonic primaries in the discrete representations of SL(2,$\mathbb{R}$)$_k$.

Let us briefly discuss the worldsheet two-point functions involving these operators. The different bosonic sectors factorize and the fermions are free, so we can express the results directly in terms of the non-trivial contributions coming from the bosonic SL(2,$\mathbb{R}$) WZW model, namely Eq. (2.114). By construction, the only non-vanishing two-point functions are the diagonal ones. For the 6D scalars coming from the NSNS sector polarizations on the $\mathbb{T}^4$, (2.148a), using the shorthands $V_i \equiv V_{j_i,m_i}$, $V_i' \equiv V'_{j_i',m_i'}$, we have

$$\langle \mathcal{V}_1^{i\bar{i}}\mathcal{V}_2^{j\bar{j}}\rangle \;=\; \langle V_1V_2\rangle\langle V_1'V_2'\rangle\left[\langle e^{-\varphi_1}e^{-\varphi_2}\rangle\langle\lambda_1^i\lambda_2^j\rangle\times \text{c.c.}\right] \;=\; \langle V_1V_2\rangle\langle V_1'V_2'\rangle\times\frac{\delta^{ij}\delta^{\bar{i}\bar{j}}}{|z_{12}|^4}\,. \tag{2.152}$$

Since we are dealing with discrete representations, the contact term in (2.114) vanishes, thus imposing $j_1=j_2\equiv j$. As discussed above Eq. (2.115), the conformal weight in the holographic CFT is to be identified with the SL(2,$\mathbb{R}$) spin, i.e. $h=j$. On the other hand, the R-charge is given by m', with $|m'|\leq j'=j-1$. Thus $h\neq|m'|$, so $\mathcal{V}^i$ cannot correspond to a chiral primary of the HCFT.

We now turn to the operators introduced in the second and third line of (2.148). When computing correlators of two $\mathcal{W}$ states, we must deal with expressions of the form

$$\langle\left(\psi_1V_{j_1}\right)_{j_1+\varepsilon_1,m_1}\left(\psi_2V_{j_2}\right)_{j_2+\varepsilon_2,m_2}\rangle=\sum_{r_{1,2}}c^{r_1}_{\varepsilon_1}c^{r_2}_{\varepsilon_2}\,\langle\psi^{r_1}\psi^{r_2}\rangle\,\langle V_{j_1,m_1-r_1}V_{j_2,m_2-r_2}\rangle\,. \tag{2.153}$$

We use the action of the bosonic currents (2.107) to express $\langle V_{j_1,m_1-r_1}V_{j_2,m_2-r_2}\rangle$ in terms of $\langle V_{j_1,m_1}V_{j_2,m_2}\rangle$, insert the coefficients (2.150), and perform the sum. We obtain

$$\langle\mathcal{W}^{\varepsilon_1}\mathcal{W}^{\varepsilon_2}\rangle=\frac{n_5^2}{4|z_{12}|^4}\langle V_1V_2\rangle\langle V_1'V_2'\rangle\times\begin{cases} j_1(1-2j_1)(j_1^2-m_1^2)\times\text{c.c.} & \varepsilon_1=\varepsilon_2=1\\ \dfrac{(j_1-1)(1-2j_1)}{(j_1-1)^2-m_1^2}\times\text{c.c.} & \varepsilon_1=\varepsilon_2=-1\\ 0 & \varepsilon_1=-\varepsilon_2\,. \end{cases} \tag{2.154}$$

From Eq. (2.114) we find that the coefficients are exactly those needed to produce the shift $j\to j+\varepsilon$ in the two-point function. Hence, Eq. (2.113) shows that the weight of the corresponding holographic dual is $h=j+\varepsilon$ [113]. In particular, the operator $\mathcal{W}^-$ with maximal SU(2) charge has $h=j-1=m'$, and thus corresponds to a chiral primary operator of the holographic CFT.

The computation of the $\langle \mathcal{X}\mathcal{X} \rangle$ correlators is analogous; we obtain

$$\langle \mathcal{X}^{\varepsilon_1}\mathcal{X}^{\varepsilon_2} \rangle = \frac{n_5^2}{4|z_{12}|^4} \langle V_1 V_2 \rangle \langle V_1' V_2' \rangle \times \begin{cases} \dfrac{(j_1'+1)(1+2j_1')}{(j_1'+1)^2 - m_1'^2} \times \text{c.c.} & \varepsilon_1 = \varepsilon_2 = 1 \\ j_1'(1+2j_1')(j_1'^2 - m_1'^2) \times \text{c.c.} & \varepsilon_1 = \varepsilon_2 = -1 \\ 0 & \varepsilon_1 = -\varepsilon_2 = \pm 1 \,. \end{cases} \tag{2.155}$$

For $\mathcal{X}^+$ at highest SU(2) weight we have $h = j = j' + 1 = m'$, leading to a second family of spacetime chiral states. We will discuss the corresponding operators in the holographic CFT theory and fix their normalization below.

So far, we have constructed chiral operators whose boundary weights $h = j - 1$ and $h = j$ are bounded from above by $h < \frac{n_5+1}{2}$, see Eq. (2.106). However, in the D1D5 CFT one can have chiral primaries in n-twisted sectors with n up to $n_1 n_5$, and where h grows linearly with n, as discussed around Eq. (2.96). Thus, it seems that so far we are missing most of the heavier chiral operators. However, as discussed in [100], such states lie in the sectors of the worldsheet theory with non-trivial spectral flow charges, as we now review.

In the supersymmetric theory, spectrally flowed primary operators are built by combining the bosonic flowed primaries introduced in Eqs. (2.117) and (2.130) with fermionic excitations. The bosons H_I allow us to express the spectral flow operation in the fermionic sectors of SL(2,$\mathbb{R}$) and SU(2), Eq. (2.133), in the following form,

$$\psi^{\pm}_{\omega} = \psi^{\pm} e^{-i\omega H_1} \,, \qquad \chi^{\pm}_{\omega'} = \chi^{\pm} e^{i\omega' H_2}, \tag{2.156}$$

while the other fermions remain unchanged. Indeed, the OPEs between the operators in (2.156) and the fermionic currents are analogous to those in (2.117) and (2.130). Once factors of $e^{-i\omega H_1}$ and $e^{i\omega' H_2}$ are included, the corresponding weights take the form in (2.118) and (2.129), with $k - 2 = n_5 = k' + 2$.

In principle, one could simply ignore the possibility of including spectral flow in SU(2) since it does not give any new representations. However, for discrete series states it is useful to do so in order to solve the modified Virasoro condition, as discussed in [52, 1]. Bosonizing the currents j^3 and k^3 with canonically normalised scalars ϕ and ϕ' respectively, we use the spectral flow operator with equal amount of spectral flow in SL(2,$\mathbb{R}$) and SU(2),

$$\exp\left(-i\omega H_1 + \omega\sqrt{\frac{n_5+2}{2}}\phi + i\omega H_2 + i\omega\sqrt{\frac{n_5-2}{2}}\phi' \right), \tag{2.157}$$

which is mutually local with the supercharges, thus producing flowed singly-excited states which will also survive the GSO projection. The corresponding Virasoro condition for the flowed vertex operators is

$$\frac{1}{2} + \frac{1}{2} - \frac{j(j-1)}{n_5} - m\omega - \frac{n_5}{4}\omega^2 + \frac{j'(j'+1)}{n_5} + m'\omega + \frac{n_5}{4}\omega^2 \ = \ 1\,. \tag{2.158}$$

We seek to solve this for general n_5. We thus impose $j = j' + 1$ as well as $m = m'$. The latter constraint is quite restrictive, since by definition we have

$$\begin{aligned} &\mathcal{V}\text{-type operators:} && |m| \geq j\,, && |m'| \leq j' = j-1\,,\\ &\mathcal{W}\text{-type operators:} && |m| \geq j-1\,, && |m'| \leq j' = j-1\,,\\ &\mathcal{X}\text{-type operators:} && |m| \geq j\,, && |m'| \leq j'+1 = j\,. \end{aligned}$$

Consequently, our only candidates are highest/lowest-weight $\mathcal{W}^-$-type operators with $m = m' = j' = j-1$ and $\mathcal{X}^+$-type operators with $m = m' = j'+1 = j$. Their explicit expressions are given by

$$\mathcal{W}^{\omega}_{j} \ = \ e^{-\varphi}\psi^- e^{-i\omega H_1} e^{i\omega H_2} V^{\omega}_{j,j} V'^{\omega}_{j-1,j-1}\,, \tag{2.159a}$$
$$\mathcal{X}^{\omega}_{j} \ = \ e^{-\varphi} e^{-i\omega H_1} \chi^+ e^{i\omega H_2} V^{\omega}_{j,j} V'^{\omega}_{j-1,j-1}\,. \tag{2.159b}$$

These flowed states are also BRST-invariant [100] since the supercurrent G can be written in the flowed frame as

$$G(z) = \tilde{G}(z) + \frac{\omega}{z}\left(\chi^3 - \psi^3\right), \tag{2.160}$$

such that the extra terms on the RHS of this equation act trivially on highest/lowest weight states.

The two-point functions of these spectrally flowed operators can be determined straightforwardly from the corresponding bosonic ones. This is because the latter impose $\omega_2 = -\omega_1$, such that the charge conservation rules for the H_I exponentials are automatically satisfied. Denoting the Hermitian conjugate operators of $\mathcal{W}^{\omega}$ and $\mathcal{X}^{\omega}$ by $\hat{\mathcal{W}}^{\omega}$ and $\hat{\mathcal{X}}^{\omega}$ respectively, we obtain

$$\langle \mathcal{W}^{\omega_1}_1 \hat{\mathcal{W}}^{\omega_2}_2 \rangle \ = \ \langle \mathcal{X}^{\omega_1}_1 \hat{\mathcal{X}}^{\omega_2}_2 \rangle \ = \ \langle V^{\omega_1}_{j_1,j_1} V^{\omega_2}_{j_1,-j_1} \rangle \langle V'^{\omega_1}_{j_1-1,j_1-1} V'^{\omega_2}_{j_1-1,1-j_1} \rangle \frac{n_5^2}{|z_{12}|^{4(1+\omega_1+\omega_1^2)}}\,. \tag{2.161}$$

Spectral flowed primaries are always annihilated by J_0^-, and are thus lowest-weight with respect to the SL(2,$\mathbb{R}$) zero mode algebra. After the supersymmetric spectral flow (2.157), similarly to the bosonic transformations (2.119), (2.130b), the spectral flowed primaries have quantum numbers h and m' that have increased by $\frac{n_5}{2}\omega$ from their values for the unflowed vertex operators. We thus conclude that these vertex operators correspond exactly to the additional chiral operators we were looking for, with

$$\mathcal{W}_j^{\omega} \ : \ h = m' = j - 1 + \frac{n_5}{2}\omega \,, \qquad \mathcal{X}_j^{\omega} \ : \ h = m' = j + \frac{n_5}{2}\omega \,. \tag{2.162}$$

These quantum numbers extend to large values, by raising ω. In the holographic CFT there are states with conformal weight of order $n_1 n_5$, however in our worldsheet models n_1 is of order g_s^{-2} and we work in perturbation theory in g_s, so finite n_1 physics is not accessible. Moreover, when considering holographic CFT operators with conformal weight of order $n_1 n_5$, the dual bulk configuration is not a light probe on the original background, but rather a different background. The rest of the modes associated to such boundary operators are obtained by acting with the global current J_0^+ as in Eq. (2.120), and do not have simple expressions in the m-basis since they are not flowed primaries.

2.6.4.2 Ramond sector

We now review the Ramond sector physical operators of the worldsheet theory, in the m-basis. To our knowledge, this construction has only been carried out explicitly in the literature for the case of highest/lowest-weight states [99, 100]; we shall present explicit expressions for more general Ramond sector operators.

We will make use of the spin fields introduced in (2.144), and distinguish the slightly more involved $\mathrm{AdS}_3 \times \mathbb{S}^3$ sector, for which we write the relevant factors as

$$S_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \;=\; e^{\frac{i}{2}(\varepsilon_1 H_1 + \varepsilon_2 H_2 + \varepsilon_3 H_3)} \,. \tag{2.163}$$

We denote the $\mathrm{AdS}_3 \times \mathbb{S}^3$ chirality by $\varepsilon \equiv \varepsilon_1 \varepsilon_2 \varepsilon_3$. We shall implement this by considering ε_3 to be fixed to be $\varepsilon_3 = \varepsilon \varepsilon_1 \varepsilon_2$. We impose the chiral GSO projection via the mutual locality condition $\prod_{I=1}^{5} \varepsilon_I = 1$, and we implement this by fixing $\varepsilon_5 = \varepsilon \varepsilon_4$. We introduce a generic linear combination of bosonic primaries and spin fields of $\mathrm{AdS}_3 \times \mathbb{S}^3$ of fixed chirality,

$$\left(SVV'\right)^{\varepsilon}_{J,m,J',m'} = \sum_{\varepsilon_1,\varepsilon_2=\pm 1} f^{\varepsilon}_{\varepsilon_1\,\varepsilon_2} S_{\varepsilon_1\varepsilon_2\varepsilon_3} V_{j,m-\frac{\varepsilon_1}{2}} V'_{j',m'-\frac{\varepsilon_2}{2}} \,, \qquad \varepsilon_3 = \varepsilon\varepsilon_1\varepsilon_2 \,, \tag{2.164}$$

where the total spins (J, J') will be related to (j, j') in various ways momentarily. Note that for highest/lowest weight states, there may be only one allowed choice of ε_1 and/or ε_2, as we shall see in an example below. In the canonical "$-\frac{1}{2}$" picture, the Ramond sector vertex operators then take the form

$$\mathcal{Y}^{\varepsilon,\varepsilon_4}_{J,m,J',m'} = e^{-\frac{\varphi}{2}} \left(SVV'\right)^{\varepsilon}_{J,m,J',m'} e^{\frac{i\varepsilon_4}{2}(H_4 + \varepsilon H_5)} \,. \tag{2.165}$$

The Clebsch–Gordan coefficients $f^{\varepsilon}_{\varepsilon_1 \varepsilon_2}$ are computed by requiring that the $\mathcal{Y}$ operators transform appropriately under the action of the currents $J^{\pm}$, $K^{\pm}$. In our conventions, this gives four linear combinations. In the equations below, the first bracket specifies how (J, J') are related to j and j'; for instance, for case A, $(J = j - 1/2, J' = j' + 1/2)$. For each case we write the coefficients as a list, $f^{\varepsilon}_{\varepsilon_1 \varepsilon_2} = (f^{\varepsilon}_{++}, f^{\varepsilon}_{+-}, f^{\varepsilon}_{-+}, f^{\varepsilon}_{--})$. We obtain

$$A\colon\ (j-\tfrac{1}{2}, j'+\tfrac{1}{2}), \quad f^{\varepsilon,A}_{\varepsilon_1\varepsilon_2} = \left(1,\, i,\, \varepsilon,\, \varepsilon i\right), \tag{2.166a}$$

$$B\colon\ (j+\tfrac{1}{2}, j'+\tfrac{1}{2}), \quad f^{\varepsilon,B}_{\varepsilon_1\varepsilon_2} = \left(f^{B_1},\, i f^{B_1},\, \varepsilon f^{B_2},\, \varepsilon i f^{B_2}\right), \tag{2.166b}$$

$$C\colon\ (j-\tfrac{1}{2}, j'-\tfrac{1}{2}), \quad f^{\varepsilon,C}_{\varepsilon_1\varepsilon_2} = \left(f^{C_1},\, -i f^{C_2},\, \varepsilon f^{C_1},\, \varepsilon(-i) f^{C_2}\right), \tag{2.166c}$$

$$D\colon\ (j+\tfrac{1}{2}, j'-\tfrac{1}{2}), \quad f^{\varepsilon,D}_{\varepsilon_1\varepsilon_2} = \left(f^{B_1}f^{C_1},\, (-i) f^{B_1}f^{C_2},\, \varepsilon f^{B_2}f^{C_1},\, \varepsilon(-i) f^{B_2}f^{C_2}\right), \tag{2.166d}$$

where

$$f^{B_1} = m + j - \frac{1}{2}, \quad f^{B_2} = m - j + \frac{1}{2}, \quad f^{C_1} = j' - m' + \frac{1}{2}, \quad f^{C_2} = j' + m' + \frac{1}{2}. \tag{2.167}$$

We note that in all cases we have $f^{+}_{\varepsilon_1\varepsilon_2} = \varepsilon_1 f^{-}_{\varepsilon_1\varepsilon_2}$. In addition, and using $j = j' + 1$, BRST-invariance gives four equations for each chirality, out of which only two are linearly independent, namely

$$f^{\varepsilon}_{-+} = \frac{1}{(j+m-\frac{1}{2})}\left[f^{\varepsilon}_{++}\left(\varepsilon m + m'\right) - i f^{\varepsilon}_{+-}\left(j' - m' + \frac{1}{2}\right)\right], \tag{2.168a}$$

$$f^{\varepsilon}_{--} = \frac{1}{(j+m-\frac{1}{2})}\left[i f^{\varepsilon}_{++}\left(j' + m' + \frac{1}{2}\right) + f^{\varepsilon}_{+-}\left(\varepsilon m - m'\right)\right]. \tag{2.168b}$$

These are satisfied by only half of the states in Eq. (2.166). The physical states in the "$-\frac{1}{2}$" picture are given by the A and D states with $\varepsilon = 1$ (and either choice of $\varepsilon_4 = \pm 1$), plus the B and C states with $\varepsilon = -1$ (and again either sign of ε_4), making the correct eight physical polarizations.

The full list of expressions in (2.166) is useful in order to construct the representatives of such operators in the "$-\frac{3}{2}$" ghost picture, necessary for computing two-point functions. To obtain the "$-\frac{3}{2}$" picture operators, we make an educated guess for their expressions, and then apply the picture *raising* operator, i.e.

$$\Phi^{(-\frac{1}{2})}(w) = \lim_{z\to w} \left(e^{\varphi} G\right)(z)\, \Phi^{(-\frac{3}{2})}(w). \tag{2.169}$$

In order to get a non-trivial propagator, and up to an overall constant, the appropriate guess is that they are given by the states with the same spins but opposite chirality. Explicitly, we have

$$\mathcal{Y}^{\varepsilon,\varepsilon_4\,(-\frac{3}{2})}_{J,m,J',m'} = \pm\frac{\sqrt{n_5}}{2j-1} e^{-\frac{3\varphi}{2}} \left(SVV'\right)^{-\varepsilon}_{J,m,J',m'} e^{\frac{i\varepsilon_4}{2}(H_4 + \varepsilon H_5)}, \tag{2.170}$$

where the negative (positive) sign holds for the cases A and B (C and D).

We can now compute the two-point functions in the unflowed Ramond sector. Only diagonal pairings are non-zero, by construction. Denoting the antiholomorphic sector contributions by "c.c.", we obtain

$$\langle \mathcal{Y}^{\varepsilon_4,(-\frac{1}{2})}_{[A]} \mathcal{Y}^{-\varepsilon_4,(-\frac{3}{2})}_{[A]} \rangle = \frac{n_5}{|z_{12}|^4} \langle V_1 V_2 \rangle \langle V_1' V_2' \rangle \left(\frac{(2j-1)}{(j-m-\frac{1}{2})(j'+m'+\frac{1}{2})} \times \text{c.c.} \right), \tag{2.171a}$$

$$\langle \mathcal{Y}^{\varepsilon_4,(-\frac{1}{2})}_{[B]} \mathcal{Y}^{-\varepsilon_4,(-\frac{3}{2})}_{[B]} \rangle = \frac{n_5}{|z_{12}|^4} \langle V_1 V_2 \rangle \langle V_1' V_2' \rangle \left(\frac{(2j-1)(j+m-\frac{1}{2})}{(j'+m'+\frac{1}{2})} \times \text{c.c.} \right), \tag{2.171b}$$

$$\langle \mathcal{Y}^{\varepsilon_4,(-\frac{1}{2})}_{[C]} \mathcal{Y}^{-\varepsilon_4,(-\frac{3}{2})}_{[C]} \rangle = \frac{n_5}{|z_{12}|^4} \langle V_1 V_2 \rangle \langle V_1' V_2' \rangle \left(\frac{(2j-1)(j'-m'+\frac{1}{2})}{(m-j+\frac{1}{2})} \times \text{c.c.} \right), \tag{2.171c}$$

$$\langle \mathcal{Y}^{\varepsilon_4,(-\frac{1}{2})}_{[D]} \mathcal{Y}^{-\varepsilon_4,(-\frac{3}{2})}_{[D]} \rangle = \frac{n_5}{|z_{12}|^4} \langle V_1 V_2 \rangle \langle V_1' V_2' \rangle \left((2j-1)(m+j-\frac{1}{2})(j'-m'+\frac{1}{2}) \times \text{c.c.} \right), \tag{2.171d}$$

where here $\langle V_1 V_2 \rangle = \langle V_{m_1-1/2} V_{-m_1+1/2} \rangle$ and $\langle V_1' V_2' \rangle = \langle V'_{m_1'-1/2} V'_{-m_1'+1/2} \rangle$. As expected, the coefficients resulting from the linear combinations effectively shift the spins $j \to J$ and $j' \to J'$ in the gamma functions coming from the bosonic correlators.

Among the states described above, the only chiral one corresponds to the SU(2) highest-weight operator of type A. To simplify notation and for later convenience, we suppress the SU(2) labels and use the label j rather than J (here $J = j - 1/2$). This operator has quantum numbers

$$\mathcal{Y}^{+,\varepsilon_4}_{j,m[A]} \; : \quad h = J = j - \frac{1}{2} = j' + \frac{1}{2} = J' = m' \, . \tag{2.172}$$

The explicit form of this operator is simpler than the generic Ramond sector operator, and is given by

$$\mathcal{Y}^{+,\varepsilon_4}_{j,m[A]} = e^{-\frac{\varphi}{2}} \left(S_{+++} V_{j,m-\frac{1}{2}} + S_{-+-} V_{j,m+\frac{1}{2}} \right)_{j-\frac{1}{2},m} V'_{j-1,j-1} e^{\frac{i\varepsilon_4}{2}(H_4+H_5)} \, . \tag{2.173}$$

As in the NS sector, the rest of the chiral operators belong to the spectrally flowed sectors. These are obtained by acting with the spectral flow operator (2.157). From the flowed Virasoro condition, similar to Eq. (2.158), the resulting operators must have $m = m'$ (together with the relations in (2.172)), and so only the second term in (2.173) is non-vanishing, giving rise to the flowed Ramond operators (we now suppress also the label $m = J = j - 1/2$)

$$\mathcal{Y}^{+,\varepsilon_4,\omega}_{j[A]} = e^{-\frac{\varphi}{2}} S^{\omega}_{-+-} V^{\omega}_{j,j} V'^{\omega}_{j-1,j-1} e^{\frac{i\varepsilon_4}{2}(H_4+H_5)} \, , \tag{2.174}$$

where

$$S^{\omega}_{-+-} \equiv e^{\frac{i}{2}[(1+2\omega)(-H_1+H_2)-H_3]}\,. \tag{2.175}$$

The corresponding two-point function is equivalent to the highest-weight case of (2.171a), up to the usual additional $\delta_{\omega_1,-\omega_2}$ factor.

2.6.4.3 Holographic dictionary for light chiral primaries

We have reviewed three sets of m-basis vertex operators corresponding to chiral primaries of the holographic CFT. Two sets are in the NS sector: $\mathcal{W}^{-}_{j,m}$ and $\mathcal{X}^{+}_{j,m}$, together with the corresponding spectral flowed operators $\mathcal{W}^{\omega}_{j}$ and $\mathcal{X}^{\omega}_{j}$. The third set is in the Ramond sector, $\mathcal{Y}^{+,\varepsilon_4}_{j,m[A]}$ and its spectral flow, $\mathcal{Y}^{+,\varepsilon_4,\omega}_{j[A]}$. From now on we shall omit the label A and the $\mathrm{AdS}_3 \times \mathrm{S}^3$ chirality $\epsilon = +$, denoting this operator by $\mathcal{Y}^{\omega,\varepsilon_4}_{j}$. Recall that in the spectral flowed sectors, the remaining states in the zero-mode algebra are obtained by acting with J^{+}_{0}, as discussed around Eqs. (2.120) and (2.162).

In order to reconstruct the corresponding local operators of the spacetime CFT, we need to combine such modes by going to the x-basis, as done in Eqs. (2.115) and (2.120) in the bosonic SL(2,$\mathbb{R}$) model.[32] For the operators at hand, the sum over m in the analog of Eqs. (2.115) and (2.120) factorizes between fermionic and bosonic contributions, leading to expressions of the following form:

$$\mathcal{W}^{\omega}_{j}(x) = e^{-\varphi}\psi^{\omega}(x)e^{i\omega H_2}V^{\omega}_{j}(x)V'^{\omega}_{j-1,j-1}\,, \tag{2.176a}$$

$$\mathcal{X}^{\omega}_{j}(x) = e^{-\varphi}\psi^{\omega-1}(x)e^{i(\omega+1)H_2}V^{\omega}_{j}(x)V'^{\omega}_{j-1,j-1}\,, \tag{2.176b}$$

$$\mathcal{Y}^{\omega,\varepsilon_4}_{j}(x) = e^{-\frac{\varphi}{2}}S^{\omega}(x)V^{\omega}_{j}(x)V'^{\omega}_{j-1,j-1}e^{\frac{i\varepsilon_4}{2}(H_4+H_5)}\,. \tag{2.176c}$$

Here $\psi^{\omega}(x)$ and $S^{\omega}(x)$ are defined as follows. First, note that the fermions ψ^{a} introduced in (2.131), which generate an affine $\widehat{\mathrm{sl}(2,\mathbb{R})}_{-2}$ algebra with level $k_{\psi} = -2$, constitute affine primaries with spin $J_{\psi} = -1$, on which, however, the zero-mode currents act as in (2.107) but with[33] $J \to 1 - J$, i.e. $J^{\pm}_{0}|J, m \pm 1\rangle = (m \pm J)|J, m \pm 1\rangle$. As a consequence, and in contrast with what happens with bosonic primaries, the action J^{+}_{0} on ψ^{-}, the lowest-weight state, is truncated. Identifying $\psi^{\omega=0}(0) = \psi^{-}$, the resulting x-basis operator has only three terms:

$$\psi^{\omega=0}(x) \equiv \psi^{-}(x) = e^{xJ^{+}_{0}}\psi^{-}e^{-xJ^{+}_{0}} = \psi^{-} - 2x\psi^{3} + x^{2}\psi^{+}\,. \tag{2.177}$$

Of course, we already knew the action of the currents on ψ^{a} from (2.131), but the advantage of the above discussion is that it extends to the spectrally flowed sectors. Indeed,

[32] In this thesis we are interested in operators of fixed R-charge. Hence, and in contrast to [99, 100], we do not introduce isospin variables in the SU(2) sector.

[33] This convention is perhaps more natural from the SL(2,$\mathbb{R}$) point of view, but we have decided to employ the conventions used in the most relevant literature for us, i.e. [101, 103, 99, 100].

$\psi^\omega(0) = \sqrt{n_5} e^{-i(1+\omega)H_1}$ is the lowest-weight component of a spin $J_\psi^\omega = -1-\omega$ field. The corresponding x-basis operator is of the form

$$\psi^\omega(x) \equiv \sqrt{n_5}\, e^{xJ_0^+} e^{-i(1+\omega)H_1} e^{-xJ_0^+} , \tag{2.178}$$

and contains $1-2J_\psi^\omega = 2\omega+3$ terms.

Similarly, the spin field (2.175) is the lowest-weight component of a representation with SL(2,$\mathbb{R}$) and SU(2) spins $(J_S^\omega, J_S'^\omega) = (-\frac{1}{2}-\omega, \frac{1}{2}+\omega)$, such that the x-basis operator is

$$S^\omega(x) \equiv e^{xJ_0^+} S^\omega_{-+-} e^{-xJ_0^+} , \tag{2.179}$$

and contains $2(1+\omega)$ terms.

We thus have three types of vertices, $\mathcal{W}_j^\omega(x,z)$, $\mathcal{X}_j^\omega(x,z)$ and $\mathcal{Y}_j^{\omega,\pm}(x,z)$, which correspond to local chiral primary operators of the boundary theory. As mentioned before, these should be completed with analogous antiholomorphic excitations, which have been omitted in the presentation. As discussed around Eqs. (2.162) and (2.172), their boundary weights are given by

$$h\left[\mathcal{W}_j^\omega\right] = j_\omega - 1 , \qquad h\left[\mathcal{Y}_j^{\omega,\pm}\right] = j_\omega - \frac{1}{2} , \qquad h\left[\mathcal{X}_j^\omega\right] = j_\omega , \tag{2.180}$$

where $j = j'+1$ and so

$$j_\omega = j + \frac{n_5}{2}\omega, \qquad j = 1, \frac{3}{2}, \ldots, \frac{n_5}{2}, \qquad \omega = 0, 1, \ldots . \tag{2.181}$$

Up to normalization, which will be fixed shortly, these operators are identified with the chiral primaries of the holographic CFT listed in Eq. (2.100). In the $\mathcal{Y}_j^{\omega,\varepsilon_4}$ tower, $\varepsilon_4 = \pm$ is identified with the boundary quantum number $\dot{A}$ in (2.99). Note also that the $\mathcal{W}_j^\omega$ tower starts with the identity operator of the boundary theory.[34] The dictionary is summarized in Table 2.3.

Worldsheet	Weight h	Twist n	Dual Operator
$\mathcal{W}_j^\omega$	$j_\omega - 1$	$2j_\omega - 1$	O_n^-
$\mathcal{Y}_j^{\omega,\varepsilon_4}$	$j_\omega - \frac{1}{2}$	$2j_\omega - 1$	$O_n^{\dot{A}}$
$\mathcal{X}_j^\omega$	j_ω	$2j_\omega - 1$	O_n^+

TABLE 2.3: Dictionary between worldsheet vertex operators and chiral primaries of the holographically dual CFT. Here $j_\omega = j + \frac{n_5}{2}\omega$, $j = 1, \frac{3}{2}, \ldots, \frac{n_5}{2}$, and $\omega = 0, 1, \ldots$

[34] As discussed in [110], this is subtle, since the operator is actually a spectral-flow-sector dependent constant. This subtlety is related to the fact that spectral flow charge is not conserved in n-point functions with $n \geq 3$, and was resolved in [114] by performing a Legendre transform to the microcanonical ensemble, in which the total number of fundamental strings is fixed.

Although most of the chiral primaries of the holographic CFT are accounted for by considering the ranges given in (2.181), it is known that those belonging to the the n-twisted sectors with $n = pn_5$ with $p \in \mathbb{N}$ are still missing [99, 100]. These would correspond to operators sitting at the boundary of the allowed range of j in Eq. (2.106), at which the spectrum becomes degenerate and the continuous representations appear [104, 115]. The absence of these states in the worldsheet spectrum has been related to the fact that the NS5-F1 model sits at a singular point in the moduli space where all RR modes are turned off [50].

The twist n of the holographic CFT operators is identified as [99, 100]

$$n = 2j - 1 + n_5\omega\,. \tag{2.182}$$

Let us make a side comment regarding the limit in which there is only a single NS5 brane sourcing the background, $n_5 = 1$. This model is special in that it corresponds to the tensionless limit of the theory. It has to be treated with care since the usual RNS formalism outlined above breaks down due to the fact that the bosonic SU(2) level would become negative. It was shown in [87, 116] that for $n_5 = 1$ the worldsheet theory is exactly dual to the supersymmetric symmetric orbifold $(\mathbb{T}^4)^{n_1}/S_{n_1}$. In this model, the discrete series is absent, the spectrum truncates to $j = 1/2$, physical states have $\omega > 0$, and the spectral flow charge is identified with n, i.e. $n = \omega$. Eq. (2.182) is the known generalization of this relation for $n_5 > 1$.

In order to fix the normalization of the operators, we compute their two-point functions. Making use of

$$\langle \psi^{\omega_1}(x_1)\psi^{\omega_2}(x_2)\rangle \times \text{c.c.} = x_{12}^{2(\omega_1+1)}\langle e^{-i(1+\omega_1)H_1}e^{i(1-\omega_2)H_1}\rangle \times \text{c.c.} = \delta_{\omega_1,-\omega_2}\frac{|x_{12}|^{4(\omega_1+1)}}{|z_{12}|^{2(\omega_1+1)^2}}\,,$$
$$\langle S^{\omega_1,+}(x_1)S^{\omega_2,-}(x_2)\rangle \times \text{c.c.} = \frac{x_{12}^{2\omega_1+2}}{z_{12}}\langle S^{\omega_1}_{-+-}S^{\omega_2}_{+-+}\rangle \times \text{c.c.} = \delta_{\omega_1,-\omega_2}\frac{|x_{12}|^{4\omega_1+2}}{|z_{12}|^{4\omega_1(\omega_1+1)+\frac{5}{2}}}\,,$$

we obtain

$$\langle \mathcal{W}^{\omega_1}_{j_1}(x_1,z_1)\mathcal{W}^{\omega_2}_{j_2}(x_2,z_2)\rangle = \frac{n_5^2 B(j_1)}{16}\frac{\delta(j_1-j_2)\delta_{\omega_1,-\omega_2}}{|x_{12}|^{4(j_1-1)+2n_5\omega_1}|z_{12}|^4}\,, \tag{2.183}$$
$$\langle \mathcal{X}^{\omega_1}_{j_1}(x_1,z_1)\mathcal{X}^{\omega_2}_{j_2}(x_2,z_2)\rangle = \frac{n_5^2 B(j_1)}{16}\frac{\delta(j_1-j_2)\delta_{\omega_1,-\omega_2}}{|x_{12}|^{4j_1+2n_5\omega_1}|z_{12}|^4}\,, \tag{2.184}$$
$$\langle \mathcal{Y}^{\omega_1,\pm}_{j_1}(x_1,z_1)\mathcal{Y}^{\omega_2,\mp}_{j_2}(x_2,z_2)\rangle = \frac{n_5 B(j_1)}{(2j_1-1+n_5\omega_1)^2}\frac{\delta(j_1-j_2)\delta_{\omega_1,-\omega_2}}{|x_{12}|^{4j_1+2(n_5\omega_1-1)}|z_{12}|^4} \tag{2.185}$$

Here we have used that in the spectrally flowed R sectors the denominator in the extra factor of the corresponding vertex operator in the "$-\frac{3}{2}$" picture is shifted as $2j-1 \to 2j-1+n_5\omega$ as compared to (2.170). Moreover, the V_{conf} factor in Eq. (2.121) is cancelled by the pole appearing in (2.114) upon setting $m_1 = j_1$.

The string two-point function is then obtained by including an extra factor $g_s^{-2} \sim n_1/n_5$ as usual in string perturbation theory, fixing $z_1 = 0$ and $z_2 = 1$, and dividing by a volume of the conformal group that leaves such worldsheet insertions fixed. As discussed in [110, 103], this cancels the divergence coming from $\delta(j_1 - j_2)$, leaving a constant j-dependent factor of the form $(2j - 1 + n_5\omega)$. As a consequence, the holographic dictionary reads

$$O_n^{--}(x,\bar{x}) \leftrightarrow A_{\rm NS}(j,\omega)\mathcal{W}_j^{\omega}(x,\bar{x}), \qquad O_n^{++}(x) \leftrightarrow A_{\rm NS}(j,\omega)\mathcal{X}_j^{\omega}(x,\bar{x}), \tag{2.186}$$

$$O_n^{\dot{A}\dot{B}} \leftrightarrow A_{\rm R}(j,\omega)\mathcal{Y}_j^{\omega,\dot{A}\dot{B}}(x,\bar{x}), \tag{2.187}$$

with n related to the worldsheet quantum numbers as in (2.182) and where [113, 99]

$$A_{\rm NS}(j,\omega) = \frac{4g_s}{\sqrt{n_5^2\, B(j)(2j-1+n_5\omega)}}\,, \qquad A_{\rm R}(j,\omega) = g_s\sqrt{\frac{(2j-1+n_5\omega)}{n_5\, B(j)}}\,. \tag{2.188}$$

Of course, this identification is only expected to hold at small string coupling, i.e. for $n_1 \gg n_5$. Analysis and comparison of boundary and worldsheet three-point functions were carried out in [117, 99, 113].

2.7 Null-gauged WZW models

In this section we review some relevant aspects of the models that we shall study in this thesis. We will aim to be brief where possible; the interested reader can find the details in the works [51, 52, 53, 54].

We consider the (10+2)-dimensional upstairs target $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{R}_t \times \mathbb{S}^1_y \times \mathbb{T}^4$, where we have introduced coordinates t and y for the timelike $\mathbb{R}$ and spacelike $\mathbb{S}^1$ factors respectively. Since the Cartan direction in $SL(2,\mathbb{R})$ is timelike, and that of $SU(2)$ is spacelike, and the levels are the same, one can form null linear combinations $J^3_{\rm sl} \pm J^3_{\rm su}$ in both holomorphic and antiholomorphic sectors of the worldsheet theory. Gauging such null currents leads [118] to the background sourced by a circular array of NS5 branes on their Coulomb branch [119, 120, 121].

The (10+2)-dimensional models have other null currents that are linear combinations of $J^3_{\rm sl}$, $J^3_{\rm su}$, ∂_t and ∂_y. It was recently found that particular linear combinations of these currents give rise to a family of backgrounds that include NS5-P and NS5-F1 BPS circular supertubes [122, 123], as well as NS5-F1-P BPS and non-BPS spectral flowed supertubes [124, 125, 37, 96, 94]. In the IR, the backgrounds become asymptotically $\mathrm{AdS}_3 \times \mathbb{S}^3$, and correspond to heavy states of the holographically dual CFT at the symmetric orbifold point [94].

2.7.1 Null-gauged sigma models

We now briefly review the null gauging formalism for general sigma models, before specialising to WZW models (see e.g. [126, 127, 128, 129]). In this passage we follow the presentation of [54]. We use units in which $\alpha' = 1$, and work at tree level in the string coupling g_s.

Consider the string worldsheet $\mathcal{M}_2$ and an embedding map φ into a pseudo-Riemannian manifold $\mathcal{N}$, namely $\varphi : \mathcal{M}_2 \to \varphi(\mathcal{M}_2) \subset \mathcal{N}$. The target manifold $\mathcal{N}$ is endowed with a metric with components G_{ij}. We wish to gauge a set of Killing vectors ξ_a generating isometries of $\mathcal{N}$, where a labels the different Killing vectors (in this thesis we will always have $a = 1, 2$). We introduce a set of independent worldsheet gauge fields $\mathcal{A}^a$, one corresponding to each Killing vector. Then the kinetic term in the string sigma model action is written in terms of the covariant derivative

$$\mathcal{D}\varphi^i = \partial\varphi^i - \mathcal{A}^a \xi^i_a \,, \tag{2.189}$$

and takes the form

$$\mathcal{L}_{\mathrm{K}} = \mathcal{D}\varphi^i \, G_{ij} \, \overline{\mathcal{D}}\varphi^j \;=\; (\partial\varphi^i - \mathcal{A}^a \xi^i_a) \, G_{ij} \, (\bar{\partial}\varphi^j - \bar{\mathcal{A}}^a \xi^j_a) \,. \tag{2.190}$$

To write the gauged Wess-Zumino (WZ) term we introduce target-space one-forms θ_a (we follow the notation of [129]), pulled back to the worldsheet. The WZ term can then be written as

$$\mathcal{L}_{\mathrm{WZ}} = B_{ij}\partial\varphi^i\bar{\partial}\varphi^j + \mathcal{A}^a\theta_{a,i}\bar{\partial}\varphi^i - \bar{\mathcal{A}}^a\theta_{a,i}\partial\varphi^i + \xi^i_{[a}\theta_{b],i}\mathcal{A}^a\bar{\mathcal{A}}^b \,, \tag{2.191}$$

where $\theta_{a,i}$ denotes the i^{th} component of the one-form θ_a. For our null-gauged models, the target-space one-forms θ_a are given by

$$\theta_a \;=\; (-1)^{a+1}\xi_a \cdot d\varphi \;\equiv\; (-1)^{a+1}\xi^i_a G_{ij} d\varphi^j \,, \qquad (a = 1, 2) \,. \tag{2.192}$$

For a consistent gauging, the following conditions must hold:

$$\imath_a H \;=\; d\theta_a \,, \qquad \imath_a \theta_b \;=\; -\imath_b \theta_a \,, \tag{2.193}$$

where $H = dB$. The expression (2.192) implies that half of the gauge field components decouple, so that they are naturally chiral. As a result, in our $U(1) \times U(1)$ gauged models, the coefficient of the term quadratic in gauge fields is proportional to the quantity

$$\Sigma \;\equiv\; -\frac{1}{2}\xi^i_1 G_{ij} \xi^j_2 \,. \tag{2.194}$$

All together, the terms in the action involving the gauge fields then reduce to

$$\mathcal{L}_{\mathcal{A}} \;=\; -2\mathcal{A}^2\xi_2^i G_{ij}\bar{\partial}\varphi^j - 2\bar{\mathcal{A}}^1\xi_1^i G_{ij}\partial\varphi^j - 4\mathcal{A}^2\bar{\mathcal{A}}^1\Sigma\,, \tag{2.195}$$

and in the following, we shall denote $\mathcal{A} \equiv \mathcal{A}^2$, $\bar{\mathcal{A}} \equiv \bar{\mathcal{A}}^1$.

We define the worldsheet currents $\mathcal{J}$, $\bar{\mathcal{J}}$ to be pull-backs of the target-space one-forms as follows[35]

$$\mathcal{J} \;\equiv\; -\theta_1 \cdot \partial\varphi \;\equiv\; -\theta_{1,i}\partial\varphi^i\,, \qquad \bar{\mathcal{J}} \;\equiv\; \theta_2 \cdot \bar{\partial}\varphi \;\equiv\; \theta_{2,i}\bar{\partial}\varphi^i\,. \tag{2.196}$$

By using (2.192), one can then write the gauge terms (2.195) as

$$\mathcal{L}_{\mathcal{A}} \;=\; 2\mathcal{A}\theta_{2,i}\bar{\partial}\varphi^i - 2\bar{\mathcal{A}}\theta_{1,i}\partial\varphi^i - 4\mathcal{A}\bar{\mathcal{A}}\Sigma \;\equiv\; 2\mathcal{A}\bar{\mathcal{J}} + 2\bar{\mathcal{A}}\mathcal{J} - 4\mathcal{A}\bar{\mathcal{A}}\Sigma\,. \tag{2.197}$$

Upon integrating out the gauge fields, the gauge terms in the action then become

$$\frac{\mathcal{J}\bar{\mathcal{J}}}{\Sigma} \;=\; -\frac{1}{\Sigma}\big(\theta_1\cdot\partial\varphi\big)\big(\theta_2\cdot\bar{\partial}\varphi\big) \;=\; \frac{1}{\Sigma}\big(\xi_1\cdot\partial\varphi\big)\big(\xi_2\cdot\bar{\partial}\varphi\big)\,, \tag{2.198}$$

where $\xi_1 \cdot \partial\varphi \equiv \xi_1^i G_{ij}\partial\varphi^j$. Thus the overall effect of the null gauging procedure is to add the term (2.198) to the ungauged sigma model lagrangian.

2.7.2 Null-gauged WZW models

We now specialise the discussion to the case where the upstairs theory is a WZW model whose target space is a Lie group $\mathcal{G}$, and thus we replace φ with a $\mathcal{G}$-valued function $g : \mathcal{M}_2 \to \mathcal{G}$. We will shortly consider $\mathcal{G}$ to be a direct product of simple and abelian factors, but for the moment we focus on one of the simple factors. We follow in places the presentation in [53].

We wish to gauge the action of a subgroup $\mathcal{H} \subset \mathcal{G}$. Its action on $\mathcal{G}$ is defined by the group homomorphism embeddings

$$\ell : \mathcal{H} \to \mathcal{G}_L\,, \qquad\qquad r : \mathcal{H} \to \mathcal{G}_R\,, \tag{2.199}$$

where $\mathcal{G}_L \times \mathcal{G}_R$ is the standard left-right isometry group, and such that we will gauge the transformations

$$g \;\mapsto\; \ell(h)\, g\, r(h)^{-1}\,, \qquad h \in \mathcal{H}\,. \tag{2.200}$$

The group embeddings ℓ and r induce corresponding Lie algebra homomorphisms. Since the meaning will be clear from the context, it is convenient to abuse notation and

[35]Note that $\mathcal{J}, \bar{\mathcal{J}}$ are related to the CFT current operators $J, \bar{J}$ by factors of i, e.g. $J = i\mathcal{J}$.

re-use the same symbols ℓ and r for the induced Lie algebra homomorphisms,

$$\ell:\ \mathfrak{h}\to\mathfrak{g}\,,\qquad\qquad r:\ \mathfrak{h}\to\mathfrak{g}\,. \tag{2.201}$$

To write the corresponding Killing vector field, let us denote the left(right)-invariant vector field corresponding to a generic $X\in\mathfrak{g}$ by X^L (X^R). Given a basis of $\mathfrak{h}$, for each element X_a there is a corresponding Killing vector field given by (see e.g. [129])

$$\xi_a\ \equiv\ -\,\ell(X_a)^R-r(X_a)^L\,. \tag{2.202}$$

Let us write the left and right Maurer-Cartan one-forms as

$$\theta_L=g^{-1}dg\,,\qquad\qquad \theta_R=-\,dg\,g^{-1}\,. \tag{2.203}$$

We denote by $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathfrak{g}$ given by the Killing form. More explicitly, for matrix groups we use the normalisation $\langle A,B\rangle=\mathrm{Tr}(AB)$. In terms of these, the one-forms θ_a introduced in Eqs. (2.191)–(2.192) take the form

$$\theta_a=\langle\ell(X_a),\theta_R\rangle-\langle r(X_a),\theta_L\rangle\ . \tag{2.204}$$

2.7.3 The models we study

We work in Type II superstring theory, however we suppress worldsheet fermions in this section for ease of presentation. Worldsheet fermions will be discussed in detail in Section 3.2 below. We consider the cosets[36]

$$\mathcal{G}/\mathcal{H}\ \times\mathbb{T}^4=\frac{SL(2,\mathbb{R})\times SU(2)\times\mathbb{R}_t\times U(1)_y}{U(1)_L\times U(1)_R}\ \times\mathbb{T}^4\,. \tag{2.205}$$

To define the action of $\mathcal{H}=U(1)_L\times U(1)_R$ we must specify the embedding into each of the four subgroups of the upstairs group $\mathcal{G}$ that participate in the gauging. Parametrising SL$(2,\mathbb{R})$ as $SU(1,1)$, we introduce coordinates for the upstairs subgroup elements as[37]

$$\begin{aligned} g\ &=\ \left(g_{\rm sl},\ g_{\rm su},\ g_t,\ g_y\right)\\ &=\ \left(e^{\frac{i}{2}(\tau-\sigma)\sigma_3}e^{\rho\sigma_1}e^{\frac{i}{2}(\tau+\sigma)\sigma_3},\ e^{\frac{i}{2}(\psi-\phi)\sigma_3}e^{i(\frac{\pi}{2}-\theta)\sigma_1}e^{-\frac{i}{2}(\psi+\phi)\sigma_3},\ e^t,\ e^{\frac{iy}{R_y}}\right)\,, \end{aligned} \tag{2.206}$$

where σ_i denotes the i^{th} Pauli matrix and $y\in[0,2\pi R_y)$. At the level of the algebra, the chiral embeddings we consider are specified by eight arbitrary real parameters l_i, r_i,

[36]More precisely, the upstairs model involves the universal cover of SL$(2,R)$, and globally we gauge $\mathbb{R}\times U(1)$, as we discuss in more detail in Section 3.2.2; see also [52].
[37]We adopt the conventions in [2, 3].

