# A relation between massive and massless string tree amplitudes 

Sitender Pratap Kashyap ${ }^{a}$, Carlos R. Mafra ${ }^{b}$, Mritunjay Verma ${ }^{b, c}$, and Luis Alberto Ypanaqué ${ }^{b}$<br>${ }^{a}$ Chennai Mathematical Institute, H1 SIPCOT IT Park, Kelambakkam, Tamil Nadu, India 603103<br>${ }^{b}$ Mathematical Sciences and STAG Research Centre, University of Southampton, Highfield, Southampton, SO17 1BJ, UK and<br>${ }^{c}$ Indian Institute of Technology Indore, Khandwa Road, Simrol, Indore 453552, India


#### Abstract

We uncover a relation between the scattering amplitudes of massive strings and the $\alpha^{\prime}$ expansion of the massless string amplitude at tree level. More precisely, the $n$-point tree amplitude of $n-1$ massless and one massive state is written as a linear combination of $n+1$ massless string amplitudes at the $\alpha^{\prime 2}$ order.


## INTRODUCTION

The spectrum of string theory contains an infinite tower of massive higher-spin states alongside the massless excitations. These massive excitations are essential for the consistency of string theory, such as perturbative unitarity. And yet, the calculation of scattering amplitudes in the massive sector remains largely unexplored.

As a first line of attack, one may wish to accumulate data in the hopes of finding all-order patterns. Using the results of [1], which hugely advanced the former, this paper takes the first steps towards accomplishing the latter. More specifically, using the Berends-Giele-like construction of the $n$-point tree-level string amplitudes involving one massive and $n-1$ massless states [1], we identify a precise relation with the $\alpha^{\prime 2}$ sector of the $(n+1)$-point massless tree-level amplitudes.

The underpinnings of this relation rely on the combinatorially-rich objects dubbed scalar BRST invariants. They played a major role in the joint analysis of the $\alpha^{\prime 2}$ sector of the massless string tree amplitudes and the low-energy limit of one-loop open string amplitudes [2]. They are naturally generated using the zero-mode saturation rules in the pure spinor formalism [3] and obey several identities [4]. In addition, they are mysteriously connected to a combinatorial algorithm [5] related to Bern-Carrasco-Johansson (BCJ) amplitude relations [6] and appear in the context of the descent algebra [7]. As we will see below, yet another relation will be added to this list.

Note that the factorization of the massless amplitudes on the massive poles implies that massive and massless amplitudes are related, see for example [8]. However, the factorization condition necessarily involves a quadratic expression of massive amplitudes and, to our knowledge, has never been used to express a single massive amplitude in terms of massless data.

To avoid index positioning gymnastics, particle labels will be written mostly downstairs and vector indices mostly upstairs. Repeated indices are summed over and $\left[m_{1} \ldots m_{N}\right.$ ] does not contain $1 / N$ !.

## STRING SCATTERING WITH MASSIVE STATES

The bosonic physical states at the first massive level of the superstring are described by a symmetric traceless tensor $g_{m n}$ and a 3 -form $b_{m n p}$ of $\mathrm{SO}(10)$ subject to $\partial^{m} g_{m n}=\partial^{m} b_{m n p}=0$ and comprising 44 and 84 degrees of freedom, respectively.

In a recent paper [1], the superstring amplitude involving $n-1$ massless states and one massive state $\underline{n}$ was packaged in terms of $(n-3)$ ! worldsheet integrals $F_{Q}^{\bar{P}}$ and partial subamplitudes $A(1, P, n-1 \mid \underline{n})$ as

$$
\begin{equation*}
\mathcal{A}(1, Q, n-1, \underline{n})=\sum_{P \in S_{n-3}} F_{Q}^{P} A(1, P, n-1 \mid \underline{n}) \tag{1}
\end{equation*}
$$

where $P$ and $Q$ are words comprised of particle labels (letters) and $F_{Q}^{P}$ have the same functional form as the string disk integrals in the massless string scattering amplitude [9-12]; the only difference stems from the massive constraint $k_{\underline{n}}^{2}=-1 / \alpha^{\prime}$ affecting the relations among Mandelstam variables. These integrals will play no role in the discussions below, and we will focus our attention in the partial amplitudes $A(1, P \mid \underline{n})$.

