MODIFIED INERTIAL TSENG METHOD FOR SOLVING VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS ON HADAMARD MANIFOLDS.

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ABSTRACT. In this article, we introduce a forward-backward splitting method with a new step size rule for finding a singularity point of an inclusion problem which is defined by means of a sum of a single-valued vector field and a multi-valued vector field on a Hadamard manifold. Using a Mann, viscosity and an inertial extrapolation method, we establish a convergence result without prior knowledge of the Lipschitz constant of the underlying operator. We present some applications of our result to variational inequality problem. Finally, we present some numerical examples to demonstrate the numerical behavior of our proposed method. The result discuss in this article extends and complements many related results in the literature.

1. INTRODUCTION

Let C be a nonempty closed geodesic convex subset of a Hadamard manifold $\mathbb{M}$, $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$ at $x \in \mathbb{M}$ and $T\mathbb{M}$ be the tangent bundle of $\mathbb{M}$. The inclusion problem is to find

$$x \in C \text{ such that } 0 \in (U+V)(x), \tag{1.1}$$

where $U : C \to T\mathbb{M}$ is a single-valued vector field, $V : C \to 2^{T\mathbb{M}}$ is a multivalued vector field and 0 denotes the zero section of $T\mathbb{M}$. We denote by Δ the solution set of VIP (1.1). Many mathematical problems such as optimization problems, equilibrium problems, variational inequality problems, saddle point problems, among others can be modeled as VIP (1.1). Thus, VIP (1.1) is central importance in nonlinear and convex analysis. The theory of variational inclusion problem has been studied by many authors in various linear spaces (see [17, 20, 27, 29, 50]) due to its wide applications in many fields such as machine learning, statistical regression, image processing and signal recovery (see [12, 18]). Due to the importance and interest of the problem, many iterative procedures have been proposed for solving VIP (1.1) in linear spaces (Hilbert and Banach spaces.)
In the case of real Banach spaces, the VIP is to find

$$x \in C \text{ such that } 0 \in (U+V)(x), \tag{1.2}$$

where C is a nonempty, closed and convex subset of a real Banach space E, $U : C \to E$ is a single-valued mapping and $V : C \to 2^E$ is a multivalued mapping.

Remark 1.1. If $U \equiv 0$, then VIP (1.1) and (1.2) becomes the Monotone Inclusion Problem (MIP) which is to find

$$x \in C \text{ such that } 0 \in Vx. \tag{1.3}$$

A simple and efficient method for solving VIP (1.2) is the forward-backward splitting algorithm introduced by Lions and Mercier [27] in a real Hilbert space H. This method is of the form:

$$x_{n+1} = J^V_{r_n}(x_n - r_n U x_n),\ \forall\, n \geq 1, \tag{1.4}$$

where $J^V_{r_n} = (I + r_n V)^{-1}$ denotes the resolvent of V, I denotes the identity mapping on H and $\{r_n\}$ is a positive real sequence. They proved that the iterative algorithm (1.3) converges weakly to an element in Ω under the assumption that U is α-inverse strongly monotone.
In 2000, Tseng [50] introduced one of the most suitable and notable iterative techniques utilized for solving VIP (1.1) known as Tseng's method. Using this method, Tseng [50] was able to dispense with the inverse strongly monotonicity which is known to be a strict assumption imposed U in (1.3). In [50], U is known to be monotone and L-Lipschitz continuous. The weakness known with Tseng method is that its stepsize requires the prior knowledge of the Lipschitz constant of the underlying operator. However, from a practical point of view,

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the Lipschitz constant is very difficult to approximate. In recent years, modifications of Tseng's method have received great attention by many authors (see [17, 20, 33, 43, 49]). Recently, Gibali and Thong [20] introduced a Mann and Viscosity method together with a new step size rule for solving VIP (1.1) in the framework of real Hilbert spaces. Under standard assumption such as the Lipschitz continuity and maximal monotonicity, they established a strong convergence of the proposed algorithms. Also, in the framework of reflexive Banach spaces, Sunthrayuth *et al.* [41] introduced two different iterative algorithms for solving a common solution of VIP (1.2) and fixed point problem of a relatively nonexpansive mapping. Under some mild assumptions, they proved the sequence generated by both algorithms converges weakly without the prior knowledge of the Lipschitz constant of the operator.

Extension of concepts and techniques from linear spaces to Riemannian manifolds has some important advantages (see [19, 23, 40]). For instance, some optimization problems with nonconvex objective functions become convex from the Riemannian geometry point of view, and some constrained optimization problems can be regarded as unconstrained ones with an appropriate Riemannian metric. In addition, the study of convex minimization problems and inclusion problems in nonlinear spaces have proved to be very useful in computing medians and means of trees, which are very important in computational phylogenetics, diffusion tensor imaging, consensus algorithms and modeling of airway systems in human lungs and blood vessels (see [8–10]). Thus, nonlinear spaces are more suitable frameworks for the study of optimization problems from linear to Riemannian manifolds. For instance, Li *et al.* [24] established the convergence of the proximal algorithm on Hadamard manifolds by using the facts that zeros of a maximal operator are fixed point of its resolvent. In another result, Li *et al.* [23] introduced the idea of a firmly nonexpansive and resolvent of the set-valued monotone operator in the framework of Hadamard manifolds. They established a strong relationship between firmly nonexpansive mappings and monotone vector fields. Using the idea of Li *et al.* [23, 24], Ansari *et al.* [7] proposed an iterative method for computing the approximate solutions of VIP (1.1) in the setting of Hadamard manifolds. They established a convergence analysis of their proposed algorithm and presented some applications of VIP (1.1) to variational inequalities and optimization problems. Very recently, Khammahawong *et al.* [21] introduced two Tseng's methods for approximating the solution of VIP (1.1). Under some suitable conditions, they established a convergence result without prior knowledge of the Lipschitz constant. Applications to variational inequality and convex minimization problems were also discussed.

In 1964, Polyak [34] introduced the inertial extrapolation method as a useful tool for speeding up the rate of convergence of iterative methods. The idea of inertial extrapolation method was inspired by an implicit discretization of a second-order in-time dissipative dynamical system, so-called "Heavy Ball with Friction"

$$h''(t) + \gamma h'(t) + \nabla f(h(t)) = 0, \tag{1.5}$$

where $\gamma > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. System (1.5) is discretized so that, having the terms x_{n-1} and x_n, the next term x_{n+1} can be determined using

$$\frac{x_{n-1} - 2x_n + x_{n-1}}{j^2} + \gamma \frac{x_n - x_{n-1}}{j} + \nabla f(x_n) = 0, \ n \geq 1, \tag{1.6}$$

where j is the step-size. Equation (1.6) yields the following iterative algorithm:

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha \nabla f(x_n), \ n \geq 1, \tag{1.7}$$

where $\beta = 1 - \gamma_j, \alpha = j^2$ and $\beta(x_n - x_{n-1})$ is called the inertial extrapolation term which is intended to speed up the convergence of the sequence generated by (1.7). The heavy ball friction is a simplified version of the differential system describing the motion of a heavy ball that rolls over the graph f and that keep rolling under its own inertia until friction stop it at a critical point of f. This nonlinear oscillation with damping, which is called the "heavy ball with friction" system has been considered by several authors from the optimization point of view, establishing different convergence results and identifying circumstances under which the rate of convergence is better than the one of the first-order-steepest descent method (see [3, 5, 34]). To be precise, (1.5) nature of heavy ball friction may be exploited in some situations in order to "accelerate" the convergence of the trajectories (or sequences in the discrete settings).

Alvarez and Attouch [4] introduced and constructed the heavy-ball method with the proximal point algorithm to solve a problem of maximal monotone operator. They defined their method as follows:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n B)^{-1} w_n, \ \forall \ n \geq 1, \end{cases} \tag{1.8}$$

where $\{\theta_n\} \subset [0,1)$ and $\{\lambda_n\}$ is nondecreasing with $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$. They established that the sequence generated by (1.8) converges weakly to a zero of the monotone operator B. For growing interests in this direction (see [1, 2, 25, 32, 36, 45, 47]).

Inspired by the results of Gibali and Thong [20], Sunthrayuth *et al.* [41], Khammahawong *et al.* [21] and Ansari *et al.* [7], we introduce a modified inertial Tseng method with a self adaptive procedure which generates dynamic step-sizes for solving an inclusion problem in the setting of an Hadamard manifolds. Using our iterative method which comprises of an inertial and viscosity techniques, we prove that the sequence generated by our iterative method converges to a common solution of inclusion and fixed point problem of a nonexpansive mapping. It is worth-mentioning that our iterative method is independent of the Lipschitz constant of the underlying operator. Our method extends and generalizes many related results from linear spaces to Riemannian manifolds.

We present some of the contributions of our result as follows:

(i) The result in this article generalizes the results in [17, 20, 21, 41, 49, 50] from linear spaces to nonlinear spaces.

(ii) We employ the inertial extrapolation method to speed up the rate of convergence of convergence of our result and also dispense with $\sum_{n=1}^{\infty} \theta_n d(x_n - x_{n-1}) < \infty$ a strong condition which has been used severally by many authors (see [22]).

(iii) We were able to dispense with the inverse strongly monotone mapping which is a strict condition imposed on the underlying operator (see [31]). Instead, we chose our operator to be monotone which is a weaker condition.

(iv) Our method requires a self-adaptive procedure which is allowed to increase from iteration to iteration and which is independent of the Lipschitz constant of the underlying operator unlike the result of Shehu [43] where the knowledge of the Lipschitz constant is required.

2. Preliminaries

Let $\mathbb{M}$ be an m-dimensional manifold, let $x \in \mathbb{M}$ and let $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$ at $x \in \mathbb{M}$. We denote by $T\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x\mathbb{M}$ the tangent bundle of $\mathbb{M}$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian metric on $\mathbb{M}$ if $\langle\cdot,\cdot\rangle_x : T_x\mathbb{M}\times T_x\mathbb{M} \to \mathbb{R}$ is an inner product for all $x \in \mathbb{M}$. The corresponding norm induced by the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x\mathbb{M}$ is denoted by $\|\cdot\|_x$. We will drop the subscript x and adopt $\|\cdot\|$ for the corresponding norm induced by the inner product. A differentiable manifold $\mathbb{M}$ endowed with a Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. In what follows, we denote the Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle$ when no confusion arises. Given a piecewise smooth curve $\gamma : [a,b] \to \mathbb{M}$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$), we define the length $l(\gamma)$ of γ by $l(\gamma) := \int_a^b \|\gamma'(t)\|dt$. The Riemannian distance $d(x,y)$ is the minimal length over the set of all such curves joining x to y. The metric topology induced by d coincides with the original topology on $\mathbb{M}$. We denote by ∇ the Levi-Civita connection associated with the Riemannian metric [42].

