North American Journal of Economics and Finance 69 (2024) 102037

Contents lists available at ScienceDirect

North American Journal of Economics and Finance

journal homepage: www.elsevier.com/locate/najef

Dynamic robust portfolio selection under market distress

Yifu Jiang [a], Jose Olmo [a,b,*], Majed Atwi [a]

[a] Department of Economic Analysis, University of Zaragoza,Gran Via 2, 50005 Zaragoza, Spain
[b] Department of Economics, University of Southampton. Highfield Campus, SO17 1BJ Southampton, United Kingdom

A R T I C L E I N F O

JEL classification:
G10
G11
G17

Keywords:
Robust portfolio selection
Dynamic multivariate copula
GJR-GARCH-EVT
Conditional value-at-risk
Financial crisis

A B S T R A C T

This article proposes a dynamic robust portfolio selection model that is based on minimizing portfolio's worst case scenarios using the Conditional Value at Risk as relevant risk measure. Our proposed empirical model for the dynamics of portfolio constituents has three main features: i) accommodates tail dependence between assets by means of a mixture of copula functions; ii) conditional heteroscedasticity and leverage effects are considered through the implementation of a GJR-GARCH model; and iii) extreme events are taken into account by considering parametric and semiparametric hybrid models for the marginal distribution of asset returns. We illustrate the performance of this portfolio before and during the COVID-19 pandemic using statistical measures such as the Sharpe ratio, cumulative returns, and volatility. The results show the outperformance of our WCVaR portfolio during the turmoil period against benchmark portfolios commonly used by practitioners. The method also exhibits good performance during calm periods.

1. Introduction

Portfolio selection is an important area in financial economics and investments. Markowitz (1952), in his pioneering work, introduced mean–variance (MV) optimal portfolios. The underlying risk of an investment position in these portfolios is determined by the portfolio variance that is, in turn, constructed from the covariance matrix of asset returns. In many settings, the fluctuations of the portfolio return around its mean may not be an appropriate measure of portfolio risk. This is, for example, the case when the interest is in minimizing portfolio's downside.

An alternative that has been widely explored in the investment literature is to construct optimal portfolios that consider quantile risk measures, a typical example being the Value-at-Risk (VaR). The literature on portfolio allocation using this measure to capture risk has grown steadily over the last twenty years. Related literature includes the mean-risk model introduced in Fishburn (1977) that can be considered as an early extension of standard mean–variance formulations. Other important contributions considering the VaR quantiles as constraints in the asset allocation optimization exercise are Basak and Shapiro (2001), Krokhmal et al. (2001), Campbell et al. (2001), Wu and Xiao (2002), Bassett et al. (2004), Engle and Manganelli (2004) and Ibragimov and Walden (2007), among others. This literature illustrates the properties of VaR-optimal portfolios while acknowledging considerable computational difficulties (Gaivoronski and Pflug, 2005; Rachev et al., 2007).

Another strand of the literature has focused on the Expected Shortfall, also denominated as Conditional VaR (CVaR) that measures the expected portfolio loss once the return on the portfolio is below the corresponding VaR. This measure has become popular because

* Corresponding author at: Department of Economic Analysis, University of Zaragoza, Gran Via 2, 50005, Zaragoza, Spain.
E-mail addresses: 849201@posta.unizar.es (Y. Jiang), j.b.olmo@soton.ac.uk, joseolmo@unizar.es (J. Olmo), matwi@unizar.es (M. Atwi).

https://doi.org/10.1016/j.najef.2023.102037
Received 11 August 2022; Received in revised form 2 August 2023; Accepted 31 October 2023
Available online 4 November 2023

of its simplicity, but more importantly, because it satisfies a set of properties that characterize the risk measure as coherent. Unfortunately, quantile risk measures such as the VaR do not satisfy these properties, see (Artzner et al., 1999). The development of investment portfolios considering the CVaR as the risk measure of interest include Topaloglou et al. (2002), Rockafellar and Uryasev (2002), among many others. A related literature (Zhu and Fukushima, 2009) applies robust optimization techniques to construct investment portfolios. The central idea of robust portfolio optimization is to use uncertainty sets for the unknown parameters characterizing the distribution functions of the asset returns instead of only point estimates, and to compute portfolios whose worst-case performance is optimal. In this scenario, worst-case performance is interpreted as portfolio performance under the least favourable combination of the model parameters within the uncertainty set. Examples of robust portfolio optimization using the CVaR as objective function are Ghaoui, Oks, and Oustry (2003) in a mean–variance setting, Zhu and Fukushima (2009), that use distribution functions within the elliptical family, Hellmich and Kassberger (2011), that consider multivariate Generalized Hyperbolic distributions. More recently, Deng and Liang (2021) explore time series copula models and Su et al. (2021) solve the portfolio selection problem using regime-switching models.

A major factor influencing the optimal portfolio decision is the behavior of the portfolio constituents in the tails and, more specifically, the likelihood of extreme events. We differentiate between two types of events that affect the risk of a portfolio when considering VaR and CVAR measures. The first type corresponds to the probability of extreme losses due to the existence of heavy tails in the marginal distribution of asset returns. The second type is concerned with the probability of joint dependence in the tails of the multivariate distribution of asset returns. A flexible specification of the multivariate distribution of the portfolio constituents is the use of copula models, see Sklar (1959). This modeling strategy is capable of capturing different forms of dependence in the tails of the multivariate return distribution. These techniques have been used for portfolio allocation problems in (Low, 2018) and Garcia and Tsafack (2011), among many others. A particular case of copula model in high dimensions is vine copulas, see Weiß and Scheffer (2015). These copulas model the multivariate dependence in a vector of random variables pairwise. Another example of copula model in portfolio allocation problems is given in Kakouris and Rustem (2014). These authors construct a mixture copula model embedded in a portfolio selection problem using CVaR and WCVaR measures as objective functions. In this model, the marginal distributions of asset returns do not consider, however, the possibility of heavy tails. Belhajjam et al. (2017) propose a multivariate extreme VaR method to obtain the optimal portfolio weights. This method applies extreme value theory (EVT) techniques to model the behavior of asset returns in the tails, and considers a copula model for the joint tail dependence between the assets in the portfolio. An important drawback of these studies is the reliance on bivariate copula models, which limits the scope of application of the methods in practice

The presence of dynamics in the conditional multivariate distribution of asset returns is a stylized fact. Under the assumption of linearity and Gaussianity of asset returns, it is sufficient to model the dynamics of the multivariate distribution through the second moments of pairs of asset returns. In particular, generalized autoregressive conditional heteroskedasticity (GARCH) type models, see Bollerslev (1986), and versions of it considering different features of the data, have been the main workhorse for modeling the presence of conditional heteroscedasticity in asset returns. The Conditional Constant Correlation model of Bollerslev (1990) and the Dynamic Conditional Correlation (DCC) model of Engle (2002) see also Aielli (2013), are among the most popular. It is well documented, though, that the dynamics of asset returns are nonlinear and exhibit stylized facts such as asymmetries and heavy tails. In this scenario, modeling the conditional variance and correlations may not be sufficient to capture the effect of tail dependencies. To correct for this, a more suitable approach is to consider dynamic copula models, see Patton (2006) as seminal contribution. Weiß (2013), Karmakar and Paul (2019) or Sun et al. (2020) combine time varying multivariate GARCH models with dynamic copulas to describe the dependence and risk of asset returns. Han, Li, and Xia (2017) and Han et al. (2020) present several dynamic robust portfolio optimization models obtained from DCC-Copula-GARCH model specifications for describing the dynamic dependence between bivariate assets. These authors capture high-dimensional dependencies using R-vine and C-vine copulas but do not consider extreme portfolio losses.

In this article, we propose a dynamic robust portfolio optimization problem that focuses on minimizing tail events. This strategy is particularly interesting during market distress episodes. The portfolio is robustified by optimizing the CVaR risk measure over a confidence set, that is, estimates of the model parameters are replaced by confidence regions for such parameters. In this way, our portfolio is optimized over the combination of model parameters that results in the worst-case scenario. In this way, our portfolio strategy can be interpreted as a maximin optimization rule in which investors optimize the conditional VaR measure under the worst-case scenario. Portfolio weights are dynamically adjusted by optimization of the WCVAR measure over rolling windows.

Our proposed specification for the dynamics of asset returns is a GJR-GARCH-EVT model, with GJR standing for the asymmetric conditional volatility model introduced in Glosten et al (1993) and EVT for the application of extreme value theory methods for modeling the tails of the marginal distribution of asset returns. This model is combined with a dynamic specification of a multivariate mixture copula model capable of describing the various potential dependence structures of multiple asset returns. Our interest is in capturing joint dynamics in the tails of the multivariate distribution of asset returns. To do this, we combine the GJR-GARCH-EVT with a dynamic mixture copula model given by a mixture of a Gaussian copula, a Student-t copula and a Clayton copula. The weights defining the mixture copula are obtained by maximum likelihood estimation. This modeling strategy allows us to capture the presence of heavy tails, conditional heteroscedasticity, and nonlinearities in the dependence structure between asset returns. The mixture of copula functions allows us to capture important stylized facts of the joint distribution of asset returns such as the presence of asymmetric dependence between the lower and upper tail of the joint distribution of asset returns. These stylized facts are particularly important during episodes of market distress. Alternative multivariate parametric models can be also used to capture the joint dynamics of asset returns, such as the multivariate Student-t distribution or the generalized hyperbolic distribution function. Copula functions are however superior modeling devices for several reasons. Copulas consider the dependency between the marginal distributions of the random variables instead of focusing directly on the dependency between the random variables themselves. This makes them more flexible than standard distributions because it is possible to separate the selection of the multivariate dependency from the selection of the univariate distributions. These functions are also capable of capturing different forms of extreme tail dependence between the left and right tails of the multivariate distribution. The parametric specification of copula functions allows natural extensions to the time-varying case by introducing dynamics in the parameters characterizing the copula function. Dynamic specifications of copula functions are a simple way of obtaining h-period ahead forecasts of the vector of asset returns that can be applied in scenario-based portfolio selection. In particular, we apply Monte-Carlo methods to simulate the predictive distribution function of multivariate asset returns necessary to obtain the robust optimal portfolio WCVaR model.

The performance of this novel portfolio allocation model is assessed during turmoil periods. To do this, we consider two differentiated periods given by the episode before the outbreak of the COVID-19 pandemic and the episode right afterwards. These periods are characterized by very different macroeconomic and financial conditions. We conduct an empirical study to compare the in-sample and out-of-sample performance of two versions of our GJR-GARCH-EVT model with a mixture copula function against popular benchmark portfolios widely used by practitioners, namely, the minimum variance portfolio, the shrinkage minimum variance portfolio, and the equally-weighted portfolio that are computed over these two periods. The empirical results show strong performance of the proposed approach in both evaluation periods using different performance measures such as the gross return and the Sharpe ratio, and highlight the ability of our flexible specification to adapt to distress episodes of the market.

