Anti-windup compensation for a class of iterative learning control systems subject to actuator saturation

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***Abstract*— This paper proposes a dynamic anti-windup scheme for a class of iterative learning control (ILC) systems. The anti-windup compensator has the same structure as a class of compensators for 1D systems and is able to guarantee similar properties: (i) that the constrained system with anti-windup compensation is exponentially stable if a certain linear matrix inequality is satisfied; and (ii) if the trajectory to be tracked by the nominal ILC controller is consistent with the control constraints, the anti-windup compensator will ensure that the behaviour of the nominal ILC controller is eventually recovered.**

I. INTRODUCTION

Iterative learning control (ILC) has evolved for application to systems that make repeated executions of the same task over a finite duration, and once complete, resetting to the starting location occurs. Each execution is known as a trial in the literature, and the duration is known as the trial length. This form of control can also be applied to a system where a trial is completed, and a stoppage occurs before the subsequent trial begins.

Once a trial is complete, all information generated, i.e., state, input, and output, is available for use in updating the control law at any instant to be applied on the subsequent trial. In ILC, it is the control input that is updated. The introduction of this form of control action is widely credited to [1]. This first work was inspired by robotic applications, where, e.g., pick and place operations are one application. In such operations, the task is to collect a sequence of items, in turn, from a fixed location, transfer each of them over a finite duration, place each of them on a moving conveyor and then return to the starting location, and so on.

Suppose that a reference trajectory is available. Then the error on each trial is the difference between this trajectory and the trial output, leading to an error sequence indexed by trial number. The control design problem is to ensure that this sequence converges in the trial number, ideally to zero or, in practical applications, to within some tolerance specified in terms of the norm on the underlying function space.

The design of ILC laws has seen application in a range of problems, e.g., printing systems [2], additive manufacturing [3], free-electron lasers [4], center-articulated industrial vehicles [5], and robotic-assisted stroke rehabilitation [6]. See also the survey papers [7], [8].

Most popular ILC algorithms assume an underlying linearity in both the system to be controlled and the controller. However, as with other types of control system they have to operate in environments where actuation is limited, and, not surprisingly, they are liable to suffer from performance degradation if actuator saturation occurs. Consequently a number of papers have investigated constrained control techniques for ILC systems - see for example [9], [10], [11]. While these approaches show some promise, none follow the anti-windup paradigm which is used extensively in conventional (non-learning) control systems.

The advantage of the anti-windup approach to handling actuator constraints is that it enables a so-called nominal controller to be designed, ignoring control constraints, and then, in a second step an anti-windup compensator is designed to assist the nominal one when actuator saturation occurs. In this way, all the properties of the nominal control system are preserved unless saturation is encountered. The anti-windup approach is rather standard in control engineering, but traditionally has been implemented without stability considerations. See [12], [13], [14], [15], [16] for further discussion for 1D systems.

For the past two decades, an appealing approach to the anti-windup problem has been to decouple the system into a *nominal linear* part and an additional nonlinear part, featuring the effects of the saturation nonlinearity and the anti-windup compensator dynamics [17], [18]. One can then see that if the baseline controller has been designed so that the system functions well in the absence of saturation, then the anti-windup compensator can be designed in a separate stage, with the goal solely to retain stability and limit performance degradation during periods of saturation. This approach is fairly straightforward for 1D linear systems, and some extensions have been made for certain classes of nonlinear 1D system, [19], [20]. The goal in this paper is to provide similar AW approaches for ILC systems.

A. Preliminary results on saturation

For simplicity, the standard element-wise saturation function is considered throughout this paper:

$$\mathrm{sat}[u] = [\mathrm{sat}_1[u_1] \quad \dots \quad \mathrm{sat}_m[u_m]]' \tag{1}$$

where $\mathrm{sat}_i[u_i] = \mathrm{sign}(u_i)\min\{|u_i|, \bar{u}_i\}$ and $\bar{u}_i > 0$ is the saturation limit in the i'th channel. The complement of the saturation function is the deadzone, defined using the identity

$$u = \mathrm{sat}[u] + \mathrm{Dz}[u] \tag{2}$$

Let $\sigma_i[\cdot]$ be the i'th element of either the saturation or the deadzone functions. It is well-known (e.g. [20]) that both functions are slope restricted; that is

$$0 \le \frac{\sigma_i[u_1] - \sigma_i[u_2]}{u_1 - u_2} \le 1 \quad \forall u_1, u_2 \neq u_1 \in \mathbb{R} \tag{3}$$

The following well-known fact is useful for slope-restricted nonlinearities and will be used to prove the main results.