Let γ be a smooth curve in $\mathbb{M}$. A vector field X along γ is said to be parallel if $\nabla_{\gamma'}X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' itself is parallel along γ, then we say that γ is a geodesic and $\|\gamma'\|$ is a constant. If $\|\gamma'\| = 1$, then the geodesic γ is said to be normalized. A geodesic joining x to y in $\mathbb{M}$ is called a minimizing geodesic if its length equals $d(x,y)$. A Riemannian manifold $\mathbb{M}$ equipped with a Riemannian distance d is a metric space $(\mathbb{M}, d)$. A Riemannian manifold $\mathbb{M}$ is said to be complete if for all $x \in \mathbb{M}$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$. The Hopf-Rinow theorem [42], posits that if $\mathbb{M}$ is complete, then any pair of points in $\mathbb{M}$ can be joined by a minimizing geodesic. Moreover, if $(\mathbb{M}, d)$ is a complete metric space, then every bounded and closed subset of $\mathbb{M}$ is compact. If $\mathbb{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ at $x \in \mathbb{M}$ is defined by

$$\exp_x v := \gamma_v(1,x) \ \forall \ v \in T_x\mathbb{M},$$

where $\gamma_v(\cdot,x)$ is the geodesic starting from x with velocity v (that is, $\gamma_v(0,x) = x$ and $\gamma'_v(0,x) = v$). Then, for any t, we have $\exp_x tv = \gamma_v(t,x)$ and $\exp_x \mathbf{0} = \gamma_v(0,x) = x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{M}$ for every $x \in \mathbb{M}$. The exponential map $\exp_x$ has an inverse $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$. For any $x,y \in \mathbb{M}$, we have $d(x,y) = \|\exp_y^{-1} x\| = \|\exp_x^{-1} y\|$ (see [42] for more details). The parallel transport $P_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{M} \to T_{\gamma(b)}\mathbb{M}$ on the tangent bundle $T\mathbb{M}$ along $\gamma : [a,b] \to \mathbb{R}$ with respect to ∇ is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}v = F(\gamma(b)), \ \forall \ a,b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}\mathbb{M},$$

where F is the unique vector field such that $\nabla_{\gamma'(t)}v = \mathbf{0}$ for all $t \in [a,b]$ and $F(\gamma(a)) = v$. If γ is a minimizing geodesic joining x to y, then we write $P_{y,x}$ instead of $P_{\gamma,y,x}$. Note that for every $a,b,r,s \in \mathbb{R}$, we have

$$P_{\gamma(s),\gamma(r)} \circ P_{\gamma(r),\gamma(a)} = P_{\gamma(s),\gamma(a)} \text{ and } P^{-1}_{\gamma(b),\gamma(a)} = P_{\gamma(a),\gamma(b)}.$$

Also, $P_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{M}$ to $T_{\gamma(b)}\mathbb{M}$, that is, the parallel transport preserves the inner product

$$\langle P_{\gamma(b),\gamma(a)}(u), P_{\gamma(b),\gamma(a)}(v)\rangle_{\gamma(b)} = \langle u,v\rangle_{\gamma(a)},\ \forall\ u,v \in T_{\gamma(a)}\mathbb{M}. \tag{2.1}$$

We now give some examples of Hadamard manifolds.

Space 1: Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{M} = (\mathbb{R}_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold equipped with the inner product $\langle x,y\rangle = xy\ \forall\ x,y \in \mathbb{R}$. Since the sectional curvature of $\mathbb{M}$ is zero [7], $\mathbb{M}$ is an Hadamard manifold. Let $x,y \in \mathbb{M}$ and $v \in T_x\mathbb{M}$ with $\|v\|_2 = 1$. Then $d(x,y) = |\ln x - \ln y|$, $\exp_x tv = xe^{\frac{vx}{t}}$, $t \in (0,+\infty)$, and $\exp_x^{-1} y = x\ln y - x\ln x$.

Space 2: Let $\mathbb{R}^m_{++}$ be the product space $\mathbb{R}^m_{++} := \{(x_1,x_2,\cdots,x_m) : x_i \in \mathbb{R}_{++},\ i = 1,2,\cdots,m\}$. Let $\mathbb{M} = ((R)_++,\langle\cdot,\cdot\rangle)$ be the m-dimensional Hadamard manifold with the Riemannian metric $\langle p,q\rangle = p^Tq$ and the distance $d(x,y) = |\ln \frac{x}{y}| = |\ln \sum\limits_{i=1}^{m} \frac{x_i}{y_i}|$, where $x,y \in \mathbb{M}$ with $x = \{x_i\}_{i=1}^m$ and $y = \{y_i\}_{i=1}^m$.

A subset $K \subset \mathbb{M}$ is said to be convex if for any two points $x,y \in K$, the geodesic γ joining x to y is contained in K. That is, if $\gamma : [a,b] \to \mathbb{M}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a+tb) \in K$ for all $t \in [0,1]$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called an Hadamard manifold. We denote by $\mathbb{M}$ a finite dimensional Hadamard manifold. Henceforth, unless otherwise stated, we represent by K a nonempty, closed and convex subset of $\mathbb{M}$.

Next, we introduce the concepts of monotonicity of vector fields in the setting of Hadamard manifolds. Suppose C is a nonempty, closed and convex subset of $\mathbb{M}$. Let $\mathcal{H}(C)$ denote the set of all single-valued vector fields $U : C \to T\mathbb{M}$ such that $U(p) \in T_p\mathbb{M}$, for each $p \in C$. Let $\Phi(C)$ denote to the set of all multivalued vector fields $V : C \to 2^{T\mathbb{M}}$ such that $V(p) \subseteq T_p\mathbb{M}$ for each $p \in C$, and the denote $Dom(V)$ the domain of V defined by $Dom(V) = \{p \in C : V(p) \neq \emptyset\}$.
We now collect some results and definitions which we shall use in the next section.

Definition 2.1. [52] A vector field $U \in \mathbb{H}(C)$ is said to be

(i) monotone, if

$$\langle U(p), \exp_p^{-1} q\rangle \leq \langle U(q), -\exp_q^{-1} p\rangle,\ \forall\ p,q \in C,$$

(ii) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|P_{p,q}U(q) - U(p)\| \leq Ld(p,q),\ \forall\ p,q \in C.$$

Definition 2.2. [15] A vector field $V \in \Phi(C)$ is said to be

(i) monotone, if for all $p,q \in Dom(G)$

$$\langle u, \exp_p^{-1} q\rangle \leq \langle v, -\exp_q^{-1} p\rangle,\ \forall\ u \in V(p) \text{ and } \forall\ v \in V(q),$$

(ii) maximal monotone if it is monotone and $\forall\ p \in C$ and $u \in T_pC$, the condition

$$\langle u, \exp_p^{-1} q\rangle \leq \langle v, -\exp_q^{-1} p\rangle,\ \forall\ q \in Dom(V) \text{ and } \forall\ v \in V(q) \text{ implies that } u \in V(p).$$

Definition 2.3. [19] Let C be a nonempty, closed and subset of $\mathbb{M}$ and $\{x_n\}$ be a sequence in $\mathbb{M}$. Then $\{x_n\}$ is said to be Fejèr convergent with respect to C if for all $p \in C$ and $n \in \mathbb{N}$,

$$d(x_{n+1},p) \leq d(x_n,p).$$

Definition 2.4. [42] Let $f : C \to \mathbb{R}$ be a geodesic convex. Let $p \in C$, then a vector $r \in T_p\mathbb{M}$ is said to be a subgradient of f at p if and only if

$$f(q) \geq f(p) + \langle r, \exp_p^{-1} q\rangle,\ \forall\ q \in C. \tag{2.2}$$

Definition 2.5. [26] Let $V \in \Phi(C)$ be a vector field and $x_0 \in C$. Then V is said to be upper Kuratowki semicontinuous at x_0 if for any sequences $\{x_n\} \subseteq C$ and $\{v_n\} \subset T\mathbb{M}$ with each $v_n \in V(x_n)$, the relations $\lim\limits_{n\to\infty} v_n = v_0$ imply that $v_0 \in V(x_0)$. Moreover, V is said to be upper Kuratowski semicontinuous on C if it is upper Kuratowski semicontinuous for each $x \in C$.

Lemma 2.6. *[19] Let C be a nonempty, closed and closed subset of $\mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ be a sequence such that $\{x_n\}$ be a Fejér convergent with respect to C. Then the following hold:*

(i) *For every $p \in C$, $d(x_n, p)$ converges,*

(ii) *$\{x_n\}$ is bounded,*

(iii) *Assume that every cluster point of $\{x_n\}$ belongs to C, then $\{x_n\}$ converges to a point in C.*

Definition 2.7. A mapping $S : \mathbb{M} \to \mathbb{M}$ is said to be

(i) μ-contractive

$$d(Sx, Sy) \leq \mu d(x, y), \forall\ x, y \in \mathbb{M}, \tag{2.3}$$

where $\mu : [0, +\infty) \to [0, +\infty)$ is a function satisfying the following condition:

(i) $\mu(s) < s$ for all $s > 0$,

(ii) μ is continuous.

Remark 2.8. (a) $\mu(s) = \frac{s}{s+1}$ for all $s \geq 0$ satisfies conditions (i) and (ii),

(b) if $\mu(s) = ks$ for all $s \geq 0$ and $k \in (0, 1)$, then S is a μ-contraction mapping with a Lipschitz constant k.

(c) Any μ-contraction mapping is nonexpansive.

If $\mu = 1$ in (2.3), then S is said to be nonexpansive.

Proposition 2.9. *[42]. Let $x \in \mathbb{M}$. The exponential mapping $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is a diffeomorphism. For any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic joining x to y, which is given by*

$$\gamma(t) = \exp_x t \exp_x^{-1} y\ \forall\ t \in [0, 1].$$

A geodesic triangle $\Delta(p, q, r)$ of a Riemannian manifold $\mathbb{M}$ is a set containing three points p, q, r and three minimizing geodesics joining these points.

Proposition 2.10. *[42]. Let $\Delta(p, q, r)$ be a geodesic triangle in $\mathbb{M}$. Then*

$$d^2(p, q) + d^2(q, r) - 2\langle \exp_q^{-1} p, \exp_q^{-1} r\rangle \leq d^2(r, q) \tag{2.4}$$

and

$$d^2(p, q) \leq \langle \exp_p^{-1} r, \exp_p^{-1} q\rangle + \langle \exp_q^{-1} r, \exp_q^{-1} p\rangle. \tag{2.5}$$

Moreover, if θ is the angle at p, then we have

$$\langle \exp_p^{-1} q, \exp_p^{-1} r\rangle = d(q, p)d(p, r)\cos\theta. \tag{2.6}$$

Also,

$$\| \exp_p^{-1} q\|^2 = \langle \exp_p^{-1} q, \exp_p^{-1} q\rangle = d^2(p, q). \tag{2.7}$$

Remark 2.11. [26] If $x, y \in \mathbb{M}$ and $v \in T_y\mathbb{M}$, then

$$\langle v, -\exp_y^{-1} x\rangle = \langle v, P_{y,x} \exp_x^{-1} y\rangle = \langle P_{x,y}v, \exp_x^{-1} y\rangle. \tag{2.8}$$

Remark 2.12. From (2.5) and Remark 2.11, let $v \in T_p\mathbb{M}$, we have

$$\langle v, \exp_p^{-1} q\rangle \leq \langle v, \exp_p^{-1} r\rangle + \langle v, P_{p,r} \exp_r^{-1} q\rangle. \tag{2.9}$$

For any $x \in \mathbb{M}$ and $C \subset \mathbb{M}$, there exists a unique point $y \in K$ such that $d(x, y) \leq d(x, z)$ for all $z \in C$. This unique point y is called the nearest point projection of x onto the closed and convex set C and is denoted $P_C(x)$.

