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Conformal invariance and Ricci-flat spacetimes

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Abstract

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Computing observables in conformal field theories (CFTs) in generic backgrounds and states represents an outstanding problem. In this thesis we develop a formalism to efficiently impose the kinematic constraints on the correlators of such theories based on the geometric construction of the ambient space by Fefferman and Graham. The latter is a Ricci-flat spacetime that can be thought of as a generalisation of the embedding space used for CFTs in vacuum and on conformally flat spaces.

We test this formalism in the case of Euclidean thermal CFTs. We find perfect agreement with results from the thermal operator product expansion. We further produce novel holographic results for thermal scalar 2-point functions, which match the predictions of the ambient space formalism and provide new insight into both the analytic structure of these correlators and the role played by the double-twist spectrum. We then apply our formalism to CFTs on squashed spheres, generating new expressions for their scalar 2-point correlators.

Finally, we establish connections of the ambient space with proposed approaches to flat holography and with the physics at spatial infinity in Beig-Schmidt gauge. By studying Einstein's equations at spatial infinity we are able to prove the antipodal matching of the asymptotic BMS charges, a crucial assumption at the basis of a well-defined gravitational scattering problem in General Relativity and celestial holography.

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

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    and parts will appear in

    - E. Parisini, K. Skenderis, and B. Withers, *The ambient space formalism* (to appear) 2023

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Contents

Chapter 1

Introduction

Gravity is a long-range force that is experimentally well-described by General Relativity (GR).[1] The theories that capture to exceptional accuracies the remaining forces of nature and matter as we know it at low energies are quantum field theories (QFTs).

It has been a self-evident fact since Newton and his apple tree that matter couples to the gravitational field. As an unavoidable consequence, also the gravitational field must exhibit a quantum nature – if the dynamics of matter is quantum, the dynamics of gravity must be quantum too. Although not apparent at any of the energy scales so far probed experimentally, we expect this quantum nature to become manifest at sufficiently high energies. This line of reasoning calls for a quantum UV completion of gravity, which is now to be meant as a low-energy effective field theory (EFT), regardless of whether such EFT is GR or a low-energy modified gravity theory.

Aside from its own theoretical relevance, there is already compelling evidence in our universe that urges us to find a consistent theory of quantum gravity. Such evidence in particular includes black holes. Data supporting their existence in our universe has been collected using several experimental approaches, recently culminating in the direct proves from gravitational waves detections [2] and interferometric images in the infrared [3]. According to GR, black holes typically involve a curvature singularity inside their horizon, a hint that the classical description of such gravitational systems breaks down there. Another prominent physical setting where classical gravity fails is realised by the early phases of the universe, whose full understanding is conventionally thought of as attainable only through a quantum gravity description.

The most promising and best developed candidate for a theory of quantum gravity is string theory, a framework which has already provided countless insights on the quantum nature of black holes (see e.g. [4, 5]) and which is able to produce predictions on the particle spectrum and cosmological observables we see in the sky [6]. Unpleasantly, most

[1] There is well-known experimental data [1] at the galactic scales that lends itself to interpretations in terms of modified theories of gravity. Reassuringly, whether one's favourite classical theory of gravity is GR or a modified gravity theory will not play any role in the following.

of the control we have of string theory is at a perturbative level; an important exception is represented by the Anti de Sitter / conformal field theory (AdS/CFT) duality.

This duality is a realisation of the so-called *holographic principle* [7, 8], which states that a theory of quantum gravity can be equivalently described as a non-gravitational quantum theory in one dimension less. Historically this assertion finds its origin in the study of black hole physics, in particular in the form of the black hole entropy in semi-classical gravity [9]. The so-called *Bekenstein-Hawking entropy* of a black hole takes the form

$$S_{BH} = \frac{A}{4G}\,, \tag{1.1}$$

where G is the gravitational coupling constant and A is the area of the black hole event horizon. It is in this sense that the information stored in a black hole in $d+1$ dimensions is effectively encoded in an observable related to a d-dimensional hypersurface.

In particular, the AdS/CFT correspondence states that any consistent quantum gravity theory on a $(d+1)$–dimensional space with negative cosmological constant (possibly times a compact space) is equivalent to a CFT on a d-dimensional manifold which can be thought of as living at the boundary of the former $(d+1)$–dimensional space [10–12]. In Section 1.3 we explain this statement in details, and it suffices for now to remark that by equivalence we mean that the physical observables in one theory can be computed using the dual theory. Moreover, in $d = 4$ AdS/CFT has a well-motivated string theory realisation.

As we review in Section 1.1, CFTs constitute a special family of quantum field theories that enjoy a large set of symmetry, *conformal symmetry*. They are ubiquitous in nature since they appear as fixed points under Renormalisation Group flow, and at second-order phase transitions. However through the eyes of AdS/CFT, CFTs and closely related theories also describe quantum gravity. The classification of consistent conformal field theories and the development of techniques to characterise their observables thus become even more important problems to address – any result on these issues directly translates into a statement about quantum gravity and its observables.

Recently some progress has been made towards mapping the space of consistent CFTs using bootstrap techniques [13, 14] (see [15] for a complete review). Conformal symmetry highly constrains CFT observables [16, 17]. In particular both 2-point and 3-point functions of primary operators are fixed by conformal symmetry up to constants, while higher-point functions are fixed up to functions of cross-ratios. To this regard, an important tool is the *embedding space formalism* [18–21], which can be used to efficiently implement the kinematical constraints on CFT n–point functions of arbitrary spin. It takes advantage of the fact that the conformal group $\mathrm{SO}(1, d+1)$ in d dimensions can be realised as the Lorentz group in $(d + 2)$ dimensions. Imposing conformal invariance on CFT observables on any d-dimensional conformally flat background[2] simply reduces to

[2]By conformally flat space we mean a manifold whose metric is related to that of flat space by a

demanding Lorentz invariance on the embedding space $\mathbb{R}^{1,d+1}$, as we review in Section 1.2. This entails that finding conformal blocks and conformally-invariant tensor structures reduces to listing tensors on $(d+2)$–dimensional Minkowski spacetime.

However, most known techniques address CFTs in the vacuum state and on conformally flat backgrounds, and much fewer results are known for more general setups such as CFTs in non-trivial states and on generic source backgrounds. One of the most relevant such examples are thermal CFTs; aside from being of high interest as condensed matter systems, via AdS/CFT they represent full quantum gravity black hole solutions.

One of the major achievements in this thesis is the development of a new framework that generalises the embedding space to CFTs on generic backgrounds and states. More specifically, the setup of the problem we addressed is the following. Given a CFT on a (generically non-conformally flat) metric background $g_{(0)}$ and in a state defined by the VEVs $\{\langle O_i \rangle\}$, we would like to encode such information in a $(d+2)$–dimensional spacetime, which is meant to be used to conveniently impose the kinematical constraints on the CFT observables.

For a fixed d–dimensional metric $g_{(0)}$, there is a canonical way to construct a $(d+2)$–dimensional spacetime with similar properties to the embedding space. Such construction is known as the *ambient space* [22, 23]. It has an important role in the mathematical literature in the context of conformal geometry, and we believe its potential in high-energy physics is far from being fully explored.

In Chapter 2 we give a detailed presentation of the ambient space geometry, reviewing its properties and how it canonically incorporates Weyl covariance. By construction, the ambient space is Ricci-flat and exhibits a nullcone structure analogous to the one of Minkowski space. The d-dimensional manifold with metric $g_{(0)}$ is recovered as a section of the ambient nullcone. Weyl transformations are realised as a special class of ambient diffeomorphisms that preserve the nullcone structure. In Section 2.2 we obtain a coordinate transformation illustrating how the ambient space can be related to the embedding space for conformally flat $g_{(0)}$ (and vanishing VEVs), and how this provides us with important guiding principles to extend the embedding space formalism to CFTs in non-trivial backgrounds.

In Chapter 3 we present new material detailing how to find and classify ambient isometries. Our results show that similarly to the embedding space, the residual conformal Killing vectors on $g_{(0)}$ are lifted to isometries on the ambient space.[3]

There is however another piece of CFT data that must be encoded in the ambient space in some way, that is the CFT state $\{\langle O_i \rangle\}$. In Chapter 4 we propose a prescription to attain this for CFTs where the multi-stress tensor operators acquire a VEV, as first presented in [24]. Such prescription is inspired by the AdS/CFT dictionary, although it does not rely on it. More specifically, the ambient space geometry can be canonically

Weyl transformation.

[3]For non-conformally flat $g_{(0)}$, conformal symmetries only form a subgroup of $\mathrm{SO}(1, d+1)$.

sliced in terms of $(d+1)$–dimensional hyperbolic spaces. According to our proposal, given a background $g_{(0)}$ and a state $\{\langle O_i \rangle\}$, the associated ambient space has the hyperbolic spaces prescribed by AdS/CFT for that background and state as slices. This requirement fully fixes the ambient space geometry in terms of $g_{(0)}$ and $\{\langle O_i \rangle\}$.

To efficiently implement the kinematical constraints imposed by Weyl invariance and by the residual conformal symmetries of a CFT in a given background and state in a similar fashion as the embedding space, we must find appropriate geometrical quantities on the ambient space. We analyse them in Section 4.1, while in Section 4.2 we illustrate how to assemble them into suitable building blocks. The latter ultimately consist of the geodesic distances between insertions as well as of a class of multi-local curvature invariants on the prescribed ambient space. In Sections 4.3 and 4.4 we write the most general form that scalar n-point functions take in terms of such ambient building blocks, while in Section 4.5 we discuss the generalisation to correlators of arbitrary spin and to different CFT states.

