

Truth Set Algebra: A New Way to Prove Undefinability

Sophia Knight
 Department of Computer Science
 University of Minnesota Duluth
 the United States

Pavel Naumov, Qi Shi, and Viganan Suntharraj
 Electronics and Computer Science
 University of Southampton
 the United Kingdom

July 4, 2023

Abstract

The article proposes a new technique for proving the undefinability of logical connectives through each other and illustrates the technique with several examples. Some of the obtained results are new proofs of the existing theorems, others are original to this work.

1 Introduction

Studying the definability (expressibility) of logical connectives in terms of one another has a long history in logic. Proving the definability of one connective through another is usually done by providing an explicit formula that expresses one connective through others. Once such a formula is found, proving definability is usually a straightforward exercise. Proving *undefinability* is significantly harder and usually requires sophisticated techniques. Different domain-specific techniques have been proposed for various logical systems. Among them, the best-known is the *bisimulation* method for modal logics [19, 1, 2, 5, 15, 18, 4, 17, 16]. It is not clear how bisimulation can be applied to non-modal logics where completely different methods have been proposed [13, 20]. In addition, even for modal logics, some proofs of undefinability use non-bisimulation methods [12, 9].

In this article, we propose a new technique for proving the undefinability of logical connectives which is applicable to a wide range of settings. The technique consists in defining the “truth set” of a formula and studying the patterns of these truth sets obtainable through the given connectives. The exact definition of “truth set” varies depending on the logical system. For example, in the context of definability of Boolean connectives through each other, the truth set is defined as a set of *valuations* that satisfy a given formula. In the context of modal logics, the truth set is the set of *worlds* of a fixed given Kripke model in which the formula is true. In the context of three-valued logics, the “truth set” is a *fuzzy* set of valuations.

We illustrate this technique on the examples from Boolean, three-valued, intuitionistic, and temporal logics. We have chosen these specific examples to make the presentation accessible to a broader logical audience: we assume that most logicians are familiar with these logical systems.

We use the Boolean logic example to introduce the basic idea behind our technique. We are not aware of any published work containing the undefinability result in that example,

but it is so simple that we assume that somebody has observed it before. Our temporal logic and intuitionistic logic examples reprove known results using the newly proposed technique. We discuss the related literature after we present these results. Our 3-valued logic results are original to this article.

2 Classical Propositional Logic

This section illustrates our technique using a simple undefinability result in propositional logic. In the rest of the article, we assume a fixed nonempty set of propositional variables. Consider language Φ_1 defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi,$$

where p is a propositional variable. As usual, we assume that constant \top is defined as $p \rightarrow p$ for some propositional variable p and constant \perp is defined as $\neg\top$. There are many well-known definability results in propositional logic:

$$\begin{aligned} \varphi \wedge \psi &\equiv \neg(\neg\varphi \vee \neg\psi), \\ \varphi \wedge \psi &\equiv \neg(\varphi \rightarrow \neg\psi), \\ \varphi \vee \psi &\equiv \neg(\neg\varphi \wedge \neg\psi), \\ \varphi \vee \psi &\equiv \neg\varphi \rightarrow \psi, \\ \varphi \rightarrow \psi &\equiv \neg\varphi \vee \psi, \\ \varphi \rightarrow \psi &\equiv \neg(\varphi \wedge \neg\psi). \end{aligned}$$

However, it is perhaps less known that disjunction can be defined through implication alone without the negation:

$$\varphi \vee \psi \equiv (\varphi \rightarrow \psi) \rightarrow \psi.$$

The last fact and the well-known symmetry between disjunction and conjunction in propositional logic naturally lead to the question of whether conjunction can be defined solely through implication. Perhaps surprisingly, the answer is negative and we prove this as our first example.

Before formally stating the result, we introduce several auxiliary notions. First, a valuation is an arbitrary assignment of Boolean values to propositional variables. Second, for any formula $\varphi \in \Phi_1$, by $\llbracket\varphi\rrbracket$ we denote the set of all valuations that satisfy formula φ . We refer to set $\llbracket\varphi\rrbracket$ as the “truth set” of formula $\varphi \in \Phi_1$. Finally, we define the semantic equivalence of formulae:

Definition 1 *Propositional formulae $\varphi, \psi \in \Phi_1$ are semantically equivalent if $\llbracket\varphi\rrbracket = \llbracket\psi\rrbracket$.*

Next is our first undefinability result.

Theorem 1 (undefinability) *The formula $p \wedge q$ is not semantically equivalent to any formula in language Φ_1 containing only connectives \vee and \rightarrow .*

Because the formula $p \wedge q$ contains only propositional variables p and q , without loss of generality, we can assume the language Φ_1 contains only propositional variables p and q . As a first step towards the proof, we introduce a way to visualise the truth set of any formula in language Φ_1 using “diagrams”. As an example, the diagram for the truth set $\llbracket p \wedge q \rrbracket$ is depicted in Figure 1. In general, a diagram is a 2×2 table whose cells represent valuations (mappings of the set $\{p, q\}$ into Boolean values). In the diagram, the cells representing elements of the given truth set are shaded grey. In other words, each element of the truth set of formula φ represents a valuation under which formula φ is true. As another example, the diagrams at the top of Figure 2 depict the truth sets $\llbracket p \rrbracket$, $\llbracket q \rrbracket$, $\llbracket p \vee q \rrbracket$, $\llbracket p \rightarrow q \rrbracket$, $\llbracket q \rightarrow p \rrbracket$, and $\llbracket \top \rrbracket$.

The next lemma is the key step in our technique.

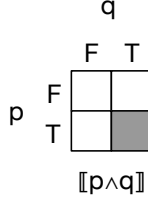


Figure 1: Truth set diagram.

Lemma 1 $\llbracket \varphi \rightarrow \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket p \vee q \rrbracket, \llbracket p \rightarrow q \rrbracket, \llbracket q \rightarrow p \rrbracket, \llbracket \top \rrbracket\}$ for any formulae $\varphi, \psi \in \Phi_1$ such that $\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket p \vee q \rrbracket, \llbracket p \rightarrow q \rrbracket, \llbracket q \rightarrow p \rrbracket, \llbracket \top \rrbracket\}$.

The lemma is proven by considering $6 \times 6 = 36$ different cases corresponding to different combinations of possible values of sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$. We show all these cases in Figure 2. For example, if $\llbracket \varphi \rrbracket = \llbracket p \rightarrow q \rrbracket$ and $\llbracket \psi \rrbracket = \llbracket q \rrbracket$, then $\llbracket \varphi \rightarrow \psi \rrbracket = \llbracket p \vee q \rrbracket$. We show this in Figure 2 by placing the diagram of the set $\llbracket p \vee q \rrbracket$ in the cell located at the intersection of the row labelled with the diagram $\llbracket p \rightarrow q \rrbracket$ and the column labelled with the diagram $\llbracket q \rrbracket$.

Lemma 2 $\llbracket \varphi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket p \vee q \rrbracket, \llbracket p \rightarrow q \rrbracket, \llbracket q \rightarrow p \rrbracket, \llbracket \top \rrbracket\}$ for any formula $\varphi \in \Phi_1$ that uses only connective \rightarrow .

PROOF. The lemma is proven by induction on the structural complexity of formula φ . The base case is true because truth sets $\llbracket p \rrbracket$ and $\llbracket q \rrbracket$ belong to the family of truth sets $\{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket p \vee q \rrbracket, \llbracket p \rightarrow q \rrbracket, \llbracket q \rightarrow p \rrbracket, \llbracket \top \rrbracket\}$. The induction step follows from Lemma 1. \square

Lemma 3 $\llbracket p \wedge q \rrbracket \notin \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket p \vee q \rrbracket, \llbracket p \rightarrow q \rrbracket, \llbracket q \rightarrow p \rrbracket, \llbracket \top \rrbracket\}$.

PROOF. See Figure 1 and the top row in Figure 2. \square The statement of Theorem 1 follows from Lemma 2, Lemma 3, and Definition 1.