Lemma 2.13. *[52]. For any $x \in \mathbb{M}$, there exists a unique nearest point projection $y = P_C(x)$. Furthermore, the following inequality holds:*

$$\langle \exp_y^{-1} x, \exp_y^{-1} z\rangle \leq 0\ \forall\ z \in C.$$

Lemma 2.14. *[26] Let $x_0 \in \mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ with $x_n \to x_0$. Then the following assertions hold:*

(i) *For any* $y \in \mathbb{M}$, *we have* $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} x_n$ *and* $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$,
(ii) *If* $v_n \in T_{x_n}\mathbb{M}$ *and* $v_n \to v_0$, *then* $v_0 \in T_{x_0}\mathbb{M}$,
(iii) *Given* $u_n, v_n \in T_{x_n}\mathbb{M}$ *and* $u_0, v_0 \in T_{x_0}\mathbb{M}$, *if* $u_n \to u_0$, *then* $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$,
(iv) *For any* $u \in T_{x_0}\mathbb{M}$, *the function* $F : \mathbb{M} \to T\mathbb{M}$, *defined by* $F(x) = P_{x,x_0}u$ *for each* $x \in \mathbb{M}$ *is continuous on* $\mathbb{M}$.

The next lemma presents the relationship between triangles in $\mathbb{R}^2$ and geodesic triangles in Riemannian manifolds (see [11]).

Lemma 2.15. *[11]. Let* $\Delta(x_1, x_2, x_3)$ *be a geodesic triangle in* $\mathbb{M}$. *Then there exists a triangle* $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ *corresponding to* $\Delta(x_1, x_2, x_3)$ *such that* $d(x_i, x_{i+1}) = \|\bar{x}_i - \bar{x}_{i+1}\|$ *with the indices taken modulo 3. This triangle is unique up to isometries of* $\mathbb{R}^2$.

The triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Lemma 2.15 is said to be the comparison triangle for $\Delta(x_1, x_2, x_3) \subset \mathbb{M}$. The points $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are called comparison points to the points x_1, x_2 and x_3 in $\mathbb{M}$.

A function $h : \mathbb{M} \to \mathbb{R}$ is said to be geodesic if for any geodesic $\gamma \in \mathbb{M}$, the composition $h \circ \gamma : [u, v] \to \mathbb{R}$ is convex, that is,

$$h \circ \gamma(\lambda u + (1-\lambda)v) \leq \lambda h \circ \gamma(u) + (1-\lambda)h \circ \gamma(v),\ u, v \in \mathbb{R},\ \lambda \in [0, 1].$$

Lemma 2.16. *[26] Let* $\Delta(p, q, r)$ *be a geodesic triangle in a Hadamard manifold* $\mathbb{M}$ *and* $\Delta(p', q', r')$ *be its comparison triangle.*

(i) *Let* α, β, γ *(resp.* α', β', γ'*) be the angles of* $\Delta(p, q, r)$ *(resp.* $\Delta(p', q', r')$*) at the vertices p,q,r (resp.* p', q', r'*). Then, the following inequalities hold:*
$$\alpha' \geq \alpha,\ \beta' \geq \beta,\ \gamma' \geq \gamma,$$
(ii) *Let z be a point in the geodesic joining p to q and* z' *its comparison point in the interval* $[p', q']$. *Suppose that* $d(z, p) = \|z' - p'\|$ *and* $d(z', q') = \|z' - q'\|$. *Then the following inequality holds:*
$$d(z, r) \leq \|z' - r'\|.$$

Lemma 2.17. *[26] Let* $x_0 \in \mathbb{M}$ *and* $\{x_n\} \subset \mathbb{M}$ *be such that* $x_n \to x_0$. *Then, for any* $y \in \mathbb{M}$, *we have* $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ *and* $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$;

The following propositions (see [19]) are very useful in our convergence analysis:

Proposition 2.18. *Let M be an Hadamard manifold and* $d : M \times M :\to \mathbb{R}$ *be the distance function. Then the function d is convex with respect to the product Riemannian metric. In other words, given any pair of geodesics* $\gamma_1 : [0, 1] \to M$ *and* $\gamma_2 : [0, 1] \to M$, *then for all* $t \in [0, 1]$, *we have*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in M$, *the function* $d(\cdot, y) : M \to \mathbb{R}$ *is a convex function.*

Proposition 2.19. *Let* $\mathbb{M}$ *be a Hadamard manifold and* $x \in \mathbb{M}$. *The map* $\Phi_x = d^2(x, y)$ *satisfying the following:*

(1) Φ_x *is convex. Indeed, for any geodesic* $\gamma : [0, 1] \to \mathbb{M}$, *the following inequality holds for all* $t \in [0, 1]$:
$$d^2(x, \gamma(t)) \leq (1-t)d^2(x, \gamma(0)) + td^2(x, \gamma(1)) - t(1-t)d^2(\gamma(0), \gamma(1)).$$
(2) Φ_x *is smooth. Moreover,* $\partial\Phi_x(y) = -2\exp_y^{-1} x$.

Proposition 2.20. *Let M be an Hadamard manifold and* $x \in M$. *Let* $\rho_x(y) = \frac{1}{2}d^2(x, y)$. *Then* $\rho_x(y)$ *is strictly convex and its gradient at y is given by*

$$\partial\rho_x(y) = -\exp_y^{-1} x.$$

Proposition 2.21. *[14]. Let* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *be Riemannian manifolds and* $fh : \mathbb{M}_1 \to \mathbb{M}_2$ *be an isometry between* $\mathbb{M}_1$ *and* $\mathbb{M}_2$. *Then, the function* $f : \mathbb{M}_2 \to \mathbb{R}$ *is convex if and only if* $f \circ h : \mathbb{M}_1 \to \mathbb{R}$ *is convex.*

Proposition 2.22. *[51]. Let C be an open geodesic convex of a Hadamard manifold* $\mathbb{M}$ *and let* $f : \mathbb{M} \to \mathbb{R}$ *be differentiable on C. Then f is convex if and only if gradf is monotone on C.*

Lemma 2.23. *[37] Let $\{u_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{v_n\}$ be a sequence of real numbers. Assume that*

$$u_{n+1} \leq (1-\alpha_n)u_n + \alpha_n v_n \ \forall\ n \geq 1.$$

If $\limsup_{k\to\infty} v_{n_k} \leq 0$ for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ satisfying the condition

$$\liminf_{k\to\infty}(u_{n_k+1} - u_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} u_n = 0$.

3. Main result

In this section, we introduce a modified inertial Tseng method for solving variational inclusion problem and fixed point problem of nonexpansive mapping in Hadamard manifolds. We state the following assumptions:

Assumption 3.1. (B1) *$U \in \mathcal{H}(C)$ is monotone and L-Lipschitz continuous, and $V \in \Phi(C)$ is maximal monotone.*

(B2) *The mapping $T : C \to C$ is a nonexpansive mapping such that $F(T) \neq \emptyset$, and $g : \mathbb{M} \to \mathbb{M}$ is a μ-contraction where $\mu : [0,+\infty) \to [0,+\infty)$ is a continuous and increasing functions satisfying $\mu(0) = 0$ and $\mu(s) < s$ for all $s > 0$.*

(B3) *The solution set $\Omega := F(T) \cap (U+V)^{-1}(0)$ is nonempty.*

Assumption 3.2. (D1) *$\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, that is, $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$,*

(D2) *Let $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,*

(D3) *$\beta_n \in (0,1)$ and $0 < \liminf \beta_n \leq \limsup \beta_n < 1$,*

(D4) *$\{\eta_n\}$ is a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \eta_n < \infty$.*

Algorithm 3.3. *Modified inertial Tseng's method for solving variational inclusion problem.*

Initialization: *Choose $\tau_0 > 0$, $\mu, \theta \in [0,1)$ and let $x_0, x_1 \in \mathbb{M}$ be arbitrary starting points.*

Iterative step: *Given x_{n-1}, x_n, and τ_n, choose $\theta_n \in [0, \bar{\theta}_n]$ where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{d(x_n, x_{n-1})}, \theta\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Calculate x_{n+1} and τ_{n+1} for each $n \geq 1$ as follows:

Step 1: *Compute*

$$\begin{cases} z_n = \exp_{x_n}(-\theta_n \exp_{x_n}^{-1} x_{n-1}) \\ \mathbf{0} \in P_{w_n,z_n} U(z_n) + V(w_n) - \frac{1}{\tau_n} \exp_{w_n}^{-1} z_n. \end{cases} \tag{3.2}$$

If $w_n = z_n$, then stop and z_n is a solution of VIP (1.1). Otherwise

Step 2: *Compute*

$$u_n = \exp_{w_n}(\tau_n(P_{w_n,z_n} U(z_n) - U(w_n))) \tag{3.3}$$

and

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\rho d(z_n, w_n)}{\|P_{w_n,z_n} U(z_n) - U(w_n)\|}, \tau_n + \eta_n\right\} & \text{if } P_{w_n,z_n} U(z_n) - U(w_n) \neq 0, \\ \tau_n + \eta_n, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 3: *Calculate*

$$\begin{cases} y_n = \exp_{u_n}(1-\delta_n)\exp_{u_n}^{-1} T(u_n), \\ x_{n+1} = \gamma_n^1(1-\alpha_n). \end{cases} \tag{3.5}$$

where γ_n^1 is a geodesic joining $g(x_n)$ to y_n.

Stopping criterion *If $w_n = z_n$ and $u_n = Tu_n$ for some $n \geq 1$ then stop. Otherwise set $n := n+1$ and return to* **Iterative step 1**.

We start by establishing a technical lemma useful to our analysis.