As it will become apparent in the following, the ambient space formalism produces strong predictions on the form of CFT correlators. In Chapter 5 we apply it to thermal CFTs, which as it was remarked, describe black holes in quantum gravity through AdS/CFT. In Section 5.4 we compare our results with the thermal operator product expansion (OPE) [25] finding perfect agreement. In Section 5.5 we produce novel results on thermal holographic scalar 2-point functions (both perturbative and non-perturbative in the temperature), in particular shedding light on the analytic structure of such correlators. Along the way we illustrate a new regime where the thermal holographic correlator can be computed exactly to arbitrarily high order in momentum space, and we show how double-twist contributions to scalar 2-point functions arise as a consequence of the periodic time direction. We then successfully test the ambient space predictions against these holographic results.

In Chapter 6 we study CFTs on squashed spheres and their correlators using the ambient space formalism. Squashed spheres are an interesting class of non-conformally flat backgrounds which display very limited isometries. The squashing can be thought of as a parameter informing us about the breaking of conformal invariance – in the limit of zero squashing, we recover a round sphere and hence full background conformal symmetry. Although CFTs on squashed spheres have been studied in the literature (see in particular [26–33]), the problem of finding the general solutions to the kinematic constraints on correlators has never been tackled due to the peculiarities of these theories as compared with CFTs on conformally flat backgrounds. We initiate such program by first discussing the general form of 1-point functions on squashed spheres. This allows us to set up the ambient formalism for this class of theories in Section 6.3, and in Section 6.4 geodesics on such ambient spaces are solved perturbatively at small squashing. In Section 6.5 we construct the relevant ambient building blocks and we give expressions for scalar 2-point functions of these theories. Interestingly, we find a mismatch with a previous Ansatz made for scalar 2-point functions on squashed spheres in [27]. We conclude in Section 6.6 with open questions regarding CFTs on squashed spheres and how the ambient formalism can be fruitfully applied to these setups.

Aside from being a useful tool to study correlators in non-trivial backgrounds and states, the ambient space has also an intriguing holographic flavour in that it encodes observables of a d-dimensional non-gravitating CFT into geometric quantities on a $(d+2)$-dimensional Ricci-flat spacetime. Supposedly being a codimension-2 kind of holography, this appears quite different from AdS/CFT. However, one of the currently best motivated bottom-up approaches to holography in spacetimes with vanishing cosmological constant is codimension-2 and is known as *celestial holography* [34–38].

As we review in Section 7.2, in celestial holography scattering processes in four dimensional Minkowski are proposed to be dual to CFT correlators on a two-dimensional section of null infinity, the so-called celestial sphere. In a scattering process on Minkowski space, data on the celestial sphere at past null infinity evolves along the nullcone up to the celestial sphere at future null infinity. Thus in this picture the holographic data at infinity is strictly connected to the physics on the nullcone, which is an essential feature of the ambient geometry. These observations motivate our endeavours in Sections 7.3 and 7.4 to relate the ambient construction to celestial holography. As a bonus, the ambient space is Ricci-flat as opposed to Riemann-flat, meaning that it may contain important information about how to generalise the celestial framework to asymptotically flat spacetimes other than four-dimensional Minkowski.

More specifically, in Sections 7.3 and 7.4 we discuss how the asymptotic symmetries of spacetimes with a vanishing cosmological constant are related to the ambient isometries presented in Chapter 3, and how the asymptotic gravitational data maps to the degrees of freedom in the ambient geometry. To this aim we also present the Beig-Schmidt gauge [39], an adapted set of coordinates to describe gravitational physics in a neighbourhood of past, spatial and future infinities. Interestingly, it can be thought of as a generalisation of the ambient geometry where the lightcone structure is generally broken. Finally, we comment at length on the relations of celestial holography and the ambient space with an alternative approach to flat holography first presented in [40, 41]. This approach uses a non-compact dimensional reduction based on the hyperbolic slicing of Minkowski space and it has recently regained a certain popularity (see among the others [42–45]).

A crucial step in the construction of celestial holography consists in the assumption of the antipodal matching of the asymptotic charges between past infinity and future infinity across spatial infinity. Although this is a trivial assumption in Minkowski spacetime, it becomes highly non-trivial when considering spacetimes which are not perturbatively close to Minkowski space, where spatial infinity is typically a non-differentiable locus, as we discuss in Section 8.1. This represents a particularly interesting problem as many of the efforts in flat holography are towards the generalisation of the existing frameworks based on Minkowski spacetime (where they often reduce to matching symmetries) to excited flat bulks, where the dynamics of the dual QFT plays an essential role [46–49].

In Chapter 8 we prove these antipodal matching conditions for the asymptotic charges in a large class of spacetimes with vanishing cosmological constant, as first presented in [50]. We do so by studying the dynamics of the gravitational field near spatial infinity. We prescribe data at past null infinity, and evolve it in a vicinity of spatial infinity up to future infinity. Note that it is the choice of the prescribed data at past null infinity that specifies to which class of spacetimes we are restricting. Since spatial infinity is in general a singular locus, the key feature of our approach is to work in a neighbourhood of spatial infinity, instead of taking the limit of all the quantities to spatial infinity. In practice, we fix data at past null infinity using the Bondi gauge (reviewed in Section 7.1), and map these degrees of freedom to the free functions in the Beig-Schmidt gauge. Such fields in Beig-Schmidt have a defined parity under the antipodal map as a consequence of Einstein's equations. This entails specific parity properties of the fields in Bondi gauge, from which the antipodal matching of the asymptotic charges follows. This is an important step towards the definition of a gravitational scattering problem in generic spacetimes with a vanishing cosmological constant, as well as a milestone towards an extension of celestial holography beyond Minkowski spacetime.

Conventions. Unless stated otherwise, in this thesis we denote $(d+2)$-dimensional indices by capital Latin letters, $(d+1)$-dimensional indices by lowercase Greek letters, and d-dimensional indices by lowercase Latin letters. In most of Chapters 7 and 8 we however use a different convention: four-dimensional indices are labelled by lowercase Greek letters, three-dimensional indices by lowercase Latin letters, and two-dimensional indices by capital Latin letters.

1.1 Conformal field theories

Euclidean CFTs in $d > 2$ dimensions are quantum field theories whose symmetries realise the conformal group $\mathrm{SO}(1, d+1)$ when defined on a conformally flat background $g_{(0)}$ and in the vacuum state [16, 17, 51–54]. Denoting by x^i the coordinates on $g_{(0)}$, the infinitesimal generators ξ of conformal transformations must satisfy the conformal Killing equations,

$$\nabla_i \xi_j + \nabla_j \xi_i = 2\psi \, g_{(0)ij}, \tag{1.2}$$

where $\psi(x) = \frac{1}{d} \nabla_l \xi^l$ is the conformal factor. These equations mean that we are allowing the metric components to change at most by a rescaling under any such diffeomorphism,

$$\mathcal{L}_\xi g_{(0)ij} = 2\psi \, g_{(0)ij} \,, \tag{1.3}$$

where $\mathcal{L}$ is the Lie derivative operator. If we specialise to flat space, $g_{(0)ij} = \delta_{ij}$, the most general infinitesimal transformation takes the form

$$\xi = \left[a^i + \omega^i{}_j x^j + \lambda x^i + b^i x^2 - 2b^k x_k x^i \right] \partial_i. \tag{1.4}$$

Here a^j parametrises the d translations P_j, while the antisymmetric ω_{ij} parametrises the rotations M_{ij}. These are proper Killing vectors and do not generate a rescaling of the metric components. The transformations parametrised by λ are dilations D, while the d parameters b^j parametrise special conformal transformations K_j. They produce a rescaling of the metric components with conformal factor

$$\psi(x) = \lambda - 2b \cdot x \,. \tag{1.5}$$

CFT operators must fall into representations of the conformal group and it is conventional to organise the operator spectrum in terms of lowest weight representations. Recalling the anti-commutators

$$[D, P_j] = P_j, \qquad\qquad [D, K_j] = -K_j, \tag{1.6}$$

we define primary operators O as the lowest weight states, satisfying at the origin $x^j = 0$

$$[D, O(0)] = \Delta\, O(0)\,, \qquad\qquad [K_j, O(0)] = 0\,, \tag{1.7}$$

and transforming in the appropriate $\mathrm{SO}(d)$ representation according to their spin. Here Δ is the scaling dimension of the primary O. Descendant operators of a given primary are constructed by acting with the translation operator P_j, and they have scaling dimensions greater than Δ.

If a quantum theory is invariant under a set of transformations, its observables must satisfy corresponding Ward Identities. In CFTs, the conformal Ward Identities for an n-point function can be written as

$$\sum_{i=1}^{n} \mathcal{L}_\xi^{(i)} \langle O_1(x_1) \dots O_n(x_n) \rangle = 0, \tag{1.8}$$

for each of the $\frac{d(d+1)}{2}$ generators ξ of the conformal group. The symbol $\mathcal{L}^{(i)}$ denotes the Lie derivative acting on the i-th insertion only. Unless Weyl anomalies are present, under a Weyl transformation of the metric $g_{(0)} \to \Omega^2(x) g_{(0)}$ CFT correlators also satisfy

$$\langle O_1(x_1) \dots O_n(x_n) \rangle_{\Omega^2 g_{(0)}} = \Omega(x_1)^{-\Delta_1} \dots \Omega(x_n)^{-\Delta_n} \langle O_1(x_1) \dots O_n(x_n) \rangle_{g_{(0)}}\,, \tag{1.9}$$

regardless of the spin of the operators.