3 Temporal Logic

In this section, we show how the truth set algebra technique can be used to prove the undefinability of one modality through another. To do this, we use several modalities from linear temporal logic. We assume that time is discrete, starts at moment 0, and runs ad infinitum. We denote the set of nonnegative integers by \mathbb{N} . In the context of temporal logic, a valuation is any function π that maps propositional variables into subsets of \mathbb{N} .

The language Φ_2 of temporal logic is defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \psi \mid F\varphi \mid X\varphi \mid \varphi U\psi \mid \varphi W\psi,$$

where p is either of the two propositional variables. We read F as ‘‘at some point in the future’’, X as ‘‘at the next moment’’, U as ‘‘until’’, and W as ‘‘weak until’’. The formal semantics of these modalities is defined below.

Definition 2 For any fixed valuation π , any integer $n \in \mathbb{N}$, and any formula $\varphi \in \Phi_2$, the satisfaction relation $n \Vdash \varphi$ is defined recursively as follows:

1. $n \Vdash p$ if $n \in \pi(p)$,
2. $n \Vdash \neg\varphi$ if $n \not\Vdash \varphi$,
3. $n \Vdash \varphi \vee \psi$ if either $n \Vdash \varphi$ or $n \Vdash \psi$,

$\begin{array}{c} \diagdown \\ \llbracket \psi \rrbracket \\ \diagup \\ \llbracket \varphi \rrbracket \end{array}$	$\llbracket p \rrbracket$	$\llbracket q \rrbracket$	$\llbracket p \vee q \rrbracket$	$\llbracket p \rightarrow q \rrbracket$	$\llbracket q \rightarrow p \rrbracket$	$\llbracket \top \rrbracket$
$\llbracket p \rrbracket$						
$\llbracket q \rrbracket$						
$\llbracket p \vee q \rrbracket$						
$\llbracket p \rightarrow q \rrbracket$						
$\llbracket q \rightarrow p \rrbracket$						
$\llbracket \top \rrbracket$						

Figure 2: Truth set $\llbracket \varphi \rightarrow \psi \rrbracket$ for different combinations of truth sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$.

4. $n \Vdash F\varphi$ if there is $m \geq n$ such that $m \Vdash \varphi$,
5. $n \Vdash X\varphi$ if $n + 1 \Vdash \varphi$,
6. $n \Vdash \varphi U \psi$ when there is $m \geq n$ such that $m \Vdash \psi$ and for each i , if $n \leq i < m$, then $i \Vdash \varphi$,
7. $n \Vdash \varphi W \psi$, when for each $m \geq n$ such that $m \not\Vdash \varphi$, there is $m' \geq n$ such that $m' \Vdash \psi$ and for each i , if $n \leq i < m'$, then $i \Vdash \varphi$.

Note that item 4 of the above definition contains inequality $m \leq n$ rather than $m < n$. Thus, informally, in our system “the future” includes the current moment. We believe that this is a common approach in temporal logic, but this choice is not significant for our results.

Definition 3 In the context of temporal logic, for any given valuation π , let the truth set $\llbracket \varphi \rrbracket$ of a formula $\varphi \in \Phi_2$ be the set $\{n \in \mathbb{N} \mid n \Vdash \varphi\}$.

Definition 4 In the context of temporal logic, formulae $\varphi, \psi \in \Phi_2$ are semantically equivalent if $\llbracket \varphi \rrbracket = \llbracket \psi \rrbracket$ for each valuation π .

3.1 Undefinability of U and W through F

In this subsection, we use the truth set algebra method to show that both versions of “until” modalities, regular U and weak W, are not definable through modality F and Boolean connectives. Without loss of generality, we assume that our language contains only propositional variables p and q . To start the proof, consider valuation π defined as follows:

$$\begin{aligned}\pi(p) &= \{n \geq 0 \mid n \equiv 1 \pmod{2}\}, \\ \pi(q) &= \{n \geq 0 \mid n \equiv 0 \pmod{4}\}.\end{aligned}$$

We visualise the truth sets of temporal formulae by drawing a one-way infinite linear sequence of cells and shading grey the cells whose position index belongs to the truth set (the left-most position corresponds to moment 0). The linear sequences in Figure 3 labelled with $\llbracket p \rrbracket$ and $\llbracket q \rrbracket$ visualise the corresponding truth sets. It is easy to verify that the other sequences also visualise the truth sets with which they are labelled.

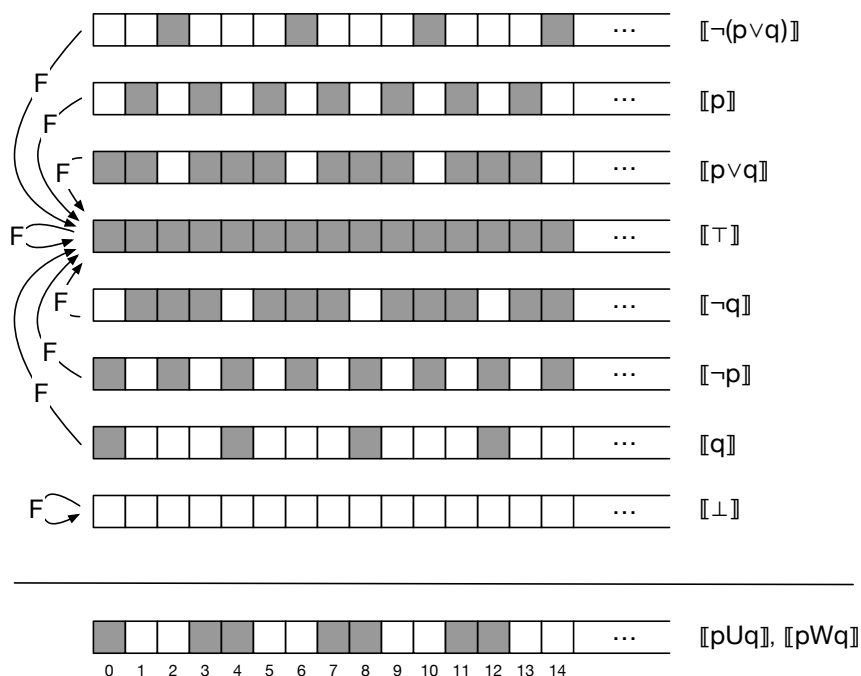


Figure 3: Visualisation of nine truth sets.

The next lemma shows that the set of eight truth sets depicted *above the horizontal bar* in Figure 3 is closed with respect to modality F.

Lemma 4 $\llbracket F\varphi \rrbracket \in \{\llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$ for any temporal formula $\varphi \in \Phi_2$ such that $\llbracket \varphi \rrbracket \in \{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}$.

PROOF. If $\llbracket \varphi \rrbracket \in \{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket\}$, then statement $n \Vdash \varphi$ holds for infinitely many values of n , see Figure 3. Thus, $n \Vdash F\varphi$ for each natural number n by item 4 of Definition 2. Therefore, $\llbracket F\varphi \rrbracket = \llbracket \top \rrbracket$ by Definition 3.

If $\llbracket \varphi \rrbracket = \llbracket \perp \rrbracket$, then $n \not\Vdash \varphi$ for each integer $n \geq 0$. Hence, $n \not\Vdash F\varphi$ for each n by item 4 of Definition 2. Therefore, $\llbracket F\varphi \rrbracket = \llbracket \perp \rrbracket$ by Definition 3. \square

Lemma 5 $\llbracket \varphi \rrbracket \in \{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}$ for any temporal formula $\varphi \in \Phi_2$ that does not contain modalities X, U, and W.