$i = 1, 2, 3, 4$, as follows (the ordering of subgroups is as in Eq. (2.206)),

$$\begin{aligned} \ell(\alpha) &= \left(i\,\mathrm{l}_1 \alpha\sigma_3,\ -i\,\mathrm{l}_2\alpha\sigma_3,\ \mathrm{l}_3\alpha,\ -i\frac{\mathrm{l}_4}{R_y}\,\alpha \right), \qquad & r(\alpha) &= 0\,, \\ r(\beta) &= -\left(i\,\mathrm{r}_1 \beta\sigma_3,\ -i\,\mathrm{r}_2\beta\sigma_3,\ \mathrm{r}_3\beta,\ -i\frac{\mathrm{r}_4}{R_y}\,\beta \right), \qquad & \ell(\beta) &= 0\,, \end{aligned} \tag{2.207}$$

where $\alpha, \beta \in \mathbb{R}$ and the signs have been chosen for later convenience, in particular for Eq. (2.215) below. The group action (2.200) being gauged is then

$$g \;\mapsto\; \left(e^{i\mathrm{l}_1\alpha\sigma_3} g_{\mathrm{sl}}\, e^{i\mathrm{r}_1\beta\sigma_3}\,,\ e^{-i\mathrm{l}_2\alpha\sigma_3} g_{\mathrm{su}}\, e^{-i\mathrm{r}_2\beta\sigma_3}\,,\ e^{\mathrm{l}_3\alpha} g_t\, e^{\mathrm{r}_3\beta}\,,\ e^{-i\frac{\mathrm{l}_4}{R_y}\alpha} g_y\, e^{-i\frac{\mathrm{r}_4}{R_y}\beta} \right) . \tag{2.208}$$

The general gauge-invariant action for such asymmetric cosets can be found in [130]. We introduce two (independent) $\mathfrak{h}$-valued worldsheet gauge fields $(\mathcal{A}_1, \bar{\mathcal{A}}_1)$ and $(\mathcal{A}_2, \bar{\mathcal{A}}_2)$. The gauged WZW action takes the form

$$\begin{aligned} S \;=\; & \sum_j \mathrm{sgn}(\kappa_j)\frac{k_j}{\pi}\Bigg(\int_{\mathcal{M}_2} \frac{1}{2}\,\mathrm{Tr}\left[g^{-1}\partial g g^{-1}\bar{\partial} g\right]_j d^2z + i \int_{\mathcal{M}_3} \frac{1}{3!}\,\mathrm{Tr}\left[g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right]_j \\ & + \int_{\mathcal{M}_2} \mathrm{Tr}\left[-\sum_{a=1}^{2}\left[\ell(\bar{\mathcal{A}}_a)\partial g g^{-1}\right]_j + \sum_{a=1}^{2}\left[r(\mathcal{A}_a)g^{-1}\bar{\partial} g\right]_j - \sum_{a,b=1}^{2}\left[g^{-1}\ell(\bar{\mathcal{A}}_a) g\, r(\mathcal{A}_b)\right]_j \right] d^2z \Bigg), \end{aligned} \tag{2.209}$$

where j runs over the Lie algebras, $\mathcal{M}_3$ is a three-dimensional auxiliary space such that $\mathcal{M}_2 = \partial\mathcal{M}_3$, k_j are the levels of the Kac-Moody algebras, and $\mathrm{sgn}(\kappa_j)$ are the signatures of the respective Killing forms, which in our conventions is positive for $\mathrm{SL}(2,\mathbb{R})$ and negative for ($SU(2)$, $\mathbb{R}_t$, $\mathbb{S}^1_y$). Here the embeddings ℓ, r should be understood as corresponding to each respective Lie subalgebra, i.e. the components of the right-hand sides of Eq. (2.207).

We now recall from the discussion of general gaugings of sigma models in Eqs. (2.189)–(2.198) that, since the gauge fields are null and chiral, one of their components simply drops out, such that we can set

$$\mathcal{A}_1 = 0\,, \qquad \bar{\mathcal{A}}_2 = 0\,. \tag{2.210}$$

The gauge field embeddings are then

$$\begin{aligned} \ell(\bar{\mathcal{A}}_1) &= \left(i\,\mathrm{l}_1 \bar{\mathcal{A}}_1\sigma_3,\ -i\,\mathrm{l}_2\bar{\mathcal{A}}_1\sigma_3,\ \mathrm{l}_3\,\bar{\mathcal{A}}_1,\ -i\frac{\mathrm{l}_4}{R_y}\,\bar{\mathcal{A}}_1 \right), \qquad & \ell(\mathcal{A}_2) &= 0\,, \\ r(\mathcal{A}_2) &= -\left(i\,\mathrm{r}_1 \mathcal{A}_2\sigma_3,\ -i\,\mathrm{r}_2\mathcal{A}_2\sigma_3,\ \mathrm{r}_3\mathcal{A}_2,\ -i\frac{\mathrm{r}_4}{R_y}\,\mathcal{A}_2 \right), \qquad & r(\bar{\mathcal{A}}_1) &= 0\,, \end{aligned} \tag{2.211}$$

consistently with (2.207). As before, in order to lighten the notation we set $\mathcal{A} = \mathcal{A}_2$, $\bar{\mathcal{A}} = \bar{\mathcal{A}}_1$ from now on.

We introduce the currents (our conventions follow [54, App. A])

$$\mathrm{j}^3_{\rm sl} = k_{\rm sl}\,\mathrm{Tr}\left(-i\frac{\sigma_3}{2}\,\partial g_{\rm sl}\,g_{\rm sl}^{-1}\right)\,, \qquad \bar{\mathrm{j}}^3_{\rm sl} = k_{\rm sl}\,\mathrm{Tr}\left(-i\frac{\sigma_3}{2}\,g_{\rm sl}^{-1}\,\bar\partial g_{\rm sl}\right)\,, \tag{2.212}$$

and similarly for $SU(2)$. Their explicit form in our coordinates is

$$\begin{aligned}
\mathrm{j}^3_{\rm sl} &= n_5\big(\cosh^2\rho\,\partial\tau + \sinh^2\rho\,\partial\sigma\big)\,, & \bar{\mathrm{j}}^3_{\rm sl} &= n_5\big(\cosh^2\rho\,\bar\partial\tau - \sinh^2\rho\,\bar\partial\sigma\big)\,,\\
\mathrm{j}^3_{\rm su} &= n_5\big(\cos^2\theta\,\partial\psi - \sin^2\theta\,\partial\phi\big)\,, & \bar{\mathrm{j}}^3_{\rm su} &= -n_5\big(\cos^2\theta\,\bar\partial\psi + \sin^2\theta\,\bar\partial\phi\big)\,.
\end{aligned} \tag{2.213}$$

We also define

$$\mathsf{P}^t_L = \partial t\,, \qquad \mathsf{P}^t_R = \bar\partial t\,, \qquad \mathsf{P}^y_L = \partial y\,, \qquad \mathsf{P}^y_R = \bar\partial y\,. \tag{2.214}$$

Note that, as usual, the bosonic subsector of the supersymmetric WZW model has $k_{\rm sl} = n_5 + 2$ and $k_{\rm su} = n_5 - 2$ while the full supersymmetric model has $k_{\rm sl} = k_{\rm su} = n_5$. As noted above, we are suppressing worldsheet fermions in the present section. The shift in the levels is important (see e.g. the discussion in [54]), and we will take care of this in detail when discussing results in the worldsheet CFT in later chapters. When discussing supergravity solutions we will work in the usual supergravity regime $n_5 \gg 1$ (and in the fivebrane decoupling limit $g_s \to 0$) and thus for our purposes in this section we can simply work with $k_{\rm sl} = k_{\rm su} = n_5$. To have canonical kinetic terms we set the (otherwise irrelevant) $\widehat{\mathfrak{u}(1)}$ levels to be $k_t = 2$, $k_y = 2R_y^2$.

The group action that we gauge, defined in Eq. (2.207), corresponds to gauging the currents

$$\begin{aligned}
\mathcal{J} &= \mathrm{l}_1\,\mathrm{j}^3_{\rm sl} + \mathrm{l}_2\,\mathrm{j}^3_{\rm su} + \mathrm{l}_3\mathsf{P}^t_{\rm L} + \mathrm{l}_4\mathsf{P}^y_{\rm L}\,,\\
\bar{\mathcal{J}} &= \mathrm{r}_1\,\bar{\mathrm{j}}^3_{\rm sl} + \mathrm{r}_2\,\bar{\mathrm{j}}^3_{\rm su} + \mathrm{r}_3\mathsf{P}^t_{\rm R} + \mathrm{r}_4\mathsf{P}^y_{\rm R}\,,
\end{aligned} \tag{2.215}$$

which we require to be null by imposing

$$n_5(\mathrm{l}_1^2 - \mathrm{l}_2^2) + \mathrm{l}_3^2 - \mathrm{l}_4^2 = 0\,, \qquad n_5(\mathrm{r}_1^2 - \mathrm{r}_2^2) + \mathrm{r}_3^2 - \mathrm{r}_4^2 = 0\,. \tag{2.216}$$

One can use these constraints to fix the overall normalization of the gauging parameters. We assume that $\mathrm{l}_1 = \mathrm{r}_1 \neq 0$ and divide through by l_1^2 and r_1^2, to work with the ratios

$$l_i \;=\; \frac{\mathrm{l}_i}{\mathrm{l}_1}\,, \qquad r_i \;=\; \frac{\mathrm{r}_i}{\mathrm{r}_1}\,, \qquad i = 2,3,4\,. \tag{2.217}$$

In practice this has the same effect as setting $\mathrm{l}_1 = \mathrm{r}_1 = 1$, however we have introduced a separate notation for later convenience. Of course, one can modify this step accordingly to deal with models in which $\mathrm{l}_1 = \mathrm{r}_1 = 0$. For later use we record that the ratio parameters l_i, r_i, $i = 2,3,4$ are subject to the constraints

$$n_5(1 - l_2^2) + l_3^2 - l_4^2 = 0\,, \qquad n_5(1 - r_2^2) + r_3^2 - r_4^2 = 0\,. \tag{2.218}$$

In the upstairs model, the line element and NSNS three-form flux are given by

$$\begin{aligned} ds^2 &= n_5\big(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\sigma^2 + d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\big) - dt^2 + dy^2, \\ H &= n_5\big(\sinh 2\rho \; d\rho\wedge d\tau\wedge d\sigma + \sin 2\theta \; d\theta\wedge d\psi\wedge d\phi\big). \end{aligned} \tag{2.219}$$

The Killing vectors associated to the group action (2.200) being gauged are

$$\begin{aligned} \xi_{\rm L} &= (\partial_\tau - \partial_\sigma) - l_2(\partial_\psi - \partial_\phi) + l_3\partial_t - l_4\partial_y\,, \\ \xi_{\rm R} &= (\partial_\tau + \partial_\sigma) + r_2(\partial_\psi + \partial_\phi) + r_3\partial_t - r_4\partial_y\,, \end{aligned} \tag{2.220}$$

and so we obtain the one-forms θ_a,

$$\begin{aligned} \theta_{\rm L} &= -n_5\left[\Big(\cosh^2\rho\, d\tau + \sinh^2\rho\, d\sigma\Big) + l_2\left(\cos^2\theta\, d\psi - \sin^2\theta\, d\phi\right)\right] - \left(l_3\, dt + l_4\, dy\right), \\ \theta_{\rm R} &= \;\; n_5\left[\Big(\cosh^2\rho\, d\tau - \sinh^2\rho\, d\sigma\Big) - r_2\left(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\right)\right] + r_3\, dt + r_4\, dy\,. \end{aligned} \tag{2.221}$$

The full null-gauged Wess-Zumino-Witten action is then

$$S = S_0^{\rm sl} + S_{\mathcal{A}}^{\rm sl} + S_0^{\rm su} + S_{\mathcal{A}}^{\rm su} + S_0^{t,y} + S_{\mathcal{A}}^{t,y}\,, \tag{2.222}$$

with

$$\begin{aligned} S_0^{\rm sl} &= \frac{n_5}{\pi}\int\Big[\partial\rho\bar\partial\rho + {\rm sh}^2\rho\,\partial\sigma\bar\partial\sigma - {\rm ch}^2\rho\,\partial\tau\bar\partial\tau - {\rm sh}^2\rho\,\big(\partial\sigma\bar\partial\tau - \partial\tau\bar\partial\sigma\big)\Big]d^2z\,, \\ S_{\mathcal{A}}^{\rm sl} &= \frac{2n_5}{\pi}\int\Big[\bar{\mathcal{A}}\Big({\rm sh}^2\rho\,\partial\sigma + {\rm ch}^2\rho\,\partial\tau\Big) + \mathcal{A}\Big({\rm ch}^2\rho\,\bar\partial\tau - {\rm sh}^2\rho\,\bar\partial\sigma\Big) - \mathcal{A}\bar{\mathcal{A}}\;{\rm ch}(2\rho)\Big]\,d^2z\,, \\ S_0^{\rm su} &= \frac{n_5}{\pi}\int\Big[\partial\theta\bar\partial\theta + c_\theta^2\partial\psi\bar\partial\psi + s_\theta^2\partial\phi\bar\partial\phi + c_\theta^2\,\big(\partial\phi\bar\partial\psi - \bar\partial\phi\partial\psi\big)\Big]\,d^2z\,, \\ S_{\mathcal{A}}^{\rm su} &= \frac{2n_5}{\pi}\int\Big[l_2\bar{\mathcal{A}}\,\big(c_\theta^2\,\partial\psi - s_\theta^2\,\partial\phi\big) - r_2\mathcal{A}\,\big(c_\theta^2\,\bar\partial\psi + s_\theta^2\,\bar\partial\phi\big) - l_2r_2\mathcal{A}\bar{\mathcal{A}}\,\cos(2\theta)\Big]\,d^2z\,, \\ S_0^{t,y} &= \frac{1}{\pi}\int\Big[-\partial t\bar\partial t + \partial y\bar\partial y\Big]d^2z\,, \\ S_{\mathcal{A}}^{t,y} &= \frac{2}{\pi}\int\Big[l_3\bar{\mathcal{A}}\partial t + r_3\mathcal{A}\bar\partial t + l_4\bar{\mathcal{A}}\partial y + r_4\mathcal{A}\bar\partial y - (l_3r_3 - l_4r_4)\mathcal{A}\bar{\mathcal{A}}\Big]d^2z\,, \end{aligned} \tag{2.223}$$

where we have used the shorthands $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$.

We note that with the Killing vectors (2.220), the quantity Σ defined in Eq. (2.194) becomes

$$\Sigma \;=\; \frac{1}{2}\Big(n_5\big[\cosh(2\rho) + l_2r_2\cos(2\theta)\big] + l_3r_3 - l_4r_4\Big)\,. \tag{2.224}$$

For convenience let us define the rescaled quantity

$$\Sigma_0 \;=\; \frac{1}{n_5}\Sigma \;=\; \sinh^2\!\rho + l_2r_2\cos^2\theta + \frac{1-l_2r_2}{2} + \frac{l_3r_3 - l_4r_4}{2n_5}\,. \tag{2.225}$$

Upon integrating out the gauge fields, the gauging procedure effectively adds a term quadratic in the currents, resulting in an action of the schematic form

$$S_{\mathrm{WZW}} + \frac{2}{\pi}\int \frac{\mathcal{J}\bar{\mathcal{J}}}{\Sigma}\, d^2 z\,. \tag{2.226}$$

As we will discuss next, one can then read off the resulting line element and B-field of the gauged model. The change in the measure also generates a non-trivial dilaton, which can be obtained by solving for the vanishing of the appropriate worldsheet one-loop beta function. It is important to note that these solutions are exact in α', and thus may receive only non-perturbative corrections: this is a direct consequence of the exact nature of the gWZW models we consider.

2.7.4 Supergravity fields

By integrating out the gauge fields and choosing the gauge $\sigma = \tau = 0$, we obtain the following line element and B-field:

$$\begin{aligned} ds^2 \;=\; & -\frac{h_t}{\Sigma_0}dt^2 + \frac{h_y}{\Sigma_0}dy^2 + \frac{(l_3 r_4 + l_4 r_3)}{n_5 \Sigma_0}dtdy \\ & + n_5(d\theta^2 + d\rho^2) + n_5\,\frac{h_\phi}{\Sigma_0}\,\sin^2\theta d\phi^2 + n_5\,\frac{h_\psi}{\Sigma_0}\,\cos^2\theta d\psi^2 \\ & - \frac{1}{\Sigma_0}\left[(l_2 r_3 + l_3 r_2)dt + (l_2 r_4 + l_4 r_2)dy\right]\sin^2\theta d\phi \\ & + \frac{1}{\Sigma_0}\left[(l_2 r_3 - l_3 r_2)dt + (l_2 r_4 - l_4 r_2)dy\right]\cos^2\theta d\psi\,, \end{aligned} \tag{2.227}$$

$$\begin{aligned} B \;=\; & \frac{(l_3 r_4 - l_4 r_3)}{2 n_5 \Sigma_0}dt \wedge dy + n_5 \frac{h_\phi}{\Sigma_0}\cos^2\theta\; d\phi \wedge d\psi \\ & + \frac{1}{2\Sigma_0}\left[(l_2 r_3 - l_3 r_2)dt + (l_2 r_4 - l_4 r_2)dy\right]\wedge \sin^2\theta d\phi \\ & - \frac{1}{2\Sigma_0}\left[(l_2 r_3 + l_3 r_2)dt + (l_2 r_4 + l_4 r_2)dy\right]\wedge \cos^2\theta d\psi\,, \end{aligned}$$

where

$$\begin{aligned} h_t &= \sinh^2\!\rho + l_2 r_2 \cos^2\theta + \frac{1 - l_2 r_2}{2} - \frac{l_3 r_3 + l_4 r_4}{2 n_5}\,, \\ h_y &= \sinh^2\!\rho + l_2 r_2 \cos^2\theta + \frac{1 - l_2 r_2}{2} + \frac{l_3 r_3 + l_4 r_4}{2 n_5}\,, \\ h_\phi &= \sinh^2\!\rho + \frac{1 + l_2 r_2}{2} + \frac{l_3 r_3 - l_4 r_4}{2 n_5}\,, \\ h_\psi &= \sinh^2\!\rho + \frac{1 - l_2 r_2}{2} + \frac{l_3 r_3 - l_4 r_4}{2 n_5}\,. \end{aligned} \tag{2.228}$$

A non-trivial dilaton Φ is generated as usual at one-loop level on the worldsheet. In the null-gauging formalism, this arises from a change in the measure in the path integral formulation. The most direct way to compute the dilaton is by considering the usual one-loop beta function (equivalently the supergravity equations of motion). This fixes $e^{2\Phi}$ to be proportional to

$$e^{2\Phi} \sim \frac{1}{\Sigma_0} . \tag{2.229}$$

The overall normalization of the dilaton can be fixed by matching to the NS5-brane decoupling limit of known solutions; we shall discuss this in detail in Section 3.4. Nevertheless, let us make some preliminary comments on this in order to highlight the physical meaning of this constant. The simplest scenario corresponds to the solution sourced by a stack of n_5 coincident fivebranes, which is described by using the harmonic function

$$H_5 = 1 + \frac{n_5}{r^2} , \tag{2.230}$$

where r is a radial coordinate, and where the dilaton is given by $e^{2\Phi} = g_s^2 H_5$. The fivebrane decoupling limit corresponds to $g_s \to 0$ with fixed r/g_s and fixed α' [131] (recall that we have set $\alpha' = 1$), which can be implemented via a scaling limit $g_s \to \epsilon$, $r \to \epsilon r$, with $\epsilon \to 0$. This brings the dilaton to the form

$$e^{2\Phi} = \frac{n_5}{r^2} . \tag{2.231}$$

Here we could have kept a fiducial rescaled $\tilde{g}_s$ (i.e. $g_s = \epsilon \tilde{g}_s$) as in [51, 52], but since the asymptotic value of $e^{2\Phi}$ is zero this has no precise physical meaning. Next, for an array of n_5 fivebranes in a circular, $\mathbb{Z}_{n_5}$ symmetric configuration, the supergravity solution sees a smeared source and the relevant harmonic function is based on the function $\tilde{\Sigma} = r^2 + a^2 \cos^2\theta$, where the scale a parametrises the radius of the circular array (see e.g. [51] and references within). In this case we take a double scaling limit given by $g_s \to 0$ with fixed r/g_s, fixed a/g_s and fixed α' [120, 121], which can be implemented via a scaling limit $g_s \to \epsilon, r \to \epsilon r, a \to \epsilon a$ with $\epsilon \to 0$. Changing variables to $r = a \sinh\rho$ in order to match the notation used above, we have $\tilde{\Sigma} = a^2(\sinh^2\rho + \cos^2\theta) \equiv a^2\tilde{\Sigma}_0$. The harmonic function in (2.230) is replaced by [119, 118, 51]

$$H_5 = 1 + \frac{n_5}{a^2\tilde{\Sigma}_0} , \tag{2.232}$$

so in the decoupling limit the dilaton takes the form

$$e^{2\Phi} = \frac{n_5}{a^2\tilde{\Sigma}_0} . \tag{2.233}$$

The backgrounds we consider will turn out to be generalisations of the circular array of fivebranes, such that, as a general expectation, the normalisation constant for the exponentiated dilaton in Eq. (2.229) should be proportional to the number of NS5 branes in the geometry. When F1 charge is also present this gets divided by n_1, giving a factor n_5/n_1 (or $\sim n_5/n_p$ in the NS5-P frame). Furthermore, there should also be a factor in the denominator given by the square of a length scale characterising the distribution of the sources. There will turn out be two lengthscales a_1, a_2 generalizing the scale a, and the decoupling limit involves scaling $g_s \to \epsilon$, $r \to \epsilon r$, $a_1 \to \epsilon a_1$, $a_2 \to \epsilon a_2$ with $\epsilon \to 0$ [52]. At this point however, we are working generally, so we do not yet know the details of the underlying bound state of branes. We postpone the precise computation until Section 3.4.

Together with the constraints (2.216) on the l_i, r_i parameters, the expressions for the supergravity fields (2.227), (2.228), (2.229) describe the most general backgrounds that can be obtained within the class of null-gauged models considered in this chapter, under our assumption $\mathrm{l}_1 \neq 0$, $\mathrm{r}_1 \neq 0$. Models in which $\mathrm{l}_1 = 0$ or $\mathrm{r}_1 = 0$ can easily be treated as a special case and we shall not consider them further.

As mentioned above, it is known that these models include the JMaRT solutions and their limits [51, 52]. In the next chapter we shall prove that these are in fact all consistent solutions in this class of null-gauged models. Moreover, we will show that this conclusion can be reached either from consistency of the worldsheet CFT or from asking that the supergravity fields (2.227), (2.228), (2.229) are free of CTCs, horizonless and smooth up to physical sources of string theory (in our cases, orbifold singularities or NS5-brane singularities).

Chapter 3

Black hole microstates from the worldsheet

3.1 Overview of the JMaRT cosets

Bound states of D1 and D5-branes, or of NS5 branes and fundamental strings (F1), possibly also carrying momentum P in a compact direction, have been a very fruitful arena in which to study black hole microstates in string theory. Taking the D1-D5 (or NS5-F1) decoupling limit gives rise to configurations that are asymptotically $AdS_3 \times \mathbb{S}^3 \times \mathcal{M}$, where $\mathcal{M}$ is $\mathbb{T}^4$ or K3. This is one of the original examples of holographic duality [16].

Configurations that have come to be known as circular supertubes [123, 122] were important early supergravity solutions describing specific microstates of the two-charge system, in particular in the D1-D5 or NS5-F1 duality frames. These solutions were generalized by Lunin and Mathur, and others, to the full class of two-charge microstates [65, 66, 67, 72, 70].

An equally important family of three-charge (D1-D5-P or NS5-F1-P) microstate solutions are known as spectral flowed circular supertubes, of which there are both BPS and non-BPS configurations [132, 124, 125, 37, 96]. In the AdS_3 decoupling limit, the general holographic description of these configurations is well understood [96, 94] and involves spectral flow in the $\mathcal{N} = (4, 4)$ superconformal algebra. Moreover, two-charge circular supertubes have proven to be important seed solutions in the construction of much more general families of "superstratum" solutions (see e.g. [133, 134, 135, 92, 136, 137, 138, 139]).

The non-BPS spectral flowed circular supertubes in the family mentioned above are known as the JMaRT solutions, after the authors of [37]. These microstates emit ergoregion radiation, which has been interpreted (via holography) as an enhanced, unitary version of Hawking radiation [140, 93, 141]. The JMaRT solutions also contain the BPS two-charge circular supertubes and BPS three-charge spectral flowed circular supertubes as (non-trivial) limits. All of these will be included in our analysis.

Recently a worldsheet description of the JMaRT three-charge NS5-F1-P configurations, in the NS5-brane decoupling limit, was constructed [51]. This regime corresponds to little string theory, an example of stringy holography which remains poorly understood. The models of [51] make use of a well-known supersymmetric WZW theory, combining it with the null-gauging formalism. More precisely, they involve an auxiliary (10+2)-dimensional group manifold, which is reduced to the physical (9+1)-dimensional target space by gauging a pair of null chiral currents. The corresponding spectrum of perturbative strings and D-branes were studied respectively in [52, 53]. The null-gauging construction was further extended to encompass more general Lunin-Mathur solutions [54], which correspond to a larger family of gauged sigma models that are generically not cosets. Other coset models that describe wrapped and/or intersecting fivebranes have also recently been studied [142].

The underlying microscopic configurations involve bound states of NS5 branes (possibly with F1 and/or P charge). Generically, the low-energy supergravity description is not reliable near the fivebrane sources, however the worldsheet theory remains under control. Indeed, these coset theories are exact in α', thus extending the description of these families of black hole microstates beyond the supergravity limit.

The models considered in [51, 52, 53] have the following basic structure. The *upstairs* theory (i.e. before gauging) involves pure NSNS fluxes and is of the form $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{T}^4$, where the $\mathbb{R}$ factor is timelike, and where the $\mathbb{S}^1$ is separated from the $\mathbb{T}^4$ because it plays a preferred role. Indeed, the gauging does not involve the $\mathbb{T}^4$ and we shall mostly work in six physical spacetime directions *downstairs* (i.e. after gauging). The non-trivial WZW model involved is that of the universal cover of $\mathrm{SL}(2,\mathbb{R})$ times $SU(2)$, which constitutes a rich and well studied example of an exactly solvable model [78, 110, 101, 102, 103]. The currents to be gauged are specific null linear combinations of the Cartan currents of $\mathrm{SL}(2,\mathbb{R}) \times SU(2)$ and the chiral momenta on $\mathbb{R} \times \mathbb{S}^1$, plus a similar (though generically not identical) linear combination of their anti-chiral counterparts.

Within this class of models, it is natural to ask what is the general family of well-behaved backgrounds that can be obtained by considering the most general null linear combination of the currents just described. This question was not addressed in [51, 52, 53]. Developing a systematic method for classifying such backgrounds is important for three reasons. First, it offers the possibility of finding novel configurations. Second, it can sharpen our understanding of the general backgrounds and how their parameters are constrained by different consistency conditions, shedding further light on the interplay between worldsheet CFT and spacetime geometry. Third, such techniques may then be applied to other similar classes of gauged models.

In this chapter we provide the answer to the above question, by proving that the JMaRT solutions and their limits represent the complete set of supergravity configurations described by this family of coset models. We do so from two complementary but independent points of view, and, in doing so, we clarify several aspects of both the worldsheet models and the supergravity backgrounds.

First, we consider the most general family of worldsheet coset theories, and derive necessary and sufficient conditions that lead to a consistent physical spectrum. These consistency conditions are obtained from analysing the gauge orbits, relating different representatives of the same physical operators, combined with worldsheet spectral flow (not to be confused with the spacetime/holographic spectral flow discussed above). This includes not only the spectral flow operation that is an essential part of the $\mathrm{SL}(2,\mathbb{R})$ WZW model [101], but also that of $SU(2)$. While $SU(2)$ spectral flow does not generate new affine representations, it has proven to be quite useful for string theory applications [118, 100, 52, 54]. As it turns out, we obtain constraints that take the form of algebraic relations for the a priori continuous gauging parameters, which imply that they can be written in terms of four integers, $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$ (of which only three are independent), plus R_y, the continuous modulus corresponding to the asymptotic proper radius of the $\mathbb{S}^1$. Furthermore, we derive restrictions on the parities of $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$.

We then show that the same conditions can equally be derived from the analysis of the set of supergravity backgrounds obtained from the general gauged models. More precisely, we show that imposing absence of horizons, absence of closed time-like curves (CTCs), and smoothness up to orbifold singularities in the corresponding classical geometries leads to an identical quantisation of the gauging parameters.

On the other hand, the JMaRT solutions are usually written in terms of their own set of seemingly continuous parameters, which however are known to be constrained by regularity and absence of CTCs to obey their own set of algebraic relations [37]. This parametrisation is quite awkward to work with, and obscures aspects of the physics.

We find that for the (NS5-decoupled) JMaRT solutions and their limits, one can completely bypass most of the seemingly continuous parameters. Let n_5, n_1, n_p denote respectively the quantised numbers of NS5 branes, fundamental strings, and units of momentum along $\mathbb{S}^1$ present in the background. We show that, in the NS5-brane decoupling limit, the six-dimensional metric and the NSNS B-field can be expressed explicitly in terms of the same set of integers $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$ introduced in the coset models, together with n_5 and R_y. Although this result is strongly inspired by our worldsheet analysis, we have derived it independently and purely within supergravity, via a non-trivial manipulation of the above-mentioned algebraic constraints.

For the dilaton, an extra parameter is necessary, which can be taken to be either n_1/V_4 or n_p/V_4, where V_4 is the volume of the $\mathbb{T}^4$. In the three-charge solutions, there is a constraint that relates n_1 and n_p, meaning that only one can be chosen independently. We take V_4 to be microscopic and fixed, and ignore it when counting parameters, so that we consider the independent ones to be $(\mathsf{m}, \mathsf{n}, n_5, R_y)$ plus one of either k or p, plus one of either n_1 or n_p. Without loss of generality, one can restrict the range of the integer parameters to be non-negative.

The resulting expressions for the supergravity fields are identical to those obtained from the general coset worldsheet actions, completing the proof that these are the unique backgrounds that arise in these coset theories. Note that for the latter, n_5 defines the level of the $\mathrm{SL}(2,\mathbb{R})$ and $SU(2)$ affine algebras.

The rewriting of these configurations in terms of the integer parametrisation significantly clarifies the properties of these solutions. In particular, it makes some of their symmetries and the action of T-duality manifest. It also sheds light on the somewhat delicate limits that lead to the two-charge configurations where either n_1 or n_p is set to zero. As a result, this allows us to derive a novel and non-trivial two-charge non-BPS NS5-P limit of the general solutions in a straightforward way.

Finally, we also comment on a potential relation to recent investigations of the so-called single-trace $T\bar{T}$ deformation of the (holographic) D1-D5 CFT [143, 144, 145]. In the worldsheet model, this can be described by using a null-gauging procedure similar to the formalism employed throughout this chapter, although in that context the $\mathrm{SL}(2,\mathbb{R})$ current involved in the gauging is not in the Cartan subalgebra.

The structure of this chapter is as follows. In Section 3.2 we review in more detail the $\mathrm{SL}(2,\mathbb{R})$ and $SU(2)$ WZW models, and analyse the consistency of the CFT spectrum in terms of the gauging parameters. In Section 3.3 we analyse absence of horizons, absence of CTCs, and smoothness in the corresponding supergravity backgrounds. In Section 3.4 we firstly match the resulting worldsheet models to the general JMaRT solutions. We then discuss in detail their various limits, including two-charge (non-BPS), BPS, and AdS_3 limits. In Section 3.5 we further discuss our results.

3.2 Consistency of the worldsheet spectrum

Here we discuss how the BRST charges are modified and under which conditions the resulting background is supersymmetric. Then, we derive a series of constraints leading to a consistent gauge-invariant spectrum. Spectral flow considerations play a key role in the analysis below.

3.2.1 Superstring Theory in null-gauged models

We now proceed to analyse the class of gauged WZW models introduced in Section 2.7. In this section we work directly at the level of the coset CFT. We will see that a number of consistency conditions can be derived, which restrict the possible values of the parameters l_i, r_i that define the embedding of the abelian subgroups being gauged.

Recall that the transformations we gauge are chiral and correspond to $g \to h_L\, g\, h_R^{-1}$, where $g \in \mathcal{G} = SL(2,\mathbb{R}) \times SU(2) \times \mathbb{R}_t \times U(1)_y$ and $h_{L(R)} \in \mathcal{H}_{L(R)} = U(1)_{L(R)}$. Keeping the notation general, we have seen that introducing two independent gauge fields $\mathcal{A}, \bar{\mathcal{A}}$ transforming as

$$\mathcal{A} \to h_L \mathcal{A}\, h_L^{-1} + \partial h_L h_L^{-1} \,, \qquad \bar{\mathcal{A}} \to h_R \bar{\mathcal{A}}\, h_R^{-1} + \bar{\partial} h_R h_R^{-1} \,, \tag{3.1}$$

leads to the gauge-invariant action [146]

$$S[g, \mathcal{A}, \bar{\mathcal{A}}] = S_{\text{WZW}}(g) + \frac{k}{\pi} \int d^2z \ \text{Tr} \left[\mathcal{A} g^{-1} \bar{\partial} g - \bar{\mathcal{A}} \partial g g^{-1} - g^{-1} \bar{\mathcal{A}} g \mathcal{A} \right] . \tag{3.2}$$

By parametrising the gauge fields as

$$\mathcal{A} = \partial H_L H_L^{-1} \,,\ \bar{\mathcal{A}} = \bar{\partial} H_R H_R^{-1} \,,\ (H_L \to h_L H_L \,,\ H_R \to h_R H_R) \,, \tag{3.3}$$

and making use of the Polyakov-Wiegmann identity

$$S_{\text{WZW}}(ab) = S_{\text{WZW}}(a) + S_{\text{WZW}}(b) + \text{sgn}(\kappa) \frac{k}{\pi} \int d^2z \ \text{Tr} \left[a^{-1} \bar{\partial} a \partial b b^{-1} \right] , \tag{3.4}$$

we can rewrite the gauged action as

$$S[g, \mathcal{A}, \bar{\mathcal{A}}] = S_{\text{WZW}} \left(H_L^{-1} g H_R \right) - S_{\text{WZW}} \left(H_L^{-1} \right) - S_{\text{WZW}} \left(H_R \right) . \tag{3.5}$$

A crucial simplification occurs when the currents being gauged are null. In this case we have $S_{\text{WZW}} \left(H_L^{-1} \right) = S_{\text{WZW}} \left(H_R \right) = 0$. Moreover, the Jacobian associated to the change of variables in Eq. (3.3) can be seen to trivialise for the same reason, except for the appearance of the usual $\tilde{b}\tilde{c}$ system [146]. Finally, one can change variables from g to the gauge-invariant $\mathcal{G}$-valued quantity $\tilde{g} = H_L^{-1} g H_R$. As a result, the path integral of the gauged theory is simply interpreted as that of the original upstairs WZW model on $\mathcal{G}$ combined with the ghost contributions. The same holds for the supersymmetric case.

The consequences of the presence of the ghosts signalling the null gaugings can be understood intuitively as follows. Upon quantisation, we find that, on top of the usual string theory contributions to the BRST charges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, it becomes necessary to include new chiral terms of the (schematic) form

$$\oint dz : \tilde{c}\, J : , \qquad \oint dz : \bar{\tilde{c}}\, \bar{J} : , \tag{3.6}$$

together with their fermionic counterparts. This ensures that, under the gauging procedure outlined above, the spectrum of the coset model is built simply out of the vertex operators of the upstairs theory that are BRST-closed. In other words, physical operators must be gauge invariant.

We can now make this construction explicit for the models under consideration. These include black hole microstate solutions with up to three charges, and we view them in the NS5-F1-P duality frame, where the relevant fields are the metric, the B-field and the dilaton. The corresponding geometries were described in Section 2.7. We know from [51] that these models include both BPS and non-BPS black hole microstates. Given that all the necessary ingredients belong to the NSNS sector we expect to have a well-defined solvable worldsheet model describing strings propagating in these backgrounds. Indeed, as shown in [51, 52] and reviewed in Section 2.7 above, the worldsheet theory associated to the propagation of strings in this context corresponds to a coset CFT of the form[38]

$$\frac{SL(2,\mathbb{R}) \times SU(2) \times \mathbb{R}_t \times U(1)_y}{U(1)_L \times U(1)_R} \times U(1)^4 . \tag{3.7}$$

Let us first characterise the upstairs twelve-dimensional model. Here we simply add the extra time direction t and spatial circle y to the matter content employed in the previous section, together with the corresponding fermionic partners λ^t and λ^y. The latter are bosonized as $i\partial H_6 = \lambda^t \lambda^y$, with $H_6^\dagger = -H_6$. They give additional free field contributions to the matter T and G in (2.135) and (2.136). The null current operators being gauged are then

$$\begin{aligned} J &= i\mathcal{J} = J^3 + l_2 K^3 + l_3 P_t + l_4 P_{y,L} , \\ \bar{J} &= i\bar{\mathcal{J}} = \bar{J}^3 + r_2 \bar{K}^3 + r_3 P_t + r_4 P_{y,R} , \end{aligned} \tag{3.8}$$

where $P_t = i\partial t$, $P_{y,L} = i\partial y$, and $P_{y,R} = i\bar{\partial} y$. Together with the extra coordinates we also include the additional set of ghosts mentioned above, together with their fermionic partners. Note that it is necessary to take $h[\tilde{c}] = 0$ and $h[\tilde{\gamma}] = 1/2$, such that the central charges $c_{\tilde{b}\tilde{c}} = -2$ and $c_{\tilde{\beta}\tilde{\gamma}} = -1$ cancel the additional matter contribution $c_{ty} = 3$. This also implies that the bosonization

$$\tilde{\beta} = e^{-\tilde{\varphi}} \partial \tilde{\xi} , \qquad \tilde{\gamma} = \tilde{\eta}\, e^{\tilde{\varphi}} \tag{3.9}$$

[38] As mentioned around Eq. (2.205), the coset involves the universal cover of SL$(2,\mathbb{R})$, and we are actually gauging the non-compact subgroup $\mathbb{R} \times U(1)$.

yields a canonically normalized scalar $\tilde{\varphi}$ with no background charge. Consequently, we can work with $\tilde{\varphi}$-independent vertex operators in the NSNS sector. On the other hand, the definition of the spin fields and the would-be spacetime supercharges is modified to [54]

$$Q_\varepsilon = \oint dz\, e^{-(\varphi-\tilde{\varphi})/2} S_\varepsilon\,, \qquad S_\varepsilon = \exp\left(\frac{i}{2}\sum_{I=1}^{6}\varepsilon_I H_I\right), \tag{3.10}$$

where the contributions to the conformal dimension of the integrand in Q_ε from the $\tilde{\varphi}$ and H_6 exponentials cancel exactly. Note that the mutual locality condition now reads $\prod_{I=1}^{6}\varepsilon_I = 1$. An analogous formula defines the anti-holomorphic counterpart $\bar{Q}_\varepsilon$.

As stressed above, the present procedure can lead to both BPS and non-BPS backgrounds. This depends on whether the charges Q_ε turn out to be BRST invariant or not, according to the precise current we choose to gauge. The left-handed BRST charge takes the form

$$\mathcal{Q} = \oint dz : \left[c\left(T + T_{\beta\gamma\tilde{\beta}\tilde{\gamma}}\right) + \gamma G + \tilde{c}J + \tilde{\gamma}\boldsymbol{\lambda} + \text{ ghosts}\right]:, \tag{3.11}$$

and similarly for the right-handed one. Here $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ are the superpartners of the currents in Eq. (3.8), that is

$$\boldsymbol{\lambda} = \psi^3 + l_2\chi^3 + l_3\lambda^t + l_4\lambda^y\,, \quad \bar{\boldsymbol{\lambda}} = \bar{\psi}^3 + r_2\bar{\chi}^3 + r_3\bar{\lambda}^t + r_4\bar{\lambda}^y. \tag{3.12}$$

This ensures that only operators satisfying the usual Virasoro and γG-invariance conditions that are also uncharged under the bosonic currents $J, \bar{J}$ and annihilated by $\tilde{\gamma}\boldsymbol{\lambda}$ and $\bar{\tilde{\gamma}}\bar{\boldsymbol{\lambda}}$ are physical. In particular, by using the γG-invariance condition $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, the Q_ε supercharges survive the gauging if and only if [54]

$$\varepsilon_1 + \varepsilon_2 l_2 = 0\,, \quad l_4 + \varepsilon_6\, l_3 = 0, \tag{3.13}$$

where the former constraint comes from the bosonic current and the latter arises from the fermionic one. Actually, only one of these two restrictions is independent due to the null condition (2.216). Analogously, for the antiholomorphic supercharges one has

$$\varepsilon_1 + \varepsilon_2 r_2 = 0\,, \quad r_4 + \varepsilon_6\, r_3 = 0, \tag{3.14}$$

For instance, the cases with $(4,4)$ and $(4,0)$ spacetime supersymmetry were considered recently in [54, App. B]. In the present work we focus mainly on the more general non-supersymmetric case, in which there is no solution to this set of constraints since $l_2 \neq \pm 1$ and $r_2 \neq \pm 1$.

We pause here to stress that we start from a (10+2)-dimensional model where all operators are taken to be mutually local. This includes the charges $Q_\varepsilon, \bar{Q}_\varepsilon$. In particular, this amounts to imposing the analogue of the GSO projection in the upstairs theory. Consequently, there are no tachyons in the resulting spectrum, nor in that of the coset model

[52, 54]. Indeed, even if no supersymmetry is preserved, the supergravity solutions we are dealing with are expected to be classically stable. Note that this is consistent with the fact that we only recover the fivebrane decoupling limit of JMaRT geometries, where there is no ergoregion [52]. The full solutions with flat asymptotics do exhibit the well-known ergoregion instability [147, 94], interpreted as an enhanced version of Hawking radiation [140, 93, 141].

3.2.2 Physical spectrum and consistency conditions

We now consider the physical states in the null-gauged theory. These are given in terms of vertex operators of the upstairs model that are BRST invariant as defined by the charge (3.11) and its anti-holomorphic counterpart.

The lightest physical states (with no winding) are given by unflowed operators with a single fermionic excitation. All such operators must satisfy the Virasoro condition

$$0 = -\frac{j(j-1)}{n_5} + \frac{j'(j'+1)}{n_5} - \frac{1}{4}E^2 + \frac{1}{4}P_y^2 \,. \tag{3.15}$$

These are automatically invariant under the action of $\boldsymbol{\lambda}$ in (3.12), so that they are BRST-invariant iff the following null-gauging[39] constraints hold:

$$0 = m + l_2\, m' + \frac{l_3}{2}E + \frac{l_4}{2}P_y \,, \qquad 0 = \bar{m} + r_2\, \bar{m}' + \frac{r_3}{2}E + \frac{r_4}{2}P_y \,. \tag{3.16}$$

The simplest states of this sort are the 6D scalars, whose spectrum was shown to coincide with that of minimally coupled massless scalar perturbations on top of the JMaRT geometries in [52]. There are also the extremal-weight states, which for $E = P_y = 0$ are half-BPS and were studied in [54] in the context of supertubes. Note that operators with $\mathbb{T}^4$ polarisations can also give BPS states in specific models with non-trivial twisted sectors [112].

On the other hand, operators with more general projections need to be combined with polarisations on the extra directions t and y in order to achieve BRST-invariance. The corresponding coefficients are determined by invariance under $\boldsymbol{\lambda}$ together with transversality in the t, y directions. The anti-holomorphic variables must satisfy analogous conditions as well, together with the gauge invariance conditions (4.12). Consequently, these constraints restrict the possible polarisations to those expected in the (9+1)- dimensional setting.

Let us now consider spectrally flowed states. A particularly simple case corresponds to the circular array of fivebranes on the Coulomb branch, which is obtained by choosing

[39] The factors of 2 in the free field terms of Eqs. (3.15) and (3.16) arise from the OPEs $\partial t(z)\partial t(0) \sim -\frac{1}{2}\frac{1}{z^2}$, $\partial y(z)\partial y(0) \sim \frac{1}{2}\frac{1}{z^2}$ (recall that we work with $\alpha' = 1$).

$l_1 = l_2 = r_1 = -r_2 = 1$ and $l_3 = l_4 = r_3 = r_4 = 0$ [118, 51]. From the worldsheet point of view, this null-gauged model is analogous to the cigar construction used in [120, 121] provided we replace the extra circle by an $\mathbb{S}^3$ and take $J_y \to K^3$. Thus, as in that example, introducing spectral flow with $\omega = \omega' = \bar{\omega}'$ does not produce new physical states since it merely amounts to a large gauge transformation. In a more general context, however, this is not true anymore, and spectrally flowed sectors contribute non-trivially to the spectrum. At this point, we also allow for both momentum n_y and winding ω_y on $\mathbb{S}^1_y$, and use the shorthands

$$P_{y,L/R} = \left(\frac{n_y}{R_y} \pm \omega_y R_y\right) \; , \; n_y, \omega_y \in \mathbb{Z}. \tag{3.17}$$

For generic states with spectral flow charges ω on $\mathrm{SL}(2,\mathbb{R})$, $(\omega', \bar{\omega}')$ on $SU(2)$, and winding ω_y on $\mathbb{S}^1_y$, the null-gauge constraints (4.12) read

$$\begin{aligned} 0 &= m + \frac{n_5}{2}\omega + l_2\left(m' + \frac{n_5}{2}\omega'\right) + \frac{l_3}{2}E + \frac{l_4}{2}P_{y,L}\,, &\text{(3.18a)}\\ 0 &= \bar{m} + \frac{n_5}{2}\omega + r_2\left(\bar{m}' + \frac{n_5}{2}\bar{\omega}'\right) + \frac{r_3}{2}E + \frac{r_4}{2}P_{y,R}, &\text{(3.18b)} \end{aligned}$$

while the Virasoro constraints take the form

$$\frac{1}{2} = \frac{j'(j'+1) - j(j-1)}{n_5} - m\omega + m'\omega' + \frac{n_5}{4}\left(\omega'^2 - \omega^2\right) - \frac{1}{4}\left(E^2 - P_{y,L}^2\right) + N\,, \tag{3.19a}$$

$$\frac{1}{2} = \frac{j'(j'+1) - j(j-1)}{n_5} - \bar{m}\omega + \bar{m}'\bar{\omega}' + \frac{n_5}{4}\left(\bar{\omega}'^2 - \omega^2\right) - \frac{1}{4}\left(E^2 - P_{y,R}^2\right) + \bar{N}. \tag{3.19b}$$

Here N and $\bar{N}$ are the excitation numbers, and we have restricted to unflowed states with no fermion excitations for simplicity.

The discussion so far does not characterise the physical spectrum in a unique way: there is a residual discrete gauge orbit connecting equivalent representatives of the same physical state. This fact was noticed in [52], so let us first recall the observations made in that work before making a set of generalisations. First, spectral flow in the null direction corresponding to the gauge current is gauge-trivial. Second, the non-compactness of $\mathbb{R}_t$ means that there cannot be independent left and right gauge spectral flow transformations, since these would shift the zero mode of t differently. Therefore, globally we work with the universal cover of $\mathrm{SL}(2,\mathbb{R})$, in which the left and right spectral flow parameters are constrained to be equal, $\omega = \bar{\omega}$. Moreover, globally we gauge $\mathbb{R} \times U(1)$, a (1+1)-dimensional cylinder composed of one compact spacelike direction and one non-compact timelike direction. The gauged model then has a single non-compact timelike direction.[40] Third, the non-compactness of the time coordinate t

[40] By contrast, in related models that do not include the $\mathbb{R}_t$ factor in the upstairs model, the single cover of $\mathrm{SL}(2,\mathbb{R})$ has been considered [118].

moreover imposes

$$l_3 = r_3 \,, \tag{3.20}$$

or, in terms of the original gauging parameters, $\mathrm{l}_3/\mathrm{l}_1 = \mathrm{r}_3/\mathrm{r}_1$. We will re-derive the condition $l_3 = r_3$ from an independent point of view in the following section by imposing smoothness, absence of horizons, and absence of CTCs in the corresponding geometry.

We now make a more general analysis of this phenomenon in the general models defined in the previous section. Let us stress that the analysis of such gauge orbits is not simply about the counting of states. Indeed, being able to identify gauge-equivalent operators in terms of the quantum numbers of the WZW model is necessary for building a consistent theory, and we will show that it further constrains the allowed values for the gauging parameters l_i, r_i.

Given a physical state, let us seek spectral flow transformations that result in the same operator. By subtracting the two equations in (3.19) we find

$$0 = \omega(\bar{m} - m) + m'\omega' - \bar{m}'\bar{\omega}' + \frac{n_5}{4}(\omega'^2 - \bar{\omega}'^2) + n_y\omega_y + N - \bar{N}, \tag{3.21}$$

which plays the role of the level-matching condition in this context. In order to find solutions of Eq. (3.21), $(\omega'^2 - \bar{\omega}'^2)$ must be a multiple of 4, so $\omega' \pm \bar{\omega}'$ must be even. Note that this preserves the statistics of the $SU(2)$ part of the state.

Let us consider a shift of the form[41]

$$\omega \to \omega + q \,, \qquad q \in \mathbb{Z} \,. \tag{3.22}$$

We shall show that this can be compensated at the level of the null-gauge constraints (3.18) without altering the weights (3.19) by shifting the remaining quantum numbers appropriately. We begin with a general shift and show that only the shift in the null gauge direction achieves this. We allow for arbitrary multiples of q to shift ω', $\bar{\omega}'$, E, n_y and ω_y as well, namely

$$(\omega', \bar{\omega}', E, P_{y,L}, P_{y,R}) \to (\omega' - a_2 q, \bar{\omega}' - b_2 q, E + a_3 q, P_{y,L} - a_4 q, P_{y,R} - b_4 q). \tag{3.23}$$

[41]We include both the bosonic and the free fermion spectral flows, so that the shifted charges and weights are written in terms of the supersymmetric level n_5 as opposed to the bosonic levels $k = n_5 + 2$ and $k' = n_5 - 2$.

For the weights (3.19) and gauge constraint (3.18) to remain unchanged for arbitrary q, we must have

$$\begin{aligned} 0 &= m + \frac{n_5}{2}\omega + a_2\left(m' + \frac{n_5}{2}\omega'\right) + \frac{a_3}{2}E + \frac{a_4}{2}P_{y,L}, \\ 0 &= n_5(1-a_2^2) + a_3^2 - a_4^2, \\ 0 &= n_5(1-l_2a_2) + l_3a_3 - l_4a_4, \end{aligned} \tag{3.24}$$

and the same with $a_{2,4} \to b_{2,4}$, $l_{2,4} \to r_{2,4}$, $m' \to \bar{m}'$, $\omega' \to \bar{\omega}'$ and $P_{y,L} \to P_{y,R}$. To satisfy the first of these three conditions for general states without over-restricting the spectrum, we must set $a_i = l_i$ and $b_i = r_i$. Indeed, in this case the first condition becomes (3.18), while the last two conditions both reduce to (2.218). The compensating shifts then take the form

$$\omega' \to \omega' - l_2\, q\,, \quad \bar{\omega}' \to \bar{\omega}' - r_2\, q, \tag{3.25}$$

for the left and right $SU(2)$ spectral flow charges, respectively,

$$E \to E + l_3\, q = E + r_3\, q, \tag{3.26}$$

for the energy, and

$$n_y \to n_y - \frac{R_y}{2}\,(l_4 + r_4)\, q\,, \quad \omega_y \to \omega_y - \frac{1}{2R_y}\,(l_4 - r_4)\, q, \tag{3.27}$$

for the S^1_y quantum numbers.

