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Holonomic quantum computation in the presence of decoherence

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We present a scheme to study non-abelian adiabatic holonomies for open Markovian systems. As an application of our framework, we analyze the robustness of holonomic quantum computation against decoherence. We pinpoint the sources of error that must be corrected to achieve a geometric implementation of quantum computation completely resilient to Markovian decoherence.

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Adiabatic holonomies in quantum mechanics are unitary transformations generated by slowly changing the Hamiltonian of a quantum system through a set of parameters in a cyclic fashion. The parameters define a manifold and the holonomies depend only on the path followed by the system in this space. In case the Hamiltonian is non-degenerate, the holonomy is an abelian phase better known as Berry's phase [1]. If the Hamiltonian is degenerate, the holonomy is a non-abelian generalization of the phase which induces transitions among states belonging to the degenerate subspace [2]. Geometric phases, i.e., Berry's phase and its various generalizations, were extensively studied in the 1980's [3] and recently became, once more, a fashionable subject since it was proposed that quantum computation (QC) could be implemented in a fault-tolerant way using geometric transformations [4, 5]. The most general scheme of this kind is called holonomic quantum computation (HQC) and it involves constructing a universal set of quantum gates by making the system acquire a succession of abelian and non-abelian geometric phases. Since holonomies depend only on geometric properties of the parameter space, HQC is robust, by construction, to any kind of error that leaves these properties invariant. This is considered to be the strongest advantage of the scheme.

In any realistic implementation of quantum computation one must consider that, in practice, quantum systems are never isolated. The interaction with the environment effects that initially pure quantum states decay into statistical mixtures (called mixed states) through a process known as decoherence [6]. It is known that this process is a severe limiting factor for quantum computation. Therefore, in a pragmatic realization of HQC schemes, it is crucial to define holonomies for mixed states, and understand how they are generated under the presence of decoherence. So far, non-abelian holonomies have been only investigated for pure quantum states evolving under unitary transformations. It is now necessary to investigate to what extent the geometric properties on which holonomies depend, are left invariant under quantum noise.

Although claims that quantum gates of geometric origin offer increased fault-tolerance can be found throughout the literature, a comprehensive analysis on the robustness to errors of these schemes has not yet been presented. Partial results, however, have been obtained: For both, abelian and non-abelian phases, some errors of classical origin have been investigated [7] and the effect of quantum noise has been analyzed only for the abelian case in [8, 9, 10]. In particular, Ref. [9] uses the quantum jump approach to study the effects of decoherence in the adiabatic and non-adiabatic case. There the spin-1/2 Berry phase was found to be robust to the depolarizing channel in the no-jump trajectory. Moreover, resilience to the phase flip error is guaranteed for any number of jumps. In case the particle is driven by a quantized field, Berry's phase is found to be robust to field decoherence to second order [10].

The robustness of the abelian phase to a number of decoherence effects suggests that non-abelian phases might behave similarly. Motivated by this, in this Letter, we investigate the effects of decoherence on non-abelian holonomies under the Markovian assumption, where environmental memory effects are negligible. We demonstrate the application of our general framework to the study of decoherence in HQC. Independently of the physical implementation considered, we estimate the errors, produced by generating a universal set of holonomic gates in the presence of an environment. We pinpoint to which specific types of errors the scheme is fallible. These results constitute the basis of a framework to perform universal HQC resilient to Markovian decoherence. Decoherence can produce two kinds of errors: those which take the system out of the degenerate subspace and those which take place within the subspace. The first kind of error can be eliminated by working in the ground state and considering a system where the energy gap with the first excited state is very large. Our analysis concerns the second kind of errors. We consider only adiabatic holonomies because the non-adiabatic case is not well understood. It is not clear how to separate, in general, dynamic from geometric evolution in non-adiabatic holonomies.

Our work focuses on the applications of holonomies in quantum computation, but geometric phases are a fundamental aspect of quantum mechanics with applications in many fields. They have manifestations [3] that range from low to high energy physics, appearing in optical and solid state systems, in molecular and atomic physics, and are at the heart of phenomena such as anomalies in gauge field theories, fractional statistics, Aharonov-Bohm and quantum Hall effect. The discussion presented here is relevant to any physical situation involving holonomies where decoherence plays a role.