PROOF. We prove the lemma by induction on the structural complexity of formula φ . For the base case, note that

$$\llbracket p \rrbracket, \llbracket q \rrbracket \in \{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}.$$

Suppose formula φ has the form $\neg\psi$. By item 2 of Definition 2 and Definition 3, the truth set $\llbracket \neg\psi \rrbracket$ is the complement of the truth set $\llbracket \psi \rrbracket$. Note that the complement of each set in the family $\{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}$ also belongs to the same family. This can be observed in Figure 3. For example, the complement of the set $\llbracket \neg(p \vee q) \rrbracket$ is the set $\llbracket p \vee q \rrbracket$. Therefore, set $\llbracket \neg\psi \rrbracket$ belongs to the family $\{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}$ by the induction hypothesis.

Assume that formula φ has the form $\psi_1 \vee \psi_2$. By item 3 of Definition 2 and Definition 3, the truth set $\llbracket \psi_1 \vee \psi_2 \rrbracket$ is the union of the truth sets $\llbracket \psi_1 \rrbracket$ and $\llbracket \psi_2 \rrbracket$. Note that the family $\{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}$ is closed with respect to union. This can also be observed in Figure 3. For example, the union of the sets $\llbracket \neg(p \vee q) \rrbracket$ and $\llbracket p \rrbracket$ is the set $\llbracket \neg q \rrbracket$. Therefore, set $\llbracket \psi_1 \vee \psi_2 \rrbracket$ belongs to the family $\{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}$ by the induction hypothesis.

If formula φ has the form $F\psi$, then the statement of the lemma follows from Lemma 4 and the induction hypothesis. \square

Lemma 6 $\llbracket pUq \rrbracket, \llbracket pWq \rrbracket \notin \{\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket\}.$

PROOF. The truth sets $\llbracket pUq \rrbracket$ and $\llbracket pWq \rrbracket$ are equal and they are visualised below the horizontal bar in Figure 3. The correctness of the visualisation follows from item 6 and item 7 of Definition 2. Observe that these sets are different from the sets $\llbracket \neg(p \vee q) \rrbracket, \llbracket p \rrbracket, \llbracket p \vee q \rrbracket, \llbracket \top \rrbracket, \llbracket \neg q \rrbracket, \llbracket \neg p \rrbracket, \llbracket q \rrbracket, \llbracket \perp \rrbracket$ visualised above the horizontal bar on the same diagram. \square

The next result follows from Definition 4 and the two lemmas above. A similar result for branching time logic is shown in [12] using a different technique. Other undefinability results for a temporal logic are given in [9].

Theorem 2 (undefinability) *Neither the formula pUq nor the formula pWq is semantically equivalent to a formula in language Φ_2 that does not contain modalities X, U, and W.*

3.2 Undefinability of F through X

In this subsection, we use a modified version of the truth set algebra method to show that modality F is not definable through modality X and Boolean connectives. Without loss of generality, in this subsection, we assume that our language contains only propositional variable p .

In Subsection 3.1, we have shown that a certain pattern can *never* be reached by applying only modality F and Boolean connectives. Here we show that a certain pattern cannot be reached *in a fixed number of steps* and use this observation to prove the undefinability.

We state and prove the undefinability result as Theorem 3 at the end of this subsection. Throughout this subsection, until the statement of that theorem, we assume that $T \geq 1$ is an arbitrary fixed positive integer. We specify the value of T in the proof of Theorem 3. Consider a valuations π defined as follows:

$$\pi(p) = \{T\}. \tag{1}$$

We visualise the truth sets in the same way as we did in the previous subsection. In Figure 4, the top linear sequence visualises the truth set $\llbracket p \rrbracket$.

For each integer t such that $1 \leq t \leq T$, we consider families of sets α_t and β_t defined as

$$\begin{aligned} \alpha_t &= \{X \mid X \subseteq \{t, \dots, T\}\}, \\ \beta_t &= \{\{0, \dots, t-1\} \cup X \cup \{T+1, \dots\} \mid X \subseteq \{t, \dots, T\}\}. \end{aligned}$$

In other words, α_t is the powerset of the set $\{t, \dots, T\}$ and β_t is the set of the complements of sets in α_t with respect to \mathbb{N} . We visualise families α_t and β_t in the middle of Figure 4. The asterisk $*$ is used as the “wildcard” to mark the integers that *may but do not have to* belong to a set in the corresponding family. It is easily seen that for any integer $t \geq 2$,

$$\alpha_t \subsetneq \alpha_{t-1} \quad \text{and} \quad \beta_t \subsetneq \beta_{t-1}. \quad (2)$$

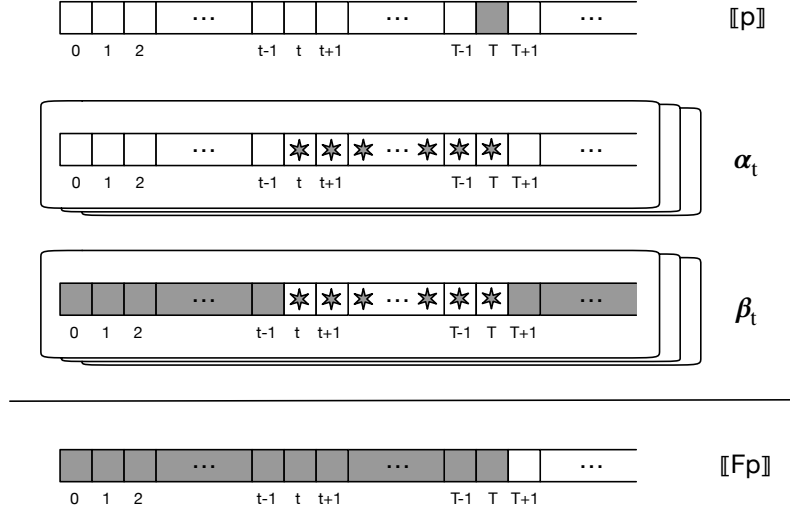


Figure 4: Visualisation of the truth sets for valuation π^T .

Lemma 7 For any formulae $\varphi, \psi \in \Phi_2$ and any $t \geq 1$, if $\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket \in \alpha_t \cup \beta_t$, then $\llbracket \varphi \vee \psi \rrbracket, \llbracket \neg \varphi \rrbracket \in \alpha_t \cup \beta_t$.

PROOF. Observe from Figure 4 that the family of sets $\alpha_t \cup \beta_t$ is closed with respect to union and complement. Then, the statement of the lemma follows from item 2 and item 3 of Definition 2. \square

Lemma 8 For any formulae $\varphi, \psi \in \Phi_2$ and any $t \geq 1$, if $\llbracket \varphi \rrbracket \in \alpha_t \cup \beta_t$, then $\llbracket X\varphi \rrbracket \in \alpha_{t-1} \cup \beta_{t-1}$.

PROOF. By item 5 of Definition 2, $\llbracket X\varphi \rrbracket = \{i \mid i \in \mathbb{N}, i+1 \in \llbracket \varphi \rrbracket\}$. Thus, if $\llbracket \varphi \rrbracket \in \alpha_t$, then $\llbracket X\varphi \rrbracket \in \alpha_{t-1}$; if $\llbracket \varphi \rrbracket \in \beta_t$, then $\llbracket X\varphi \rrbracket \in \beta_{t-1}$. Therefore, when $\llbracket \varphi \rrbracket \in \alpha_t \cup \beta_t$, $\llbracket X\varphi \rrbracket \in \alpha_{t-1} \cup \beta_{t-1}$. \square

Lemma 9 For any integer $k \leq T$ and any formula $\varphi \in \Phi_2$ that contains only modality X and Boolean connectives, if formula φ contains at most k occurrences of modality X , then $\llbracket \varphi \rrbracket \in \alpha_{T-k} \cup \beta_{T-k}$.

PROOF. We prove the statement of this lemma by structural induction on formula φ . If φ is a propositional variable p , then

$$\llbracket \varphi \rrbracket = \llbracket p \rrbracket = \pi(p) = \{T\} \in \alpha_T \subseteq \alpha_{T-k}$$

by item 1 of Definition 2, statement (1), and statement (2).

If formula φ is a disjunction or a negation, then the statement of this lemma follows from the induction hypothesis by Lemma 7.