For Eqs. (3.22) and (3.25)–(3.27) to make sense in terms of integer spectral flows and momentum/winding numbers, the gauging parameters must be quantised in a specific way. On the one hand, taking into account that, as argued above, $\omega' \pm \bar{\omega}'$ must be even, for the $SU(2)$ sector we find $l_2 \pm r_2 \in 2\mathbb{Z}$. We can thus write as a first pass (to be refined momentarily)

$$l_2 = \mathsf{m} + \mathsf{n}\,,\; r_2 = -(\mathsf{m} - \mathsf{n})\,, \qquad \mathsf{m},\mathsf{n} \in \mathbb{Z}\,, \tag{3.28}$$

where the signs are chosen for later convenience. Furthermore, recall that in the $\mathrm{SL}(2,\mathbb{R})$ and $SU(2)$ sectors the spectral flow operations do not act solely on the bosonic subalgebras. Indeed, they also shift the fermionic modes as in Eqs. (2.133). At the level of the vertex operators, this is accounted for by including the $H_{1,2}$ exponentials introduced in (2.156), which were taken into account for computing the weights (3.19). However, if as in the computation above we start from an unflowed state with no fermionic excitations, and use the shifts (3.22) and (3.25) with, say, $q = 1$ (or any other odd value), the presence of these exponentials also indicates that the fermion numbers on the left- and right-handed components will *not* be preserved for arbitrary values of $\mathsf{m} \pm \mathsf{n}$. Thus, we see that it is necessary to make Eq. (3.28) more precise by restricting to

$$l_2 = \mathsf{m} + \mathsf{n} \in 2\mathbb{Z} + 1\,, \qquad r_2 = -(\mathsf{m} - \mathsf{n}) \in 2\mathbb{Z} + 1\,, \qquad \mathsf{m}, \mathsf{n} \in \mathbb{Z}\,. \tag{3.29}$$

On the other hand, from (3.27) we also must have

$$\frac{1}{2R_y}\,(r_4 - l_4) = \mathsf{k} \in \mathbb{Z}\,, \qquad \frac{R_y}{2}\,(l_4 + r_4) = \mathsf{p} \in \mathbb{Z}\,, \tag{3.30}$$

or equivalently

$$l_4 \;=\; -\Big(\mathsf{k}R_y - \frac{\mathsf{p}}{R_y}\Big)\,, \qquad r_4 \;=\; \mathsf{k}R_y + \frac{\mathsf{p}}{R_y}\,. \tag{3.31}$$

By plugging the expressions (3.29) and (3.31) into the null constraints (2.216), we now solve for l_3, r_3 and p. Firstly, we obtain

$$l_3 \;=\; r_3 \;=\; -\sqrt{\mathsf{k}^2 R_y^2 + \frac{\mathsf{p}^2}{R_y^2} + n_5\,(\mathsf{m}^2 + \mathsf{n}^2 - 1)}, \tag{3.32}$$

where we have chosen the negative square root for l_3, r_3, so that we gauge away the difference of the upstairs time directions. This fixes the energy shift (3.26) in terms of $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$. More interestingly, we also get

$$\mathsf{k}\,\mathsf{p} = n_5\,\mathsf{m}\,\mathsf{n}, \tag{3.33}$$

which shows that only three of the integers $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$ are actually independent. Moreover, either k or p (or both) must be even.

We will show below that the integers m and n introduced above control the angular momenta of the classical configuration along the $\mathbb{S}^3$ coordinates ϕ and ψ, respectively. We will further argue in the following that the absolute value of the integer k is to be interpreted as an orbifold parameter. The meaning of the remaining integer p is slightly complicated to interpret in classical terms. This is due to its stringy nature, and it can be understood either holographically or in terms of T-duality, as follows.

For $\mathsf{k} \neq 0$, from (3.33) we find that p is n_5 times the momentum per strand mn/k in the holographic description of JMaRT states, as noted in [52]. This must be an integer since the holographic CFT is a symmetric product orbifold theory (see the discussions in [96, 94]). On the other hand, it is well known that the worldsheet theory is invariant under T-duality along a circular direction. In the language of gauged WZW models, T-dual models arise due to the equivalence of vector and axial gaugings. In the present context, T-duality along y amounts to $l_4 \to -l_4$. Together with the usual radius redefinition $R_y \to 1/R_y$, this exchanges the role of the integers k and p. Thus, depending on the choice of duality frame, either k or p are interpreted as an orbifold parameter, while the remaining integer is fixed in terms of n_5, m and n, and it controls the momentum charge.

While on the subject of the T-duality, let us also observe that (3.31) has special features at the self-dual radius, which in $\alpha' = 1$ units is at $R_y = 1$. At the self-dual radius, in string theory one observes an enhancement of the abelian $U(1)$ symmetry to a non-abelian $SU(2)$ symmetry due to the presence of new massless fields. In our case, when $R_y = 1$ the expressions of the gauging parameters l_4, r_4 corresponding to the $U(1)_y$ component resemble those of their $SU(2)$ counterparts l_2, r_2. In addition, we see that it becomes possible to set either l_4 or r_4 to zero while keeping the other one non-trivial, which is not possible for generic values of R_y. Since the self-dual radius is associated with the appearance of new massless states (in the upstairs theory), this might lead to new solutions. We leave a more detailed exploration of such configurations for future work.

Note that the values of the gauging parameters (3.29)–(3.32) imply that

$$\begin{aligned} \Sigma_0\left(\rho=0,\theta=0\right) &= \frac{1-l_2r_2}{2}+\frac{l_3r_3-l_4r_4}{n_5} &&= \mathsf{m}^2+\frac{\mathsf{k}^2R_y^2}{n_5}\,, \\ \Sigma_0\left(\rho=0,\theta=\frac{\pi}{2}\right) &= \frac{1+l_2r_2}{2}+\frac{l_3r_3-l_4r_4}{n_5} &&= \mathsf{n}^2+\frac{\mathsf{k}^2R_y^2}{n_5}\,. \end{aligned} \tag{3.34}$$

The combinations in (3.34) correspond to the minimal values of the quantity Σ_0 appearing in the denominator of various components of the supergravity fields (2.227). We have just shown that the consistency conditions of the spectrum imply that they are both non-negative quantities. This will be important when studying the corresponding geometry in the following section.

Let us summarise our results so far. Starting with a class of generic null gauged models, the gauged currents are defined in terms of eight parameters, namely $\mathrm{l}_i, \mathrm{r}_i$, $i = 1,2,3,4$. Since only the direction of the gauging matters, the overall scale becomes irrelevant. This means that we can work directly with the six ratios l_i, r_i. These must satisfy the two null conditions (2.218) and also $l_3 = r_3$ from the non-compactness of t. Finally, by focusing on the worldsheet CFT and relating the action of spectral flow in $\mathrm{SL}(2,\mathbb{R})$ to that of the gauge orbits, we have shown that the theory is consistent only if the remaining three parameters can be written in terms of three integers (in addition to R_y), which we can take to be m, n and k (when $\mathsf{k} \neq 0$). Moreover, these must be chosen so that $\mathsf{p} = n_5\mathsf{mn}/\mathsf{k}$ is also integer-valued. It is possible that the quantization conditions on $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$ could alternatively be obtained by analyzing the global consistency of the gauging, see e.g. [148, 149, 150, 151, 152]. In the next section we will instead proceed to analyze the global geometry of the gauged target space.

The set of conditions (3.29)–(3.32) is one of the main results of this chapter. It will allow us to rewrite the general supergravity fields in Eqs. (2.227)–(2.229) in a simple way, making their main physical features and some of their symmetries manifest. Furthermore, this will lead to a complete characterisation of the full set of consistent solutions.

3.3 Analysis of the supergravity backgrounds

In the previous section we have shown that the gauging parameters l_i, r_i, $i = 1,2,3$ can be defined in terms of $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$, all of which are integers that have a clear physics meaning. We now perform an independent supergravity analysis of the metrics introduced in Eq. (2.227), and show that imposing smoothness and absence of closed timelike curves provides an alternative derivation and complementary interpretation of the constraints (3.29)–(3.32).

3.3.1 Eliminating potential closed timelike curves

To investigate potential closed timelike curves we complete the squares successively in the periodic variables ψ, ϕ and y, to rewrite the line element (2.227). We obtain

$$\begin{aligned} ds^2 = &- T(\rho)dt^2 + Y(\rho)\,\left[dy + A_y(\rho;dt)\right]^2 + n_5(d\rho^2 + d\theta^2) + \\ &+ n_5\,\sin^2\theta\,\frac{h_\phi}{\Sigma_0}\,\left[d\phi + A_\phi(\rho;dt,dy)\right]^2 + n_5\,\cos^2\theta\,\frac{h_\psi}{\Sigma_0}\,\left[d\psi + A_\psi(\rho;dt,dy)\right]^2, \end{aligned} \tag{3.35}$$

where the functions Σ_0, h_ϕ and h_ψ were defined in (2.224) and (2.228) respectively, A_y, A_ψ, and A_ϕ are one-forms depending only on the radial variable ρ and with legs in the appropriate arguments, while $T(\rho)$ will turn out to be a non-negative function whose explicit expression we will not need. In order to ensure the absence of CTCs we must require the functions multiplying the squares in the periodic variables to be non-negative, i.e.

$$\frac{h_\phi}{\Sigma_0} \geq 0\,, \qquad \frac{h_\psi}{\Sigma_0} \geq 0\,, \qquad Y(\rho) \geq 0. \tag{3.36}$$

We now show that asking for the inequalities (3.36) to hold everywhere in geometry is equivalent to imposing

$$l_3 = r_3. \tag{3.37}$$

Let us first see that (3.36) implies (3.37). By combining the first two inequalities we find that the product $h_\phi h_\psi$ is non-negative. On the other hand, the explicit expression of $Y(\rho; l_i, r_i)$ reads

$$\begin{aligned} Y(\rho) \;&=\; \frac{4\sinh^2\rho\,\left(n_5\cosh^2\rho + l_3 r_3\right) + (n_5 + l_3 r_3)^2 - (n_5 l_2^2 + l_4^2)(n_5 r_2^2 + r_4^2)}{4n_5^2 h_\phi h_\psi} \\ &=\; \frac{4n_5\sinh^2\rho\,\left(n_5\cosh^2\rho + l_3 r_3\right) - (l_3 - r_3)^2}{4n_5 h_\phi h_\psi}\,, \end{aligned} \tag{3.38}$$

where in the second line we have used (2.218). It follows from this last expression that the third inequality in (3.36) can only be satisfied at the origin $\rho = 0$ if $l_3 = r_3$.

It remains to be seen that the implication holds in the other direction as well. For this, we note that the minimal value of Σ_0 is given by

$$\Sigma_0^{\min} = \frac{1}{2n_5}\big[n_5(1-|l_2 r_2|) + l_3 r_3 - l_4 r_4\big]. \tag{3.39}$$

Using $l_3 = r_3$ we can rewrite the null conditions (2.218) as

$$n_5\left(l_2^2 - r_2^2\right) + r_4^2 - l_4^2 = 0\,, \qquad l_3^2 + r_3^2 = 2l_3^2 = l_4^2 + r_4^2 + n_5\left(l_2^2 + r_2^2 - 2\right), \tag{3.40}$$

so that

$$2\big[n_5(1 \pm l_2 r_2) + l_3 r_3 - l_4 r_4\big] = n_5(l_2 \pm r_2)^2 + (l_4 - r_4)^2\,. \tag{3.41}$$

It follows that $\Sigma_0 \geq 0$ everywhere. Given that $h_\phi = \Sigma_0(\rho, \theta = 0)$ and $h_\psi = \Sigma_0(\rho, \theta = \pi/2)$, the same holds for these functions and the first two inequalities in (3.36) are thus satisfied. The third one also holds, as can be checked from (3.38).

This proves that in the asymptotically linear dilaton geometry the necessary and sufficient condition for avoiding CTCs is precisely $l_3 = r_3$, Eq. (3.37). This constraint was also obtained in the worldsheet analysis of Section 3.2 from the non-compactness of the t direction, Eq. (3.20). Moreover, once this is imposed we find that, as advertised above, $T(\rho)$ is non-negative.

3.3.2 Absence of horizons

Here and in the following subsection we perform an analysis which closely follows that of [37]. The determinant of the metric (2.227) reads

$$\det g = -\left(\frac{n_5^2 \sin(2\theta)\sinh(2\rho)}{4\Sigma_0}\right)^2, \tag{3.42}$$

where we have used the null-gauge constraints (2.218). Besides the usual zeros at the poles of the $\mathbb{S}^3$, this only vanishes at $\rho = 0$. Given that the determinant of the induced metric on surfaces of constant ρ is simply (3.42) divided by n_5, we see that $\rho = 0$ corresponds to either a horizon or an origin of higher codimension. In order to distinguish between these two cases, we further compute the determinant of the induced metric on surfaces of constant ρ and t, and we evaluate it at $\rho = 0$, giving

$$\lim_{\rho\to 0} \det g\big|_{(y,\theta,\phi,\psi)} = -\left(\frac{n_5(l_3 - r_3)\,\sin(2\theta)}{4\Sigma_0(0,\theta)}\right)^2. \tag{3.43}$$

Hence, in order to obtain a horizonless and possibly smooth geometry we again need to impose $l_3 = r_3$, Eq. (3.37), such that (3.43) vanishes. Smoothness will then be achieved if some circle direction shrinks appropriately when $\rho \to 0$, as we discuss next.

3.3.3 Smoothness and quantisation

When the background contains F1 charge, the supergravity solutions must be smooth up to possible orbifold singularities. In the absence of F1 charge, NS5-brane singularities will be present. We begin by treating the more general case in which F1 charge is present, and treat the latter as a special case.

We therefore focus on the periodic directions and consider a generic Killing vector of the form

$$\xi = \partial_y + \alpha\,\partial_\psi - \beta\,\partial_\phi\,, \qquad \alpha,\beta\in\mathbb{R}. \tag{3.44}$$

where the signs have been chosen for later convenience. To find smooth solutions we seek pairs of coefficients (α,β) such that the norm of ξ vanishes $\forall\,\theta\in[0,\frac{\pi}{2}]$ when we approach $\rho=0$. From the metric (2.227) we find that this is indeed the case when

$$\alpha = \frac{l_2 r_4 - l_4 r_2}{2n_5\Sigma_0(0,\frac{\pi}{2})}\,, \qquad \beta = \frac{l_2 r_4 + l_4 r_2}{2n_5\Sigma_0(0,0)}\,, \tag{3.45}$$

since for these values, and upon using the null constraint (4.12), we obtain

$$\lim_{\rho\to 0} g_{ij}\,\xi^i\,\xi^j = -\frac{(l_3-r_3)^2}{4n_5\Sigma_0(0,0)\;\Sigma_0(0,\frac{\pi}{2})} = 0\,, \tag{3.46}$$

where $i,j=y,\psi,\phi$. In the last step we have used the no-CTC condition (3.37). Then, we define the following shifted coordinates

$$\hat{\psi} = \psi + \alpha\,y\,, \qquad \hat{\phi} = \phi - \beta\,y\,, \tag{3.47}$$

where the signs are chosen for later convenience. By examining the integral curves of ξ, we see that the direction that shrinks at $\rho=0$ is y at fixed $\hat\psi$, $\hat\phi$. We find that near $\rho=0$ the line element at fixed $(t,\theta,\hat\psi,\hat\phi)$ is of the form

$$ds^2_{\rho\to 0} \simeq n_5\left[d\rho^2+\rho^2\,d\left(\frac{y}{R}\right)^2\right]\,, \qquad R^2 = \left[\frac{2n_5^2\Sigma_0(0,0)\,\Sigma_0(0,\frac{\pi}{2})}{\left(n_5+l_3^2\right)\,(l_4-r_4)}\right]^2\,, \tag{3.48}$$

where we have used (2.216) and (3.37), and assumed $l_4\neq r_4$ (for now). Strictly speaking, a smooth geometry will be obtained only if the radius R coincides with R_y. Given that string theory is well-defined on orbifold backgrounds, as usual we allow for possible $\mathbb{Z}_\mathsf{k}$ orbifold singularities, k being the corresponding orbifold parameter. Thus, we relax this condition and impose $R^2=\mathsf{k}^2R_y^2$ for some positive integer k instead, i.e.

$$\mathsf{k}^2R_y^2 = \left[\frac{2n_5^2\Sigma_0(0,0)\,\Sigma_0(0,\frac{\pi}{2})}{\left(n_5+l_3^2\right)\,(l_4-r_4)}\right]^2\,. \tag{3.49}$$

Making use of the intuition developed in Section 3.2, we further rewrite the values of the parameters l_i, r_i in terms of new quantities m, n, k and p as in Eqs. (3.28) and (3.30), namely

$$l_2 = \mathsf{m} + \mathsf{n}\ ,\ r_2 = -(\mathsf{m} - \mathsf{n})\ ,\ \frac{1}{2}(l_4 - r_4) = -\mathsf{k}R_y\ ,\ \frac{1}{2}(l_4 + r_4) = \frac{\mathsf{p}}{R_y} = \frac{n_5}{R_y}\frac{\mathsf{mn}}{\mathsf{k}}. \quad (3.50)$$

In (3.50) we could a priori have written k' instead of k. However, if we were to then substitute (3.50) into (3.49), we would find that $\mathsf{k}' = \pm\mathsf{k}$. In other words, within this parametrisation Eq. (3.49) is trivially satisfied.

Up to this point, the reparametrisation (3.50) does not assume that m and n are integers. However, the periodicities of the new angular variables $\hat{\phi}$ and $\hat{\psi}$ should be consistent with that of y. The corresponding quantisation conditions read

$$\alpha\,(\mathsf{k}\,R_y) = \mathsf{m} \in \mathbb{Z}\,, \qquad \beta\,(\mathsf{k}\,R_y) = \mathsf{n} \in \mathbb{Z}\,. \quad (3.51)$$

From the classical point of view, the values of the integers m and n seem otherwise unrestricted. However, we know from the discussion around Eq. (3.29) that one of m, n must be even and the other one must be odd. A geometric argument leading to this restriction was put forward in [37] in the JMaRT context. This is based on discussing the periodicity of the fermions along the $\mathbb{S}^1_y$ circle and the associated spin structure of the target space. Although it should be possible, we will not attempt to extend these arguments to the case with general gauging parameters. This is because in the next section we will directly match the JMaRT solutions to the supergravity fields of our coset models.

Moreover, out of the four parameters $\mathsf{m}, \mathsf{n}, \mathsf{k}$ and $\mathsf{p} = n_5\mathsf{mn}/\mathsf{k}$, a priori only the first three appear to be required to be integers from the above smoothness analysis, and nothing seems to prevent p from being a rational (not necessarily integer) number. The stringy nature of this parameter manifests itself in the fact that T-duality along $\mathbb{S}^1_y$, namely $l_4 \to -l_4$ and $R_y \to 1/R_y$, maps $\mathsf{k} \leftrightarrow \mathsf{p}$. Since in the T-dual geometries p is an orbifold parameter, it must also be quantised. Moreover, given that m and n may vanish or be non-vanishing integers with arbitrary signs, so can p, and consequently, the same applies to k. In due course we will restrict to non-negative values of $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{p}$, without loss of generality.

Recall that in the previous passage we assumed $l_4 \neq r_4$. The case $l_4 - r_4 = 0$, which corresponds to $\mathsf{k} = 0$ in the parametrisation (3.50), needs to be treated separately. When $\mathsf{k} = 0$, Σ_0 goes to zero at $\rho = 0$ and either $\theta = 0$ or $\theta = \pi/2$, see e.g. Eq. (3.34). The metric is singular as $\Sigma_0 \to 0$. We choose conventions in which the zero is at $\theta = \pi/2$. This corresponds to the location of the (smeared) NS5 brane source. As we shall see in the next section, this is because the F1 charge vanishes and the solutions are two-charge NS5-P (see Eqs. (3.69) and (3.76)–(3.77) below). Let us analyse the geometry away from the source. The region of interest is the neighbourhood of $\rho = 0$ for $\theta \neq \pi/2$. Note that, assuming $l_3 = r_3$, the null conditions (2.218) imply $l_2 = \pm r_2$, so that either $\mathsf{m} = 0$ or $\mathsf{n} = 0$. To have the source at $\theta \neq \pi/2$, we take $l_2 = -r_2$, i.e. $\mathsf{n} = 0$. Then the norm of the Killing vector (3.44) is always non-vanishing. However, the ψ circle shrinks as $\rho \to 0$. Indeed, in this neighbourhood the line element at fixed (t, y, θ, ϕ) reads

$$ds^2_{\rho\to 0} \simeq n_5 \left[d\rho^2 + \rho^2 \, d\left(\frac{\psi}{\mathsf{m}}\right)^2 \right] , \tag{3.52}$$

where we have used the parametrisation (3.50) without imposing the last equality, so that p is unconstrained. We conclude that for generic values of θ and $\mathsf{m} = \pm 1$ the geometry is smooth. As before, we allow for orbifold singularities, so that m is again quantised: any other non-zero $\mathsf{m} \in \mathbb{Z}$ leads to a $\mathbb{Z}_{|\mathsf{m}|}$ orbifold structure. Had we chosen $l_2 = r_2$ instead, the source would have been at $\theta = 0$, the ϕ circle would have been the one shrinking at small ρ, and the parameter m would have been replaced by n.

3.3.4 Killing Spinors

Finally, we discuss the relation that the embedding coefficient must satisfy in order to preserve a certain amount of supersymmetry. For simplicity, we work directly in the $AdS_3 \times S^3$ limit, which will be shown to exist in all consistent cases. In order to achieve this, we first consider the Killing spinor in global AdS_3, as given in [37]:

$$\epsilon_L^{\pm} = e^{\pm\frac{i}{2}\tilde{\phi}_L} \, e^{-\frac{i}{2}y} \epsilon_0 \, , \qquad \epsilon_R^{\pm} = e^{\pm\frac{i}{2}\tilde{\phi}_R} \, e^{-\frac{i}{2}y} \epsilon_0 \, , \tag{3.53}$$

where ϵ_0 is a constant AdS_3 spinor. The dependence on the spacetime coordinates was derived in [153] and, in particular, the y dependence is such that the Killing spinors are regular near the origin (see also [123] and [154, App. D,E]). By a large gauge transformation one can induce the S^3 angular momenta starting from global $AdS_3 \times S^3$ which, in terms of the $\tilde{\phi}_{L,R}$ coordinates, translates into the following diffeomorphism

$$\tilde{\phi}_L = \phi + (\mathsf{m} + \mathsf{n})\, y = \phi + l_2 \, y \, , \qquad \tilde{\phi}_R = \phi + (\mathsf{m} - \mathsf{n})\, y = \phi - r_2 \, y \, . \tag{3.54}$$

This is known as spacetime spectral flow (see e.g. [96, 94]). Focusing on solutions that admit an asymptotically flat completion, the Killing spinor equations demand that the spinors be independent of y after performing the above large gauge transformation. This is obtained by imposing

$$|l_2| = 1 \quad \text{or} \quad |r_2| = 1. \tag{3.55}$$

By virtue of the null condition (2.216), the above constraint implies an analogous one for the remaining embedding coefficients, namely

$$|l_3| = |l_4| \quad \text{or} \quad |r_3| = |r_4| \,. \tag{3.56}$$

We thus conclude that the above constraints must be satisfied in order to have non-trivial Killing spinors in spacetime. This is consistent with the discussion around Eqs. (3.13) and (3.14) above.

Summarising, we have imposed absence of CTCs, absence of horizons, and smoothness up to physical sources (corresponding to orbifold singularities or NS5 branes) of the general background (2.227). These conditions imply a set of constraints on the group-theoretic embedding coefficients l_i, r_i parametrising the space of solutions in which the string propagates without pathologies. These consistency conditions take exactly the same form as those obtained from the worldsheet CFT analysis in Section 3.2, Eqs. (3.29)–(3.32). In passing, we note that recently a similar relation between consistency of the worldsheet theory and a well-behaved geometry was found in a related context in [155, 156].

3.4 Matching to JMaRT and two-charge limits

In the previous sections we have shown that the class of null-gauged models defined in Section 2.7 in terms of the gauging parameters l_i, r_i are consistent iff the latter can be written simply in terms of the four integers $\mathsf{k}, \mathsf{m}, \mathsf{n}$ and p. Out of these, only three are independent. Here we show that the resulting models correspond precisely to the full family of JMaRT solutions [37] and their various limits. The metric and B-field take a simple form in terms of these parameters, all of which have a clear physical meaning. Moreover, starting from the generic three-charge solution, we describe in detail the delicate limits that lead to the two-charge configurations and exhibit novel non-BPS NS5-P solutions.

3.4.1 JMaRT metric and B-field

Let us recall the form of the NS5-decoupled limit of the NS5-F1-P JMaRT solutions, that is, the S-duals of the smooth and horizonless non-supersymmetric D1-D5-P backgrounds obtained in [37]. Note that we are working in units where $\alpha' = 1$, hence

$Q_5 = n_5$. In our conventions, these geometries take the form [52]

$$\begin{aligned}
ds^2 \;=\; & \frac{f}{\tilde{H}_1}\left(-dt^2+dy^2\right) + \frac{M}{\tilde{H}_1}\left(c_p dt - s_p dy\right)^2 + n_5\left(d\rho^2+d\theta^2\right) \\
& + \frac{n_5}{\tilde{H}_1}\Big[\left(r_+^2-r_-^2\right)\cosh^2\rho + r_-^2 + a_2^2 + M s_1^2\Big]\,\sin^2\theta\, d\phi^2 \\
& + \frac{n_5}{\tilde{H}_1}\Big[\left(r_+^2-r_-^2\right)\sinh^2\rho + r_+^2 + a_1^2 + M s_1^2\Big]\,\cos^2\theta\, d\psi^2 \\
& + \frac{2\sqrt{M\,n_5}}{\tilde{H}_1}\Big[\left(a_2 c_1 c_p - a_1 s_1 s_p\right)dt + \left(a_1 s_1 c_p - a_2 c_1 s_p\right)dy\Big]\,\sin^2\theta\, d\phi \\
& + \frac{2\sqrt{M\,n_5}}{\tilde{H}_1}\Big[\left(a_1 c_1 c_p - a_2 s_1 s_p\right)dt + \left(a_2 s_1 c_p - a_1 c_1 s_p\right)dy\Big]\,\cos^2\theta\, d\psi \\
& + ds^2_{\mathbb{T}^4}\,,
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
B \;=\; & -\frac{M s_1 c_1}{\tilde{H}_1}\,dt\wedge dy + \frac{n_5\cos^2\theta}{\tilde{H}_1}\Big[(r_+^2-r_-^2)\sinh^2\rho + r_+^2 + a_2^2 + M s_1^2\Big]d\phi\wedge d\psi \\
& + \frac{\sqrt{M n_5}}{\tilde{H}_1}\Big[(a_1 c_1 c_p - a_2 s_1 s_p)dt + (a_2 s_1 c_p - a_1 c_1 s_p)dy\Big]\wedge \sin^2\theta\, d\phi \\
& + \frac{\sqrt{M n_5}}{\tilde{H}_1}\Big[(a_2 c_1 c_p - a_1 s_1 s_p)dt + (a_1 s_1 c_p - a_2 c_1 s_p)dy\Big]\wedge \cos^2\theta\, d\psi,
\end{aligned} \tag{3.58}$$

together with the dilaton

$$e^{2\Phi} = g_s^2\,\frac{n_5}{\tilde{H}_1}. \tag{3.59}$$

Here the charges are given in terms of the boost parameters $\delta_{1,p}$, that is

$$Q_i = M s_i c_i\,,\; c_i = \cosh(\delta_i)\,,\; s_i = \sinh(\delta_i)\,, \qquad i = 1, p\,, \tag{3.60}$$

and

$$\begin{aligned}
& f = \frac{1}{2}\left[(r_+^2-r_-^2)\cosh(2\rho) + (a_2^2-a_1^2)\cos(2\theta) + r_+^2 + r_-^2 + a_1^2 + a_2^2\right]\,, \\
& \tilde{H}_1 = f + M s_1^2 = \frac{1}{2}\left[(r_+^2-r_-^2)\cosh(2\rho) + (a_2^2-a_1^2)\cos(2\theta) + M^2\right] + M s_1^2\,, \\
& r_\pm^2 = \frac{1}{2}\Big[(M-a_1^2-a_2^2) \pm \sqrt{(M-a_1^2-a_2^2)^2 - 4a_1^2 a_2^2}\Big] = -a_1 a_2\left(\frac{s_1 s_p}{c_1 c_p}\right)^{\pm 1}.
\end{aligned} \tag{3.61}$$

In these formulas, parameters such as $a_{1,2}$ are a priori thought of as continuous. As will be clear shortly, this is potentially misleading since they are constrained by the smoothness conditions and the absence of horizons. In this setup, these constraints read

$$\begin{aligned}
a_1 a_2 \;&=\; \frac{Q_1 Q_5}{\mathsf{k}^2 R_y^2}\,\frac{s_1^2 c_1^2 s_p c_p}{(c_1^2 c_p^2 - s_1^2 s_p^2)}, \\
M \;&=\; a_1^2 + a_2^2 + r_+^2 + r_-^2 \;=\; a_1^2 + a_2^2 - a_1 a_2\,\frac{c_1^2 c_p^2 + s_1^2 s_p^2}{c_1 c_p s_1 s_p}\,,
\end{aligned} \tag{3.62}$$

and

$$\mathsf{m} = \sqrt{\frac{M}{n_5}} \frac{\mathsf{k} R_y s_p c_p}{(a_2 s_1 s_p - a_1 c_1 c_p)} \quad \in \mathbb{Z}\,, \qquad \mathsf{n} = \sqrt{\frac{M}{n_5}} \frac{\mathsf{k} R_y s_p c_p}{(a_2 c_1 c_p - a_1 s_1 s_p)} \quad \in \mathbb{Z}. \tag{3.63}$$

Here the integer numbers m and n again parametrise the angular momenta on $\mathbb{S}^3$. Moreover, these constraints imply the important identity

$$\frac{Q_p}{Q_1} = \frac{n_5\, \mathsf{m}\, \mathsf{n}}{\mathsf{k}^2 R_y^2} = \frac{\mathsf{p}}{\mathsf{k}} \frac{1}{R_y^2} \tag{3.64}$$

with $\mathsf{k}\,\mathsf{p} = n_5\, \mathsf{m}\, \mathsf{n}$ as before, which relates non-trivially the three charges sourcing the configuration. We now show that the above set of conditions leads to a further set of relations which enable us to rewrite the JMaRT solutions in terms of the three integers k, m and n, together with a single dimensionful scale set by R_y. A related but different calculation was carried out in [157]. Defining

$$b^2 = r_+^2 - r_-^2 \quad \Rightarrow \quad f = b^2 f_0 \quad \Rightarrow \quad \tilde{H}_1 = b^2 \Sigma_0\,, \tag{3.65}$$

the most useful relations are of the following form:

$$a_2^2 - a_1^2 = b^2(\mathsf{m}^2 - \mathsf{n}^2)\,,\; f + M\, s_p^2 = b^2\, h_y\,,\; f - M\, c_p^2 = b^2\, h_t\,, \tag{3.66}$$

$$M(c_1^2 + s_1^2) = b^2 \left(\mathsf{m}^2 + \mathsf{n}^2 - 1 + \frac{2\mathsf{k}^2 R_y^2}{n_5} \right),\; M(c_p^2 + s_p^2) = b^2 \left(\mathsf{m}^2 + \mathsf{n}^2 - 1 + \frac{2\mathsf{p}^2}{n_5 R_y^2} \right),$$

$$a_1^2 + r_+^2 + M s_1^2 = b^2 \left(\mathsf{n}^2 + \frac{\mathsf{k}^2 R_y^2}{n_5} \right),\; a_2^2 + r_+^2 + M s_1^2 = b^2 \left(\mathsf{m}^2 + \frac{\mathsf{k}^2 R_y^2}{n_5} \right),$$

$$\sqrt{M\, n_5}\, \big(a_2 c_1 s_p - a_1 s_1 c_p\big) = b^2 \left(\mathsf{m} \frac{\mathsf{p}}{R_y} + \mathsf{n}\, \mathsf{k} R_y \right),\; \sqrt{M\, n_5}\, \big(a_2 s_1 s_p - a_1 c_1 c_p\big) = b^2\, \mathsf{n}\, \Delta\,,$$

$$\sqrt{M\, n_5}\, \big(a_2 s_1 c_p - a_1 c_1 s_p\big) = b^2 \left(\mathsf{n} \frac{\mathsf{p}}{R_y} + \mathsf{m}\, \mathsf{k} R_y \right),\; \sqrt{M\, n_5}\, \big(a_2 c_1 c_p - a_1 s_1 s_p\big) = b^2\, \mathsf{m}\, \Delta\,,$$

where we have defined

$$\Sigma_0 = \sinh^2 \rho + (\mathsf{m}^2 - \mathsf{n}^2) \cos^2 \theta + \mathsf{n}^2 + \frac{\mathsf{k}^2 R_y^2}{n_5}\,, \tag{3.67}$$

$$\Delta = \sqrt{n_5(\mathsf{m}^2 + \mathsf{n}^2 - 1) + \mathsf{k}^2 R_y^2 + \frac{\mathsf{p}^2}{R_y^2}}\,, \tag{3.68}$$

such that Σ_0 is the same quantity as in previous sections. Finally, we have

$$\frac{Q_1}{b^2} = \frac{\mathsf{k} R_y}{n_5}\, \Delta\,, \qquad \frac{Q_p}{b^2} = \frac{\mathsf{p}}{n_5 R_y}\, \Delta\,. \tag{3.69}$$

We note that for the metric and the B-field we do not need the individual charges Q_1 and Q_p, but only the ratios (3.69). By using these formulas, the b^2 factor cancels out completely, and we finally obtain the six-dimensional fields

$$
\begin{aligned}
ds^2 \;=\; & n_5(d\theta^2 + d\rho^2) + \frac{1}{\Sigma_0}\Bigg[-\left(\sinh^2\!\rho + (\mathsf{m}^2 - \mathsf{n}^2)\cos^2\theta + 1 - \mathsf{m}^2 - \frac{\mathsf{p}^2}{n_5 R_y^2}\right) dt^2 \\
& + \left(\sinh^2\!\rho + (\mathsf{m}^2 - \mathsf{n}^2)\cos^2\theta + \mathsf{n}^2 + \frac{\mathsf{p}^2}{n_5 R_y^2}\right) dy^2 - 2\frac{\mathsf{p}}{n_5 R_y}\Delta\, dt dy \\
& + \left(n_5 \sinh^2\!\rho + n_5 \mathsf{m}^2 + \mathsf{k}^2 R_y^2\right)\sin^2\theta\, d\phi^2 + \left(n_5 \sinh^2\!\rho + n_5 \mathsf{n}^2 + \mathsf{k}^2 R_y^2\right)\cos^2\theta\, d\psi^2 \\
& + 2\left(\mathsf{m}\Delta dt - \left(\mathsf{m}\frac{\mathsf{p}}{R_y} + \mathsf{n}\mathsf{k}R_y\right)dy\right)\sin^2\theta d\phi - 2\left(\mathsf{n}\Delta dt - \left(\mathsf{n}\frac{\mathsf{p}}{R_y} + \mathsf{m}\mathsf{k}R_y\right)dy\right)\cos^2\theta\, d\psi\Bigg], \\
B \;=\; & \frac{1}{\Sigma_0}\Bigg[-\frac{\mathsf{k}R_y}{n_5}\Delta dt \wedge dy + \left(n_5 \sinh^2\!\rho + n_5\, \mathsf{m}^2 + \mathsf{k}^2 R_y^2\right)\cos^2\!\theta\, d\phi \wedge d\psi \\
& + \left(\mathsf{m}\Delta dt - \left(\mathsf{m}\frac{\mathsf{p}}{R_y} + \mathsf{n}\mathsf{k}R_y\right)dy\right)\wedge \cos^2\theta d\psi - \left(\mathsf{n}\Delta dt - \left(\mathsf{n}\frac{\mathsf{p}}{R_y} + \mathsf{m}\mathsf{k}R_y\right)dy\right)\wedge \sin^2\theta d\phi\Bigg].
\end{aligned}
\tag{3.70}
$$

This is exactly the geometry we get from the null-gauge construction studied in the previous sections when inserting the parametrisation (3.50) for the l_i, r_i gauging parameters in Eqs. (2.227).

It is worth discussing some interesting facts about the expressions we have presented in (3.70). First, we note the trivial symmetry associated to exchanging the two $\mathbb{S}^3$ angular momenta. This corresponds to the re-labelling $\mathsf{m} \leftrightarrow \mathsf{n}$ and $\phi \leftrightarrow -\psi$, which must be accompanied by the shift $\theta \to \pi/2 - \theta$. On the other hand, we note that while the usual JMaRT geometry is obtained by replacing $\mathsf{p} = n_5\mathsf{mn}/\mathsf{k}$ in the expressions in (3.70), here we have chosen a slightly more general form by keeping p explicit. As a result, we easily find a symmetry that corresponds to exchanging $\mathsf{k} \leftrightarrow \mathsf{p}$ and $R_y \to 1/R_y$, which we have identified above as T-duality. At the classical level, we can now see that this operation is equivalent the well-known Buscher rules [158], where $g_{yy} \to 1/g_{yy}$, $g_{ty} \to B_{ty}/g_{yy}$, etc.

Importantly, by keeping p explicit in (3.70) we have presented expressions that are valid even for solutions where $\mathsf{k} = 0$. As will be reviewed below, this includes the limit associated to the BPS and non-BPS two-charge NS5-P configurations.

3.4.2 The dilaton

As described above, the JMaRT dilaton is of the form (3.59). The only coordinate-dependent part of this expression corresponds to Σ_0 as defined in (3.67), where we have used (3.65). This matches exactly with the expression obtained in Section 2.7, see Eq. (2.229), by considering the supergravity equations of motion, which provide the dilaton up to a multiplicative constant. The matching with the JMaRT backgrounds thus gives a criteria for choosing this constant appropriately: it is given by n_5/b^2, i.e.

$$e^{2\Phi} = \frac{n_5}{b^2\Sigma_0}. \tag{3.71}$$

In order to make the expression for the dilaton more transparent we proceed as follows. First, we introduce the canonical expressions for the charges

$$Q_1 = n_1 \frac{g_s^2}{V_4}, \qquad Q_p = \frac{n_p}{R_y^2}\frac{g_s^2}{V_4}, \tag{3.72}$$

where V_4 is the volume of the internal $\mathbb{T}^4$, while n_1 is the number of fundamental string sources and n_p is the integer momentum charge. We observe that the key property (3.64) is equivalent to

$$\frac{\mathsf{k}R_y}{Q_1} = \frac{\mathsf{p}}{Q_p R_y}. \tag{3.73}$$

This ties in nicely with the fact that, as discussed above, T-duality interchanges $Q_1 \leftrightarrow Q_p$ and $\mathsf{k}R_y \leftrightarrow \mathsf{p}/R_y$. It also justifies referring to p as being related to the momentum charge as we did in previous sections. We have seen that the parameters m and n are associated to the angular momenta of the geometry. Here we find in (3.73) that p and k relate to the momentum and F1 winding charges of the black hole microstate in question along the asymptotic y-circle.

Furthermore, while the b^2 factor is irrelevant for writing down the metric and B-field, it does appear when computing the dilaton. This means that we need to work with the individual charges Q_1 and Q_p as opposed to the ratios. By making use of Eqs. (3.69) we obtain two equivalent expressions for the dilaton, namely

$$e^{2\Phi} = \frac{\Delta}{\Sigma_0}\frac{\mathsf{k}R_y}{Q_1} = \frac{\Delta}{\Sigma_0}\frac{\mathsf{p}/R_y}{Q_p}, \tag{3.74}$$

where Δ was defined in (3.68). This shows that for the dilaton the Buscher rule $\Phi \to \Phi - \frac{1}{2}\log g_{yy}$ is once again equivalent to the simultaneous replacements $Q_1 \leftrightarrow Q_p$ and $\mathsf{k}R_y \leftrightarrow \mathsf{p}/R_y$ since the constant prefactor in front of Σ_0 is invariant by itself.

3.4.3 JMaRT uniqueness

Having rewritten the NS5-decoupled JMaRT solutions in the form given in Eq. (3.70), we observed that these supergravity fields are exactly those we obtained from the null-gauge construction studied in the previous sections when inserting the parametrisation (3.29)–(3.32) for the l_i, r_i gauging parameters in the general solutions in Eq. (2.227).

We have thus shown that we are able to reproduce the full family of supergravity backgrounds collectively denoted as JMaRT. We now argue that these solutions exhaust the full set of consistent null-gauged models considered in this chapter, by scrutinising the allowed ranges of the parameters and identifying physically equivalent solutions.

At first sight, by looking at the metric, B-field and dilaton given in Eqs. (3.70) and (3.74), one would expect that $\mathsf{k}, \mathsf{m}, \mathsf{n}$ and p could take any integer value. Additionally, in writing these expressions we have fixed an extra degree of freedom by choosing a positive sign for Δ. However, this parameter space is constrained. In the general case, the parameter p is fixed in terms of the other three as in Eq. (3.33), though we will see below that in the limit in which the fundamental string charge Q_1 vanishes, its value becomes arbitrary. On the other hand, choosing the opposite sign for Δ simply results in an equivalent time-reversed configuration. This leaves us with arbitrary m, n and k (for negative k, the orbifold parameter is identified with its absolute value). Changing the sign of m, n or k can be compensated by choosing the orientation of the circle coordinate y or the $\mathbb{S}^3$ angles ϕ and ψ, respectively. Therefore we can take all of $\mathsf{m}, \mathsf{n}, \mathsf{k}$ to be non-negative. In addition, due to the $\mathsf{m} \leftrightarrow \mathsf{n}$ symmetry described below (3.70), we are free to restrict to $\mathsf{m} \geq \mathsf{n}$. Finally, based on spectral flow considerations we have argued in Section 3.2 that we must restrict to angular momenta such that $\mathsf{m} \pm \mathsf{n}$ are odd, see the discussion below Eq. (3.28). This clarifies the discussion about spin structures in [37]. This excludes $\mathsf{m} = \mathsf{n}$ and so we conclude that the set of inequivalent configurations is given by

$$\mathsf{k} \geq 0\,, \; \mathsf{m} > \mathsf{n} \geq 0, \qquad \mathsf{m} \pm \mathsf{n} \in 2\mathbb{Z}+1, \tag{3.75}$$

which is precisely the principal range of values considered in [37]. As we shall discuss below, for $\mathsf{k} = 0$ we take the limit such that one of the angular momenta vanishes and p is generically kept non-zero and finite.

3.4.4 Two-charge limits and novel non-BPS NS5-P solutions

The expressions (3.70) and (3.74) we have obtained for the metric, B-field and dilaton generated in the classical limit of the null-gauged models match exactly the solutions given in [37], which have $\mathsf{k} > 0$ (and where $\mathsf{p} = n_5 \mathsf{m}\mathsf{n}/\mathsf{k}$). However, and as discussed above, they are presented in a form that is slightly more general and can be used to access somewhat delicate limits. In particular, we now examine two-charge limits.

There are two such limits that we can access. The first of these corresponds to NS5-F1 solutions, obtained by setting $Q_p = 0$, which were analysed in [37]. As shown by the identity (3.73), in order to keep Q_1 finite and arbitrary we need to do this carefully. More precisely, we also need to take the limit $\mathsf{p} \to 0$ in such a way that the ratio $\mathsf{p}/(R_y Q_p) = \mathsf{k} R_y / Q_1$ is finite. An analogous conclusion for taking $\mathsf{n} \to 0$ with $\mathsf{p}/\mathsf{n} = n_5 \mathsf{m}/\mathsf{k}$ fixed is obtained by considering (3.33).

On the other hand, we can now similarly access a different limit leading to novel non-BPS NS5-P configurations. In this case, we take $Q_1 = \mathsf{k} = \mathsf{n} = 0$ while keeping the ratios $\mathsf{k}R_y/Q_1 = \mathsf{p}/(R_y Q_p)$ and $\mathsf{k}/\mathsf{n} = n_5\mathsf{m}/\mathsf{p}$ fixed. This allows Q_p to take arbitrary values as needed. To the best of our knowledge, the metric and B-field for the non-BPS NS5-P solutions have not been presented in the literature. They take the following form:

$$\begin{aligned} ds^2 &= \frac{1}{\Sigma_0}\Bigg[-\left(\sinh^2\!\rho + \mathsf{m}^2(\cos^2\theta - 1) + 1 - \frac{\mathsf{p}^2}{n_5 R_y^2}\right) dt^2 - 2\,\frac{\mathsf{p}}{n_5 R_y}\,\Delta\, dt dy \\ &\quad + \left(\sinh^2\!\rho + \mathsf{m}^2\cos^2\theta + \frac{\mathsf{p}^2}{n_5 R_y^2}\right) dy^2 + 2\mathsf{m}\left(\Delta\, dt - \frac{\mathsf{p}}{R_y}\, dy\right)\sin^2\theta\, d\phi \\ &\quad + n_5\left(\sinh^2\!\rho + \mathsf{m}^2\right)\sin^2\theta\, d\phi^2 + n_5 \sinh^2\!\rho\, \cos^2\theta\, d\psi^2\Bigg] + n_5(d\theta^2 + d\rho^2)\,, \\ B &= \frac{n_5}{4\Sigma_0}\Big[\mathsf{m}^2 - 1 + \cosh(2\rho)\Big]\cos(2\theta)\, d\phi\wedge d\psi + \frac{\mathsf{m}}{\Sigma_0}\left(\Delta\, dt - \mathsf{m}\frac{\mathsf{p}}{R_y}\, dy\right)\wedge\cos^2\theta\, d\psi\,, \end{aligned} \tag{3.76}$$

with the dilaton given by the second expression in (3.74), and where we now have

$$\Sigma_0 = \sinh^2\rho + \mathsf{m}^2\cos^2\theta\,, \qquad \Delta = \sqrt{n_5(\mathsf{m}^2 - 1) + \frac{\mathsf{p}^2}{R_y^2}}\,. \tag{3.77}$$

Recall that, as discussed around Eq. (3.52), these solutions involve a fivebrane source at $\rho = 0, \theta = \pi/2$ and a $\mathbb{Z}_{|m|}$ orbifold singularity at $\rho = 0, \theta \neq \pi/2$.

Finally, we can restrict to the BPS cases by setting $\mathsf{m} = 1$, as indicated by the supersymmetry conditions (3.13) and (3.14). Thus, in this limit we find $\Delta = \mathsf{k}R_y$ for the BPS NS5-F1 configuration, and $\Delta = \mathsf{p}/R_y$ for the NS5-P one. In the former case, the first expression for the dilaton in (3.74) then gives

$$e^{2\Phi}|_{\mathrm{NS5-F1}} = \frac{1}{Q_1}\,\frac{\mathsf{k}^2 R_y^2}{\sinh^2\rho + \cos^2\theta + \mathsf{k}^2 R_y^2/n_5}\,, \tag{3.78}$$

which coincides with that of [51], Eq. (4.13), while for the latter the alternative expression in (3.74) yields

$$e^{2\Phi}|_{\mathrm{NS5-P}} = \frac{1}{Q_p}\,\frac{(\mathsf{p}/R_y)^2}{\sinh^2\rho + \cos^2\theta}\,, \tag{3.79}$$

which coincides with that of [51], Eq. (4.2). One can check that in both cases the metric and B-field match as well.