When all external states are bosonic, the partial amplitude with $|P|=n-1$ massless states and one state from the first massive multiplet are given by [1]

$$
\begin{equation*}
A(P \mid \underline{n})=\phi_{P}^{m n} g_{\underline{n}}^{m n}+\phi_{P}^{m n p} b_{\underline{n}}^{m n p}, \tag{2}
\end{equation*}
$$

where $g_{m n}$ and $b_{m n p}$ are the massive polarizations while the $n-1$ massless states are encoded in (note the modified normalization conventions compared to [1]):

$$
\begin{align*}
& \phi_{P}^{m n}=\alpha^{\prime} \sum_{X Y=P} f_{X}^{m a} f_{Y}^{a n}+\operatorname{cyc}(P)  \tag{3}\\
& \phi_{P}^{m n p}=2 i \sum_{X Y=P} e_{X}^{m} k_{Y}^{n} e_{Y}^{p}-\frac{4 i}{3} \sum_{X Y Z=P} e_{X}^{m} e_{Y}^{n} e_{Z}^{p}+\operatorname{cyc}(P) .
\end{align*}
$$

The notation $+\operatorname{cyc}(P)$ instructs to add the cyclic permutations of the letters in $P, X Y=P$ denote the deconcatenations of $P$ into non-empty words $X$ and $Y$, and $k_{i j \ldots p}^{m}=k_{i}^{m}+k_{j}^{m}+\cdots+k_{p}^{m}$. The multiparticle polarizations in (3) obey the recursion [13] (equivalent to [14])

$$
e_{P}^{m}=\frac{1}{k_{P}^{2}} \sum_{X Y=P}\left[e_{Y}^{m}\left(k_{Y} \cdot e_{X}\right)+f_{X}^{m n} e_{Y}^{n}-(X \leftrightarrow Y)\right]
$$

$$
\begin{equation*}
f_{P}^{m n}=k_{P}^{m} e_{P}^{n}-k_{P}^{n} e_{P}^{m}-\sum_{X Y=P}\left(e_{X}^{m} e_{Y}^{n}-e_{X}^{n} e_{Y}^{m}\right) \tag{4}
\end{equation*}
$$

starting with the single-particle $e_{i}^{m}$ gluon polarization vector and its field strength $f_{i}^{m n}=k_{i}^{m} e_{i}^{n}-k_{i}^{n} e_{i}^{m}$.

Example amplitudes for $\underline{n}=3,4$ read

$$
\begin{align*}
A(1,2 \mid \underline{3}) & =i e_{1}^{m} f_{2}^{n p} b_{\underline{3}}^{m n p}+\alpha^{\prime} f_{1}^{m p} f_{2}^{p n} g_{\underline{3}}^{m n}+\operatorname{cyc}(12) \\
A(1,2,3 \mid \underline{4}) & =i\left(2 e_{1}^{m} k_{23}^{n} e_{23}^{p}+2 e_{12}^{m} k_{3}^{n} e_{3}^{p}-\frac{4}{3} e_{1}^{m} e_{2}^{n} e_{3}^{p}\right) b_{\underline{4}}^{m n p} \\
& +\alpha^{\prime}\left(f_{1}^{m a} f_{23}^{a n}+f_{12}^{m a} f_{3}^{a n}\right) g_{\underline{4}}^{m n}+\operatorname{cyc}(123) \tag{5}
\end{align*}
$$

Massless strings at $\alpha^{\prime 2}$ order: Recall the definition of $A^{F^{4}}$ as the massless disk amplitudes at $\alpha^{\prime 2}$ order [2]

$$
\begin{equation*}
A(Q)=A^{\mathrm{YM}}(Q)+\alpha^{\prime 2} \zeta_{2} A^{F^{4}}(Q)+\cdots \tag{6}
\end{equation*}
$$