Our paper is closely related to Zhu and Fukushima (2009) and Han et al (2017). These authors also consider optimal portfolios obtained from maximizing the WCVaR risk measure. However, in contrast to these studies, our model adopts the mixture copula model for the dynamics of multivariate returns and the GJR-GARCH-EVT model to capture the occurrence of extreme events and the presence of conditional heteroscedasticity. These additional features of our model prove instrumental to show the strong performance of the method during stress periods.

The paper is organized as follows: Section 2 presents the related methodology and techniques, containing the GJR-GARCH-EVT model, mixture copula model, and dynamic robust WCVaR portfolio method. Section 3 describes the data set. Section 4 presents an empirical application with real data from asset markets. Section 5 concludes.

2. Econometric model

This section is divided into four blocks. The first subsection discusses the GJR-GARCH-EVT model to describe the dynamics of asset returns. The second block introduces a mixture copula model as a modeling device that improves the estimation of the dependence structure for multiple asset returns. The third subsection presents a dynamic robust portfolio selection method that uses the WCVaR as objective function. The last section details an algorithm that implements the proposed methodology.

2.1. The GJR-GARCH-EVT model

An interesting feature of financial markets is that asset price volatility is more sensitive to bad news than good news. This is reflected in a strong negative correlation between current returns and future volatility. Volatility decreases when returns increase and increases when returns decrease. This trend is generally referred to as the leverage effect. Traditional GARCH models are not able to capture this stylized fact of the data. This phenomenon is captured, though, by Glosten et al. (1993) in the GJR-GARCH model and Zakoian (1994) in the threshold-GARCH model. These models can reflect the asymmetry inherent in the volatility process through the leverage coefficient. Asset returns have other particular characteristics such as conditional heteroscedasticity, heavy tails and negative skewness of the distribution (Low, 2018). In this case, the AR(1)-GJR-GARCH(1,1) model with student-t distribution is an interesting time series process to model the dynamics of asset returns.

In this paper, we consider the optimal portfolio allocation for a universe of n financial assets, and let $r_{i,t}(i = 1,\cdots n; t = 1,\cdots T)$ denote the continuously compounded daily returns of the i-th asset, defined as

$$r_{i,t} = 100 \times \left(ln\left(pr_{i,t}\right) - ln\left(pr_{i,t-1}\right)\right) \tag{1}$$

where $r_{i,t}$ and $pr_{i,t}$ denote the percentage daily returns and the closing price of the i-th asset at the t-th day, respectively. Mathematically, the AR(1)-GJR-GARCH(1, 1) model is

$$\begin{cases} r_{it} = \varphi_0 + \varepsilon_{it} \\ \varepsilon_{it} = h_{it}^{\frac{1}{2}} z_{it}, z_{it} \sim F_i(\cdot) \\ h_{it} = \alpha_{i0} + \alpha_i \varepsilon_{i,t-1}^2 + \gamma_i \varepsilon_{i,t-1}^2 I_{\varepsilon_{i,t-1}} + \beta_i h_{i,t-1}, \end{cases} \tag{2}$$

where ε_{it} and z_{it} denote the errors and innovations, respectively, of the time series process $r_{it}(i = 1,\cdots n; t = 1,\cdots T)$; $F_i(\cdot)$ denotes the cumulative distribution function of the innovations, and φ_0 and φ_1 are the parameters characterizing the dynamics of the autoregressive process. The conditional volatility process is denoted as h_{it} and $I_{\varepsilon_{i,t-1}}$ is an indicator function which equals 1 if $\varepsilon_{i,t-1}$ is positive and zero, otherwise. Here $\alpha_{i0} \geq 0$, $\gamma_i \geq 0$, $\alpha_i \geq 0$, and $\beta_i \geq 0$ are the intercept, ARCH, GARCH and leverage coefficients characterizing the conditional volatility process. The coefficient γ_i determines the asymmetric effect on conditional volatility. If $\gamma_i > 0$, negative shocks (bad news) have larger impact $(\alpha_i + \gamma_i)\varepsilon_{i,t-1}^2$ on the conditional variance h_{it} than positive shocks (good news) given by $\alpha_i \varepsilon_{i,t-1}^2$. If $\gamma_i < 0$, the stock market is more responsive to good news. An additional condition to ensure the weak stationarity of the above conditional volatility process is $\alpha_i + 0.5\gamma_i + \beta_i < 1$.

To complete the dynamics of asset returns we propose a flexible parametric model for the marginal distribution $F_i(\cdot)$ of the innovations in (2). The distribution in (3) is obtained by applying the conditional probability theorem and exploiting extreme value theory (EVT) results for the tails of the distribution. For the middle domain defined by the interval $[\eta^L, N_{\eta^R}]$, with η^L and η^R the lower and upper thresholds characterizing the tails of the distribution, we consider two modeling strategies: (a) the empirical distribution function[1] and (b) a standard Normal distribution. The distribution in (3) focuses on the semiparametric model (a). More importantly, the conditional distribution in both tails is modeled as a Generalized Pareto distribution (GPD). This parametric choice is motivated by EVT theory and, in particular, by a theoretical result in Pickands (1975) that shows that the conditional distribution of a random variable in the tail can be approximated by a GPD distribution for sufficiently large values of the thresholds η^L and N_{η^R}. This is done to capture the presence of extreme events beyond the range provided by standard parametric models such as the Normal or Student-t distributions, see also Alexander and Rüdiger (2000) and Embrechts et al (2001). Mathematically,

$$F_i(z) = \begin{cases} \frac{N_{\eta^L}}{N}\left\{1+\varsigma^L\frac{\eta^L - z}{\chi^L}\right\}^{-\frac{1}{\varsigma^L}}, z < \eta^L, \\ \frac{N_z}{N}, \eta^L \le z \le \eta^R, \\ \frac{N_{\eta^R}}{N}\left\{1+\varsigma^R\frac{z-\eta^R}{\chi^R}\right\}^{-\frac{1}{\varsigma^R}}, z > \eta^R, \end{cases} \tag{3}$$

where ς^L and ς^R denote the shape parameters (tail index coefficients) corresponding to the left and right tails of the distribution of the sequence of innovations z; χ^L and χ^R are, respectively, the left and right tail scale parameters. Similarly, N denotes the total number of observations and N_z the number of observations below or equal to z, thus, N_z/N is the empirical distribution function.

Estimation of the model parameters is carried out using maximum likelihood methods. In a first step, we fit the GJR-GARCH model to the return prices and obtain estimates of the conditional mean and variance equations. In a second step, the tail indices and scale parameters of the GPD tail distributions are fitted by maximum likelihood as in McNeil and Frey (2000). In particular, the tail indices characterizing the decay of the distribution function in each tail are fitted using the Hill estimator, see Hill (1975), after suitable selection of the threshold estimates η^L and η^R.

2.2. Dynamic mixture copula model

Copulas have been widely adopted to model the dependence structures between financial markets with the purpose of monitoring risk and modeling returns in portfolios of assets. Popular examples of copula functions in multivariate settings are the Gaussian, the multivariate Student-t, and the multivariate Archimedean copula. These copulas are simple to represent and capture different stylized facts (e.g. asymmetric tail dependence across tails, asymptotic dependence in the tails) of multivariate returns.

The theory on copula functions was developed in Sklar (1959). This author shows that the multivariate distribution function G(z) of a vector of random variables $z = (z_1, \cdots, z_n)$ with marginal distributions $F_1, F_2, \cdots, F_n$ can be expressed in terms of a copula function C such that

$$G(z) = C(F_1(z_1), F_2(z_2), ..., F_n(z_n)) = C(u_1, ..., u_n), \tag{4}$$

where $u_i = F(z_i) \in [0,1]$ for $i = 1,\ldots,n$.

In this paper, we adopt a mixture of three copula functions: Gaussian, Student-t, and Clayton copula functions. This flexible approach allows us to capture complex dependence structures among asset returns. The Gaussian copula is aimed to capture dependence in the middle regimes, the Student-t copula is sensitive to extreme events in the tails of the distributions, but it does not exhibit extreme tail dependence for very large values of asset returns. The Clayton copula function is used to describe asymptotic tail dependence in the lower tail of the joint distribution of asset returns. The distribution function of the multivariate Gaussian copula is expressed as

$$C^{\text{Gau}}(u_1, ..., u_n; \rho^g) = \Psi_{\rho^g}\left(\Psi^{-1}(u_1), ..., \Psi^{-1}(u_n)\right) \tag{5}$$

where $\Psi_{\rho^g}(\cdot)$ is the standard multivariate Normal distribution function with correlation matrix ρ^g; $\Psi(\cdot)$ is the univariate Normal distribution and $\Psi^{-1}(\cdot)$ its inverse function. The distribution function of the Student-t copula is expressed as

$$C^{\text{Stu}}(u_1, ..., u_n; \nu, \rho^s) = \Phi_{\nu,\rho^s}\left(\Phi^{-1}_{\nu_1}(u_1), ..., \Phi^{-1}_{\nu_n}(u_n)\right) \tag{6}$$

where $\Phi_{\nu,\rho^s}(\cdot)$ is the standard multivariate Student-t distribution function with correlation matrix ρ^s and degrees of freedom parameter ν. The model allows for different degrees of freedom across marginal distributions. Thus, $\Phi^{-1}_{\nu_i}(\cdot)$ denotes the inverse of the distribution function $\Phi_{\nu_i}(\cdot)$. Finally, the distribution function of the Clayton copula is expressed as

[1] See McNeil and Frey (2000) and Engle and Gonzalez-Rivera (2001) for semiparametric formulations of the distribution of asset returns.

$$C^{\text{Cla}}(u_1,...,u_n;\theta)=\left(\sum_{i=1}^{n}u_i^{-\theta}-n+1\right)^{-1/\theta},\theta\in(0,\infty) \tag{7}$$

Thus, the multivariate mixture copula model can be expressed as

$$MC(u_1,...,u_n;\rho^g,\rho^s,\gamma,\nu,\theta)=\lambda^G C^{\text{Gau}}(u_1,...,u_n;\rho^g)+\lambda^S C^{\text{Stu}}(u_1,...,u_n;\nu,\rho^s)$$

$$+\lambda^C C^{\text{Cla}}(u_1,...,u_n;\theta) \tag{8}$$

where λ^G, λ^S, and λ^C are respectively the weights of Gaussian, Student-t, and Clayton copula functions, satisfying $\lambda^G,\lambda^S,\lambda^C\geq 0$ and $\lambda^G+\lambda^S+\lambda^C=1$.