3.4.5 AdS_3 limit and holography

The AdS limit of the geometries under consideration is obtained by taking the large R_y limit, while keeping the charge Q_1 fixed. This describes the region of small radial distances (as compared with Q_1 and R_y). The energy and momenta ER_y and P_yR_y also stay fixed, such that the coordinates

$$\tilde{t} = t/R_y \,,\; \tilde{y} = y/R_y \tag{3.80}$$

are better suited for this region. The six-dimensional metric (3.70) then takes the form of an orbifolded $\mathrm{AdS}_3 \times \mathbb{S}^3$, namely

$$\begin{aligned} ds^2 \;=\; n_5 \Big[& -\frac{1}{\mathsf{k}^2}\cosh^2\rho \, d\tilde{t}^2 + \frac{1}{\mathsf{k}^2}\sinh^2\rho \, d\tilde{y}^2 + d\rho^2 \\ & + d\theta^2 + \sin^2\theta\left(d\phi - \frac{\mathsf{n}}{\mathsf{k}}d\tilde{t} + \frac{\mathsf{m}}{\mathsf{k}}d\tilde{y}\right)^2 + \cos^2\theta\left(d\psi + \frac{\mathsf{m}}{\mathsf{k}}d\tilde{t} - \frac{\mathsf{n}}{\mathsf{k}}d\tilde{y}\right)^2\Big] . \end{aligned} \tag{3.81}$$

The orbifold singularity structure near $\tilde{y} = 0$ depends on the common divisors between $\mathsf{m}, \mathsf{n}, \mathsf{k}$ and is described in [37, 94, 52]. By means of the large gauge transformation

$$\tilde{\psi} \;=\; \psi + \frac{\mathsf{m}}{\mathsf{k}}\tilde{t} - \frac{\mathsf{n}}{\mathsf{k}}\tilde{y} \,, \qquad \tilde{\phi} \;=\; \phi - \frac{\mathsf{n}}{\mathsf{k}}\tilde{t} + \frac{\mathsf{m}}{\mathsf{k}}\tilde{y} \,, \tag{3.82}$$

one can formally re-absorb the contributions from the angular momenta, that is, the terms depending on m and n. This is related to the general holographic description of such configurations. They are interpreted as excited states in the holographic symmetric orbifold CFT which can be constructed by considering $n_1 n_5/\mathsf{k}$ identical strands of length k in their NS vacuum state and performing left-right asymmetric fractional spectral flow [94]. The spectral flow charges are of the form

$$\alpha = \frac{\mathsf{m}+\mathsf{n}}{\mathsf{k}} \,, \qquad \bar{\alpha} = \frac{\mathsf{m}-\mathsf{n}}{\mathsf{k}} \,, \tag{3.83}$$

which matches the intuition derived from (3.82) and provides yet another interpretation for the gauging parameters l_2 and r_2. Note that this is distinct from the *worldsheet* spectral flow that was used in Section 3.2. Interestingly, within this description the momentum per strand is given by mn/k and must be an integer number,

$$\frac{\mathsf{mn}}{\mathsf{k}} \in \mathbb{Z} \,, \tag{3.84}$$

which is a slightly more restrictive condition than the quantisation of p discussed above.

Furthermore, we also note that in this AdS limit there seems to be no particular issues with the solutions with even $\mathsf{m} \pm \mathsf{n}$, see the discussion around Eq. (3.29). In particular, the case $\mathsf{m} = \mathsf{n} = 0$ takes us back to the global AdS_3 vacuum. The present perspective shows that the cases that extend consistently to the full linear dilaton geometry, namely $\mathsf{m} \pm \mathsf{n} \in 2\mathbb{Z}+1$, belong to the RR sector (in the covering space) of the holographic

CFT, while those that do not are characterised by having spectral flow charges (2.89) with even numerators, such that they correspond to NSNS states.

Moreover, in this AdS_3 limit the dilaton becomes constant, as can be seen from (3.74) since $\Delta \to \mathsf{k}R_y$ and $\Sigma_0 \to \mathsf{k}^2 R_y^2/n_5$. In other words, the rescaled harmonic function $\tilde{H}_1$ associated to the fundamental string charges approaches the constant value Q_1, see Eq. (3.69). In terms of the actual harmonic function H_1, this roughly corresponds to the usual *dropping the "1+"* term, as is usual in such decoupling limits (see e.g. [136]).

In terms of the null-gauged description, there is an intuitive way of understanding this $\mathrm{AdS}_3 \times \mathbb{S}^3$ limit. Indeed, the upstairs model already contains an $\mathrm{SL}(2,\mathbb{R}) \times SU(2)$ factor, complemented by the novel $\mathbb{R}_t \times \mathbb{S}^1_y$ factor. For large R_y, the gauging parameters associated to the former are $l_{1,2} \sim r_{1,2} \sim \mathcal{O}(1)$, while those corresponding to the latter grow parametrically large as $l_{3,4} \sim r_{3,4} \sim \mathcal{O}(R_y)$. Thus, we are mostly gauging away the extra directions t and y.

3.4.6 From AdS back to the linear dilaton background

At this point, it is interesting to go back to the intuition developed within the sigma model description presented at the beginning of Section 2.7.1. There, we argued that, after integrating out the gauge fields, the gauging procedure induces a deformation given by including an additional term of the form $\mathcal{J}\bar{\mathcal{J}}/\Sigma$ to the action, see Eq. (2.198). We can simplify the discussion by working in the $t = y = 0$ gauge, such that the currents $\mathcal{J}$ and $\bar{\mathcal{J}}$ are nothing but linear combinations of the diagonal currents of the $\mathrm{SL}(2,\mathbb{R})$ and $SU(2)$ WZW models. In the AdS limit, we have seen that the coefficient $1/\Sigma$ approaches a constant value, such that the induced contribution becomes $\mathcal{J}\bar{\mathcal{J}}$, giving a simple marginal deformation of the worldsheet theory.

The situation is to be contrasted with that of [143, 144, 145]. There, the authors make use of the null-gauging formalism to introduce a $J\bar{J}$ deformation for the AdS_3 worldsheet theory, the crucial difference being that the current under consideration corresponds to the J^- instead of J^3. For related work, see [159]. Based on [110], this procedure is interpreted as the dual of the so-called single-trace $T\bar{T}$ irrelevant deformation of the holographic CFT. Such deformation triggers a controlled flow to the UV, which is realised in the dual geometry by effectively *reinserting the "1+" term* in the harmonic function associated to the fundamental string sources. This produces an asymptotically linear dilaton geometry, i.e. an NS5-decoupled background, such that the above UV flow is thought of as leading to a realisation of little string theory.

The parallel to our construction can thus be made more general. As described above, for large R_y we know that Σ becomes constant, and our $\mathcal{J}\bar{\mathcal{J}}$ deformation on the worldsheet made up of J^3 and K^3 (together with their antiholomorphic counterparts) has a much less dramatic effect, as it produces a sort of large gauge transformation which, however, does not further modify the AdS_3 asymptotics, where the dilaton stays constant. On the other hand, when moving away from the AdS limit by keeping R_y finite one recovers the full non-trivial coordinate dependence of the function Σ, which sits at the denominator of various terms in the supergravity fields. This modifies the effect of the $\mathcal{J}\bar{\mathcal{J}}/\Sigma$, which now *does* take us back to the full asymptotically linear dilaton background described by Eqs. (3.70) and (3.74). This effect has also recently been observed in a larger class of solutions in [54].

3.5 Discussion

In this chapter we have analysed all consistent backgrounds within a general class of null-gauged WZW models. We showed that the (NS5-decoupled) JMaRT family, and limits thereof, are the unique supergravity backgrounds that arise in these models. We also showed that the metric and B-field can be written explicitly in terms of the integers $\mathsf{k}, \mathsf{m}, \mathsf{n}, n_5$ and the modulus R_y, while for the dilaton one needs further include the ratio n_1/V_4 (or n_p/V_4).

Our analysis makes the connection between the worldsheet and geometric descriptions quite explicit. In the supergravity solutions, imposing absence of CTCs implies $l_3 = r_3$, which excludes horizons. The converse statements are also true: excluding horizons implies $l_3 = r_3$, which excludes CTCs. On the other hand, the condition $l_3 = r_3$ is necessary but not sufficient for smoothness (up to orbifold singularities), which further requires the quantization of $\mathsf{k}, \mathsf{m}, \mathsf{n}$. This is consistent with the interpretation of the allowed configurations as a family of black hole microstates.

At the level of the worldsheet CFT, we have shown that a consistent spectrum is obtained if and only if the gauging parameters are given in terms of three independent integers $\mathsf{k}, \mathsf{m}, \mathsf{n}$ together with n_5 and R_y, as in (3.29)–(3.33). We explicitly rewrote the JMaRT metric and B-field in terms of these quantities, which enabled us to completely bypass the usual, somewhat cumbersome, supergravity parametrisation in Eqs. (3.57), (3.58).

Our parametrisation also provides a clear and important physical understanding of the quantity $\mathsf{p} = n_5\mathsf{mn}/\mathsf{k}$. In the AdS_3 decoupling limit, the quantity mn/k has previously been interpreted as being the momentum per strand in the holographically dual symmetric product orbifold CFT [94]. We have uncovered the direct role of p in the (asymptotically linear dilaton) supergravity solutions, as being the quantity that is T-dual to k, where the T-duality is performed along the y circle.

As we mentioned in the Introduction, one of the motivations of our systematic analysis was the possibility of finding new backgrounds. Although our uniqueness proof means that the set of models we analysed does not have more general backgrounds than the JMaRT family, we have exhibited a novel sub-family of two-charge non-BPS NS5-P backgrounds that arise from a non-trivial limit, see Eqs. (3.76) and (3.77). We observed that in the core of the solutions but away from the fivebrane source, the solutions involve a $\mathbb{Z}_m$ orbifold singularity. To our knowledge, these solutions have not appeared before in the literature.

We expect that our results will be useful in analysing generalisations of the models studied here, either by changing the currents being gauged to include non-Cartan generators of the non-Abelian factors of the upstairs group, or by changing the upstairs group, or both. Our systematic approach should enable generalisations to be investigated in a similar way. For instance, there are multi-centre non-BPS generalizations of the JMaRT family [160, 161, 162].

Besides this, within the models considered in this work there remain several unanswered questions. For instance, since we have control over these theories exactly in α', there are many interesting correlation functions that can be computed. We intend to report an analysis of such correlators in the near future.

The results we have obtained, and the possibilities they open up for future work, offer the prospect of improving our understanding of little string theory and the corresponding non-AdS holography. Furthermore, it is tempting to wonder about extending some of these ideas beyond the fivebrane decoupling limit into the full asymptotically flat regime.

Having an exact worldsheet description of heavy pure states, far from the vacuum of the theory, is rare and valuable. Such models allow us to study aspects of black hole microstates that are smeared out in supergravity, and so cannot be studied with supergravity techniques. This offers the tantalising prospect of obtaining a quantitative understanding of the microscopic degrees of freedom of black holes.

Chapter 4

Worldsheet Correlators in Black Hole Microstates

String Theory provides a microscopic description of black holes as being bound states of strings and branes with an exponentially large number of internal microstates [19]. Amongst these microstates, there are coherent pure states, large families of which have been shown to be well-described by smooth and horizonless supergravity solutions, see e.g. [65, 66, 135, 92, 137, 138, 163]. Upon taking an appropriate AdS decoupling limit, these solutions are proposed to correspond to specific families of pure states in the holographically dual CFT (HCFT); precision holography has provided sharp evidence supporting this correspondence [164, 72, 165, 139, 166].

While supergravity constructions provide valuable insight into the structure of black hole microstates, it is natural to expect that string-theoretic physics beyond supergravity will be necessary to obtain a complete description of black hole microstructure. A fruitful arena in which to investigate such stringy physics is provided by bound states of NS5 branes carrying fundamental string (F1) and/or momentum charge (P). More specifically, we work in Type IIB compactified on $\mathbb{S}^1 \times \mathbb{T}^4$, with n_5 NS5 branes wrapped on $\mathbb{S}^1 \times \mathbb{T}^4$, n_1 units of F1 winding on $\mathbb{S}^1$, and n_P units of momentum charge along $\mathbb{S}^1$.

Upon taking the fivebrane decoupling limit, one obtains asymptotically linear dilaton configurations, which are holographically dual to (doubly scaled) Little String Theory [120, 115]. In an appropriate region of the parameter space, there is an AdS_3 regime in the IR, and one can take a further AdS_3 decoupling limit [16]. Upon doing so, one obtains the well-studied NS5-F1 instance of $\mathrm{AdS}_3/\mathrm{CFT}_2$ holography [78, 110].

The NSNS vacuum of the holographic CFT corresponds to the global $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ background, whose worldsheet theory involves an SL(2,$\mathbb{R}$)$\times$SU(2) Wess-Zumino-Witten (WZW) model [109, 106, 101, 102, 103]. In recent work, a family of gauged

WZW models involving the same Lie groups has been constructed and studied, providing an exact worldsheet description of a set of NS5-F1-P black hole microstates [51, 52, 53, 54, 1].

Processes in which light probes interact with a heavy background such as a black hole or a black hole microstate give rise to interesting dynamical observables, in particular mixed heavy-light (HL) correlators. Such correlators have been previously studied in holographic systems, see e.g. [167, 168, 169, 170, 171]. In the NS5-F1 system, there is a locus in moduli space at which the holographic CFT is conjectured to be an $\mathcal{N} = (4,4)$ symmetric product orbifold CFT with target space $\left(\mathbb{T}^4\right)^N/S_N$, where $N = n_1 n_5$. There is now a substantial body of evidence for this conjecture, see e.g. [164, 72, 165, 139, 166, 172, 113, 99, 100]. For recent discussions of holography in related systems, see [90, 173, 174].

For instance, heavy-light-light-heavy (HLLH) four-point functions have been computed in the supergravity approximation and/or in the symmetric product orbifold CFT, for particular sets of heavy and light operators [167, 170, 171]. Having solvable worldsheet models associated to black hole microstates means we can go much further by taking into account α' corrections [51]. Given a worldsheet model describing string dynamics on a heavy background, the relevant quantities correspond to integrated correlators of light operators in the worldsheet vacuum.

Worldsheet correlators in global AdS_3 were first studied in [103], building in part on [175, 106], and the role of the vertex operators associated with spectrally flowed representations was highlighted. Further studies include [176, 177, 178, 38, 179]. The spectrum of chiral primaries and their three-point functions in global $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ were computed in [172, 113, 99, 100], and shown to match those of the symmetric product orbifold CFT, as studied in [180, 97].

The supergravity backgrounds we consider are known as NS5-F1 circular supertubes and spectral flows thereof [132, 124, 125, 37, 96, 94]. This includes non-BPS spectrally flowed supertubes, known as the JMaRT solutions, after the authors of [37]. The associated worldsheet models are null-gauged WZW models, where before gauging one considers a (10+2)-dimensional target space $AdS_3 \times \mathbb{S}^3 \times \mathbb{R}_t \times \mathbb{S}^1_y \times \mathbb{T}^4$, and after gauging one obtains a (9+1)-dimensional spectrally flowed supertube solution. In the asymptotic linear dilaton region, the gauging is concentrated mostly in the time and angular directions of SL(2,$\mathbb{R}$), while in the IR AdS_3 region, the gauging is concentrated mostly along the t and y directions.

These coset models can also be thought of as marginal current-current deformations of the worldsheet theory for strings in AdS_3. These are instances of a larger class of deformations that undo the decoupling limit with respect to the F1 harmonic function, i.e. they *"add back the 1+"* in that function, leading to linear dilaton asymptotics; see e.g. [54]. At the level of the dual field theory, a closely related procedure has been argued to correspond to the so-called single-trace $T\bar{T}$ irrelevant deformation of the original holographic CFT [145, 143], flowing towards a non-local Little String Theory.

To study string correlators in these highly excited backgrounds, we first compute a large set of physical vertex operators, in both NSNS and RR sectors, building on [52, 54]. These describe linearized perturbations of the background configurations. We focus primarily on coset states in discrete series representations, including worldsheet spectral flow, that are dual to chiral primary operator excitations in the HCFT. When the background is BPS, a subset of these are BPS fluctuations.

The currents being gauged in these cosets are linear combinations of the Cartan generators of the symmetry algebra. Therefore the "m-basis" for vertex operators, in which the actions of these currents are diagonalized, is the natural framework to use. In the IR AdS_3 limit, we describe how these operators are related to their global $\mathrm{AdS}_3 \times \mathbb{S}^3$ counterparts.

We then compute a large set of correlators in the AdS_3 limit. It is well known that in worldsheet models of global AdS_3, one can define an "x" variable that corresponds to the local coordinate of the holographic CFT [110]. One of the main novelties of our approach is the identification of the analogous x variable in the coset models we study. This identification requires some care due to the gauging. Indeed, the construction of [110] breaks down, because the $SL(2,\mathbb{R})$ raising and lowering operators do not commute with the BRST charge. A considerable amount of interesting physics follows from this step. It leads, for instance, to the combination of seemingly simple m-basis two-point functions into spacetime-local x-basis correlators with highly non-trivial x-dependence.

Our first main result is of a family of HLLH correlators, for which we obtain fully explicit expressions. In doing so, we show that these correlators assume a remarkably simple structure when written in terms of a covering space related only to the heavy states. From this observation, we obtain our second main result: a closed-form expression for a set of HL worldsheet correlators with an arbitrary number n of massless insertions, in terms of a correlator consisting of n light insertions in global $\mathrm{AdS}_3 \times \mathbb{S}^3$. For $n = 3$ this result can be made completely explicit, and we present a particular example in full detail. This constitutes the first correlator in the literature involving three light worldsheet vertices on a black hole microstate background, dual to a heavy-light five-point function of the holographic CFT.

A priori, our worldsheet correlators give predictions for correlators of the dual holographic CFT at strong coupling. Generically, four-point (and higher-point) correlators are not protected across moduli space, however, a specific set of HLLH correlators have been shown to precisely agree between supergravity and the symmetric product orbifold CFT [167]. Similarly, the emission spectrum and rate for the unitary analog of

Hawking radiation from the JMaRT solutions agrees between supergravity and symmetric product orbifold CFT [140, 93, 141, 94]. Thus it is natural to investigate more generally which HL correlators are protected (at large N) between worldsheet and symmetric product orbifold CFT, and which are not.

We carry out this comparison for three sub-families of our worldsheet correlators. Firstly, we compare various sets of HLLH correlators to the symmetric product orbifold CFT, finding exact agreement in all cases for which the orbifold CFT correlator is available in the literature. Importantly, this matching holds at leading order in large N, but exactly in α'. This comparison includes a substantial generalization of the supergravity and holographic CFT correlators computed in [167]. Our comparisons notably include an example in which the light operators in the symmetric orbifold CFT are twist-two. In this case, and as shown recently in [181, 182], the Lunin-Mathur covering map used in the symmetric orbifold computation is different to the one appearing in the worldsheet computation, making the comparison highly non-trivial. Remarkably, both results agree exactly in the large N limit.

Secondly, we compute the five-point HLLLH symmetric orbifold CFT correlator corresponding to the three-point worldsheet correlator mentioned above, and also find exact agreement.

Finally, we compute the analogue of the Hawking radiation rate for the JMaRT solutions. Once again, we find perfect agreement with the dual symmetric product orbifold CFT, extending the supergravity and holographic CFT results of [140, 93, 141, 94].

A likely explanation for this remarkable agreement is that the heavy states we consider are quite special. Specifically, the heavy backgrounds are related to the global $\mathrm{AdS}_3 \times \mathbb{S}^3$ vacuum via orbifolding and fractional spectral flow [96, 94]. This fact also underlies our general formula for the HL correlators with n light insertions. When $n > 3$, we do not expect these HL correlators to be generically protected across moduli space; we shall discuss this in detail in due course.

The structure of the chapter is as follows. In Section 4.1 we construct a large set of vertex operators of the worldsheet cosets we study, both in the NS and in the R sectors. We then examine their AdS_3 limit and relate these vertices to those constructed in Sec. 2.5.1. In Sections 4.2 and 4.3, we present our main results. We identify the "x" variable dual to the local coordinate of the holographic CFT, and obtain an extensive set of novel HLLH correlators, including massless insertions with arbitrary spacetime weights and charges. The final results are presented in Eqs. (4.81) and (4.93). We then compare a subset of these results to the symmetric product orbifold CFT, finding exact agreement for all correlators available in the literature. We present a closed formula for a large class of worldsheet correlators with an arbitrary number of massless insertions, Eq. (4.94). We compute a five-point correlator in the symmetric orbifold CFT and find agreement with our general worldsheet formula. Finally, we compute the amplitude describing the unitary analogue of Hawking radiation for the JMaRT microstates. We discuss our results in Section 4.4.

4.1 Null-gauged description and worldsheet spectrum

We now proceed to describe massless excitations in the worldsheet theories associated with the heavy backgrounds we study. We describe in detail how to obtain the low-lying physical states via BRST quantization in these null-gauged models, in both the NSNS and RR sectors. In the subset of the backgrounds that preserve some supersymmetry, we discuss the BPS light excitations.

In the full null-gauged models, these massless vertex operators describe linearized fluctuations around the full asymptotically linear dilaton solutions describing the heavy states, and so can be thought of as worldsheet representatives of light states belonging to the Little String Theory living on the NS5 branes.

Our main interest in this work will be computing correlators in the IR AdS_3 limit, in which we have reviewed the fact that the backgrounds are related to orbifolded $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ via a spacetime spectral flow large coordinate transformation. We describe how the AdS_3 limit can be taken on the worldsheet vertex operators. This leads to states that can be understood holographically, in the spacetime (fractionally) spectrally flowed frame defining the heavy background, as discussed around Eq. (2.89).

4.1.1 BRST quantization

We start by reviewing the quantization of the class of worldsheet coset models introduced in Section 2.7, which describe the propagation of superstrings in the JMaRT backgrounds and their (BPS and/or two-charge) limits [51, 52, 54, 1].

Before gauging, we have the WZW model associated to the (10+2)-dimensional group manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathbb{R}_t \times \mathrm{U}(1)_y \times \mathrm{U}(1)^4$ as introduced in (2.205). This is described simply by adding the extra time direction t and spatial circle y to the matter content employed in the previous section, together with the corresponding fermionic partners λ^t and λ^y. The latter are bosonized using a canonically normalised scalar H_6 as

$$\lambda_t = \frac{1}{2}\left(e^{iH_6} - e^{-iH_6}\right) , \qquad \lambda_y = \frac{1}{2}\left(e^{iH_6} + e^{-iH_6}\right) , \tag{4.1}$$

$$i\partial H_6 = 2\,\lambda^t \lambda^y , \qquad H_6^\dagger = -H_6 . \tag{4.2}$$

The holomorphic parts of their OPEs are

$$-t(z)t(w) \sim y(z)y(w) \sim -\frac{1}{2}\log(z-w)\,, \qquad -\lambda^t(z)\lambda^t(w) \sim \lambda^y(z)\lambda^y(w) \sim \frac{1}{2}\frac{1}{(z-w)}\,, \tag{4.3}$$

and they give additional free field contributions to the matter T and G in (2.135) and (2.136),

$$T_{(ty)} \;=\; \partial t\partial t - \partial y\partial y\,, \qquad G_{(ty)} \;=\; 2i\left(-\lambda^t\partial t + \lambda^y\partial y\right). \tag{4.4}$$

From now on, T and G will denote the stress tensor and supercurrent of the full model, including the above terms. Introducing the operators $\hat{P}_t = i\partial t$, $\hat{\bar{P}}_t = i\bar{\partial} t$, $\hat{P}_y = i\partial y$, and $\hat{\bar{P}}_y = i\bar{\partial} y$, we gauge the chiral null currents[42]

$$J \;=\; i\mathcal{J} \;=\; J^3 + l_2K^3 + l_3\hat{P}_t + l_4\hat{P}_y\,, \quad \bar{J} \;=\; i\bar{\mathcal{J}} \;=\; \bar{J}^3 + r_2\bar{K}^3 + r_3\hat{\bar{P}}_t + r_4\hat{\bar{P}}_y\,, \tag{4.5}$$

which are the quantum operator versions of the classical currents in Eq. (2.215). The supersymmetric partners of the currents J and $\bar{J}$ are given in Eq. (3.12).

To perform the null gauging, one introduces additional fermionic and bosonic first-order ghosts, denoted by $(\tilde{b},\tilde{c})$ and $(\tilde{\beta},\tilde{\gamma})$, with conformal weights $\Delta[\tilde{c}] = 0$ and $\Delta[\tilde{\gamma}] = 1/2$ [54]. The central charges $c_{\tilde{b}\tilde{c}} = -2$ and $c_{\tilde{\beta}\tilde{\gamma}} = -1$ cancel the additional matter contribution $c_{ty} = 3$. The $(\tilde{\beta},\tilde{\gamma})$ system has no background charge and is bosonized via

$$\tilde{\beta} = e^{-\tilde{\varphi}}\partial\tilde{\xi}\,, \qquad \tilde{\gamma} = \tilde{\eta}\, e^{\tilde{\varphi}}\,. \tag{4.6}$$

We shall momentarily introduce a modified BRST charge that imposes invariance under the action of the null currents (3.8) and their supersymmetric partners (3.12). Physical operators in 9+1 dimensions will be given by states of the ungauged (10+2)-dimensional WZW model that survive the gauging procedure [54]. Of course, and as we shall see shortly, the Virasoro conditions and the expressions of the BRST-exact states will be modified accordingly. We consider a set of mutually local operators before the gaugings, i.e. we perform the analog of the GSO projection in the (10+2)-dimensional model. We thereby obtain a tachyon-free spectrum in the gauged models.

Underlying this procedure is the fact that for the case of chiral null gaugings, the Polyakov-Wiegmann identity allows one to rewrite the gauged action of the *downstairs* model into a form identical to that of the *upstairs* model in terms of a new gauge-invariant variables [146, 118], see also [1]. This is achieved at the level of the path integral by means of a field redefinition with a Jacobian that is almost trivial except for a factor which, when exponentiated, gives rise to the additional ghost fields described above.

[42] The symbol J for the current operators should not be confused with the total SL(2,$\mathbb{R}$) spin that has appeared in Section 2.6.4. The meaning should be clear from the context.

Explicitly, physical operators in the coset model are defined by the cohomology classes of the BRST charge [54]

$$\mathcal{Q} = \oint dz \left[c \left(T + T_{\beta\gamma\bar{\beta}\bar{\gamma}} \right) + \gamma G + \tilde{c} J + \tilde{\gamma} \boldsymbol{\lambda} + \text{ghosts} \right] , \tag{4.7}$$

where the last two terms implement the null-gauging procedure. Whether the resulting spectrum is supersymmetric or not depends on whether some linear combination(s) of the following supercharges are BRST-invariant [54],

$$Q_\varepsilon = \oint dz \, e^{-(\varphi - \tilde{\varphi})/2} S_\varepsilon \, , \qquad S_\varepsilon = \exp\left(\frac{i}{2} \sum_{I=1}^{6} \varepsilon_I H_I \right) . \tag{4.8}$$

For these to be mutually local, we impose the analog of the GSO projection in (10+2) dimensions,

$$\prod_{I=1}^{6} \varepsilon_I = 1 \, , \tag{4.9}$$

where the sign choice is dictated by requiring compatibility with the (9+1)-d GSO projection (2.145) and our other conventions, as we shall see below. We shall discuss the conditions for spacetime supersymmetry after we have analyzed more general Ramond sector vertex operators, around Eq. (4.29). For now, we emphasize that only a subset of the backgrounds we consider preserve some spacetime supersymmetry.

4.1.2 The unflowed NS sector

We now analyze physical NS sector states of the gauged models, focusing on states with no spectral flow charges in SL(2,$\mathbb{R}$) or SU(2), and no winding charge ω_y around the y-circle. As usual, the lightest physical operators come with a single fermionic excitation on top of the tachyon state

$$\mathcal{T}_{j,m,j',m'} \;=\; e^{-\varphi} V_{j,m} V'_{j',m'} e^{i(-Et + P_y y)} \, . \tag{4.10}$$

Note that since t is a non-compact direction and $\omega_y = 0$, both E and P_y are identical on the left and on the right sectors. For massless states, the L_0 and $\bar{L}_0$ Virasoro constraints both read

$$0 \;=\; -\frac{j(j-1)}{n_5} + \frac{j'(j'+1)}{n_5} - \frac{1}{4}E^2 + \frac{1}{4}P_y^2 \, . \tag{4.11}$$

Moreover, operators are uncharged with respect to the null-currents $J, \bar{J}$ in (3.8) if and only if their quantum numbers are related by

$$0 = m + l_2 \, m' + \frac{l_3}{2} E + \frac{l_4}{2} P_y \, , \qquad 0 = \bar{m} + r_2 \, \bar{m}' + \frac{r_3}{2} E + \frac{r_4}{2} P_y \, . \tag{4.12}$$

We will work in the canonical "-1" picture for the φ ghost. On the other hand, the fact that $\tilde{\varphi}$ has background charge $Q_{\tilde{\varphi}} = 0$ allows us to build NS states directly at $\tilde{\varphi}$-picture zero. BRST-closed operators must then have a vanishing second-order pole in their OPE with the supercurrent G, and vanishing first-order pole in their OPE with the fermionic current $\boldsymbol{\lambda}$ given in (3.12).

As can be expected from the fact that the $\mathbb{T}^4$ is untouched by the gaugings, the simplest solutions are the 6D scalars

$$\mathcal{V}^i_{j,m,j',m'} = e^{-\varphi}\lambda^i V_{j,m} V'_{j',m'}\, e^{i(-Et+P_y y)}\,, \qquad i = 6,\ldots,9. \tag{4.13}$$

These are direct analogs of the global AdS_3 states defined in Eq. (2.148a). They were considered in detail in [52], and their energies were matched with those of the minimally coupled scalar perturbations on top of the JMaRT background as computed in supergravity.

The remaining massless vertex operators will constitute the beginning of the main new results of this work. They are slightly more involved to construct, due to the fact that their polarization lies in a direction in which the null currents act non-trivially. An important consequence is that now the raising/lowering operators $J_0^{\pm}$ and $K_0^{\pm}$ do not commute with the BRST charge $\mathcal{Q}$. So, unlike in global $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ as reviewed in Section 2.6, physical states need not have definite SL(2,$\mathbb{R}$) and SU(2) spins. They will, however, have definite projections $m, \bar{m}, m'$ and $\bar{m}'$, and also well-defined energy E and momentum P_y.

This situation is a consequence of the fact that the $AdS_3 \times \mathbb{S}^3$ isometries are absent in the asymptotically linear dilaton geometry. Nevertheless, these isometries are restored in the IR, by taking R_y large while keeping ER_y and P_yR_y fixed. In this regime, the vertex operators of the gauged models will reduce to the $AdS_3 \times \mathbb{S}^3$ expressions in Eqs. (2.148b) and (2.148c).

Let us consider a generic linear combination of NS sector vertex operators,

$$e^{-\varphi}\left[\left(c^r\psi^r V_{j,m-r}V'_{j',m'} + d^r\psi^r V_{j,m}V'_{j',m'-r}\right) + \left(c^t\lambda^t + c^y\lambda^y\right)V_{j,m}V'_{j',m'}\right]e^{i(-Et+P_y y)}, \tag{4.14}$$

where the notation mirrors that of the $AdS_3 \times \mathbb{S}^3$ expressions in Eq. (2.149); in particular, summation over $r = +1, -1, 0$ is implicit, with "0" corresponding to the "3" direction of the respective algebras. Of these eight degrees of freedom, two are removed by the conditions arising from the G and $\boldsymbol{\lambda}$ terms in the BRST charge, which respectively read

$$0 = mc^3 + (m-j)c^+ + (m+j)c^- + m'd^3 + (j'+m')d^+ + (j'-m')d^- + c^t\frac{E}{2} + c^y\frac{P_y}{2}, \tag{4.15}$$

and

$$0 = n_5\left(-c^3 + l_2 d^3\right) - l_3 c^t + l_4 c^y. \tag{4.16}$$

This leaves six states, of which two turn out to be BRST exact. The first exact state comes, as usual, from the action of G on the tachyon operator (4.10), while the second one has no global AdS_3 counterpart and appears due to the action of $\boldsymbol{\lambda}$ on the same state. Their explicit expressions are

$$\begin{aligned}\Phi_G = e^{-\varphi} \quad & \Big[\frac{2}{n_5}\frac{1}{2}V'_{j',m'}\left((m-j+1)\psi^- V_{j,m+1} + (m+j-1)\psi^+ V_{j,m-1} - 2m\psi^3 V_{j,m}\right) \\ & + \frac{2}{n_5}\frac{1}{2}V_{j,m}\left((j'+m'+1)\chi^- V'_{j',m'+1} + (j'-m'+1)\chi^+ V'_{j',m'-1} + 2m'\chi^3 V'_{j',m'}\right) \\ & + \frac{1}{2}\left(-E\lambda^t + P_y\lambda^y\right) V_{j,m}V'_{j',m'}\Big]\, e^{i(-E\,t+P_y y)}, \end{aligned} \tag{4.17}$$

and

$$\Phi_\lambda = e^{-\varphi}\left[\psi^3 + l_2\chi^3 + l_3\lambda^t + l_4\lambda^y\right] V_{j,m}V'_{j',m'}\, e^{i(-E\,t+P_y y)}, \tag{4.18}$$

respectively. Such states are trivially BRST invariant since G and $\boldsymbol{\lambda}$ square to the Virasoro constraint (4.11) and the null condition (2.216), and the relevant term in their product is $G(z)\,\boldsymbol{\lambda}(0) \sim \boldsymbol{\lambda}(z)\,G(0) \sim J(0)/z$, whose action vanishes by means of the condition (4.12). In the end, we are left with four physical vertex operators to add to the four from the $\mathbb{T}^4$ directions to give the correct eight polarizations in the holomorphic sector in 9+1 dimensions.

We choose a basis for these four physical vertex operators such that, in the AdS_3 limit, they reduce to the basis of global AdS_3 vertex operators described around Eq. (2.149). We thus obtain

$$\mathcal{W}^\varepsilon = e^{-\varphi}\left[(\psi V_j)_{j+\varepsilon,m}V'_{j',m'} + \left(c^t_\varepsilon\,\lambda^t + c^y_\varepsilon\,\lambda^y\right)V_{j,m}V'_{j',m'}\right] e^{i\left(-E\,t+P_y y\right)}, \tag{4.19a}$$

$$\mathcal{X}^\varepsilon = e^{-\varphi}\left[V_{j,m}(\chi V'_{j'})_{j'+\varepsilon,m'} + \left(d^t_\varepsilon\,\lambda^t + d^y_\varepsilon\,\lambda^y\right)V_{j,m}V'_{j',m'}\right] e^{i\left(-E\,t+P_y y\right)}, \tag{4.19b}$$

where the SL(2,$\mathbb{R}$) and SU(2) coefficients are those given in (2.149)–(2.150), while the novel ones are[43]

$$c^t_\varepsilon = -c^3_\varepsilon\frac{n_5P_y}{l_4E+l_3P_y}\,, \qquad c^y_\varepsilon = c^3_\varepsilon\frac{n_5E}{l_4E+l_3P_y}\,, \tag{4.20}$$

$$d^t_\varepsilon = d^3_\varepsilon\frac{n_5l_2P_y}{l_4E+l_3P_y}\,, \qquad d^y_\varepsilon = -d^3_\varepsilon\frac{n_5l_2E}{l_4E+l_3P_y}\,. \tag{4.21}$$

By construction, the resulting states are polarized transverse to the gauge directions. As anticipated, they are built out of a linear combination of terms of spin j and $j+\varepsilon$ (j' and $j'+\varepsilon$). Moreover, at leading order in the large R_y expansion, the coefficients in the t,y directions go to zero, since $E,P_y \sim \mathcal{O}(1/R_y)$, $l_{3,4} \sim \mathcal{O}(R_y)$, and $l_2 \sim \mathcal{O}(1)$.

[43] The coefficients $c^{t,y}$ and $d^{t,y}$ were reported in the letter [2] with a slightly different notation, related by $c^{t,y}_{\text{there}} = c^{t,y}_{\varepsilon,\text{here}}/c^3_\varepsilon$, and likewise for $d^{t,y}$.

4.1.3 The unflowed R sector

We now describe the physical states in the R sector of the null-gauged model. The computation turns out to be more involved than in the NS sector, since the spin fields necessarily involve all ε-chiralities. As a consequence, we will not find a situation akin to (4.19) in which a subset of coefficients are exactly those of the global $\mathrm{AdS}_3 \times \mathbb{S}^3$ operators. However, we will again show that in the AdS_3 limit the vertex operators will reduce to their global $\mathrm{AdS}_3 \times \mathbb{S}^3$ counterparts.

We introduce $\mathrm{AdS}_3 \times \mathbb{S}^3$ and $\mathbb{R}_t \times \mathbb{S}^1_y \times \mathbb{T}^4$ spin fields,[44]

$$S_{\varepsilon_1\varepsilon_2\varepsilon_3} = e^{\frac{i}{2}(\varepsilon_1 H_1 + \varepsilon_2 H_2 + \varepsilon_3 H_3)} , \qquad \mathcal{S}_{\varepsilon_6\varepsilon_4\varepsilon_5} = e^{\frac{i}{2}(\varepsilon_6 H_6 + \varepsilon_4 H_4 + \varepsilon_5 H_5)} . \tag{4.22}$$

Recalling the definition of the $\mathrm{AdS}_3 \times \mathbb{S}^3$ chirality ε and the mutual locality / chiral GSO projections in 9+1 and 10+2 dimensions, (2.145), (4.9), we substitute away ε_3 and ε_6 via

$$\varepsilon_3 = \varepsilon\varepsilon_1\varepsilon_2 , \qquad \varepsilon_6 = \varepsilon\varepsilon_4\varepsilon_5 . \tag{4.23}$$

The $H_{4,5}$ exponentials are spectators under the action of $\mathcal{Q}$, so the parameters $\varepsilon_4, \varepsilon_5$ will label the vertex operators. For fixed $\varepsilon_4, \varepsilon_5$, we consider ε_6 to be controlled by ε through the second equation in (4.23), and we will form linear combinations of different values over ε_1, ε_2, ε.

We work with vertex operators in ghost pictures $(q_\varphi, q_{\tilde\varphi}) = (-\frac{1}{2}, +\frac{1}{2})$, for which the $\boldsymbol{\lambda}$-constraint is non-trival, while there is no need to worry about BRST-exact states. We thus make an ansatz for R sector vertices of the following form:

$$\mathcal{Y}^{\varepsilon_4,\varepsilon_5} \;=\; e^{-(\varphi-\tilde\varphi)/2} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon} F^{\varepsilon}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} S_{\varepsilon_1\varepsilon_2\varepsilon_3} \mathcal{S}_{\varepsilon_6\varepsilon_4\varepsilon_5} V_{j,m-\frac{\varepsilon_1}{2}} V'_{j',m'-\frac{\varepsilon_2}{2}} e^{i(-Et+P_y y)} . \tag{4.24}$$

Note that the coefficients $F^{\varepsilon}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5}$ are not determined by the representation theory of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$, since the states will not in general have definite spin.

The cT and $\tilde{c}J$ terms of the BRST operator $\mathcal{Q}$ (4.7) act as in the NS sector. Hence the unflowed, non-winding states of the R sector also satisfy both the Virasoro condition (4.11) and the bosonic null-gauge constraint (4.12).

Next, the $e^{\tilde\varphi}\boldsymbol{\lambda}$ term in $\mathcal{Q}$ leaves $\varepsilon_{1,2,4,5}$ unchanged, so for this term we can treat $\varepsilon_{1,2,4,5}$ as fixed, and focus on the sum over $\varepsilon = \pm$. The resulting constraints on $F^{\pm}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5}$ form a two-dimensional homogeneous linear system, which is degenerate due to the null

[44] The order of the spin fields in $\mathcal{S}_{\varepsilon_6\varepsilon_4\varepsilon_5}$ has been chosen for convenience in order to reduce clutter in computations involving cocycle factors.

condition on the gauge parameters, Eq. (2.216). For each choice of $\varepsilon_{1,2,4,5}$, we have

$$\begin{aligned}(l_3+\varepsilon_4\varepsilon_5 l_4)F^{-}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} \;-\; i\sqrt{n_5}\,(1-\varepsilon_1\varepsilon_2 l_2)F^{+}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} \;&=\; 0\,,\\ i\sqrt{n_5}(1+\varepsilon_1\varepsilon_2 l_2)F^{-}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} \;-\; (l_3-\varepsilon_4\varepsilon_5 l_4)F^{+}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} \;&=\; 0\,.\end{aligned} \tag{4.25}$$

These constraints halve the degrees of freedom. When $|l_2| = 1$ (and so $|l_3| = |l_4|$), some of the $F^{\varepsilon}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5}$ get set to zero. For a given $\varepsilon_{1,2,4,5}$, when neither of $F^{\pm}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5}$ get set to zero, their ratio $F^{-}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5}/F^{+}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5}$ becomes determined. So the 32 d.o.f. remaining after imposing GSO in (10+2) dimensions have now become 16, corresponding to $\varepsilon_{1,2,4,5}$ in our parameterization.

Let us pause to discuss how Eq. (4.25) behaves in the large R_y limit. We have $l_2 \sim \mathcal{O}(1)$ and generically $|l_2| \neq 1$, while $l_3 + l_4 \sim \mathcal{O}(R_y)$ and $l_3 - l_4 \sim \mathcal{O}(1/R_y)$, from (3.29)–(3.32). When $\varepsilon_4\varepsilon_5 = +1$, we obtain $F^+ \sim \mathcal{O}(1)$ and $F^- \sim \mathcal{O}(1/R_y)$, so at leading order in large R_y we obtain an operator of purely positive $\mathrm{AdS}_3 \times \mathbb{S}^3$ chirality ε. Similarly, when $\varepsilon_4\varepsilon_5 = -1$, at leading order in large R_y we obtain a purely negative chirality operator. So we obtain operators of definite $\mathrm{AdS}_3 \times \mathbb{S}^3$ chirality ε, with $\varepsilon_4\varepsilon_5 = \varepsilon$, exactly as in Section 2.6, see Eq. (2.165). This explains the sign choice of the (10+2)-d GSO projection (4.9). As before, one of ε_4 or ε_5 remains unfixed, say ε_4.

We now examine the action of $e^{\varphi}G$ on the R vertex operator ansatz (4.24). This will reduce the remaining 16 degrees of freedom to the correct 8 physical polarizations in the holomorphic sector. It leads to the following set of equations (we suppress the $\varepsilon_4, \varepsilon_5$ subscripts on the RHS for ease of notation):

$$\begin{aligned}\mathcal{B}^{\varepsilon}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} \;\equiv\; \Big(m+\varepsilon_1 j-\frac{\varepsilon_1}{2}\Big)\,F^{\varepsilon}_{(-\varepsilon_1)\varepsilon_2} + i\,\varepsilon_1\varepsilon_2\,\Big(j'-\varepsilon_2 m'+\frac{1}{2}\Big)\,F^{\varepsilon}_{\varepsilon_1(-\varepsilon_2)}&\\ -\,(\varepsilon m+\varepsilon_1\varepsilon_2 m')F^{\varepsilon}_{\varepsilon_1\varepsilon_2} + \frac{i\sqrt{n_5}}{2}\,(\varepsilon_4\varepsilon_5 P-\varepsilon\,E)\,F^{(-\varepsilon)}_{\varepsilon_1\varepsilon_2} \;&=\; 0\,.\end{aligned} \tag{4.26}$$

Comparing to the $\mathrm{AdS}_3 \times \mathbb{S}^3$ BRST condition, the only new term is the fourth and final one, proportional to $F^{(-\varepsilon)}_{\varepsilon_1\varepsilon_2}$, which has the effect of mixing the ε chiralities. The first three terms are unchanged from the $\mathrm{AdS}_3 \times \mathbb{S}^3$ BRST condition, so the $\mathrm{AdS}_3 \times \mathbb{S}^3$ limit of this condition is simply to drop the fourth term. Note that Eq. (4.25) implies that half of these equations are redundant, and allows us to decouple the F^+ from the F^- coefficients. Moreover, by using the Virasoro constraint (4.11), the bosonic null-gauge condition (4.12), and the null constraint on the gauge parameters (2.216), one can show that for fixed ε_4 and ε_5, actually only two equations are linearly independent. For generic values of quantum numbers, such that all denominators appearing below are nonzero, the linearly independent equations can be taken to be

$$\begin{aligned} F^{+}_{-+} &= -i\,\frac{j'-m'+\frac{1}{2}}{j+m-\frac{1}{2}}\,F^{+}_{+-} + \frac{-j(j-1)+j'(j'+1)+m^2-m'^2}{(j+m-\frac{1}{2})\left[m-m'+\frac{l_4-\varepsilon_4\varepsilon_5 l_3}{2(l_2-1)}\left(\varepsilon_4\varepsilon_5 E-P_y\right)\right]}F^{+}_{++}\,, \\ F^{+}_{--} &= \frac{m-m'+\frac{l_4-\varepsilon_4\varepsilon_5 l_3}{2(l_2-1)}\left(\varepsilon_4\varepsilon_5 E-P_y\right)}{j+m-\frac{1}{2}}F^{+}_{+-} + i\,\frac{j'+m'+\frac{1}{2}}{j+m-\frac{1}{2}}\,F^{+}_{++}\,. \end{aligned} \tag{4.27}$$

Alternatively, the two linearly independent equations can generically be taken to be

$$F^{-}_{-+} = -i\,\frac{j'-m'+\frac{1}{2}}{j+m-\frac{1}{2}}\,F^{-}_{+-} + \frac{-j(j-1)+j'(j'+1)+m^2-m'^2}{(j+m-\frac{1}{2})\left[-m-m'+\frac{n_5(l_2-1)}{2(l_4-\varepsilon_4\varepsilon_5 l_3)}\left(\varepsilon_4\varepsilon_5 E+P_y\right)\right]}F^{-}_{++}\,,$$

$$F^{-}_{--} = \frac{-m-m'+\frac{n_5(l_2-1)}{2(l_4-\varepsilon_4\varepsilon_5 l_3)}\left(\varepsilon_4\varepsilon_5 E+P_y\right)}{j+m-\frac{1}{2}}F^{-}_{+-} + i\,\frac{j'+m'+\frac{1}{2}}{j+m-\frac{1}{2}}\,F^{-}_{++}\,. \tag{4.28}$$

Let us pause again to check consistency with the $\mathrm{AdS}_3 \times \mathbb{S}^3$ limit. Setting $\varepsilon_4\varepsilon_5 = \varepsilon$ and taking the large R_y limit of Eqs. (4.27), we indeed find that a solution is given by setting (the $\mathrm{AdS}_3 \times \mathbb{S}^3$ limit of) $F^{\varepsilon}_{\varepsilon_1\varepsilon_2}$ to be equal to the values $f^{\varepsilon}_{\varepsilon_1\varepsilon_2}$ specified in Eqs. (2.166)–(2.168).

Similarly to $\mathrm{AdS}_3 \times \mathbb{S}^3$, Eqs. (4.27) are two equations for four unknowns, so for each ε_4, ε_5 there is a two-parameter family of solutions, which we take to be parameterized by the values of $F^{+}_{+\pm}$. If working with Eqs. (4.28), we take the two-parameter family of solutions to be parameterized by the values of $F^{-}_{+\pm}$. Together with ε_4, ε_5, this gives 8 physical polarizations.

In Section 2.6, for $\mathrm{AdS}_3 \times \mathbb{S}^3$ these unfixed coefficients were chosen such that the vertex operators transform appropriately under the action of the currents $J^{\pm}$ and $K^{\pm}$. However, in the null-gauged worldsheet theory associated to the full asymptotically linear dilaton geometry this need not necessarily be the case.

In the cosets we fix these coefficients by requiring a reasonable IR limit. We treat the different ε chiralities separately. For $\varepsilon = 1$ we set the particular components $F^{+}_{+\pm}$ equal to their values in the AdS limit, $F^{+}_{+\pm} = f^{+}_{+\pm}$. The rest of the coefficients are then obtained using Eqs. (4.27) and (4.25). Alternatively, for $\varepsilon = -1$ we set $F^{-}_{+\pm} = f^{-}_{+\pm}$ and again solve for the remaining coefficients using Eqs. (4.28) and (4.25).

We now turn to the analysis of the spacetime supercharges preserved by the null gauging, following on from the initial discussion around Eq. (4.8) (see also [54]). The supercharge analysis corresponds to the limit of the Ramond sector analysis in which we take $j = j' = E = P_y = 0$, $m = \frac{\varepsilon_1}{2}$, $m' = \frac{\varepsilon_2}{2}$, as can be seen by comparing Eqs. (4.8) and (4.24). In this limit, the center-of-mass wavefunction trivializes, and we are left with integrated vertex operators involving only the spin fields. As before, we parameterize the ε_i according to (4.23); ε_4 and ε_5 are spectators that will label the supercharges; and we will sum over $\varepsilon, \varepsilon_1, \varepsilon_2$ as in (4.24). The J constraint (4.12) reduces to

$$\varepsilon_1 + \varepsilon_2\, l_2 = 0 \qquad \Rightarrow \qquad \varepsilon_1 \varepsilon_2 = -l_2\,, \tag{4.29}$$

so supersymmetry is preserved in the holomorphic sector if and only if $|l_2| = 1$, and thus $|l_3| = |l_4|$. The γG constraint (4.26) reduces directly to

$$\varepsilon = -1\,, \tag{4.30}$$

so all supercharges have negative $\mathrm{AdS}_3 \times \mathrm{S}^3$ chirality. Then the λ constraint in the first line of (4.25), with $F^+ = 0$, reduces to

$$l_3 + \varepsilon_4 \varepsilon_5\, l_4 = 0\,. \tag{4.31}$$

So when $|l_2| = 1$ and $|l_3| = |l_4| \neq 0$, there are four holomorphic supercharges, labelled by say ε_2 and ε_4.

Combining this analysis with the corresponding one in the antiholomorphic sector, we observe consistency with the passage below Eq. (3.82) describing which subset of the backgrounds are supersymmetric. In terms of the spacetime spectral flow parameters s, $\bar{s}$ introduced in Eq. (2.89), we have $l_2 = 2s+1$, $r_2 = 2\bar{s}+1$. The circular supertube backgrounds of [122, 123] have $s = \bar{s} = 0$ and preserve supersymmetry in both holomorphic and antiholomorphic sectors; the backgrounds of [132, 124, 125, 96] have $\bar{s} = 0$, $s \neq 0$ and so preserve supersymmetry only in the antiholomorphic sector; the general JMaRT backgrounds [37] have s and $\bar{s}$ both nonzero, and preserve no supersymmetry. As we will see in the next chapter, the absence of supersymmetry will not obstruct us in the computation of correlation functions in the AdS_3 limit. However, given that the embedding coefficients l_i, r_i enter the bosonic null gauge constraints Eq. (4.12), the number of admissible BRST-invariant spectrum of excitations on a given background will change.