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 and the sequence $\{\tau_n\}$ is generated by (3.4). Then we have that $\lim_{n\to\infty} \tau_n = \tau$ and $\tau \in \left[\min\left\{\frac{\rho}{L},\ \tau_0\right\}, \tau_0 + \eta\right]$, where $\eta = \sum_{n=0}^{\infty} \eta_n$.*

Proof. It is obvious that U is L-Lipschitz continuous with constant $L > 0$, then in the case of $P_{w_n,z_n}(U(z_n) - U(w_n)) \neq 0$, we obtain

$$\frac{\rho d(z_n, w_n)}{\|P_{w_n,z_n} U(z_n) - U(w_n)\|} \geq \frac{\rho d(z_n, w_n)}{L d(z_n, w_n)} = \frac{\rho}{L}. \tag{3.6}$$

By the definition of τ_{n+1} in (3.4) and mathematical induction, we have that the sequence $\{\tau_n\}$ has upper bound of $\tau_0 + \eta$ and lower bound $\min\left\{\frac{\rho}{L}, \tau_0\right\}$. The rest of the proof is similar to Lemma 3.1 in [28], so we omit it. □

Remark 3.5. It is obvious that the stepsize in Algorithm (3.3) is allowed to increase from iteration to iteration and so (3.4) reduces the dependence on the initial stepsize τ_0. Also, since $\{\eta_n\}$ is summable, we obtain $\lim_{n\to\infty} \eta_n = 0$. Thus the stepsize τ_n may be non-increasing when n is large. If $\eta_n \equiv 0$, the step size in (3.3) reduces to the one in [21].

Lemma 3.6. *Suppose that Assumptions (B1)-(B3) holds and let $\{u_n\}$ be a sequence generated by Algorithm 3.3, then*

$$d^2(u_n, q) \leq d^2(z_n, p) - (1 - \rho^2 \frac{\tau_n^2}{\tau_{n+1}^2}) d^2(z_n, w_n), \ \forall\ q \in \Omega.$$

Proof. Let $q \in \Omega$, then $-U(q) \in V(q)$. From (3.2) of Algorithm 3.3, we get $\frac{1}{\tau_n} \exp_{w_n}^{-1} z_n - P_{w_n,z_n} U(z_n) \in V(w_n)$. By applying the monotonicity of V, we obtain

$$\begin{aligned} \langle \frac{1}{\tau_n} \exp_{w_n}^{-1} z_n - P_{w_n,z_n} U(z_n), \exp_{w_n}^{-1} q\rangle &\leq \langle -U(q), -\exp_{x^*}^{-1} w_n\rangle \\ &= \langle U(q), \exp_q^{-1} w_n\rangle. \end{aligned} \tag{3.7}$$

Since U is a monotone vector field, then

$$\langle U(q), exp_q^{-1} w_n\rangle \leq \langle -U(w_n), \exp_{w_n}^{-1} q\rangle. \tag{3.8}$$

By combining (3.7) and (3.8), we have

$$\langle \frac{1}{\tau_n} \exp_{w_n}^{-1} z_n - P_{w_n,z_n} U(z_n), \exp_{w_n}^{-1} q\rangle \leq \langle -U(w_n), \exp_{w_n}^{-1} q\rangle,$$

and hence

$$\langle \exp_{w_n}^{-1} z_n, \exp_{w_n}^{-1} q\rangle \leq \tau_n \langle P_{w_n,z_n} U(z_n) - U(w_n), \exp_{w_n}^{-1} q\rangle. \tag{3.9}$$

Now, for $n \in \mathbb{N}$. Let $\Delta(z_n, w_n, q) \subseteq \mathbb{M}$ be a geodesic triangle with vertices z_n, w_n and q and let $\Delta(z_n', w_n', q') \subset \mathbb{R}^2$ be the corresponding comparison triangle, thus we have $d(z_n, q) = \|z_n' - q'\|$, $d(w_n, q) = \|w_n' - q'\|$ and $d(w_n', z_n') = \|w_n' - z_n'\|$. Also, let $\Delta(u_n, w_n, q) \subseteq \mathbb{M}$ be a geodesic triangle with vertices $u_n,\ w_n$ and q, then $\Delta(u_n', w_n', q') \subseteq \mathbb{R}^2$ is the corresponding comparison triangle. Hence, we have $d(u_n, q) = \|u_n' - q'\|$, $d(w_n, q) = \|w_n' - q'\|$ and

$d(u_n, w_n) = \|u'_n - w'_n\|$.
Now,

$$\begin{aligned}
d^2(u_n, q) &\le \|u'_n - q'\| \\
&= \|u'_n - w'_n + w'_n - q'\| \\
&= \|w'_n - q'\|^2 + \|u'_n - w'_n\|^2 + 2\langle u'_n - w'_n, w'_n - q'\rangle \\
&= \|(w'_n - z'_n) + (z'_n - q')\|^2 + \|u'_n - w'_n\|^2 + 2\langle u'_n - w'_n, w'_n - q'\rangle \\
&= \|w'_n - z'_n\|^2 + \|z'_n - q'\|^2 + \|u'_n - w'_n\|^2 + 2\langle w'_n - z'_n, z'_n - q'\rangle \\
&+ 2\|z'_n - q'\|^2 - 2\|z'_n - q'\|^2 + 2\langle u'_n - w'_n, w'_n - q'\rangle + 2\|w'_n - q'\|^2 \\
&- 2\|w'_n - q'\|^2 \\
&= d^2(z_n, q) + d^2(w_n, z_n) + \|u'_n - w'_n\|^2 + 2\langle w'_n - z'_n, z'_n - q'\rangle + 2\langle z'_n - q', z'_n - q'\rangle \\
&- 2d^2(z_n, q) + 2\langle u'_n - w'_n, w'_n - q'\rangle + 2\langle w'_n - q', w'_n - q'\rangle - 2d^2(w_n, q) \\
&= d^2(z_n, q) + d^2(w_n, z_n) + \|u'_n - w'_n\|^2 + 2\langle w'_n - q', z'_n - q'\rangle - 2d^2(z_n, q) \\
&+ 2\langle u'_n - q', w'_n - q'\rangle - 2d^2(w_n, q). \qquad (3.10)
\end{aligned}$$

Let ϕ and ϕ' be the angles of the vertices w_n and w'_n respectively. By Lemma 2.16 (i), we get $\phi' \ge \phi$. Therefore, we obtain from Lemma 2.15 and (2.6) that

$$\begin{aligned}
\langle w'_n - q', z'_n - q'\rangle &= \|w'_n - q'\| \; \|z'_n - q'\| cos\phi' \\
&= d(w_n, q)d(q, z_n) cos\phi' \\
&\le d(w_n, q)d(q, z_n) cos\phi \\
&= \langle \exp_q^{-1} w_n, \exp_q^{-1} z_n\rangle. \qquad (3.11)
\end{aligned}$$

Also, by applying Lemma 2.15 and (2.6), we have

$$\begin{aligned}
\langle u'_n - q', w'_n - q'\| &= \|u'_n - q'\| \; \|w'_n - q'\| cos\psi' \\
&= d(u_n, q)d(q, w_n) cos\psi' \\
&\le d(u_n, q)d(q, w_n) cos\psi \\
&= \langle \exp_q^{-1} u_n, \exp_q^{-1} w_n\rangle \qquad (3.12)
\end{aligned}$$

where ψ and ψ' are the angles at the vertices of u_n and u'_n respectively. It is obvious from (3.3) that

$$\|u'_n - w'_n\|^2 \le \tau_n^2 \|P_{w_n, z_n} U(z_n) - U(w_n)\|^2. \qquad (3.13)$$

From Remark (2.12), (3.11) and (3.12), we have

$$\begin{aligned}
\langle w'_n - q', w'_n - q'\rangle &\le \langle \exp_q^{-1} w_n, \exp_q^{-1} z_n\rangle \\
&\le \langle \exp_q^{-1} z_n, \exp_q^{-1} z_n\rangle + \langle P_{q, z_n} \exp_{z_n}^{-1} w_n, \exp_q^{-1} w_n\rangle \\
&= d^2(z_n, q) + \langle P_{q, z_n} \exp_{z_n}^{-1} w_n, \exp_q^{-1} z_n\rangle, \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
\langle u'_n - q', w'_n - q'\rangle &\le \langle \exp_q^{-1} u_n, \exp_q^{-1} w_n\rangle \\
&\le \langle \exp_q^{-1} w_n, \exp_q^{-1} w_n\rangle + \langle P_{q, w_n} \exp_{w_n}^{-1} u_n, \exp_q^{-1} w_n\rangle \\
&= d^2(w_n, q) + \langle P_{q, w_n} \exp_{w_n}^{-1} u_n, \exp_q^{-1} w_n\rangle. \qquad (3.15)
\end{aligned}$$

On substituting (3.13), (3.14) and (3.15) in (3.10), we have

$$\begin{aligned}
d^2(u_n, q) &\le d^2(z_n, q) + d^2(w_n, z_n) + \tau_n^2 \|P_{w_n, z_n} U(z_n) - U(w_n)\|^2 \\
&+ 2\langle P_{q, z_n} \exp_{z_n} \exp_{z_n}^{-1} w_n, \exp_q^{-1} z_n\rangle + 2\langle P_{q, w_n} \exp_{w_n}^{-1} u_n, \exp_q^{-1} w_n\rangle. \qquad (3.16)
\end{aligned}$$

In view of Step 3.3 of Algorithm 3.3, we have $\exp_{w_n}^{-1} u_n = \tau_n (P_{w_n, z_n} U(z_n) - U(w_n))$, thus (3.16) becomes

$$\begin{aligned}
d^2(u_n, q) &\le d^2(z_n, q) + d^2(w_n, z_n) + \tau_n^2 \|P_{w_n, z_n} U(z_n) - U(w_n)\|^2 + 2\langle P_{q, w_n} \exp_{z_n}^{-1} w_n, \exp_q^{-1} z_n\rangle \\
&+ 2\tau_n \langle U(w_n) - P_{w_n, z_n} U(z_n), \exp_{w_n}^{-1} q\rangle. \qquad (3.17)
\end{aligned}$$