The estimation of the mixture copula parameters ($\rho^g,\rho^s,\nu,\theta,\lambda^G,\lambda^S,\lambda^C$) is done by applying maximum likelihood estimation (MLE) methods. In practice, it is difficult to obtain an analytical solution using MLE. To overcome this, we resort to simulation methods by combining the MLE approach with the expectation maximization (EM) algorithm. The EM algorithm for missing data transforms the likelihood function maximization problem for incomplete data into the maximization of the likelihood function for complete data. This is done by assuming the presence of latent variables in the maximization exercise. Through the iteration of the expectation and maximization processes up to convergence, we obtain an optimal solution of the likelihood function. More details of the estimation of the mixture copula parameters can be found in the recent studies by Sahamkhadam et al. (2018) and Ben Nasr and Chebana (2022).

Considering the fact that the dependence of multiple assets is time varying, we combine the GJR-GARCH-EVT model with a mixture copula to dynamically describe the dependence and risk of asset returns, where the parameters λ^{Gaus}, $\lambda^{\text{stu-t}}$, and λ^{Cla} also change over time. The implementation of the dynamic mixture copula model is detailed in Section 2.4.

2.3. Dynamic robust portfolio optimization

To set up the framework for the robust optimization problem, we define the loss function $L(w,r)$ associated to a decision $w\in\mathbb{R}^n$ and a random return $r\in\mathbb{R}^n$. For the optimal portfolio allocation problem, the loss function is $L(w,r)=-w\hat{A}r$ but other configurations of the loss function are also possible. For the sake of generality, the following expressions define the relevant tail risk measures as a function of the general loss function $L(w,r)$. By doing so, we stay close to the formulations of the CVaR in Rockafellar and Uryasev (2002) and Kakouris and Rustem (2014).

The portfolio is characterized by the allocation to the different assets. This allocation is determined by the vector $w=(w_1,w_2,\cdots,w_n)$ and a random return vector $r=(r_1,r_2,\cdots,r_n)$. For each w, we denote by $\Psi(w,a)$ the distribution function for the loss function $L(w,r)$ such that $\Psi(w,\xi)=\int_{L(w,r)\leq\xi}g(r)dr$, where $g(\cdot)$ is the joint density function of the vector of random returns r and ξ denotes a tolerance level for the loss function. Furthermore, let $\alpha\in(0,1)$ denote a confidence level associated to the maximum loss of the portfolio. In applications, this value is in the range [0.95,0.99]. The $VaR_\alpha(w)$ associated to the loss function is defined as

$$VaR_\alpha(w)=min\{\zeta:\Psi(w,\zeta)\geq\alpha\}. \tag{9}$$

Similarly, the $CVaR_\alpha(w)$ is defined as the expected loss once the risk measure $VaR_\alpha(w)$ is exceeded. More formally,

$$CVaR_\alpha(w)=\frac{1}{1-\alpha}\int_{L(w,r)\geq VaR_\alpha(w)}L(w,r)g(r)dr. \tag{10}$$

As mentioned above, the $CVaR_\alpha(w)$ measure has some appealing properties not shared by the VaR such as the sub-additivity and coherence, see Artzner et al. (1999), Topaloglou et al. (2002), Guo et al. (2019). Rockafellar and Uryasev (2002), in their seminal contribution, show that the $VaR_\alpha(w)$ and $CVaR_\alpha(w)$ of the loss function associated to the vector $w=(w_1,w_2,\cdots,w_n)$ can be calculated simultaneously by solving the following convex optimization problem:

$$G_\alpha(w,\zeta)=\zeta+\frac{1}{1-\alpha}\int[L(w,r)-\xi]^+g(r)dr, \tag{11}$$

such that $CVaR_\alpha(w)=\min_{\xi\in\Re}G_\alpha(w,\zeta)$. The associated $VaR_\alpha(w)$ is defined as the value $\xi\in\Re$ that solves the minimization problem. More formally, $\xi_\alpha(w)=\underset{\xi\in\Re}{argmin}G_\alpha(w,\zeta)$ such that $CVaR_\alpha(w)=G_\alpha(w,VaR_\alpha(w))$. Note that throughout we use $VaR_\alpha(w)$ and $\xi_\alpha(w)$ indistinctively.

Kakouris and Rustem (2014) extend this approach to obtain the VaR_α and $CVaR_\alpha$ of a portfolio of assets with multivariate dependence modeled using copula models. To do this, these authors transform the loss function such that $L(w,r)=\widetilde{L}(w,u)$, with $u=(u_1,...,u_n)=(F_1(r_1),...,F_n(r_n))$. The loss function $\widetilde{L}(w,u)$ maps the domain of the loss function from $\Re^n$ to $[0,1]^n$. The relevant tail risk measures with the copula representation are defined as

$$\begin{cases} VaR_{\alpha}(w) = \min\left\{\xi : C\left(u|\widetilde{L}(w,u)\leqslant\zeta\right)\geqslant\alpha\right\}, \\ CVaR_{\alpha}(w) = \frac{1}{1-\alpha}\int_{\widetilde{L}(w,u)\geqslant VaR_{\alpha}(w)}\widetilde{L}(w,u)c(u)du. \end{cases} \tag{12}$$

Note that $C(u\big|\widetilde{L}(w,u)\leq\zeta)$ denotes the multivariate distribution function associated to the copula function for the vector $u = (u_1,...,u_n)$ and conditional on the event $\widetilde{L}(w,u)\leq\zeta$. Similarly,$c(u)$ denotes the corresponding copula density. The $CVaR_{\alpha}(w)$ is obtained, as before, from minimizing $G_{\alpha}(w,\zeta)$ that, under the transformation of the loss function, it can be expressed as $G_{\alpha}(w,\zeta) = \zeta + \frac{1}{1-\alpha}\int_{u\in[0,1]^n}\left[\widetilde{L}(w,u)-\xi\right]^{+}c(u)du$.

These formulations of the tail risk measures involve knowledge of the copula function modeling the multivariate dependence structure between the asset returns. A robust approach to compute the CVaR that incorporates uncertainty about the specific choice of the copula function is the worst-case CVaR. This risk measure considers the worst outcome from a set of possible scenarios. These scenarios are modelled by different choices of the copula density function. Following Kakouris and Rustem (2014), we define the worst-case CVaR as $WCVaR_{\alpha}(w) = \underset{c(\cdot)\in\mathbb{C}}{sup}\ CVaR_{\alpha}(w)$, where $\mathbb{C}$ denotes a set of candidate copulas within the list introduced above. This approach can be extended to consider the mixture copula $MC(u_1,...,u_n;\rho^g,\rho^s,\gamma,\nu,\theta)$ introduced in expression (8). In this case the $WCVaR_{\alpha}$ is obtained from minimizing the function

$$G_{\alpha}(w,\zeta,\lambda^g,\lambda^s,\lambda^c) = \lambda^g G^g_{\alpha}(w,\zeta) + \lambda^s G^s_{\alpha}(w,\zeta) + \lambda^c G^c_{\alpha}(w,\zeta). \tag{13}$$

More formally, $WCVaR_{\alpha}(w) = \underset{\xi}{min}\underset{\lambda}{max}G_{\alpha}(w,\zeta,\lambda^g,\lambda^s,\lambda^c)$, with $\Delta = \{\lambda^g,\lambda^s,\lambda^c : \lambda^g + \lambda^s + \lambda^c = 1, \lambda^g,\lambda^s,\lambda^c\geq 0\}$.

In this setup, the optimal portfolio allocation is obtained from $\underset{w\in W}{min}WCVaR_{\alpha}(w)$, with W the set of feasible portfolio weights. More specifically, the results in Zhu and Fukushima (2009) allow us to obtain the optimal portfolio allocation as the solution of the following robust optimization problem:

$$WCVaR_{\zeta}(w) = \underset{w\in\mathbb{W}}{min}\underset{\xi}{min}\underset{\lambda}{max}G_{\alpha}(w,\xi,\lambda)$$

$$= min_{(w,\xi,\phi)}\left\{\phi : \lambda^g G^g_{\alpha}(w,\xi) + \lambda^s G^s_{\alpha}(w,\xi) + \lambda^c G^c_{\alpha}(w,\xi) \leq \phi, \forall\lambda\right\},$$

with ϕ satisfying that $G^i_{\alpha}(w,\xi)\leq\phi$, for i = g,s,c. The objective function can be reduced to $min_{(w,\xi,\phi)\in\mathbb{W}\times\mathbb{R}\times\mathbb{R}}\left\{\phi : G^i_{\alpha}(w,\xi)\leq\phi, i = g,s,c\right\}$. The solution to this problem can be obtained using Monte Carlo simulation methods. Rockafellar and Uryasev (2002) provides an approximation of the functions $G^i_{\alpha}(w,\xi)$ based on simulation of S^i random samples. Then, the optimal portfolio allocation is obtained from the minimization of the simulated function

$$\min_{(w,\xi,\phi)}\left\{\phi : \xi + \frac{1}{S_i(1-\alpha)}\sum_{k=1}^{S^i}\left[\widetilde{L}\left(w,u_i^{[k]}\right)-\xi\right]^{+}\leqslant\phi, i = g,s,c\right\}, \tag{14}$$

where $u_i^{[k]}$ is the kth sample vector associated to copula $C^i(\cdot)$. Following Zhu and Fukushima (2009), the minimization problem can be expressed as

$$\begin{aligned} & min\phi \\ & s.t. \\ & \xi + \frac{1}{S^i(1-\alpha)}\left(1^i\right)^T v_i\leqslant\phi,\ \ i = g,s,c \\ & v_i^k\geqslant\widetilde{L}\left(w,u_i^{[k]}\right)-\xi, k = 1,2,\cdots,S_i,\ \ i = g,s,c, \\ & v_i^k\geq 0, k = 1,2,\cdots,S_i, k = 1,2,\cdots,S_i, i = g,s,c, \end{aligned} \tag{15}$$

where $v = (v^1,v^2,v^3)$ with $m = S_1 + S_2 + S_3$ and $1^i = (1,1,\cdots,1)^T\in\mathbb{R}^{S^i}$. The set of feasible portfolio weights $\mathbb{W}$ is defined as $\mathbb{W} = \left\{\sum_{j=1}^n w_j = 1, w_j\geqslant\mathbf{0},\ \mathrm{j} = 1,...,\mathrm{n}\right\}$.