Picture changing in the R sector

In order to compute two-point functions of operators $\mathcal{Y}$ in the Ramond sector of the gauged model, we need to define their picture-changed versions. Propagators will be non-vanishing only if the total ghost charges add up to $-Q_\varphi = -2$ and $-Q_{\tilde{\varphi}} = 0$. The picture-changing operators are given by $P_{+1} \sim e^{\varphi} G$ and $\tilde{P}_{+1} \sim e^{\tilde{\varphi}} \lambda$. One possible natural choice would be to compute the two-point function (superscripts denote $(q_\varphi, q_{\tilde{\varphi}})$ charges)

$$\langle \mathcal{Y}^{\left(-\frac{3}{2},-\frac{1}{2}\right)}(z)\, \mathcal{Y}^{\left(-\frac{1}{2},+\frac{1}{2}\right)}(w) \rangle\ . \tag{4.32}$$

However, it turns out that looking for an explicit expression for the state $\mathcal{Y}^{\left(-\frac{3}{2},-\frac{1}{2}\right)}$ is not the simplest way to go. This is due to the fact that such a state is automatically BRST closed, so that it must be determined by the somewhat cumbersome procedure of removing all the BRST-exact contributions. To avoid this issue, one can distribute the ghost charges in a different way inside the correlator, and consider instead the equivalent two-point function

$$\langle \mathcal{Y}^{\left(-\frac{3}{2},+\frac{1}{2}\right)}(z)\, \mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(w) \rangle \ . \tag{4.33}$$

Here $\mathcal{Y}^{\left(-\frac{3}{2},+\frac{1}{2}\right)}$ is in the canonical $\tilde{\varphi}$-picture, while $\mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$ is in the canonical φ-picture. Thus, although this forces us to compute two additional R-sector operators instead of only one, these are constrained by the $\tilde{\gamma}\lambda$ and γG BRST constraints, respectively. The procedure is then similar to that employed above to construct $\mathcal{Y}^{\left(-\frac{1}{2},+\frac{1}{2}\right)}$. We thus make the Ansätze

$$\begin{aligned}
\mathcal{Y}^{\left(-\frac{3}{2},+\frac{1}{2}\right),\varepsilon_4\varepsilon_5} &= e^{-\left(\frac{3}{2}\varphi-\frac{1}{2}\tilde{\varphi}\right)} \sum_{\varepsilon,\varepsilon_1,\varepsilon_2} \mathcal{L}^{\varepsilon}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} S_{\varepsilon_1\varepsilon_2\varepsilon_3} \mathcal{S}_{\varepsilon_6\varepsilon_4\varepsilon_5} V_{j,m-\frac{\varepsilon_1}{2}} V'_{j',m'-\frac{\varepsilon_2}{2}} e^{i(-Et+P_y y)} \,, \\
\mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right),\varepsilon_4\varepsilon_5} &= e^{-\left(\frac{1}{2}\varphi+\frac{1}{2}\tilde{\varphi}\right)} \sum_{\varepsilon,\varepsilon_1,\varepsilon_2} \mathcal{G}^{\varepsilon}_{\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5} S_{\varepsilon_1\varepsilon_2\varepsilon_3} \mathcal{S}_{\varepsilon_6\varepsilon_4\varepsilon_5} V_{j,m-\frac{\varepsilon_1}{2}} V'_{j',m'-\frac{\varepsilon_2}{2}} e^{i(-Et+P_y y)} \,.
\end{aligned}$$

where again ε_3, ε_6 are substituted away using (4.23). These must satisfy

$$\begin{aligned}
\oint dz : e^{\tilde{\varphi}}\lambda :(z)\, \mathcal{Y}^{\left(-\frac{3}{2},+\frac{1}{2}\right),\varepsilon_4\varepsilon_5}(w) &= 0\,, \\
\oint dz : e^{\varphi} G :(z)\, \mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right),\varepsilon_4\varepsilon_5}(w) &= 0\,,
\end{aligned} \tag{4.34}$$

and

$$\begin{aligned}
: e^{\varphi} G :(z)\, \mathcal{Y}^{\left(-\frac{3}{2},+\frac{1}{2}\right),\varepsilon_4\varepsilon_5}(w) &= \mathcal{Y}^{\left(-\frac{1}{2},+\frac{1}{2}\right),\varepsilon_4\varepsilon_5}(w)\,, \\
: e^{\tilde{\varphi}}\lambda :(z)\, \mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right),\varepsilon_4\varepsilon_5}(w) &= \mathcal{Y}^{\left(-\frac{1}{2},+\frac{1}{2}\right),\varepsilon_4\varepsilon_5}(w)\,.
\end{aligned} \tag{4.35}$$

By solving the above constraints, all the coefficients $\mathcal{L}^{\varepsilon}$ and $\mathcal{G}^{\varepsilon}$ can be expressed explicitly in terms of the F^{ε} coefficients in Eq. (4.27).

In addition, one can explicitly check that in the AdS limit they correctly reproduce the expected behaviour. From the definition of the corresponding coset states $\mathcal{Y}^{\left(-\frac{3}{2},+\frac{1}{2}\right)}$ and $\mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$, one might reasonably expect that they would reduce to the states $\mathcal{Y}^{\left(-\frac{3}{2}\right)}$ and $\mathcal{Y}^{\left(-\frac{1}{2}\right)}$ of Section 2.6.4.2 respectively. However, care is needed when comparing both chiralities and normalisations in the UV and IR. To explain this, let us consider for instance $\mathcal{Y}_A^{\left(-\frac{1}{2}\right)}$, whose ε-chirality is $\varepsilon = +1$. (Analogous comments hold for other operators.) First of all, recall that already in the case of $AdS_3 \times S^3 \times \mathbb{T}^4$, the picture-changing operator induces a change in chirality of the state. Indeed, in case "A" of the analysis in (2.166), the physical states in the "$-\frac{3}{2}$" picture have *negative* ε-chirality.

In the full coset models, an analogous pattern occurs with the two picture-changing operators P_{+1}, $\tilde{P}_{+1}$. The coset state $\mathcal{Y}^{\left(-\frac{3}{2},+\frac{1}{2}\right)}$ that correctly reduces to $\mathcal{Y}_A^{\left(-\frac{3}{2}\right),\varepsilon=-1}$ in the AdS limit indeed has *positive* ε-chirality. This means that, in our case under study, the coefficients $\mathcal{L}^{+}_{\varepsilon_1\varepsilon_2}$ reduce to $-\frac{\sqrt{n_5}}{j+j'}f^{-}_{\varepsilon_1\varepsilon_2}$. Similarly, the coset state $\mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$ with negative ε-chirality reduces to the $\mathcal{Y}_A^{\left(-\frac{1}{2}\right)}$ state. However, in the latter case there is a normalisation factor $(i\,\mathsf{k}R_y)^{-1}$ to account for. This is removed by taking into account the normalisation of the picture-changing operator, which indeed contains a term which is dominant in the AdS limit, $l_3\lambda_t + l_4\lambda_y \sim (\mathsf{k}R_y)(\lambda_t + \lambda_y)$.

We illustrate the example of the $\mathcal{G}^{-}_{++}$ coefficient, appearing in the coset state $\mathcal{Y}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$. The argument holds analogously for all the other coefficients. The explicit expression of $\mathcal{G}^{-}_{++}$ in terms of positive coefficients $F^{+}_{\varepsilon_1\varepsilon_2}$ is obtained by solving the constraints Eq. (4.34) and Eq. (4.35), without using Eq. (4.11) and Eq. (4.12). For generic quantum numbers such that the denominators below are non-zero, one finds

$$\begin{aligned}
&\frac{(l_3+l_4)^2}{i(E-P_y)n_5(1+l_2)}\,\mathcal{G}^{-}_{++} \\
&= \frac{(m+j-\frac{1}{2})F^{+}_{-+} + i(j'-m'+\frac{1}{2})F^{+}_{+-} - \left(\frac{2(j-j'+1)(j+j')(l_3+l_4)}{(E-P_y)n_5(1+l_2)} + \frac{(-1+l_2)}{(1+l_2)}(m-m')\right)F^{+}_{++}}{\left(-j(j-1)+j'(j'+1)+n_5\frac{(m+l_2m')(E-P_y)}{l_3+l_4} + n_5\frac{n_5(-1+l_2^2)(E-P_y)^2}{4(l_3+l_4)^2}\right)},
\end{aligned} \tag{4.36}$$

and thus for $R_y \gg 1$ one has $\mathcal{G}^{-}_{++} \simeq (i\mathsf{k}R_y)^{-1}f^{+}_{++} + \mathcal{O}(R_y^{-2})$, as claimed above.

The expressions for the coefficients $\mathcal{G}, \mathcal{L}$ are quite lengthy, and we leave the computation of correlators in the full coset model for future work. Nevertheless, it is easy to check that in all cases the coefficients reduce to the expected expressions when going into the IR regime. Consequently, we find that, to leading order in R_y, the coset two-point functions in the RR sector reproduce the m-basis expressions in Eq. (2.171), as they should. As will become clear in Section 4.2 below, this does *not* mean that the physics in the IR regime of the coset model is that of global $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$; the bosonic null gauge condition (4.12) will lead to substantially different correlators in the appropriately defined x-basis.

4.1.4 Flowed/winding sectors

We now briefly discuss the states with non-trivial spectral flow we will be interested in, that is, those corresponding to the description of the higher-weight chiral primaries described in Section 2.6.

A generic state with excitation numbers $(\frac{1}{2},\frac{1}{2})$ in the null-gauged worldsheet theory must satisfy the the L_0 and $\bar{L}_0$ Virasoro constraints

$$0 = \frac{j'(j'+1)-j(j-1)}{n_5} - m\omega + m'\omega' + \frac{n_5}{4}\left(\omega'^2 - \omega^2\right) - \frac{1}{4}\left(E^2 - P_y^2\right), \tag{4.37a}$$

$$0 = \frac{j'(j'+1)-j(j-1)}{n_5} - \bar{m}\omega + \bar{m}'\bar{\omega}' + \frac{n_5}{4}\left(\bar{\omega}'^2 - \omega^2\right) - \frac{1}{4}\left(E^2 - \bar{P}_y^2\right), \tag{4.37b}$$

where

$$P_y = \frac{n_y}{R_y} - \omega_y R_y \,, \qquad \bar{P}_y = \frac{n_y}{R_y} + \omega_y R_y \,, \tag{4.38}$$

and we recall that $\omega_y \in \mathbb{Z}$ is the winding on the y-circle. The level-matching $L_0 - \bar{L}_0$ constraint thus reads

$$0 = \omega(\bar{m} - m) + m'\omega' - \bar{m}'\bar{\omega}' + \frac{n_5}{4}(\omega'^2 - \bar{\omega}'^2) - n_y\omega_y \,. \tag{4.39}$$

Let us try to follow the global AdS$_3$ procedure as close as possible, and consider states with $\omega = \omega' = \bar{\omega}'$ and $m = m' = \bar{m} = \bar{m}'$. This will be general enough to describe all coset states corresponding to chiral primary operators of the spacetime theory. Then, (4.39) forces us to set $\omega_y = 0$. Moreover, for states constructed by spectrally flowing highest/lowest weight primaries, the discussion around Eq. (2.160) shows that the $e^{\varphi}G$ part of the BRST charge acts as in the unflowed case. Moreover, the action of $e^{\tilde{\varphi}}\lambda$ is left unchanged. The derivation of the coefficients involved in the definition of the vertex operators constructed above then goes through without changes, and we only need to restrict to the highest (lowest) possible values of m (m') in each case.

Regarding the gauge constraints, recall that, both in SL(2,$\mathbb{R}$) and in SU(2), the different modes of the spectrally flowed operators are obtained by acting with the raising/lowering operators $J_0^{\pm}$ and $K_0^{\pm}$ on the flowed primary. Although this does not lead to operators that can be expressed in a simple way in the m-basis, the presence of these different modes is crucial in order to obtain the set of physical modes that satisfy the gauge constraints. Focusing on (lowest-weight) discrete states corresponding to operators of spacetime weight h in the chiral multiplets, this gives modes described by worldsheet operators with projections $m_\omega = J + \frac{n_5}{2}\omega + n = h + n$ and $m'_\omega = J' + \frac{n_5}{2}\omega - n'$, with $n, n' \in \mathbb{N}_0$, and similarly in the antiholomorphic sector. The bosonic gauge constraints now read

$$0 = m_\omega + l_2\, m'_\omega + \frac{l_3}{2}E + \frac{l_4}{2}P_y \,, \qquad 0 = \bar{m}_\omega + r_2\, \bar{m}'_\omega + \frac{r_3}{2}E + \frac{r_4}{2}P_y \,. \tag{4.40}$$

As discussed below Eq. (4.13), this implies that $J_0^{\pm}$ and $K_0^{\pm}$ do not commute with the BRST charge. Consequently and importantly, only the subset of modes satisfying Eqs. (4.40) will be physical. The implications will be discussed at length in the following section. Combining the gauge constraints with the conditions discussed below Eq. (4.39) is then sufficient to obtain the set of BRST invariant states we are interested in.

Before concluding this section, let us review the fact that there is a residual discrete gauge symmetry in these models, which implies that operators related by shifts of the following form describe the same physical state [52], see also [1]. Parameterizing (l_4, r_4) through p and k as in (3.31), the symmetry is

$$\delta\left(\omega, \omega', \bar{\omega}', E, n_y, \omega_y\right) = (1, -l_2, -r_2, l_3, -\mathsf{p}, -\mathsf{k}) \,. \tag{4.41}$$

In particular, one can trade a unit of SL(2,$\mathbb{R}$) spectral flow (i.e. shift ω by $\delta\omega = -1$) for k units of y-winding ω_y, together with the corresponding other shifts implied by Eq. (4.41). In particular, the energy acquires a term linear in R_y, namely $\delta E = \mathsf{k} R_y + \mathcal{O}(1/R_y)$. The origin of the factor k relating ω and ω_y can be traced back to the $\mathbb{Z}_{\mathsf{k}}$ orbifold appearing in the IR, see Eq. (3.81). It reflects the fact that the CFT state associated with the background lives in the k-twisted sector of the D1D5 CFT.

The operators discussed in this section do not exhaust the spectrum of the worldsheet model; for instance we have not discussed operators that do not not satisfy $\omega_y \equiv 0 \mod \mathsf{k}$, which were analyzed in [52]. However, the operators described above comprise a large set of light operators in spectral flowed sectors, in parallel to the analysis of global $AdS_3 \times \mathbb{S}^3$, which will be general enough for our purposes in the present work.

4.2 Novel heavy-light correlators from the worldsheet

In this section we describe the computation of two-point correlators in the null gauged models, corresponding to HLLH correlators of the holographic CFT. To do so, we take a set of physical coset operators derived in the previous section and flow them to the IR, in which the geometry is locally an orbifold of $AdS_3 \times \mathbb{S}^3$. We develop a proposal to define coset operators in an appropriate x-basis corresponding to local operators of the holographic CFT. We then use this definition to compute a large set of HLLH correlators. We observe precise agreement between a subset of these and known results computed in supergravity and holographic CFT, and significantly extend these results.

4.2.1 Light states in the AdS_3 regime

We begin by describing in more detail the vertex operators of the null-gauged model in the AdS_3 limit. We send $R_y \to \infty$, keeping $\tilde{t} = t/R_y$ and $\tilde{y} = y/R_y$ fixed. After choosing the gauge $\tau = \sigma = 0$, this leads to a geometry described by the six-dimensional metric (3.81), which is related to $\mathbb{Z}_{\mathsf{k}}$-orbifolded $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ by the large coordinate transformation Eq. (3.82).

We focus initially on light states with no winding or worldsheet spectral flow. As we have argued in the previous section, the different polarizations and the associated coefficients simply reduce to those described in Section 2.6 in the AdS_3 limit. Here we further describe what happens to their quantum numbers in the regime of interest. In general, the Virasoro condition (4.11) determines j via the solution

$$j = \frac{1}{2} + \sqrt{\left(j' + \frac{1}{2}\right)^2 + \frac{n_5}{4}\left(P_y^2 - E^2\right)}\,, \tag{4.42}$$

where we have fixed the sign in order to have j in the range (2.106). As $R_y \to \infty$, we hold fixed the rescaled energy and momentum

$$\mathcal{E} = E R_y \,, \qquad n_y = P_y R_y \,. \tag{4.43}$$

Hence, the second term inside the square root in (4.42) is $\mathcal{O}\left(1/R_y^2\right)$, and at large R_y the solution becomes $j = j' + 1 + \mathcal{O}\left(1/R_y^2\right)$, which to leading order is the usual $\mathrm{AdS}_3 \times \mathbb{S}^3$ relation. The $\mathcal{O}\left(1/R_y^2\right)$ corrections to j are non-zero when $|\mathcal{E}| \neq |n_y|$, which is generically the case. To see this, note that at large R_y the gauging parameters associated to the t and y directions become

$$l_3 = r_3 = l_4 = -r_4 = -\mathsf{k} R_y + \mathcal{O}\left(1/R_y\right). \tag{4.44}$$

On the other hand, those associated to the $\mathbb{S}^3$ angular directions do not scale with R_y, and remain $l_2 = 2s+1$ and $r_2 = 2\bar{s}+1$. Hence, at leading order at large R_y, Eqs. (4.12) take the form

$$0 = m + (2s+1)\, m' - \frac{\mathsf{k}}{2}\left(\mathcal{E} + n_y\right) \,, \qquad 0 = \bar{m} + (2\bar{s}+1)\, \bar{m}' - \frac{\mathsf{k}}{2}\left(\mathcal{E} - n_y\right) \,, \tag{4.45}$$

which fix $\mathcal{E}$ and n_y in terms of m, m', $\bar{m}$, $\bar{m}'$, such that indeed generically $\mathcal{E} \neq \pm n_y$.

Although for simplicity we restricted to light states with no winding or worldsheet spectral flow in Eqs. (4.42) and (4.45), the present discussion and the computations in the rest of this section are analogous for winding states after replacing the projections $m \to m_\omega$, etc, where m_ω was defined above Eq. (4.40). Our results will be valid for the full set of chiral primaries that can be described within the usual $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ worldsheet theory, as well as their descendants under the global part of the chiral algebra, that fill out the short multiplet. We shall comment further on states with non-trivial winding and/or worldsheet spectral flow in due course.

4.2.2 Identifying the spacetime modes

Let us discuss the identification of the spacetime modes. We shall work in a gauge in which the upstairs SL(2,$\mathbb{R}$) time τ and angular direction σ are fixed. Then, importantly, the asymptotic boundary of the physical downstairs AdS_3 is parameterized by t/R_y and y/R_y, at a fixed point on the $\mathbb{S}^3$. We therefore define

$$m_y = \frac{1}{2}\left(\mathcal{E} + n_y\right) \,, \qquad \bar{m}_y = \frac{1}{2}\left(\mathcal{E} - n_y\right) \,, \tag{4.46}$$

and we interpret these as the asymptotic mode labels. We will see that this gives rise to a rich set of correlators that agree with and extend previous results.

Let us first consider $\mathsf{k} = 1$, and continue to focus mainly on the holomorphic sector. Given a holographic CFT chiral primary with spacetime weight h and definite (left) R-charge m', we wish to construct the dual worldsheet operator by summing over the corresponding modes. As reviewed in Section (2.6), in global $\mathrm{AdS}_3 \times \mathbb{S}^3$ this leads to x-basis operators of the form $\mathcal{O}_{h,m'}(x) = \sum_m x^{m-h}\mathcal{O}_{h,m,m'}$, where the modes $\mathcal{O}_{h,m,m'}$ are identified as either $\mathcal{W}_{j,m,j',m'}$, $\mathcal{X}_{j,m,j',m'}$ or $\mathcal{Y}_{j,m,j',m'}$, where j and h are related by Eq. (2.180). For simplicity, we collect all of these modes under the notation $\mathcal{V}_{j,m,\bar{m}}\mathcal{V}'_{j',m',\bar{m}'}$. In the null-gauged models, we replace $m \to m_y$ in the exponent of x in the sum, defining the worldsheet operator

$$\mathcal{O}_{h,m'}(x,\bar{x}) \quad \equiv \quad \frac{1}{\mathsf{k}^{h+\bar{h}}} \sum_{m_y,\bar{m}_y} x^{m_y-h}\bar{x}^{\bar{m}_y-\bar{h}}\mathcal{V}_{j,m,\bar{m}}\mathcal{V}'_{j',m',\bar{m}'}e^{-im_y(\tilde{t}-\tilde{y})}e^{-i\bar{m}_y(\tilde{t}+\tilde{y})}\,, \quad (4.47)$$

where we have temporarily included the antiholomorphic dependence to emphasize the coupling between the left- and right-moving sectors due to the null gauge constraints, Eq. (4.12). The normalization factors of k have been introduced for later convenience and will be discussed below Eq. (4.81). We emphasize that, from the point of view of the worldsheet theory, x is an auxiliary complex variable, while $\tilde{t}$ and $\tilde{y}$ are scalar fields.

Note that combining the bosonic null constraint (4.45) and the definition of the modes (4.46), we obtain $m_y = m + (2s+1)m'$. This relation parallels the supergravity spectral flow large gauge transformation (3.82), and we will make use of this observation when discussing the relation to the holographic CFT in due course.

For $\mathsf{k} > 1$ we follow the same logic, and make the same definition (4.47). This time however, combining Eqs. (4.45) and (4.46) we obtain

$$m_y = \frac{1}{\mathsf{k}}\left(m + (2s+1)m'\right)\,, \qquad \bar{m}_y = \frac{1}{\mathsf{k}}\left(\bar{m} + (2\bar{s}+1)\bar{m}'\right)\,. \qquad (4.48)$$

We shall shortly see that this gives rise to an important technical complication relating the holomorphic and antiholomorphic sectors.

Before computing our first example of a HLLH correlator, let us briefly return to operators with non-zero spectral flow and/or winding charge. In Section 4.1.4 we analyzed a set of coset vertex operators in sectors with non-zero worldsheet spectral flow, corresponding to chiral primaries of higher twists, similar to those in global $\mathrm{AdS}_3 \times \mathbb{S}^3$. In that section we worked with $\omega_y = 0$, and reviewed that the large gauge spectral flow transformation (4.41) relates these to operators with $\omega_y \in \mathsf{k}\mathbb{Z}$.

When $\omega_y \neq 0$, in general one should be careful both when examining whether the states survive the AdS_3 limit, and also when defining the AdS_3 energy $\mathcal{E}$ and angular momentum n_y, since fundamental string y-winding charge can be exchanged for background flux [52]. However, for operators which have $\omega_y \in \mathsf{k}\mathbb{Z}$, the situation is straightforward: we shall simply use gauge spectral flow to always work in a frame in which $\omega_y = 0$, and then use the definitions of $\mathcal{E}$ and n_y in (4.43) and the modes m_y, $\bar{m}_y$ in (4.46).

The discussion becomes more complicated when considering global SL(2,$\mathbb{R}$)$\times$SU(2) descendants of the spectrally flowed primaries. The isomorphism between the affine modules gives a simple identification for the highest/lowest weight states, but this structure becomes more complicated for rest of the multiplet. Indeed, global descendants of the spectrally flowed affine primary state are identified with affine/Virasoro descendants of the corresponding unflowed equivalent state with non-trivial y-winding, such that one needs to include string oscillator excitations. The situation is similar to what happens for the usual series identification $V^{\omega=0}_{j,j} \sim V^{\omega=1}_{\frac{k}{2}-j,j-\frac{k}{2}}$ in bosonic SL(2,$\mathbb{R}$), where, for instance, one has $V^{\omega=0}_{j,m} \sim (j_0^+)^{m-j} V^{\omega=0}_{j,j} \sim (j_{-1}^+)^{m-j} V^{\omega=1}_{\frac{k}{2}-j,j-\frac{k}{2}}$, see [54]. We leave a more detailed exploration of these features in the coset models for future work. In the remainder of this chapter we will work with operators that have $\omega_y = 0$.

4.2.3 HLLH correlators: first example

We now compute a first explicit example of a worldsheet two-point function in the cosets corresponding to the heavy backgrounds under consideration. For this purpose, we focus on a particular light operator probing the backgrounds with $\bar{s} = 0$ (hence $r_2 = 1$), but general s. These describe supersymmetric spectral flowed supertubes [132, 124, 125, 37, 96]. We shall demonstrate that this worldsheet correlator agrees with both the supergravity and symmetric product orbifold CFT HLLH four-point functions computed in [167]. We will also significantly extend beyond the set of correlators computed in [167].

The light operator in this first example is a massless RR operator of $\mathcal{Y}_{[A]}$ type with $h = m' = 1/2$ and hence $j = 1$, see Eq. (2.172). We shall denote this operator by $\mathcal{Y}_{\frac{1}{2}}$. Together with the analogous antiholomorphic part, this vertex operator is dual to a particular $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ chiral primary of the HCFT denoted by O^{++}, which we introduced in Eq. (2.92). In the six-dimensional supergravity arising from reduction on $\mathbb{T}^4$, this corresponds to a particular combination of fluctuations of a scalar and anti-self-dual two-form potential, in a tensor multiplet that is not turned on in the backgrounds we consider. In type IIB supergravity, in the present NS5-F1-P duality frame, these fields correspond to a particular combination of supergravity fluctuations of the RR axion and certain components of the RR two-form and four-form potentials. (In the D1-D5-P frame, the fluctuations are of the RR axion and certain components of the RR four-form and NSNS two-form potentials). The light operator corresponds to a particular scalar spherical harmonic of these fields [167, 139, 166].

In the large R_y limit, we have shown that the $\mathcal{Y}$ operators simply reduce to their $\mathrm{AdS}_3 \times \mathrm{S}^3$ cousins of Section 2.6, times the additional exponentials in t and y. These exponentials give trivial contributions to the two-point functions $\langle \mathcal{Y}_{\frac{1}{2}} \mathcal{Y}_{\frac{1}{2}} \rangle$ once the charge conservation conditions $m_{y,1} = -m_{y,2}$ and $\bar{m}_{y,1} = -\bar{m}_{y,2}$ are imposed. So do the SU(2) parts of the vertex operators for $m_1' = -m_2'$, upon using the appropriate normalization. Hence, at the level of the two-point function of m-basis operators, i.e. the mode correlators from the spacetime point of view, the only non-trivial contribution comes from the $\mathrm{SL}(2,\mathbb{R})$ part of the upstairs theory.

To illustrate this more explicitly and also more generally, we introduce the following notation: $\mathcal{O}_{h,m'}$ will denote a generic massless worldsheet vertex operator in the AdS limit of the null-gauged model with spacetime weight h and spacetime R-charge m'. When $m' = h$, we shall suppress the label m' and write $\mathcal{O}_h$. The corresponding holographic CFT chiral primaries will be denoted by O_h. Let us then consider the following worldsheet correlator,

$$\begin{aligned} &\langle s, \bar{s}, \mathsf{k} | \mathcal{O}_{h_1,m_1'}(x_1, z_1) \mathcal{O}_{h_2,m_2'}(x_2, z_2) \, | s, \bar{s}, \mathsf{k} \rangle \\ &\qquad \equiv \frac{1}{\mathsf{k}^{2h_1+2h_2}} \sum_{m_{y,i}, \bar{m}_{y,i}} \prod_{i=1,2} x_i^{m_{y,i}-h_i} \bar{x}_i^{\bar{m}_{y,i}-h_i} \lim_{R_y \to \infty} \langle \mathcal{V}_1(z_1) \mathcal{V}_2(z_2) \rangle \,, \end{aligned} \tag{4.49}$$

where $\mathcal{V}$ denotes a generic m-basis massless vertex operator of the full coset model. The spacetime correlator corresponds to the worldsheet-integrated version of (4.49).

As discussed below (2.185), the normalization of the vertex operators is chosen so that the overall factors coming from the worldsheet integration procedure cancel out. After setting $h_1 = h_2 = h$, $x_1 = 1$ and $x_2 = x$, computing the free-field correlators, and imposing the various charge conservations, the integrated correlator becomes

$$\langle s, \bar{s}, \mathsf{k} | \, \mathcal{O}_{h,m'}(1) \mathcal{O}^\dagger_{h,m'}(x) \, | s, \bar{s}, \mathsf{k} \rangle \; = \; \frac{1}{\mathsf{k}^{4h}} \sum_{m_y, \bar{m}_y} x^{m_y - h} \bar{x}^{\bar{m}_y - h} \lim_{R_y \to \infty} \langle \mathcal{V}_1 \mathcal{V}_2 \rangle \,, \tag{4.50}$$

where $\langle \mathcal{V}_1 \, \mathcal{V}_2 \rangle$ stands for the m-basis two-point function with the z-dependence stripped out, c.f. Eqs. (2.154), (2.171). Note that on the right-hand side, the sum over m_y involves the R-charge and other quantum numbers of the operator located at x.

In turn, the remaining correlator is particularly simple in our context. As argued around Eq. (2.171a), the m-basis two-point functions of worldsheet chiral primary operators reduce to the Gamma functions expression in the bulk term of (2.114) with the replacement $j \mapsto J = h$. These are simply the coefficients obtained by Mellin-transforming the usual propagator $|1-x|^{-4h}$, which further all become equal to one when $h = \frac{1}{2}$. Thus, for $\mathcal{Y}_{\frac{1}{2}}$ in the $\bar{s} = 0$ backgrounds, we obtain

$$\langle s, \mathsf{k} | \, \mathcal{Y}_{\frac{1}{2}}(1) \mathcal{Y}^\dagger_{\frac{1}{2}}(x) \, | s, \mathsf{k} \rangle \; = \; \frac{1}{\mathsf{k}^2} \sum_{m_y, \bar{m}_y} x^{m_y - \frac{1}{2}} \bar{x}^{\bar{m}_y - \frac{1}{2}} \,. \tag{4.51}$$

To be fully explicit, we take the operator at $x_2 = x$ to be a discrete series state $\mathcal{D}_j^+$ corresponding to an anti-chiral primary which, having set $J = h = \bar{h}$, has $m = h + n$, $\bar{m} = h + \bar{n}$ and $m' = \bar{m}' = -h$, where $n, \bar{n}$ are non-negative integers. For the supersymmetric backgrounds with $\bar{s} = 0$ and setting $h = \frac{1}{2}$, the relation (4.48) becomes

$$m_y = \frac{n-s}{\mathsf{k}}\,, \qquad \bar{m}_y = \frac{\bar{n}}{\mathsf{k}}\,. \tag{4.52}$$

Hence, the correlator takes the form

$$\langle s, \mathsf{k}|\, \mathcal{Y}_{\frac{1}{2}}(1)\mathcal{Y}^\dagger_{\frac{1}{2}}(x)\,|s, \mathsf{k}\rangle \;=\; \frac{1}{\mathsf{k}^2}\sum_{n,\bar{n}}{}' x^{\frac{n-s}{\mathsf{k}}-\frac{1}{2}}\,\bar{x}^{\frac{\bar{n}}{\mathsf{k}}-\frac{1}{2}}\,, \tag{4.53}$$

where we have denoted the sum with a prime because the range of summation over $n, \bar{n}$ is constrained. We now determine this constraint. Subtracting the two equations in (4.52) and comparing with (4.46), we obtain

$$\begin{aligned} m_y - \bar{m}_y \;&=\; \frac{1}{\mathsf{k}}(n - \bar{n} - s) \;=\; n_y \\ &\Rightarrow n - \bar{n} \;=\; \mathsf{k} n_y + s\,. \end{aligned} \tag{4.54}$$

Thus the allowed values of $n - \bar{n}$ are constrained by $n_y \in \mathbb{Z}$. To see this in detail, let us first write $s = \mathsf{k}p + \hat{s}$ with $0 \le \hat{s} < \mathsf{k}$ and $p \in \mathbb{N}$. For convenience here and later, we define $a \equiv \mathsf{k} - \hat{s}$, so that $s = \mathsf{k}p - a$, and $1 \le a \le \mathsf{k}$. Then the sum in (4.53) is restricted to be over non-negative integers $n, \bar{n}$ satisfying

$$\bar{n} - n \;\equiv\; a \mod \mathsf{k}. \tag{4.55}$$

We now demonstrate that the correlator (4.53), (4.55) agrees precisely with the supergravity and orbifold CFT expressions derived in [167]. In the holographic CFT, we have a HLLH four-point function $\langle O_H(x_3) O_L(x_1) O_L^\dagger(x_2) O_H^\dagger(x_4)\rangle$. Using Möbius symmetry we set $x_3 = 0$ and $x_4 \to \infty$, in which case the heavy operators are interpreted as in/out states which we similarly denote as $|s, \mathsf{k}\rangle$, such that the four-point function becomes a two-point function in the heavy background. We further set $x_1 = 1$, while $x_2 = x$ parametrizes the usual cross-ratio. Then the supergravity result of [167, Eq. (4.25)] is

$$\langle s, \mathsf{k}|O_{\frac{1}{2}}(1)O^\dagger_{\frac{1}{2}}(x)|s, \mathsf{k}\rangle \;=\; \frac{x^{(\hat{s}-s)/\mathsf{k}}}{|x||1-x|^2}\,\frac{1 - |x|^{2(1-\hat{s}/\mathsf{k})} + \bar{x}\left(|x|^{-2\hat{s}/\mathsf{k}} - 1\right)}{1 - |x|^{2/\mathsf{k}}}\,, \tag{4.56}$$

where the overall normalization of the supergravity amplitude was not fixed in [167]. Eq (4.56) was further shown to coincide with the corresponding symmetric orbifold CFT calculation for the particular cases $\hat{s} = 0$ and $\hat{s} = \mathsf{k} - 1$.

To demonstrate agreement between (4.53) and (4.56), it is easier to work from (4.56) towards our expression (4.53). We start by rewriting (4.56) in terms of sums akin to those involved in the definition of the x-basis, i.e. the mode expansion of local operators in spacetime as seen from the worldsheet theory. Recalling that $\hat{s} = \mathsf{k} - a$, the correlator becomes

$$\begin{aligned} \langle s, \mathsf{k}|O_{\frac{1}{2}}(1)O^{\dagger}_{\frac{1}{2}}(x)|s, \mathsf{k}\rangle &= \frac{x^{1-p}}{|x||1-x|^2}\frac{1-|x|^{2a/\mathsf{k}}+\bar{x}\left(|x|^{2a/\mathsf{k}-2}-1\right)}{1-|x|^{2/\mathsf{k}}} \\ &= \frac{x^{-p}}{|x|}\left[\frac{1}{|1-x|^2}\frac{1-|x|^2}{1-|x|^{2/\mathsf{k}}}-\frac{1}{1-\bar{x}}\frac{1-|x|^{2a/\mathsf{k}}}{1-|x|^{2/\mathsf{k}}}\right]. \end{aligned} \tag{4.57}$$

Assuming $|x| < 1$, the RHS can then be expressed as

$$\sum_{\hat{n}=0}^{\infty}\left[\sum_{\hat{n}=0}^{\infty}\sum_{\delta=0}^{\mathsf{k}-1} - \delta_{\hat{n},0}\sum_{\delta=0}^{a-1}\right] x^{\hat{n}+\frac{\delta}{\mathsf{k}}-p-\frac{1}{2}}\,\bar{x}^{\hat{\bar{n}}+\frac{\delta}{\mathsf{k}}-\frac{1}{2}}\,. \tag{4.58}$$

The second term in (4.58) is understood as subtracting the $\hat{n} = 0$ and $\delta = 0, \ldots, a-1$ coming from the first term. As a consequence, we can further rewrite the sum over $\hat{n}, \hat{\bar{n}}$ and δ in Eq. (4.58) as a restricted double sum,

$$\sum_{n,\bar{n}}{}' x^{\frac{n-s}{\mathsf{k}}-\frac{1}{2}}\,\bar{x}^{\frac{\bar{n}}{\mathsf{k}}-\frac{1}{2}}\,, \tag{4.59}$$

over pairs of non-negative integers $(n, \bar{n})$ satisfying the following restrictions

$$\bar{n} - n \equiv a \bmod \mathsf{k}\,, \qquad n, \bar{n} \in \mathbb{N}_0\,. \tag{4.60}$$

Indeed, for $a = \mathsf{k}$ we simply parametrize $n = \mathsf{k}\hat{n} + \delta$ and $\bar{n} = \mathsf{k}\hat{\bar{n}} + \delta$, which enforces the mod k condition (4.60), so that the sum gives the first term in (4.58). On the other hand, for $a < \mathsf{k}$ we can take $n = \mathsf{k}\hat{n} + \delta - a$ and $\bar{n} = \mathsf{k}\hat{\bar{n}} + \delta$, so that the factors of a coming from s and n cancel each other out in the exponent. However, in this case we need to explicitly subtract the contribution of all pairs $(\hat{n}, \delta)$ which lead to $n < 0$, thus again giving (4.58). Therefore we observe agreement between (4.53), (4.55) and (4.56), up to the overall normalization that was not fixed in the supergravity calculation of [167]. Here and throughout the chapter, we shall not keep track of overall normalization factors.

For $\hat{s} = 0$ and $\hat{s} = \mathsf{k} - 1$, since Eq. (4.56) matches the corresponding symmetric orbifold CFT correlator, we have demonstrated an explicit match between the worldsheet and symmetric orbifold CFT correlators. This is striking, as it is an agreement across moduli space for a correlator that a priori is not covered by an existing non-renormalization theorem.

This agreement is almost certainly due to the special nature of the heavy states we consider. Indeed, let us compare the worldsheet x-basis operator Eq. (4.47) with the discussion of holographic CFT spectral flow in [93, App. A]. Spectral flow in the holographic CFT is an automorphism of the small $(4, 4)$ superconformal algebra, that is a useful tool to relate different states and operators. For instance, the heavy backgrounds we consider are related by fractional spectral flow to the k-orbifolded NSNS vacuum, as discussed around Eq. (2.89).

Given a symmetric orbifold CFT correlator, one can perform spectral flow on both the operators and the background states. The value of the correlator is invariant under this operation. One can use this to map the correlator in one of our heavy backgrounds to a correlator in the k-orbifolded NSNS vacuum. Of course, for $\mathsf{k} = 1$, after undoing the spectral flows one obtains a vacuum correlator.

Taking for simplicity $\mathsf{k} = 1$ and $\bar{s} = 0$, the transformation of a chiral primary operator under this operation is [93, App. A]

$$\tilde{O}_h(x) \;=\; x^{(2s+1)m'} O_h(x)\,. \tag{4.61}$$

The exponent of x directly parallels the x factors appearing in Eq. (4.47). This observation generalizes straightforwardly to $\bar{s} \neq 0$ and to $\mathsf{k} > 1$, whereupon operators have fractional modes taking values in $\mathbb{Z}/\mathsf{k}$. We will comment further on the relation between worldsheet and symmetric product orbifold CFT correlators in due course.

4.2.4 Non-BPS HLLH correlators for $h = \frac{1}{2}$

The correlator presented in the previous subsection can be readily generalized to compute a set of novel HLLH correlators involving the same light operators, but probing the more general class of non-supersymmetric backgrounds given by the JMaRT solutions, in which both spacetime (fractional) spectral flow parameters s and $\bar{s}$ are non-trivial. Note that the parameters $s, \bar{s}, \mathsf{k}$ defining the background must satisfy $s(s+1) - \bar{s}(\bar{s}+1) \in \mathsf{k}\,\mathbb{Z}$, from combining Eqs. (3.84) and (2.89).

The same steps as described in the previous subsection lead directly to the following generalization of Eq. (4.53):

$$\langle s, \bar{s}, \mathsf{k}|\, \mathcal{Y}_{\frac{1}{2}}(1) \mathcal{Y}^{\dagger}_{\frac{1}{2}}(x)\, |s, \bar{s}, \mathsf{k}\rangle = \frac{1}{\mathsf{k}^2} {\sum_{n,\bar{n}}}' x^{\frac{n-s}{\mathsf{k}} - \frac{1}{2}}\, \bar{x}^{\frac{\bar{n}-\bar{s}}{\mathsf{k}} - \frac{1}{2}}\,. \tag{4.62}$$

To make precise the restricted summation, analogously to $s = \mathsf{k}p - a$ we write $\bar{s} = \mathsf{k}\bar{p} - \bar{a}$, with $1 \leq \bar{a} \leq \mathsf{k}$. We parametrize $n = \mathsf{k}\hat{n} + \delta - a$ and $\bar{n} = \mathsf{k}\hat{\bar{n}} + \delta - \bar{a}$ in order to satisfy the condition coming from the subtracted gauge constraint generalizing Eq. (4.55), namely

$$\bar{n} - n \;\equiv\; (a - \bar{a}) \bmod \mathsf{k}\,. \tag{4.63}$$

Then, by summing over all possible values of $\hat{n}$, $\hat{\bar{n}}$ and δ such that n and $\bar{n}$ are non-negative and satisfy (4.63), defining $b \equiv \min(a, \bar{a})$ we obtain

$$\begin{aligned}\langle s, \bar{s}, k | \mathcal{Y}_{\frac{1}{2}}(1) \mathcal{Y}^{\dagger}_{\frac{1}{2}}(x, \bar{x}) | s, \bar{s}, k \rangle &= \frac{1}{\mathsf{k}^2} \frac{x^{-p} \bar{x}^{-\bar{p}}}{|x|} \times \\ \Bigg[\frac{1}{|1-x|^2} \frac{1-|x|^2}{1-|x|^{2/\mathsf{k}}} - \frac{1}{1-\bar{x}} \frac{1-|x|^{2a/\mathsf{k}}}{1-|x|^{2/\mathsf{k}}} &- \frac{1}{1-x} \frac{1-|x|^{2\bar{a}/\mathsf{k}}}{1-|x|^{2/\mathsf{k}}} + \frac{1-|x|^{2b/\mathsf{k}}}{1-|x|^{2/\mathsf{k}}} \Bigg] .\end{aligned} \tag{4.64}$$

As before, the second and third terms remove contributions for which either n or $\bar{n}$ become negative, while the fourth one compensates for the over-counting of cases in which both n and $\bar{n}$ are negative.

At first sight, Eq. (4.64) may seem to depend on the values of a and $\bar{a}$ separately, in apparent contradiction with the fact that, as is implied by (4.63), only their difference matters. However, the RHS of Eq. (4.64) can be rewritten as

$$\begin{aligned}&\frac{1}{\mathsf{k}^2} \frac{x^{-s/\mathsf{k}} \bar{x}^{-\bar{s}/\mathsf{k}}}{|x||1-x|^2} \frac{1}{(1-|x|^{2/\mathsf{k}})} \left(\frac{x}{\bar{x}}\right)^{(\bar{a}-a)/2\mathsf{k}} \times \\ &\qquad \Big[(1-x)|x|^{(a-\bar{a})/\mathsf{k}} + (1-\bar{x})|x|^{-(a-\bar{a})/\mathsf{k}} - |1-x|^2 |x|^{-|a-\bar{a}|/\mathsf{k}} \Big] ,\end{aligned} \tag{4.65}$$

which explicitly depends only the orbifold parameter k, the spectral flow parameters s and $\bar{s}$, and the difference $a - \bar{a} \equiv \bar{s} - s \bmod \mathsf{k}$, as expected. Note that (4.65) is symmetric under the simultaneous replacements $x \leftrightarrow \bar{x}$ and $s \leftrightarrow \bar{s}$.

The worldsheet correlators (4.64)–(4.65) are one of the main results of this thesis. Unlike the $\bar{s} = 0$ supersymmetric example of the previous subsection, generically the corresponding supergravity or holographic CFT correlators have not been computed in the literature.

Since the backgrounds are non-supersymmetric when s and $\bar{s}$ are both non-zero, again a priori there is no obvious reason to expect the correlators to be protected across moduli space.

However, better-than-expected agreement between supergravity and holographic CFT has already been observed for a closely related observable describing the analog of the Hawking radiation process [140, 141, 94]; we shall comment further on this in due course.

Moreover, for the particular cases in which s and $\bar{s}$ are congruent to either 0 or $\mathsf{k} - 1$ mod k, the holographic CFT correlator follows straightforwardly from the techniques in App. A of [167], providing another exact match even for the non-supersymmetric backgrounds. We shall generalize this further in the next section.

4.3 More general heavy-light correlators

In this section we present HLLH correlators for generic chiral primaries of conformal weights $h > 1/2$. We then progress to describe higher-point heavy-light correlators. As an application, we compute the analogue of the Hawking radiation process from the backgrounds under consideration. We also compute a five-point HLLLH correlator of the symmetric product orbifold CFT, and demonstrate precise agreement with the corresponding worldsheet correlator.

4.3.1 HLLH correlators for general h

We now consider HLLH correlators where the light operators are chiral primaries with $h \geq 1$. For these correlators the m-basis $\mathrm{SL}(2,\mathbb{R})$ two-point functions do not trivialize, and the resulting sums become more complicated.

By following the method outlined in the previous sections, the worldsheet computation leads to restricted sums of the form

$$\frac{1}{\mathsf{k}^{4h}} \sum_{n,\bar{n}}{}' x^{\frac{n-2hs}{\mathsf{k}}-h} \bar{x}^{\frac{\bar{n}-2h\bar{s}}{\mathsf{k}}-h} \frac{\Gamma(2h+n)\Gamma(2h+\bar{n})}{\Gamma(2h)^2\Gamma(n+1)\Gamma(\bar{n}+1)} \,. \tag{4.66}$$

For $\mathsf{k} = 1$, there are no restrictions on the allowed values of the mode numbers n and $\bar{n}$, and so the sum can be performed straightforwardly to obtain

$$\langle s, \bar{s}, \mathsf{k} = 1 | \mathcal{O}_h(1) \mathcal{O}_h^\dagger(x,\bar{x}) | s, \bar{s}, \mathsf{k} = 1 \rangle \;=\; \frac{x^{-2hs}\bar{x}^{-2h\bar{s}}}{|1-x|^{4h}} \,. \tag{4.67}$$

This expression agrees with the corresponding HLLH correlator of the symmetric product orbifold CFT, as a direct consequence of the discussion around Eq. (4.61).

On the other hand, for $\mathsf{k} > 1$ the sum becomes more difficult to carry out explicitly since n and $\bar{n}$, which appear in the arguments of the Gamma functions, must satisfy the constraint

$$\bar{n} - n \;\equiv\; 2h\,(\bar{s} - s) \;\bmod \mathsf{k} \,, \tag{4.68}$$

which is the direct generalization of Eq. (4.63). We shall first briefly describe a method to construct these correlators iteratively, starting from the $h = \frac{1}{2}$ case obtained above, then present an improved method.

Our iterative construction operates by expressing the additional coefficients in the sum in Eq. (4.66) in terms of differential operators acting on results for lower values of h. Let us illustrate how this works for the simplest non-trivial case $h = 1$. From the general expression (4.66), we have

$$\begin{aligned}\langle s,\bar{s},\mathsf{k}|\mathcal{O}_{h=1}(1)\mathcal{O}^{\dagger}_{h=1}(x,\bar{x})|s,\bar{s},\mathsf{k}\rangle &= \frac{1}{\mathsf{k}^4}\sum_{n,\bar{n}}{}' x^{\frac{n-2s}{\mathsf{k}}-1}\bar{x}^{\frac{\bar{n}-2\bar{s}}{\mathsf{k}}-1}(n+1)(\bar{n}+1) \\ &= \frac{1}{\mathsf{k}^4}\frac{x^{-\frac{2s}{\mathsf{k}}}\bar{x}^{-\frac{2\bar{s}}{\mathsf{k}}}}{|x|^2}(\mathsf{k}x\partial_x+1)(\mathsf{k}\bar{x}\partial_{\bar{x}}+1)\sum_{n,\bar{n}}{}' x^{\frac{n}{\mathsf{k}}}\bar{x}^{\frac{\bar{n}}{\mathsf{k}}}\,.\end{aligned} \tag{4.69}$$

Hence, the differential operators act on a sum similar to the one analyzed in the previous section. For the general case, the procedure iterates. We redefine a and $\bar{a}$ to be generalizations of the a and $\bar{a}$ used in the $h = 1/2$ correlators (see above Eqs. (4.55) and (4.63)), where we replace $s \mapsto 2hs$, $\bar{s} \mapsto 2h\bar{s}$, such that $a - \bar{a} \equiv 2h(\bar{s}-s) \bmod \mathsf{k}$. Then we obtain

$$\langle s,\bar{s},\mathsf{k}|\mathcal{O}_{h}(1)\mathcal{O}^{\dagger}_{h}(x,\bar{x})|s,\bar{s},\mathsf{k}\rangle = \frac{1}{\mathsf{k}^{4h}}\frac{x^{-\frac{2hs}{\mathsf{k}}}\bar{x}^{-\frac{2h\bar{s}}{\mathsf{k}}}}{|x|^{2h}}\frac{D_{h,\mathsf{k}}\bar{D}_{h,\mathsf{k}}}{\Gamma(2h)^2}\sum_{n,\bar{n}}{}' x^{\frac{n}{\mathsf{k}}}\bar{x}^{\frac{\bar{n}}{\mathsf{k}}}\,, \tag{4.70}$$

where we have introduced differential operators of order $2h-1$ defined as

$$D_{h,\mathsf{k}} \equiv (\mathsf{k}x\partial_x + 2h - 1)\cdots(\mathsf{k}x\partial_x+1)\,, \tag{4.71}$$

and where

$$\sum_{n,\bar{n}}{}' x^{\frac{n}{\mathsf{k}}}\bar{x}^{\frac{\bar{n}}{\mathsf{k}}} = \frac{(1-x)|x|^{(a-\bar{a})/\mathsf{k}} + (1-\bar{x})|x|^{-(a-\bar{a})/\mathsf{k}} - |1-x|^2|x|^{-|a-\bar{a}|/\mathsf{k}}}{|1-x|^2\,(1-|x|^{2/\mathsf{k}})}\left(\frac{x}{\bar{x}}\right)^{(\bar{a}-a)/2\mathsf{k}}. \tag{4.72}$$

Although it leads to correct results, the procedure outlined above quickly becomes cumbersome, and leads to seemingly complicated expressions for higher values of h. In addition, it does not appear to give any insight into whether the results are likely to match with computations in the symmetric product orbifold CFT. However, we can improve on both these points with a different method. We now describe this method by first rederiving the $h = 1/2$ correlators of the previous section, and then generalizing the improved method to arbitrary values of h, and also to higher-point heavy-light correlators.