If formula φ has the form $X\psi$, then formula ψ contains at most $k-1$ occurrences of modality X . The statement of this lemma follows from Lemma 8. \square

Lemma 10 *If $T \geq 1$, then $0 \in \llbracket \mathbf{F}p \rrbracket$ and $T + 1 \notin \llbracket \mathbf{F}p \rrbracket$.*

PROOF. Since $\llbracket p \rrbracket = \pi(p) = \{T\}$, by item 4 of Definition 2. Then, $\llbracket \mathbf{F}p \rrbracket = \{0, \dots, T\}$, see the bottom linear sequence in Figure 4. Therefore, $0 \in \llbracket \mathbf{F}p \rrbracket$ and $T + 1 \notin \llbracket \mathbf{F}p \rrbracket$. \square

The next theorem shows that modality \mathbf{F} is not definable through modality \mathbf{X} and Boolean connectives.

Theorem 3 (undefinability) *The formula $\mathbf{F}p$ is not semantically equivalent to any formula in language Φ_2 that does not contain modalities \mathbf{F} , \mathbf{U} , and \mathbf{W} .*

PROOF. Assume there is a formula $\varphi \in \Phi_2$ that contains only modality \mathbf{X} and Boolean connectives which is semantically equivalent to $\mathbf{F}p$. Suppose k to be the number of occurrences of modality \mathbf{X} in formula φ . Let $T = k + 1$. Then, $\llbracket \varphi \rrbracket \in \alpha_{k+1-k} \cup \beta_{k+1-k} = \alpha_1 \cup \beta_1$ by Lemma 9. However, $\llbracket \mathbf{F}p \rrbracket \notin \alpha_1 \cup \beta_1$ by Lemma 10. Therefore, $\llbracket \mathbf{F}p \rrbracket \neq \llbracket \varphi \rrbracket$, which contradicts the assumption that formulae $\mathbf{F}p$ and φ are semantically equivalent by Definition 4. \square

4 Intuitionistic Logic

In this section, we illustrate the truth set algebra method by proving the mutual undefinability of connectives in Heyting [6] calculus for intuitionistic logic. These results¹ were independently obtained by McKinsey [13] and Wajsberg [20] in 1939. Note that there were no Kripke semantics [8] for intuitionistic logic at the time [13, 20] were written. Our proof of definability uses Kripke models and, thus, is also significantly different from the original proofs in [13, 20].

We start by recalling the standard Kripke semantics for intuitionistic logic [14]. As usual, by “partial order” we mean a reflexive, transitive, and antisymmetric binary relation.

Definition 5 *An intuitionistic Kripke model is a tuple (W, \preceq, π) , where*

1. W is a (possibly empty) set of “worlds”,
2. \preceq is a partial order on set W ,
3. for each propositional variable p , valuation $\pi(p) \subseteq W$ is a set of worlds such that for any worlds $w, u \in W$, if $w \in \pi(p)$ and $w \preceq u$, then $u \in \pi(p)$.

In this section, we use the same language Φ_1 as defined in Section 2.

Definition 6 *For any world $w \in W$ of a Kripke model (W, \preceq, π) and any formula $\varphi \in \Phi_1$, the satisfaction relation $w \Vdash \varphi$ is defined as follows:*

1. $w \Vdash p$, if $w \in \pi(p)$,
2. $w \Vdash \neg\varphi$, if there is no world $u \in W$ such that $w \preceq u$ and $u \Vdash \varphi$,
3. $w \Vdash \varphi \wedge \psi$, if $w \Vdash \varphi$ and $w \Vdash \psi$,
4. $w \Vdash \varphi \vee \psi$, if either $w \Vdash \varphi$ or $w \Vdash \psi$,
5. $w \Vdash \varphi \rightarrow \psi$, when for each world $u \in W$ if $w \preceq u$ and $u \Vdash \varphi$, then $u \Vdash \psi$.

Note that item 3 of Definition 5 and items 2 and 5 of Definition 6 capture the intuitionistic nature of this semantics.

Definition 7 *For any given intuitionistic Kripke model (W, \preceq, π) , the truth set $\llbracket \varphi \rrbracket$ of an arbitrary formula $\varphi \in \Phi_1$ is the set $\{w \in W \mid w \Vdash \varphi\}$.*

Definition 8 *In the context of intuitionistic logic, formulae $\varphi, \psi \in \Phi_1$ are semantically equivalent if $\llbracket \varphi \rrbracket = \llbracket \psi \rrbracket$ for each intuitionistic Kripke model.*

¹McKinsey [13] and Wajsberg [20] talk about definability in terms of *provable* equivalence not *semantical* equivalence that we use in this article. The provable equivalence is equal to semantical equivalence due to the completeness theorem for intuitionistic logic proven by Kripke [8] in 1965.

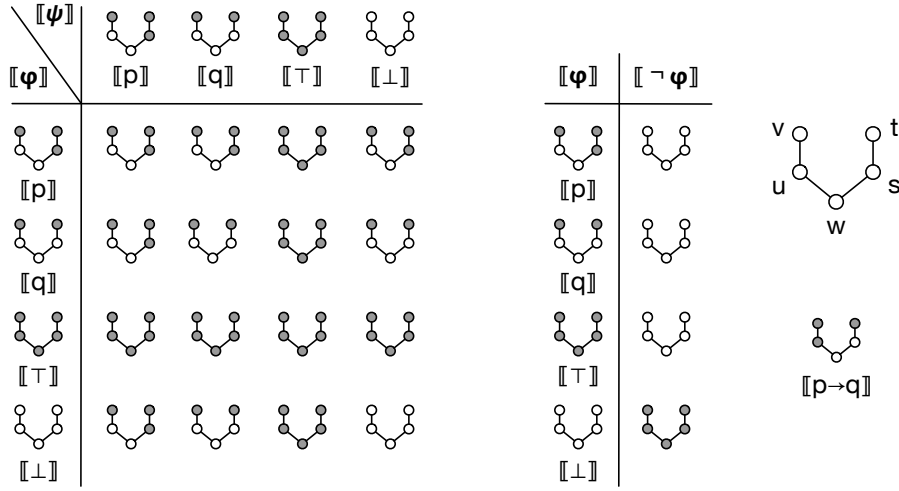


Figure 5: Truth set $\llbracket \varphi \vee \psi \rrbracket$ for different combinations of truth sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$ (left). Truth set $\llbracket \neg \varphi \rrbracket$ for different truth sets $\llbracket \varphi \rrbracket$ (centre). Hasse diagram for a Kripke model and the truth set $\llbracket p \rightarrow q \rrbracket$ (right).

4.1 Undefinability of \rightarrow through \neg , \wedge , and \vee

In this subsection, we use our truth set algebra method to prove that implication \rightarrow is not definable in intuitionistic logic through negation \neg , conjunction \wedge , and disjunction \vee . Without loss of generality, assume that language Φ_1 contains only propositional variables p and q . Let us consider the Kripke model whose Hasse diagram is depicted in the upper-right corner of Figure 5. It contains five worlds, w , u , v , s , and t . The partial order \preceq on these worlds is given by the diagram. For example, $w \preceq v$ because the diagram contains an upward path from w to v . We assume that $\pi(p) = \{v, s, t\}$ and $\pi(q) = \{v, t\}$.

Recall that we define constant \top as $p \rightarrow p$ and constant \perp as $\neg \top$. We visualise the truth set of a formula in language Φ_1 by shading the worlds that belong to the set. For example, the rows and the columns in the left-most table in Figure 5 are labelled by the diagrams visualising the truth sets $\llbracket p \rrbracket$, $\llbracket q \rrbracket$, $\llbracket \top \rrbracket$, and $\llbracket \perp \rrbracket$.