By applying Remark 2.11, 2.12 and (2.1), we obtain

$$
\begin{aligned}
d^2(u_n,q) &\le d^2(z_n,q)+d^2(w_n,z_n)+\tau^2\|P_{w_n,z_n}U(z_n)-U(w_n)\|^2\\
&+2\tau_n\langle U(w_n)-P_{w_n,z_n}U(z_n),\exp^{-1}_{w_n}q\rangle+2\langle P_{w_n,z_n}\exp^{-1}_{w_n}z_n,\exp^{-1}_{z_n}q\rangle\\
&\le d^2(z_n,q)+d^2(w_n,z_n)+\tau_n^2\|P_{w_n,z_n}U(z_n)-U(w_n)\|^2\\
&+2\tau_n\langle U(w_n)-P_{w_n,z_n}U(z_n),\exp^{-1}_{w_n}q\rangle+2\langle P_{z_n,w_n}\exp^{-1}_{w_n}z_n,\exp^{-1}_{z_n}w_n\rangle\\
&+2\langle P_{z_n,w_n}\exp^{-1}_{w_n}z_n,P_{z_n,w_n}\exp^{-1}_{w_n}q\rangle\\
&= d^2(z_n,q)+d^2(w_n,z_n)+\tau_n^2\|P_{w_n,z_n}U(z_n)-U(w_n)\|^2\\
&+2\tau_n\langle U(w_n)-P_{w_n,z_n}U(z_n),\exp^{-1}_{w_n}q\rangle+2\langle\exp^{-1}_{w_n}z_n,P_{w_n,z_n}\exp^{-1}_{z_n}w_n\rangle\\
&+2\langle\exp^{-1}_{w_n}z_n,P_{w_n,z_n}(P_{z_n,w_n}\exp^{-1}_{w_n}q)\rangle\\
&= d^2(z_n,q)+d^2(w_n,z_n)+\tau_n^2\|P_{w_n,z_n}U(z_n)-U(w_n)\|^2\\
&+2\tau_n\langle U(w_n)-P_{w_n,z_n}U(z_n),\exp^{-1}_{w_n}q\rangle-2\|\exp^{-1}_{w_n}z_n\|+2\langle\exp^{-1}_{w_n}z_n,\exp^{-1}_{w_n}q\rangle\\
&= d^2(z_n,q)-d^2(w_n,z_n)+\tau_n^2\|P_{w_n,z_n}U(z_n)-U(w_n)\|^2+2\tau_n\langle U(w_n)-P_{w_n,z_n}U(z_n),\exp^{-1}_{w_n}q\rangle\\
&+2\langle\exp^{-1}_{w_n}z_n,\exp^{-1}_{w_n}q\rangle. \qquad (3.18)
\end{aligned}
$$

On substituting (3.4) and (3.9) in (3.18), we get

$$
\begin{aligned}
d^2(u_n,q) &\le d^2(z_n,q)-d^2(w_n,z_n)+\rho^2\frac{\tau_n^2}{\tau_{n+1}^2}d^2(w_n,z_n)\\
&+2\tau_n\langle U(w_n)-P_{w_n,z_n}U(z_n),\exp^{-1}_{w_n}q\rangle-2\tau_n\langle U(w_n)-P_{w_n,z_n}U(z_n),\exp^{-1}_{w_n}q\rangle\\
&= d^2(z_n,q)-(1-\rho^2\frac{\tau_n^2}{\tau_{n+1}^2})d^2(w_n,z_n) \qquad (3.19)\\
&\le d^2(z_n,q). \qquad (3.20)
\end{aligned}
$$

Hence, the proof is completed. □

Lemma 3.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 and $g:C\to C$ be a μ-contraction. Assume that*

$$0<\epsilon:=\sup\{\frac{\mu(d(x_n,q))}{d(x_n,q)}:x_n\neq q,n\ge 0,q\in\Omega\}<1,$$

then the sequence $\{x_n\}$ is bounded.

Proof. Let $q\in\Omega$, $\gamma_n^2:[0,1]\to\mathbb{M}$ be a geodesic space such that $\gamma_n^2(0)=u_n$ and $\gamma_n^2(1)=Tu_n$. Then using Proposition 2.19 and Algorithm 3.3, we obtain

$$
\begin{aligned}
d^2(y_n,q) &= d(\gamma_n^2(1-\beta_n),q)\\
&\le (1-\beta_n)d^2(\gamma_n^2(0),q)+\beta_n d^2(\gamma_n^2(1),q)-\beta_n(1-\beta_n)d^2(\gamma_n^2(0),\gamma_n^2(1))\\
&\le (1-\beta_n)d^2(u_n,q)+\beta_n d^2(T(u_n),Tq)-\beta_n(1-\beta_n)d^2(u_n,Tu_n)\\
&\le (1-\beta_n)d^2(u_n,q)+\beta_n d^2(u_n,q)-\beta_n(1-\beta_n)d^2(u_n,Tu_n)\\
&= d^2(u_n,q)-\beta_n(1-\beta_n)d^2(u_n,Tu_n) \qquad (3.21)\\
&\le d^2(u_n,q). \qquad (3.22)
\end{aligned}
$$

By utilizing the geodesic triangles $\Delta(z_n,x_n,q)$ and $\Delta(x_n,x_{n-1},q)$ with their respective comparison triangle $\Delta(z_n^{'},x_n^{'},q^{'})\subseteq\mathbb{R}^2$. Then by Lemma 2.15, we have $d(z_n,x_n)=\|z_n^{'}-x_n^{'}\|$, $d(z_n,q)=\|z_n^{'}-q^{'}\|$ and $d(x_n,x_{n-1})=\|x_n^{'}-x_{n-1}^{'}\|$. Now, by applying step 1 of Algorithm 3.3, we have

$$
\begin{aligned}
d(z_n,q) &= \|z_n^{'}-q^{'}\|\\
&= \|x_n^{'}+\theta_n(x_n^{'}-x_{n-1}^{'})-q^{'}\|\\
&\le \|x_n^{'}-q^{'}\|+\theta_n\|x_n^{'}-x_{n-1}^{'}\|\\
&= \|x_n^{'}-q^{'}\|+\alpha_n\cdot\frac{\theta_n}{\alpha_n}\|x_n^{'}-x_{n-1}^{'}\|. \qquad (3.23)
\end{aligned}
$$

Since $\frac{\theta_n}{\alpha_n}\|x_n^{'} - x_{n-1}^{'}\| = \frac{\theta_n}{\alpha_n} d(x_n, x_{n-1}) \to 0$ as $n \to \infty$, then there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n} d(x_n, x_{n-1}) \leq M_1$. Thus, we obtain from (3.23) that

$$d(z_n, q) \leq d(x_n, q) + \alpha_n M_1. \tag{3.24}$$

By simple computation, it is obvious that

$$d^2(z_n, q) = \|x_n^{'} - q^{'}\|^2 + \theta_n\|x_n^{'} - x_{n-1}^{'}\|\big[2\|x_n^{'} - q^{'}\| + \theta_n\|x_n^{'} - x_{n-1}^{'}\|\big]. \tag{3.25}$$

Since $2\|x_n^{'} - q^{'}\| + \theta_n\|x_n^{'} - x_{n-1}^{'}\| = 2d(x_n, q) + \theta_n d(x_n, x_{n-1}) \leq M_2$ for some constant $M_2 > 0$. Thus, we obtain from (3.25), that

$$d^2(z_n, q) \leq d^2(x_n, q) + \theta_n d(x_n, x_{n-1}) M_2. \tag{3.26}$$

From (3.20), (3.22) and (3.24), we have

$$\begin{aligned} d(y_n, q) &\leq d(u_n, q) \\ &\leq d(z_n, q) \\ &\leq d(x_n, q) + \alpha_n M_1. \end{aligned} \tag{3.27}$$

From Step 3 of Algorithm 3.3 and (3.27), we obtain

$$\begin{aligned} d(x_{n+1}, q) &= d(\gamma_n^1(1 - \alpha_n), q) \\ &\leq \alpha_n d(\gamma_n^1(0), q) + (1 - \alpha_n) d(\gamma_n^1(1), q) \\ &\leq d(g(x_n), q) + (1 - \alpha_n) d(y_n, q) \\ &\leq \alpha_n\big[d(g(x_n), g(q)) + d(g(p), q)\big] + (1 - \alpha_n) d(y_n, q) \\ &\leq \alpha_n\big[\mu d(x_n, q) + d(g(q), q)\big] + (1 - \alpha_n) d(y_n, q). \end{aligned} \tag{3.28}$$

Since

$$0 < \epsilon := \sup\{\frac{\mu(d(x_n, q))}{d(x_n, q)} : x_n \neq q, n \geq 0, q \in \Omega\} < 1,$$

we get from (3.24) that

$$\begin{aligned} d(x_{n+1}, q) &\leq \alpha_n \epsilon d(x_n, q) + \alpha_n d(q(q), q) + (1 - \alpha_n) d(y_n, q) \\ &\leq \alpha_n \epsilon d(x_n, q) + \alpha_n d(q(q), q) + (1 - \alpha_n)\big[d(x_n, q) + \alpha_n M_1\big] \\ &= (1 - \alpha_n(1 - \epsilon)) d(x_n, q) + \alpha_n \left[(1 - \epsilon)\frac{d(g(q), q) + M_1}{1 - \epsilon}\right] \\ &\vdots \\ &\leq \max\left\{d(x_n, q), \frac{d(g(q), q) + M_1}{1 - \epsilon}\right\}. \end{aligned}$$

By induction, we obtain that

$$d(x_{n+1}, q) \leq \max\left\{d(x_1, q), \frac{d(g(q), q) + M_1}{1 - \epsilon}\right\}.$$

Hence, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{w_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ and $\{Tu_n\}$ are bounded. □

Theorem 3.8. *Suppose $g : C \to C$ is a μ-contraction, and assume that conditions $(B1)$-$(B3)$ and $(D1) - (D4)$ holds. If $0 < \epsilon := \sup\{\frac{\mu(d(x_n,q))}{d(x_n,q)} : x_n \neq q, n \geq 0, q \in \Omega\} < 1$ holds, then the sequence $\{x_n\}$ generated by Algorithm 3.3 converges to $q \in \Omega$, where $q = P_\Omega g(q)$ and P_Ω is the nearest point projection of C onto Ω.*

Proof. Let $q \in \Omega$, then by substituting (3.19) and (3.26) into (3.21), we have

$$\begin{aligned} d^2(y_n, q) &\le d^2(u_n, q) - \beta_n(1-\beta_n)d^2(u_n, Tu_n) \\ &\le d^2(z_n, q) - (1 - \tau_n^2 \frac{\rho^2}{\tau_{n+1}^2})d^2(w_n, z_n) - \beta_n(1-\beta_n)d^2(u_n, Tu_n) \\ &\le d^2(x_n, p) + \theta_n d(x_n, x_{n-1})M_2 - (1 - \tau_n^2 \frac{\rho^2}{\tau_{n+1}^2})d^2(w_n, z_n) \\ &- \beta_n(1-\beta_n)d^2(u_n, Tu_n). \end{aligned} \tag{3.29}$$