Let us thus re-examine the general expression Eq. (4.66), and consider the case in which $a - \bar{a} = 0$ for simplicity. We can take into account the restriction on the allowed values of n and $\bar{n}$ by considering an *unrestricted* sum over arbitrary positive integers by making use of a "Kronecker comb". In other words, we impose that $n - \bar{n} = 0 \bmod \mathsf{k}$ by including extra coefficients of the form

$$\sum_{q\in\mathbb{Z}}\delta_{n-\bar{n},\mathsf{k}q} = \frac{1}{\mathsf{k}}\sum_{r=0}^{\mathsf{k}-1}e^{2\pi i r\,\frac{n-\bar{n}}{\mathsf{k}}}\,, \tag{4.73}$$

where the final equality is obtained by Fourier transformation, and represents a simple form of the discrete Poisson summation formula. The RHS in Eq. (4.73) is interesting, because the exponentials can be absorbed into terms involving powers of x and $\bar{x}$. Explicitly, we can rewrite the expression (4.66) with $a-\bar{a}=0$ as

$$\frac{1}{\mathsf{k}^{4h+1}}\frac{x^{-\frac{2hs}{\mathsf{k}}}\bar{x}^{-\frac{2h\bar{s}}{\mathsf{k}}}}{|x|^{2h}}\sum_{n,\bar{n}\geq 0}\sum_{r=0}^{\mathsf{k}-1}u_r^n\bar{u}_r^{\bar{n}}\frac{\Gamma(2h+n)\Gamma(2h+\bar{n})}{\Gamma(2h)^2\Gamma(n+1)\Gamma(\bar{n}+1)}\,, \tag{4.74}$$

where u_r, $\bar{u}_r$ are the k^{th} roots of x and $\bar{x}$, respectively; writing $x^{\frac{1}{\mathsf{k}}}\equiv|x|^{\frac{1}{\mathsf{k}}}e^{2\pi i\frac{\mathrm{Arg}(x)}{\mathsf{k}}}$,

$$u_r\equiv x^{\frac{1}{\mathsf{k}}}e^{2\pi i\frac{r}{\mathsf{k}}}\,,\qquad \bar{u}_r\equiv\bar{x}^{\frac{1}{\mathsf{k}}}e^{-2\pi i\frac{r}{\mathsf{k}}}\,. \tag{4.75}$$

Thus, inside the convergence region $|x|<1$ we can exchange the order of the sums, such that the *unrestricted* sum over integers n and $\bar{n}$ leads to the usual expression for the CFT two-point function. However, it is evaluated at the different values of u_r, instead of the insertion point itself. Thus, the expression (4.74) becomes

$$\frac{1}{\mathsf{k}^{4h+1}}\frac{x^{-\frac{2hs}{\mathsf{k}}}\bar{x}^{-\frac{2h\bar{s}}{\mathsf{k}}}}{|x|^{2h}}\sum_{r=0}^{\mathsf{k}-1}\frac{1}{|1-u_r|^{4h}}\,. \tag{4.76}$$

In fact, we can rewrite this in a slightly more general form. Indeed, it is easy to "unfix" the first insertion point and write the full expression of the HLLH correlator in terms of x_1 and x_2. To do so, we introduce k^{th} roots of x_1, x_2, and x_2/x_1 via $u_{1,r_1}^{\mathsf{k}}=x_1$, $u_{1,r_2}^{\mathsf{k}}=x_2$, and $u_{21,r}^{\mathsf{k}}=x_2/x_1$, and then make use of the identity

$$\frac{1}{|x_1|^{\frac{4h}{\mathsf{k}}}}\sum_{r=0}^{\mathsf{k}-1}\frac{1}{|1-u_{21,r}|^{4h}}=\frac{1}{\mathsf{k}}\sum_{r_1,r_2=0}^{\mathsf{k}-1}\frac{1}{|u_{1,r_1}-u_{2,r_2}|^{4h}}\,. \tag{4.77}$$

This gives

$$\begin{aligned}&\langle s,\bar{s},\mathsf{k}|\mathcal{O}_h(x_1,\bar{x}_1)\mathcal{O}_h^\dagger(x_2,\bar{x}_2)|s,\bar{s},\mathsf{k}\rangle\Big|_{2h(s-\bar{s})\,=\,0\,\mathrm{mod}\,\mathsf{k}}\\&=\frac{1}{\mathsf{k}^{4h+2}}\left(\frac{x_2}{x_1}\right)^{-h\frac{(2s+1)}{\mathsf{k}}}\left(\frac{\bar{x}_2}{\bar{x}_1}\right)^{-h\frac{(2\bar{s}+1)}{\mathsf{k}}}|x_1x_2|^{2h\left(\frac{1}{\mathsf{k}}-1\right)}\sum_{r_1,r_2=0}^{\mathsf{k}-1}\frac{1}{|u_{1,r_1}-u_{2,r_2}|^{4h}}\,.\end{aligned} \tag{4.78}$$

The general case is computed entirely analogously. We must simply replace $n-\bar{n}\mapsto n-\bar{n}+(a-\bar{a})$ in Eq. (4.73), which induces some extra phases. The appropriate generalization of Eq. (4.77) is given by

$$\frac{1}{|x_1|^{\frac{4h}{\mathsf{k}}}}\sum_{r=0}^{\mathsf{k}-1}\frac{e^{2\pi i r(a-\bar{a})/\mathsf{k}}}{|1-u_{21,r}|^{4h}}=\frac{1}{\mathsf{k}}\sum_{r_1,r_2=0}^{\mathsf{k}-1}\frac{e^{2\pi i(r_2-r_1)(a-\bar{a})/\mathsf{k}}}{|u_{1,r_1}-u_{2,r_2}|^{4h}}\,. \tag{4.79}$$

Then the HLLH correlator with generic values of the orbifold parameter k, the spectral flow parameters s and $\bar{s}$, and the weight of the light chiral primary operator h, takes the form

$$\langle s, \bar{s}, \mathsf{k}|\mathcal{O}_h(x_1,\bar{x}_1)\mathcal{O}_h^\dagger(x_2,\bar{x}_2)|s,\bar{s},\mathsf{k}\rangle$$
$$= \frac{1}{\mathsf{k}^{4h+2}}\left(\frac{x_2}{x_1}\right)^{-h\frac{(2s+1)}{\mathsf{k}}}\left(\frac{\bar{x}_2}{\bar{x}_1}\right)^{-h\frac{(2\bar{s}+1)}{\mathsf{k}}}|x_1x_2|^{2h\left(\frac{1}{\mathsf{k}}-1\right)}\sum_{r_1,r_2=0}^{\mathsf{k}-1}\frac{e^{2\pi i(r_2-r_1)(a-\bar{a})/\mathsf{k}}}{|u_{1,r_1}-u_{2,r_2}|^{4h}}\,, \tag{4.80}$$

where the sum is over the k-th roots of the insertion points x_1 and x_2, as defined above (4.77), and where $2h(\bar{s}-s) \equiv a-\bar{a} \mod \mathsf{k}$.

Note that we can relax the chiral primary condition and consider operators in which $m' \neq \pm h$. We shall continue to focus on massless vertex operators, however this could be generalized further. In addition, by making use of the phases and the $x_{1,2}$ powers on the RHS of (4.80), we can rewrite the result in a cleaner form,

$$\langle s, \bar{s}, \mathsf{k}|\mathcal{O}_{h,m'}(x_1,\bar{x}_1)\mathcal{O}_{h,m'}^\dagger(x_2,\bar{x}_2)|s,\bar{s},\mathsf{k}\rangle$$
$$= \frac{1}{\mathsf{k}^2}\sum_{r_1,r_2=0}^{\mathsf{k}-1}\left(\frac{u_{2,r_2}}{u_{1,r_1}}\right)^{-m'(2s+1)}\left(\frac{\bar{u}_{2,r_2}}{\bar{u}_{1,r_1}}\right)^{-\bar{m}'(2\bar{s}+1)}\frac{|u_{1,r_1}u_{2,r_2}|^{2h(1-\mathsf{k})}}{\mathsf{k}^{4h}|u_{1,r_1}-u_{2,r_2}|^{4h}}\,. \tag{4.81}$$

4.3.2 Matching between worldsheet and symmetric product orbifold

The appearance of the k^{th} roots of the physical insertions $x_{1,2}$ in Eq. (4.81) is related to the fact that the holographic description of the heavy backgrounds involves heavy states in k-twisted sectors of the boundary CFT. The same feature appears in certain computations performed using the Lunin-Mathur covering space technique [97], specifically when there are operators of twist k inserted at the origin and infinity of the CFT plane, and when there are other untwisted operators in the correlator. Then the coordinate transformation to the k-fold covering space is precisely $x = u^{\mathsf{k}}$.

Thus, when the light worldsheet operators correspond to untwisted operators of the symmetric product orbifold CFT, it is natural to identify u with the coordinate on the k-fold covering space that trivializes the twist operators involved in the definition of the heavy states.

The sum over the different roots generates the usual phases included in the definition of fractional modes by summing over the different copies of the theory [97],

$$O_{\frac{m}{\mathsf{k}}} = \oint \frac{dx}{2\pi i}\sum_{r=1}^{\mathsf{k}} O_{(r)}(x)e^{\frac{2\pi i m}{\mathsf{k}}(r-1)}x^{h+\frac{m}{\mathsf{k}}-1}\,. \tag{4.82}$$

Moreover, the *fractional* spectral flow defining the background, when mapped to a k-fold covering space, becomes *integer* spectral flow with parameters $2s+1$ and $2\bar{s}+1$ [96, 94]. Hence, one can generalize the discussion around Eq. (4.61) and simply consider the appropriate powers of u_{i,r_i} to arise from performing spacetime spectral flow on the operators, in the k-fold covering space. Finally, the last factor on the RHS of (4.81) corresponds to the usual two-point function evaluated at the roots, including the necessary Jacobian factors arising from mapping to the k-fold covering space, $|\partial u/\partial x|^{2h}$. Obtaining precisely this Jacobian is the justification for the factors of k introduced in the definition of the x-basis operators in Eq. (4.47). Thus, we see that symmetric orbifold CFT HLLH correlators for which the covering space is $x = u^{\mathsf{k}}$ agree in both structure and value with the worldsheet correlator (4.81).

By contrast, for twisted operators, the interpretation of our worldsheet result (4.81) is more involved: the Lunin-Mathur covering map for such correlators is *not* $x = u^{\mathsf{k}}$. To understand the precise relation, we focus on light operators of twist two, and show that Eq. (4.81) nevertheless matches with the symmetric orbifold CFT also in this case. The relevant four-point function was studied recently in [183, 181, 182] in the $\mathrm{Sym}^N(\mathbb{T}^4)$ CFT. At leading order in large N, the correlator is dominated by a contribution from a covering space with genus zero, where the copy indices of the light twist-two operator act on different k-strands corresponding to the heavy state. One of the light insertions effectively joins together two k-strands into a $2\mathsf{k}$-strand, and the other light insertion effectively cuts the $2\mathsf{k}$-strand back to two strands of length k. For this process, and setting for simplicity $s = \bar{s} = 0$ as done in [181], the relation between the physical-space cross-ratio x and the covering-space cross ratio v is[45]

$$x(v) = \left(\frac{v+1}{v-1}\right)^{2\mathsf{k}} . \tag{4.83}$$

The correlator of interest is written in terms $v(x)$ as defined through Eq. (4.83), and involves the sum over the $2\mathsf{k}$ pre-images of x and also an N- and k-dependent overall factor. However, due to the $v \to 1/v$ symmetry of the map (4.83), there are actually only k distinct contributions [182], corresponding to distinct ramified coverings of the base space. In more detail, up to overall normalization the correlator takes the form [182, Eq. (4.17)]

$$\langle \mathsf{k}|\mathcal{O}_{h,m'}(x,\bar{x})\mathcal{O}^{\dagger}_{h,m'}(1)|\mathsf{k}\rangle \;\sim\; \sum_{r=0}^{\mathsf{k}-1}\Big[v_r^{K_0}(v_r-1)^{K_-}(v_r+1)^{K_+}\,\times\,\text{c.c.}\Big] , \tag{4.84}$$

where $|\mathsf{k}\rangle = |s = \bar{s} = 0, \mathsf{k}\rangle$. The exponents K_0 and $K_\pm$ are defined in [182, Eq. (4.17b)]; for this particular correlator, they evaluate to $K_0 = -2h$ and $K_\pm = 2h(1 \mp \mathsf{k}) \pm 2m'$. Upon inserting the explicit solutions

$$v_r(x) \;=\; \frac{x^{\frac{1}{2\mathsf{k}}}e^{\frac{i\pi r}{\mathsf{k}}}+1}{x^{\frac{1}{2\mathsf{k}}}e^{\frac{i\pi r}{\mathsf{k}}}-1} , \qquad r = 0,\ldots,\mathsf{k}-1 , \tag{4.85}$$

[45] Note that in [181, 182], the base (physical) space coordinates are denoted by z or u rather than our x, while the covering space coordinates are denoted by t or x rather than our v.

where, as before, $x^{\frac{1}{2\mathsf{k}}}$ stands for a particular $(2\mathsf{k})^{\text{th}}$ root of x, the final expression remarkably coincides with Eq. (4.81). The analysis for the JMaRT states and for more general light insertions can be carried out analogously.

Recall that, as reviewed in Section 2.5.1, at a generic spacetime dimension h there is a degeneracy in the twist n of light states in the symmetric product orbifold CFT. An interesting feature of the worldsheet correlator (4.81) is that it is independent of this twist n. Recall also that, for untwisted light operators, the worldsheet correlator has the same structure as the covering space method of the symmetric product orbifold CFT. The fact that the agreement of HLLH correlators extends to (at least some) twisted light operators is thus remarkable from the point of view of the holographic CFT. Despite the more complicated covering map, the above discussion demonstrates how, for these correlators, the end result agrees with an expression whose structure is that of the simple map $x = u^{\mathsf{k}}$.

4.3.3 Higher-point heavy-light correlators

Our general expression for HLLH correlators, Eq. (4.81), together with the matching to the symmetric product orbifold CFT that we have observed so far, motivate a deeper exploration. Thus, we now describe how local x-basis operators are seen from the spectrally flowed frame as indicated by our null-gauged worldsheet models. This will allow us to extract consequences for worldsheet three-point and higher-point functions, corresponding to holographic CFT correlators with two heavy states and three or more light operators.

The AdS_3 limit of the holomorphic gauge condition, (4.45), upon using the definition of m_y in Eq. (4.46), reads

$$0 = m + (2s+1)m' - \mathsf{k}m_y\,. \tag{4.86}$$

We wish to re-interpret this constraint in the local coordinate basis of the holographic CFT. A priori, it is perhaps not obvious that this is a useful thing to do, since the usual x-basis operators are constructed by resumming the action of $J_0^{\pm}$, which does not commute with the BRST charge in the coset theory. However we shall see that it will be very useful.

Let us observe that there are two notions of x-type local coordinates in the worldsheet model. The one used so far in Section 4.2 and the present section is the physical x coordinate of the gauged models. However, before gauging, there is an analogous coordinate for the *upstairs* SL(2,$\mathbb{R}$) algebra. We will denote the associated coordinate by the complex variable u; we will see momentarily that $u^{\mathsf{k}} = x$, so that there will be no clash with the u used above.

The differential operator $x\partial_x + h$ corresponds to the quantity m_y, as can be seen by comparing Eqs. (2.110), (2.115), (4.47). On the other hand, the upstairs SL(2,$\mathbb{R}$) projection m corresponds to an analogous operator in the u variable: we write this as $u\partial_u + h - \beta$, where we have allowed for a shift β, whose precise form will become clear shortly, as will the reason for its existence. Then Eq. (4.86) can be expressed in terms of these differential operators as

$$\mathsf{k}x\partial_x \;=\; u\partial_u + (2s+1)m' + h(1-\mathsf{k}) - \beta\,. \tag{4.87}$$

In order that this condition is solved by $u^{\mathsf{k}} = x$, we choose

$$\beta = h(1-\mathsf{k}) + (2s+1)m'\,. \tag{4.88}$$

Thus the role of β is two-fold. On the one hand, the first term in (4.88) effectively replaces the weight by $h \to h_u \equiv \mathsf{k}h$, which further supports the discussion above about u corresponding to a covering space coordinate in the holographic CFT. It also generates the Jacobian factor obtained in Eq. (4.81). On the other hand, the second term in (4.88) takes into account the shift arising from spacetime spectral flow.

We now use this to obtain an improved construction of gauge-invariant operators directly in the x-basis, built upon u-basis operators of the upstairs SL(2,$\mathbb{R}$), i.e. without relying on their spacetime Virasoro mode expansion as in (4.47). Although such a construction gives equivalent results at the level of worldsheet two-point functions (4.81), its importance for higher-point functions was highlighted recently in [38]. The construction proceeds as follows:

1. We consider an operator whose upstairs SL(2,$\mathbb{R}$) part is expressed in the usual local SL(2,$\mathbb{R}$) basis, $\mathcal{V}_h(u,\bar{u})$, where for simplicity we set $h=\bar{h}$. We multiply this by an SU(2) vertex operator $\mathcal{V}'_{j',m',\bar{m}'}$. We suppress the exponentials of t and y, since they have weight zero in the AdS$_3$ limit, and their only effect is taken into account through (4.86) and its antiholomorphic counterpart. We introduce the notation
$$\hat{\mathcal{O}}_{h,m',\bar{m}'}(u,\bar{u}) \;\equiv\; \mathcal{V}_h(u,\bar{u})\mathcal{V}'_{j',m',\bar{m}'}\,. \tag{4.89}$$
2. We introduce the above β-shift by multiplying by an extra factor $u^{\beta}\bar{u}^{\bar{\beta}}$.
3. We sum the resulting operator over all insertion points u such that $u^{\mathsf{k}} = x$.

Explicitly, we define

$$\mathcal{O}_{h,m',\bar{m}'}(x,\bar{x}) \;\equiv\; \frac{1}{\mathsf{k}^{2h+1}}\sum_{u^{\mathsf{k}}=x} u^{\beta}\bar{u}^{\bar{\beta}}\hat{\mathcal{O}}_{h,m',\bar{m}'}(u,\bar{u})\,. \tag{4.90}$$

Comparing with Eq. (4.47), using the Kronecker comb (4.73) to impose the constraints as above, we indeed have

$$\begin{aligned}
\mathcal{O}_{h,m',\bar{m}'}(x,\bar{x}) &\equiv \frac{1}{\mathsf{k}^{h+\bar{h}}} \sum_{m_y,\bar{m}_y} x^{m_y-h}\bar{x}^{\bar{m}_y-h}\,\mathcal{V}_{j,m,\bar{m}}\mathcal{V}'_{j',m',\bar{m}'} \\
&= \frac{1}{\mathsf{k}^{2h}} \sum_{m,\bar{m}}{}' \, x^{\frac{1}{\mathsf{k}}[m+(2s+1)m']-h}\,\bar{x}^{\frac{1}{\mathsf{k}}[\bar{m}+(2\bar{s}+1)\bar{m}']-h}\,\mathcal{V}_{j,m,\bar{m}}\mathcal{V}'_{j',m',\bar{m}'} \\
&= \frac{1}{\mathsf{k}^{2h+1}} \sum_{u^{\mathsf{k}}=x}\sum_{m,\bar{m}} u^{m-h+(2s+1)m'+h(1-\mathsf{k})}\,\bar{u}^{\bar{m}-h+(2\bar{s}+1)\bar{m}'+h(1-\mathsf{k})}\,\mathcal{V}_{j,m,\bar{m}}\mathcal{V}'_{j',m',\bar{m}'} \\
&= \frac{1}{\mathsf{k}^{2h+1}} \sum_{u^{\mathsf{k}}=x} u^{\beta}\bar{u}^{\bar{\beta}}\,\hat{\mathcal{O}}_{h,m',\bar{m}'}(u,\bar{u})\,.
\end{aligned} \tag{4.91}$$

We note that in the symmetric product orbifold CFT, when mapping the S_{k}-invariant untwisted operators $O(x) = \sum_{r=1}^{\mathsf{k}} O_{(r)}(x)$ to the k-fold covering space, using the inverse relation to (4.82),

$$O_{(r)}(x) \;=\; \frac{1}{\mathsf{k}}\sum_m O_{\frac{m}{\mathsf{k}}}\, x^{-\frac{m}{\mathsf{k}}-h}e^{-\frac{2\pi i m}{\mathsf{k}}(r-1)}, \tag{4.92}$$

one obtains an expression closely analogous to Eq. (4.90).

We now exploit the expression (4.91) to study higher-point functions. We first rewrite the HLLH correlator (4.81) in the simple form

$$\langle s,\bar{s},\mathsf{k}|\mathcal{O}_{h_1,m_1'}(x_1,\bar{x}_1)\mathcal{O}^{\dagger}_{h_2,m_2'}(x_2,\bar{x}_2)|s,\bar{s},\mathsf{k}\rangle \;=\; \frac{1}{\mathsf{k}^{4h+2}}\sum_{u_i^{\mathsf{k}}=x_i}\frac{u_1^{\beta_1}\bar{u}_1^{\bar{\beta}_1}u_2^{\beta_2}\bar{u}_2^{\bar{\beta}_2}}{|u_1-u_2|^{4h}}\,, \tag{4.93}$$

with $\beta_i = h_i(1-\mathsf{k})+(2s+1)m_i'$, $\bar{\beta}_i = h_i(1-\mathsf{k})+(2\bar{s}+1)\bar{m}_i'$, and $h_1 = h_2 = h$, and where the charge conservation $m_1'+m_2' = 0$ is understood. We then deduce that the worldsheet correlator with n light insertions with weights h_i and charges $m_i', \bar{m}_i'$ is given by the following straightforward generalization of (4.93) (in which we partially suppress antiholomorphic quantities):

$$\begin{aligned}
&\langle s,\bar{s},\mathsf{k}|\mathcal{O}_{h_1,m_1'}(x_1)\ldots\mathcal{O}_{h_n,m_n'}(x_n)|s,\bar{s},\mathsf{k}\rangle \\
&\qquad = \frac{1}{\mathsf{k}^{H+\bar{H}+n}}\sum_{u_i^{\mathsf{k}}=x_i}\left(\prod_{\ell=1}^{n} u_\ell^{\beta_\ell}\bar{u}_\ell^{\bar{\beta}_\ell}\right)\langle\hat{\mathcal{O}}_{h_1,m_1'}(u_1)\ldots\hat{\mathcal{O}}_{h_n,m_n'}(u_n)\rangle\,,
\end{aligned} \tag{4.94}$$

where $H = h_1+\cdots+h_n$, and $\langle\hat{\mathcal{O}}_{h_1,m_1'}(u_1)\ldots\hat{\mathcal{O}}_{h_n,m_n'}(u_n)\rangle$ stands for the global $\mathrm{AdS}_3\times\mathbb{S}^3$ vacuum n-point function evaluated at the roots of the original insertion points.

The expression (4.94), which holds for generic values of $s,\bar{s},\mathsf{k}$ and generic light weights and charges $h_i, m_i', \bar{m}_i$, constitutes one of the main results of this thesis. In Eq. (4.94), the $n=3$ case can be made quite explicit, as we shall do so in the next subsection.

The above result can straightforwardly seen to include spectrally-flowed vertex operators, as follows. Setting $\omega' = \bar{\omega}' = \omega$ for simplicity, the bosonic null-gauge condition

Eq. (4.40) in the AdS limit becomes

$$0 = m_\omega + (2s+1)m'_\omega - \mathsf{k}m_y \,, \qquad 0 = \bar{m}_\omega + (2\bar{s}+1)\bar{m}'_\omega - \mathsf{k}\bar{m}_y \,, \tag{4.95}$$

where, for discrete states in the lowest weight representation, $m_\omega = h_\omega + n$, $h_\omega = J + n_5\,\omega/2$ and $m'_\omega = h'_\omega + n_5\,\omega/2 - n'$. As a consequence, the exponent β of the covering space coordinate u gets replaced by $\beta \mapsto \beta_\omega = h_\omega(1-\mathsf{k}) + (2s+1)m'_\omega$, and the power of k in the normalisation factor is modified accordingly. Thus, for the vertex operators in the coset models, the net effects of the spectral flow procedure are the replacements $h \mapsto h_\omega, m \mapsto m'_\omega$. This is understood by the fact that when a boundary light operator has the spacetime dimension $h = J$ that renders the SL(2,$\mathbb{R}$) spin above the unitary bound Eq. (2.106), it corresponds holographically to a spectrally-flowed worldsheet vertex operator [101]. This implies that the structure of the correlator in Eq. (4.94) is not drastically modified when $\omega \neq 0$.

It is important to note, however, that the entire computational complication due to worldsheet spectral flow remains present in the resulting *vacuum* correlator. Indeed, we see that the n-point function on the heavy state is now written in terms of a vacuum n-point function of spectrally-flowed states. It is thus natural to expect that the AdS$_3$ selection rules carry over to n-point functions in the JMaRT microstates. We conclude that the generalisation of Eq. (4.94) to the case of worldsheet spectrally-flowed states reads

$$\begin{aligned}
&\langle s, \bar{s}, \mathsf{k}| \mathcal{O}^{\omega_1}_{h_1,m'_1}(x_1)\dots\mathcal{O}^{\omega_n}_{h_n,m'_n}(x_n)|s,\bar{s},\mathsf{k}\rangle = \\
&\qquad \frac{1}{\mathsf{k}^{H_\omega+\bar{H}_\omega+n}} \sum_{u_i^{\mathsf{k}}=x_i} \left(\prod_{\ell=1}^{n} u_\ell^{\beta_{\omega,\ell}}\bar{u}_\ell^{\bar{\beta}_{\omega,\ell}}\right) \langle \hat{\mathcal{O}}^{\omega_1}_{h_1,m'_1}(u_1)\dots\hat{\mathcal{O}}^{\omega_n}_{h_n,m'_n}(u_n)\rangle \,,
\end{aligned} \tag{4.96}$$

where $H_\omega = \sum_i h_{\omega,i}$ and the light operators $\mathcal{O}^{\omega_i}_{h_i,m'_i}$ are x-basis spectrally-flowed worldsheet vertex operators.

We emphasize that the construction we have outlined in this section only holds in the IR AdS$_3 \times \mathbb{S}^3$ limit. In the full asymptotically linear dilaton geometry, the identification of the modes m_y and $\bar{m}_y$ as defined in (4.46) breaks down, and the t and y exponentials can no longer be ignored. This is consistent with the fact that in the UV the dual holographic theory is not a CFT, but is instead a little string theory. Since little string theories are non-local, it is correct that the above definition of local operators does not apply. Note, however, that the *mode* correlators computed in the m-basis still make perfect sense, and carry information about string perturbation theory in the full geometry.

Let us speculate on which subset of the above correlators can be expected to agree with those of the symmetric product orbifold theory. Since our expressions for the general correlators (4.94), (4.96) involve vacuum correlators, it is natural to conjecture that for these particular heavy backgrounds, the heavy-light correlator is protected whenever the global $AdS_3 \times \mathbb{S}^3$ vacuum correlator appearing in (4.94), (4.96) is protected. Recall that, in the global $AdS_3 \times \mathbb{S}^3$ vacuum, two-point and three-point correlation functions of chiral primaries are protected [184], while four-point and higher-point functions are generically renormalized. So heavy-light correlators with two or three light insertions on these backgrounds may be protected between worldsheet and symmetric product orbifold CFT. It may even be possible to prove a non-renormalization theorem generalizing [184]; work in this direction is in progress. For now however, we next compute a heavy-light correlator with three light insertions in both worldsheet and holographic CFT.

4.3.4 An HLLLH correlator in worldsheet and holographic CFT

We now investigate the general expression for our worldsheet correlator (4.94), in a particular example with three light insertions, and compare it to the symmetric product orbifold CFT. We shall observe another highly non-trivial agreement.

We consider three light insertions living in the untwisted sector of the holographic CFT, with weights $(h_1, h_2, h_3) = (\frac{1}{2}, \frac{1}{2}, 1)$. In the dual CFT notation, we are then interested in computing the correlator $\langle O_{\frac{1}{2}}(x_1) O_{\frac{1}{2}}(x_2) O_1^{\dagger}(x_3)\rangle_H$. We further focus on heavy backgrounds with $s = \mathsf{k}p$ with $p \in \mathbb{Z}$ and $\bar{s} = 0$.

We start by evaluating the general expression (4.94) for this particular worldsheet correlator. In the worldsheet theory associated to the global $AdS_3 \times \mathbb{S}^3$, the $O_{\frac{1}{2}}$ correspond to two RR states, while O_1 is an NSNS state polarized on the $\mathbb{S}^3$ directions. The (integrated) vacuum three-point functions for these chiral primaries were studied in [99]. In our notation, they take the form

$$\langle \mathcal{O}_{h_1}^{\rm RR}(x_1) \mathcal{O}_{h_2}^{\rm RR}(x_2) \mathcal{O}_{h_3}^{\dagger\,\rm NSNS}(x_3)\rangle \;=\; \frac{1}{N^{1/2}} \frac{\sqrt{(2j_1-1)(2j_2-1)(2j_3-1)^{-1}}}{|x_{12}|^{2(h_1+h_2-h_3)}|x_{13}|^{2(h_1+h_3-h_2)}|x_{23}|^{2(h_2+h_3-h_1)}}\,, \tag{4.97}$$

where $j_1 = h_1 + \frac{1}{2}$, $j_2 = h_2 + \frac{1}{2}$, and $j_3 = h_3$. The relevant values for us are simply $j_i = 1$, and upon a global SL(2,$\mathbb{C}$) transformation to set $x_3 = 1$, we have

$$\langle \mathcal{O}_{\frac{1}{2}}(x_1) \mathcal{O}_{\frac{1}{2}}(x_2) \mathcal{O}_1^{\dagger}(1)\rangle \;=\; \frac{1}{N^{1/2}} \frac{1}{|1-x_1|^2|1-x_2|^2}\,. \tag{4.98}$$

In order to compute the HLLLH correlator in the worldsheet coset models corresponding to the JMaRT backgrounds, we must sum (4.98) evaluated at all $\mathsf{k}^{\rm th}$-roots of the insertion points. An explicit expression can be obtained following the arguments of Sec. 4.2.3, using the following generalisation of Eq. (4.77) and Eq. (4.79) for the case of three insertions,

$$\frac{\mathsf{k}}{|x_3|^{\frac{2(\alpha+\beta)}{\mathsf{k}}}} \sum_{r_{1,2}=0}^{\mathsf{k}-1} \frac{1}{|1-u_{13,r}|^{2\alpha}|1-u_{23,r}|^{2\beta}} \;=\; \sum_{r_{1,2,3}=0}^{\mathsf{k}-1} \frac{1}{|u_{3,r_3}-u_{1,r_1}|^{2\alpha}|u_{3,r_3}-u_{2,r_2}|^{2\beta}}\,, \tag{4.99}$$

where $u_{i,r_i} = x_i^{1/\mathsf{k}} e^{2\pi i\, r_i/\mathsf{k}}$ and $u_{j\ell,r} = (x_j/x_\ell)^{1/\mathsf{k}} e^{2\pi i\, r/\mathsf{k}}$ with $\alpha = \beta = 1$, one obtains

$$\langle \mathcal{O}_{\frac{1}{2}}(x_1)\mathcal{O}_{\frac{1}{2}}(x_2)\mathcal{O}_1^\dagger(1)\rangle_H \;=\; \frac{1}{\mathsf{k}^7}\frac{(x_1x_2)^p\,|x_1x_2|^{\frac{2}{\mathsf{k}}-1}}{|1-x_1|^2|1-x_2|^2}\,\frac{1-|x_1|^2}{1-|x_1|^{\frac{2}{\mathsf{k}}}}\,\frac{1-|x_2|^2}{1-|x_2|^{\frac{2}{\mathsf{k}}}}\,. \tag{4.100}$$

The result (4.100) constitutes the first computation of a heavy-light worldsheet correlator with three light insertions probing a black hole microstate.

We now show that the same result can be obtained from the HCFT at the symmetric orbifold point. We follow the method used in [167, App. A] for the HLLH correlator reviewed in Section 4.2.3. The heavy states we use indicate that we should work in the k-twisted sector of the theory. The operators can be written in terms of the fermions introduced in Eq. (2.97). For the $h = \frac{1}{2}$ chiral primaries, and for each strand of k copies of the theory, this simply reads

$$O_{\frac{1}{2}} \;=\; \sum_{r=1}^{\mathsf{k}} O_{\frac{1}{2},(r)} \;=\; -\frac{i}{\sqrt{2}}\sum_{r=1}^{\mathsf{k}} \psi_{(r)}^{+\dot{A}}\tilde{\psi}_{(r)}^{+\dot{B}}\epsilon_{\dot{A}\dot{B}} \;=\; -\frac{i}{\sqrt{2}}\sum_{\rho=0}^{\mathsf{k}-1}\psi_\rho^{+\dot{A}}\tilde{\psi}_\rho^{+\dot{B}}\epsilon_{\dot{A}\dot{B}}\,, \tag{4.101}$$

while for the $h = 1$ operator we find

$$\begin{aligned} O_1^\dagger \;=\; \sum_{r=1}^{\mathsf{k}} O_{1,(r)} \;&=\; \frac{1}{4}\sum_{r=1}^{\mathsf{k}} \psi_{(r)}^{-\dot{A}}\psi_{(r)}^{-\dot{B}}\tilde{\psi}_{(r)}^{-\dot{C}}\tilde{\psi}_{(r)}^{-\dot{D}}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} \\ &=\; \frac{1}{4\mathsf{k}}\sum_{\rho_i=0}^{\mathsf{k}-1}\delta_{\rho_1+\rho_2,\rho_3+\rho_4}\,\psi_{\rho_1}^{-\dot{A}}\psi_{\rho_2}^{-\dot{B}}\tilde{\psi}_{\rho_3}^{-\dot{C}}\tilde{\psi}_{\rho_4}^{-\dot{D}}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}}. \end{aligned} \tag{4.102}$$

We will work in the bosonized language, in which

$$\psi_\rho^{+\dot{1}} = ie^{iH_\rho}\,, \qquad \psi_\rho^{-\dot{2}} = ie^{-iH_\rho}\,, \qquad \psi_\rho^{+\dot{2}} = e^{iK_\rho}\,, \qquad \psi_\rho^{-\dot{1}} = e^{-iK_\rho}\,, \tag{4.103}$$

Here H_ρ and K_ρ are canonically normalized bosonic fields, in terms of which the (unit normalized) heavy states take the form [167]

$$|H\rangle = |s = \mathsf{k}p,\mathsf{k}\rangle = \left[\Sigma_{\mathsf{k}}\tilde{\Sigma}_{\mathsf{k}}\prod_{\rho=0}^{\mathsf{k}-1} e^{i\left(p+\frac{1}{2}-\frac{\rho}{\mathsf{k}}\right)(H_\rho+K_\rho)}e^{i\left(\frac{1}{2}-\frac{\rho}{\mathsf{k}}\right)(\tilde{H}_\rho+\tilde{K}_\rho)}\right]^{\frac{N}{\mathsf{k}}}|0\rangle\,, \tag{4.104}$$

where Σ_{k} and $\tilde{\Sigma}_{\mathsf{k}}$ are the twist operators. Note that the contribution of Σ_{k} and $\tilde{\Sigma}_{\mathsf{k}}$ to the correlators will simply factorize, since the ψ_ρ fermions diagonalize the twisted boundary conditions. Choosing the labelling of the insertion points for later convenience, the correlator to be computed is then

$$\langle O_{\frac{1}{2}}(x_1)O_{\frac{1}{2}}(x_2)O_1^\dagger(x_3)O_H(x_4)O_H^\dagger(x_5)\rangle = \frac{1}{\mathsf{k}}\prod_{\rho,\rho'=0}^{\mathsf{k}-1}\sum_{\rho_i=0}^{\mathsf{k}-1}\delta_{\rho_3+\rho_4,\rho_5+\rho_6} \tag{4.105}$$
$$\Big\langle \left[\psi_{\rho_1}^{+\dot{A}_1}\tilde{\psi}_{\rho_1}^{+\dot{B}_1}\epsilon_{\dot{A}_1\dot{B}_1}\right](x_1)\left[\psi_{\rho_2}^{+\dot{A}_2}\tilde{\psi}_{\rho_2}^{+\dot{B}_2}\epsilon_{\dot{A}_2\dot{B}_2}\right](x_2)\left[\psi_{\rho_3}^{-\dot{A}_3}\psi_{\rho_4}^{-\dot{B}_3}\tilde{\psi}_{\rho_5}^{-\dot{C}_3}\tilde{\psi}_{\rho_6}^{-\dot{D}_3}\epsilon_{\dot{A}_3\dot{B}_3}\epsilon_{\dot{C}_3\dot{D}_3}\right](x_3)$$
$$e^{i\left(p+\frac{1}{2}-\frac{\rho}{\mathsf{k}}\right)(H_\rho+K_\rho)}(x_4)e^{-i\left(p+\frac{1}{2}-\frac{\rho'}{\mathsf{k}}\right)(H_{\rho'}+K_{\rho'})}(x_5)e^{i\left(\frac{1}{2}-\frac{\rho}{\mathsf{k}}\right)(\tilde{H}_\rho+\tilde{K}_\rho)}(x_4)e^{-i\left(\frac{1}{2}-\frac{\rho'}{\mathsf{k}}\right)(\tilde{H}_{\rho'}+\tilde{K}_{\rho'})}(x_5)\Big\rangle.$$

Clearly, charge conservation implies $\rho = \rho'$. For the same reason, the correlator vanishes unless $\rho_1 = \rho_3$ and $\rho_2 = \rho_4$, or $\rho_1 = \rho_4$ and $\rho_2 = \rho_3$, or both. An analogous statement holds with ρ_3, ρ_4 replaced by ρ_5, ρ_6, hence all contributions trivially satisfy the $\rho_3 + \rho_4 = \rho_5 + \rho_6$ constraint. Consequently, we only really need to sum over all possible values of, say, ρ_1 and ρ_2, and also compute the product over ρ. In this way, up to an irrelevant numerical factor, the holomorphic free field contractions give

$$\sum_{\rho_1,\rho_2=0}^{\mathsf{k}-1}\frac{1}{|x_{45}^{2h_H}x_{13}x_{23}|^2}\left(\frac{x_{41}x_{35}}{x_{51}x_{34}}\right)^{p+\frac{1}{2}-\frac{\rho_1}{\mathsf{k}}}\left(\frac{x_{42}x_{35}}{x_{52}x_{34}}\right)^{p+\frac{1}{2}-\frac{\rho_2}{\mathsf{k}}}\left(\frac{\bar{x}_{41}\bar{x}_{35}}{\bar{x}_{51}\bar{x}_{34}}\right)^{\frac{1}{2}-\frac{\rho_1}{\mathsf{k}}}\left(\frac{\bar{x}_{42}\bar{x}_{35}}{\bar{x}_{52}\bar{x}_{34}}\right)^{\frac{1}{2}-\frac{\rho_2}{\mathsf{k}}} \tag{4.106}$$

where h_H is the weight of the heavy state.

We can now take $x_3 \to 1$, $x_4 \to 0$ and $x_5 \to \infty$, and perform the sums over ρ_1 and ρ_2 explicitly. Upon doing so, we find that the structure of this orbifold CFT correlator Eq. (4.106) precisely matches the worldsheet correlator (4.100).

4.3.5 Hawking radiation from the worldsheet

As a final application of our results, we now use the HLLH correlator (4.81) to compute the amplitude that describes the analogue of the Hawking radiation process for the JMaRT backgrounds [140, 141, 93, 94]. In the bulk, this process is ergoregion radiation, which is a feature of the full asymptotically flat JMaRT solutions [147]. The ergoregion does not survive the fivebrane decoupling limit [52] or AdS_3 decoupling limit, however aspects of the process can still be studied quantitatively in those limits. This process has been interpreted as an enhanced analogue of Hawking radiation, since both are described by the same microscopic process in the holographic CFT [140]. Indeed, acting on a thermal state, this vertex operator gives precisely the spectrum and rate of Hawking radiation of the corresponding black hole, while acting on the states dual to the JMaRT solutions yields their characteristic spectrum and rate of emission [140, 141, 93, 94].

The emission spectrum and rate for general $\mathsf{k}, s, \bar{s}$ was computed in supergravity and symmetric product orbifold CFT in [94], building on the results of [140, 141, 93]. We will reproduce these results from the worldsheet CFT.

We start with a specific HLLH correlator in which the light operators are given by minimally coupled scalars in six dimensions, after reducing on the $\mathbb{T}^4$. The corresponding vertex operators were defined in Eq. (4.13). These are not chiral primaries of the boundary theory, but are their superdescendants within the short multiplet, so the holographic correlator arising in the AdS_3 limit is easily computed by using the techniques outlined in the previous sections. The amplitude of interest involves an initial state consisting of a probe excitation on top of the JMaRT background, a vertex operator $\mathcal{V}$ associated to a light insertion, and a final state given by the black hole microstate. Schematically we have

$$\langle s, \bar{s}, \mathsf{k} | \, \mathcal{O}(x) \, | s, \bar{s}, \mathsf{k} + \text{probe} \rangle = \langle s, \bar{s}, \mathsf{k} | \, \mathcal{O}(x) \, \mathcal{O}^\dagger(0) \, | s, \bar{s}, \mathsf{k} \rangle \ . \tag{4.107}$$

To begin with, we work with $\mathsf{k} = 1$. Up to an overall sign, and considering the lowest energy state, the holomorphic part of the amplitude for the Hawking emission of a single quanta of dimension $h = \frac{l}{2} + 1$ and whose corresponding vertex operator has charge $m' = k - \frac{l}{2}$ reads [93]

$$\mathcal{A}_L(x) = \frac{1}{x^{(1+\alpha)\frac{l}{2} - \alpha k + 1}} = \frac{1}{x^{\frac{l}{2} + 1 - \alpha(k - \frac{l}{2})}} \,. \tag{4.108}$$

Here $\frac{l}{2}$ denotes the total angular momentum of the probe on the $\mathbb{S}^3$ part of the geometry, while k is the number of $\mathbb{J}_0^+$ operators acting on the state with the lowest projection, appearing in the definition of the vertex operator. To compare their computation with our worldsheet result, one uses the following (notation) map:

$$k - \frac{l}{2} \mapsto m' \ , \qquad \frac{l}{2} + 1 \mapsto h \ , \qquad \alpha \mapsto l_2 = \mathsf{m} + \mathsf{n} = 2s + 1 \,. \tag{4.109}$$

Taking care of the cylinder-to-plane conversion factor $x^{-\frac{l}{2}-1}$, one obtains

$$\mathcal{A}_L(x) = \frac{1}{x^{2h - m'(2s+1)}} \,. \tag{4.110}$$

We now perform the analogous computation in the worldsheet cosets. From Eq. (4.107), in the worldsheet formalism all we need to do is to insert the second operator at the boundary origin, i.e. to take the $x_2 \to 0$ limit in Eq. (4.81) (with $\mathsf{k} = 1$ for now). This gives

$$\begin{aligned} \lim_{x_2 \to 0} \ x_2^{m'(2s+1)} \bar{x}_2^{\bar{m}'(2\bar{s}+1)} \ \langle s, \bar{s}, 1 | \mathcal{O}_h^{(m', \bar{m}')}(x_1) \mathcal{O}_h^{(m', \bar{m}')\dagger}(x_2) | s, \bar{s}, 1 \rangle \\ = \frac{1}{x^{2h - m'(2s+1)}} \frac{1}{\bar{x}^{2h - \bar{m}'(2\bar{s}+1)}} \,, \end{aligned} \tag{4.111}$$

in agreement with (4.110) upon including the antiholomorphic contribution.

The procedure is analogous for general $\mathsf{k}, s, \bar{s}$. We again evaluate the amplitude for $x_2 \to 0$ by including the appropriate Jacobian factor for the light state, and obtain

$$\begin{aligned}
&\lim_{x_2\to 0} \mathsf{k}^{2h} x_2^{m_2' \frac{(2s+1)}{\mathsf{k}}+h(1-\frac{1}{\mathsf{k}})}\, \bar{x}_2^{\bar{m}_2' \frac{(2\bar{s}+1)}{\mathsf{k}}+h(1-\frac{1}{\mathsf{k}})} \langle s, \bar{s}, \mathsf{k}| \mathcal{O}_h^{(m',\bar{m}')}(x_1) \mathcal{O}_h^{(m',\bar{m}')\dagger}(x_2)|s, \bar{s}, \mathsf{k}\rangle \\
&\quad = \frac{1}{\mathsf{k}^{2h+2}} \sum_{r_1=0}^{\mathsf{k}-1} \frac{e^{2\pi i \frac{r_1}{\mathsf{k}}(m'(2s+1)-\bar{m}'(2\bar{s}+1))}}{x_1^{h(1+\frac{1}{\mathsf{k}})-m'\frac{(2s+1)}{\mathsf{k}}}\, \bar{x}_1^{h(1+\frac{1}{\mathsf{k}})-\bar{m}'\frac{(2\bar{s}+1)}{\mathsf{k}}}} \sum_{r_2=0}^{\mathsf{k}-1} e^{2\pi i \frac{r_2}{\mathsf{k}}(-m'(2s+1)+\bar{m}'(2\bar{s}+1))} \\
&\quad = \frac{1}{\mathsf{k}^{2h}} \frac{1}{x^{h(1+\frac{1}{\mathsf{k}})-m'\frac{(2s+1)}{\mathsf{k}}}\, \bar{x}^{h(1+\frac{1}{\mathsf{k}})-\bar{m}'\frac{(2\bar{s}+1)}{\mathsf{k}}}} \sum_{\ell\in\mathbb{Z}} \delta_{m'(2s+1)-\bar{m}'(2\bar{s}+1),\, \mathsf{k}\ell}\,, \qquad (4.112)
\end{aligned}$$

where in the first equality we have exchanged the finite sum with the limit, and $x = x_1$, $m = m_1 = -m_2$. When $\mathsf{k} = 1$, this reduces to Eq. (4.111). The Kronecker comb enforces the constraint $(2s+1)m' - (2\bar{s}+1)\bar{m}' \in \mathsf{k}\,\mathbb{Z}$, which is a direct consequence of the difference beween left and right null-gauge constraints (4.45) in the regime of interest. Moreover, by first multiplying the correlator in Eq. (4.81) by $x^n\, \bar{x}^{\bar{n}}$ we can also consider descendant insertions. This condition is in agreement with the results present in [141, 94] (see also [52]), where our n_y has to be identified with their λ from the supergravity analysis.

When considering the case of multi-particle emission, the above amplitude must be multiplied by a combinatorial factor, as explained in [140, 93, 141]. To obtain the emission rate, one needs to consider the unit amplitude evaluated at $(x, \bar{x}) = (1, 1)$, implying that the spatial dependence trivialises. Nevertheless, the crucial feature related to the presence of the prefactor k^{-2h}, which enters the final expression of the emission rate[46], is reproduced by (4.112).

Even though the spatial dependence of the two-point function Eq. (4.112) plays a trivial role in the emission rate, the power of x has a precise meaning in terms of the energy spectrum of the nearly unstable Hawking quanta [93, 94]. Indeed, consider the holomorphic part of the energy of these modes. In the conventions of [94], the corresponding spectrum reads

$$\omega\, \mathsf{k} R_y = \frac{1}{2}\alpha \mathsf{k}(m_\phi - m_\psi) - \frac{1}{2}\bar{\alpha}\mathsf{k}(m_\phi + m_\psi) - 2\left(\frac{l}{2}+1\right). \qquad (4.113)$$

In our notation, $(m_\phi - m_\psi) = 2m'$, $(m_\phi + m_\psi) = -2\bar{m}'$, and $\frac{l}{2}+1 = h$, so this becomes

$$-\omega R_y = \frac{2h}{\mathsf{k}} - \alpha\, m' - \bar{\alpha}\, \bar{m}'\,, \qquad (4.114)$$

[46] To compare to the final results of [141, 94] one must include the additional factor $\sqrt{\mathsf{k}\nu}$, where ν is related to the Bose enhancement, which is not visible for a single-particle process.

where $\alpha, \bar{\alpha}$ are the same as in Eq. (2.89). Finally, taking care of the cylinder-to-plane conversion factor for a field of spacetime conformal dimension h, we obtain

$$-\omega R_y \,=\, 2h\left(1+\frac{1}{\mathsf{k}}\right) - \alpha\, m' - \bar{\alpha}\, \bar{m}' \,. \tag{4.115}$$

The RHS is exactly the sum of the exponents of x and $\bar{x}$ in Eq. (4.112). Furthermore, we note that this relation is precisely the sum of the left and right bosonic null gauge constraints Eq. (4.45) for discrete states with $n = \bar{n} = 0$.