Lemma 11 *For any formulae $\varphi, \psi \in \Phi_1$, if $\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$, then $\llbracket \varphi \vee \psi \rrbracket, \llbracket \varphi \wedge \psi \rrbracket, \llbracket \neg \varphi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$.*

PROOF. Let us first prove that $\llbracket \varphi \vee \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$ if $\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$. We do this in the left table depicted in the left of Figure 5. The proof consists of explicitly constructing the truth set $\llbracket \varphi \vee \psi \rrbracket$ for each possible combination of sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$.

Alternatively, one can also see, by Definition 6 and Definition 7, that $\llbracket \varphi \vee \psi \rrbracket = \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket$. Then, $\llbracket \varphi \vee \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$ because the family of truth sets $\{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$ is closed with respect to union.

The proof for the truth set $\llbracket \varphi \wedge \psi \rrbracket$ is similar: either by building a table or observing that $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$ and that the family of truth sets $\{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$ is closed with respect to intersection.

Finally, for the truth set $\llbracket \neg \varphi \rrbracket$, see the middle table in Figure 5. It shows the truth set $\llbracket \neg \varphi \rrbracket$ for each formula φ such that $\llbracket \varphi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$. The validity of this table can be verified using item 2 of Definition 6. \square

Lemma 12 $\llbracket \varphi \rrbracket \in \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$ for any formula $\varphi \in \Phi_1$ that does not use implication.

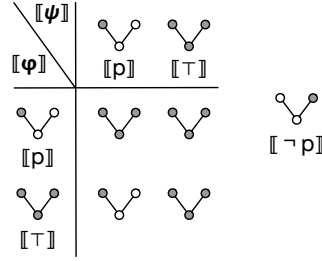


Figure 6: Truth set $\llbracket \varphi \rightarrow \psi \rrbracket$ for different combinations of truth sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$ (left). Truth set $\llbracket \neg p \rrbracket$ (right).

PROOF. We prove the statement of the lemma by induction on the structural complexity of formula φ . In the base case, the statement of the lemma is true because the truth sets $\llbracket p \rrbracket$ and $\llbracket q \rrbracket$ are elements of the family $\{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$.

In the induction case, the statement of the lemma follows from Lemma 11 and the induction hypothesis. \square

Lemma 13 $\llbracket p \rightarrow q \rrbracket \notin \{\llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket \top \rrbracket, \llbracket \perp \rrbracket\}$.

PROOF. We visualise the truth set $\llbracket p \rightarrow q \rrbracket$ on the right of Figure 5. The validity of this visualisation can be verified using item 5 of Definition 6. \square

The next theorem follows from the two lemmas above.

Theorem 4 (undefinability) *Formula $p \rightarrow q$ is not semantically equivalent to any formula in language Φ_1 that does not use implication.*

4.2 Undefinability of \neg through \wedge , \vee , and \rightarrow

In this subsection, we show that, in intuitionistic logic, negation is not definable through conjunction, disjunction, and implication. Because negation is a unary connective, in this section, without loss of generality, we assume that language Φ_1 contains a single propositional variable p .

The proof follows the same pattern as the one in the previous subsection, but it uses a simpler Kripke model. In this case, the Hasse diagram of the model is a tree consisting of a root node and two child nodes: the left child and the right child. Set $\pi(p)$ contains only the left child node. In Figure 6, we show the truth sets $\llbracket p \rrbracket$, $\llbracket \top \rrbracket$, and $\llbracket \neg p \rrbracket$ for this model.

Lemma 14 *For any two formulae $\varphi, \psi \in \Phi_1$, if $\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket \top \rrbracket\}$, then $\llbracket \varphi \vee \psi \rrbracket, \llbracket \varphi \wedge \psi \rrbracket, \llbracket \varphi \rightarrow \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket \top \rrbracket\}$.*

PROOF. Suppose that $\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket \top \rrbracket\}$. Then, $\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket, \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket \top \rrbracket\}$, see visualisation of the truth sets $\llbracket p \rrbracket$ and $\llbracket \top \rrbracket$ in Figure 6. Hence, $\llbracket \varphi \vee \psi \rrbracket, \llbracket \varphi \wedge \psi \rrbracket \in \{\llbracket p \rrbracket, \llbracket \top \rrbracket\}$ by items 3 and 4 of Definition 6 and Definition 7.

On the left of Figure 6, we visualise the truth set $\llbracket \varphi \rightarrow \psi \rrbracket$ as a function of the truth sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$. The validity of this table can be verified using item 5 of Definition 6 and Definition 7. \square

The proof of the next lemma is similar to the proof of Lemma 12, but instead of Lemma 11 it uses Lemma 14.

Lemma 15 $\llbracket \varphi \rrbracket \in \{\llbracket p \rrbracket, \llbracket \top \rrbracket\}$ for any formula $\varphi \in \Phi_1$ that does not use negation.

Lemma 16 $\llbracket \neg p \rrbracket \notin \{\llbracket p \rrbracket, \llbracket \top \rrbracket\}$.

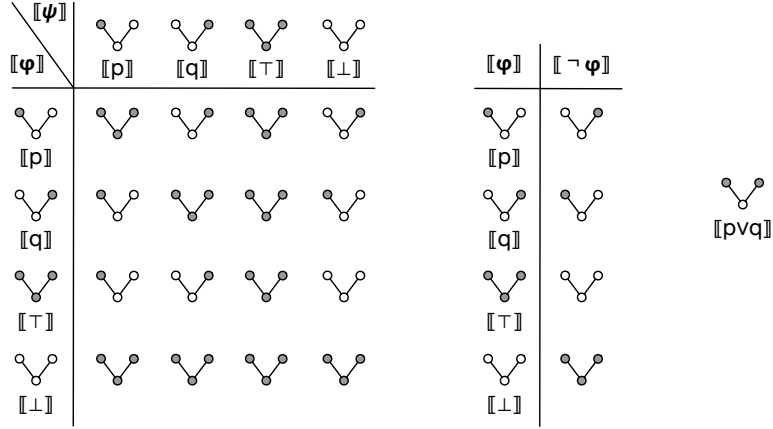


Figure 7: Truth set $\llbracket \varphi \rightarrow \psi \rrbracket$ for different combinations of truth sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$ (left). Truth set $\llbracket \neg \varphi \rrbracket$ for different truth sets $\llbracket \varphi \rrbracket$ (centre). Truth set $\llbracket p \vee q \rrbracket$ (right).

PROOF. We visualise the truth set $\llbracket \neg p \rrbracket$ on the right of Figure 6. The validity of this visualisation can be verified using item 2 of Definition 6. \square

The next theorem follows from the two lemmas above.

Theorem 5 (undefinability) *Formula $\neg p$ is not semantically equivalent to any formula in language Φ_1 that does not use negation.*

4.3 Undefinability of \vee through \neg , \wedge , and \rightarrow

The proof of the next theorem is similar to the proof of Theorem 5 except that it uses Figure 7 instead of Figure 6.

Theorem 6 (undefinability) *Formula $p \vee q$ is not semantically equivalent to any formula in language Φ_1 that does not use disjunction.*

4.4 Undefinability of \wedge through \neg , \vee , \rightarrow

The proof of the next theorem is similar to the proof of Theorem 5 except that it uses Figure 8 instead of Figure 6.