Solving the differential equations provided by these Ward Identities strongly constrains the form of low-point functions. 1-point functions are all vanishing, while 2- and 3-point functions are fixed up to an overall constant. For instance, indicating $|x_i - x_j| = x_{ij}$, on flat space scalar 2-point functions take the form

$$\langle O(x_1) O(x_2) \rangle = \frac{C_\Delta}{(x_{12})^{2\Delta}}, \tag{1.10}$$

non-vanishing only for $\Delta_1 = \Delta_2 = \Delta$, while for scalar 3-point functions

$$\langle O(x_1)O(x_2)O_3(x_3)\rangle = \frac{C_{123}}{(x_{12})^{v_{123}}(x_{13})^{v_{132}}(x_{23})^{v_{231}}}, \qquad v_{ijk} = \Delta_i + \Delta_j - \Delta_k\,. \tag{1.11}$$

Generic scalar n-point functions with $n \geq 4$ are instead only fixed up to functions of the so-called cross-ratios. We can write them as

$$\langle O_1(x_1)\dots O_n(x_n)\rangle = \left(\prod_{i<j}(x_{ij})^{2\alpha_{ij}}\right) f\left(y\right)\,, \tag{1.12}$$

where α_{ij} are defined through $\Delta_i = -\sum_{j=1}^{n}\alpha_{ij}$, and y denotes the set of cross-ratios

$$y_{[pqrs]} = \frac{x_{pr}x_{qs}}{x_{pq}x_{rs}}\,. \tag{1.13}$$

The constants C_Δ, C_{ijk} and the function $f(y)$ are not fixed by the kinematic constraints and they encode the dynamics of the CFT. In the next section we review how to conveniently implement these constraints on correlators using a geometric approach.

A particularly powerful statement in CFT is the operator product expansion (OPE). Inside a correlator, the product of two operators $O_1(x_1)$ and $O_2(x_2)$ evaluated at non-coincident insertion points $x_1 \neq x_2$ can be decomposed as a sum over the primaries ϕ of the theory and all their descendants,

$$O_1(x_1)O_2(x_2) = \sum_{\phi\in O\times O} h_{12\phi}(x_i,\partial_i)\phi(x_2)\,. \tag{1.14}$$

This sum can be proven [55] to be convergent, with radius of convergence defined by the position of the other closest insertion.

1.2 The embedding space

The key idea at the root of the embedding space construction is that conformal transformations on $\mathbb{R}^d$ form the group $\mathrm{SO}(1,d+1)$, and hence can be realised as Lorentz transformations in the embedding space, $\mathbb{R}^{1,d+1}$. The advantage of this perspective is that conformally-covariant quantities on $\mathbb{R}^d$ can be easily represented as Lorentz tensors [18–21, 56]. In this section we review how to embed $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$ and how this can be used to efficiently constrain CFT correlation functions.

To find an embedding we parameterise $\mathbb{R}^{1,d+1}$ with a set of coordinates $X^M = \left(X^0, X^i, X^{d+1}\right)$ and the Minkowski metric, $ds^2 = \eta_{MN}dX^M dX^N$. A Lorentz invariant locus of $\mathbb{R}^{1,d+1}$ is given by $X^2 = \text{const}$. This gives a $d+1$ dimensional space which we need to reduce further to d dimensions. This is achieved by restricting to the lightcone, $X^2 = 0$ and picking a section $X^+ = \mathcal{F}(X^i)$, where $X^\pm = X^0 \pm X^{d+1}$.

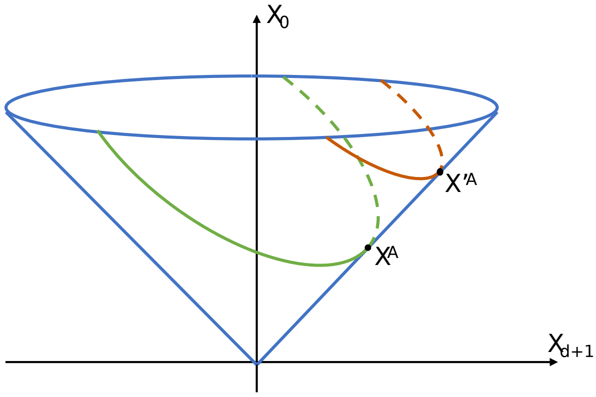

Figure 1.1: The points X^A and X'^A lie on different lightcone sections but on the same light-ray. Hence they are represented by the same point on the projective slice.

The only sectional choice which gives $\mathbb{R}^d$ and preserves conformal transformations $\tilde{X}^A = \Lambda^A{}_B X^B$ is given by a constant function, $X^+ = t$, with the embedding map

$$X^M = t\left(\frac{1+x^2}{2}, x^i, \frac{1-x^2}{2}\right), \tag{1.15}$$

where x^i denote coordinates on $\mathbb{R}^d$ with the induced metric $g_{(0)ij} = t^2\delta_{ij}$. Here, changing the choice of constant t can be viewed as a gauge transformation. More precisely, one can define an equivalence of points in embedding space, based on whether they are connected by a light-ray,

$$X^A \sim X'^A \qquad \Longleftrightarrow \qquad X'^A = tX^A, \tag{1.16}$$

for some non-vanishing real t (see Figure 1.1). This amounts to describing $\mathbb{R}^d$ with projective coordinates,

$$x^i = \frac{X^i}{X^+}, \tag{1.17}$$

and one can simply work on this projective slice. This is a particularly useful perspective that will be adopted in the ambient space construction.

The more general choice of lightcone section $X^+ = \mathcal{F}(X^i) = \Omega(x^i)$ allows one to describe manifolds other than $\mathbb{R}^d$. In this case, the embedding map takes the form

$$X^M = \Omega(x)\left(\frac{1+x^2}{2}, x^i, \frac{1-x^2}{2}\right), \tag{1.18}$$

with a conformally flat induced metric, $g_{(0)ij} = \Omega(x)^2\delta_{ij}$. This is the most general class of $d-$dimensional spacetimes that can be embedded in the Minkowski lightcone preserving its structure. Thus, global rescalings of the embedding coordinates (generated by the dilation vector $X^M\partial_M$) end up describing the same projective slice, while local rescalings induce Weyl transformations on the CFT background.

The embedding space machinery outlined above allows one to write all kinematic constraints on conformal correlators in a simple and convenient fashion. In particular, conformal invariance is realised by Lorentz invariance in the embedding space, while Weyl covariance is realised by the freedom in the choice of the lightcone section, $\mathcal{F}(X^+)$. In what follows, we treat correlators on the embedding space as multi-local conformal densities depending on the insertion points on the lightcone $\{X_i\}$ and with dimensions $\{\Delta_i\}$, where $i = 1 \ldots n$ labels the insertion.

Invariance under Lorentz transformations, generated by $J_{MN} = X_M\partial_N - X_N\partial_M$, result in the following Ward Identities

$$\sum_{i=1}^{n} J_{MN}^{(i)} \langle O_1(X_1) \ldots O_n(X_n) \rangle = 0, \tag{1.19}$$

where $J_{MN}^{(i)}$ acts on X_i. This is simply a rewriting of the conformal Ward Identities (1.8) in embedding language. Thus, finding the form of correlators on the embedding space reduces to enumerating the compatible Lorentz tensor structures. For 2- and 3-point functions of scalar primaries, the only available invariants consist in the pairwise products of the insertion points,

$$X_{ij} = -2X_i \cdot X_j, \tag{1.20}$$

which are equal to the square distances $x_{ij}^2 = |x_i - x_j|^2$ once reduced onto a d-dimensional section.

For Weyl transformations, correlators of a CFT on a background $g_{(0)}$ transform as (1.9). In the embedding space, the correlator on the left hand side is simply the embedding space correlator in a different lightcone section. Thus the transformation (1.9) is realised by an adjustment to the function $\mathcal{F}(X^i)$, giving different embedding maps (1.18). For instance the invariants X_{ij} transform as

$$X_{ij} \to \Omega(x_i)\Omega(x_j)X_{ij} \tag{1.21}$$

and consequently constrain the form of the correlator.

Note that in the above discussion, we had to take into account the whole lightcone and not just the projective slice so as to make correlators well-defined on every $d-$dimensional conformally flat space. Being defined exclusively on the lightcone, correlators in the embedding space are determined up to contributions $\sim X^2$. This gauge redundancy will play an interesting role when discussing the ambient space.

Let us consider some simple examples for illustration. For scalar 2-point functions with embedding insertions X_1 and X_2, Lorentz invariance implies that it must be a function of the invariant X_{12}. Furthermore, Weyl covariance fixes this function up to a multiplicative constant, and makes the 2-point function non-vanishing only for identical operators,

$$\langle O(X_1)O(X_2)\rangle = \frac{C_\Delta}{(X_{12})^\Delta}, \tag{1.22}$$

where O is an operator of dimension Δ. This expression returns the correct CFT correlator (1.10) once reduced back onto the lightcone section. Following similar arguments for scalar 3-point functions, Lorentz invariance and Weyl covariance determine

$$\langle O_1O_2O_3\rangle = \frac{C_{123}}{(X_{12})^{\alpha_{123}}(X_{13})^{\alpha_{132}}(X_{23})^{\alpha_{231}}}, \qquad \alpha_{ijk} = \frac{\Delta_i+\Delta_j-\Delta_k}{2}, \tag{1.23}$$

again matching the form of (1.11). Finally, as we know scalar higher-point functions are only fixed by conformal symmetry up to functions of the cross-ratios. We can conveniently express them on the embedding space as

$$\langle O_1(X_1)\dots O_n(X_n)\rangle = \left(\prod_{i<j}(X_{ij})^{\alpha_{ij}}\right) f(u)\ , \tag{1.24}$$

where α_{ij} are defined as in (1.12), while now we denote the cross-ratios by

$$u_{[pqrs]} = \frac{X_{pr}X_{qs}}{X_{pq}X_{rs}}\,. \tag{1.25}$$

Note that the expression (1.24) automatically satisfies the requirement of Weyl-covariance (1.9) as a consequence of the scaling property (1.21).

Without going into details, we point out that the embedding space formalism is particularly powerful for dealing with spinning correlators, since elaborate conformal tensor structures can be written as simple tensors on the embedding space, where useful differential operators can also be constructed [20, 21, 57].