Fix $n \ge 1$ and let $v = g(x_n), u = y_n$ and $w = g(q)$. Consider the following geodesic triangles with their respective comparison triangles $\Delta(v, u, w)$ and $\Delta(v^{'}, u^{'}, w^{'}), \Delta(w, u, v)$ and $\Delta(w^{'}, u^{'}, v^{'}), \Delta(w, u, q)$ and $\Delta(w^{'}, u^{'}, q^{'})$. Applying Lemma 2.15, we get $d(v, u) = \|v^{'} - u^{'}\|, d(v, w) = \|v^{'} - w^{'}\|, d(v, q) = \|v^{'} - q^{'}\|, d(u, w) = \|u^{'} - w^{'}\|$ and $d(w, q) = \|w^{'} - q^{'}\|$. From Algorithm (3.3), the comparison point of $x_{n+1} \in \mathbb{R}^2$ is $x^{'}_{n+1} = \alpha_n v^{'} + (1-\alpha_n)u^{'}$. Let ϕ and $\phi^{'}$ denote the angle and comparison angle at q and $q^{'}$ in the triangles $\Delta(w, x_{n+1}, q)$ and $\Delta(y^{'}, x^{'}_{n+1}, q^{'})$ respectively. Therefore, $\phi \le \phi^{'}$ and $\cos\phi^{'} \le \cos\phi$.
By applying Lemma 2.16 and the property of g, we obtain

$$\begin{aligned} d^2(x_{n+1}, q) &\le \|x^{'}_{n+1} - q^{'}\|^2 \\ &= \|\alpha_n(v^{'} - q^{'}) + (1-\alpha_n)(u^{'} - q^{'})\|^2 \\ &\le \|\alpha_n(v^{'} - w^{'}) + (1-\alpha_n)(u^{'} - q^{'})\|^2 + 2\alpha_n\langle x^{'}_{n+1} - q^{'}, w^{'} - q^{'}\rangle \\ &\le (1-\alpha_n)\|u^{'} - q^{'}\|^2 + \alpha_n\|v^{'} - w^{'}\|^2 + 2\alpha_n\|x^{'}_{n+1} - q^{'}\| \, \|w^{'} - q^{'}\| \cos\phi^{'} \\ &\le (1-\alpha_n)d^2(u, q) + \alpha_n d^2(v, w) + 2\alpha_n d(x_{n+1}, q)d(w, q)\cos\phi \\ &= (1-\alpha_n)d^2(y_n, q) + \alpha_n d^2(g(x_n), g(q)) + 2\alpha_n d(x_{n+1}, q)d(w, q)\cos\phi. \end{aligned} \tag{3.30}$$

It is obvious that $d(x_{n+1}, q)d(g(q), q)\cos\phi = \langle \exp_q^{-1} g(q), \exp_q^{-1} x_{n+1}\rangle$, then by substituting (3.29) into (3.30), we obtain

$$\begin{aligned} d^2(x_{n+1}, q) &\le (1-\alpha_n)d^2(y_n, q) + \alpha_n d^2(g(x_n), g(q)) + 2\alpha_n\langle \exp_q^{-1} g(q), \exp_q^{-1} x_{n+1}\rangle \\ &\le (1-\alpha_n)d^2(x_n, q) + (1-\alpha_n)\theta_n d(x_n, x_{n-1})M_2 - (1-\alpha_n)(1 - \tau_n^2\frac{\rho^2}{\tau_{n+1}^2})d^2(w_n, z_n) \\ &+ \alpha_n d^2(g(x_n), g(q)) + 2\alpha_n\langle \exp_q^{-1} g(q), \exp_q^{-1} x_{n+1}\rangle - (1-\alpha_n)\beta_n(1-\beta_n)d^2(u_n, Tu_n) \\ &= \big(1 - \alpha_n(1-\epsilon)\big)d^2(x_n, q) + \alpha_n(1-\epsilon)\left[\frac{\frac{\theta_n}{\alpha_n}d(x_n, x_{n-1})M_2 + 2\langle \exp_q^{-1} g(q), \exp_q^{-1} x_{n+1}\rangle}{(1-\epsilon)}\right] \\ &- (1-\alpha_n)(1 - \tau_n^2\frac{\rho^2}{\tau_{n+1}^2})d^2(w_n, z_n) - (1-\alpha_n)\beta_n(1-\beta_n)d^2(u_n, Tu_n) \qquad (3.31) \\ &= \big(1 - \alpha_n(1-\epsilon)\big)d^2(x_n, q) + \alpha_n(1-\epsilon)H_n, \qquad (3.32) \end{aligned}$$

where $H_n = \left[\frac{\frac{\theta_n}{\alpha_n}d(x_n, x_{n-1})M_2 + 2\langle \exp_q^{-1} g(q), \exp_q^{-1} x_{n+1}\rangle}{(1-\epsilon)}\right] - (1-\alpha_n)(1-\tau_n^2\frac{\rho^2}{\tau_{n+1}^2})d^2(w_n, z_n) - (1-\alpha_n)\beta_n(1-\beta_n)d^2(u_n, Tu_n)$.
From (3.31), we obtain

$$(1-\alpha_n)(1 - \tau_n^2\frac{\rho^2}{\tau_{n+1}^2})d^2(w_n, z_n) - (1-\alpha_n)\beta_n(1-\beta_n)d^2(u_n, Tu_n) \le d^2(x_n, p) - d^2(x_{n+1}, p) + \alpha_n(1-\epsilon)M_3, \tag{3.33}$$

where $M_3 := \sup_{n\in\mathbb{N}} H_n$.
To show that $d(x_n, p) \to 0$ as $n \to \infty$. Let $a_n = d(x_n, p)$ and $d_n = \beta_n(1-\epsilon)$. It is very easy to see that the inequality (3.32) satisfies

$$a_{n+1} \le (1-d_n)a_n + d_n b_n.$$

In view of Lemma 2.23, we claim that $\limsup_{k\to\infty} H_{n_k} \le 0$ for a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{n\to\infty}(a_{n_{k+1}} - a_{n_k}) \ge 0.$$

Now, from (3.33), we get

$$\begin{aligned}\limsup_{k\to\infty}\Big[(1-\alpha_{n_k})\Big(1-\tau_{n_k}\frac{\rho^2}{\tau_{n_k+1}}\Big)d^2(w_{n_k},z_{n_k})+(1-\alpha_{n_k})\beta_{n_k}d^2(u_{n_k},T(u_{n_k}))\Big]&\\ \leq \limsup_{k\to\infty}\left[d^2(x_{n_k},p)-d^2(x_{n_k+1},p)+\beta_{n_k}(1-\epsilon)M_3\right]&\\ = -\liminf_{k\to\infty}(d^2(x_{n_{k+1}},p)-d^2(x_{n_k},p))&\\ \leq 0.&\end{aligned} \tag{3.34}$$

By applying the condition on α_{n_k},β_{n_k} and the fact that

$$\lim_{k\to\infty}\left(1-\tau_{n_k}\frac{\rho^2}{\tau^2_{n_k+1}}\right)=1-\rho^2>0,$$

thus, we obtain that

$$\lim_{k\to\infty}d(w_{n_k},z_{n_k})=0=\lim_{k\to\infty}d(u_{n_k},T(u_{n_k})). \tag{3.35}$$

From Algorithm 3.3 and (3.23), it is clear that

$$\begin{aligned}\lim_{k\to\infty}d(z_{n_k},x_{n_k})&\leq\lim_{k\to\infty}\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|x^{'}_{n_k}-x^{'}_{n_k-1}\|\\ &\leq\lim_{k\to\infty}\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}d(x_{n_k},x_{n_k-1})\\ &=0.\end{aligned} \tag{3.36}$$

Using Algorithm 3.3, (3.35) and (3.36), the following are easy to establish:

$$\begin{cases}\lim_{k\to\infty}d(w_{n_k},x_{n_k})=0,\\ \lim_{k\to\infty}d(u_{n_k},z_{n_k})=0,\\ \lim_{k\to\infty}d(u_{n_k},w_{n_k})=0,\\ \lim_{k\to\infty}d(u_{n_k},x_{n_k})=0,\\ \lim_{k\to\infty}d(y_{n_k},u_{n_k})=0,\\ \lim_{k\to\infty}d(y_{n_k},x_{n_k})=0,\\ \lim_{k\to\infty}d(x_{n_k+1},y_{n_k})=0,\\ \lim_{k\to\infty}d(x_{n_k+1},x_{n_k})=0.\end{cases} \tag{3.37}$$

Since $\{x_{n_k}\}$ is Fejér convergent, then from Lemma 2.6 (ii) we obtain that $\{x_{n_k}\}$ is bounded. Hence, there exists a subsequence $\{x_{n_{k_l}}\}$ which converges to a cluster point p. Also, from (3.37), there exists a subsequence $\{w_{n_{k_l}}\}$ of $\{w_{n_k}\}$ which converges weakly to $p\in\mathbb{M}$. By Algorithm 3.3, we get

$$\Upsilon_{n_{k_l}}:=-P_{w_{n_{k_l}},z_{n_{k_l}}}U(z_{n_{k_l}})-\frac{1}{\tau_{n_{k_l}}}\exp^{-1}_{w_{n_{k_l}}}z_{n_{k_l}}\in V(w_{n_{k_l}}). \tag{3.38}$$

Thus, using (3.35)

$$\lim_{l\to\infty}\frac{1}{\tau_{n_{k_l}}}\|\exp^{-1}_{w_{n_{k_l}}}z_{n_{k_l}}\|=\lim_{l\to\infty}\frac{1}{\tau_{n_{k_l}}}d(w_{n_{k_l}},z_{n_{k_l}})=0,$$

so

$$\lim_{l\to\infty}\frac{1}{\tau_{n_{k_l}}}\exp^{-1}_{w_{n_{k_l}}}z_{n_{k_l}}=0. \tag{3.39}$$

Since U is a Lipschitz continuous vector field and $z_{n_{k_l}}\to p$ as $l\to\infty$. By combining (3.38) and (3.39), we obtain

$$\lim_{l\to\infty}\Upsilon_{n_{n_{k_l}}}=-U(p). \tag{3.40}$$

Since V is a maximal monotone vector field, so it is upper Kuratowski semi-continuous. Hence $-U(p)\in V(p)$, which implies that p solves Δ. Also, since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ which converges to $p \in \mathbb{M}$ such that

$$\begin{aligned}\lim_{l\to\infty}\langle \exp_q^{-1} g(q), \exp_q^{-1} x_{n_{k_l}}\rangle &= \limsup_{k\to\infty}\langle \exp_q^{-1} g(q), \exp_q^{-1} x_{n_k}\rangle \\ &= \langle \exp_q^{-1} g(q), \exp_q^{-1} p\rangle \\ &\leq 0. \end{aligned} \tag{3.41}$$

By substituting (3.41) into (3.32) and applying Lemma 2.23, we conclude that $\{x_n\}$ converges to $q \in \Omega$. □

4. Application

4.1. **Variational Inequalities.** Let $U : C \to T\mathbb{M}$ be a single-vector field, the Variational Inequality Problem (VP) introduced by [30] is to find $q \in C$ such that

$$\langle U(q), \exp_q^{-1} s\rangle \geq 0, \ \forall\ s \in C. \tag{4.1}$$

We denote by VP(U,C) the solution set of (4.1). Let $N_C(p)$ denote the normal cone of the set C at $p \in C$:

$$N_C(p) := \{r \in T_p\mathbb{M} : \langle r, \exp_p^{-1} s\rangle \leq 0, \ \forall\ s \in C\}.$$

Let δ_C be the indicator function of C, that is

$$\delta_C(p) = \begin{cases} 0, & \text{if } p \in C, \\ +\infty, & \text{if } p \notin C. \end{cases} \tag{4.2}$$