2.4. Dynamic algorithm

The above methods are implemented using the algorithm developed in Weiß (2013), Nikoloulopoulos et al. (2012), Han, Li, and Xia (2017) and Sahamkhadam et al. (2018). The parameters of the GJR-GARCH-EVT model with mixture copula $MC(u_1,...,u_n;\rho^g,\rho^s,\gamma,\nu,\theta)$

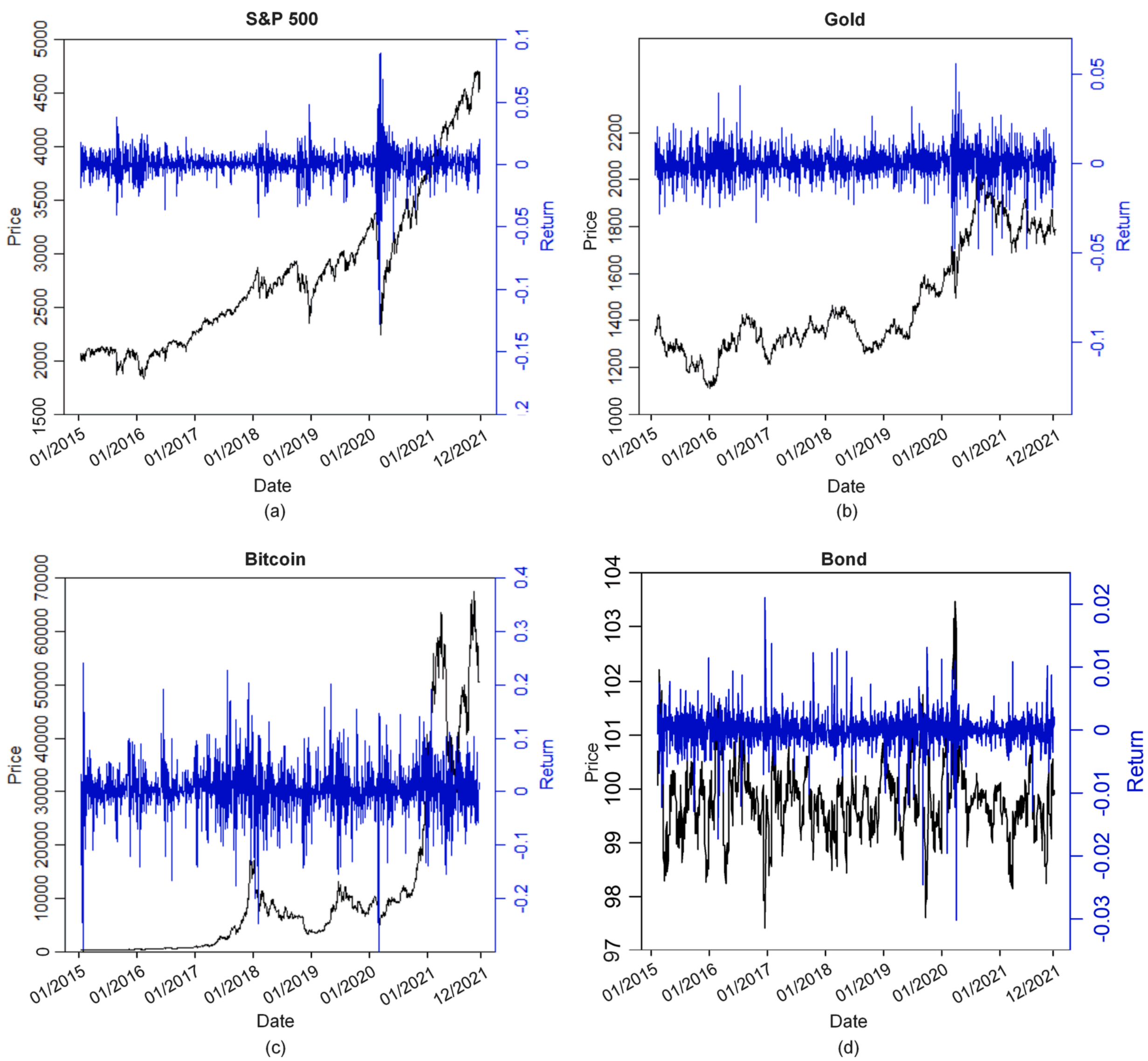


Fig. 1. Daily prices and returns for the four portfolio constituents.

introduced in expressions (3) and (8) are estimated over rolling windows of T observations from the vector of daily log returns $r_t = (r_{1t}, \cdots, r_{nt})$. We consider a Monte Carlo procedure to simulate one-period ahead returns of the assets in the portfolio. The algorithm is as follows;

Step 1: The vector of parameters characterizing the conditional volatility processes $h_{i,t}$ and the marginal distribution function in (3) are estimated by maximum likelihood to obtain estimates of the standardized residuals z_{it} in expression (2) and marginal distribution functions $u_{it} = \widehat{F}_i(z_{it})$ for $i = 1,\ldots,n$.

Step 2: The vector of dependence parameters $(\rho^g, \rho^s, \gamma, \nu, \theta)$ characterizing the mixture copula is estimated by maximizing the log-likelihood of the mixture copula function for the vectors $(u_{1t}, \ldots, u_{nt}) = (\widehat{F}_1(z_{1t}), \ldots, \widehat{F}_n(z_{nt}))$ for $t = 1,\ldots,T$. This procedure also includes the estimation of the vector $(\lambda^g, \lambda^s, \lambda^c)$ defining the mixture copula.

Step 3: The parameter estimates $(\widehat{\rho^g}, \widehat{\rho^s}, \widehat{\gamma}, \widehat{\nu}, \widehat{\theta})$ and $(\widehat{\lambda}^g, \widehat{\lambda}^s, \widehat{\lambda}^c)$ obtained from Step 2 are used to simulate random samples of vectors of residuals $(z_{1t}^{[k]}, \ldots, z_{nt}^{[k]})$ indexed by $k = 1,\ldots,S$ from the estimated mixture copula $MC(u_1, \ldots, u_n; \widehat{\rho^g}, \widehat{\rho^s}, \widehat{\gamma}, \widehat{\nu}, \widehat{\theta})$; S denotes the number of simulated samples obtained from the Monte Carlo exercise.

Step 4: Plug in the simulated vector of observations $(z_{1t}^{[k]}, \ldots, z_{nt}^{[k]})$ obtained from Step 3 into the location-scale process introduced in expression (2);

$$r_{it}^{[k]} = \widehat{\varphi}_0 + \widehat{h}_{it}^{1/2} z_{it}^{[k]}$$

Table 1
Descriptive statistics of daily logarithmic returns for the portfolio constituents.

Index	Mean	Std. Dev.	Min	Max	Skewness	Kurtosis	Jarque–Bera
In-sample (pre-COVID-19 period): 01/02/2015–01/29/2020							
S&P 500	0.00036	0.00846	−0.0418	0.0484	−0.5283	3.8517	97.8501 (*)
Gold	0.00015	0.00763	−0.0337	0.0439	0.16057	2.7220	9.57671 (*)
Bitcoin	0.00266	0.04579	−0.2942	0.2408	−0.2024	5.5266	347.847 (*)
Bond	$-4.6*10^{-6}$	0.00281	−0.0245	0.0212	−0.4671	13.2992	5681.55(*)
Out-of sample (during/post COVID-19 period): 01/30/2020–12/13/2021							
S&P 500	0.00076	0.01693	−0.1277	0.0897	−1.0424	14.316	2609.51 (*)
Gold	0.00021	0.01183	−0.0508	0.0563	−0.4365	4.2782	47.2222 (*)
Bitcoin	0.00347	0.05090	−0.4973	0.1918	−1.9932	20.215	6154.12 (*)
Bond	$-1.9*10^{-5}$	0.00286	−0.0301	0.0110	−0.2.914	32.426	17847.5(*)
Overall period: 01/02/2015–12/13/2021							
S&P 500	0.00047	0.01138	−0.1277	0.0897	−1.0468	21.1361	24289.4 (*)
Gold	0.00017	0.00896	−0.0508	0.0563	−0.1942	5.12853	341.170 (*)
Bitcoin	0.00287	0.04720	−0.4973	0.2408	−0.8048	10.8708	4703.36 (*)
Bond	$-7.6*10^{-7}$	0.00278	−0.0301	0.0212	−1.020	18.5688	17967.61(*)

Note: This table applies the natural logarithmic returns for the indexes. (*) denotes rejection of the null hypothesis of normality at a 1% significance level.

for i = 1,…,n and t = 1,…,T, where $\widehat{\varphi}_0$ denotes the intercept of the location-scale process estimated in Step 1 and $\widehat{h}_{it}$ are the forecasts of the conditional volatility process in (3) (i.e, GJR-GARCH) constructed from the corresponding parameter estimates.

Step 5: Input the forecasted returns $(r_{1t}^{[k]}, r_{2t}^{[k]}, \ldots, r_{nt}^{[k]}]$ into the WCVaR optimization problem in expression (15) and compute the optimal portfolio weights w^* by solving the minimization problem.

Step 6: The in-sample rolling window is extended by one day such that the in-sample vector of returns covers observations from 2 to *T*. This sample is used to estimate the model parameters, compute the portfolio returns, and construct the optimal portfolio allocation obtained from minimizing the worst-case CVaR at a confidence level α.

3. Data

Daily closing prices of four representative assets given by the S&P500, gold, Bitcoin, and United States 5-Year Treasury Bond are used in this study. Daily data from 2 January 2015 to 13 December 2021 are obtained from https://www.investing.com/ and Bloomberg. The sample period is divided into three sub-periods: the Pre-COVID-19 period (from January 2, 2015 to January 29, 2020); the COVID-19 period (from January 29, 2020 to July 31, 2020), and the After-COVID-19 period (from August 1, 2020 to December 13, 2021). Similar periods are used in Ali et al (2021) and Raj et al (2022), among many others. In our work, we mainly evaluate portfolio allocation during and after the epidemic outbreak periods but the whole sample period is used for estimation of the model parameters. We estimate the model from the daily logarithmic returns using data from 01/02/2015 to 01/29/2020 and use data from 01/30/2020 to 12/13/2021 for the out-of-sample evaluation exercise. The latter period covers the outbreak of the COVID-19 pandemic and a period that we will denominate After-COVID-19.

Fig. 1 plots the daily prices (black color) and returns (blue color) for S&P500, gold, Bitcoin, and 5-year bond for the constituents of the portfolio. The S&P500 index and gold price have a similar positive trend in the Pre-COVID-19 period, while bond prices fluctuate considerably. It is also noticeable the large drops experienced by the S&P500 index and Bitcoin at the start of the COVID-19 pandemic. In contrast, the After-COVID-19 period witnesses large increases in the price of these assets that exhibit a strong positive trend whereas gold and bond prices remain stable or even show a slight decline during this period. All the assets exhibit strong volatility clustering that motivates the application of conditional volatility models such as the GJR-GARCH to capture these dynamics and remove serial dependence from the squared returns.