Fact 1: Assume $\sigma[u] : \mathbb{R}^m \mapsto \mathbb{R}^m$ is a decentralised slope-restricted function in the sense that inequality (3) holds for

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$$\begin{bmatrix} \xi_{k+1}(p+1) \\ \tilde{\xi}_{k+1}(p+1) \\ \hline \bar{e}_{k+1}(p) \end{bmatrix} = \left[\begin{array}{cc|c} A+BK_1 & -BK_1 & BK_2 \\ 0 & (A-LC) & 0 \\ \hline CA^{r-1}(A+BK_1) & CA^{r-1}BK_1 & I+CA^{r-1}BK_2 \end{array}\right] \begin{bmatrix} \xi_{k+1}(p) \\ \tilde{\xi}_{k+1}(p) \\ \hline \bar{e}_k(p) \end{bmatrix} \tag{13}$$

each of its elements. Then for all positive definite diagonal matrices $W \in \mathbb{D}_+^m$, the following inequality holds

$$\big(\sigma[u]-\sigma[v]\big)'W\big(u-v-\sigma[u]+\sigma[v]\big) \geq 0 \quad \forall u,v \in \mathbb{R}^m$$

II. Problem formulation

Consider the discrete-plant

$$\Sigma_P \sim \begin{cases} x_k(p+1) \;=\; Ax_k(p)+B\nu_k(p) \\ \quad y_k(p) \;=\; Cx_k(p) \end{cases} \quad x_k(0)=x_0 \quad \forall k \tag{4}$$

where $p \in \{0,1,N-1\}$ is the time index and $k \geq 0$ is the trial index. $x_k(p) \in \mathbb{R}^n$ is the plant state at time p, trial k, $y_k(p) \in \mathbb{R}^{n_y}$ is the output which must track a profile $y_{\text{ref}}(p) \in \mathbb{R}^{n_y}$ and $\nu_k(p) \in \mathbb{R}^m$ is the plant input. The following assumption is made throughout.

Assumption 1: The plant Σ_P is such that for some $r<n$,

$$CA^{r-1}B \neq 0, \quad CA^iB=0 \quad i=\{1,2,\dots,r-2\} \tag{5}$$

Similar to [21], the following ILC control law is considered.

$$u_k(p) = u_k^{nom}(p) - v_k^{[1]}(p) \tag{6}$$

$$u_k^{nom}(p) = u_{k-1}^{nom}(p) + K_1(\hat{x}_k(p)-\hat{x}_{k-1}(p)) + K_2 e_{k-1}(p+r) \tag{7}$$

where K_1 and K_2 are the controller gains, $\hat{x}_k(p)$ represents the state-estimate, $e_k(p)$ the tracking error (defined below) and $v_k^{[1]}(p)$ is a signal whose role will be defined shortly. The estimated state is generated by the observer

$$\Sigma_O \sim \begin{cases} \hat{x}_k(p+1) \;=\; (A-LC)\hat{x}_k(p)+Bu_k^{nom}(p) \\ \qquad\qquad\qquad +L\big(y_k(p)+v_k^{[2]}(p)\big) \end{cases} \tag{8}$$

where L is the observer gain and again $v_k^{[2]}(p)$ is a signal whose role will be clarified shortly.

The reference tracking error is given by

$$e_k(p) = y_{ref}(p) - y_k(p) \tag{9}$$

The evolution of this error over k trials is given by

$$e_{k+1}(p) = e_k(p) - \big(y_{k+1}(p)-y_k(p)\big) \tag{10}$$

A. Nominal linear dynamics

The system is said to be *nominal* when no saturation is present at the plant input, and hence no anti-windup compensator is required to be active. This corresponds to the interconnection conditions

$$\begin{cases} \nu_k(p) = u_k(p) \\ v_k^{[1]}(p) = 0 \\ v_k^{[2]}(p) = 0 \end{cases} \tag{11}$$

Defining

$$\begin{cases} \xi_{k+1}(p) = x_{k+1}(p)-x_k(p) \\ \tilde{\xi}_{k+1}(p) = \tilde{x}_{k+1}(p)-\tilde{x}_k(p) \\ \bar{e}_k(p) = e_k(p+r) \end{cases} \tag{12}$$

where $\tilde{x}_k(p) = x_k(p)-\hat{x}_k(p)$, the dynamics of the system (4),(6),(7),(8),(10) and (11) can be written as in equation (13) above, where Assumption 1 has been used. The dynamics (13) are to be kept in mind for the subsequent development and the reader's attention is drawn to the following equivalent representation of the control law (6) when saturation is absent

$$u_k(p) = u_{k-1}(p) + K_1(\xi_k(p)-\tilde{\xi}_{k-1}(p)) + K_2\bar{e}_{k-1}(p) \tag{14}$$

Since equation (13) is a linear system, K_1, K_2 and L can be designed such that it is exponentially stable and such that the error $\bar{e}_k(p)$ converges to zero exponentially. The following assumption [22] is therefore made.