Finally we note that the embedding space is a useful tool for treating holographic duals of CFTs. This is because aside from the lightcone $X^2=0$ discussed above, another Lorentz-invariant locus in the embedding space is Euclidean AdS$_{d+1}$, given by the upper half-hyperboloid

$$X^2 = -R^2 \qquad \text{with } X^0>0, \tag{1.26}$$

and the Poincaré patch $ds^2=\frac{R^2}{r^2}\left[dr^2+\delta_{ij}dx^idx^j\right]$ via the map

$$X^M = \left(X^0, X^i, X^{d+1}\right) = R\left(\frac{1+x^2+r^2}{2r}, \frac{x^i}{r}, \frac{1-x^2-r^2}{2r}\right). \tag{1.27}$$

A key observation is that the embedding space allows one to represent bulk and boundary point covariantly in the same language. Denoting by X and P the bulk and boundary point respectively, the scalar bulk-to-boundary propagator reads

$$K_\Delta(X,P) = \frac{C'_\Delta}{(-2P\cdot X)^\Delta}. \tag{1.28}$$

Note that modulo the normalization, its form matches that of scalar 2-point functions (1.22).

1.3 Overview of AdS/CFT

In this section we provide an introduction to the key concept of AdS/CFT that will be used in the following chapters. The AdS/CFT duality [10–12, 58, 59] can be stated as an equality between the partition functions of two theories. The first is a gravitating theory with negative cosmological constant, and the second one is a QFT in one dimension less,

$$Z_{grav}\left[\phi_{(0)}\right] = Z_{QFT}\left[\phi_{(0)}\right] , \tag{1.29}$$

where for the gravitational theory $\phi_{(0)}$ represents the prescribed values of the bulk fields at the boundary, while for the dual QFT $\phi_{(0)}$ plays the role of the sources for the dual operators. Denoting the bulk by M and its boundary by ∂M,

$$Z_{grav}\left[\phi_{(0)}\right] = \int_{\phi_{(0)}} \mathcal{D}\phi\, e^{-S[\phi]} , \tag{1.30}$$

$$Z_{QFT}\left[\phi_{(0)}\right] = \left\langle \exp\left(- \int_{\partial M} d^d x\, O\phi_{(0)} \right) \right\rangle_{QFT} . \tag{1.31}$$

Several checks have been performed for the duality (1.29) in the classical bulk limit, where the gravitational partition function reduces to the bulk onshell action, while the QFT partition function becomes the generating function of connected correlators,

$$S_{\text{onshell}}\left[\phi_{(0)}\right] = -W_{QFT}\left[\phi_{(0)}\right] . \tag{1.32}$$

In this thesis we will work in this limit, where gravity in the bulk is classical.

One of the best understood setups realising (1.29) is type IIB string theory on $\text{AdS}_5 \times \mathbb{S}^5$ backgrounds, which is conjectured to be dual to a $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills gauge theory. For generic coupling and number of colors, (1.29) becomes the only available fully-non-perturbative definition of string theory on $\text{AdS}_5 \times \mathbb{S}^5$ backgrounds. When the boundary theory is strongly coupled and has a large number of colors, it reduces to the classical bulk limit (1.32), becoming the equivalence between classical type IIB supergravity dynamics on an ALAdS_5 space and strongly-coupled $\mathcal{N} = 4$ Super Yang-Mills at large N.

This realisation of the AdS/CFT conjecture has passed several tests. Historically, the first evidence in support of this duality came from the matching of quantities that are independent of the coupling of the theories since they are protected by non-renormalisation theorems due to supersymmetry. These include the matching of correlators involving BPS operators [59–61] and the matching of conformal anomalies [62–64].

In the following subsections we discuss how to extract boundary observables using the bulk theory in the classical regime (1.32). This framework goes under the name of *holographic renormalisation* [62,63,65–67].

1.3.1 ALAdS asymptotics

The first step to extract boundary observables from the bulk dynamics consists in studying the relation between bulk and boundary geometry [22, 65–67]. We formally define an asymptotically locally AdS (ALAdS) spacetime as the interior of a manifold with boundary, whose metric can be written in the form

$$g^{+}_{\mu\nu} = \frac{L^2}{F(x^\mu)^2}\bar{g}^{+}_{\mu\nu}, \tag{1.33}$$

where $\bar{g}^{+}$ is regular in the limit to the boundary and $F(x^\mu)$ has a simple zero there. We further demand it to be a solution to Einstein's equations with a negative cosmological constant,

$$R^{+}_{\mu\nu} + \frac{d}{L^2} g^{+}_{\mu\nu} = 0. \tag{1.34}$$

Given an ALAdS bulk, one can ask how to construct a boundary metric. From (1.33), the easiest way to construct a boundary metric is to multiply the bulk metric by a function r which has a simple zero at the boundary, so as to compensate the divergence in the prefactor. The resulting metric

$$g_{(0)} = r^2 g^{+}_{\mu\nu}\big|_{r=0} \tag{1.35}$$

is well-defined at the boundary. Observe that $r = F(x^\mu)$ is an acceptable choice, however one can equivalently pick any function which is a regular local rescaling of $F(x^\mu)$, e.g. $r = e^{\Omega(x)}F(x^\mu)$. Thus with the prescription (1.35) we are not constructing a single boundary metric. Instead, we are providing the boundary with a conformal class of metrics $\left[g_{(0)}\right]$, defined as

$$\hat{g} \in \left[g_{(0)}\right] \quad \text{iff} \quad \hat{g} = e^{2\Omega(x)} g_{(0)}, \tag{1.36}$$

for some non-singular $\Omega(x)$, all related by a Weyl transformation. This procedure is the conformal compactification of ALAdS manifolds.

Conversely, one could ask to what extent a fixed conformal class at the boundary $r = 0$ determines the bulk geometry. In [22] it was shown that after picking a representative $g_{(0)}$ from the conformal class $\left[g_{(0)}\right]$, one can think of it as an initial condition which can be evolved near the boundary using the bulk Einstein's equations (1.34). One can choose suitable coordinates $x^\mu = (r, x^i)$ near the boundary of the bulk (normally referred to as the *Fefferman-Graham gauge*) in which the metric takes the form

$$ds^2 = g^{+}_{\mu\nu}dx^\mu dx^\nu = \frac{L^2}{r^2}\left(dr^2 + g_{ij}(x,r)dx^i dx^j\right), \tag{1.37}$$

with boundary at $r \to 0$. The metric $g_{ij}(x,r)$ induced on a constant-r slice is not fixed a priori, except for its boundary value $g_{ij}(x,0) = g_{(0)ij}(x)$. One can evolve such boundary condition using the bulk equations by solving them perturbatively near the boundary. In full generality, the expansion is of the form

$$g(x,r) = g_{(0)} + r^2 g_{(2)} + \cdots + r^d \left(g_{(d)} + h_{(d)} \log r\right) + \ldots \tag{1.38}$$

for even d, while for odd d it reads

$$g(x,r) = g_{(0)} + r^2 g_{(2)} + \cdots + r^d g_{(d)} + \ldots . \tag{1.39}$$

The most important feature of this expansion is that fixed a $g_{(0)}$, all the coefficients of order up to $O(r^d)$ are fully determined, except for the trace-free part of $g_{(d)}$. More precisely, Einstein's equations fix the trace $g_{(0)}^{ij} g_{(d)ij}$ and the divergence $\nabla^i g_{(d)ij}$ only. For odd d, $g_{(d)ij}$ is traceless and divergenceless. Holographically the role of $g_{(0)}$ is the metric background of the CFT and thus the source for the dual stress tensor. In the following subsection we delve into the holographic interpretation of $g_{(d)}$ and $h_{(d)}$.

Finally, we mention that in this framework it is possible to show that boundary Weyl transformations correspond to a specific class of bulk diffeomorphisms [68, 69], while boundary conformal symmetries are mapped to bulk asymptotic isometries [70]. For instance, this entails that in pure Euclidean AdS$_{d+1}$ with $d \geq 3$ the boundary conformal transformations SO$(1, d+1)$ are realised as bulk Killing vectors. This fact plays an important role in the embedding space as we illustrate in Section 1.2, as well as in the ambient space (see Section 2.4 and Chapter 3).

1.3.2 The holographic stress tensor

Through equation (1.32), boundary correlators can be found by taking functional derivatives of the bulk onshell action with respect to the corresponding boundary sources,

$$\langle O(x_1) \ldots O(x_n) \rangle = (-1)^n \left. \frac{\delta^n S_{\text{onshell}}\left[\phi_{(0)}\right]}{\delta\phi_{(0)}(x_1) \cdots \delta\phi_{(0)}(x_n)} \right|_{\phi_{(0)}=0} . \tag{1.40}$$

For correlators of stress tensor operators we thus have to set the bulk gravitational action

$$S = \frac{1}{16\pi G} \left[\int_M d^{d+1}x \sqrt{g^+} \left(R[g^+] + 2\Lambda\right) - \int_{\partial M} d^d x \sqrt{\gamma}\, 2K \right], \tag{1.41}$$

onshell, and then take derivatives with respect to the boundary metric $g_{(0)}$. The first piece in (1.41) is the Einstein-Hilbert action, while the second piece is the Gibbons-Hawking-York boundary term [71,72]. Here γ_{ij} is the metric induced onto ∂M by g^+ and K is the trace of the extrinsic curvature of ∂M.

As it is, the action (1.41) however diverges because of the double-pole in the metric g^+ at $r=0$, which leads to a divergence in the induced metric γ as well as in the bulk volume term. In order to extract a finite bulk onshell action, we first evaluate (1.41) on a regulated surface close to the boundary, then we renormalise it via the addition of *local* and *covariant* bulk counterterms. This is the procedure of holographic renormalisation for the bulk gravity action [62, 63, 65–67].