To introduce the concept of holonomy we consider a Hamiltonian $H(\lambda_0) = H_0$ describing an energy degenerate N-dimensional quantum system. The Hilbert space has k subspaces with corresponding energy E_k . Our analysis will be restricted to one of the degenerate subspaces with n-fold degeneracy and energy E. The system is initially prepared in a state belonging to this subspace $|\psi_i^0\rangle = |\psi_i(0)\rangle$ and the Hamiltonian of the system varied through a set of parameters λ on a control manifold \mathcal{M} in an adiabatic way. The adiabatic theorem states that if the Hamiltonian is varied sufficiently slowly with respect to any time-scale associated with the dynamics, the system will remain in a state belonging to the subspace corresponding to energy E. In other words, there is no level-crossing. The degeneracy structure of the Hamiltonian is preserved throughout the evolution which we can write as $H(\lambda) = V(\lambda)H_0V^{\dagger}(\lambda)$, where $V(\lambda)$ is unitary. If the Hamiltonian returns to its initial value after a time T, $H(T) = H_0$, describing a closed curve C in \mathcal{M} , the state $|\psi_i^0\rangle$ is mapped to $e^{-iET}U_C(\lambda)|\psi_i^0\rangle$. The transformation $U_C(\lambda)$ is called the holonomy and in the following, we will derive it.

By changing the Hamiltonian through the set of parameters λ , the state of the system is parallel transported in parameter space. A rule for parallel transport in the manifold, i.e. a connection, must be specified since there is no unique way of parallel transporting a vector. A connection is provided by requiring that the state $|\psi_{\alpha}(\lambda)\rangle$ remains normalized through parallel transport, $\langle \psi_{\beta} | \frac{\partial}{\partial \lambda_{\mu}} | \psi_{\alpha} \rangle = 0.$

In terms of the local reference basis of the degenerate subspace $\{|\phi_{\alpha}(\lambda)\rangle\}$, the state at any point of the adiabatic path in parameter space is expressed as, $|\psi_{\alpha}(\lambda)\rangle = U_{\alpha\beta}(\lambda)|\phi_{\beta}(\lambda)\rangle$, where U is unitary and $|\psi_{\alpha}(\lambda)\rangle$ corresponds to the solution of the Schrödinger equation for the initial condition $|\psi_{\alpha}^{o}\rangle = |\phi_{\alpha}(0)\rangle$. The parallel transport condition then reads,

$$U_{\gamma\delta}^{\dagger} \frac{\partial U_{\alpha\beta}}{\partial \lambda_{\mu}} P_{\beta\gamma} + U_{\gamma\delta}^{\dagger} U_{\alpha\beta} A_{\beta\gamma} = 0, \qquad (1)$$

with $P_{\beta\gamma} = \langle \phi_{\gamma} | \phi_{\beta} \rangle$ and $A_{\beta\gamma} = \langle \phi_{\gamma} | \frac{\partial}{\partial \lambda_{\mu}} | \phi_{\beta} \rangle$, in the Wilczek-Zee [2] notations. The matrix P is hermitian and in the case when the states of the local basis are orthogonal it is equal to the identity. In the presence of decoherence, the latter is not true in general. We ignored the constant term $EP_{\beta\gamma}$ which produces the global dynamical phase e^{-iET} . Solving for U when P is invertible, $U^{-1}\dot{U} = -AP^{-1}$, and integrating over the closed path C

we obtain the holonomy,

$$U_C(\lambda) = \mathbf{P}e^{-\int_C AP^{-1}}.$$
 (2)

where **P** is the path ordering operator. In terms of the eigenstates of the initial Hamiltonian H_0 , we have $A_{\beta\gamma} = \langle \psi_{\gamma}^0 | V^{\dagger} \dot{V} | \psi_{\beta}^0 \rangle$. The dimensionality *n* of the holonomy equals the degree of degeneracy of the eigenspace. Berry phase is the special case when the eigenspace is non-degenerate, and the unitary transformation is then one dimensional, i.e. a complex number. The holonomy (2) depends only on the path followed in parameter space, and transforms, under gauge transformation *g*, as $U_C(A) \to gU_C(A)g^{-1}$ [2]. We point out that all previous considerations can be extended to the case of an open curve without any modifications since gauge invariance is not relevant in our context.