It is easy to see that δ_C is a proper lower semicontinuous and geodesic convex function on a Hadamard manifold $\mathbb{M}$. By Lemma 4.4 in [21], we have that $\partial\delta_C$ is a multivalued vector field.
Now, we present an inertial Tseng method for solving VP (4.1) as follows:

Algorithm 4.1. *Modified inertial Tseng's method for solving variational inequality problem.*

Initialization: *Choose $\tau_0 > 0$, $\mu, \theta \in (0, 1)$ and let $x_0, x_1 \in \mathbb{M}$ be arbitrary starting points.*

Iterative step: *Given x_{n-1}, x_n, and τ_n, choose $\theta_n \in [0, \bar\theta_n]$ where*

$$\bar\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{d(x_n, x_{n-1})}, \theta\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{4.3}$$

Calculate x_{n+1} and τ_{n+1} for each $n \geq 1$ as follows:

Step 1: *Compute*

$$\begin{cases} z_n = \exp_{x_n}(-\theta_n \exp_{x_n}^{-1} x_{n-1}) \\ \mathbf{0} \in P_{w_n,z_n} U(z_n) + \partial\delta_C(w_n) - \frac{1}{\tau_n}\exp_{w_n}^{-1} z_n. \end{cases} \tag{4.4}$$

If $w_n = z_n$, then stop and z_n is a solution of VIP (1.1). Otherwise

Step 2: *Compute*

$$u_n = \exp_{w_n}(\tau_n(P_{w_n,z_n}U(z_n) - U(w_n))) \tag{4.5}$$

and

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\rho d(z_n, w_n)}{\|P_{w_n,z_n}U(z_n) - U(w_n)\|}, \tau_n + \eta_n\right\} & \text{if } P_{w_n,z_n}U(z_n) - U(w_n) \neq 0, \\ \tau_n + \eta_n, & \text{otherwise.} \end{cases} \tag{4.6}$$

Step 3: *Calculate*

$$y_n = \exp_{u_n}(1 - \delta_n)\exp_{u_n}^{-1} T(u_n) \tag{4.7}$$

Step 4: *Calculate x_{n+1} and α_{n+1} by*

$$x_{n+1} = \gamma_n^1(1 - \alpha_n) \tag{4.8}$$

Stopping criterion *If $w_n = z_n$ and $u_n = Tu_n$ for some $n \geq 1$ then stop. Otherwise set $n := n + 1$ and return to* **Iterative step 1**.

Proof. From (4.2), $\delta_C(p) = 0$ for all $p \in C$ and hence from (2.2), we get

$$\begin{aligned}\partial\delta_C(p) &= \{r \in T_p\mathbb{M} : \langle r, \exp_p^{-1} q\rangle \le \delta_C(q) - \delta_C(p)\},\\ &= \{r \in T_p\mathbb{M} : \langle r, \exp_p^{-1} q\rangle \le 0\}.\end{aligned} \tag{4.9}$$

Thus, $\partial\delta_C p = N_C(p)$. For every $p \in C$ and $U \in \mathcal{H}$,applying (4.9), we obtain that

$$\begin{aligned}p \in (U + \partial\delta_C)^{-1}(\mathbf{0}) &\Leftrightarrow -U(p) \in \partial\delta_C(p)\\ &\Leftrightarrow \langle -U(p), \exp_p^{-1} q\rangle \le 0,\ \forall\ q \in C\\ &\Leftrightarrow p \in VP(U, C).\end{aligned}$$

□

By replacing V by $\partial\delta_C$ in Algorithm 3.3 and take $\Gamma := F(T) \cap VP(U, C)$ to be nonempty, we obtain the following result.

Theorem 4.2. *Suppose $g : C \to C$ is a μ-contraction, and assume that conditions $(A1)$-$(A3)$ and $(B1) - (B4)$ holds. If $0 < \epsilon := \sup\{\frac{\mu(d(x_n,q))}{d(x_n,q)} : x_n \ne q, n \ge 0, q \in \Gamma\} < 1$ holds, then the sequence $\{x_n\}$ generated by Algorithm 4.1 converges to $q \in \Gamma$, where $q = P_\Gamma g(q)$ and P_Γ is the nearest point projection of C onto Γ.*

Proof. It is obvious that if $V = \partial\delta_C$ in (3.3) and Assumption (A1), then the result follows. Hence, we the proof completes . □

5. Numerical example

In this section, we report some numerical examples to illustrate the efficiency of our method.

Example 5.1. Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{M} = (\mathbb{R}_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold with Riemannian metric defined by $\langle u, v\rangle = \frac{1}{x^2}uv,\ \in \mathbb{R}_{++},\ u, v \in T_x\mathbb{M}$. The Riemannian distance $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}_+$ is given by $d(x, y) = |\ln \frac{y}{x}|$ for all $x, y \in \mathbb{M}$. Let $x \in \mathbb{M}$, then the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is defined by $\exp_x tv = xe^{\frac{vt}{x}}$ for all $v \in T_x\mathbb{M}$. The inverse of the exponential map, $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$ is defined by $\exp_x^{-1} y = x \ln \frac{y}{x}$ for all $x, y \in \mathbb{M}$. The parallel transport is the identity on $T\mathbb{M}$. Let $C = (0, 1]$, $V : C \to \mathbb{R}$ and $U : C \to T\mathbb{M}$ be defined by $Vx = x \ln x$ and $Ux = x(1 + \ln x)$ respectively. Then V is maximal monotone on C and U is a continuous and monotone vector field on C. It is not difficult to see that w_n in Algorithm 3.3 can be expressed as

$$w_n = \left(\frac{z_n}{e^{\tau_n}}\right)^{\frac{1}{1+\tau_n}},\ \tau_n > 0$$

and $(U + V)^{-1}(0) = \frac{1}{\sqrt{e}}$. Now, let $T : C \to C$ be defined by $Tx = x$. Define a mapping $g : C \to C$ by $g(x) = \frac{x^{\frac{1}{4}}}{4}$ for all $x \in C$, then g is a μ-contraction mapping with a continuous function $\mu(s) = \frac{s}{2}$. For this experiment, we choose $\alpha_n = \frac{1}{n+1}$, $\frac{n}{5n+7}$, $\eta_n = \frac{1}{n\sqrt{n}}$, $\rho = \theta = \frac{1}{2}$ and $\tau_0 = 3$. We terminate the execution of the process at $E_n = d(x_{n+1}, x_n) = 10^{-3}$ and make a comparison of Algorithm 3.3 with an unaccelerated version of it (i.e $\theta_n = 0$). The result of this experiment is shown in Figure 1.

Example 5.2. Let $\mathbb{R}^3_{++} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0, i = 1, 2, 3\}$, $\mathbb{M} = (\mathbb{R}^3_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold with the Riemannian metric is defined by

$$\langle u, v\rangle = uG(x)v^T, \quad x \in \mathbb{R}^3_{++},\ u, v \in T_x\mathbb{R}^3_{++} = \mathbb{R}^3,$$

where $G(x)$ is a diagonal matrix defined $G(x) = diag(x_1^{-2}, x_2^{-2}, x_3^{-2})$. The Riemannian $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}_+$ is defined by

$$d(x, y) = \sqrt{\left(\sum_{i=1}^{3} \ln^2 \frac{x_i}{y_i}\right)}\ \forall\ x, y \in \mathbb{M}.$$

The sectional curvature of the Riemannian manifold $\mathbb{M}$ is 0. Thus $\mathbb{M} = (\mathbb{R}^3_{++}, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold. Let $x = (x_1, x_2, x_3) \in \mathbb{M}$. Then, the exponential map $\exp_x : T_x\mathbb{M} \to$ is defined by

$$\exp_x(u) = (x_1 e^{\frac{u_1}{x_1}}, x_2 e^{\frac{u_2}{x_2}}, x_3 e^{\frac{u_3}{x_3}})$$

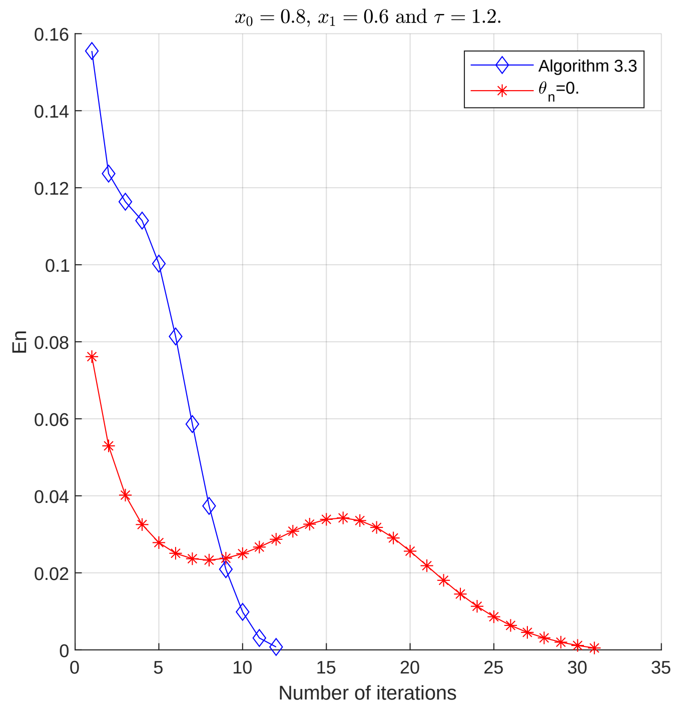

Figure 1. Numerical report for Example 5.1.

for all $u = (u_1, u_2, u_3) \in T_x\mathbb{M}$. The inverse of the exponential map, $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$ is defined by

$$\exp_x^{-1} y = \left(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2}, x_3 \ln \frac{y_3}{x_3} \right)$$

for all $x, y \in \mathbb{M}$. The parallel transport $P_{y,x} : T_x\mathbb{M} \to T_y\mathbb{M}$ is defined by

$$P_{y,x}(u) = \left(u_1 \frac{y_1}{x_1}, u_2 \frac{y_2}{x_2}, u_3 \frac{y_3}{x_3} \right)$$

for all $u = (u_1, u_2, u_3) \in T_xM$. Let $C = \{x = (x_1, x_2, x_3) \in \mathbb{M} : 0 < x_i \leq 1,\ \text{for } i = 1, 2, 3\}$ be the geodesic convex subset of M. Let $U : \mathbb{M} \to T\mathbb{M}$ be defined by

$$V(x) = (-x_1, x_2 \ln x_2, 3x_3)\ \forall\ (x_1, x_2, x_3) \in \mathbb{M}$$

and $U : \mathbb{M} \to T\mathbb{M}$ be defined by

$$U(x_1, x_2, x_3) = (x_1 + x_1 \ln x_1, x_2, -3x_1 + 2x_3 \ln 2x_3)\ \forall\ (x_1, x_2, x_3) \in M.$$

Then V is maximal monotone vector field on C and U is continuous and monotone vector field on C (see [6, Example 1]). By simple calculation, we see that w_n in Algorithm 3.3 can be expressed as

$$w_n = (w_n^1 e^{\tau_n}, (w_n^2)^{\frac{1}{1+\tau_n}}, w_n^3 e^{-3\tau_n}).$$

Note that $(V + U)^{-1}(0) = \{(1, \frac{1}{e}, \frac{1}{2})\}$. Define a mapping $g : C \to C$ by $g(x) = \frac{x^{\frac{1}{4}}}{4}$ for all $x = (x_1, x_2, x_3) \in C$, then g is a μ-contraction mapping with a continuous function $\mu(s) = \frac{s}{2}$. Choose $\alpha_n = \frac{1}{n+3}$, $\delta = \frac{1}{3n+7}$, $\eta_n = \frac{1}{n\sqrt{n}}$, $\mu = \frac{1}{2}$ and $\theta = \frac{1}{5}$. We terminate the execution of the process at $E_n = d(x_{n+1}, x_n) = 10^{-3}$ and make a comparison of Algorithm 3.3 with an unaccelerated version of it (i.e $\theta_n = 0$). The result of this experiment is shown in Figure 2.