Assumption 2: The dynamics (13) are exponentially stable; that is there exist real numbers $\kappa>0$ and $\lambda \in (0,1)$ such that

$$\|\xi_k(p)\|^2+\|\tilde{\xi}_k(p)\|^2+\|\bar{e}_k(p)\|^2 \leq \kappa\lambda^{k+p} \;\; \forall \xi_k,\tilde{\xi}_k \in \mathbb{R}^n, \bar{e}_k \in \mathbb{R}^{n_y} \tag{15}$$

B. Dynamics with input saturation

When the system contains input saturation, the dynamics become nonlinear and can lead to severe performance and stability degradation. To temper this degradation, an anti-windup compensator similar to that used in 1D anti-windup compensation is used (see e.g. [18]). In this case the interconnection conditions become

$$\begin{cases} \nu_k(p) = \text{sat}[u_k(p)] \\ v_k^{[1]}(p) = Fw_k(p) \\ v_k^{[2]}(p) = Cw_k(p) \end{cases} \tag{16}$$

where the signal $w_k(p) \in \mathbb{R}^n$ is the anti-windup compensator state vector. The first condition of (16) captures the presence of saturation with the remaining two dictating how the anti-windup compensator exerts its influence on the system. The anti-windup compensator has the typical property that it is not active unless the control signal $u_k(p)$ exceeds the saturation values i.e. $\text{Dz}[u_k(p] \neq 0$ only when the saturation bounds $\bar{u}_i$ are exceeded.

$$\Sigma_A \sim \big\{ \; w_k(p+1) = (A+BF)w_k(p) + B\text{Dz}[u_k(p)] \tag{17}$$

The same form of control signal (6) is used but in this case $v_k^{[1]}(p)$ may not be zero, and the nominal control signal (7) is replaced with

$$u_k^{nom}(p) = u_{k-1}^{nom}(p) + K_1(\hat{x}_k(p)-\hat{x}_{k-1}(p)) + K_2\tilde{e}_{k-1}(p+r) \tag{18}$$

This has an identical form to the control law generated by the linear system without saturation (equation (7)), except that error $e_k(p)$ is replaced by the *modified tracking error*,

$$\tilde{e}_k(p) := y_{ref}(p) - (y_k(p)+Cw_k(p)) \tag{19}$$

meaning that the relationship between this and the original error $e_k(p)$, in equation (9), is

$$e_k(p) = \tilde{e}_k(p) + Cw_k(p) \tag{20}$$

This modified tracking error is introduced to assist the development of the forthcoming stability proofs.

$$\left[\begin{array}{c}\xi_{k+1}(p+1)\\ \tilde{\xi}_{k+1}(p+1)\\ \alpha_{k+1}(p+1)\\ \hline \check{e}_{k+1}(p)\end{array}\right]=\left[\begin{array}{cccc} A+BK_1 & -BK_1 & 0 & BK_2\\ 0 & A-LC & 0 & 0\\ 0 & 0 & A+BF & 0\\ \hline -CA^{r-1}(A+BK_1) & CA^{r-1}BK_1 & 0 & I-CA^{r-1}BK_2\end{array}\right]\left[\begin{array}{c}\xi_{k+1}(p)\\ \tilde{\xi}_{k+1}(p)\\ \alpha_{k+1}(p)\\ \hline \check{e}_k(p)\end{array}\right]+\left[\begin{array}{c}0\\0\\B\\ \hline 0\end{array}\right]\phi_{k+1}(p) \quad (32)$$

$$\left[\begin{array}{c}\xi_{k+1}(p+1)\\ \tilde{\xi}_{k+1}(p+1)\\ \hline \check{e}_{k+1}(p)\end{array}\right]=\left[\begin{array}{cc|c} A+BK_1 & -BK_1 & BK_2\\ 0 & (A-LC) & 0\\ \hline CA^{r-1}(A+BK_1) & CA^{r-1}BK_1 & I+CA^{r-1}BK_2\end{array}\right]\left[\begin{array}{c}\xi_{k+1}(p)\\ \tilde{\xi}_{k+1}(p)\\ \hline \check{e}_k(p)\end{array}\right] \quad (33)$$

$$\alpha_{k+1}(p+1)=(A+BF)\alpha_{k+1}(p)+B\phi_{k+1} \quad (34)$$

Remark 1:

- The anti-windup dynamics are identical to the 1D full-order anti-windup compensator case [18]: in particular, the compensator becomes active, on trial k, if the control signal on that trial exceeds the saturation bounds.
- A crucial element of the anti-windup approach is the concept of the *nominal control signal* defined in (18). This has an identical form to the linear nominal control law (see also (6)). It will be shown that a return to nominal "unsaturated" behaviour will occur if (and only if) the nominal control signal $u_k^{nom}(p)$ eventually lies within the control bounds. Under these conditions, the error (10) will then converge to zero, although not necessarily exponentially. □

Similar to the case of 1D systems (see [17], [14], [19]) it is illuminating to write the closed-loop dynamics (4),(6),(8),(16),(17), and (18) in a different set of coordinates to emphasize the decoupling offered by the anti-windup compensator (17).