To investigate the effects of decoherence in the nonabelian geometric evolution of states described above, we employ the quantum jump approach. The master equation $(\hbar = 1)$

$$\dot{\rho} = \frac{1}{i} [H(\lambda), \rho] - \frac{1}{2} \sum_{k=1}^{n} \{ L_k^{\dagger} L_k \rho + \rho L_k^{\dagger} L_k - 2L_k \rho L_k^{\dagger} \}$$
(3)

dictates, in the Markovian approximation, the evolution of the system described by the density operator ρ . The commutator generates, through the Hamiltonian $H(\lambda)$, the coherent part of the evolution and the second part represents the effect of the environment on the dynamics of system. The operators L_k are called Lindbladian, and by prescribing them, one models different decohering processes. Equation (3) is in general very difficult to solve but the quantum jump approach provides an ingenious solution to this problem. Consider that the time evolution of the density matrix, for small time intervals Δt , can be written as

$$\rho(t + \Delta t) \approx \sum_{k=0}^{n} W_k \rho(t) W_k^{\dagger}, \qquad (4)$$

where the operators W_k are complete positive maps fulfilling the completeness relation $\sum_{k=0}^{n} W_k^{\dagger} W_k = \mathbf{1}$. By setting $W_0 = \mathbf{1} - i\tilde{H}\Delta t$ and $W_k = \sqrt{\Delta t}L_k$ with \tilde{H} a non-Hermitian effective Hamiltonian,

$$\tilde{H} = H - \frac{i}{2} \sum_{k=1}^{n} L_k^{\dagger} L_k, \qquad (5)$$

the dynamics of the system is approximated by dividing the total evolution time T into a sequence of discrete intervals $\Delta t = \frac{T}{N}$. W_0 and W_k are called the "no-jump" and jump operators respectively. According to Eq. (4), the state of the system, after any time step $t_m = m\Delta t$, evolves into $\rho(t_{m+1}) = W_k \rho(t_m) W_k^{\dagger}$ (up to first order in Δt), with probability $p_k = Tr W_k \rho(t_m) W_k^{\dagger}$. The time evolution of the system is then calculated for a set of possible trajectories containing, each one of them, different numbers of jumps occurring at different times. Each trajectory is defined as a chain of states obtained by the action of a sequence of operators W_k on the initial state. For example, for an initial pure state $|\psi_0\rangle$, the (nonnormalized) state of the system, after the *m*-th step along the *i*-th trajectory, is given by:

$$|\psi_m^{(i)}\rangle = \prod_{l=1}^m W_{i(l)}|\psi_0\rangle,\tag{6}$$

where i(l) stands for the *l*-th element of a sequence of indexes with values from 0...n. Each trajectory is represented by a discrete sequence of pure states $\{\psi_0, \psi_0^{(i)}, \ldots, \psi_N^{(i)}\}$. The dynamics given by the master equation is recovered by summing incoherently all the states associated to each trajectory, and taking the continuous limit $\Delta t \to 0$.

The no-jump trajectory corresponds to the case in which no decay occurs. The evolution of a quantum state along this trajectory is obtained by the repeated action of the operator W_0 and taking the continuous limit $N \to \infty$. This yields a dynamics governed by the complex effective Hamiltonian \tilde{H} :

$$i\frac{d}{dt}|\psi(t)\rangle = \tilde{H}|\psi(t)\rangle, \quad |\psi(0)\rangle = |\psi_0\rangle.$$
(7)

Since the Hamiltonian is non-hermitian, the corresponding eigenstates are non-orthogonal.

We are now ready to consider the case in which a non-abelian holonomy is generated by the transformation $H(\lambda) = V(\lambda)H_0V^{\dagger}(\lambda)$ when the system's dynamics governed by (3). Note that it is only possible to generate the holonomy when the interaction of the system with the environment is such that the degeneracy structure of the Hamiltonian is preserved. This is the case when $\kappa = \sum_{k=1}^{n} L_k^{\dagger} L_k = \alpha \mathbf{1}$ or $\kappa = \alpha H$. The holonomy in the no-jump trajectory is then the same as the one acquired by an isolated system evolving under the same Hamiltonian H. This is because the eigenstates of H coincide with the eigenstates of H, so that P is proportional to the identity and the connection in the no-jump trajectory $(AP^{-1})^{(0)}$ is equal to $A = V^{\dagger} \dot{V}$. The interaction with the environment only produces an overall visibility factor $e^{\alpha t/2}$ for $\kappa = \alpha \mathbf{1}$ and $e^{\alpha E t/2}$ for $\kappa = \alpha H$. The factor is small for low decoherence rates and in the second case, it is eliminated by working in the ground state. In other words, for sources of decoherence that satisfy these conditions, the holonomy is robust in the no-jump trajectory.