The expansions (1.38) and (1.39) are in terms of even powers of r (at least up to $O(r^d)$ in odd d) and it is thus convenient to turn to the new radial coordinate $\rho = r^2$. We pick $\rho = \epsilon$ as a regulating surface so that the regulated bulk action reads

$$S_{\text{reg}} = \frac{1}{16\pi G}\left[\int_{\rho\geq\epsilon} d^{d+1}x\sqrt{g^+}\left(R[g^+]+2\Lambda\right) - \int_{\rho=\epsilon} d^dx\sqrt{\gamma}\,2K\right], \tag{1.42}$$

where now γ and K are the induced metric and scalar extrinsic curvature on the $\rho=\epsilon$ surface. Plugging the onshell expansions (1.38) and (1.39) into S_{reg}, we find a number of poles dependent on the dimension d as well as a logarithmic divergence (appearing only for even d),

$$S_{\text{reg}} = \frac{1}{16\pi G}\int d^dx\sqrt{g_{(0)}}\left[\epsilon^{-d/2}a_{(0)} + \cdots + \epsilon^{-1}a_{(d-2)} - \log\epsilon\, a_{(d)}\right] + O(\epsilon^0)\,, \tag{1.43}$$

where all the coefficients $a_{(k)}$ are local and covariant in terms of $g_{(0)}$, and they do not involve $g_{(d)}$.

One can now define a renormalised onshell bulk action by subtracting these divergent terms from the regulated action and then taking the limit $\epsilon \to 0$,

$$S_{\text{ren}} = \lim_{\epsilon\to 0}\frac{1}{16\pi G}\left[S_{\text{reg}} - \int d^dx\sqrt{g_{(0)}}\left(\epsilon^{-d/2}a_{(0)} + \cdots + \epsilon^{-1}a_{(d-2)} - \log\epsilon\, a_{(d)}\right)\right]. \tag{1.44}$$

The 1-point function of the boundary stress tensor can then be computed as

$$\langle T_{ij}(x)\rangle = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta g^{ij}_{(0)}(x)} = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d/2-1}}T_{ij}[\gamma]\right), \tag{1.45}$$

where $T_{ij}[\gamma]$ is the classical stress tensor of the classical theory living on the surface $\rho=\epsilon$ and with action S_{ren}.

Evaluating (1.45) yields

$$\langle T_{ij}(x)\rangle = \frac{d}{16\pi G}\left[g_{(d)ij} + Y^{(d)}_{ij}\right], \tag{1.46}$$

where $Y^{(d)}_{ij}$ are scheme-dependent terms which are local functions of $g_{(0)}$ and whose expression depends on the dimension d. Up to $d=5$,

$$Y^{(2k+1)}_{ij} = 0\,, \qquad Y^{(2)}_{ij} = -g_{(0)ij}\,\mathrm{tr}\,g_{(2)}\,, \tag{1.47}$$

$$Y^{(4)}_{ij} = -\frac{1}{8} g_{(0)ij} \left[(\operatorname{tr} g_{(2)})^2 - \operatorname{tr} g^2_{(2)} \right] - \frac{1}{2} (g^2_{(2)})_{ij} + \frac{1}{4} g_{(2)ij} \operatorname{tr} g_{(2)} \,. \tag{1.48}$$

Therefore $g_{(d)}$ encodes the VEV of the operator dual to the background metric $g_{(0)}$. Note that the fact that $g_{(d)ij}$ is traceless for odd d ensures that $\langle T_{ij}(x) \rangle$ is traceless too, a result in agreement with the absence of Weyl anomalies for the metric sector in odd d.

The regulated action S_{reg} in (1.44) as well as the power-law divergences are invariant under the full set of bulk diffeomorphisms, while the logarithmically divergent piece breaks the class of bulk diffeomorphisms that induce Weyl transformations at the boundary. Thus from the boundary perspective we see the appearance of the boundary Weyl anomaly through (1.32),

$$S_{\text{ren}} \left[e^{2\Omega(x)} g_{(0)} \right] = S_{\text{ren}} \left[g_{(0)} \right] + \mathcal{A} \left[g_{(0)}, \Omega \right] . \tag{1.49}$$

S_{ren} and the boundary generating function thus exhibit a dependence on the boundary representative $g_{(0)}$. For infinitesimal Ω the anomaly $\mathcal{A}\left[g_{(0)}, \Omega \right]$ is proportional to the coefficient $a_{(d)}$ in the logarithmically divergent piece in (1.43) [62]. One can subsequently show [69] that the coefficient $h_{(d)}$ appearing in the bulk metric expansion contains the metric variation of the anomaly,

$$h_{(d)ij} = \frac{\upsilon_{(d)}}{\sqrt{g_{(0)}}} \frac{\delta}{\delta g^{ij}_{(0)}} \int d^d x \sqrt{g_{(0)}}\, \mathcal{A} \,, \tag{1.50}$$

where $\upsilon_{(d)}$ is a constant dependent on the dimension d.

1.3.3 The holographic scalar

A similar process of holographic renormalisation can be carried out for bulk matter fields, either coupled to the metric or uncoupled, at least as long as they are dual to a relevant or marginal operator in the boundary QFT. In this subsection we present the procedure and the results for a probe massive scalar on an ALAdS bulk [67,69], a case which will be extensively used in several parts of this thesis, including Section 2.5 and Chapter 5.

The relevant action for this case is

$$S = \frac{1}{2} \int d^{d+1} x \sqrt{g^+} \left(g^{+\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2 \right) \tag{1.51}$$

on a fixed Euclidean ALAdS background g^+. We start by studying its solutions space. Solving the bulk equation

$$(-\Box^+ + m^2)\Phi = 0 \tag{1.52}$$

order by order at small ρ, one obtains a near-boundary expansion of the form

$$\Phi(x,\rho) = \rho^{\frac{d-\Delta}{2}} \phi(x,\rho) = \rho^{\frac{d-\Delta}{2}} \phi_{(0)} + \rho^{\frac{d-\Delta}{2}+1} \phi_{(2)} + \cdots + \rho^{\frac{\Delta}{2}} \left(\phi_{(2\Delta-d)} + \psi_{(2\Delta-d)} \log \rho \right) + \ldots \,. \tag{1.53}$$

Here Δ will play the role of the conformal dimension of the boundary operator O dual to Φ, and it is related to the mass by

$$m^2 = \Delta(\Delta - d)\,. \tag{1.54}$$

All the orders up to $O(\rho^{\Delta/2})$ are fully determined by $\phi_{(0)}$ except for $\phi_{(2\Delta-d)}$ which is not fixed by the Dirichlet boundary condition. The logarithmic term is only present when $\Delta = \frac{d}{2} + \kappa$, $\kappa = 0, 1 \ldots$. Using a holographic renormalisation prescription, we intend to provide a dual interpretation for $\phi_{(0)}$, $\phi_{(2\Delta-d)}$ and $\psi_{(2\Delta-d)}$.

Also in this case, the onshell action is divergent and must be renormalised. Considering a regularisation hypersurface at $\rho = \epsilon$, the regularised onshell action takes the form

$$S_{\text{reg}} = -\int_{\rho=\epsilon} d^dx\sqrt{G(\epsilon)}G^{\rho\rho}\Phi\partial_\rho\Phi = -\int_{\rho=\epsilon} d^dx\sqrt{g}\rho^{\frac{d}{2}-\Delta}\left[\frac{d-\Delta}{2}\phi^2 + \rho\phi\partial_\rho\phi\right] \tag{1.55}$$

$$= \int d^dx\sqrt{g_{(0)}}\left[\epsilon^{-\Delta+d/2}a^M_{(0)} + \cdots + \epsilon^{-1}a^M_{(2\Delta-d+2)} - \log\epsilon\, a^M_{(2\Delta-d)}\right] + O(\epsilon^0)\,, \tag{1.56}$$

where the coefficients $a^M_{(k)}$ are local in $\phi_{(0)}$. The logarithmic divergence appears for $\Delta = \frac{d}{2} + \kappa$, $\kappa = 0, 1 \ldots$.

As in the gravitational case, we define a renormalised onshell action by subtracting these divergent terms from the onshell action,

$$S_{\text{ren}} = \lim_{\epsilon\to 0}\frac{1}{16\pi G}\left[S_{\text{reg}} - \int d^dx\sqrt{g_{(0)}}\left(\epsilon^{-\Delta+d/2}a^M_{(0)} + \cdots + \epsilon^{-1}a^M_{(2\Delta-d+2)} - \log\epsilon\, a^M_{(2\Delta-d)}\right)\right]\,. \tag{1.57}$$

As they are, however, the coefficients $a^M_{(k)}$ (and hence the counterterms in (1.57)) are not covariant under bulk diffeomorphisms, and for this reason we have to re-express them in terms of the bulk field Φ, formally inverting the expansions $\Phi(\phi_{(0)})$ and $g^+(g_{(0)})$ in ρ to sufficiently high order (according to the dimension d).

Through (1.32), the 1-point function of the operator dual to the bulk field Φ in the presence of the source $\phi_{(0)}$ can thus be computed as

$$\langle O(x)\rangle = -\frac{1}{\sqrt{g_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta\phi_{(0)}(x)}\,, \tag{1.58}$$

which yields

$$\langle O(x)\rangle = (2\Delta - d)\phi_{(2\Delta-d)}(x) + \mathcal{F}(\phi_{(0)}(x), g_{(0)}(x), g_{(d)}(x))\,, \tag{1.59}$$

where $\mathcal{F}(\phi_{(0)}, g_{(0)}, g_{(d)})$ is in general a scheme-dependent local function of the metric coefficients in the near-boundary expansion, as well as the source $\phi_{(0)}$. Thus the coefficient

$\phi_{(2\Delta-d)}$ in the near-boundary expansion of the scalar field encodes the 1-point function of the dual scalar operator in the presence of sources.