We will now propose a map that replaces the massive external state $\underline{n}$ by two massless states $n$ and $n+1$ whose momenta satisfy $2 \alpha^{\prime}\left(k_{n} \cdot k_{n+1}\right)=-1$. It turns the massive $n$-point amplitude $A(P \mid \underline{n})$ into sums of massless $\alpha^{\prime 2} A^{F^{4}}$ at $n+1$ points. For convenience, let us use the shorthand $H$ for this map. More precisely,

$$
H: \begin{cases}\left(g_{\underline{n}}^{r s}, b_{\underline{n}}^{r s t}\right) & \rightarrow\left(g_{n, n+1}^{r s}, b_{n, n+1}^{r s t}\right),  \tag{7}\\ \alpha^{\prime} k_{\underline{n}}^{2}=-1 & \rightarrow 2 \alpha^{\prime}\left(k_{n} \cdot k_{n+1}\right)=-1\end{cases}
$$

with

$$
\begin{align*}
& g_{n, n+1}^{r s}= \frac{1}{8}\left(e_{n}^{r} e_{n+1}^{s}+e_{n}^{s} e_{n+1}^{r}-\frac{1}{3} \delta^{r s}\left(e_{n} \cdot e_{n+1}\right)\right)  \tag{8}\\
&+ \frac{\alpha^{\prime}}{12}\left(\left(k_{n}^{r} k_{n}^{s}-2 k_{n}^{r} k_{n+1}^{s}\right)\left(e_{n} \cdot e_{n+1}\right)\right. \\
&+3\left(k_{n+1}^{r} e_{n}^{s}+k_{n+1}^{s} e_{n}^{r}\right)\left(k_{n} \cdot e_{n+1}\right) \\
&+(n \leftrightarrow n+1)) \\
&-\frac{\alpha^{\prime}}{12} \delta^{r s}\left(k_{n} \cdot e_{n+1}\right)\left(k_{n+1} \cdot e_{n}\right) \\
&+ \frac{\alpha^{\prime 2}}{6} k_{n, n+1}^{r} k_{n, n+1}^{s}\left(k_{n} \cdot e_{n+1}\right)\left(k_{n+1} \cdot e_{n}\right) \\
& b_{n, n+1}^{r s t}= \frac{i \alpha^{\prime}}{16}\left(k_{n}^{[r} e_{n}^{s} e_{n+1}^{t]}+k_{n+1}^{[r} e_{n+1}^{s} e_{n}^{t]}\right) \\
&+ \frac{i \alpha^{\prime 2}}{8}\left(k_{n}^{[r} k_{n+1}^{s} e_{n+1}^{t]}\left(k_{n+1} \cdot e_{n}\right)\right. \\
&\left.+k_{n+1}^{[r} k_{n}^{s} e_{n}^{t]}\left(k_{n} \cdot e_{n+1}\right)\right)
\end{align*}
$$

For example, with $s_{i j}=\left(k_{i} \cdot k_{j}\right)$

$$
\begin{align*}
\left.A(1,2 \mid \underline{3})\right|_{H} & =-\alpha^{\prime 2} A^{F^{4}}(1,2,3,4), \quad s_{34}=-\frac{1}{2 \alpha^{\prime}}  \tag{9}\\
\left.A(1,2,3 \mid \underline{4})\right|_{H} & =\alpha^{\prime 2} A^{F^{4}}(1,3,4,2,5)-{\alpha^{\prime}}^{2} A^{F^{4}}(1,4,2,3,5) \\
& -\alpha^{\prime 2} A^{F^{4}}(1,2,5,3,4), \quad s_{45}=-\frac{1}{2 \alpha^{\prime}}
\end{align*}
$$

In general,

$$
\begin{equation*}
\left.A(1, P \mid \underline{n})\right|_{H}=-\frac{\alpha^{\prime 2}}{6} A^{F^{4}}\left(\gamma_{1 \mid P, n, n+1}\right), \quad s_{n, n+1}=-\frac{1}{2 \alpha^{\prime}} \tag{10}
\end{equation*}
$$

where $\gamma_{1 \mid P, n, n+1}$ are the BRST-invariant permutations related to the descent algebra defined in [7]. ${ }^{1}$ We have explicitly [15] checked the validity of (10) up to $\underline{n}=6$.