Let us consider a more interesting case, the trajectory in which there is one jump W_i at λ_1 . During the adiabatic evolution, the system evolves under a no-jump trajectory from λ_0 to λ_1 , then the jump occurs instantaneously and the system continues to evolve by the transformation $V(\lambda)$ until time T, corresponding to λ_f . Using the composition rule for holonomies, $U_1 = (\mathbf{P}e^{-\Gamma_1})(\mathbf{P}e^{-\Gamma_0})$ where

$$\Gamma_0 = \int_0^{\lambda_1} (AP^{-1})^{(0)} = \int_0^{\lambda_1} A, \qquad (8)$$

$$\Gamma_{1} = \int_{\lambda_{1}}^{\lambda_{f}} (AP^{-1})^{(1)}.$$
 (9)

Since the one-jump connection for $W_i^{\dagger}W_i = \alpha_i \mathbf{1}$ is given by,

$$(AP^{-1})^{(1)}_{\beta\gamma} = \langle \phi'_{\delta} | V^{\dagger}(\lambda) \dot{V}(\lambda) | \phi'_{\beta} \rangle P^{-1}_{\delta\gamma}, \qquad (10)$$

$$|\phi_{\gamma}'\rangle = W_i e^{\Gamma_o} |\phi_{\gamma}^0\rangle, \qquad (11)$$

$$P_{\beta\gamma} = \langle \phi_{\gamma}' | \phi_{\beta}' \rangle = \alpha_i \delta_{\beta\gamma}, \qquad (12)$$

the holonomy after the jump is

$$\mathbf{P}e^{-\Gamma_1} = \mathbf{P}e^{-\frac{1}{\alpha_i}\int_{\lambda_1}^{\lambda_f} W_i^{\dagger}AW_i}.$$
 (13)

So in the one-jump trajectory $U_1 = \mathbf{P}e^{-\frac{1}{\alpha_i}W_i\Gamma'_0W_i}\mathbf{P}e^{-\Gamma_0}$. This result can then be generalized to a trajectory for which n jumps occur

$$U_{n}^{i} = \prod_{l=1}^{n} \mathbf{P} e^{-\frac{1}{\alpha_{i}} W_{i(l)}^{\dagger} \Gamma_{0}^{i(l)} W_{i(l)}} \mathbf{P} e^{-\Gamma_{0}}$$
(14)

$$\Gamma_0^{i(l)} = \int_{\lambda_l}^{\lambda_{l+1}} A. \tag{15}$$

Note that we can write,

$$\mathbf{P}e^{-\frac{1}{\alpha_i}W_{i(l)}^{\dagger}\Gamma_0^{i(l)}W_{i(l)}} = \frac{1}{\alpha_i}W_{i(l)}^{\dagger}\left(\mathbf{P}e^{-\int_{\lambda_1}^{\lambda_f}A}\right)W_{i(l)}.$$
 (16)

We now consider the application of our results in QC, where the idea of using quantum systems to perform computations more efficiently than classical computers is investigated [12]. Geometric phases play an important role in this field since quantum gates can be implemented in a geometric way. Jones *et. al.* [4] proposed a scheme for generating quantum single and two-qubit phase gates using Berry's phase.

The generalized scheme to implement QC by geometric means is HQC [5]. In this scheme the input information is encoded in the *n* base states of a given degenerate subspace at $H(\lambda_0)$. By renormalizing one can choose the energy of the subspace E = 0. The gates are generated by the holonomic evolutions described above (this includes abelian and non-abelian phases) which are produced by slowly changing the initial Hamiltonian through the set of parameters λ . After λ completes a loop C in \mathcal{M} rooted at λ_0 , the initially prepared state $|\psi_{in}\rangle$ in which the information was encoded is mapped to an output state $|\psi_{out}\rangle = U_C(\lambda)|\psi_{in}\rangle$ where $U_C(\lambda)$ is the quantum gate. To perform a given gate it is necessary to find the path or succession of paths in parameter space which yields such a gate. A quantum algorithm is built from a sequence of gates such that the final state corresponds to the solution of the computational problem. Any unitary transformation, thus any algorithm, can be approximated by a strategical succession of closed paths in the control manifold \mathcal{M} . As long as the adiabatic condition holds, the computation does not depend on the rate at which the control loops are driven. The gates performed in such a way depend only on geometric properties of the parameter space. Thus, any computational error which is pathpreserving does not change the transformation. Hence, errors such as fluctuations in the driving parameters and systematic errors are automatically avoided. However, the environment might induce some fluctuations which are not path-preserving; it is then important to quantify them.