Theorem 7 (undefinability) *Formula $p \wedge q$ is not semantically equivalent to any formula in language Φ_1 that does not use conjunction.*

5 Three-Valued Logic

In this section, we apply our technique to investigate the definability of logical connectives in 3-valued logic. This logic contains three truth values: 0, $\frac{1}{2}$, and 1, often referred to as “false”, “unknown”, and “true”, respectively. The meanings of propositional connectives \wedge , \vee , and \neg in 3-valued logic are a straightforward generalisation of their meanings in Boolean logic: $p \wedge q = \min\{p, q\}$, $p \vee q = \max\{p, q\}$, and $\neg p = 1 - p$. Thus, for example, if the value of p is “unknown”, then the value of the expression $p \vee \neg p$ is also “unknown”. In this article, we visualise values “false”, “unknown”, and “true” as a white square, a diagonally crossed square, and a grey square, respectively. The first two diagrams in Figure 9 show truth tables for connectives \wedge and \vee . For example, in the left-most diagram, the crossed

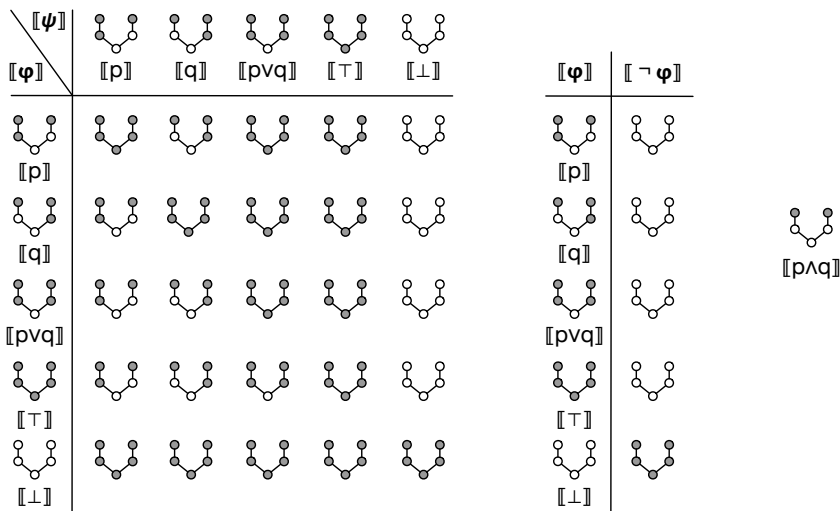


Figure 8: Truth set $\llbracket \varphi \rightarrow \psi \rrbracket$ for different combinations of truth sets $\llbracket \varphi \rrbracket$ and $\llbracket \psi \rrbracket$ (left). Truth set $\llbracket \neg \varphi \rrbracket$ for different truth sets $\llbracket \varphi \rrbracket$ (centre). Truth set $\llbracket p \wedge q \rrbracket$ (right).

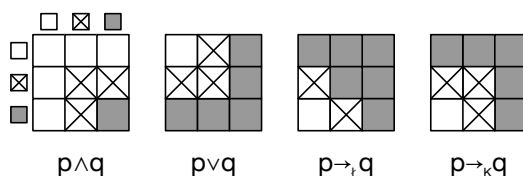


Figure 9: Truth tables for binary connectives in 3-valued logic.

cell in the middle of the last row represents the fact that if $p = 1$ (the third row) and $q = \frac{1}{2}$ (the second column), then $p \wedge q = \frac{1}{2}$.

Defining the meaning of implication in 3-valued logic is less straightforward. Two such definitions are suggested: one by Lukasiewicz [10, p.213] and the other by Kleene [7]. We denote their implications by \rightarrow_L and \rightarrow_K , respectively. The truth tables for these implications are shown in the two right-most diagrams in Figure 9. In this section, we study interdefinability of 3-valued connectives \neg , \wedge , \vee , \rightarrow_K , and \rightarrow_L .

By Φ_3 we denote the language defined by the following grammar:

$$\varphi := p \mid \neg \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow_L \varphi \mid \varphi \rightarrow_K \varphi,$$

where p is a propositional variable. Because each of the connectives has at most two arguments, for the purposes of proving undefinability, it suffices to assume that there are only two propositional variables, p and q .

5.1 Fuzzy Truth Sets

To apply the truth set algebra technique in the setting of 3-valued logic, we need to make one small modification to this technique. Namely, instead of regular truth sets, we consider *fuzzy truth sets* of formulae. In our case, a fuzzy set can have only three degrees of membership: an element can belong, half-belong, or not belong to a fuzzy set.

We consider operations union, intersection, and complement on fuzzy sets. We define the degree of membership in a *union* of two fuzzy sets as the *maximum* of the degrees of

membership in the two original fuzzy sets. For example, suppose fuzzy set X contains an apple and half-contains a banana. In addition, let fuzzy set Y half-contain a banana and contain a carrot. In that case, the union of fuzzy sets X and Y contains an apple, a carrot, and half-contains a banana.

Similarly, we define the degree of membership in an *intersection* of two fuzzy sets as the *minimum* of the degrees of membership in the two original fuzzy sets. In our example, the intersection of fuzzy sets X and Y half-contains a banana and nothing else.

Finally, consider any regular (not fuzzy) set U and any fuzzy set S of elements from set U . We define a complement of the fuzzy set S with respect to the universe U . The degree of the membership of an element in the complement is $1 - d$, where d is the degree of membership of the same element in the fuzzy set S . In our example, assuming that the universe consists of an apple, a banana, and a carrot, the complement of the fuzzy set X is the fuzzy set Y .

Recall our assumption that language Φ_3 contains only propositional variables p and q . For any formula $\varphi \in \Phi_3$ and any values $b_1, b_2 \in \{0, \frac{1}{2}, 1\}$, by $\varphi[b_1, b_2]$ we denote the value of the formula φ when p has value b_1 and q has value b_2 . We are now ready to define a fuzzy truth set.

Definition 9 *For any formula $\varphi \in \Phi_3$, the fuzzy truth set $\llbracket \varphi \rrbracket$ is a fuzzy set of all pairs $(b_1, b_2) \in \{0, \frac{1}{2}, 1\}^2$ such that*

1. (b_1, b_2) belongs to the fuzzy set $\llbracket \varphi \rrbracket$ if $\varphi[b_1, b_2] = 1$,
2. (b_1, b_2) half-belongs to the fuzzy set $\llbracket \varphi \rrbracket$ if $\varphi[b_1, b_2] = \frac{1}{2}$.

We visualise the fuzzy truth set $\llbracket \varphi \rrbracket$ of an arbitrary formula φ as a 3×3 table. A cell (b_1, b_2) is coloured white if the pair (b_1, b_2) does not belong to $\llbracket \varphi \rrbracket$, it is crossed if the pair (b_1, b_2) half-belongs to $\llbracket \varphi \rrbracket$, and it is coloured grey if the pair (b_1, b_2) belongs to $\llbracket \varphi \rrbracket$. For example, the four diagrams in Figure 9 visualise the fuzzy truth sets $\llbracket p \wedge q \rrbracket$, $\llbracket p \vee q \rrbracket$, $\llbracket p \rightarrow_L q \rrbracket$, and $\llbracket p \rightarrow_K q \rrbracket$.

Definition 10 *In the context of 3-valued logic, formulae $\varphi, \psi \in \Phi_3$ are semantically equivalent if $\llbracket \varphi \rrbracket = \llbracket \psi \rrbracket$.*

Next, we state and prove a very simple undefinability result about 3-valued logic that does not require the truth set algebra technique.

Theorem 8 *Formula $\neg p$ is not semantically equivalent to any formula containing only connectives $\wedge, \vee, \rightarrow_K$, and \rightarrow_L .*

PROOF. Observe that if all propositional variables are assigned value 1, then the value of any formula that contains only connectives $\wedge, \vee, \rightarrow_K$, and \rightarrow_L is 1, see Figure 9. At the same time, the value of $\neg p$ is 0. \square

5.2 Expressive Power of Kleene's Implication

In this subsection, we illustrate how the truth set algebra method can be used to prove undefinability results in 3-valued logic. Namely, we show a relatively simple observation that neither of the other connectives can be defined through Kleene's implication.

In the rest of this subsection, we use names A, \dots, R to refer to the 18 fuzzy truth sets depicted in Figure 10. Note that $P = \llbracket p \rrbracket$ and $Q = \llbracket q \rrbracket$. Let \mathcal{S} be the family $\{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R\}$ of these 18 fuzzy truth sets.