The consistency of (8) can be verified from $k_{i j}^{m} g_{i j}^{m n}=$ $k_{i j}^{m} b_{i j}^{m n p}=0$, and that $g_{m n}$ is traceless symmetric while $b_{m n p}$ is totally antisymmetric. To see this, one uses the transversality $\left(k_{i} \cdot e_{i}\right)=0$ and the mass $k_{i}^{2}=k_{j}^{2}=0$ of the gluon states and the constraint $2 \alpha^{\prime}\left(k_{i} \cdot k_{j}\right)=-1$.

## DERIVATION

The derivation of the relations (8) and (10) are the result of an alternative construction of a superstring massive vertex operator and its subsequent use in an amplitude calculation at tree level using the pure spinor formalism. In the following discussions we will briefly outline the techniques and reasoning that led to those relations. More details will appear in a longer paper [16].

CFT basics of the pure spinor formalism: The pure spinor formalism [3] is based on a conformal field theory (CFT) on the two-dimensional string worldsheet. As such, the prescription to compute tree-level amplitudes of string states is given by a correlation function of vertex operators inserted at points $z_{i}$ on a genus-zero Riemann surface

$$
\begin{equation*}
A=\left\langle V_{1}\left(z_{1}\right) V_{2}\left(z_{2}\right) V_{3}\left(z_{3}\right) \prod_{i=4}^{n} \int d z_{i} U_{i}\left(z_{i}\right)\right\rangle \tag{11}
\end{equation*}
$$

where the brackets $\langle-\rangle$ indicate a CFT correlation function (see [17] for a review). The integrated (unintegrated) vertices $\int U_{i}\left(V_{i}\right)$ for physical states at the mass level $n$ are ghost-number zero (one) expressions in the cohomology of the pure spinor BRST charge, $Q=\oint \lambda^{\alpha} d_{\alpha}$, with conformal weight $n+1(n)$ at zero momentum. $\lambda^{\alpha}$ is a bosonic spinor satisfying the pure spinor constraint $\left(\lambda \gamma^{m} \lambda\right)=0$ and $d_{\alpha}$ is the supersymmetric Green-Schwarz constraint. Finally, after integrating out the variables of non-vanishing conformal weight (see below), the amplitude prescription (11) reduces to a correlation involving only the zero-modes of $\lambda^{\alpha}$ and $\theta^{\alpha}$. They are integrated out using the prescription $\left\langle\left(\lambda \gamma^{m} \theta\right)\left(\lambda \gamma^{n} \theta\right)\left(\lambda \gamma^{p} \theta\right)\left(\theta \gamma_{m n p} \theta\right)\right\rangle=2880 \alpha^{\prime 2}$.
Massless vertices: The vertex operators for the massless states are given by [3]

$$
\begin{align*}
& V=\lambda^{\alpha} A_{\alpha}  \tag{12}\\
& U=\partial \theta^{\alpha} A_{\alpha}+\Pi^{m} A_{m}+2 \alpha^{\prime} d_{\alpha} W^{\alpha}+\alpha^{\prime} N^{m n} F_{m n}
\end{align*}
$$

[^0]where $A_{\alpha}, A^{m}, W^{\alpha}$ and $F^{m n}$ are the ten-dimensional ${ }^{2}$ linearized superfields describing the SYM multiplet while $\Pi^{m}$ is a supersymmetric momentum and $N^{m n}$ is the Lorentz current of the pure spinor. The superfields satisfy [18]
\[

$$
\begin{align*}
& D_{(\alpha} A_{\beta)}=\gamma_{\alpha \beta}^{m} A_{m}, \quad D_{\alpha} A_{m}=\left(\gamma^{m} W\right)_{\alpha}+\partial_{m} A_{\alpha}  \tag{13}\\
& D_{\alpha} W^{\beta}=\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}^{\beta} F_{m n}, \quad D_{\alpha} F_{m n}=\partial_{[m}\left(\gamma_{n]} W\right)_{\alpha}
\end{align*}
$$
\]