The emission takes place when the energy is positive, $\omega > 0$, and corresponds to quanta leaving the AdS region; in a near-decoupling limit, these quanta escape into an asymptotically flat region. Indeed, the exponent of x in Eq. (4.112) becomes positive and the amplitude diverges at large x, such that the energy indeed turns from negative to positive. This is consistent with the description of the ergoregion radiation process as pair creation [185].

4.4 Discussion

In this chapter we have computed a large set of worldsheet correlators describing the dynamics of light modes probing a class of highly excited supergravity backgrounds, the JMaRT solutions, in the fivebrane decoupling limit. The results are exact in α' and were obtained by exploiting the solvability of the null-gauged WZW models corresponding to these backgrounds.

These coset models provide a powerful method to calculate HLLH correlators, since the heavy states are already taken into account in the worldsheet CFT itself. Thus spacetime HLLH correlators are two-point functions on the worldsheet, which can be computed once the vertex operators have been constructed.

We constructed physical vertex operators in both NS and R sectors, and then computed several families of correlators in the full coset models. We primarily focused on short strings belonging to discrete representations of the affine SL(2,$\mathbb{R}$) algebra, as well as a tower of modes generated by worldsheet spectral flow. Our main techniques can also be employed in more general sectors of the theory.

In the IR AdS_3 limit, due to the non-trivial gauging, the identification of the x variable dual to the local coordinate of the holographic CFT requires some care. Once we made this identification, we computed several non-trivial HLLH correlators explicitly, and analyzed them in the context of AdS_3/CFT_2.

Vertex operators that are local on the AdS_3 boundary are constructed by summing over all allowed values of the spacetime modes. An important step in our analysis consists of identifying these modes. We chose a gauge in which the IR AdS_3 boundary coordinates are (t, y) of the timelike $\mathbb{R}$ and spacelike $\mathbb{S}^1$ directions of the (10+2)-dimensional model before gauging. We therefore identified the spacetime mode indices with the quantum numbers m_y and $\bar{m}_y$ defined in (4.46). Then the gauge constraints (4.12) satisfied by the physical states imply that the m_y mode numbers take values in $\mathbb{Z}/\mathsf{k}$. This is how the worldsheet coset models capture the fact that when $\mathsf{k} > 1$, the heavy background states of the symmetric product orbifold CFT are in the k-twisted sector [96, 94].

We observed that, at large N, several correlators agree exactly between worldsheet and symmetric product orbifold CFT. The fact that our correlators are exact in α' significantly strengthens previous results that compared HLLH correlators between the separate supergravity and symmetric product orbifold CFT regimes.

To demonstrate our method, we presented a detailed example with an $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ chiral primary. The worldsheet correlator involves a non-trivial structure in terms of the boundary coordinate x, Eq. (4.56). When the background is BPS, the correlator agrees precisely with the supergravity and symmetric product orbifold CFT correlators computed in [167]. The non-BPS JMaRT backgrounds were not considered in [167], however we demonstrated that the agreement extends also to those backgrounds.

Similarly to correlators on the background of the global AdS_3 vacuum, the holomorphic and antiholomorphic sectors are related through the constraint $m_y - \bar{m}_y = n_y$, where n_y is the quantized momentum on the y circle. Thus, while the spacetime modes m_y, $\bar{m}_y$ are fractional, their difference must be an integer. This mirrors the $m - \bar{m} \in \mathbb{Z}$ condition in the SL(2,$\mathbb{R}$)/U(1) cigar coset and in global AdS_3, which ensures that the wavefunctions are single-valued. In our models, the difference of the left and right gauge constraints leads to the mod k condition in Eq. (4.68), constraining which of the SL(2,$\mathbb{R}$) modes can contribute. The HLLH correlator is then obtained by summing over a specific linear combination of m-basis worldsheet two-point functions.

Our analysis of these correlators involving the $h = 1/2$ light operator indicated a way to obtain similar expressions for more general correlators. We considered general massless vertex operators, which correspond to symmetric product orbifold CFT operators in short multiplets whose top component is a chiral primary of arbitrary weight h, including those that live in twisted sectors. We computed all HLLH correlators where the light operators are massless, and where the heavy states correspond to any of the general family of orbifolded JMaRT configurations, including their BPS limits. The result assumes a remarkably simple form, presented in Eq. (4.81). It is built from three distinct factors: (1) the global $AdS_3 \times \mathbb{S}^3$ *vacuum* two-point function of the light operators inserted at the k-th roots of the original insertion points x_i, (2) the Jacobian factor associated with the corresponding change of coordinates, and (3) an additional factor coming from the way in which operators of definite R-charge transform under spectral flow. The product of these factors is then summed over all such roots. This structure reflects that one can formulate the computation in a k-fold covering space of the target space.

We then obtained a similar expression for all higher-point functions of the schematic form $\langle H|\mathcal{O}_1(x_1,\bar{x}_1)\ldots\mathcal{O}_n(x_n,\bar{x}_n)|H\rangle$, with heavy JMaRT states, and n massless insertions. This is presented in Eq. (4.94). We expect this to be valid for an arbitrary number of massless insertions of weights h_i and charges m_i' and $\bar{m}_i'$, and also arbitrary parameters $(\mathsf{k},s,\bar{s})$ for which a consistent background exists. In this way, we have provided a recipe for computing such $(n+2)$-point heavy-light correlators in terms of n-point global $\mathrm{AdS}_3\times\mathbb{S}^3$ *vacuum* correlation functions of the corresponding light insertions.

It is known that vacuum two- and three-point functions of chiral primary operators are protected [184]. We therefore conjectured that heavy-light correlators in JMaRT heavy states are protected whenever the corresponding vacuum correlator in our general formula (4.94) is protected. We investigated a particular HLLLH five-point function—the first of its kind in the literature—and found that worldsheet and symmetric product results agree. We leave a more general investigation of this proposal to future work.

As an application, we have shown that our results describe the analog of the Hawking radiation process for the general family of non-BPS JMaRT black hole microstates, generalizing the analysis in [93, 46, 94].

In addition to these main results, our work has clarified some important technical details. For instance, since the full asymptotically linear dilaton JMaRT backgrounds do not have $\mathrm{AdS}_3\times\mathbb{S}^3$ isometries, the SL(2,$\mathbb{R}$) and SU(2) raising and lowering operators $J^\pm$, $K^\pm$ of the (10+2)-dimensional ungauged worldsheet model do not commute with the gauging. Thus, the NS sector vertex operators of the cosets do not have well-defined SL(2,$\mathbb{R}$) or SU(2) spins, see for instance Eq. (4.19). The same holds for the chirality quantum number ε in the R sector, as discussed around Eq. (4.24).

The absence of the SL(2,$\mathbb{R}$) spin has important implications also from the holographic point of view. It underlines the fact that the construction of x-basis operators is only appropriate in the AdS_3 limit, and breaks down otherwise. The breakdown of the x coordinate is a signal of the non-locality of the non-gravitational little string theory that lives on the worldvolume of the NS5 branes, dual to the full asymptotically linear-dilaton models. Thus the states we have constructed contain valuable information about the dual LST and, more generally, about non-AdS holography [145, 143, 144]. We have nevertheless demonstrated how, in the AdS_3 limit, our vertex operators acquire definite spins and reduce to the appropriate expressions.

Our results suggest several directions for future investigations. First, it would be interesting to compute more general worldsheet correlators, both in the AdS_3 limit and in the full models. Our correlators are likely to generalize to a larger set of worldsheet vertex operators that correspond to operators in the symmetric product orbifold CFT that transform nicely under spacetime spectral flow [186]. In global AdS_3, correlators are known to involve a highly non-trivial structure related to the non-conservation of the spectral flow number [103]. More generally, one would like to describe the physics of long/winding strings and their correlators in these backgrounds. A number of interesting techniques recently developed in [38, 179] (for the bosonic case) are likely to have interesting implications for computations in the coset theories, for which SL(2,$\mathbb{R}$) constitutes a crucial building block.

It would also be interesting to study such correlators by using conformal perturbation theory on top of a putative dual CFT explicitly associated to the NSNS singular point [50] of the moduli space, defined along the lines of [174, 90]. Doing so would require an understanding of how to define the JMaRT heavy states in such a theory. Separately, it would be interesting to investigate the case $n_5 = 1$, which would require going beyond the RNS formalism, as done in related recent developments [187, 87, 188]. Here one should go though the coset construction starting with the supergroup PSU(1,1—2).

In the full asymptotically linear dilaton models, more general correlators can be computed by using the vertex operators constructed in Section 4.1. However, a shift in perspective will be needed, since the x coordinate seems unlikely to be of any use in this regime. Although *a priori* in our case it is more natural to work in the m-basis, it seems plausible to relate our results to the momentum-space correlators studied in [143, 145], see also [189]. In those papers, the authors work with a related null-gauged model, and further interpret their holographic LST correlators in terms of an irrelevant (single-trace) $T\bar{T}$-deformation of the IR CFT_2.

Separately, it will be interesting to investigate our proposal for the subset of heavy-light correlators that we expect to be protected by considering the dual computations in the symmetric product orbifold CFT.

Last, but not least, one would like to explore further how these correlation functions encode more detailed information about the physics of the microstate backgrounds we are working with. For instance, two-point functions are expected to probe the multipole ratios of the geometry [190, 191], while certain worldsheet three-point functions should be related to the Penrose process in the JMaRT backgrounds [192].

Although the JMaRT backgrounds are atypical microstates, the HLLH correlators we have computed approach black-hole-like behaviour at large k, reflecting the properties of the backgrounds in this limit. We expect that the techniques developed in this work will help further the study of more typical black hole microstates in string theory.

Chapter 5

A proof for string three-point functions in AdS_3

In most known cases of the AdS/CFT duality, the bulk theory is approximated by supergravity, due to the notorious difficulty in performing stringy computations. The case of string theory in AdS_3 with pure NS fluxes is a notable exception. The dynamics of closed strings propagating on this background can be described at the worldsheet level in terms of the Wess-Zumino-Witten (WZW) model built upon the universal cover of $SL(2,\mathbb{R})$. This model is believed to be exactly solvable, hence providing a concrete scenario in which the AdS/CFT duality might be proven at finite 't Hooft coupling.

The worldsheet CFT, being both Lorentzian and non-compact, enjoys a number of unusual features. Perhaps the most important one is the non-trivial action of the so-called spectral flow outer-automorphisms of the affine $sl(2,\mathbb{R})_k$ algebra. Spectral flow plays a central role in the determination of the string spectrum and partition function [101, 102]. In a semi-classical description of long string states, the spectral flow charge ω can be thought of as a winding number around the asymptotic boundary of AdS_3. However, it is not a conserved quantity since the associated circle becomes contractible in the AdS_3 interior.

Spectral flow also introduces important complications, especially concerning the computation of worldsheet correlation functions [103]. Indeed, while being Virasoro primaries, spectrally flowed vertex operators are *not* affine primaries. Their operator product expansions (OPEs) with the conserved currents become increasingly complicated with growing ω, and contain many unknown terms. Hence, some conventional techniques of two-dimensional CFT can not be applied directly.

The interest in correlation functions of the $SL(2,\mathbb{R})$-WZW model is explained by their important holographic applications, especially in the context of superstring theory in $AdS_3 \times \mathbb{S}^3 \times \mathbb{T}^4$ (or K3) [78, 110, 100, 113], and for the dynamics of light probes in black hole microstates [51, 52, 53, 54, 193, 1, 2, 3]. Worldsheet vertex operators are functions of the worldsheet coordinate z, and of an additional continuous parameter x, which plays the role of the holomorphic coordinate on the conformal boundary. Hence, z-integrated correlators are identified with n-point functions of local operators in the holographic CFT. Although the precise definition of the latter remains elusive, a concrete albeit perturbative proposal was put forward recently in [90, 174].

The generators of the spacetime Virasoro algebra are in one-to-one correspondence with those of the $\mathrm{sl}(2,\mathbb{R})_k$ algebra, and the worldsheet quantum numbers determine the conformal weight h of dual CFT operators [78]. Suppressing the anti-holomorphic quantities, we denote the spectrally-flowed worldsheet vertex operators in the so-called x-basis as $V^{\omega}_{jh}(x,z)$, where j is the $\mathrm{SL}(2,\mathbb{R})$ spin, while h is the spacetime weight.

In this chapter, we consider genus-zero worldsheet correlators of spectrally flowed vertex operators. Recently, progress in the computation of three- and four-point functions has been made in [88, 38, 179], thanks to the systematic use of the constraints deriving from global and local symmetries. In particular, the latter imply a set of complicated linear recursion relations among correlators involving different assignments of spectral flow charges. These are difficult to derive in general. However, in [38] it has been shown that they can be recast in the form of partial differential equations thanks to the introduction of the so-called 'y-variable'. In this new basis, a vertex operator is now defined as a coherent superposition of states with different spacetime weights, denoted as $V^{\omega}_{j}(x,y,z)$. The authors of [38] were able to infer a closed-form expression for all three-point functions

$$\langle V^{\omega_1}_{j_1}(x_1;y_1;z_1)V^{\omega_2}_{j_2}(x_2;y_2;z_2)V^{\omega_3}_{j_3}(x_3;y_3;z_3)\rangle \ . \tag{5.1}$$

This proposal can be understood intuitively from the existence, in some cases, of a holomorphic mapping of the worldsheet to the AdS_3 boundary, which has branching points of order ω_i at each of the corresponding insertions. The same authors provided a closed-form expression for four-point functions in a follow-up work [179]. These proposals satisfy a number of highly non-trivial consistency checks. However, in both cases, these expressions were derived on a case-by-case basis, and for a finite set of sufficiently low spectral flow charges.

In this work, we provide a proof for the conjecture of [38] concerning the three-point functions (5.1). Our methods rely heavily on the so-called 'series identifications' of SL(2,$\mathbb{R}$), originally formulated in [175, 101], which constitute a set of isomorphisms among affine modules of the $\mathrm{sl}(2,\mathbb{R})_k$ algebra with adjacent spectral flow charges. The corresponding identities at the level of y-basis operators were obtained recently in [39]. This allows us to derive the differential equations satisfied by (and the resulting y-dependence of) all spectrally flowed three-point functions, including those that do not admit a holomorphic cover.

The structure of the chapter is as follows. In section 5.1, we establish our conventions and review the derivation of the recursion relations satisfied by the SL(2,$\mathbb{R}$) correlation functions of [88]. We also introduce the y-basis operators of [38] together with their conjecture for three-point functions. Our main results are presented in section 5.2, where derive the partial differential equations satisfied by all y-basis three-point functions, and show that all the corresponding solutions are compatible with the proposal of [38]. In particular, generic odd parity correlators are considered in section 5.2.1, while even parity correlators are obtained in section 5.2.2. Edge cases and correlators with unflowed insertions are treated in sections 5.2.3 and 5.2.4, respectively. Finally, in section 5.2.5 we fix the y-independent normalisation factors, following the arguments presented in [39]. We conclude with some discussions and outlook for future work in section 5.3.

5.1 Conventions and brief review of the conjecture

Let us consider the bosonic SL(2,$\mathbb{R}$)-WZW model at level $k > 3$. In this section, we introduce the spectrum and the associated vertex operators. We also briefly review the analysis of [88] leading to the recursion relations satisfied by the correlation functions of the model, and present the conjecture put forward in [38] for spectrally flowed three-point functions.

5.1.1 Vertex operators

We focus mainly on the holomorphic sector. Compared to previous chapters, the present focuses on the purely bosonic sector of the theory. For this reason, in this chapter the bosonic currents will be denoted by J.

The conserved currents satisfy

$$J^a(z)J^b(w) \sim \frac{\eta^{ab}k/2}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{z-w}\,, \tag{5.2}$$

with $a = +,-,3$, $\eta^{+-} = -2\eta^{33} = 2$, $f^{+-}{}_3 = -2$ and $f^{3+}{}_+ = -f^{3-}{}_- = 1$. In the so-called x-basis, vertex operators are denoted as $V^{\omega}_{jh}(x,z)$. They depend both on the worldsheet coordinate z and on x, the coordinate on the boundary of AdS_3 associated with the holographic CFT. Moreover, h and j denote the spacetime weight and the SL(2,$\mathbb{R}$) spin, respectively. The worldsheet conformal weight is

$$\Delta = -\frac{j(j-1)}{k-2} - h\omega + \frac{k}{4}\omega^2\,, \tag{5.3}$$

while $\omega \geq 0$ is the spectral flow charge.

Two types of states must be taken into account [101]. For operators defined upon unflowed states in the continuous representations $\mathcal{C}_j^\alpha$, the relevant quantum numbers are

$$j \in \frac{1}{2} + i\mathbb{R} \quad \text{and} \quad m = \alpha \pm n\,, \quad \text{with} \quad \alpha \in [0,1) \quad \text{and} \quad n \in \mathbb{N}_0\,, \tag{5.4}$$

where m stands for the spin projection of the corresponding unflowed state. On the other hand, for the unflowed discrete highest/lowest-weight representations $\mathcal{D}_j^\pm$ we have

$$\frac{1}{2} < j < \frac{k-1}{2} \quad \text{and} \quad m = \pm(j+n)\,, \quad \text{with} \quad j \in \mathbb{R} \quad \text{and} \quad n \in \mathbb{N}_0. \tag{5.5}$$

In both cases we have $h = m + \frac{k}{2}\omega$ when $\omega > 0$, while $h = j$ for $\omega = 0$.

The vertex operators are defined by means of their OPEs with the currents:

$$J^+(w)V^\omega_{jh}(x,z) = \sum_{n=1}^{\omega+1} \frac{\left(J^+_{n-1}V^\omega_{jh}\right)(x,z)}{(w-z)^n} + \cdots\,, \tag{5.6a}$$

$$J^3(w)V^\omega_{jh}(x,z) = x\sum_{n=2}^{\omega+1} \frac{\left(J^+_{n-1}V^\omega_{jh}\right)(x,z)}{(w-z)^n} + \frac{\left(J^3_0V^\omega_{jh}\right)(x,z)}{(w-z)} + \cdots\,, \tag{5.6b}$$

$$J^-(w)V^\omega_{jh}(x,z) = x^2\sum_{n=2}^{\omega+1} \frac{\left(J^+_{n-1}V^\omega_{jh}\right)(x,z)}{(w-z)^n} + \frac{\left(J^-_0V^\omega_{jh}\right)(x,z)}{(w-z)} + \cdots\,, \tag{5.6c}$$

where the ellipsis indicates higher order terms in $(w-z)$. Unflowed vertex operators will be denoted by $V_j(x,z)$. The zero modes act as differential operators in x,

$$\left(J^+_0V^\omega_{jh}\right)(x,z) = \partial_x V^\omega_{jh}(x,z)\,, \tag{5.7a}$$

$$\left(J^3_0V^\omega_{jh}\right)(x,z) = (x\partial_x + h)V^\omega_{jh}(x,z)\,, \tag{5.7b}$$

$$\left(J^-_0V^\omega_{jh}\right)(x,z) = (x^2\partial_x + 2hx)V^\omega_{jh}(x,z)\,, \tag{5.7c}$$

while

$$\left(J^\pm_{\pm\omega}V^\omega_{jh}\right)(x,z) = \left[h - \frac{k}{2}\omega \pm (1-j)\right] V^\omega_{j,h\pm1}(x,z)\,. \tag{5.8}$$

Importantly, in terms of the currents

$$J^+(x,z) = J^+(z)\,, \quad J^3(x,z) = J^3(z) - xJ^+(z)\,, \quad J^-(x,z) = J^-(z) - 2xJ^3(z) + x^2J^+(z) \tag{5.9}$$

we get

$$J^3(x,w)V^{\omega}_{jh}(x,z) = \frac{h}{(w-z)}V^{\omega}_{jh}(x,z) + \cdots\,, \tag{5.10a}$$

$$J^-(x,w)V^{\omega}_{jh}(x,z) = (w-z)^{\omega-1}\left(J^-_{-w}V^{\omega}_{jh}\right)(x,z) + \cdots\,. \tag{5.10b}$$

An alternative (equivalent) definition was given recently in [39], based on [103]. This is to be understood as a point-splitting procedure between the corresponding unflowed vertex and the so-called generalized spectral flow operator $V^{\omega-1}_{\frac{k}{2},\frac{k}{2}\omega}(x,z)$, and reads

$$V^{\omega}_{jh}(x,z) = \lim_{\varepsilon,\bar{\varepsilon}\to 0}\varepsilon^{m\omega}\bar{\varepsilon}^{\bar{m}\omega}\int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1}V_j(x+y,z+\varepsilon)V^{\omega-1}_{\frac{k}{2},\frac{k}{2}\omega}(x,z)\,. \tag{5.11}$$

We will come back to this shortly in section 5.1.3.

5.1.2 Recursion relations among correlators

We now discuss the correlation functions of the model. It was shown in [88] that they must satisfy a set of recursion relations. Let us briefly review how this works for three-point functions of the form

$$F = \left\langle \prod_{j=1}^{3} V^{\omega_j}_{j_j h_j}(x_j,z_j)\right\rangle. \tag{5.12}$$

We define

$$F^i_n = \left\langle \left(J^+_n V^{\omega_i}_{j_i h_i}\right)(x_i,z_i)\prod_{j\neq i}V^{\omega_j}_{j_j h_j}(x_j,z_j)\right\rangle, \tag{5.13}$$

so that, in particular,

$$F^i_0 = \partial_{x_i}F\,, \tag{5.14}$$

and

$$F^i_{\omega_i} = \left(h_i - \frac{k}{2}\omega_i + 1 - j_i\right)\left\langle V^{\omega_i}_{j_i,h_i+1}(x_i,z_i)\prod_{j\neq i}V^{\omega_j}_{j_j h_j}(x_j,z_j)\right\rangle, \tag{5.15}$$

as implied by Eqs. (5.7a) and (5.8) respectively. Note that all F^i_n with $n = 1,\ldots,\omega_i - 1$ are, in principle, unknown.

By using the OPEs in Eq. (5.6), one finds that correlators involving a current insertion can be expanded as

$$\left\langle J^+(z)\prod_{j=1}^{3} V^{\omega_j}_{j_j h_j}(x_j,z_j)\right\rangle = \sum_{i=1}^{3}\left[\frac{\partial_{x_i}F}{z-z_i}+\sum_{n=1}^{\omega_i}\frac{F^i_n}{(z-z_i)^{n+1}}\right]+\cdots, \tag{5.16a}$$

$$\left\langle J^3(z)\prod_{j=1}^{3} V^{\omega_j}_{j_j h_j}(x_j,z_j)\right\rangle = \sum_{i=1}^{3}\left[\frac{(h_i+x_i\partial_{x_i})F}{z-z_i}+\sum_{n=1}^{\omega_i}\frac{x_iF^i_n}{(z-z_i)^{n+1}}\right]+\cdots, \tag{5.16b}$$

$$\left\langle J^-(z)\prod_{j=1}^{3} V^{\omega_j}_{j_j h_j}(x_j,z_j)\right\rangle = \sum_{i=1}^{3}\left[\frac{(2h_ix_i+x_i^2\partial_{x_i})F}{z-z_i}+\sum_{n=1}^{\omega_i}\frac{x_i^2F^i_n}{(z-z_i)^{n+1}}\right]+\cdots \tag{5.16c}$$

Combining these expressions, we get

$$G_j(z)\equiv\left\langle J^-(x_j,z)\prod_{l=1}^{3} V^{\omega_l}_{j_l h_l}(x_l,z_l)\right\rangle = \sum_{i\neq j}\left[\frac{(2h_ix_{ij}+x_{ij}^2\partial_{x_i})F}{z-z_i}+\sum_{n=1}^{\omega_i}\frac{x_{ij}^2F^i_n}{(z-z_i)^{n+1}}\right]+\cdots \tag{5.17}$$

where $x_{ij}=x_i-x_j$.

On the other hand, Eq. (5.10b) imposes stringent restrictions on the behavior of $G_j(z)$ when z is close to z_j. More precisely, in this regime, we must have

$$(z-z_j)^{1-\omega_j}G_j(z)=\left(h_j-\frac{k}{2}\omega_j-1+j_j\right)\left\langle V^{\omega_j}_{j_j,h_j-1}(x_j,z_j)\prod_{i\neq j}V^{\omega_i}_{j_ih_i}(x_i,z_i)\right\rangle+\cdots, \tag{5.18}$$

where we have used (5.8). As it turns out, the regularity of $(z-z_j)^{1-\omega_j}G_j(z)$ at $z=z_j$ as implied by (5.18) for all $j=1,2,3$ provides enough information to solve for all the unknown F^i_n in terms of F, its x_i-derivatives, and $F^i_{\omega_i}$. Upon inserting the resulting expressions into Eq. (5.18), one obtains a set of complicated linear relations between correlators involving spectrally flowed vertex operators with consecutive spacetime weights and their x_i derivatives. Note that the latter are under control since the global Ward identities uniquely fix the x_i-dependence of all such correlators.

5.1.3 The conjectured solution

The non-trivial OPEs in Eqs. (5.6) render the computation of correlation functions involving spectrally flowed insertions quite complicated. It will be useful to work with somewhat unusual linear combinations of the operators $V^{\omega}_{jh}(x,z)$ with the same j and ω but different values of h. These are the y-basis operators $V^{\omega}_{j}(x,y,z)$ introduced in [38]. In [39], these were shown to be precisely the integrands on the RHS of Eq. (5.11), namely

$$V^{\omega}_{j}(x,y,z)\equiv\lim_{\varepsilon,\bar{\varepsilon}\to 0}|\varepsilon|^{2j\omega}V_j(x+y\varepsilon^{\omega},z+\varepsilon)V^{\omega-1}_{\frac{k}{2},\frac{k}{2}\omega}(x,z)\,. \tag{5.19}$$

This can be understood directly from Eq. (5.11) above. Indeed, upon changing variables $y \to y\epsilon^{\omega}$ this can be re-written as

$$V^{\omega}_{jh}(x,z) = \int d^2y \, y^{j-m-1}\bar{y}^{j-\bar{m}-1} V^{\omega}_{j}(x,y,z)\,, \tag{5.20}$$

which coincides with the so-called y-transform of [38]. More details can be found in [38, 39].

The OPEs of y-basis operators with the conserved currents are

$$\begin{aligned} J^{+}(w)V^{\omega}_{j}(x,y,z) &= \sum_{n=1}^{\omega+1} \frac{\left(J^{+}_{n-1}V^{\omega}_{jh}\right)(x,y,z)}{(w-z)^{n}} + \cdots\,, && (5.21a)\\ J^{3}(x,w)V^{\omega}_{j}(x,y,z) &= \frac{y\partial_y + j + \frac{k}{2}\omega}{(w-z)} V^{\omega}_{j}(x,y,z) + \cdots\,, && (5.21b)\\ J^{-}(x,w)V^{\omega}_{j}(x,y,z) &= (w-z)^{\omega-1}\left(J^{-}_{-w}V^{\omega}_{j}\right)(x,y,z) + \cdots\,. && (5.21c)\end{aligned}$$

While the zero modes still act as in (5.7), the main motivation for using the y-variable is that $J^{\pm}_{\pm\omega}$ act as differential operators in y. More precisely, we have

$$\begin{aligned} \left(J^{+}_{\omega}V^{\omega}_{j}\right)(x,y,z) &= \partial_y V^{\omega}_{j}(x,y,z)\,, && (5.22a)\\ \left(J^{-}_{-\omega}V^{\omega}_{j}\right)(x,y,z) &= (y^2\partial_y + 2jy)V^{\omega}_{j}(x,y,z)\,. && (5.22b)\end{aligned}$$

This allows one to re-write the recursion relations derived formally in the previous section as differential equations for correlators of the form

$$F_y \equiv \langle V^{\omega_1}_{j_1}(x_1,y_1,z_1)V^{\omega_2}_{j_2}(x_2,y_2,z_2)V^{\omega_3}_{j_3}(x_3,y_3,z_3)\rangle\,. \tag{5.23}$$

The x-basis correlators in Eq. (5.12) can be obtained from these by means of (5.20)[47]. This integration procedure can be complicated, and was only carried out explicitly in [38] for a subset of cases.

In order to discuss the structure of these differential equations and their solutions, it will be useful to make use of the conformal invariance on the worldsheet and boundary CFT to fix $x_1 = z_1 = 0$, $x_2 = z_2 = 1$ and $x_3 = z_3 = \infty$, and consider

$$\hat{F}_y = \langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle \equiv \langle V^{\omega_1}_{j_1}(0,y_1,0)V^{\omega_2}_{j_2}(1,y_2,1)V^{\omega_3}_{j_3}(\infty,y_3,\infty)\rangle\,. \tag{5.24}$$

[47] The procedure is slightly different for flowed discrete and continuous states. The former arise as residues from poles of the integrand in (5.20) around $y = 0$ or $y = \infty$, depending on whether the corresponding unflowed vertex operator belongs to a lowest- or highest-weight representation. For the flowed continuous states one must integrate over the full complex plane.

The latter is related to the original correlator in Eq. (5.23) by

$$\langle V_{j_1}^{\omega_1}(x_1,y_1,z_1)V_{j_2}^{\omega_2}(x_2,y_2,z_2)V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle = \frac{x_{21}^{h_3^0-h_1^0-h_2^0}x_{31}^{h_2^0-h_1^0-h_3^0}x_{32}^{h_1^0-h_2^0-h_3^0}}{z_{21}^{\Delta_1^0+\Delta_2^0-\Delta_3^0}z_{31}^{\Delta_1^0+\Delta_3^0-\Delta_2^0}z_{32}^{\Delta_2^0+\Delta_3^0-\Delta_1^0}}\times$$
$$\left\langle V_{j_1}^{\omega_1}\left(0,y_1\frac{x_{32}z_{21}^{\omega_1}z_{31}^{\omega_1}}{x_{21}x_{31}z_{32}^{\omega_1}},0\right)V_{j_2}^{\omega_2}\left(1,y_2\frac{x_{31}z_{21}^{\omega_2}z_{32}^{\omega_2}}{x_{21}x_{32}z_{31}^{\omega_2}},1\right)V_{j_3}^{\omega_3}\left(\infty,y_3\frac{x_{21}z_{31}^{\omega_3}z_{32}^{\omega_3}}{x_{31}x_{32}z_{21}^{\omega_3}},\infty\right)\right\rangle, \tag{5.25}$$

where

$$h_i^0 = j_i + \frac{k}{2}\omega_i\,, \qquad \Delta_i^0 = -\frac{j_i(j_i-1)}{k-2} - j_i\omega_i - \frac{k}{4}\omega_i^2\,. \tag{5.26}$$

Except for a certain subfamily of correlators which will be discussed below, spectrally flowed y-basis three-point functions and their associated differential equations were studied in [38] on a case-by-case basis. This was done for sufficiently low values of the spectral flow charges ω_i, thus leading the authors to conjecture a general solution for the y-dependence of these correlators. The proposed expressions (not including the right-movers, and up to an overall normalization constant to be discussed below) read as follows. For *odd parity* correlators, namely when $\omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}+1$, one has

$$\left\langle V_{j_1}^{\omega_1}(0,y_1,0)\,V_{j_2}^{\omega_2}(1,y_2,1)\,V_{j_3}^{\omega_3}(\infty,y_3,\infty)\right\rangle = X_{123}^{\frac{k}{2}-j_1-j_2-j_3}\prod_{i=1}^{3}X_i^{-\frac{k}{2}+j_1+j_2+j_3-2j_i}\,, \tag{5.27}$$

while for the *even parity* case, i.e. when $\omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}$,

$$\left\langle V_{j_1}^{\omega_1}(0,y_1,0)\,V_{j_2}^{\omega_2}(1,y_2,1)\,V_{j_3}^{\omega_3}(\infty,y_3,\infty)\right\rangle = X_{\varnothing}^{j_1+j_2+j_3-k}\prod_{i<\ell}X_{i\ell}^{j_1+j_2+j_3-2j_i-2j_\ell}\,. \tag{5.28}$$

Here, for any subset $I \subset \{1,2,3\}$,

$$X_I(y_1,y_2,y_3) \equiv \sum_{i\in I:\ \varepsilon_i=\pm1} P_{\boldsymbol{\omega}+\sum_{i\in I}\varepsilon_i e_i}\prod_{i\in I}y_i^{\frac{1-\varepsilon_i}{2}}\,. \tag{5.29}$$

The spectral flow parameters are chosen as $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$, while $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. The numbers $P_{\boldsymbol{\omega}}$ are defined as

$$P_{\boldsymbol{\omega}} = 0 \qquad \text{for} \qquad \sum_j \omega_j < 2\max_{i=1,2,3}\omega_i \quad \text{or} \quad \sum_i \omega_i \in 2\mathbb{Z}+1 \tag{5.30}$$

and

$$P_{\boldsymbol{\omega}} = S_{\boldsymbol{\omega}}\frac{G\left(\frac{-\omega_1+\omega_2+\omega_3}{2}+1\right)G\left(\frac{\omega_1-\omega_2+\omega_3}{2}+1\right)G\left(\frac{\omega_1+\omega_2-\omega_3}{2}+1\right)G\left(\frac{\omega_1+\omega_2+\omega_3}{2}+1\right)}{G(\omega_1+1)G(\omega_2+1)G(\omega_3+1)}\,, \tag{5.31}$$

otherwise, where $G(n)$ is the Barnes G function

$$G(n) = \prod_{i=1}^{n-1}\Gamma(i) \tag{5.32}$$

for positive integer values, while S_ω is a phase depending on $\boldsymbol{\omega}$ mod 2. For more details, see [38].

Regarding the overall constants, their precise form was also conjectured in [38] and later proven in [39]. These structure constants are

$$C_\omega(j_1,j_2,j_3) = \begin{cases} C(j_1,j_2,j_3), & \text{if} \quad \omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}, \\ \mathcal{N}(j_1) C\left(\frac{k}{2}-j_1,j_2,j_3\right), & \text{if} \quad \omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}+1, \end{cases} \tag{5.33}$$

where $C(j_1,j_2,j_3)$ are the structure constants of the unflowed three-point functions defined in terms of Barnes double Gamma functions in [106, 103]. Finally, $\mathcal{N}(j_1)$ is defined in terms of the reflection coefficient appearing the unflowed two-point functions, namely

$$\mathcal{N}(j) = \sqrt{\frac{B(j)}{B\left(\frac{k}{2}-j\right)}}, \tag{5.34}$$

with[48]

$$B(j) = \frac{2j-1}{\pi} \frac{\Gamma[1-b^2(2j-1)]}{\Gamma[1+b^2(2j-1)]} \nu^{1-2j}, \quad \nu = \frac{\Gamma[1-b^2]}{\Gamma[1+b^2]}, \quad b^2 = (k-2)^{-1}. \tag{5.35}$$

As shown in [38], the proposal given in Eqs. (5.27)-(5.33) passes a number of non-trivial consistency checks, including bosonic exchange symmetry and reflection symmetry for continuous states. However, no general expression for the y-basis differential equations is known. Hence, so far, this conjecture remains to be proven.

In the remainder of the chapter, we prove that this solution is indeed correct. In doing so, we highlight the role of holomorphic covering maps. Although these maps appear to be essential for the study of four-point functions [179], we show that they also play a key role in the present context. Furthermore, when treating cases where there is no well-defined covering map available, we will make use of the so-called series identifications for spectrally flowed vertex operators constructed upon states belonging to the discrete representations of SL(2,$\mathbb{R}$) [175, 39].

5.2 The proof for y-basis three-point functions

In this section we prove the conjecture put forward in [38]. It was shown in [103] that all non-vanishing spectrally flowed three-point functions in the SL(2,$\mathbb{R}$) model must

[48]To be precise, ν is actually a free parameter of the model, which essentially plays the role of the string coupling. Here we simply reproduce the value originally advocated in [106]. For a related discussion, see [99, 90]. We thank A. Dei and L. Eberhardt for pointing this out.

satisfy the following condition:

$$\omega_i + \omega_j \geq \omega_k - 1 \qquad \forall\, i \neq j \neq k. \tag{5.36}$$

We first consider the subfamily of odd parity correlators for which an associated holomorphic covering map exists [88, 38]. We then show how to compute all remaining non-vanishing correlators with three non-trivial spectral flow charges. This includes even parity correlators, and also those we denote as edge correlators. The latter correspond to correlators for which either the inequality in (5.36) for odd parity assignments or the analogous inequality for even parity cases saturate. These need to be treated with special care. Finally, we also discuss correlators with unflowed insertions and the overall normalizations.

5.2.1 Holomorphic covering maps and odd parity correlators

For concreteness, and with no loss of generality, we take ω_3 to be the largest spectral flow charge, i.e. $\omega_3 \geq \omega_{1,2}$. Let us consider correlators satisfying

$$\omega_1 + \omega_2 + \omega_3 \in 2\mathbb{Z} + 1\,, \qquad \omega_1 + \omega_2 > \omega_3 - 1\,, \qquad \omega_i \geq 1\,,\ \forall\, i. \tag{5.37}$$

It was shown in [180] that there exists a unique holomorphic covering map $\Gamma[\omega_1, \omega_2, \omega_3](z) \equiv \Gamma(z)$ from the worldsheet to the AdS_3 boundary such that

$$\Gamma(z) \sim x_i + a_i (z - z_i)^{\omega_i} + \cdots \qquad \text{when} \quad z \sim z_i\,, \tag{5.38}$$

where the ellipsis indicates higher order terms in $(z - z_i)$. This is a rational function which approaches a constant Γ_∞ as $z \to \infty$. One can show that it develops N single poles, counted by the Riemann-Hurwitz formula

$$N = \frac{1}{2}\left(\omega_1 + \omega_2 + \omega_3 - 1\right). \tag{5.39}$$

The coefficients a_i appearing in Eq. (5.38) take the form

$$a_i = \begin{pmatrix} \frac{\omega_i + \omega_{i+1} + \omega_{i+2} - 1}{2} \\ \frac{-\omega_i + \omega_{i+1} + \omega_{i+2} - 1}{2} \end{pmatrix} \begin{pmatrix} \frac{-\omega_i + \omega_{i+1} - \omega_{i+2} - 1}{2} \\ \frac{\omega_i + \omega_{i+1} - \omega_{i+2} - 1}{2} \end{pmatrix}^{-1} \frac{x_{i,i+1} x_{i+2,i}}{x_{i+1,i+2}} \left(\frac{z_{i+1,i+2}}{z_{i,i+1} z_{i+2,i}} \right)^{\omega_i}, \tag{5.40}$$

where the subscripts are understood mod 3. Note that the last two factors in (5.40) simplify to 1 upon setting $x_1 = z_1 = 0$, $x_2 = z_2 = 1$ and $x_3 = z_3 = \infty$. We will use the same notation a_i for the resulting purely numerical coefficients.

We now derive the differential equations satisfied by y-basis three-point functions satisfying (5.37). Although the presentation here is slightly different, this was already done in [88]. Consider the operator $J^-(\Gamma(z),z)$, where we use the notation of Eq. (5.10b), namely

$$J^-(\Gamma(z),z) = J^-(z) - 2\Gamma(z)J^3(z) + \Gamma^2(z)J^+(z)\,. \tag{5.41}$$

In order to obtain the recursion relations for the correlators under consideration, we compute following the contour integral:

$$\oint_{z_i} \frac{dz}{(z-z_i)^{\omega_i}} \left\langle J^-(\Gamma(z),z)\, V_{j_1}^{\omega_1}(x_1,y_1,z_1) V_{j_2}^{\omega_2}(x_2,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3) \right\rangle . \tag{5.42}$$

Similarly to what was done in Sec. 5.1.2 above, we do this in two different ways. First, we note that, near z_i, we have

$$J^-(\Gamma(z),z) = J^-(x_i,z) - 2a_i(z-z_i)^{\omega_i}J^3(x_i,z) + a_i^2(z-z_i)^{2\omega_i}J^+(z) + \cdots . \tag{5.43}$$

Hence, by using (5.21) we find that

$$(5.42) = \left[\left(2j_iy_i + y_i^2\partial_{y_i}\right) - 2a_i\left(j_i + \frac{k}{2}\omega_i + y_i\partial_{y_i}\right) + a_i^2\partial_{y_i}\right] F_y\,, \tag{5.44}$$

where F_y was defined in Eq. (5.23). On the other hand, using (5.41) together with the OPEs in Eq. (5.6), one finds that

$$\begin{aligned}\left\langle J^-(\Gamma(z),z)\, V_{j_1}^{\omega_1}(x_1,y_1,z_1) V_{j_2}^{\omega_2}(x_2,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3) \right\rangle = \\ = \sum_{j=1}^{3}\left\{-\frac{2[\Gamma(z)-x_j](y_j\partial_{y_j}+j_j+\frac{k}{2}\omega_j)}{z-z_j}F_y + \sum_{n=1}^{\omega_j}\frac{[\Gamma(z)-x_j]^2}{(z-z_j)^{n+1}}F_{y,n}^i\right\},\end{aligned} \tag{5.45}$$

where $F_{y,n}^i$ stands for the y-basis analogues of the F_n^i defined in Eq. (5.13). As discussed in [88], the RHS of (5.45) is a rational function of z which, as implied by the constraint equations, has zeros of order $\omega_i - 1$ at all z_i. It also has double poles at the N simple poles of $\Gamma(z)$, and further goes to zero as z^{-2} for $z \to \infty$ due to the global Ward identities. This implies that it must be proportional to the derivative of the covering map, namely

$$\left\langle J^-(\Gamma(z),z)\, V_{j_1}^{\omega_1}(x_1,y_1,z_1) V_{j_2}^{\omega_2}(x_2,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3) \right\rangle = \alpha\,\partial\Gamma(z)\,, \tag{5.46}$$

where α must be independent of z. This coefficient was also computed in [88]. When working in the y-basis it takes the following form:

$$\alpha = -\frac{1}{N}\sum_{j=1}^{3}\left((y_j-a_j)\partial_{y_j} + j_j + \frac{k}{2}\omega_j\right)F_y\,. \tag{5.47}$$

This allows us to provide an alternative expression for the contour integral (5.42). Indeed, the behavior of the covering map near the insertion points showcased in (5.38) implies that

$$(5.42) = -\frac{a_i\omega_i}{N}\sum_{j=1}^{3}\left((y_j - a_j)\partial_{y_j} + j_j + \frac{k}{2}\omega_j\right) F_y \,. \tag{5.48}$$

By combining the results in Eqs. (5.44) and (5.48), and further fixing the insertion points as in (5.24), one obtains the following differential equations:

$$\left\{(y_i - a_i)^2\partial_{y_i} + 2j_i(y_i - a_i) + \frac{a_i\omega_i}{N}\left[\sum_{j=1}^{3}\left((y_j - a_j)\partial_{y_j} + j_j\right) + \frac{k}{2}\right]\right\}\hat{F}_y = 0, \tag{5.49}$$

for $i = 1, 2, 3$. This was derived originally in this form in [38][49]. In this way, the use of the covering map and its derivative allows one to avoid dealing with the cumbersome unknowns discussed in Sec. 5.1.2.

The system of equations encoded in (5.49) uniquely fixes the dependence of the corresponding correlators on y_1, y_2 and y_3. Up to some overall normalization, which will be discussed in section 5.2.5 below, the solution of (5.49) is

$$\begin{aligned}\langle V_{j_1}^{\omega_1}(y_1)V_{j_2}^{\omega_2}(y_2)V_{j_3}^{\omega_3}(y_3)\rangle \quad &= \quad (y_1 - a_1)^{j_2+j_3-j_1-\frac{k}{2}}(y_2 - a_2)^{j_3+j_1-j_2-\frac{k}{2}}(y_3 - a_3)^{j_1+j_2-j_3-\frac{k}{2}} \\ &\times\left[\sum_{\varepsilon_{1,2,3}=\pm 1} 2\frac{N_1^{\varepsilon_1,\varepsilon_2,\varepsilon_3}}{N}\prod_{i=1}^{3} a_i^{\frac{\varepsilon_i+1}{2}} y_i^{\frac{\varepsilon_i-1}{2}}\right]^{\frac{k}{2}-j_1-j_2-j_3} .\end{aligned} \tag{5.50}$$

An equivalent, perhaps simpler expression for the same solution is given by

$$\begin{aligned}\langle V_{j_1}^{\omega_1}(y_1)V_{j_2}^{\omega_2}(y_2)V_{j_3}^{\omega_3}(y_3)\rangle &= (y_1 - a_1)^{-2j_1}(y_2 - a_2)^{-2j_2}(y_3 - a_3)^{-2j_3} \\ &\times\left(\omega_1\frac{y_1 + a_1}{y_1 - a_1} + \omega_2\frac{y_2 + a_2}{y_2 - a_2} + \omega_3\frac{y_3 + a_3}{y_3 - a_3} - 1\right)^{\frac{k}{2}-j_1-j_2-j_3} .\end{aligned} \tag{5.51}$$

Note that this formulation makes the bosonic exchange symmetry manifest.

At first sight, this expression might seem very different from that in Eq. (5.27). The connection comes from the fact that while the P_ω defined in (5.31) are somewhat complicated, their ratios are actually much simpler. For instance, consider the X_i term appearing in (5.27). From Eqs. (5.29) and (5.31), up to an overall sign, we find that

$$X_i = P_{\omega-e_i}y_i + P_{\omega+e_i} = P_{\omega-e_i}\left(y_i + \frac{P_{\omega+e_i}}{P_{\omega-e_i}}\right) = P_{\omega-e_i}\left(y_i - a_i\right) . \tag{5.52}$$

A similar result holds for X_{123}, showing that the y-dependence of the expression in (5.51) is consistent with that of Eq. (5.27).

As shown in Eq. (5.51), y-basis three-point functions diverge whenever a variable y_i approaches the corresponding coefficient a_i. Thus, the a_i are very special points in the

[49]Note that we have corrected a couple of typos in their presentation.

y-plane, which signal the existence of an appropriate holomorphic covering map. An even more extreme situation takes place in the tensionless limit, which corresponds to $k = 3$ in the bosonic language [194, 187, 87, 88]. There, spectrally flowed correlators are non-vanishing *only* when $y_i = a_i$ for all i. The corresponding recursion relations (5.49) are then satisfied provided[50] $j_1 + j_2 + j_3 = \frac{3}{2}$. It would be interesting to fully understand the relation between y-variables and covering map coefficients in the general case.

We end this section by noting that the same discussion can not be applied directly to odd parity correlators saturating the inequality in Eq. (5.36), i.e. those with $\omega_3 = \omega_1 + \omega_2 + 1$. These will be discussed in detail in section 5.2.3 below. Correlators with unflowed insertions are further considered in section 5.2.4.

5.2.2 Series identification and even parity correlators

We now prove that the conjecture of [38] also holds for correlators satisfying

$$\omega_1 + \omega_2 + \omega_3 \in 2\mathbb{Z}\,, \qquad \omega_1 + \omega_2 > \omega_3\,, \qquad \omega_i \geq 1\,,\ \forall\, i. \tag{5.53}$$

These include all non-vanishing spectrally flowed three-point functions for which the total spectral flow charge is even, except for the edge cases where $\omega_3 = \omega_1 + \omega_2$ and/or some of the vertex operators are unflowed, which will be treated separately.

When the ω_i are as in (5.53), it is not possible to construct a covering map such as the one used in the previous section. Thus, one might wonder if differential equations similar to those in (5.49) can be deduced in this context. Indeed, the corresponding recursion relations have only been obtained on a case-by-case basis and for sufficiently low values of the spectral flow charges [38].

Nevertheless, we observe that the procedure outlined in Sec. 5.1.2 guarantees that, provided the system is compatible, and once the F^i_n have been solved for, the resulting y-basis recursions must take the form

$$\left[y_i(y_i\partial_{y_i} + 2j_i) + \sum_{j=1}^{3}(A_{ij}y_j - B_{ij})\partial_{y_j} + C_i\right] \langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle = 0\,, \tag{5.54}$$

where A_{ij}, B_{ij} and C_i are some numerical constants to be determined, which depend on the spins j_i and the charges ω_i. The rationale behind the structure of Eq. (5.54), which is the y-basis version of Eq. (5.18), goes as follows. First, note that, upon using Eq. (5.22b), the operator $y_i(y_i\partial_{y_i} + 2j_i)$ is identified with the RHS of (5.18). Second, recall that the recursion relations were derived by expressing the OPEs of the vertex operators with the conserved currents in terms of the unknowns coming from the action of the modes

[50] In the supersymmetric case, the RNS formalism breaks down for the tensionless theory. It was shown in [87], using the so-called hybrid formalism, that in this case only vertex operators with $j = \frac{1}{2}$ are allowed.

of $J^+(z)$, see Eq. (5.17). This implies that the term involving the action of $J^-(x,z)$ does not mix with the rest. By using the Möbius-fixed expression for the three-point function in Eq. (5.25), one can see that the terms in the recursion relations involving unknowns and x-derivatives of the correlator are mapped to operators of the schematic form $y\partial_y$ and ∂_y, as well as y-independent multiplicative factors. In Eq. (5.54) we have allowed for generic coefficients A_{ij}, B_{ij} and C_i in front of the corresponding contributions.