In explicit computations on a given background, the coefficient $\phi_{(2\Delta-d)}$ is fixed non-locally with respect to the Dirichlet value $\phi_{(0)}$ by the solution of the bulk differential problem. This entails that to obtain the scalar 2-point function $\langle OO \rangle$ it is sufficient to differentiate the 1-point function (1.59) once more and set $\phi_{(0)} = 0$,

$$\langle O(x)O(y)\rangle = (2\Delta - d)\frac{\delta\phi_{(2\Delta-d)}(x)}{\delta\phi_{(0)}(y)} + \frac{\delta}{\delta\phi_{(0)}(y)}\mathcal{F}(\phi_{(0)}(x), g_{(0)}(x), g_{(d)}(x))\Big|_{\phi_{(0)}=0} . \quad (1.60)$$

We will discuss an explicit example of such a computation in Section 5.5, where we study the holographic scalar 2-point function of a massive scalar field on a planar AdS black hole bulk.

Observe that similarly to the gravitational case, the renormalised action (1.57) transforms anomalously under Weyl transformations because of the logarithmically divergent piece. The coefficient $a^M_{(2\Delta-d)}$ can be related to the scalar Weyl anomaly for a given d and Δ. Note that such term is present only when $\Delta = \frac{d}{2} + \kappa$, $\kappa = 0, 1\ldots$ as expected for anomalies in the scalar sector.

Chapter 2

The ambient space

In this chapter we present the geometric construction of the ambient space. Our aim here as well as in Chapters 3 to 6 is to extend the embedding formalism to more general settings where conformal invariance may be broken, including non-conformally flat CFT backgrounds and generic states, so that we may usefully constrain the form of correlators in such settings. To achieve this we adopt the ambient space [22,23] as our principal tool, a $(d+2)$-dimensional spacetime that replaces the embedding space.

In Section 2.1 we discuss how to construct ambient spaces given a CFT background $g_{(0)}$, in such a way that the key features of the embedding space are maintained. These consist in a notion of local flatness, as well as a nullcone structure (which is closely related to the presence of the dilation vector $X \cdot \partial$ on the embedding space). When the CFT background $g_{(0)}$ is conformally flat and the state is trivial, the ambient space reduces to the embedding space, as we establish in Section 2.2. In Section 2.3 we present several other special classes of ambient spaces and their properties, while in Section 2.4 one of the crucial features of the ambient construction is discussed, that is how it canonically encodes Weyl covariance. Finally, in Section 2.5 we review a family of Weyl-covariant differential operators that can be constructed on the metric $g_{(0)}$ using the ambient space.

2.1 Constructing ambient spaces

There are two key defining features of the ambient space. The first is the existence of a null scaling isometry T (called a homothetic vector in mathematics literature), obeying $\mathcal{L}_T\tilde{g} = 2\tilde{g}$ where $\tilde{g}$ is metric of the $(d+2)-$dimensional ambient space. Note that T is a conformal Killing vector of the ambient space and is non-Killing. The existence of T reflects the fact that CFT correlators on any background and state satisfy Weyl-covariance constraints, playing a role analogous to the embedding space's $X \cdot \partial_X$ dilation vector.

The second defining feature is Ricci-flatness. To depart from the embedding space we must depart from $\mathbb{R}^{1,d+1}$. Riemann-flatness is too restrictive, as this results in a formalism locally equivalent to the embedding space, leaving Ricci-flatness is the next most natural class of spacetimes. One may wish to consider further relaxing this by introducing matter with a stress-tensor, but we will not do so for the present discussion; we will comment on the role of such extensions in Section 4.5.

Given a d-dimensional conformal manifold with coordinates x^i and a representative $g_{(0)ij}(x)$ of the conformal class of metrics $[g_{(0)ij}(x)]$, one is able to construct a new $d+2$-dimensional spacetime with these two requirements, the ambient space [22, 23]. Parameterising the $d+2$ ambient space directions with the coordinates $\widetilde{X}^M = (t, x^i, \rho)$, the most general ambient space metric is given by

$$\widetilde{g} = 2\rho dt^2 + 2tdtd\rho + t^2 g_{ij}(x,\rho)dx^i dx^j, \tag{2.1}$$

where $g_{ij}(x,0) = g_{(0)ij}(x)$ and where $g_{ij}(x,\rho)$ is such that $\widetilde{R}_{MN} = 0$. In these coordinates the homothetic vector is given by

$$T = t\partial_t, \tag{2.2}$$

and we note the useful property $\widetilde{\nabla}_A T_B = \tilde{g}_{AB}$.

The meaning t and ρ is depicted in Figure 2.1. The coordinate t is related to ambient scale transformations, generated by the homothety T. Intuitively, the coordinate ρ describes the distance from the nullcone.[1] While t is taken to be strictly positive, ρ is real and we place the nullcone at $\rho = 0$. Hence, projecting onto the boundary amounts to setting $t = 1$ and $\rho = 0$ where one recovers $g_{(0)ij}(x)$. As in the embedding space, the nullcone is obtained by rescaling $g_{(0)ij}$ and as such it is covered by the coordinates t and x^i, in analogy with (1.15). Choosing a specific t corresponds to restricting to a specific section of the nullcone.

Knowing the d-dimensional metric together with the presence of the dilations T completely fixes the geometry on the nullcone. The specific form of (2.1) follows from a convenient gauge choice such that ρ and t are geodesic coordinates in a vicinity of the nullcone. By this we mean that starting at a fixed point $(t, x, 0)$ on the nullcone, the curve $\gamma(\rho) = (t, x, \rho)$ is a geodesic for the ambient metric $\widetilde{g}$. Similarly, any curve $\gamma(t) = (t, x, \rho)$ starting at (t_0, x, ρ) is a geodesic for the ambient metric $\widetilde{g}$. This fixes the ambient geometry to take the form of a Gaussian null foliation [73, 74] near the ambient lightcone, resulting in (2.1). Observe that the t-dependence is completely fixed by the choice of gauge and homogeneity. In particular, at $\rho = 0$ a dilation generated by $T = t\partial_t$ will produce a rescaling of the boundary metric as desired.

Solving the equations $\widetilde{R}_{MN} = 0$ determines the components $g_{ij}(x,\rho)$, with boundary

[1] We call it nullcone to keep in mind the relation to the corresponding surface of the embedding space. As we will see shortly, in the ambient space it is a null submanifold with a degenerate induced metric, which can be represented as a cone space of the arbitrary manifold $g_{(0)}$.

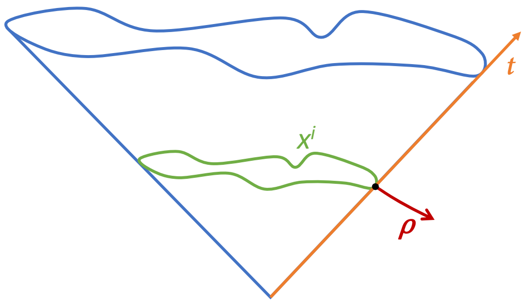

Figure 2.1: This cartoon illustrates qualitatively the meaning of the ambient coordinates: t and x^i span the nullcone, while ρ describes the distance from it.

conditions given by the d-dimensional metric $g_{ij}(x,0) = g_{(0)ij}(x)$. These are,

$$\widetilde{R}_{ij} = \rho g''_{ij} - \rho g^{kl} g'_{ik} g'_{lj} + \frac{1}{2}\rho g^{kl} g'_{kl} g'_{ij} - \left(\frac{d}{2} - 1\right) g'_{ij} - \frac{1}{2} g^{kl} g'_{kl} g_{ij} + R_{ij} = 0\,, \tag{2.3a}$$

$$\widetilde{R}_{i\rho} = \frac{1}{2} g^{kl}\left(\nabla_k g'_{il} - \nabla_i g'_{kl}\right) = 0\,, \tag{2.3b}$$

$$\widetilde{R}_{\rho\rho} = -\frac{1}{2} g^{kl} g''_{kl} + \frac{1}{4} g^{lk} g^{pq} g'_{kp} g'_{ql} = 0\,, \tag{2.3c}$$

where the primes denote derivatives in ρ, while R_{ij} and ∇_i indicate the Ricci tensor and the covariant derivative of $g_{ij}(x,\rho)$ evaluated at fixed ρ.

General properties of $g_{ij}(x,\rho)$ maybe studied through solutions of (2.3a), (2.3b), (2.3c) obtained in a perturbative expansion at small ρ, i.e. a near-nullcone expansion. In terms of the boundary metric $g_{(0)ij}(x)$, one has

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + 2P_{ij}\,\rho + \cdots + \rho^{\frac{d}{2}}\left(g_{(d)ij} + h_{(d)ij}\log\rho\right) + \ldots \tag{2.4}$$

for even dimensions d, while for odd d one has

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + 2P_{ij}\,\rho + \cdots + \rho^{\frac{d}{2}} g_{(d)ij} + \ldots, \tag{2.5}$$

where the coefficient of the expansion only depend on x, and P_{ij} is the boundary Schouten tensor

$$P_{ij} = \frac{1}{d-2}\left(R_{ij} - \frac{R}{2(d-1)} g_{(0)ij}\right). \tag{2.6}$$

Remarkably, all the coefficients including h_{ij} are completely fixed by the boundary metric up to order $O\left(\rho^{d/2}\right)$, while only the trace and divergence of $g_{(d)ij}$ are determined by $g_{(0)ij}$. The remaining transverse traceless piece of $g_{(d)ij}$ constitutes the second piece of boundary data required for general solutions of the set of second order differential equations (2.3). Note that $g_{ij}(x,\rho)$ is in general non-analytic at $\rho=0$ since at order $O\left(\rho^{d/2}\right)$ logarithmic contributions are present for even d and for non-conformally flat $g_{(0)}$, while half-odd powers of ρ appears for odd d starting at $O\left(\rho^{d/2}\right)$.