Table 1 reports summary statistics for the logarithmic returns for the four assets, including the Jarque–Bera test of normality. With the only exception of the bond market, the other three assets have positive mean returns in both the in-sample as well as out-of-sample periods. Importantly, the volatility in asset returns, captured by the standard deviation, is significantly higher during/after the COVID-19 pandemic outbreak than in the pre-COVID-19 period for S&P 500, gold, and the US bond. Table 1 also shows the high excess kurtosis for all four assets in both periods, especially for Bitcoin, but the skewness differs considerably across periods. Bitcoin achieves the highest average log returns, which might have attracted investment, but it has the highest volatility leading to high portfolio risk. The Jarque–Bera test statistics applied to all assets in both periods reject the null hypothesis of normality at 1 % significance level.

4. Empirical results

This section presents an empirical exercise that compares portfolio performance (cumulative returns, volatility, Sharpe ratio, maximum drawdown) between the following five investment strategies: 1) The dynamic robust WCVaR portfolio, which uses the GJR-GARCH-EVT model and multivariate dynamic mixture copula model. This portfolio is denoted throughout as G-E-D-M−C−WCVaR; 2)

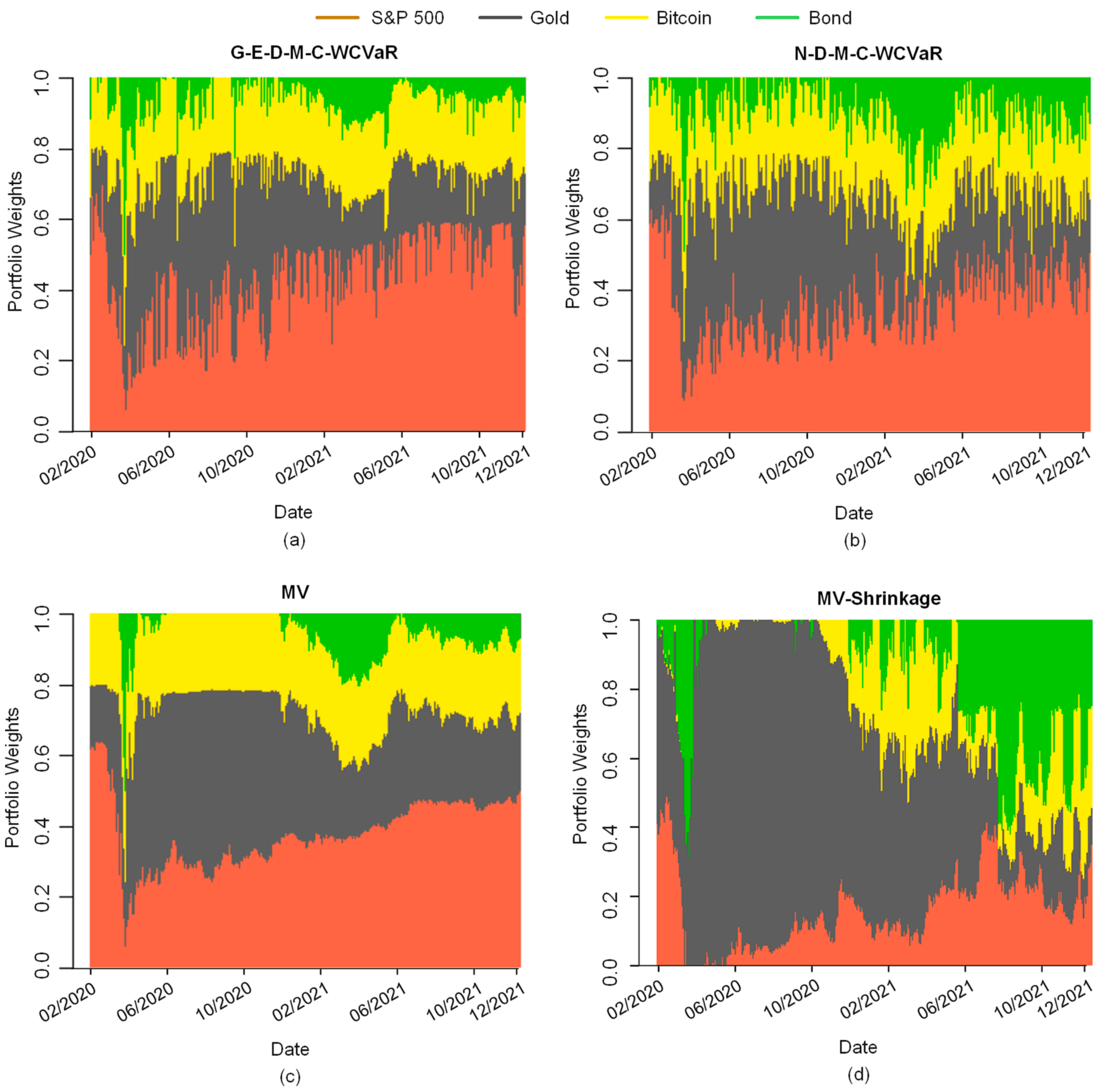


Fig. 2. Panels (a) to (d) present the time-varying portfolio allocation to each of the four assets in the portfolio. The evaluation period is January 29, 2020 to December 13, 2021.

The dynamic robust WCVaR portfolio, which uses the dynamic multivariate mixture copula model but fits a Normal distribution for the marginal distribution of asset returns. This portfolio is denoted as N-D-M−C−WCVaR; 3) The minimum-variance portfolio (MV). This procedure is a popular investment strategy that minimizes the global portfolio variance without setting a target portfolio return. The choice of this portfolio instead of the mean–variance portfolio introduced in Markowitz (1952) is because the expected return is usually measured with error and can lead to noisy portfolios with significant estimation error, see the seminal contribution of Jagannathan and Ma (2003). The variance of the portfolio is given by:

$$Var(R) = w^T \sum w \tag{17}$$

where $\sum$ is the covariance matrix of asset returns. The global MV portfolio allocation is obtained by solving the standard Markowitz asset allocation problem defined as

$$\min\left\{w^T \sum w\right\}$$

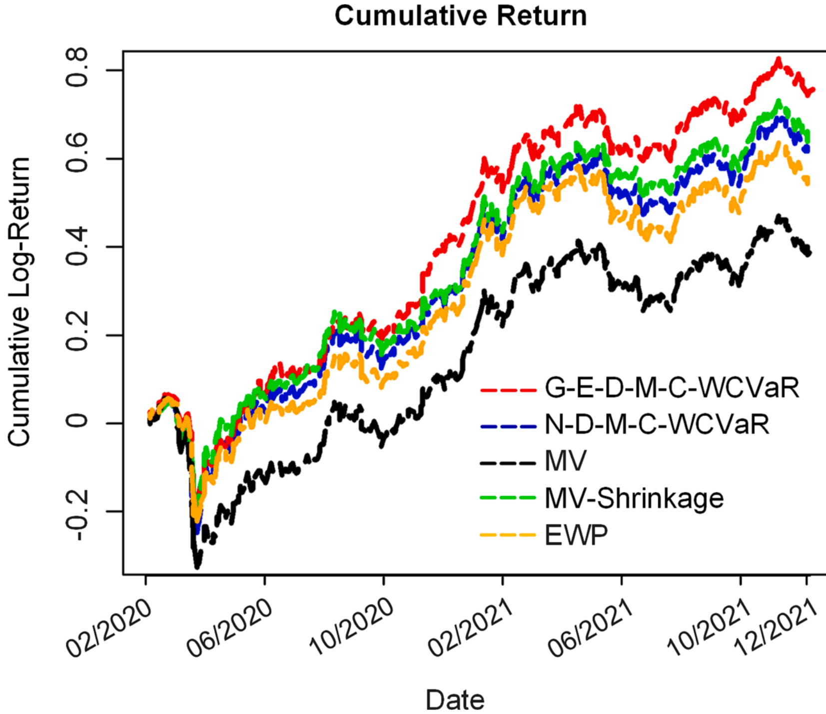


Fig. 3. The cumulative return process for the five investment portfolios at $\alpha = 0.95$ during the evaluation period January 29, 2020 to December 13, 2021.

$$s.t. \sum_{i=1}^{n} w_i = 1$$

$$0 \le w_i \le 1, i \in \{1, \cdots, n\} \tag{18}$$

where $w = (w_1, w_2, \cdots, w_n)$ and n is the number of assets in the portfolio. 4) The fourth proposed investment strategy is the MV portfolio with shrinkage estimation. In practice, the population covariance matrix $\sum$ is not known and is replaced by its sample counterpart, denoted by $\widehat{S}$. The sample covariance matrix has appealing properties, such as being maximum likelihood under normality. However, when the number of assets in the portfolio is larger than the sample size the estimation of the covariance matrix is very noisy. Ledoit and Wolf (2003, 2004, 2012, 2017, 2022) in a sequel of seminal contributions propose shrinkage methods to correct the effect of estimation error. These methods based on the ideas of Stein (1956) and James and Stein (1961) impose some structure on the estimator of the covariance matrix. A simple approach is to use a multiple of the identity matrix as shrinkage target, see Ledoit and Wolf (2004), such that $\sum^* = \alpha\, \mathrm{I_n} + (1-\alpha)\, \widehat{S}$ where $\mathrm{I_n}$ is the $n\ x\ n$ identity matrix and α is the shrinkage parameter obtained from minimizing the quadratic loss function $E\left[\|\Sigma^* - \Sigma\|^2\right]$. In this context, estimation error is not an issue given that the sample size is much larger than the number of candidate assets in the portfolio. Nevertheless, for completeness and to assess the role of shrinkage, we also consider this investment strategy. 5) The last investment strategy is the equally-weighted portfolio (EWP), characterized by the same allocation (1/n) to all assets in the portfolio. The size of the rolling window T is set to 475 throughout the empirical section.

4.1. Performance evaluation during/after COVID-19

To show the superiority and robustness of the proposed method, this subsection uses the data during/after the COVID-19 period (January 29, 2020 to December 13, 2021) as evaluation period. This period is characterized by the presence of large uncertainty in the economy and financial markets.

Fig. 2 reports the composition of the portfolios for the different investment strategies. The allocation to the S&P500 index is in red, the allocation to Gold is in brown, the allocation to Bitcoin is in yellow, and the allocation to the US 5-year Treasury bond is in green. A rapid inspection of the different panels reveals the importance of the S&P500 index and gold for the four investment strategies. Portfolios in panels (a) to (c) are quite diversified in the sense that the share of investment to the S&P500, gold and Bitcoin in each asset is similar. The smallest allocation is to the US Treasury bond but this choice is consistent across strategies. This is due to the poor performance of the bond during this period. The strategies based on the G-E-D-M−C−WCVaR and N-D-M−C−WCVaR portfolios in Panels 2(a) and 2(b) are constructed using a confidence level of α = 0.95. These strategies show more variation in the portfolio allocation than the MV portfolio. The comparison of Panels 2(c) and 2(d) also reveals the effect of shrinkage in the optimal allocation of

Table 2
Out-of-sample performance comparisons for the five competing portfolios at $\alpha = 0.95$ during the evaluation period January 29, 2020 to December 13, 2021.