$$\begin{cases} \tilde{w}_k(p) &= x_k(p)+w_k(p)\\ \tilde{x}_k(p) &= x_k(p)+w_k(p)-\hat{x}_p\\ w_k(p) &= w_k(p)\end{cases} \quad (21)$$

After lengthy algebra, the dynamics can be re-written as

$$\begin{aligned}\tilde{w}_k(p+1)=&(A+BK_1)\tilde{w}_k(p)-BK_1\tilde{x}_k(p)+Bu_{k-1}^{nom}(p)\\ &+BK_2\tilde{e}_{k-1}(p+r)-BK_1\tilde{w}_{k-1}(p)+BK_1\tilde{x}_{k-1}(p)\end{aligned} \quad (22)$$

$$\tilde{x}_k(p+1)=(A-LC)\tilde{x}_k(p) \quad (23)$$

$$w_k(p+1)=(A+BF)w_k(p)+B\mathrm{Dz}[u_k(p)] \quad (24)$$

A further change of coordinates, similar to the linear case ([21]), is also defined

$$\begin{cases} \xi_{k+1}(p) &= \tilde{w}_{k+1}(p)-\tilde{w}_k(p)\\ \tilde{\xi}_{k+1}(p) &= \tilde{x}_{k+1}(p)-\tilde{x}_k(p)\\ \check{e}_k(p) &= \tilde{e}_k(p+r)\end{cases} \quad (25)$$

and an extra change of coordinates is defined for the trial-to-trial difference in anti-windup compensator state:

$$\alpha_{k+1}(p)=w_{k+1}(p)-w_k(p) \quad (26)$$

Using the coordinate transformations above, and the dynamics (22)-(24), after some algebra it follows that

$$\xi_{k+1}(p+1)=(A+BK_1)\xi_{k+1}(p)-BK\xi_{k+1}(p)+BK_2\check{e}_k(p) \quad (27)$$

$$\tilde{\xi}_{k+1}(p+1)=(A-LC)\tilde{\xi}_{k+1}(p) \quad (28)$$

$$\alpha_{k+1}(p+1)=(A+BF)\alpha_{k+1}(p)+B\phi_{k+1}(p) \quad (29)$$

where $\phi_{k+1}(p):=\mathrm{Dz}[u_{k+1}(p)]-\mathrm{Dz}[u_k(p)]$. Note also, that the dynamics of $\check{e}_k(p)$ can be written as

$$\begin{aligned}\check{e}_{k+1}(p)&=y_{ref}(p+r)-C\tilde{w}_k(p+r) \quad (30)\\ &=\check{e}_k(p)-C\xi_{k+1}(p+r) \quad (31)\end{aligned}$$

meaning that the dynamics of the closed-loop system with input saturation and anti-windup can be written as in equation (32) above. Note further that the states $\xi_k(p)$, $\tilde{\xi}_k(p)$ and $\check{e}_k(p)$ evolve *independently* of both $\alpha_k(p)$ and $\phi_k(p)$. Therefore, it is possible to *decouple* the dynamics (32) into (33) and (34).

Finally, observe that the dynamics (33) have *exactly the same form* as equation (13) i.e. the nominal linear system dynamics. Therefore, by Assumption 2, it follows that

$$\|\check{e}_{k+1}\|^2+\|\xi_{k+1}(p)\|^2+\|\tilde{\xi}_{k+1}(p)\|^2\leq\kappa\lambda^{k+p} \quad (35)$$

for some $\kappa>0$ and $\lambda\in(0,1)$. Therefore, the system (33)-(34) will be stable if the dynamics (34) are stable. Note that (34) is driven by the output of (33) since the control law (6) and (18) can be written as

$$\begin{aligned}u_{k+1}(p)=u_k^{nom}(p)+K_1\left(\xi_{k+1}(p)-\tilde{\xi}_{k+1}(p)\right)\\ +K_2\check{e}_k(p)-Fw_{k+1}(p)\end{aligned} \quad (36)$$

III. Anti-windup compensator design

It is assumed throughout that K_1, K_2 and L have been designed such that the dynamics (33) are exponentially stable: Assumption 2. The goal of this section is to give a procedure for computing the anti-windup gain F such that the overall system in equation (32) is exponentially stable, and then to use this result to prove that the "real" error $e_k(p)$ converges asymptotically to zero, under a natural condition: that the steady state control signal $u_k^{nom}(p)$ lies below the saturation limits for all $p>\bar{p}$, for some $\bar{p}$, and all $k>\bar{k}-1$ for some $\bar{k}$. This quite logical condition is similar to what one expects of anti-windup in 1D systems.