Lemma 17 *For any formulae $\varphi, \psi \in \Phi_3$, if $\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket \in \mathcal{S}$, then $\llbracket \varphi \rightarrow_K \psi \rrbracket \in \mathcal{S}$.*

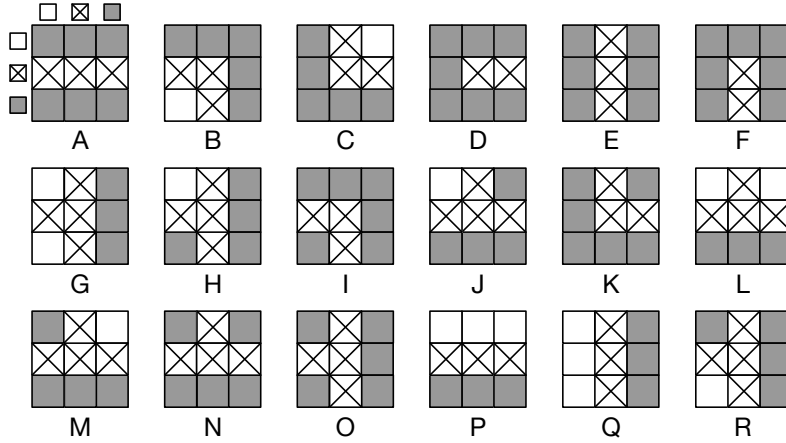


Figure 10: Towards the proof of Theorem 9.

PROOF. Consider first the case when $\llbracket \varphi \rrbracket = A$ and $\llbracket \psi \rrbracket = Q$. To compute the fuzzy truth set $\llbracket \varphi \rightarrow_{\kappa} \psi \rrbracket$, we compute the degree of membership for each pair (b_1, b_2) in this fuzzy set. Consider, for example, the case $b_1 = \frac{1}{2}$ and $b_2 = 0$ which is visualised as the middle-left cell in each diagram. Note that the middle-left cells in the diagrams of fuzzy sets A and Q are crossed and white, respectively, see Figure 10. Hence, pair (b_1, b_2) half-belongs to the fuzzy truth sets $\llbracket \varphi \rrbracket = A$ and does not belong to the fuzzy truth set $\llbracket \psi \rrbracket = Q$. Thus, by Definition 9, the values $\varphi[\frac{1}{2}, 0]$ and $\psi[\frac{1}{2}, 0]$ are $\frac{1}{2}$ and 0, respectively. Observe that the value of $\frac{1}{2} \rightarrow_{\kappa} 0$ is $\frac{1}{2}$, see the last diagram in Figure 9. Hence, $(\varphi \rightarrow_{\kappa} \psi)[\frac{1}{2}, 0] = \frac{1}{2}$. Then, by Definition 9, the pair (b_1, b_2) half-belongs to the fuzzy truth set $\llbracket \varphi \rightarrow_{\kappa} \psi \rrbracket$. Thus, the middle-left cell in the diagram visualising the fuzzy truth set $\llbracket \varphi \rightarrow_{\kappa} \psi \rrbracket$ is crossed. By repeating the same computation for each pair (b_1, b_2) , one can see that the fuzzy truth set $\llbracket \varphi \rightarrow_{\kappa} \psi \rrbracket$ is fuzzy set G , see Figure 10. We show this result by placing the letter G in row A , column Q of Table 1. Therefore, $\llbracket \varphi \rightarrow_{\kappa} \psi \rrbracket \in \mathcal{S}$.

The other cases are similar. We show the corresponding fuzzy sets $\llbracket \varphi \rightarrow_{\kappa} \psi \rrbracket$ in Table 1. The statement of the lemma holds because all sets in Table 1 belong to family \mathcal{S} . \square

Lemma 18 $\llbracket \varphi \rrbracket \in \mathcal{S}$ for any formula $\varphi \in \Phi_3$ that uses connective \rightarrow_{κ} only.

PROOF. We prove the statement of the lemma by induction on the structural complexity of formula φ . If φ is propositional variable p , then $\llbracket p \rrbracket = P \in \mathcal{S}$, see Figure 10. Similarly, if φ is propositional variable q , then $\llbracket q \rrbracket = Q \in \mathcal{S}$. If formula φ has the form $\varphi_1 \rightarrow_{\kappa} \varphi_2$, then the statement of the lemma follows from Lemma 17 and the induction hypothesis. \square

Theorem 9 (undefinability) Each of the formulae $p \wedge q$, $p \vee q$, and $p \rightarrow_{\perp} q$ is not 3-valued-equivalent to a formula that uses connective \rightarrow_{κ} only.

PROOF. The fuzzy truth sets $\llbracket p \wedge q \rrbracket$, $\llbracket p \vee q \rrbracket$, and $\llbracket p \rightarrow_{\perp} q \rrbracket$ are depicted in Figure 9. Note that none of them belongs to the family \mathcal{S} , see Figure 10. Thus, the statement of the theorem follows from Lemma 18 and Definition 10. \square

In the rest of this section, we present our main technical results about the connectives \neg , \wedge , \vee , \rightarrow_{κ} , and \rightarrow_{\perp} .

5.3 Undefinability of Conjunction

In this subsection, we focus on the definability of conjunction \wedge through the rest of the connectives. First, let us start with three definability facts. Each of them is easily verifiable

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	G	R
B	A	I	C	D	E	F	H	H	I	J	K	L	M	N	O	P	H	O
C	A	B	K	D	E	F	G	H	I	J	K	J	N	N	O	J	Q	R
D	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
E	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	L	Q	R
F	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
G	A	I	C	D	E	F	O	O	I	N	K	M	M	N	O	M	O	O
H	A	B	C	D	E	F	R	O	I	N	K	M	M	N	O	M	R	R
I	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	G	R
J	A	B	C	D	E	F	R	O	I	N	K	M	M	N	O	M	R	R
K	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	L	Q	R
L	A	B	K	D	E	F	R	O	I	N	K	N	N	N	O	N	R	R
M	A	B	K	D	E	F	G	H	I	J	K	J	N	N	O	J	G	R
N	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	L	G	R
O	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	L	G	R
P	A	B	D	D	F	F	B	I	I	A	D	A	A	A	I	A	B	B
Q	D	F	C	D	E	F	E	E	F	K	K	C	C	K	E	C	E	E
R	A	I	C	D	E	F	H	H	I	J	K	L	M	N	O	L	H	O

Table 1: The fuzzy truth set $\llbracket \varphi \rightarrow_K \psi \rrbracket$, where $\llbracket \varphi \rrbracket$ is the row label and $\llbracket \psi \rrbracket$ is the column label.

using Figure 9 and the definition of negation. In the theorem below and the rest of this section, by \equiv we denote 3-value-equivalence of formulae in language Φ_3 .

Theorem 10 *The following equivalences hold in 3-valued logic:*

1. $p \wedge q \equiv \neg(\neg p \vee \neg q)$,
2. $p \wedge q \equiv \neg(p \rightarrow_K \neg q)$,
3. $p \wedge q \equiv \neg(p \rightarrow_L \neg(p \rightarrow_L q))$.

The first two equivalences in the above theorem are well-known. We are not aware of the third equivalence being mentioned in the literature. It was discovered by our computer program while trying to prove the undefinability of \wedge through \neg and \rightarrow_K . All three equivalences could be easily verified using the definitions of the connectives.

Let us now discuss the undefinability results about the conjunction. Note that *binary* connective \wedge cannot be defined through *unary* connective \neg . If \neg is combined with any one of the remaining connectives, then \wedge becomes definable, see Theorem 10. To completely answer the question about the definability of conjunction, it suffices to show that it cannot be defined without the use of negation. We prove this in the next theorem.

Theorem 11 *Formula $p \wedge q$ is not 3-value-equivalent to any formula in language Φ_3 containing only connectives \vee , \rightarrow_K , and \rightarrow_L .*

The proof of the above theorem follows the same pattern as the proof of Theorem 9. However, instead of the 18 fuzzy truth sets depicted in Figure 10, it uses 176 fuzzy truth sets. The equivalent of Table 1 in the new proof is a table containing 176 rows and 176 columns. We used a computer program written in Python to find 176 diagrams like the ones in Figure 10. The same program also verifies, similarly to how we do in Table 1, that the set of 176 diagrams is closed with respect to the operations \vee , \rightarrow_K , and \rightarrow_L . Finally, it checks that this set does not contain the diagram for the fuzzy truth set $\llbracket p \wedge q \rrbracket$. The algorithm that

we used starts with fuzzy truth sets $\llbracket p \rrbracket$ and $\llbracket q \rrbracket$ and applies the operations \vee , \rightarrow_K , and \rightarrow_L until no new diagrams could be generated.