Investigating the effects of decoherence in HQC becomes crucial for the scheme to be realizable. Our quantum jump formalism can be used to investigate decoherence in a universal set of gates generated by holonomies. By universal we mean that any unitary evolution can be constructed from this set of elementary gates [11]. The set consists of two 1-qubit gates of the form $U_i =$ $e^{i\theta_i\sigma_i}$ where σ_i are Pauli matrices, and one 2-qubit gate $U_3 = e^{i\phi\sigma_i\otimes\sigma_j}$. For example, $U_1 = e^{i\theta_1\sigma_1}$, $U_2 = e^{i\theta_2\sigma_2}$ and $U_3 = e^{i\phi\sigma_1 \otimes \sigma_1}$ are universal since any transformation in SU(4) can be generated from them. Consider that these gates are performed by holonomic evolution as described in [13] but the system interacts with the environment. Errors within the subspace for one qubit gates are proportional to σ_{-} , σ_{+} and σ_{i} , with i = 1, 2, 3. For $L_i = \sqrt{\alpha}\sigma_i$ and α the decoherence rate of the error, we obtain $L_i^{\dagger}L_i = \alpha \mathbf{1}$. Thus the universal set of gates is robust to these errors in the no-jump trajectory. When there is a jump in the trajectory, it is easy to see from eq. (16) that the gate U_i is robust to errors L_j when i = j and for errors with $j \neq i$ one obtains a change in sign (since $\sigma_i^{\dagger} \sigma_i \sigma_j = -\sigma_i$). For example, if one jump at occurs at λ_1 , the gate U_i becomes $U_i = \exp i(\theta_1 - \theta_2)\sigma_i$ where

$$\theta_1 = \int_{\lambda_0}^{\lambda_1} \Omega, \quad \theta_2 = \int_{\lambda_1}^{\lambda_f} \Omega. \tag{17}$$

Here Ω corresponds to the solid angle subtended by the evolution of the system in parameter space. The angle $\theta = \theta_1 + \theta_2$ corresponds to the gate without errors. Generalizing to n errors of the type L_j we find that $U_i(n) = e^{i\theta_e\sigma_i}$ with $\theta_e = \sum_m (-1)^m \theta_m$ and $\theta_m = \int_{\lambda_m}^{\lambda_m+1} \Omega$. When errors σ_{\pm} occur, the degeneracy structure of the Hamiltonian is preserved only for initial Hamiltonians which are proportional to σ_3 since $\sigma_{\pm}\sigma_{\mp} = \frac{1}{2}(\mathbf{1} \pm \sigma_3)$. In this case the holonomy is robust in the no-jump trajectory. For a single jump σ_{\pm} , P is not invertible but we can calculate the gates from eq. 1. Unfortunately, the gates are completely lost, $e^{i\theta\sigma_i}$ becomes a U(1) phase after one jump. This is the main source of error that must be corrected.

The analysis of the two-qubit gates follows directly from our previous conclusions. As an example we consider errors of the form $\sigma_i \otimes \sigma_j$. For simplicity let us consider the gate $U_3 = e^{i\phi\sigma_1\otimes\sigma_1}$, any other gate can be analyzed in the same way. The gate is robust to any number of jumps of the type $\sigma_1 \otimes \sigma_1$, $\sigma_1 \otimes \mathbf{1}$, $\mathbf{1} \otimes \sigma_1$ and $\sigma_2 \otimes \sigma_2$ but changes in signs occur for $\mathbf{1} \otimes \sigma_2$, $\sigma_2 \otimes \mathbf{1}$, $\sigma_1 \otimes \sigma_2$ and $\sigma_2 \otimes \sigma_1$. Nevertheless in general the most common non local error is $\sigma_i \otimes \sigma_i$, to which the 2-qubits gate is robust.

We have presented a general scheme to study the effects of Markovian decoherence in the generation of nonabelian adiabatic holonomies. We applied it to analyze the effects of the environment on a universal set of holonomic quantum gates. A scheme for quantum computation completely robust to Markovian decoherence can be constructed by using HQC and finding a way to correct for the errors pointed out in this Letter by other means. This could possibly be achieved using error correction techniques. An example of the use of geometric phases to produce an error correcting code is explored in [14]. We are currently working on the details of such a scheme. Since holonomies are a main ingredient in loop quantum gravity it is likely our scheme can be applied to study decoherence in this field.

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