Method	AR (10^{-2})	Vol (10^{-2})	SP	CVaR	MD	TR
G-E-D-M−C−WCVaR	0.1593	0.0229	0.0239	0.0666	0.1851	0.7581
D-M−C−WCVaR	0.1294	0.0243	0.0149	0.0866	0.1878	0.6158
MV	0.0815	0.0257	0.0089	0.0921	0.1988	0.3881
MV-Shrinkage	0.1357	0.0198	0.0217	0.0624	0.1486	0.6459
EWP	0.1146	0.0238	0.0133	0.0860	0.1769	0.5457

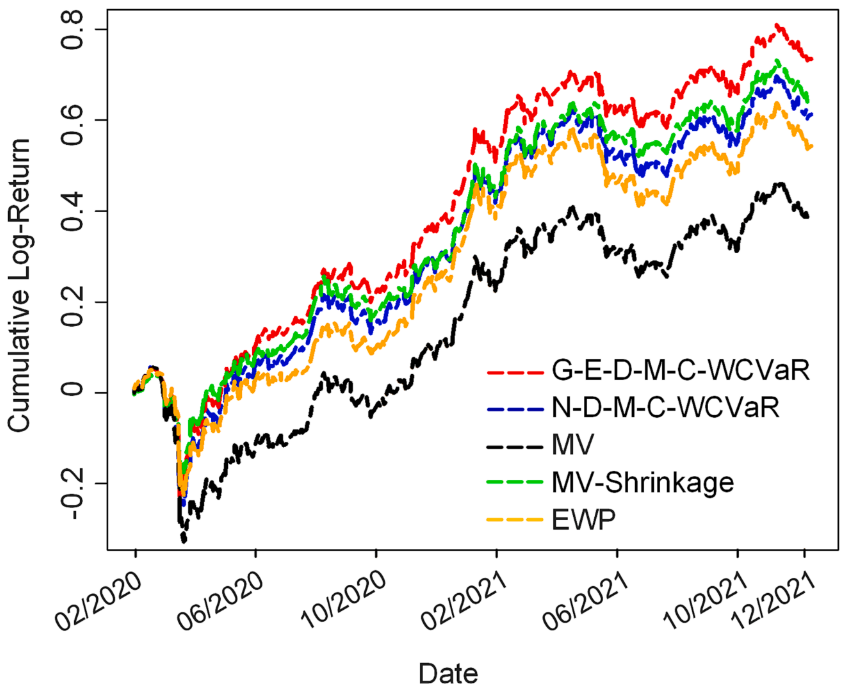


Fig. 4. The cumulative return process for the five investment portfolios at $\alpha = 0.975$ during the evaluation period January 29, 2020 to December 13, 2021.

assets to the investment portfolios. The contribution of Gold gains importance at the expense of the S&P500 index in the MV-Shrinkage portfolio throughout the COVID-19 period. This may be due to the improved ability of this portfolio to get exposure to Gold - that acts as a safe haven in market distress episodes. On the downside, we note that the portfolio weights of the MV-Shrinkage method fluctuate more than for the MV strategy, which results in higher portfolio turnover and transaction costs.

Fig. 3 reports the cumulative returns of the five investment portfolios over the evaluation period January 29, 2020 to December 13, 2021 at a confidence level $\alpha = 0.95$. Similarly, Table 2 presents performance statistics given by the average daily return (AR), volatility (Vol), Sharpe ratio (SP), CVaR, maximum drawdown (MD), and total return (TR). The performance of the G-E-D-M−C−WCVaR portfolio selection method is superior to the performance of the remaining competitors across all metrics. The profitability is higher as shown by the AR and TR measures. Risk is lower as shown by Vol, MD and CVaR. Correspondingly, the tradeoff between risk and return is also superior as shown by the SP out-of-sample statistic. Interestingly, the dynamics in Fig. 3 and the results in Table 2 also reveal the outperformance of the MV-Shrinkage method compared to the standard MV methodology and confirm the suitability of shrinkage methods for the MV strategy even in small dimensions. As shown in Panel 2(d), the outperformance during the crisis period may be due to a larger exposure to Gold compared to the MV portfolio that has similar exposure to the S&P500 index and Gold.

To assess the robustness of the results to different definitions of the tail of the distribution of the portfolio return, we consider different values of the confidence level α. Figs. 4 and 5 report the same performance analysis for higher confidence levels ($\alpha = 0.975$ and $\alpha = 0.99$). These values correspond to investment portfolios more concerned with downside risk events. Note that the MV and EWP strategies are identical to Fig. 2. The cumulative returns for the other two strategies show slight variations compared to $\alpha = 0.95$.

Tables 3 and 4 present the corresponding statistics for the performance measures for $\alpha = 0.975$ and $\alpha = 0.99$. The results confirm the outperformance of the G-E-D-M−C−WCVaR portfolio selection method for all metrics (return, risk and return/risk tradeoff) during this period under different tail risk tolerance levels. In contrast, the MV method obtains the lowest Sharpe ratio, as well as the highest volatility, downside risk, and maximum drawdown in the out-of-sample evaluation period. Importantly, this result highlights the poor

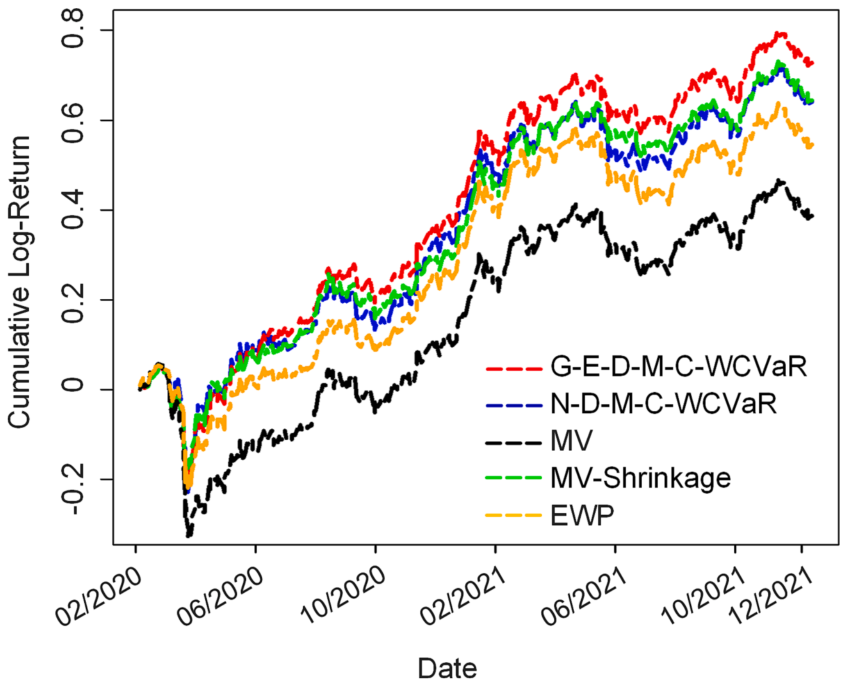


Fig. 5. The cumulative return process for the four investment portfolios at $\alpha = 0.99$ during the evaluation period January 29, 2020 to December 13, 2021.

Table 3
The out-of-sample performance statistics for the five investment strategies at $\alpha = 0.975$ during the evaluation period January 29, 2020 to December 13, 2021.

Method	AR (10^{-2})	Vol (10^{-2})	SP	CVaR	MD	TR
G-E-D-C-WCVaR	0.1548	0.0239	0.0195	0.0795	0.1866	0.7368
N-D-C-WCVaR	0.1287	0.0243	0.0150	0.0856	0.1885	0.6125
MV	0.0815	0.0257	0.0089	0.0921	0.1988	0.3881
MV-Shrinkage	0.1357	0.0198	0.0217	0.0624	0.1486	0.6459
EWP	0.1146	0.0238	0.0133	0.0860	0.1769	0.5457

Table 4
The out-of-sample performance statistics for the five investment strategies at $\alpha = 0.99$ during the evaluation period January 29, 2020 to December 13, 2021.

Method	AR (10^{-2})	Vol (10^{-2})	SP	CVaR	MD	TR
G-E-D-M−C−WCVaR	0.1517	0.0240	0.0187	0.0810	0.1875	0.7222
N-D-C-WCVaR	0.1347	0.0246	0.0166	0.0812	0.1791	0.6410
MV	0.0815	0.0257	0.0089	0.0921	0.1988	0.3881
MV-Shrinkage	0.1357	0.0198	0.0217	0.0624	0.1486	0.6459
EWP	0.1146	0.0238	0.0133	0.0860	0.1769	0.5457

performance of mainstream investment strategies such as the MV and EWP widely used by practitioners during distress episodes of the market. Interestingly, the MV-Shrinkage portfolio manages to report superior performance measures than the simple MV approach.

4.2. Performance evaluation in the Pre-COVID-19 period

This section studies portfolio performance over the period preceding the pandemic that we denominate as Pre-COVID 19 (from January 2, 2019 to January 28, 2020). The cumulative portfolio return of the five investment strategies is shown in Fig. 6 and the corresponding performance statistics are reported in Table 5. Fig. 6 shows very similar performance for the five investment strategies during the Pre-COVID-19 period. The EWP slightly outperforms, but in all cases the cumulative return has a clear positive trend. Interestingly, though, the proposed G-E-D-M−C−WCVaR strategy maintains its performance showing a robust positive trend during the COVID-19 turmoil period whereas the other four methods exhibit a negative trend during the second semester of 2019.

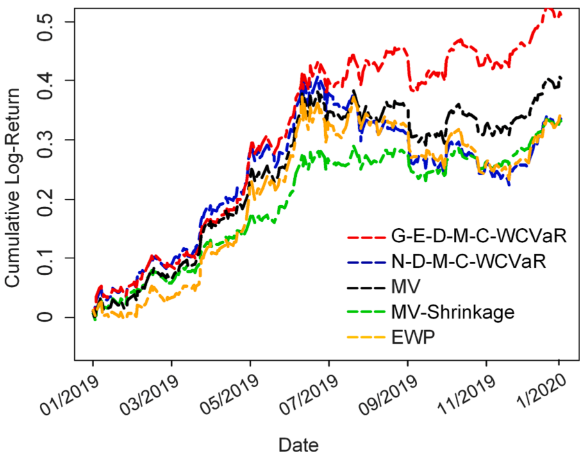


Fig. 6. The cumulative return process for the four investment portfolios at $\alpha = 0.95$ during the evaluation period January 2, 2019 to January 28, 2020.