In what follows, we give an example to illustrate the application of our method to the variational inequalities as given in Section 4.1. We make a comparison our method to the methods reported in We first give this important Remark.

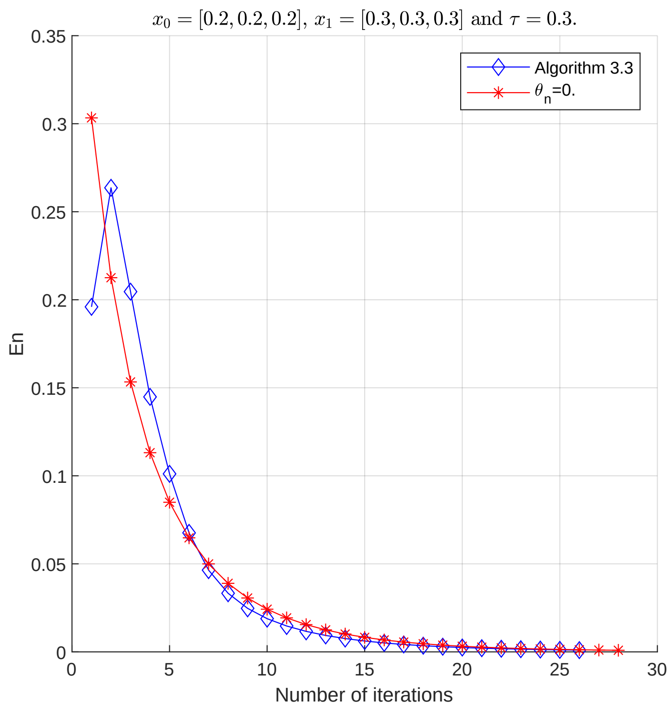

Figure 2. Numerical report for Example 5.2

Remark 5.3. [35]. Let $\mathbb{M} = (\mathbb{R}^n_{++}, \langle\cdot,\cdot\rangle)$be a Hadamard manifold with the Riemannian metric $\langle u, v\rangle = uG(x)v^T$ for all $x \in \mathbb{R}_{++}$ and $T_x\mathbb{R}^n_{++}$, where $G(x)$ is an $n \times n$ matrix. Let $\varphi : \mathbb{M} \to \mathbb{R}$ be a differentiable function. Then $grad\varphi(x) = \nabla\varphi(x)G(x)^{-1}$ for all $x \in \mathbb{M}$, where $G(x)^{-1}$ is the inverse of the matrix $G(x)$ and $\nabla\varphi(x)$ is the gradient of φ in the Euclidean sense.

Proposition 5.4. *[30]. Let C be a geodesic convex subset of a Hadamard manifold $\mathbb{M}$ and $\varphi : C \to \mathbb{R}$ be a differentiable convex function. Then, x is a solution to the minimization problem:*

$$\min_{x\in C} \varphi(x)$$

if and only if x is a solution to $VP(grad\varphi, C)$.

Example 5.5. This example was used in [38]. Consider the manifold $M = \mathbb{R}^3_{++}$ with the same description as in Example 5.2. Let $\phi : \mathbb{R}^3_{++} \to \mathbb{R}$ be given by

$$\phi(x) = \frac{1}{2}\|Dx - b\|^2, \ \forall\ x \in \mathbb{R} \tag{5.1}$$

where D is a 3×3 real matrix and $b \in \mathbb{R}^3$. Let $\psi : \mathbb{R}^3_{++} \to \mathbb{R}$ be the function defined by

$$\psi(x) = (\ln(x_1), \ln(x_2), \ln(x_3)), \ \forall\ x = (x_1, x_2, x_3) \in \mathbb{R}^3_{++}.$$

It follows that ψ is an isometry between $\mathbb{R}^3_{++}$ and $\mathbb{R}$. Assume

$$D = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and let $\varphi : \mathbb{R}^3_{++} \to \mathbb{R}$ be a function defined by

$$\varphi(x) = \phi(\psi(x)), \ \forall\ x \in \mathbb{R}^3_{++}.$$

Now, the gradient of φ in the Euclidean sense is given as

$$\nabla\varphi(x) = \left(\frac{3}{x_1}\ln\frac{x_1x_2}{x_3}, \frac{3}{x_2}\ln\frac{x_1x_2}{x_3}, -\frac{3}{x_3}\ln\frac{x_1x_2}{x_3}\right).$$

It follows from Remark 5.3, that

$$grad\varphi(x)\left(3x_1\ln\frac{x_1x_2}{x_3}, 3x_2\ln\frac{x_1x_2}{x_3}, -3x_3\ln\frac{x_1x_2}{x_3}\right).$$

Now, let $C = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3_{++} : 1 \le x_i \le 2 \text{ for all } i = 1, 2, 3\}$. Then, C is a closed, godesic convex subset of $\mathbb{R}^3_{++}$. Observe that ϕ is a proper convex function on $\mathbb{R}^3$ (see [39]). Also, since ψ is an isometry, it follows from Proposition 2.21 that φ is a proper continuous geodesic convex function. By Proposition 2.22, we have that $grad\varphi$ is a monotone vector field and by Proposition 5.4, we get

$$\begin{aligned} VP(grad\varphi, C) &= \{x \in C : grad\varphi(x) = 0\} \\ &= \{x = (x_1, x_2, x_3) \in C : \frac{x_1x_2}{x_3} = 1\}. \end{aligned}$$

Let the mapping $g : C \to C$ be defined by $g(x) = \frac{x^{\frac{1}{4}}}{4}$ for all $x = (x_1, x_2, x_3) \in C$, then g is a μ-contraction mapping with a continuous function $\mu(s) = \frac{s}{2}$. Suppose $T : C \to C$ is defined by $Tx = x$. Choose $\alpha_n = \frac{1}{3}$, $\eta_n = \frac{1}{n\sqrt{n}}$, $\mu = \frac{1}{2}$ and $\theta = \frac{1}{5}$. We terminate the execution of the process using $\|r(x, \lambda)\| \le 10^{-4}$, where $r(x, \lambda) = \exp_x^{-1} P_C(\exp_x(-\lambda Fx))$. We compare our method with the methods reported in [38, Algorithm 2] and [48, Algorithm 4.1]. For Algorithm 2 and Algorithm 4.1, we let $\alpha = \frac{1}{3}$, $\sigma = 0.7$ and $\lambda = 1$. The result of this experiment for some initial values of x_0 and x_1 is shown in Figure 2.

Case 1 $x_0 = [1.5, 1.2, 1.5]$ and $x_1 = [1.4, 1.3, 1.4]$.
Case 2 $x_0 = [1, 1, 3]$ and $x_1 = [2, 3, 1]$.

Data availability: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests The authors declare that there is no conflicting interests.

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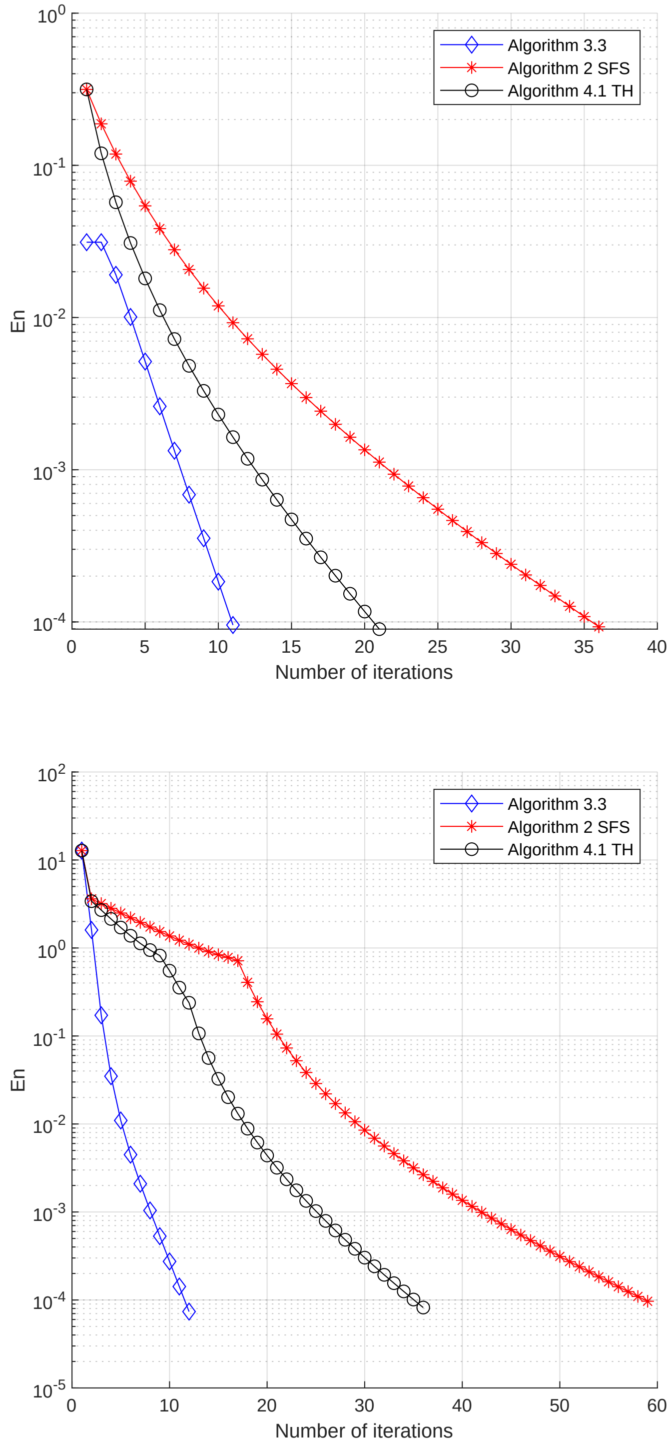

Figure 3. Numerical report for Example 5.5

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