It is interesting to point out that the 18 diagrams depicted in Figure 10, as well as the L^AT_EX code for Table 1, are also generated by the same program.

5.4 Undefinability of Disjunction

In this subsection, we analyse the definability of disjunction through the rest of the connectives in 3-valued logic. Let us start with the following observation which can be verified using the definitions of the connectives.

Theorem 12 *The following equivalences hold in 3-valued logic:*

1. $p \vee q \equiv \neg(\neg p \wedge \neg q)$,
2. $p \vee q \equiv \neg p \rightarrow_K q$,
3. $p \vee q \equiv (p \rightarrow_L q) \rightarrow_L q$.

All of the above equivalences are well-known in 3-valued logic. In fact, the last of them is the 3-valued version of Boolean equivalence $\varphi \vee \psi \equiv (\varphi \rightarrow \psi) \rightarrow \psi$ that we used in Section 2 of this article. Note that Theorem 12 shows that the disjunction is definable through \rightarrow_L alone or also when \neg is used with any other connective. The only case not covered by Theorem 12 is resolved in the next theorem.

Theorem 13 *Formula $p \vee q$ is not 3-value-equivalent to any formula containing only connectives \wedge and \rightarrow_K .*

The computer-generated proof of the above theorem uses 36 fuzzy truth sets.

5.5 Undefinability of Kleene Implication

Let us again start with three definability results verifiable through the definitions of the connectives.

Theorem 14 *The following equivalences hold in 3-valued logic:*

1. $p \rightarrow_K q \equiv \neg p \vee q$,
2. $p \rightarrow_K q \equiv \neg(p \wedge \neg q)$,
3. $p \rightarrow_K q \equiv p \rightarrow_L \neg(p \rightarrow_L \neg q)$.

The first two equivalences are well-known. The third equivalence was discovered by our computer program. We are not aware of it ever being mentioned in the literature. The only question about the definability of \rightarrow_K , which is not answered by the above theorem, is answered by the one below.

Theorem 15 *Formula $p \rightarrow_K q$ is not semantically equivalent to any formula containing only connectives \wedge , \vee , and \rightarrow_L .*

The computer proof of this theorem uses 72 diagrams.

5.6 Undefinability of Łukasiewicz Implication

Out of the five connectives that we study only negation (see Theorem 8) and Łukasiewicz implication are not definable through the others.

Theorem 16 *Formula $p \rightarrow_L q$ is not semantically equivalent to any formula containing only connectives \neg , \wedge , \vee , and \rightarrow_K .*

The computer proof of the above result uses 82 diagrams. However, in this case, there is a simple argument that does not require the use of a computer. Indeed, if the value of all variables is set to $\frac{1}{2}$ (“unknown”), then the value of any expression that uses only connectives \neg , \wedge , \vee , and \rightarrow_K is $\frac{1}{2}$. At the same time the value of $\frac{1}{2} \rightarrow_L \frac{1}{2}$ is 1, see Figure 9. Therefore, connective \rightarrow_L is not definable through \neg , \wedge , \vee , and \rightarrow_K .

Although our fuzzy truth sets technique is not required to prove Theorem 16, this technique could be used to strengthen the theorem. Namely, we can show that connective \rightarrow_L is not definable through connectives \neg , \wedge , \vee , and \rightarrow_K and 3-valued constants 0 (“false”), $\frac{1}{2}$ (“unknown”), and 1 (“true”). The computer proof of this fact already uses 197 diagrams. This is the largest proof mentioned in this section.

Definability results for many other 3-valued connectives are discussed in [3]. We are not aware of any existing proofs of undefinability in 3-valued logic besides the two proofs mentioned above that don’t use fuzzy truth sets: the proof of Theorem 8 and the proof of the original (without constants) version of Theorem 16.

6 Conclusion

In this work, we introduced a new method for proving the undefinability of logical connectives and demonstrated it on examples from Boolean logic, temporal logic, intuitionistic logic, and three-valued logic. Although the technique is potentially applicable to other, more modern logical systems, we have chosen to use these classical examples to make the work accessible to a wider logical audience.

References

- [1] Alexandru Baltag and Giovanni Ciná. Bisimulation for conditional modalities. *Studia Logica*, 106(1):1–33, 2018.
- [2] Patrick Blackburn and Johan Van Benthem. Modal logic: a semantic perspective. In *Studies in logic and practical reasoning*, volume 3, pages 1–84. Elsevier, 2007.
- [3] Davide Ciucci and Didier Dubois. A map of dependencies among three-valued logics. *Information Sciences*, 250:162–177, 2013.
- [4] Kaya Deuser and Pavel Naumov. Strategic knowledge acquisition. *ACM Transactions on Computational Logic (TOCL)*, 22(3):1–18, 2021.
- [5] Jie Fan. A unified logic for contingency and accident. *Journal of Philosophical Logic*, pages 1–28, 2022.
- [6] A. Heyting. Die formalen regeln der intuitionistischen logik. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, pages 42–56, 1930.
- [7] Stephen Cole Kleene. On notation for ordinal numbers. *The Journal of Symbolic Logic*, 3(4):150–155, 1938.
- [8] Saul A. Kripke. Semantical analysis of intuitionistic logic I. In J.N. Crossley and M.A.E. Dummett, editors, *Formal Systems and Recursive Functions*, volume 40 of *Studies in Logic and the Foundations of Mathematics*, pages 92–130. Elsevier, 1965.

- [9] François Laroussinie. About the expressive power of CTL combinators. *Information Processing Letters*, 54(6):343–345, 1995.
- [10] Clarence Irving Lewis and Cooper Harold Langford. *Symbolic Logic*. The Century Company, 1932.
- [11] P. Mancosu, editor. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*. Oxford University Press, 1998.
- [12] Alan Martin. Adequate sets of temporal connectives in CTL. *Electronic Notes in Theoretical Computer Science*, 52(1):21–31, 2002.
- [13] John Charles Chenoweth McKinsey. Proof of the independence of the primitive symbols of heyting’s calculus of propositions. *The journal of symbolic logic*, 4(4):155–158, 1939.
- [14] Joan Moschovakis. Intuitionistic Logic. In Edward N. Zalta and Uri Nodelman, editors, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2022 edition, 2022.
- [15] Pavel Naumov and Anna Ovchinnikova. An epistemic logic of preferences. *Synthese*, 201(77), 2023.
- [16] Pavel Naumov and Jia Tao. Two forms of responsibility in strategic games. In *30th International Joint Conference on Artificial Intelligence (IJCAI-21)*, 2021.
- [17] Pavel Naumov and Rui-Jie Yew. Ethical dilemmas in strategic games. In *Proceedings of Thirty-Fifth AAAI Conference on Artificial Intelligence (AAAI-21)*, 2021.
- [18] Pavel Naumov and Yuan Yuan. Intelligence in strategic games. *Journal of Artificial Intelligence Research*, 71:521–556, 2021.
- [19] Johan van Benthem, Sieuwert van Otterloo, and Olivier Roy. Preference logic, conditionals and solution concepts in games. In Henrik Lagerlund, Sten Lindström, and Rysiek Sliwinski, editors, *Modality matters: twenty-five essays in honour of Krister Segerberg*, pages 61–77. Uppsala Univ., Dept. of Philosophy, 2006. (Uppsala Philosophical Studies 53).
- [20] Mordchaj Wajsberg. Untersuchungen über den Aussagenkalkül von A. Heyting. *Wiadomości Matematyczne*, 46:45–101, 1939.