Moreover, we note that the way in which these equations are derived only depends on the values of the spectral flow charges involved in a given correlator. In other words, for a given set of ω_i, the recursion relations are independent of whether the vertex operators involved belong to spectrally flowed discrete or continuous representations. These two observations will allow us to obtain all y-basis differential equations associated with even parity correlators in closed form.

As it turns out, even and odd parity cases are not completely disconnected. For the discrete representations, affine modules in spectral flow sectors with one unit of difference in the spectral flow charge are identifiable. For y-basis operators the corresponding series identifications read [39]

$$V_j^{\omega}(x,y=0,z) = \mathcal{N}(j) \lim_{y\to\infty} y^{k-2j} V^{\omega+1}_{\frac{k}{2}-j}(x,y,z), \tag{5.55}$$

and

$$\lim_{y\to\infty} y^{2j} V_j^{\omega}(x,y,z) = \mathcal{N}(j) V^{\omega-1}_{\frac{k}{2}-j}(x,y=0,z), \tag{5.56}$$

where $\mathcal{N}(j)$ was defined in Eq. (5.34). It was shown recently in [39] that, assuming the y-dependence proposed in [38] for all three-point functions, these relations fix the y-basis structure constants in terms of the unflowed ones, which can be found in [103]. Here we show that Eqs. (5.55) and (5.56) are actually much more powerful: they allow us to fix the y-dependence as well. More explicitly, we use them to derive all unknown coefficients A_{ij}, B_{ij}, and C_i appearing in (5.54).

By means of Eqs. (5.55) and (5.56), we find that all even parity correlators can be related to (at least) three *different* situations where a covering map satisfying (5.38) and (5.40) exists. Explicitly, given $(\omega_1,\omega_2,\omega_3)$ satisfying (5.53), it follows that the adjacent assignments $(\omega_1+1,\omega_2,\omega_3)$, $(\omega_1,\omega_2+1,\omega_3)$ and $(\omega_1,\omega_2,\omega_3-1)$ satisfy all conditions in (5.37). Let us denote the corresponding covering maps as follows:

$$\Gamma_1^+ \equiv \Gamma[\omega_1+1,\omega_2,\omega_3]\,, \qquad \Gamma_2^+ \equiv \Gamma[\omega_1,\omega_2+1,\omega_3]\,, \qquad \Gamma_3^- \equiv \Gamma[\omega_1,\omega_2,\omega_3-1]\,. \tag{5.57}$$

Then, the relations (5.55) and (5.56) provide the following set of identities:

$$\langle V^{\omega_1}_{j_1}(0)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle = \lim_{y_1\to\infty} y_1^{k-2j_1}\mathcal{N}(j_1)\langle V^{\omega_1+1}_{\frac{k}{2}-j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle \tag{5.58a}$$

$$\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(0)V^{\omega_3}_{j_3}(y_3)\rangle = \lim_{y_2\to\infty} y_2^{k-2j_2}\mathcal{N}(j_2)\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2+1}_{\frac{k}{2}-j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle \tag{5.58b}$$

$$\lim_{y_3\to\infty} y_3^{2j_3}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle = \mathcal{N}(j_3)\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3-1}_{\frac{k}{2}-j_3}(0)\rangle . \tag{5.58c}$$

Having excluded the even edge cases, the same holds for the adjacent assignments $(\omega_1-1,\omega_2,\omega_3)$, $(\omega_1,\omega_2-1,\omega_3)$ and $(\omega_1,\omega_2,\omega_3+1)$, the corresponding maps being

$$\Gamma_1^- \equiv \Gamma[\omega_1-1,\omega_2,\omega_3]\,, \qquad \Gamma_2^- \equiv \Gamma[\omega_1,\omega_2-1,\omega_3]\,, \qquad \Gamma_3^+ \equiv \Gamma[\omega_1,\omega_2,\omega_3+1]\,. \tag{5.59}$$

These lead to

$$\lim_{y_1\to\infty} y_1^{2j_1}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle = \mathcal{N}(j_1)\langle V^{\omega_1-1}_{\frac{k}{2}-j_1}(0)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle \tag{5.60a}$$

$$\lim_{y_2\to\infty} y_2^{2j_2}\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle = \mathcal{N}(j_2)\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2-1}_{\frac{k}{2}-j_2}(0)V^{\omega_3}_{j_3}(y_3)\rangle , \tag{5.60b}$$

$$\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(0)\rangle = \lim_{y_3\to\infty} y_3^{k-2j_3}\mathcal{N}(j_3)\langle V^{\omega_1}_{j_1}(y_1)V^{\omega_2}_{j_2}(y_2)V^{\omega_3+1}_{\frac{k}{2}-j_3}(y_3)\rangle \tag{5.60c}$$

All expressions on the RHS of Eqs. (5.58) and (5.60) are limits of correlators discussed in the previous section. Hence, they must satisfy the appropriate limits of the differential equations given in (5.49). For instance, $\langle V^{\omega_1-1}_{\frac{k}{2}-j_1}(0)V^{\omega_2}_{j_2}(y_2)V^{\omega_3}_{j_3}(y_3)\rangle$ is annihilated by the differential operators

$$\begin{aligned} &y_2(y_2\partial_{y_2}+2j_2)+(\omega_1-\omega_2-\omega_3)^{-1}\big\{(\omega_1+\omega_2-\omega_3)a_2[\Gamma_1^-]^2\partial_{y_2} \\ &-2a_2[\Gamma_1^-]\,\big[(\omega_1-\omega_3)(j_2+y_2\partial_{y_2})+\omega_2(j_3+y_3\partial_{y_3}-a_3[\Gamma_1^-]\partial_{y_3}+j_1+j_3)\big]\,\big\} \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} &y_3(y_3\partial_{y_3}+2j_3)+(\omega_1-\omega_2-\omega_3)^{-1}\big\{(\omega_1-\omega_2+\omega_3)a_3[\Gamma_1^-]^2\partial_{y_3} \\ &-2a_3[\Gamma_1^-]\,\big[(\omega_1-\omega_2)(j_3+y_3\partial_{y_3})+\omega_3(j_2+y_2\partial_{y_2}-a_2[\Gamma_1^-]\partial_{y_2}+j_1+j_2)\big]\,\big\}, \end{aligned} \tag{5.62}$$

where $a_i[\Gamma_1^-]$ denotes the coefficient a_i associated with the map Γ_1^-. One can obtain analogous equations from the other odd parity correlators involved in (5.58). This leads to twelve differential operators, which must coincide with the appropriate limits of those provided in Eqs. (5.54). As an example, in the situation considered above we should match (5.61) and (5.62) with

$$\begin{aligned} &y_2(y_2\partial_{y_2}+2j_2)+A_{22}y_2\partial_{y_2}+A_{23}y_3\partial_{y_3}-B_{22}\partial_{y_2}-B_{23}\partial_{y_3}-2A_{21}j_1+C_2, \\ &y_3(y_3\partial_{y_3}+2j_3)+A_{33}y_3\partial_{y_3}+A_{32}y_2\partial_{y_2}-B_{33}\partial_{y_3}-B_{32}\partial_{y_2}-2A_{31}j_1+C_3. \end{aligned} \tag{5.63}$$

After identifying all linearly independent terms in these 12 equations we find a total of 60 conditions. In the end, 21 of these 60 conditions can be used to solve explicitly for all the coefficients A_{ij}, B_{ij} and C_i in (5.54). Consistency of the system demands that the remaining 39 conditions must hold. We emphasize that the fact that these are identically satisfied is a highly non-trivial check of the logic behind our proof, and also a remarkable consequence of the identities relating the a_i coefficients of the different covering maps involved.

There are many equivalent ways to write the resulting coefficients. We find that the simplest one is as follows:

$$A = \begin{pmatrix} \frac{2(\omega_3-\omega_2)}{\omega_1+\omega_2-\omega_3} a_1[\Gamma_3^-] & \frac{2\omega_1}{\omega_1+\omega_2-\omega_3} a_1[\Gamma_3^-] & \frac{2\omega_1}{\omega_1-\omega_2+\omega_3} a_1[\Gamma_2^-] \\ \frac{2\omega_2}{\omega_1+\omega_2-\omega_3} a_2[\Gamma_3^-] & \frac{2(\omega_1-\omega_3)}{-\omega_1+\omega_2+\omega_3} a_2[\Gamma_1^-] & \frac{2\omega_2}{-\omega_1+\omega_2+\omega_3} a_2[\Gamma_1^-] \\ \frac{2\omega_3}{\omega_1-\omega_2+\omega_3} a_3[\Gamma_2^-] & \frac{2\omega_3}{-\omega_1+\omega_2+\omega_3} a_3[\Gamma_1^-] & \frac{2(\omega_1-\omega_2)}{-\omega_1+\omega_2+\omega_3} a_3[\Gamma_1^-] \end{pmatrix}, \tag{5.64}$$

$$B = \begin{pmatrix} \frac{(\omega_1-\omega_2+\omega_2)}{\omega_1+\omega_2-\omega_3} a_1[\Gamma_3^-]^2 & \frac{2\omega_1}{\omega_1+\omega_2-\omega_3} a_1[\Gamma_3^-] a_2[\Gamma_3^-] & \frac{2\omega_1}{\omega_1-\omega_2+\omega_3} a_1[\Gamma_2^-] a_3[\Gamma_2^-] \\ \frac{2\omega_2}{\omega_1+\omega_2-\omega_3} a_1[\Gamma_3^-] a_2[\Gamma_3^-] & \frac{(\omega_1+\omega_2-w_2)}{-\omega_1+\omega_2+\omega_3} a_2[\Gamma_1^-]^2 & \frac{2\omega_2}{-\omega_1+\omega_2+\omega_3} a_2[\Gamma_1^-] a_3[\Gamma_1^-] \\ \frac{2\omega_3}{\omega_1-\omega_2+\omega_3} a_1[\Gamma_2^-] a_3[\Gamma_2^-] & \frac{2\omega_3}{-\omega_1+\omega_2+\omega_3} a_2[\Gamma_1^-] a_3[\Gamma_1^-] & \frac{(\omega_1-\omega_2+w_2)}{-\omega_1+\omega_2+\omega_3} a_3[\Gamma_1^-]^2 \end{pmatrix}, \tag{5.65}$$

and

$$C = \begin{pmatrix} \frac{4\omega_1 j_3}{\omega_1-\omega_2+\omega_3} a_1[\Gamma_2^-] + \frac{2\omega_1(j_2+j_3)+2j_1(\omega_3-\omega_2)}{\omega_1+\omega_2-\omega_3} a_1[\Gamma_3^-] \\ \frac{4\omega_2 j_1}{\omega_1+\omega_2-\omega_3} a_2[\Gamma_3^-] - \frac{2\omega_2(j_1+j_3)+2j_2(\omega_1-\omega_3)}{\omega_1-\omega_2-\omega_3} a_2[\Gamma_1^-] \\ \frac{4\omega_3 j_1}{\omega_1-\omega_2+\omega_3} a_3[\Gamma_2^-] - \frac{2\omega_3(j_1+j_2)+2j_3(\omega_1-\omega_2)}{\omega_1+\omega_2-\omega_3} a_3[\Gamma_1^-] \end{pmatrix}. \tag{5.66}$$

Consequently, we can write the differential equations for the even parity correlators as

$$\Big\{ \left(y_1 - a_1[\Gamma_3^-]\right)^2 \partial_{y_1} + 2j_1 \left(y_1 - a_1[\Gamma_3^-]\right) + \frac{2a_1[\Gamma_3^-]\omega_1}{\omega_1+\omega_2-\omega_3} \Big[(y_1 - a_1[\Gamma_3^-])\partial_{y_1} + j_1 \\ + (y_2 - a_2[\Gamma_3^-])\partial_{y_2} + j_2 - (y_3 - a_3[\Gamma_2^-])\partial_{y_3} - j_3 \Big] \Big\} \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3) \rangle = 0\,, \tag{5.67}$$

$$\Big\{ \left(y_2 - a_2[\Gamma_3^-]\right)^2 \partial_{y_2} + 2j_2 \left(y_2 - a_2[\Gamma_3^-]\right) + \frac{2a_2[\Gamma_3^-]\omega_2}{\omega_1+\omega_2-\omega_3} \Big[(y_1 - a_1[\Gamma_3^-])\partial_{y_1} + j_1 \\ + (y_2 - a_2[\Gamma_3^-])\partial_{y_2} + j_2 - (y_3 - a_3[\Gamma_1^-])\partial_{y_3} - j_3 \Big] \Big\} \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3) \rangle = 0\,, \tag{5.68}$$

and

$$\Big\{ \left(y_3 - a_3[\Gamma_1^-]\right)^2 \partial_{y_3} + 2j_3 \left(y_3 - a_3[\Gamma_1^-]\right) + \frac{2a_3[\Gamma_1^-]\omega_3}{\omega_1 - \omega_2 - \omega_3} \Big[(y_1 - a_1[\Gamma_2^-])\partial_{y_1} + j_1 \\ - (y_2 - a_2[\Gamma_1^-])\partial_{y_2} - j_2 - (y_3 - a_3[\Gamma_1^-])\partial_{y_3} - j_3 \Big] \Big\} \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3) \rangle = 0\,. \tag{5.69}$$

The structural similarity of these equations with those of the odd case in Eq. (5.49) is striking. This suggest that there must be a way to derive Eqs. (5.67), (5.68) and (5.69) directly from the *adjacent* covering maps along the lines of Sec. 5.2.1. We will not attempt to do this here.

By solving Eqs. (5.67), (5.68) and (5.69), we find that, up to an overall constant, even parity correlators satisfying (5.53) take the form

$$\begin{aligned} \langle V_{j_1}^{\omega_1}(y_1) V_{j_2}^{\omega_2}(y_2) V_{j_3}^{\omega_3}(y_3) \rangle \;=\; & \left(1 - \frac{y_2}{a_2[\Gamma_3^+]} - \frac{y_3}{a_3[\Gamma_2^+]} + \frac{y_2 y_3}{a_2[\Gamma_3^-]a_3[\Gamma_2^+]}\right)^{j_1-j_2-j_3} \\ \times & \left(1 - \frac{y_1}{a_1[\Gamma_3^+]} - \frac{y_3}{a_3[\Gamma_1^+]} + \frac{y_1 y_3}{a_1[\Gamma_3^-]a_3[\Gamma_1^+]}\right)^{j_2-j_3-j_1} \\ \times & \left(1 - \frac{y_1}{a_1[\Gamma_2^+]} - \frac{y_2}{a_2[\Gamma_1^+]} + \frac{y_1 y_2}{a_1[\Gamma_2^+]a_2[\Gamma_1^-]}\right)^{j_3-j_1-j_2} . \end{aligned} \tag{5.70}$$

As in the odd case, one can check that this is consistent with the conjectured expressions in Eq. (5.28) by using the relations between covering map coefficients and ratios of the numbers P_ω defined in (5.31). It is also straightforward to see that (5.70) matches all relevant limits of the corresponding odd cases in Eq. (5.50), as implied by the different series identifications in Eqs. (5.58) and (5.60). We also note that, upon using some identities among the a_i coefficients such as $a_1[\Gamma_2^+]a_2[\Gamma_1^-] = a_1[\Gamma_2^-]a_2[\Gamma_1^+]$, the solution in Eq. (5.70) manifestly enjoys bosonic exchange symmetry.

5.2.3 Edge cases

In this subsection, we consider spectral flow assignments lying at the "edge" of the inequalities displayed in Eqs. (5.37) and (5.53). More explicitly, we consider three-point functions with spectral flow charges satisfying either

$$\omega_3 = \omega_1 + \omega_2 \quad \text{or} \quad \omega_3 = \omega_1 + \omega_2 + 1\,, \qquad \omega_i \geq 1\,,\, \forall\, i\,. \tag{5.71}$$

Following the nomenclature of the previous sections, we refer to these as the even and odd edge cases, respectively. Note that, according to the fusion rules in Eq. (5.36), these include all possibly non-vanishing correlators which were not included in sections 5.2.1 and 5.2.2 above, except for correlators with unflowed insertions, which will be discussed later on.

The treatment of the edge cases is slightly different from what we have discussed so far. Indeed, the general method based on series identifications employed in Sec. 5.2.2 breaks down when considering the even edge cases. We find that many of the coefficients in Eqs. (5.64), (5.65) and (5.66) become either divergent or indeterminate when $\omega_1 + \omega_2 = \omega_3$, related to the fact that three of the six adjacent maps cease to exist, namely those in (5.59). Although, in principle, the existence of the three maps in (5.57) could give enough constraints, in practice one runs into similar problems with divergent or indeterminate coefficients. As for the odd edge cases, it turns out that the associated covering map does not exist. Moreover, the former are only related to even edge cases by the SL(2,$\mathbb{R}$) series identifications.

We now show that alternative techniques involving current insertions can be used to derive the relevant differential equations satisfied by the y-basis edge correlators. Some of these equations are easier to derive in the limit where the first two vertex operators collide. As discussed in [38], one can take the vertex operators to be inserted at $(x_1, x_2, x_3) = (0, x, \infty)$, and then consider the limit $x \to 0$. More explicitly, we have [38]

$$\begin{aligned}&\langle V_{j_1}^{\omega_1}(0, y_1, 0) V_{j_2}^{\omega_2}(x, y_2, 1) V_{j_3}^{\omega_3}(\infty, y_3, \infty)\rangle \\ &= x^{-j_1-j_2+j_3+\frac{k}{2}(-\omega_1-\omega_2+\omega_3)} \langle V_{j_1}^{\omega_1}\left(0, \frac{y_1}{x}, 0\right) V_{j_2}^{\omega_2}\left(1, \frac{y_2}{x}, 1\right) V_{j_3}^{\omega_3}(\infty, y_3 x, \infty)\rangle \,. \qquad (5.72)\end{aligned}$$

The three-point functions that remain well-defined in this limit are those for which

$$|\omega_1 + \omega_2 - \omega_3| \leq 1\,, \qquad (5.73)$$

including both edge cases in (5.71). In these instances, it is possible to obtain the correlator with vertex operators inserted at generic points (x_1, x_2, x_3) from the one in the colliding limit with (x, x, x_3) by means of the global Ward identities. We will derive some of the relevant differential equations satisfied by the edge correlators in the above collision limit, and only recover the full correlators at the end. Note that taking $x \to 0$ and $x_3 \to \infty$ precisely corresponds to the limit in which these correlators can be interpreted as m-basis correlators of flowed primaries as in [103, 178], where they were denoted as spectral flow *conserving* and spectral flow *violating* three-point functions, depending on the overall parity of the spectral flow charges.

To illustrate how this works, let us derive a constraint that will be satisfied by both edge cases. We set $x_1 = x_2 = x$ and consider the integral

$$\oint_{\mathcal{C}} \langle J^3(x,z) V_{j_1}^{\omega_1}(x, y_1, z_1) V_{j_2}^{\omega_2}(x, y_2, z_2) V_{j_3}^{\omega_3}(x_3, y_3, z_3)\rangle \, dz\,, \qquad (5.74)$$

where $\mathcal{C}$ denotes a contour encircling all three insertion points. This trivially vanishes as there is no residue at infinity. On the other hand, the action of the current on the vertex operators at x can be read off directly from Eq. (5.21), while for the one inserted at x_3 we use

$$J^3(x,z) = J^3(x_3,z) + (x_3 - x)J^+(x_3,z)\,, \tag{5.75}$$

which follows from (5.9). We obtain the following differential equation:

$$\left[\sum_{i=1}^{3}\left(y_i\partial_{y_i} + j_i + \frac{k}{2}\omega_i\right) + (x_3 - x)\partial_{x_3}\right]\langle V_{j_1}^{\omega_1}(x,y_1,z_1)V_{j_2}^{\omega_2}(x,y_2,z_2)V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle = 0\,. \tag{5.76}$$

We now Möbius-fix the worldsheet insertions to $(0,1,\infty)$, and further consider the limit $(x,x_3) \to (0,\infty)$. The y_i coordinates then get rescaled according to Eq. (5.25). This leads to

$$\begin{aligned}\Big[&y_1\partial_{y_1} + y_2\partial_{y_2} - y_3\partial_{y_3} + j_1 + j_2 - j_3 \\ &+ \frac{k}{2}(\omega_1 + \omega_2 - \omega_3)\Big]\langle V_{j_1}^{\omega_1}(0,y_1,0)V_{j_2}^{\omega_2}(0,y_2,1)V_{j_3}^{\omega_3}(\infty,y_3,\infty)\rangle = 0\,.\end{aligned} \tag{5.77}$$

We find that Eq. (5.77) is the y-basis version of the usual charge-conservation equation for m-basis three-point functions of spectrally flowed primaries. This holds for all correlators satisfying (5.73), including both edge cases.

5.2.3.1 Even edge cases

In this subsection we focus on the even edge cases, where $\omega_3 = \omega_1 + \omega_2$. We will derive the remaining two differential equations by considering correlators with an extra insertion of the $J^-(x,z)$ current multiplied by the appropriate ratio of worldsheet coordinates. This is similar to what was used in [107] when discussing the proof of the m-basis spectral flow *violation* rules, and also more recently in [195] in the context of the supersymmetric version of this model.

The integral

$$\oint_C \langle J^-(x_3,z)V_{j_1}^{\omega_1}(x_1,y_1,z_1)V_{j_2}^{\omega_2}(x_2,y_2,z_2)V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle\,\frac{(z-z_1)^{\omega_1}(z-z_2)^{\omega_2}}{(z-z_3)^{\omega_3}}dz \tag{5.78}$$

vanishes since there is no pole at infinity. Using the OPEs of $J^-(x_3,z)$ with the vertex operators, this yields

$$\begin{aligned}&\left[x_{31}^2\frac{z_{12}^{\omega_2}}{z_{13}^{\omega_3}}\partial_{y_1} + x_{32}^2\frac{z_{21}^{\omega_1}}{z_{23}^{\omega_3}}\partial_{y_2} + z_{31}^{\omega_1}z_{32}^{\omega_2}(y_3^2\partial_{y_3} + 2j_3y_3)\right] \\ &\qquad\langle V_{j_1}^{\omega_1}(x_1,y_1,z_1)V_{j_2}^{\omega_2}(x_2,y_2,z_2)V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle = 0\,.\end{aligned} \tag{5.79}$$

Proceeding similarly with

$$\oint_C \langle J^-(x,z) V_{j_1}^{\omega_1}(x,y_1,z_1) V_{j_2}^{\omega_2}(x,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle \frac{(z-z_3)^{\omega_3}}{(z-z_1)^{\omega_1}(z-z_2)^{\omega_2}} dz\,, \tag{5.80}$$

where we have imposed the collision limit mentioned above, we find

$$\begin{aligned}\Big[\frac{z_{13}^{\omega_3}}{z_{12}^{\omega_2}}(y_1^2\partial_{y_1}+2j_1y_1)+\frac{z_{23}^{\omega_3}}{z_{21}^{\omega_2}}(y_2^2\partial_{y_2}+2j_2y_2)+\frac{(x-x_3)^2}{z_{31}^{\omega_1}z_{32}^{\omega_2}}\partial_{y_3}\Big]&\\ \langle V_{j_1}^{\omega_1}(x,y_1,z_1) V_{j_2}^{\omega_2}(x,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle = 0\,.&\end{aligned} \tag{5.81}$$

We now fix the worldsheet coordinates to $(0,1,\infty)$ while sending $x\to 0$ and $x_3\to\infty$, and use Eq. (5.25) for the corresponding rescaling of the y variables. Including the charge conservation condition (5.77), the system of differential equations satisfied by the even edge correlator in the collision limit is then

$$\begin{aligned}0&=\Big[j_1+j_2-j_3+y_1\partial_{y_1}+y_2\partial_{y_2}-y_3\partial_{y_3}\Big]\langle\ldots\rangle\,,\\ 0&=\Big[(-1)^{\omega_1}\partial_{y_1}+(-1)^{\omega_3}\partial_{y_2}+(y_3^2\partial_{y_3}+2j_3y_3)\Big]\langle\ldots\rangle\,,\\ 0&=\Big[(-1)^{\omega_1}(y_1^2\partial_{y_1}+2j_1y_1)+(-1)^{\omega_3}(y_2^2\partial_{y_2}+2j_2y_2)+\partial_{y_3}\Big]\langle\ldots\rangle\,.\end{aligned} \tag{5.82}$$

where $\langle\ldots\rangle$ stands for $\langle V_{j_1}^{\omega_1}(0,y_1,0) V_{j_2}^{\omega_2}(0,y_2,1) V_{j_3}^{\omega_3}(\infty,y_3,\infty)\rangle$. Note that only the second equation in (5.82) remains valid away from the collision limit.

Up to an overall y-independent constant, the general solution of the system (5.82) can be written as follows:

$$\begin{aligned}&\langle V_{j_1}^{\omega_1}(0,y_1,0) V_{j_2}^{\omega_2}(0,y_2,1) V_{j_3}^{\omega_3}(\infty,y_3,\infty)\rangle\\ &=((-1)^{\omega_1}y_1-(-1)^{\omega_3}y_2)^{j_3-j_1-j_2}(1+(-1)^{\omega_3}y_2y_3)^{j_1-j_2-j_3}(1+(-1)^{\omega_1}y_1y_3)^{j_2-j_1-j_3}\,.\end{aligned} \tag{5.83}$$

This matches the result of [38], see their Eq. (5.37b).

As mentioned above, for more general values of the insertion points the corresponding three-point functions follow from the global Ward identities. As it turns out, we can infer the result in a heuristic way by looking at the general expression given in Eq. (5.70). Indeed, one can verify that, upon setting $\omega_3=\omega_1+\omega_2$, the coefficients $a_1(\Gamma_3^+), a_2(\Gamma_3^+)$ and $a_2(\Gamma_1^-)$ diverge, which is a manifestation of the fact that the associated covering maps do not exist. Since all other coefficients remain finite, we obtain

$$\begin{aligned}\langle V_{j_1}^{\omega_1}(y_1)V_{j_2}^{\omega_2}(y_2)V_{j_3}^{\omega_3}(y_3)\rangle\ &=\left(1+(-1)^{\omega_1}\frac{(\omega_1+\omega_2-1)!}{(\omega_1-1)!\omega_2!}y_3+(-1)^{\omega_3}y_2y_3\right)^{j_1-j_2-j_3}\\ \times&\quad\left(1+(-1)^{\omega_1+1}\frac{(\omega_1+\omega_2-1)!}{\omega_1!(\omega_2-1)!}y_3+(-1)^{\omega_1}y_1y_3\right)^{j_2-j_3-j_1}\\ \times&\quad\left(1+(-1)^{\omega_1+1}\frac{\omega_1!\omega_2!}{(\omega_1+\omega_2)!}((-1)^{\omega_1}y_1-(-1)^{\omega_3}y_2)\right)^{j_3-j_1-j_2}\,.\end{aligned} \tag{5.84}$$

Up to the normalisation, to be discussed below, this precisely matches the conjecture (5.28).

5.2.3.2 Odd edge cases

We now turn to the odd edge cases, where $\omega_3 = \omega_1 + \omega_2 + 1$. Since the procedure is analogous to what we just described we will skip some of the intermediate steps.

In addition to (5.77), we find two differential equations by considering contour integrals very similar to that in Eq. (5.78). We first take

$$\oint_{\mathcal{C}} \langle J^-(x_3,z) V_{j_1}^{\omega_1}(x_1,y_1,z_1) V_{j_2}^{\omega_2}(x_2,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle \, \frac{(z-z_1)^{\omega_1+1}(z-z_2)^{\omega_2}}{(z-z_3)^{\omega_3}} dz\,, \tag{5.85}$$

which again vanishes due to the absence of a residue at infinity. The same holds for

$$\oint_{\mathcal{C}} \langle J^-(x_3,z) V_{j_1}^{\omega_1}(x_1,y_1,z_1) V_{j_2}^{\omega_2}(x_2,y_2,z_2) V_{j_3}^{\omega_3}(x_3,y_3,z_3)\rangle \, \frac{(z-z_1)^{\omega_1}(z-z_2)^{\omega_2+1}}{(z-z_3)^{\omega_3}} dz\,. \tag{5.86}$$

Hence, we find the following system of differential equations:

$$\begin{aligned} 0 &= \left[j_1 + j_2 - j_3 - \frac{k}{2} + y_1\partial_{y_1} + y_2\partial_{y_2} - y_3\partial_{y_3} \right] \langle \ldots \rangle \,, \\ 0 &= \left[(-1)^{\omega_3}\partial_{y_2} + (y_3^2\partial_{y_3} + 2j_3y_3) \right] \langle \ldots \rangle \,, \\ 0 &= \left[(-1)^{\omega_1}\partial_{y_1} + (y_3^2\partial_{y_3} + 2j_3y_3) \right] \langle \ldots \rangle \,. \end{aligned} \tag{5.87}$$

where $\langle \ldots \rangle$ again denotes $\langle V_{j_1}^{\omega_1}(0,y_1,0) V_{j_2}^{\omega_2}(0,y_2,1) V_{j_3}^{\omega_3}(\infty,y_3,\infty)\rangle$. In this case, only the first of these equations gets eventually modified away from the collision limit.

We find that, up to an overall normalization, the odd edge three-point functions read

$$\begin{aligned} &\langle V_{j_1}^{\omega_1}(0,y_1,0) V_{j_2}^{\omega_2}(0,y_2,1) V_{j_3}^{\omega_3}(\infty,y_3,\infty)\rangle \\ &= y_3^{j_1+j_2-j_3-\frac{k}{2}} (1 + (-1)^{\omega_1} y_1 y_3 + (-1)^{\omega_3} y_2 y_3)^{\frac{k}{2}-j_1-j_2-j_3}\,, \end{aligned} \tag{5.88}$$

in the collision limit, thus matching the result in [38], see their Eq. (5.37c). Moreover, as in the even edge case, we can infer the solution for generic insertion points from the general expression in Eq. (5.51). For this, we set $\omega_3 = \omega_1 + \omega_2 + 1$ and carefully take the limit $a_3 \to 0$, $a_{1,2} \to \infty$ with the products $a_1 a_3$ and $a_2 a_3$ fixed. This yields

$$\begin{aligned} &\langle V_{j_1}^{\omega_1}(0,y_1,0) V_{j_2}^{\omega_2}(1,y_2,1) V_{j_3}^{\omega_3}(\infty,y_3,\infty)\rangle = y_3^{j_1+j_2-j_3-\frac{k}{2}} \\ &\times \left(1 + (-1)^{\omega_1+1} \frac{(\omega_1+\omega_2)!}{\omega_1!\omega_2!} y_3 + (-1)^{\omega_1} y_1 y_3 + (-1)^{\omega_3} y_2 y_3 \right)^{\frac{k}{2}-j_1-j_2-j_3} . \end{aligned} \tag{5.89}$$

One can check that this matches the y-dependence given in the conjecture of Eq. (5.27) for correlators with appropriate spectral flow assignments. Moreover, upon using Eq. (5.55) and (5.56) we also see that, as expected, the expressions in Eq. (5.70) and Eq. (5.89) are related via series identifications.

5.2.4 Three-point functions with unflowed insertions

So far, we have considered three-point functions where all vertex operators had non-zero spectral flow charges. However, it is natural to expect that the above results include the special cases where some of the insertions are unflowed. We now show how the latter are obtained. Note that we still assume $\omega_3 \geq \omega_{1,2}$, as in the previous sections.

Let us start by discussing the case of a single unflowed insertion, namely $\omega_1 = 0$. The fusion rules in Eq. (5.36) can then be satisfied iff $\omega_3 = \omega_2$ or $\omega_3 = \omega_2 + 1$. These are exactly the two cases that were computed in [178] in full generality from m-basis techniques. In this sense, the results presented in this section are not new, but we include them for completeness. Hence, the relevant correlators correspond to the spectral flow assignments $(0, \omega, \omega + 1)$ and $(0, \omega, \omega)$. By means of

$$V_j(x,z) = \mathcal{N}(j)\, V^1_{\frac{k}{2}-j,h=j}(x,z) = \mathcal{N}(j) \lim_{y\to\infty} y^{k-2j} V^1_{\frac{k}{2}-j}(x,y,z)\,, \tag{5.90}$$

which is a particular case of (5.55), these can be obtained from the three-point functions with charges $(1, \omega, \omega + 1)$ and $(1, \omega, \omega)$, respectively. More precisely, we have

$$\langle V_{j_1} V^{\omega}_{j_2}(y_2) V^{\omega+1}_{j_3}(y_3)\rangle = \mathcal{N}(j_1) \lim_{y_1\to\infty} y_1^{k-2j_1} \langle V^1_{\frac{k}{2}-j_1}(y_1) V^{\omega}_{j_2}(y_2) V^{\omega+1}_{j_3}(y_3)\rangle\,, \tag{5.91}$$

and

$$\langle V_{j_1} V^{\omega}_{j_2}(y_2) V^{\omega}_{j_3}(y_3)\rangle = \mathcal{N}(j_1) \lim_{y_1\to\infty} y_1^{k-2j_1} \langle V^1_{\frac{k}{2}-j_1}(y_1) V^{\omega}_{j_2}(y_2) V^{\omega}_{j_3}(y_3)\rangle\,, \tag{5.92}$$

where we have abbreviated $V_{j_1}(0,0) \equiv V_{j_1}$.

Focusing on (5.91), the RHS involves an even edge correlator. We thus need to consider the appropriate limit of Eq. (5.84), which gives

$$\langle V_{j_1} V^{\omega}_{j_2}(y_2) V^{\omega+1}_{j_3}(y_3)\rangle = y_3^{j_1+j_2-j_3-\frac{k}{2}} \left(1 - y_3 - (-1)^{\omega} y_2 y_3\right)^{\frac{k}{2}-j_1-j_2-j_3}\,. \tag{5.93}$$

up to an overall constant. On the other hand, the RHS of (5.92) is obtained as the appropriate limit of the solution in Eq. (5.50), namely

$$\langle V^1_{j_1}(y_1)V^{\omega}_{j_2}(y_2)V^{\omega}_{j_3}(y_3)\rangle = (y_1-\omega)^{j_2+j_3-j_1-\frac{k}{2}}(y_2+(-1)^{\omega})^{j_1+j_3-j_2-\frac{k}{2}}(y_3-1)^{j_1+j_2-j_3-\frac{k}{2}}$$
$$\times\Big((-1)^{\omega+1}(\omega+1)+(-1)^{\omega}y_1-y_2+(-1)^{\omega}y_3-(\omega-1)y_2y_3+y_1y_2y_3\Big)^{\frac{k}{2}-j_1-j_2-j_3}. \tag{5.94}$$

Hence, we get

$$\langle V_{j_1}V^{\omega}_{j_2}(y_2)V^{\omega}_{j_3}(y_3)\rangle = (y_2+(-1)^{\omega})^{j_3-j_1-j_2}\,(y_3-1)^{j_2-j_1-j_3}\,((-1)^{\omega}+y_2y_3)^{j_1-j_2-j_3}. \tag{5.95}$$

up to the overall constant. Finally, we consider correlators with exactly two unflowed insertions, $\omega_1=\omega_2=0$. According to (5.36) this can only be non-trivial for $\omega_3=1$. By using again Eq. (5.90) we get

$$\begin{aligned}\langle V_{j_1}V_{j_2}V^1_{j_3}(y_3)\rangle &= \mathcal{N}(j_2)\lim_{y_2\to\infty}y_2^{k-2j_2}\langle V_{j_1}V^1_{\frac{k}{2}-j_2}(y_2)V^1_{j_3}(y_3)\rangle\\ &= y_3^{j_1+j_2-j_3-\frac{k}{2}}(y_3-1)^{\frac{k}{2}-j_1-j_2-j_3},\end{aligned} \tag{5.96}$$

where in the last line we have ignored an overall normalization factor. In this way, we match all the corresponding results of [38], where the authors showed that this further reproduces the original computations of [103, 178].

5.2.5 Normalization

So far we have focused on the dependence of the y-basis spectrally flowed correlators on the variables y_1, y_2 and y_3, and shown that it matches precisely the predictions of [38]. We now describe how the overall normalizations in Eqs. (5.27)-(5.33) are obtained[51].

Once again, the argument relies on the SL(2,$\mathbb{R}$) series identifications. Indeed, identities such as those in Eqs. (5.58) and (5.60) must hold exactly, including the normalization factors. Having fixed the y-dependence, we can thus determine the normalizations recursively, starting from the unflowed three-point functions of [106, 103]. For instance, we consider the following identity:

$$\lim_{y_3\to\infty}y_3^{2j_3}\left\langle V^{\omega_1}_{j_1}(y_1)\,V^{\omega_2}_{j_2}(y_2)\,V^{\omega_3}_{j_3}(y_3)\right\rangle = \mathcal{N}(j_3)\left\langle V^{\omega_1}_{j_1}(y_1)\,V^{\omega_2}_{j_2}(y_2)\,V^{\omega_3-1}_{\frac{k}{2}-j_3}(0)\right\rangle, \tag{5.97}$$

which will give us a recursion relation for $C_\omega(j_1,j_2,j_3)$. Since the latter is independent of the y_i, we can set $y_1=y_2=0$. Using the y-dependence derived above, written as in

[51]This was already discussed in [39], assuming the y-dependence of the correlators was as in [38]. We reproduce the argument here for completeness

Eqs. (5.27) and (5.28), one finds that the product of X_I factors on the left- an right-hand sides of (5.97), both reduce to either

$$P_{\omega}^{j_1+j_2+j_3-k}\, P_{\omega+e_1+e_2}^{j_3-j_1-j_2}\, P_{\omega+e_2-e_3}^{j_1-j_2-j_3}\, P_{\omega+e_1-e_3}^{j_2-j_3-j_1}, \tag{5.98}$$

or

$$P_{\boldsymbol{\omega+e_1+e_2-e_3}}^{\frac{k}{2}-j_1-j_2-j_3}\, P_{\omega+e_1}^{-j_1+j_2+j_3-\frac{k}{2}}\, P_{\omega+e_2}^{j_1-j_2+j_3-\frac{k}{2}}\, P_{\omega-e_3}^{j_1+j_2-j_3-\frac{k}{2}}, \tag{5.99}$$

depending on the overall parity of the spectral flow charges. Consequently, in both cases we find that Eq. (5.97) holds iff

$$C_\omega(j_1,j_2,j_3) = \mathcal{N}(j_3)C_{\omega-e_3}\left(j_1,j_2,\frac{k}{2}-j_3\right). \tag{5.100}$$

Analogous statements can be derived by shifting the spectral flow charges ω_1 and ω_2 instead. Moreover, one has the identity

$$\mathcal{N}(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right) = \mathcal{N}(j_2)C\left(j_1,\frac{k}{2}-j_2,j_3\right) = \mathcal{N}(j_3)C\left(j_1,j_2,\frac{k}{2}-j_3\right). \tag{5.101}$$

Since $\mathcal{N}(j)\mathcal{N}(\frac{k}{2}-j) = 1$, it follows that, as stated in (5.33), $C_\omega(j_1,j_2,j_3)$ can only be $C(j_1,j_2,j_3)$, i.e. the unflowed three-point function, or $\mathcal{N}(j_1)C(\frac{k}{2}-j_1,j_2,j_3)$, depending on the parity of $\omega_1+\omega_2+\omega_3$. To be precise, this argument is valid for discrete representations, although we expect that it holds also for the continuous series by analytic continuation in j [103, 196]. This concludes our computation of three-point functions with arbitrary spectral flow charges.

5.3 Discussion

In this chapter, we have computed the y-basis string three-point function in AdS_3 involving vertex operators with arbitrary spectral flow charges. This provides a proof for the conjecture put forward recently in [38], thus establishing integral expressions for all (primary) three-point functions of the SL(2,$\mathbb{R}$)-WZW model at level k, for all $k>3$.

The subfamily of (odd parity) correlators for which a holomorphic covering map from the worldsheet to the AdS_3 boundary exists had been obtained in [88, 38]. Here we have relied on the general structure of local Ward identities (in their y-basis formulation) and made extensive use of the SL(2,$\mathbb{R}$) series identifications, whose importance was recently highlighted in [39]. This allowed us to extend the methods based on covering maps to all other non-vanishing correlators, as defined by the fusion rules computed in [103].

Our strategy can be summarised as follows. We first argued that the differential equations satisfied by all y-basis three-point functions must take the form given in Eq. (5.54). Obtaining the general expressions for these equations for all even parity correlators then reduces to computing all unknown coefficients in (5.54). We have provided the relations among correlators with adjacent spectral flow assignments that follow from SL(2,$\mathbb{R}$) series identifications in Eqs. (5.58) and (5.60). These provide a considerable number of identities between even and odd parity correlators in the limit where one of the y variables is taken to either zero or infinity. This allowed us to evaluate all relevant coefficients A_{ij}, B_{ij} and C_i in closed form, as given in Eqs. (5.64)-(5.66). The derivation of these 21 coefficients involves solving a total of 60 conditions, of which 39 can be taken as consistency checks. The latter turn out to be satisfied in a highly non-trivial manner, related to the existence of a set of identities relating the behaviour of different covering maps in the vicinity of the insertion points.

The resulting differential equations satisfied by even parity correlators are provided in Eqs. (5.67)-(5.69). These show a striking similarity with the cases of odd total spectral flow, a hallmark of the existence of a more direct derivation by means of *adjacent* covering maps. We leave this for future work. Here we have shown that the general solution to these equations, namely Eq. (5.70), is compatible with the proposal of [38].

We have also discussed the so-called edge cases, whose spectral flow assignments saturate the fusion rules in Eqs. (5.37) and (5.53). Some subtleties arise when trying to apply the general method described above in this context. For these cases we have provided an alternative approach, based on an improved version of the m-basis methods [103, 178]. We have then described how to obtain correlators involving unflowed insertions. Finally, we fixed the overall normalization of all y-basis three-point functions, following the arguments of [39].

At this point, it is natural to ask if an analogous story holds for four-point functions, which encode crucial dynamical information about the theory. A closed formula for four-point functions in the y-basis with arbitrary spectral flow assignments in terms of the corresponding unflowed correlator was conjectured in [179]. Here the situation is more subtle: on top of the four y_i variables, four-point functions also depend non-trivially on the worldsheet and spacetime cross-ratios, and must satisfy the corresponding Knizhnik-Zamolodchikov equations. It has been known for some time [197, 198] that the latter intertwine non-trivially with the recursion relations of the type described in section 5.1.2. If the structure put forward in [179] is correct, its proof is likely to work in two steps. First, one should use arguments similar to those we have considered in this chapter to show that solutions to the y-basis differential equations associated with the four-point functions consist of various prefactors given by powers of the generalized differences X_I, defined in Eq. (5.29), multiplied by an arbitrary function of the so-called generalized cross-ratio. Second, one should prove that this arbitrary function must satisfy the same Knizhnik-Zamolodchikov equation as the corresponding *unflowed* four-point function. The extension of the proof for the case of correlation functions involving four spectrally-flowed vertex operators is a work in progress.

Chapter 6

Conclusions

In this final chapter we summarise the findings presented above, and we will discuss some possible future projects and generalisations.

In Chapter 3 we have shown, among other things, how the consistency of the worldsheet CFT is in one to one correspondence with absence of CTCs, singularities and horizon. We have proven that the quantisation conditions derived from the worldsheet BRST analysis are exactly the same as those obtained in the gravity description when demanding absence of singularities, horizons and CTCs. Using these results, we showed that the metric, B-field and dilaton obtained from the coset model can be rewritten in terms of the integers $\mathsf{k}, \mathsf{m}, \mathsf{n}, n_5$ and the modulus R_y. These background fields precisely coincide with the NS5-decoupled JMaRT background [37], thus proving that the latter is the most general consistent supergravity background described by the coset Eq. (2.205). Despite this 'no-go theorem', we were able to derive a novel subfamily of two-charge non-BPS NS5-P backgrounds not appeared before in the literature, see Eqs. (3.76) and (3.77).

The computation of stringy observables involving light probes in the NS5-decoupled JMaRT background was presented in Chapter 4. Here we first constructed the physical vertex operators (in both NS and R sectors) in the AdS limit of the coset model, and we carefully identified the worldsheet x variable dual to the local coordinate of the holographic CFT. Our findings heavily generalise the results available in the literature [167, 183, 181]. We first extended the computation of HLLH correlators to the full non-supersymmetric AdS limit of the JMaRT solution. The light massless probes involved in the correlation function possess any conformal dimension and R-charge. Secondly, we further generalised the results to an arbitrary number of light insertions, providing an elegant formula that express the coset correlator as a function of the vacuum $\mathrm{AdS}_3 \times \mathbb{S}^3$ correlator, see Eq. (4.94). This is valid for an arbitrary number of massless insertions of weights h_i and charges m_i' and $\bar{m}_i'$, and also arbitrary parameters $(\mathsf{k}, s, \bar{s})$ for which a consistent background exists. We also observed that, at large N, certain worldsheet correlators exactly match the that of the dual orbifold CFT. The non-BPS JMaRT backgrounds were not considered in [167], however we demonstrated that the agreement extends also to those backgrounds. It is known that vacuum two- and three-point functions of chiral primary operators are protected [184]. We thus conjectured that heavy-light correlators in JMaRT heavy states are protected whenever the corresponding vacuum correlator in our general formula (4.94) is protected. We investigated a particular HLLLH five-point function—the first of its kind in the literature—and found that worldsheet and symmetric product results agree.

Given its importance for coset correlators and more broadly for the AdS_3/CFT_2 correspondence, in Chapter 5 we presented the computation of the y-basis string three-point function in AdS_3 involving vertex operators with arbitrary spectral flow charges. This proves the conjectured integral expressions for all primary three-point functions of the SL(2,$\mathbb{R}$)-WZW model at level k, for all $k > 3$, first appeared in [38]. In order to achieve this goal we exploited the general y-basis structure of local Ward identities and, crucially, made extensive use of the y-basis SL(2,$\mathbb{R}$) series identifications [39].

A number of possible ideas for future research projects can originate from this thesis. Let us discuss some possible developments.

First of all, it would be tempting to generalise the results of Chapter 3 to different backgrounds. The fascinating relation between the worldsheet consistency and absence of singularities, horizons and CTCs is very suggestive and perfectly in line with the Fuzzball proposal mentioned in the Introduction. However, great care is due. At the point of writing it is far from obvious how to generalise these results to all the backgrounds that admit a worldsheet description of some sort. Hence, an obvious task would be that of finding more general (and more typical) microstate backgrounds and verify the absence of horizons, singularities and CTCs. Finding new worldsheet models is notoriously difficult. However, a natural extension of our work is immediate when involving the $\mathbb{T}^4$ in the gauging. In the latter case, we strongly believe that the majority of the results will be unchanged: the background will be absent of singularities, horizons and CTCs. Most of the computations presented in Chapter 3 will follow through.

Nevertheless, we expect that our results will be useful in analysing generalisations of the models studied here, either by changing the currents being gauged to include non-Cartan generators of the non-Abelian factors of the upstairs group, or by changing the upstairs group, or both. Our systematic approach should enable generalisations to be investigated in a similar way. For instance, there are multi-centre non-BPS generalizations of the JMaRT family [160, 161, 162] or the newly discovered NS-NS superstrata [163].

Once new worldsheet descriptions for microstate backgrounds are found, it is natural to try to generalise the results we have presented in Chapter 4. The computation of heavy-light correlators will provide valuable information about possible non-trivial properties of the background and be of great interest in the light of holography in backgrounds different from the global vacuum. We expect that our results may provide a guideline for these possible developments.

The reader should note that the computations of Chapter 4 were restricted to the AdS limit of the coset model. However, a natural question to ask is how to extend the results to the UV theory. In the full asymptotically linear dilaton models, correlators can be computed by using the vertex operators of Section 4.1. However, the x coordinate seems unlikely to be of any use in this regime given the non-local nature of the dual theory.

The results in Chapter 5 concern the proof of the conjecture for three point functions. It is very natural to ask if an analogous story holds for four-point functions, which encode crucial dynamical information about the theory. A closed formula for four-point functions in the y-basis with arbitrary spectral flow assignments in terms of the corresponding unflowed correlator was conjectured in [179]. Here the situation is more subtle: on top of the four y_i variables, four-point functions also depend non-trivially on the worldsheet and spacetime cross-ratios, and must satisfy the corresponding Knizhnik-Zamolodchikov equations. It has been known for some time [197, 198] that the latter intertwine non-trivially with the recursion relations of the type described in Section 5.1.2.

In this thesis we have shown that worldsheet techniques, when available, are tremendously effective and powerful. By contrast, trying to use a field theory description for black hole interior and horizon physics has proven so far to be largely unsuccessful. Being able to compute observables to all orders in α' allows us to depart from (super-)gravity results, and hence be able to directly address problems associated to black hole physics with the power of string theory. Indeed, on top of the infinite degrees of freedom provided by stringy excitations, it is the extended nature of strings and branes that allows us to tame issues related to curvature singularities and black hole paradoxes.

Despite its success, string theory is nowadays still in large development. Many aspects of it are still not clear, especially in strong coupling and less supersymmetric settings, two regimes that may be crucial for an appropriate description of black holes and their interior. However, there are indications that string theory will shed a definite light on problems involving short distance physics, strong gravity, singularities, and various other puzzles and paradoxes. A deeper understanding of the whole theoretical framework will necessarily help in the resolution of the information paradox and the entropy puzzle: these issues afflict black holes since the very first computations that followed from the tremendous insight and intuition of Stephen Hawking.

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