To ease the proof of the results, note that equations (33) and (34), along with the control law (36), may be written as

$$\left[\begin{array}{c}\eta_{k+1}(p+1)\\ \alpha_{k+1}(p+1)\\ \hline \check{e}_{k+1}(p)\end{array}\right]=\left[\begin{array}{cc|c}\bar{A}_{11} & 0 & \bar{B}\\ 0 & (A+BF) & 0\\ \hline \bar{A}_{21} & 0 & \bar{A}_{22}\end{array}\right]\left[\begin{array}{c}\eta_{k+1}(p)\\ \alpha_{k+1}(p)\\ \hline \check{e}_k(p)\end{array}\right]+\left[\begin{array}{c}0\\B\\ \hline 0\end{array}\right]\phi_{k+1} \quad (37)$$

$$u_{k+1}(p)=u_k^{nom}(p)+\bar{K}_1\eta_{k+1}(p)+K_2\check{e}_k(p)-Fw_{k+1}(p) \quad (38)$$

where the various matrices and vectors have obvious definitions. It will also be useful to keep in mind the *nominal* control signal can be written as

$$u_{k+1}^{nom}(p)=u_k^{nom}(p)+\bar{K}_1\eta_{k+1}(p)+K_2\check{e}_k(p) \quad (39)$$

and also that

$$u_k(p)=u_k^{nom}(p)-Fw_k(p) \quad (40)$$

The first result establishes conditions ensuring exponential stability of the origin of system (33)-(34) (and thus (32)).

Proposition 1: Let Assumptions 1 and 2 hold and further assume there exist matrices $Q>0$, $U>0$ and diagonal, and

$$\begin{bmatrix}\eta_{k+1}(p)\\ \alpha_{k+1}(p)\\ \check{e}_k(p)\end{bmatrix}' \begin{bmatrix}\bar{A}_{11}'P_\eta\bar{A}_{11}+\bar{A}_{21}'P_\eta\bar{A}_{21}-P_\eta & 0 & \bar{A}_{11}'P_\eta\bar{B}+\bar{A}_{21}'P_e\bar{A}_{22}\\ \star & (A+BF)'P_\alpha(A+BF)-P_\alpha & 0\\ \star & \star & \bar{A}_{22}'P_e\bar{A}_{22}+\bar{B}'P_\eta\bar{B}_\eta-P_e\end{bmatrix}\begin{bmatrix}\eta_{k+1}(p)\\ \alpha_{k+1}(p)\\ \check{e}_k(p)\end{bmatrix}$$
$$+2\alpha_{k+1}(p)'(A+BF)'P_\alpha B\phi_{k+1}+\phi_{k+1}'B'P_\alpha B\phi_{k+1}(p)<0 \quad (44)$$

$$\begin{bmatrix}\bar{A}_{11}'P_\eta\bar{A}_{11}+\bar{A}_{21}'P_\eta\bar{A}_{21}-P_\eta & 0 & \bar{A}_{11}'P_\eta\bar{B}+\bar{A}_{21}'P_e\bar{A}_{22} & \bar{K}'W\\ \star & (A+BF)'P_\alpha(A+BF)-P_\alpha & 0 & (A+BF)'PB-F'W\\ \star & \star & \bar{A}_{22}'P_e\bar{A}_{22}+\bar{B}'P_\eta\bar{B}_\eta-P_e & K_2'W\\ \star & \star & \star & -2W+B'P_\alpha B\end{bmatrix}<0 \quad (45)$$

$$\begin{bmatrix}\bar{A}_{11}'P_\eta\bar{A}_{11}+\bar{A}_{21}'P_e\bar{A}_{21}-P_\eta & \bar{A}_{11}'P_\eta\bar{B}+\bar{A}_{21}'P_e\bar{A}_{22} & 0 & \bar{K}'W\\ \star & \bar{A}_{22}'P_e\bar{A}_{22}+\bar{B}'P_\eta\bar{B}_\eta-P_e & 0 & K_2'W\\ \star & \star & (A+BF)'P_\alpha(A+BF)-P_\alpha & (A+BF)'PB-F'W\\ \star & \star & \star & -2W+B'P_\alpha B\end{bmatrix}<0 \quad (46)$$

an unstructured matrix Y such that the following linear matrix inequality holds

$$\begin{bmatrix}-Q & Y' & QA'+Y'B'\\ \star & -2U & UB'\\ \star & \star & -Q\end{bmatrix}<0 \quad (41)$$

Then the origin of system (33)-(34) is exponentially stable if $F=YQ^{-1}$.

Proof: Exponential stability of (33)-(34) is equivalent to exponential stability of system (37)-(38). Hence consider this and choose the Vector Lyapunov function:

$$V(\eta_{k+1}(p),\alpha_{k+1}(p),\check{e}_k(p))= \begin{bmatrix}\eta_{k+1}(p)'P_\eta\eta_{k+1}(p)+\alpha_{k+1}(p)'P_\alpha\alpha_{k+1}(p)\\ \check{e}_k(p))'P_e\check{e}_k(p))\end{bmatrix} \quad (42)$$

where $P_i>0$ for $i=\{\eta,\alpha,e\}$. It follows from the results of [21], [22] that if