The variables $\lambda^{\alpha}, \theta^{\alpha}\left(\partial \theta^{\alpha}, \Pi^{m}, d_{\alpha}\right.$ and $\left.N^{m n}\right)$ have conformal weight zero (one). Thus, the massless vertices (12) have conformal weights zero and one, respectively. Furthermore, the equations of motion (13) imply $Q V=0$ and $Q U=\partial V$.
Massive unintegrated vertex: The unintegrated vertex operator for the first massive level was constructed using ten-dimensional superspace in [19],

$$
\begin{gather*}
V=\lambda^{\alpha} \partial \theta^{\beta} B_{\alpha \beta}+\lambda^{\alpha} \Pi^{m} H_{\alpha m}+2 \alpha^{\prime} \lambda^{\alpha} d_{\beta} C_{\alpha}^{\beta}  \tag{14}\\
+\alpha^{\prime} N^{m n} \lambda^{\alpha} F_{\alpha m n}
\end{gather*}
$$

where $B_{\alpha \beta}, H_{\alpha}^{m}, C^{\beta}{ }_{\alpha}$ and $F_{\alpha m n}$ are superfields encoding the massive polarization tensors (and spinors) of the first massive supermultiplet. Their equations of motion were spelled out in [19] and they were gauge fixed to

$$
\begin{gather*}
B_{\alpha \beta}=\gamma_{\alpha \beta}^{m n p} B_{m n p}, \quad \partial^{m} B_{m n p}=0  \tag{15}\\
\gamma^{m \alpha \beta} H_{m \beta}=0, \quad \partial^{m} H_{m \alpha}=0 \\
C^{\alpha}{ }_{\beta}=\frac{1}{4}\left(\gamma^{m p n q}\right)^{\alpha}{ }_{\beta} \partial_{m} B_{n p q}, \quad \gamma^{m \alpha \beta} F_{\alpha m n}=0 .
\end{gather*}
$$

The construction in [19] followed a general ansatz with the correct conformal weight and ghost number and BRST invariance $Q V=0$. Alternatively, one can derive a mass-level $n$ unintegrated vertex $V$ using the OPEs between the massless vertices. The prescription is [20, 21]

$$
\begin{equation*}
V_{3}(z)=\oint_{z} d w U_{1}(w) V_{2}(z), \quad 2 \alpha^{\prime}\left(k_{1} \cdot k_{2}\right)=-n \tag{16}
\end{equation*}
$$

where $U_{1}$ and $V_{2}$ are integrated and unintegrated massless vertices containing the plane waves $e^{i k_{1} \cdot X}$ and $e^{i k_{2} \cdot X}$ with $k_{1}^{2}=k_{2}^{2}=0$. Under the OPE, the plane waves of $U_{1}$ and $V_{2}$ combine to the plane wave $e^{i\left(k_{1}+k_{2}\right) \cdot X}$ of $V_{3}$, and the $\mathrm{n}^{\text {th }}$ mass-level condition $\left(k_{1}+k_{2}\right)^{2}=-n / \alpha^{\prime}$ gives rise to the constraint $2 \alpha^{\prime}\left(k_{1} \cdot k_{2}\right)=-n$, ensuring that the contour integral picks up the correct conformal weight. It follows from (16) with $n=1$ that $V_{3}(z)$ is BRST invariant, has ghost number one, and has conformal weight one at zero momentum. Therefore, it qualifies to be an unintegrated vertex operator for the first massive level.

[^1]Long but straightforward calculations using the OPEs between massless vertices yield an expression for the massive vertex (16) with the following massless SYM representation for the massive superfields:

$$
\begin{align*}
B_{\alpha \beta} & =-2 \alpha^{\prime} i k_{2}^{m}\left(\gamma^{m} W_{1}\right)_{\beta} A_{\alpha}^{2}-\alpha^{\prime} i k_{1}^{m}\left(\gamma^{n} W_{1}\right)_{\beta}\left(\gamma^{m n} A_{2}\right)_{\alpha} \\
& -\frac{\alpha^{\prime}}{2} F_{1}^{m n}\left(\gamma_{m n} D\right)_{\beta} A_{\alpha}^{2}  \tag{17}\\
H_{\alpha}^{m} & =A_{1}^{m} A_{\alpha}^{2}+2 \alpha^{\prime} k_{1}^{m}\left(k^{2} \cdot A^{1}\right) A_{\alpha}^{2} \\
& -2 i \alpha^{\prime} k_{1}^{m} W_{1}^{\beta} D_{\beta} A_{\alpha}^{2}-\frac{\alpha^{\prime}}{2} i k_{1}^{m} F_{1}^{n p}\left(\gamma_{n p} A_{2}\right)_{\alpha}, \\
C_{\alpha}^{\beta} & =W_{1}^{\beta} A_{\alpha}^{2} \\
F_{\alpha m n} & =F_{m n}^{1} A_{\alpha}^{2} .
\end{align*}
$$

Gauge fixing: While the vertex operator (16) with the explicit SYM realization (17) of its superfields is a legitimate unintegrated vertex operator, it still contains gauge redundancies due to $V_{3} \rightarrow V_{3}+Q \Omega$ that need to be fixed. Following the gauge-fixing procedures of [19], a long set of redefinitions detailed in [16] yields the massless SYM representation of the massive superfields satisfying the gauge conditions (15):

$$
\begin{aligned}
B_{m n p}= & \frac{1}{18} \alpha^{\prime}\left(W_{1} \gamma_{m n p} W_{2}\right)+\frac{1}{9} \alpha^{\prime 2} k_{[m}^{1} k_{n}^{2}\left(W_{1} \gamma_{p]} W_{2}\right) \\
+ & \frac{1}{18} i \alpha^{\prime 2}\left[k^{2 q} F_{q[m}^{1} F_{n p]}^{2}+(1 \leftrightarrow 2)\right] \\
H_{m \alpha}= & \frac{i \alpha^{\prime}}{6}\left(-5 i F_{m n}^{1}\left(\gamma^{n} W_{2}\right)_{\alpha}-2 k_{m}^{12} A_{n}^{1}\left(\gamma^{n} W_{2}\right)_{\alpha}\right. \\
& +k_{p}^{1} A_{n}^{1}\left(\gamma^{m n p} W_{2}\right)_{\alpha} \\
& \left.\quad-4 \alpha^{\prime} k_{m}^{12}\left(k^{2} \cdot A^{1}\right) k_{n}^{1}\left(\gamma^{n} W_{2}\right)_{\alpha}+(1 \leftrightarrow 2)\right) \\
C_{\alpha}^{\beta}= & \frac{1}{4}\left(\gamma_{m n p q}\right)^{\beta}{ }_{\alpha} i k_{12}^{m} B^{n p q} \\
F_{\alpha m n}= & \frac{1}{16}\left(7 i k_{[m}^{12} H_{n] \alpha}+i k_{q}^{12}\left(\gamma_{q[m}\right)_{\alpha}^{\beta} H_{n] \beta}\right)
\end{aligned}
$$

with $B_{\alpha \beta}=\gamma_{\alpha \beta}^{m n p} B_{m n p}$.
The massive polarization map (8): We are now in a position to explain the origin of the prescription (8). According to the $\theta$ expansion analysis of the massive superfields [22], the massive polarizations $g_{m n}$ and $b_{m n p}$ can be extracted from the massive superfields as

$$
\begin{equation*}
g^{m n}=\left.\frac{1}{64}\left(D \gamma^{(m} H^{n)}\right)\right|_{\theta=0}, \quad b^{m n p}=\left.\frac{9}{8} B^{m n p}\right|_{\theta=0} \tag{19}
\end{equation*}
$$

where the overall normalizations were chosen for later convenience. The expressions in (8) follow from the above definitions using the massless representations (18).