The similarity to the usual holographic expansion for asymptotically locally AdS spaces is striking, and we can make this relation more precise by performing the following coordinate transformation,

$$\rho = -\frac{r^2}{2}\,, \quad t = \frac{s}{r}\,, \tag{2.7}$$

with $r,s>0$, covering the region $\rho<0$. The ambient metric becomes

$$\widetilde{g} = -ds^2 + s^2\left(\frac{dr^2 + g_{ij}(x,r)dx^i dx^j}{r^2}\right), \tag{2.8}$$

where the piece in parentheses must solve the vacuum Einstein equations with a negative cosmological constant in $d+1$ dimensions, as a consequence of Ricci-flatness in $d+2$.

Let us prove this statement. Denoting the $(d+1)$-dimensional metric withing the brackets in (2.8) by $g^+_{\mu\nu}$, we start by noting that (2.8) can be seen as conformal to a metric $\bar{g}$ through $\tilde{g}=s^2\bar{g}$, with

$$\bar{g} = -\frac{ds^2}{s^2} + \frac{1}{r^2}\left(dr^2 + g_{ij}(x,r)dx^i dx^j\,.\right) \tag{2.9}$$

Using the properties of the Riemann tensor under Weyl rescaling one can show that the Riemann and Ricci tensor components for $\tilde{g}$ are related to the ones for $\bar{g}$ by

$$\begin{aligned}\widetilde{R}_{ABMN} &= s^2\left[\bar{R}_{ABMN} + \bar{g}_{AM}\left(\bar{g}_{NB} + \frac{\delta^0_N\delta^0_B}{s^2}\right) - \bar{g}_{AN}\left(\bar{g}_{MB} + \frac{\delta^0_M\delta^0_B}{s^2}\right)\right. \\ &\quad \left. +\bar{g}_{A0}\left(\delta^0_N\bar{g}_{MB} - \delta^0_M\bar{g}_{NB}\right)\right], \end{aligned}\tag{2.10}$$

$$\widetilde{R}_{MN} = \bar{R}_{MN} + d\left[\frac{\delta^0_M\delta^0_N}{s^2} + \bar{g}_{MN}\right]. \tag{2.11}$$

The metric $\bar{g}$ is in diagonal block form, meaning that the the corresponding Riemann and Ricci tensors are analogously factorized,

$$R_{MN} = g^{AB}R_{AMBN} = R_{cd}\delta^c_M\delta^d_N + R_{kl}\delta^k_M\delta^l_N\,. \tag{2.12}$$

If we then restrict to the directions $x^\mu=(x^i,r)$, equation (2.11) reduces to

$$\widetilde{R}_{\mu\nu} = R^+_{\mu\nu} + dg^+_{\mu\nu}\,, \tag{2.13}$$

so that requiring Ricci-flatness for an ambient metric yields the Einstein condition for the corresponding ALAdS sections g^+.

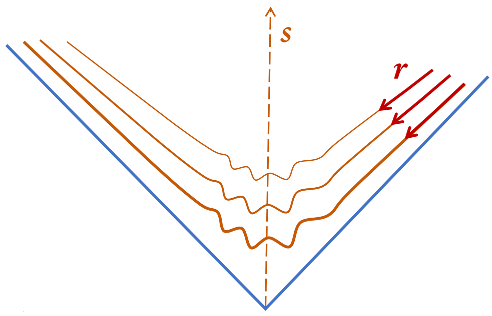

Figure 2.2: This cartoon illustrates qualitatively the meaning of the ambient coordinates: t and x^i span the nullcone, while ρ describes the distance from it.

We thus conclude that hypersurfaces at fixed s are ALAdS metrics of radius s in $d{+}1$ dimensions and we recognise $g_{ij}(x,r)$ as the usual near-boundary Fefferman-Graham expansion. This is a generalization of the AdS slicing of the embedding space (1.27), as sketched in Figure 2.2. Note also that the $d-$dimensional manifold is recovered in the limit $r \to 0$ and $s \to 0$, keeping fixed $t = \frac{s}{r} = 1$. The homothetic vector reads $T = s\partial_s$ in these coordinates.

This hyperbolic slicing illustrates several interesting properties of the ambient space. First, it tells us that the coefficients in (2.4) and (2.5) contain precisely the same information as the corresponding ones in the usual holographic expansion[2] as presented in Section 1.3. In particular, $g_{(d)ij}$ is related to the VEV of the boundary stress tensor, while the boundary metric $g_{(0)ij}(x)$ plays the role of its source. Finally, $h_{(d)ij}$ is proportional to the metric variation of the boundary Weyl anomaly. Therefore the ambient space geometrically encodes both the generic CFT background as well as its possibly non-trivial state. Importantly, it does so in a Weyl-covariant way as we will remark later.

Another important consequence of this slicing is that exact ambient solutions can be found starting from ALAdS geometries in the Fefferman-Graham gauge by considering the AdS radius as a new coordinate s and fibering it according to (2.8). This automatically solves the Ricci-flatness equations.

Note that the change of coordinates (2.7) only covers negative ρ's. Alternatively we can change coordinates with

$$\rho = +\frac{r^2}{2}, \quad t = \frac{s}{r}, \tag{2.14}$$

[2]Observe that the ambient coordinate ρ is proportional to the holographic ρ_{Holo} as defined in Subsection 1.3.2 according to $\rho = -\frac{1}{2}\rho_{Holo}$.

leading to the metric

$$\widetilde{g} = ds^2 + s^2 \left(\frac{-dr^2 + g_{ij}(x,r)dx^i dx^j}{r^2} \right). \tag{2.15}$$

As expected from the analogy with Minkowski, here we are covering the $\rho > 0$ region of the ambient space with a foliation in terms of $(d+1)-$dimensional asymptotically locally dS (ALdS) spaces. One can move from the positive ρ region to the negative ρ region by taking the analytic continuation $s \to is$ and $r \to ir$ of the radius and Fefferman-Graham coordinate, recovering the well-known map from Euclidean AdS to dS spaces. Similarly to the case of negative ρ, we can find exact ambient geometries in this patch by plugging ALdS metrics in the brackets of (2.15).

2.2 Relation to the embedding space

To illustrate the relationship between the embedding space presented in Section 1.2 and the more general ambient metric (2.1), we start by rewriting the $(d+2)-$dimensional Minkowski metric $ds^2 = \eta_{MN} dX^M dX^N$ in terms of the Gaussian null coordinates $\widetilde{X}^M = (t, x^i, \rho)$. In view of a comparison with the Poincaré slicing of the embedding space (1.27) we consider a flat boundary $g_{(0)ij} = \delta_{ij}$. Defining $X^{\pm} = X^0 \pm X^{d+1}$, a suitable change of coordinates[3] is

$$t = X^+, \qquad \rho = \frac{\eta_{MN} X^M X^N}{2(X^+)^2}, \qquad x^i = \frac{X^i}{X^+}, \tag{2.16}$$

with inverse map

$$X^0 = \frac{t}{2}\left(1 - 2\rho + x^2\right), \qquad X^i = t x^i, \qquad X^{d+1} = \frac{t}{2}\left(1 + 2\rho - x^2\right). \tag{2.17}$$

The resulting ambient metric is

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \delta_{ij} dx^i dx^j, \tag{2.18}$$

which is simply Minkowski space in Gaussian null coordinates. From the map (2.17) it is clear that fixing a value of t determines a single slice of the nullcone, where the boundary directions x^i play the role of the projective coordinates (1.17), being independent of the section one picks. The coordinate ρ tells us how far from the nullcone we are, with $\rho > 0$ the region with a timelike separation from the origin, and $\rho < 0$ spacelike. As a consistency check, note that the map (2.17) reduces to the embedding (1.15) taking $\rho = 0$ and matches the AdS slicing (1.27) using the change of coordinates (2.7).

Comparing the ambient metric (2.18) to the general expansion at small ρ, we immediately recognise in which sense the embedding formalism can be generalised via the

[3] A similar set of coordinates for Minkowski was used in [75].

ambient space. The latter can describe non-trivial states at the boundary in case of a non-vanishing stress tensor VEV turned on. In addition to this, we remark that a similar map from the embedding space to the ambient formulation as in (2.16) can be found for any conformally flat boundary metric, not only for $g_{(0)ij}(x) = \delta_{ij}$. Assuming the boundary data $g_{(d)ij}$ vanishes, one can check that the ambient space is locally Minkowski (i.e. its Riemann tensor vanishes) if and only if the boundary metric $g_{(0)ij}(x)$ is conformally flat. A special case is $d = 2$, where all boundaries are conformally flat, where one can show [23] that the 4-dimensional ambient space is automatically Riemann-flat, even for non-vanishing stress tensor VEVs. In Section 2.3 we provide more details and proofs regarding these statements.

Let us now turn our attention to embedding space correlation functions. Focussing on scalar 2-point functions on $\mathbb{R}^d$, their extension to the ambient space must also be a scalar and this entails that one has to find building blocks which are scalars under ambient diffeomorphisms.[4]

In the embedding space one can simply use insertion coordinates X_i^M to construct scalars as in (1.20), since Minkowski space is in fact locally isomorphic to its tangent space. In ambient space however, (t, x^i, ρ) are merely coordinates on a curved manifold, and cannot be directly contracted at different insertion points to construct scalars, as these belong to different tangent spaces. Fortunately, by definition, the ambient space comes equipped with the homothetic vector $T = t\partial_t$. Since T coincides with $X = X^M \partial_M$ in the flat case, it is natural to replace the positions of the insertions in the ambient space is the vector field T evaluated at the insertion points.

Consider two ambient insertion points $\widetilde{X}_i, \widetilde{X}_j$. In order to construct a scalar quantity under ambient diffeomorphisms we need to contract and sum the vectors $T(\widetilde{X}_i)$ and $T(\widetilde{X}_j)$ belonging to different tangent spaces. To this aim we use parallel transport to move one vector to another tangent space along a geodesic. One now has two vectors belonging to the same tangent space and their contraction with the ambient metric at that point results in a well-defined scalar.