Table 5
The out-of-sample performance statistics for the five investment strategies at $\alpha = 0.95$ during the evaluation period January 2, 2019 to January 28, 2020.

Method	AR (10^{-2})	Vol (10^{-2})	SP	CVaR	MD	TR
G-E-D-M−C−WCVaR	0.1892	0.00093	0.1152	0.0164	0.0783	0.5126
N-D-C-WCVaR	0.1237	0.01117	0.0550	0.0225	0.0905	0.3352
MV	0.1491	0.00095	0.0665	0.0224	0.0641	0.4042
MV-Shrinkage	0.1255	0.0124	0.0545	0.0231	0.0888	0.3400
EWP	0.1255	0.01241	0.0545	0.0230	0.0888	0.3400

Table 5 confirms these findings for different performance measures. The dynamic G-E-D-M−C−WCVaR portfolio significantly outperforms the different competitors in terms of Sharpe ratio, cumulative return, and CVaR. These results highlight the importance of considering the occurrence of extreme events and tail dependence along with the dynamics in the conditional volatility and correlation processes present in most modern portfolio allocation methods.

4.3. Portfolio turnover and transaction costs

The above analysis highlights the strong performance of the portfolios proposed in this study. On the downside, we observe in Fig. 2 higher turnover in the optimal allocation of assets comprising our proposed optimal portfolios. The turnover in the EWP is zero, by construction. This rapid reaction to unexpected market movements may be the reason for the strong performance of the proposed portfolios. However, introducing additional flexibility to the portfolio comes at a price. Dynamic portfolios incur significant transaction costs due to the need of rebalancing the portfolios more often. Following the investment literature and, in particular, Kirby and Ostdiek (2012), Shen et al. (2014), Li et al. (2017) and Zhang et al. (2022), these costs are introduced as percentage rates of the total investment.

Let $\widehat{w}_{i,t}$ denote the portfolio weight before rebalancing the portfolio. At time t, the weight in asset i before the portfolio is rebalanced is

$$\widehat{w}_{i,t} = \frac{w_{i,t-1}\left(1 + r_{i,t}\right)}{1 + \sum_{j=1}^{n} w_{j,t-1} r_{j,t}} \tag{19}$$

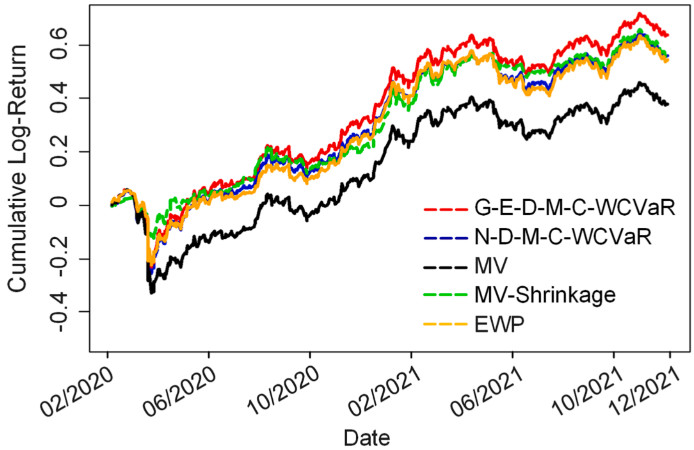
with $w_{i,t-1}(1 + r_{i,t})$ the portfolio wealth invested on asset i and $w_{i,t}$ is the portfolio weight at time t after rebalancing. Portfolio turnover (TO) at time t is defined as

$$TO_t = \sum_{i=1}^{n} \left| w_{i,t} - \widehat{w}_{i,t} \right|, \tag{20}$$

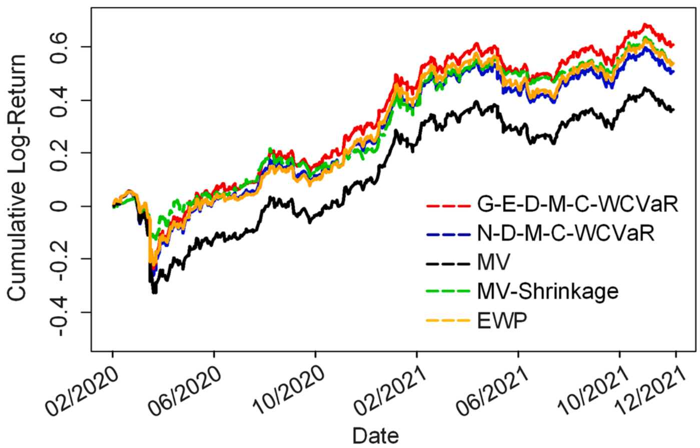
and the associated transaction costs are defined as $TC_t = \pi TO_t$ where π is the assumed level of proportional costs per transaction. Kirby and Ostdiek (2012) show that the portfolio return, $\overline{r}_{pt}$, net of rebalancing transaction costs for period t is given by

$$\overline{r}_{pt} = (1 + w_{t-1} r_t)(1 - TC_t) - 1, \tag{21}$$

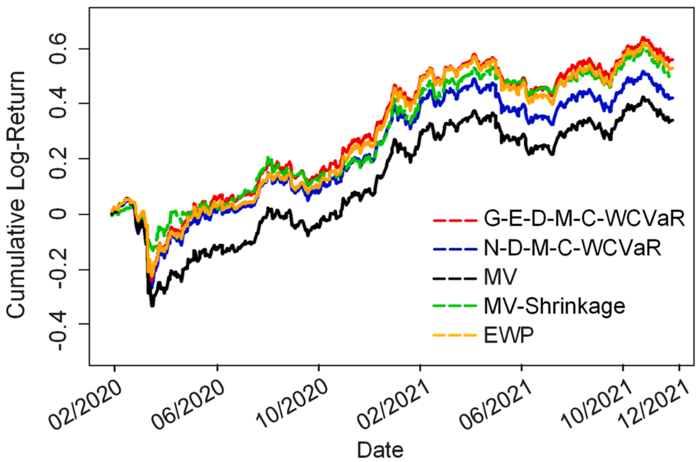
where w_{t-1} denotes the vector of portfolio weights after rebalancing and r_t is the vector of asset returns.



(a) Transaction cost rate 10 basis point



(b) Transaction cost rate 25 basis point



(c) Transaction cost rate 50 basis point

Fig. 7. The cumulative return process for the five investment portfolios during the evaluation period January 29, 2020 to December 13, 2021 at $\alpha = 0.95$. T ransaction cost rates are 10, 25, and 50 basis points, respectively.

Table 6
Average portfolio turnover (TO) for the four investment portfolios for $\alpha = 0.95,\ 0.975,\ 0.99$ when the level of proportional costs per transaction is $\pi = 10$ bps.

Method	G-E-D-M−C−WCVaR	D-M−C−WCVaR	MV	MV-Shrinkage	EWP
$\alpha = 0.95$	0.1192	0.1461	0.0410	0.0808	0.0143
$\alpha = 0.975$	0.1490	0.1446	0.0410	0.0808	0.0143
$\alpha = 0.99$	0.1311	0.2071	0.0410	0.0808	0.0143

In order to control the amount of portfolio turnover during investment, we set a constraint TR, with $TO_t < TR$, for all t, that establishes the maximum portfolio turnover in each period. Thus, dynamics in the portfolio weights over time are not allowed if TR = 0 whereas a large amount of flexibility is allowed if TR increases. The empirical application below considers TR = 1 and penalize such flexibility with different cost rates $\pi = 10, 25 \text{and} 50$ basis points (bp). The constraint on portfolio turnover is used to smooth the portfolio weights over time and avoid wide fluctuations across consecutive periods.

Fig. 7 reports the cumulative portfolio returns after considering transaction costs $TC_t = \pi TO_t$, with $\pi = 10, 25 \text{and} 50$ bps, respectively.[2] As expected, compared with the results in Fig. 3, the cumulative return process considering transaction costs, $\bar{r}_{pt}$, declines considerably for the three dynamic portfolios but most notably for the G-E-D-M−C−WCVaR and D-M−C−WCVaR investment portfolios. For example, the cumulative returns of the proposed G-E-D-M−C−WCVaR method are 0.6379, 0.6075 and 0.5567, respectively, for $\pi = 10, 25 \text{ and} 50$ bps, showing a monotonic decrease in the profitability of the portfolio as transaction costs rise. Nevertheless, our proposed G-E-D-M−C−WCVaR method still outperforms the other four methods. Interestingly, the bottom panel of Fig. 7 shows the level of transaction costs that evens the profitability of our proposed strategy and the EWP.

Table 6 reports the portfolio turnover statistic for the five investment portfolios and confidence levels $\alpha = 0.95,\ 0.975,\ 0.99$. As shown in Fig. 2, both the G-E-D-M−C−WCVaR and D-M−C−WCVaR strategies have much higher portfolio turnover rates than the other methods. These figures indicate the flexibility of the dynamic robust portfolios to adapt to market conditions. Unsurprisingly, this flexibility comes at a cost that is incorporated in the portfolio comparison through the presence of transaction costs. In addition, as the MV and EWP approaches do not take the confidence level constraint α into account the portfolio optimization problem, and thus their corresponding portfolio weights, do not change under different confidence levels, leading to the same TO value with $\alpha = 0.95,\ 0.975,\ 0.99$ Shen et al. (2014), Li et al. (2017) and Zhang et al. (2022).

5. Conclusion

This article proposes an optimal portfolio allocation based on the minimization of a tail risk measure in a worst case scenario. The optimization under the worst case adds robustness to the portfolio that is shielded against the presence of parameter uncertainty. Conceptually, the portfolio is optimized over a confidence set rather than over maximum likelihood point estimates. This investment strategy is particularly fruitful over distress episodes of the market that are characterized by abrupt changes in asset returns. The asset allocation obtained from our robust optimal portfolio strategy is dynamic and obtained by considering rolling windows over which to minimize the objective risk measure WCVAR. The optimization is achieved in several steps. First, the existence of conditional volatility clustering is filtered out using the GJR-GARCH-EVT model for the dynamics of returns. In a second stage, we model the joint dependence between the asset returns using a multivariate mixture copula model with particular emphasis on capturing tail events and negative extreme dependence.