The origin of (10) will become clear in the following discussion of the three-point amplitude.
Three-point tree amplitude: The string three-point amplitude with one massive and two massless states was
firstly computed in the pure spinor formalism in [23] and simplified in [24]:

$$
\begin{equation*}
A(1,2 \mid \underline{3})=\frac{i}{2 \alpha^{\prime}}\left\langle V_{1}\left(\lambda \gamma_{m} W_{2}\right)\left(\lambda H_{3}^{m}\right)\right\rangle, \tag{20}
\end{equation*}
$$

where particles 1 and 2 are massless SYM states and 3 is massive. The component expansion in terms of polarization and momenta of (20) can be evaluated in two different ways:

1. Using the theta expansion of the massive superfield $H_{3 \alpha}^{m}$ in terms of $g_{m n}$ and $b_{m n p}$ derived in [22]. This yields the expression in (5).
2. Using the massless SYM representation of $H_{3 \alpha}^{m}$ and performing the calculations as a regular four-point pure spinor superspace expression, while imposing the constraint $2 \alpha^{\prime}\left(k_{3} \cdot k_{4}\right)=-1$ after the last step. This yields (with $s_{i j}=\left(k_{i} \cdot k_{j}\right)$ ),

$$
\begin{gather*}
\frac{1}{\alpha^{\prime 2}} A(1,2 \mid \underline{3})=  \tag{21}\\
s_{23}\left(\left(k_{1} \cdot e_{2}\right)\left(k_{1} \cdot e_{3}\right)\left(e_{1} \cdot e_{4}\right)-\left(k_{1} \cdot e_{2}\right)\left(k_{1} \cdot e_{4}\right)\left(e_{1} \cdot e_{3}\right)\right. \\
+\left(k_{1} \cdot e_{2}\right)\left(k_{2} \cdot e_{3}\right)\left(e_{1} \cdot e_{4}\right)-\left(k_{1} \cdot e_{2}\right)\left(k_{2} \cdot e_{4}\right)\left(e_{1} \cdot e_{3}\right) \\
-\left(k_{1} \cdot e_{2}\right)\left(k_{3} \cdot e_{1}\right)\left(e_{3} \cdot e_{4}\right)-\left(k_{1} \cdot e_{3}\right)\left(k_{2} \cdot e_{1}\right)\left(e_{2} \cdot e_{4}\right) \\
+\left(k_{1} \cdot e_{3}\right)\left(k_{2} \cdot e_{4}\right)\left(e_{1} \cdot e_{2}\right)+\left(k_{1} \cdot e_{4}\right)\left(k_{2} \cdot e_{1}\right)\left(e_{2} \cdot e_{3}\right) \\
-\left(k_{1} \cdot e_{4}\right)\left(k_{2} \cdot e_{3}\right)\left(e_{1} \cdot e_{2}\right)-\left(k_{2} \cdot e_{1}\right)\left(k_{2} \cdot e_{3}\right)\left(e_{2} \cdot e_{4}\right) \\
+\left(k_{2} \cdot e_{1}\right)\left(k_{2} \cdot e_{4}\right)\left(e_{2} \cdot e_{3}\right)+\left(k_{2} \cdot e_{1}\right)\left(k_{3} \cdot e_{2}\right)\left(e_{3} \cdot e_{4}\right) \\
\left.-\left(e_{1} \cdot e_{2}\right)\left(e_{3} \cdot e_{4}\right) s_{23}\right) \\
+s_{12}\left(\left(k_{1} \cdot e_{2}\right)\left(k_{2} \cdot e_{3}\right)\left(e_{1} \cdot e_{4}\right)-\left(k_{1} \cdot e_{3}\right)\left(k_{3} \cdot e_{2}\right)\left(e_{1} \cdot e_{4}\right)\right. \\
+\left(k_{1} \cdot e_{4}\right)\left(k_{2} \cdot e_{1}\right)\left(e_{2} \cdot e_{3}\right)-\left(k_{1} \cdot e_{4}\right)\left(k_{2} \cdot e_{3}\right)\left(e_{1} \cdot e_{2}\right) \\
+\left(k_{1} \cdot e_{4}\right)\left(k_{3} \cdot e_{2}\right)\left(e_{1} \cdot e_{3}\right)-\left(k_{2} \cdot e_{1}\right)\left(k_{2} \cdot e_{3}\right)\left(e_{2} \cdot e_{4}\right) \\
+\left(k_{2} \cdot e_{1}\right)\left(k_{2} \cdot e_{4}\right)\left(e_{2} \cdot e_{3}\right)+\left(k_{2} \cdot e_{1}\right)\left(k_{3} \cdot e_{2}\right)\left(e_{3} \cdot e_{4}\right) \\
-\left(k_{2} \cdot e_{3}\right)\left(k_{3} \cdot e_{1}\right)\left(e_{2} \cdot e_{4}\right)+\left(k_{2} \cdot e_{4}\right)\left(k_{3} \cdot e_{1}\right)\left(e_{2} \cdot e_{3}\right) \\
\left.+\left(k_{3} \cdot e_{1}\right)\left(k_{3} \cdot e_{2}\right)\left(e_{3} \cdot e_{4}\right)-\left(e_{1} \cdot e_{4}\right)\left(e_{2} \cdot e_{3}\right) s_{12}\right) \\
+s_{12} s_{23}\left(\left(e_{1} \cdot e_{3}\right)\left(e_{2} \cdot e_{4}\right)-\left(e_{1} \cdot e_{2}\right)\left(e_{3} \cdot e_{4}\right)\right. \\
\left.-\left(e_{1} \cdot e_{4}\right)\left(e_{2} \cdot e_{3}\right)\right)
\end{gather*}
$$