$$\mathscr{D}\{V(\eta_{k+1}(p),\alpha_{k+1}(p),\check{e}_k(p))\}< -c_3(\|\eta_{k+1}(p)\|^2+\|\alpha_{k+1}(p)\|^2+\|\check{e}_k(p)\|^2) \quad (43)$$

where $\mathscr{D}(\cdot)$ represents the divergence operator, then the system (37)-(38) will be exponentially stable. First note that the left hand side of inequality (43) simplifies to inequality (44) above. Next, because the deadzone nonlinearity is slope restricted, Fact 1 implies that, for all diagonal matrices $W>0$,

$$\phi_{k+1}'W\left(u_{k+1}(p)-u_k(p)-\phi_{k+1}\right)\geq 0$$
$$\Leftrightarrow \phi_{k+1}'W\left(\bar{K}\eta_{k+1}(p)+K_2\check{e}_k(p)-F\alpha_{k+1}(p)-\phi_{k+1}\right)\geq 0$$

which is a quadratic constraint. Using the S-procedure, and after some algebra, inequality (44) holds if the matrix inequality (45) holds; interchanging the second and third rows/columns, this inequality is equivalent to inequality (46). Letting $P_\eta=\delta\tilde{P}_\eta$ and $P_e=\delta\tilde{P}_e$, it follows that there always exists a sufficiently large δ such that inequality (46) holds if the following two matrix inequalities hold:

$$\begin{bmatrix}\bar{A}_{11}'\tilde{P}_\eta\bar{A}_{11}+\bar{A}_{21}'\tilde{P}_e\bar{A}_{21}-\tilde{P}_\eta & \bar{A}_{11}'\tilde{P}_\eta\bar{B}+\bar{A}_{21}'\tilde{P}_e\bar{A}_{22}\\ \star & \bar{A}_{22}'\tilde{P}_e\bar{A}_{22}+\bar{B}'\tilde{P}_\eta\bar{B}_\eta-\tilde{P}_e\end{bmatrix}<0 \quad (47)$$

$$\begin{bmatrix}(A+BF)'P_\alpha(A+BF)-P_\alpha & (A+BF)'PB-F'W\\ \star & -2W+B'P_\alpha B\end{bmatrix}<0 \quad (48)$$

The first matrix inequality is satisfied by assumption since (Assumption 2) the nominal closed loop is exponentially stable. Hence, exponential stability of the system (38) is achieved if the second inequality holds. The second inequality holds, via the Schur complement, if

$$\begin{bmatrix}-P_\alpha & -F'W & (A+BF)'\\ \star & -2W & B'\\ \star & \star & -P_\alpha^{-1}\end{bmatrix}<0 \quad (49)$$

Using several congruence transformations and defining $Q=P_\alpha^{-1}$, $Y=FQ$, the LMI in the proposition then follows. □□

Proposition 1 gives a computational procedure to pick F, and hence design the anti-windup compensator (17), such that the modified error $\check{e}_k(p)$ converges exponentially . However, from equation (20) the *real* tracking error is

$$e_k(p)=\tilde{e}_k(p)+Cw_k(p) \quad (50)$$

Hence, since $\tilde{e}_k(p)$ exponentially converges, then $e_k(p)$ will also converge if $w_k(p)$ converges. The following result gives conditions which ensure that $e_k(p)$ does indeed converge.

Proposition 2: Consider the control signal $u_k^{nom}(p)$ and assume that there exists integers $\bar{k}>0$ and $\bar{p}>0$ such that $|u_{k-1}^{nom}(p)|\preceq\bar{u}$ for all $k\geq\bar{k}-1$ and $p\geq\bar{p}$. Then, under the conditions of Proposition 1,

$$\lim_{k,p\to\infty} e_k(p)=0$$

Proof: From equation (50) it is clear, that since $\tilde{e}_k(p)$ is exponentially stable by Proposition 1, it remains to investigate the convergence of $w_k(p)$; that is the state of the anti-windup compensator. The anti-windup compensator state evolves as

$$w_k(p+1)=(A+BF)w_k(p)+B\mathrm{Dz}[u_k(p)] \quad (51)$$
$$=(A+BF)w_k(p)+B\underbrace{\left(\mathrm{Dz}[u_k(p)]-\mathrm{Dz}[u_{k-1}^{nom}(p)]\right)}_{\psi_k(p)}+B\mathrm{Dz}[u_{k-1}^{nom}(p)] \quad (52)$$

where $\psi_k(p)$, by Fact 1, is such that

$$2\psi_k(p)'W\left(u_k(p)-u_{k-1}^{nom}(p)-\psi_k(p)\right)>0 \quad (53)$$
$$2\psi_k(p)'W\left(\bar{K}_1\eta_k(p)+K_2\check{e}_{k-1}(p)-Fw_k(p)-\psi_k(p)\right)>0 \quad (54)$$