Using this model, we have examined the portfolio performance of different portfolios constructed from a set of four representative assets (S&P 500, Gold, Bitcoin, and US 5-year Treasury bond) during two periods characterized by market turmoil and the occurrence of extreme events caused by the COVID-19 pandemic crisis. The empirical investigation shows that our dynamic WCVaR model allocates more weight to the assets with lower tail risk and lower tail dependencies. Over the COVID-19 distress episode, this strategy achieves superior performance in terms of higher cumulative returns, Sharpe ratio, lower maximum drawdown, and less risk compared to several benchmark portfolios widely used in the investment literature such as the minimum-variance and the equally-weighted portfolio. The strategy is also competitive before the outbreak of the pandemic suggesting that portfolios focused on minimizing tail risk are also useful in calm periods in which extreme returns are not a cause of concern.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Aielli, G. P. (2013). Dynamic conditional correlation: On properties and estimation. *Journal of Business and Economic Statistics, 31*, 282–299.

[2] Transaction costs of 10, 25 and 50 basis points are standard choices in the literature on portfolio investments, see DeMiguel et al (2009) and Kirby and Ostdiek (2012).

Alexander, J. M., & Rüdiger, F. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach. *Journal of Empirical Finance, 7*(3), 271–300.

Ali, F., & Y., Jiang, Y., A., Sensoy, A.,. (2021). Downside risk in Dow Jones Islamic equity indices: Precious metals and portfolio diversification before and after the COVID-19 bear market. *Research in International Business and Finance, 58*, Article 101502.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance, 9*(3), 203–228.

Basak, S., & Shapiro, A. (2001). Value-at-risk based risk management: Optimal policies and asset prices. *Review of Financial Studies, 14*, 371–405.

Bassett, G. W., Koenker, R., & Kordas, G. (2004). Pessimistic portfolio allocation and choquet expected utility. *Journal of Financial Economics, 2*, 477–492.

Belhajjam, A., Belbachir, M., & El Ouardirhi, S. (2017). Robust multivairiate extreme value at risk allocation. *Finance Research Letters, 23*, 1–11.

Ben Nasr, I., & Chebana, F. (2022). Estimation method for mixture copula models in hydrological context. *Journal of Hydrology, 615, Part A*, Article 128603.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Economics, 31*(3), 307–327.

Bollerslev, T. (1990). Modeling the coherence in short-run nominal exchange rates: A multivariate generalized ARCH model. *Review of Economics and Statistics, 72*, 498–505.

Campbell, R., Huisman, R., & Koedijk, K. (2001). Optimal portfolio selection in a value-at-risk framework. *Journal of Banking and Finance, 25*, 1789–1804.

DeMiguel, V., Garlappi, L., & Uppal, R. (2009). Optimal versus Naive Diversification: How inefficient Is the 1/n Portfolio Strategy? *Review of Financial Studies, 22*, 1915–1953.

Deng, X., & Liang, Y. (2021). Robust portfolio optimization based on semi-parametric ARMA-TGARCH-EVT model with mixed copula using WCVaR. *Computer Economics*. https://doi.org/10.1007/s10614-021-10207-5

Engle, R. F., & Manganelli, S. (2004). CAViaR: Conditional autoregressive value at risk by regression quantiles. *Journal of Business and Economic Statistics, 22*, 367–381.

Embrechts, P., Lindskog, F., & McNeil, A. (2001). *Modeling dependence with Copulas and applications to risk management*. P Embrechts: Handbook of Heavy Tailed Distributions in Finance.

Engle, R. F. (2002). Dynamic conditional correlation – A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business and Economic Statistics, 20*, 339–350.

Engle, R. F., & Gonzalez-Rivera, G. (2001). Semiparametric ARCH models. *Journal of Business and Economic Statistics, 9*(4), 345–359.

Fishburn, P. C. (1977). Mean-risk analysis with risk associated with below-target returns. *American Economic Review, 67*, 116–126.

Gaivoronski, A., & Pflug, G. (2005). Value-at-risk in portfolio optimization: Properties and computational approach. *Journal of Risk, 7*(2), 1–31.

Garcia, R., & Tsafack, G. (2011). Dependence structure and extreme comovements in international equity and bond markets. *Journal of Banking and Finance, 35*(8), 1954–1970.

Ghaoui, L. E., Oks, M., & Oustry, F. (2003). Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operational Research, 51*(4), 543–556.

Glosten, L. R., Jagannathan, R., & Runkle, D. E. (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance, 48*, 1779–1801.

Guo, X., Chan, R. H., Wong, W. K., & Zhu, L. X. (2019). Mean–variance, mean–VaR, and mean–CVaR models for portfolio selection with background risk. *Risk Management, 21*, 73–98.

Han, Y., Li, P., & Xia, Y. (2017). Dynamic robust portfolio selection with copulas. *Finance Research Letters, 21*, 190–200.

Han, Y. W., Li, P., Li, J., & Wu, S. M. (2020). Robust portfolio selection based on copula change analysis. *Emerging Markets Finance and Trade, 56*, 3635–3645.

Hellmich, M., & Kassberger, S. (2011). Efficient and robust portfolio optimization in the multivariate Generalized Hyperbolic framework. *Quantitative Finance, 11*(10), 1503–1516.

Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. *Annals of Statistics, 3*, 1163–1174.

Ibragimov, R., & Walden, D. (2007). The limits of diversification when losses may be large. *Journal of Banking and Finance, 31*, 2551–2569.

Jagannathan, R., & Ma, T. (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *Journal of Finance, 58*, 1651–1684.

James, W., & Stein, C. (1961). In *Estimation with quadratic loss* (p. 452). Berkeley: University of California Press.

Kakouris, I., & Rustem, B. (2014). Robust portfolio optimization with copulas. *European Journal of Operational Research, 235*(1), 28–37.

Karmakar, M., & Paul, S. (2019). Intraday portfolio risk management using VaR and CVaR: A CGARCH-EVT-Copula approach. *International Journal of Forecasting, 35*, 699–709.

Kirby, C., & Ostdiek, B. (2012). Optimizing the performance of sample mean-variance efficient portfolios. *AFA 2013 San Diego Meetings Paper, Available at SSRN*.

Krokhmal, P., Uryasev, S., & Palmquist, J. (2001). Portfolio optimization with conditional value- at-risk objective and constraints. *Journal of Risk, 4*(2), 43–68.

Ledoit, O., & Wolf, M. (2003). Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance, 10*, 603–621.

Ledoit, O., & Wolf, M. (2004). A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis, 88*, 365–411.

Ledoit, O., & Wolf, M. (2012). Nonlinear Shrinkage Estimation of Large Dimensional Covariance Matrices. *Annals of Statistics, 40*, 1024–1060.

Ledoit, O., & Wolf, M. (2017). Nonlinear shrinkage of the covariance matrix for portfolio selection: Markowitz meets Goldilocks. *Review of Financial Studies, 30*, 4349–4388.

Ledoit, O., & Wolf, M. (2022). The Power of (Non-)Linear Shrinking: A Review and Guide to Covariance Matrix Estimation. *Journal of Financial Econometrics, 20*, 187–218.

Li, B., Wang, J., Huang, D., & Hoi, S. C. (2017). Transaction cost optimization for online portfolio selection. *Quantitative Finance, 18*(8), 1411–1424.

Low, R. K. Y. (2018). Vine copulas: Modelling systemic risk and enhancing higher-moment portfolio optimisation. *Accounting Finance, 58*, 423–463.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance, 7*(1), 77–91.

McNeil, A. J., & Frey, R. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach. *Journal of Empirical Finance., 7*(3), 271–300.

Nikoloulopoulos, A. K., Joe, H., & Li, H. (2012). Vine copulas with asymmetric tail dependence and applications to fifinancial return data. *Computational Statistics and Data Analysis, 56*(11), 3659–3673.

Patton, A. J. (2006). Modelling asymmetric exchange rate dependence. *International Economic Review (Philadelphia), 47*(2), 527–556.

Pickands, J. (1975). Statistical inference using extreme order statistics. *Annals of Statistics, 3*, 119–131.

Rachev, S. T., Stoyanov, S., & Fabozzi, F. J. (2007). *Advanced stochastic models, Risk assessment, and portfolio optimization: The ideal risk, uncertainty, and performance measures*. Finance: John Wiley.

Raj, A., Mukherjee, A. A., Jabbour, A., & Srivastava, S. K. (2022). Supply chain management during and post-COVID-19 pandemic: Mitigation strategies and practical lessons learned. *Journal of Business Research, 1*(42), 1125–1139.

Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance, 26*(7), 1443–1471.

Sahamkhadam, M., Stephan, A., & Östermark, R. (2018). Portfolio optimization based on GARCH-EVT-Copula forecasting models. *International Journal of Forecasting, 34*(3), 497–506.

Shen, W., Wang, J., & Ma, S. (2014). Doubly regularized portfolio with risk minimization. *Proceedings of the AAAI Conference on Artifificial Intelligence, 28*, 1286–1292.

Sklar, A. (1959). Fonctions de Repartition an Dimensions et Leurs Marges. *Publications de l'Institut de Statistique deL'Universite de Paris, 8*, 229–231.

Stein, C. (1956). In *Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution* (pp. 197–206). Berkeley: Univ. of California Press.

Su, X. S., Bai, M. Y., & Han, Y. W. (2021). Robust portfolio selection with regime switching and asymmetric dependence. *Economic Modelling*, 1–34.

Sun, X. L., Liu, C., Wang, J., & Li, J. P. (2020). Assessing the extreme risk spillovers of international commodities on maritime markets: A GARCH-Copula-CoVaR approach. *International Review of Financial Analysis, 68*(C).

Topaloglou, N., Vladimirou, H., & Zenios, S. A. (2002). Cvar models with selective hedging for international asset allocation. *Journal of Banking and Finance, 26*(7), 1535–1561.

Weiß, G. N. (2013). Copula–GARCH versus dynamic conditional correlation: An empirical study on var and ES forecasting accuracy. *Review of Quantitative Finance and Accounting, 41*(2), 179–202.

Weiß, G. N., & Scheffer, M. (2015). Mixture pair-copula-constructions. *Journal of Banking and Finance, 54*, 175–191.

Wu, G., & Xiao, Z. (2002). An analysis of risk measures. *Journal of Risk, 4*(4), 53–75.

Zakoian, J. (1994). Threshold heteroskedastic models. *Journal of Economic Dynamics and Control, 18*(5), 931–955.

Zhang, Y., Zhao, P., Wu, Q., Li, B., Huang, J., & Tan, M. (2022). Cost-sensitive portfolio selection via deep reinforcement learning. *IEEE Transactions on Knowledge and Data Engineering, 34*(1), 236–248.

Zhu, S., & Fukushima, M. (2009). Worst-case conditional value-at-risk with application to robust portfolio management. *Operational Research, 57*(5), 1155–1168.