Further evidence for (23) stems from the fact that both sides are annihilated by shuffling $P=R \amalg S$ for nonempty $R$ and $S$; the left-hand side due to the KleissKuijf identity of the massive partial amplitude [1], and the right-hand side by construction [2, 5]. Therefore, we uncovered a hidden relation between the massive string tree amplitude with one massive external state and the $\alpha^{\prime 2}$ sector of the purely massless tree-level string amplitudes.

## CONCLUSION AND OUTLOOK

In this paper we found a relation between the $n$-point string tree amplitude with one massive and $n-1$ massless states and linear combinations of $n+1$ massless string tree amplitudes at $\alpha^{\prime 2}$ order. To see this, we defined a map that replaces the massive polarizations of one massive leg by the polarizations and momenta of two massless gluons. Then, after being transformed by this map, the partial amplitudes $A(P \mid \underline{n})$ of the full string tree amplitude (1), are written in terms of the $\alpha^{\prime 2}$ correction of the purely massless string disk amplitude.

It is not the first time that relations were discovered where some string states are replaced by others: the prime example being the KLT relations at tree level trading one graviton for two gluons [26], see also [27-30] for relations along the same lines. However, the relation found in this paper not only trades massive for massless polarizations but also connects amplitudes at different orders of $\alpha^{\prime}$ expansions.

It will be interesting to extend the observations here to more external massive states as they will probably give rise to linear combinations of amplitudes at higher $\alpha^{\prime}$ orders. How to characterize the associated permutations? Another question to investigate is related to the factorization on massive poles of the massless tree amplitudes [8]. Using the results presented here could lead to some sort of self consistency built in in the massless tree amplitudes via their $\alpha^{\prime}$ expansion. Moreover, similar relations
are also expected to hold in bosonic string amplitudes, where a wealth of data is available [31, 32].

Also, it is worth noting that there are more "topologies" of the scalar BRST invariants starting at multiplicity six; for example, $C_{1 \mid 234,5,6}$ and $C_{1 \mid 23,45,6}$. They have different combinatorial properties and their expansions in terms of $A F^{F^{4}}$ are completely different. As already explicitly checked in (22), only one topology appears at (massless) multiplicities six and seven. In general, what happens to the other topologies? Do they map to something meaningful?

Finally, it would be desirable to invert the map (7) as a means of obtaining the massive string amplitudes starting from their massless counterparts. If this is achieved and extensions with more massive legs and higher orders in $\alpha^{\prime}$ are found, it would mean that all massive amplitudes could be simply extracted from the massless amplitudes computed in [9-12].

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[^0]:    ${ }^{1}$ Note that (10) is not written in a minimal basis of $A^{F^{4}}$ amplitudes. Additional KK-like relations [2, 7] were used to arrive at the examples (9).

[^1]:    ${ }^{2}$ Recall that ten-dimensional superspace is described by $X^{m}$ with $m=1, \ldots, 10$ and $\theta^{\alpha}$ with $\alpha=1, \ldots, 16$.