Choosing $V_k(p)=w_k(p)P_\alpha w_k(p)$ as a Lyapunov function, computing its increment, and adding the quadratic constraint (54) via the S-procedure, gives inequality (55). Now, since it is assumed there exists a $\bar{k}$ and $\bar{p}$ such that

$$\mathrm{Dz}[u_k^{nom}(p)]=0 \quad \forall k>\bar{k}-1, p>\bar{p}$$

$$\Delta_p V_k(p) \leq \begin{bmatrix} w_k(p) \\ \psi_k(p) \end{bmatrix}' \begin{bmatrix} (A+BF)'P_\alpha(A+BF) - P_\alpha & (A+BF)'P_\alpha B - F'W \\ \star & B'P_\alpha B - 2W \end{bmatrix} \begin{bmatrix} w_k(p) \\ \psi_k(p) \end{bmatrix} + 2w_k(p)'(A+BF)'P_\alpha B\mathrm{Dz}[u_k^{nom}(p)] + 2\psi_k(p)'W\left(\bar{K}_1\eta_k(p) + K_2\check{e}_{k-1}(p)\right) \quad (55)$$

then it follows from this and the matrix inequality in Proposition 1 that there exist constants $c_1, c_2 > 0$ such that

$$\Delta_p V_k(p) \leq -c_1\|w_k(p)\|^2 + c_2(\|\eta_k(p)\|^2 + \|\check{e}_{k-1}(p)\|^2) \quad \forall k > \bar{k}-1, p > \bar{p} \quad (56)$$

Now since $\eta_k(p)$ and $\check{e}_k(p)$ are exponentially convergent (Assumption 2), from the above inequality it thus follows $V_k(p)$ is convergent for all $k \geq \bar{k}-1, p \geq \bar{p}$ and thus that $w_k(p)$ is also convergent eventually. This then implies $e_k(p)$ converges as claimed in the proposition. □□

Proposition 2 ensures that, if the nominal control signal (that produced by the system with no saturation) eventually falls to levels within the saturation limits, the error $e_k(p)$ will converge to zero, despite the presence of the constraints. This condition is reminiscent of the 1D case, where "linear performance" is recovered asymptotically if, in steady-state, the nominal control signal lies within the saturation bounds.

Remark 2: The LMI in Proposition 1 is solvable if and only if the matrix A is Schur; this can be proved using the Projection Lemma [23]. Again this mirrors the 1D case. If the matrix A is *not* Schur, a local version of the anti-windup problem is solvable, whereby stability only holds with a certain region of attraction. □

IV. Simulation Example

The following continuous-time system is considered

$$G_c(s) = 23.7\frac{s+661.2}{(s+0.05)(s+426.7s+1.74\times10^5)} \quad (57)$$

This system is stable and, when discretized, the "A" matrix is Schur. The system was discretized using the zero-order hold method with a sample time of 0.01s.

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0.7855 & 0.3997 & 0.2243 & 0.0312 \\ 0.5000 & 0 & 0 & 0 \\ 0 & 0.1250 & 0 & 0 \\ \hline 0.0261 & 0.0164 & 0.0169 & 0 \end{array}\right] \quad (58)$$

A nominal ILC control system (6) was designed with

$$L = [0.5437 \quad 0.0031 \quad 0.0113]' \quad (59)$$

$$K_1 = 0.04[-20.8 \quad -16.9 \quad -6889] \quad K_2 = 210 \quad (60)$$

Without saturation this controller provides good performance and the error $\tilde{e}_k(p)$ converges rapidly over trials.

A. Saturated system

For demonstration purposes, the reference depicted by the dashed line in Figure 2 is considered and it is assumed that the control signal saturates at ±50 units. Figure 1 shows the system response (nominal controller with saturation) over 50 trials. Although at first it appears that the nominal ILC control is robust to saturation, as the trials continue, the tracking of the reference actually degrades. For clarity, Figures 2 and 3 show the output response and the saturated control signal after 50 trials: observe the poor tracking and transient periods of rapid oscillation in the control signal.

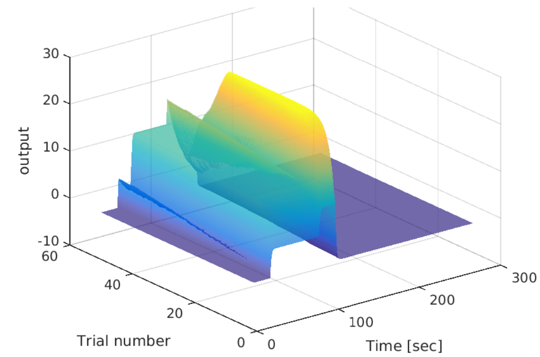

Fig. 1. Output $y_k(p)$ response over 50 trials, with saturated control input

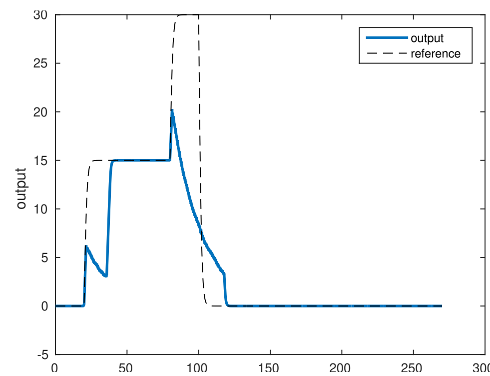

Fig. 2. Output $y_k(p)$ response at trial 50, with saturated control input

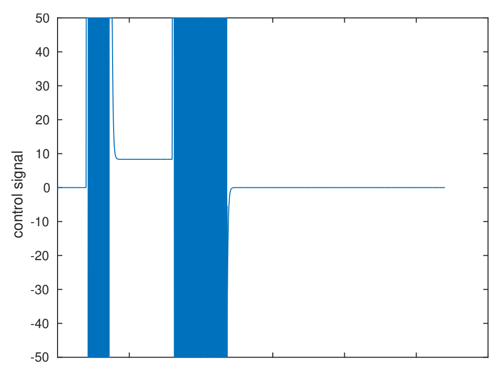

Fig. 3. Saturated control $\hat{u}_k(p)$ response at trial 50

B. Anti-windup design

An anti-windup compensator of the form (17) was designed using Proposition 1, to try to recover the behaviour of the un-saturated system as far as possible. However, instead of solving the LMI (41), the following LMI (61) was solved:

$$\begin{bmatrix} -Q & Y' & 0 & QC' & QA'+Y'B' \\ \star & -2U & I & 0 & UB' \\ \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -Q \end{bmatrix} < 0 \quad (61)$$

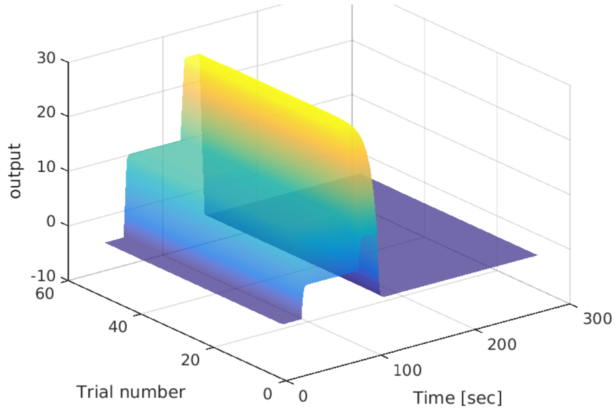

Fig. 4. Output $y_k(p)$ response over 50 trials, with saturated control input and anti-windup

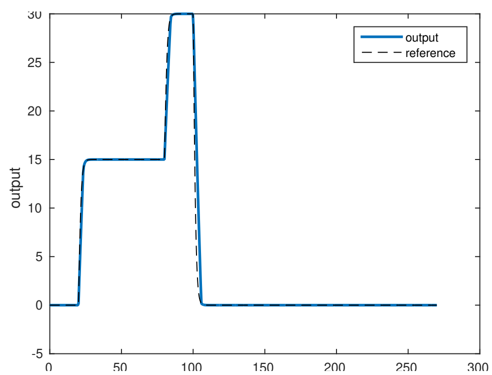

Fig. 5. Output $y_k(p)$ response at trial 50 with saturation and anti-windup

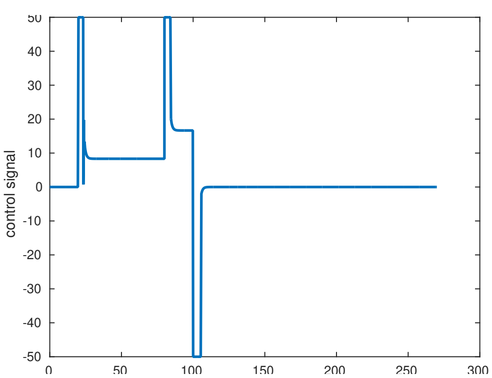

Fig. 6. Control $u_k(p)$ response at trial 50 with saturation and anti-windup

in the variables Q, Y, U and scalar $\gamma > 0$. This LMI guarantees that (41) is satisfied, but also (in 1D systems) guarantees an $\mathscr{L}_2$ gain bound which measures the deviation from linear behaviour. This seems to be useful in the ILC context too.

Figure 4 shows the the system output over 50 trials: the system response gradually converges to the reference as the trials progress. Again, for clarity, Figures 5 and 6 show the time histories of the output and control signal at trial 50: the addition of anti-windup improves both the transient response in the trials, and also the convergence of the error over trials.

V. CONCLUSION

This paper has proposed an anti-windup algorithm for ILC systems. The anti-windup compensator has a similar form to 1D systems and is able to ensure that the error between the reference and the output converges as $k, p \to \infty$, despite the presence of control signal constraints. Currently, the technique is restricted to stable systems since these systems are globally null controllable. Extensions to unstable systems are envisaged but this will require careful translation of local stability results [15] to 2D systems.

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