This prescription allowing one to generalise X_{ij} is valid for any ambient space. We will discuss it at length in Chapter 4. For now let us check it reproduces the known embedding space invariant in (1.20) in the case of a flat boundary with ambient metric (2.18). Given two points not necessarily on the lightcone

$$\widetilde{X}_0 = (t_0, x_0, \rho_0), \qquad \widetilde{X}_1 = (t_1, x_1, \rho_1), \tag{2.19}$$

as a first step parallel transport requires solving the geodesic equations from $\widetilde{X}_0$ to $\widetilde{X}_1$

$$\ddot{\widetilde{X}}^M(\lambda) + \Gamma^M_{AB}(\lambda)\dot{\widetilde{X}}^A(\lambda)\dot{\widetilde{X}}^B(\lambda) = 0, \tag{2.20}$$

[4]We will explain in detail this statement in Subsection 2.4, where in particular it will be shown which class of diffeomorphisms is relevant in this context and why we are requiring full diffeomorphism invariance.

where Γ^M_{AB} here refers to the ambient connection. On the flat ambient space (2.18) they can be easily solved,

$$t(\lambda) = A\lambda + B, \tag{2.21a}$$

$$x^m(\lambda) = \frac{E^m}{A\lambda + B} + F^m, \tag{2.21b}$$

$$\rho(\lambda) = \frac{E^m E_m}{2(A\lambda + B)^2} + \frac{G}{A\lambda + B} + H, \tag{2.21c}$$

with a total of $2(d+2)$ integration constants accounting for the components of the initial and final point (2.19) of the geodesic. We set the endpoints to correspond to the values of the affine parameter $\lambda = 0$ and $\lambda = 1$ respectively, fixing the integration constants to

$$A = -t_0 + t_1, \qquad B = t_0, \qquad E^m = t_0 t_1 \frac{x_1^m - x_0^m}{t_0 - t_1}, \tag{2.22a}$$

$$F^m = \frac{t_1 x_1^m - t_0 x_0^m}{t_1 - t_0} \qquad H = -\frac{E^m E_m + 2G t_0}{2t_0^2}, \tag{2.22b}$$

$$G = -\frac{t_0 t_1}{(t_0 - t_1)^2} \left[(t_0 + t_1)(x_1 - x_0)^2 + 2(t_0 - t_1)(\rho_0 - \rho_1) \right]. \tag{2.22c}$$

These geodesics are of course simply straight lines on Minkowski in disguise. We now have to evolve the initial condition $T(\lambda = 0) = T_0 = (t_0, 0, 0)$ at $\widetilde{X}_0$ to the point $\widetilde{X}_1$ along these geodesics using the parallel transport equations

$$\dot{\widetilde{X}}^M(\lambda)\, \widetilde{\nabla}_M T^A(\lambda) = 0. \tag{2.23}$$

In this case also these equations can be solved exactly and after imposing the boundary conditions at the endpoints, at $\lambda = 1$ one finds

$$\hat{T}_0 \equiv T_0(1) = \left(t_0, -\frac{t_0}{2t_1} \left[(x_0^i - x_1^i)^2 - 2(\rho_0 - \rho_1) \right], \frac{t_0}{t_1}(x_0^i - x_1^i) \right). \tag{2.24}$$

We define the ambient analogue $\widetilde{X}_{ij}$ of the embedding space invariant X_{ij} as the contraction of $T_1 = (t_1, 0, 0)$ with $\hat{T}_0$ using the ambient metric evaluated at $\widetilde{X}_1$. This leads to

$$\widetilde{X}_{01} \equiv -2\, \hat{T}_0 \cdot T_1 = t_0 t_1 \left((x_0^i - x_1^i)^2 - 2(\rho_0 + \rho_1) \right). \tag{2.25}$$

Placing the insertions on the lightcone section $\rho_{0,1} = 0, t_{0,1} = 1$, one recovers the expected value of the embedding space invariant X_{01}. For more general ambient spaces, we treat the calculation of this invariant in more details in Section 4.2 and in Appendix B.

As already discussed when constructing the ambient space, and as we will see in more detail in Section 2.4, the conformal dimension under Weyl transformations of an ambient object coincides with minus its weight in t. This fixes scalar 2-point functions of dimension Δ and bulk-to-boundary propagators to the known forms,

$$\langle O_1(\widetilde{X}_1)O_2(\widetilde{X}_2)\rangle = \frac{C_\Delta}{(\widetilde{X}_{12})^\Delta}\Bigg|_{\substack{t_{1,2}\to 1\\ \rho_{1,2}\to 0}} = \frac{C_\Delta}{(x_{12})^{2\Delta}} \tag{2.26}$$

$$K_\Delta(r_0, x_1; x_2) = \frac{C'_\Delta}{(\widetilde{X}_{12})^\Delta}\Bigg|_{\substack{t_1\to 1\\ \rho_1\to 0}} = \frac{C'_\Delta}{R^\Delta}\left(\frac{r_0}{(x_{12})^2 + r_0^2}\right)^\Delta, \tag{2.27}$$

where we set $t_1 = \frac{R}{r_1}$ and $\rho_1 = -\frac{r_1^2}{2}$, and R is the AdS radius.

2.3 Special classes of ambient spaces

In this section we describe special families of ambient spaces that will be relevant for the applications of the ambient space formalism appearing in Chapters 5 and 6. Restricting to specific classes of d-dimensional metrics $g_{(0)}$ yields further properties and simplifications in the form of the corresponding ambient metric. Vice versa, additional conditions on the ambient space impose constraints on $g_{(0)}$ and typically constrains the quantity $g_{(d)}$ (and hence the stress tensor VEV) in terms of $g_{(0)}$.

In what follows we discuss flat ambient spaces, which are locally equivalent to Minkowski space; ambient spaces associated to conformally Einstein metrics $g_{(0)}$, where the logarithmic piece $h_{(d)ij}$ appearing in (2.4) is identically zero; ambient spaces satisfying (anti)self-duality conditions on their Weyl tensor, for which the stress tensor VEV piece in the ambient expansion is fixed in terms of the boundary geometry $g_{(0)}$.

2.3.1 Conformally flat $g_{(0)}$

In this section we study the implications of the requirement of either the ambient space or the metric $g_{(0)}$, and we describe how these two conditions are related. We first discuss the case of $d \geq 3$ and then the special case of $d = 2$.

$d \geq 3$

As a preliminary observation note that due to the Ricci-flatness of the ambient space, an ambient space is conformally flat if and only if it is Riemann-flat,

$$\widetilde{W}_{ABMN} = 0 \qquad \Longleftrightarrow \qquad \widetilde{R}_{ABMN} = 0, \tag{2.28}$$

where we denote the ambient Weyl tensor by $\widetilde{W}$.

Given a (conformally) flat ambient space, it is easy to show that also its ALAdS sections must be conformally flat. Starting from equation (2.10), once projected onto the $x^\mu = (x^i, r)$ components along the ALAdS sections, the condition $\widetilde{R}_{ABMN} = 0$ implies that

$$R^+_{\alpha\beta\mu\nu} + g^+_{\alpha\mu} g^+_{\beta\nu} - g^+_{\alpha\nu} g^+_{\beta\mu} = 0. \tag{2.29}$$

Using then $(d+1)$-dimensional Einstein's equations $R^+_{\mu\nu} + d g^+_{\mu\nu} = 0$, from (2.29) it follows that $W^+_{\alpha\beta\mu\nu} = 0$. Hence we have shown that any flat ambient metric is made of conformally flat ALAdS sections. Furthermore, the equations $\widetilde{R}_{ABMN} = 0$ do not contain additional conditions besides $W^+_{\alpha\beta\mu\nu} = 0$, since $\widetilde{R}_{0BMN} = 0$ is guaranteed by the homothetic symmetry of the ambient space $t\partial_t = s\partial_s$ (where by the index 0 we indicate the s component). We conclude that an ambient space is flat if and only if its ALAdS sections are conformally flat.

Let us now turn to the implications of ambient Riemann-flatness for the metric $g_{(0)}$. One can show [23, 76] that $g_{(0)}$ must be conformally flat, and the ambient ρ expansion truncates,

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \left[g_{(0)ij} + 2P_{ij}\rho + P_i{}^k P_{kj} \rho^2 \right] dx^i dx^j \,, \tag{2.30}$$

where as before P_{ij} is the Schouten tensor associated to $g_{(0)}$. Thus there is no freedom in the choice of the stress tensor VEV, which is fully fixed by the requirement of ambient flatness.

Vice versa, an ambient space associated to a conformally flat $g_{(0)}$ has a small-ρ expansion of the form,

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \left[g_{(0)ij} + 2P_{ij}\rho + P_i{}^k P_{kj} \rho^2 + O\left(\rho^{\frac{d}{2}}\right) \right] dx^i dx^j \,, \tag{2.31}$$

where the order $O\left(\rho^{\frac{d}{2}}\right)$ encodes a possible non-vanishing stress tensor VEV. An ambient space of this form will be object of study in Chapter 5. As we discuss in Section 4.1, an ambient space with conformally flat $g_{(0)}$ has an associated Riemann tensor of particularly simple form.

As an illustration of these properties, let us consider a CFT on a Euclidean AdS$_d$ background, with metric

$$g_{(0)ij} dx^i dx^j = \frac{dz^2 + dx_a dx^a}{z^2}, \tag{2.32}$$

with $a = 2 \ldots d$. For a vanishing stress tensor VEV, using 2.31 the ambient expansion reads

$$\widetilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \left(1 - \frac{\rho}{2}\right)^2 \frac{dz^2 + dx_a dx^a}{z^2}. \tag{2.33}$$

and the change of coordinates mapping this space to Minkowski is

$$X^0 = \frac{t}{2\sqrt{2}z}\left(2 + z^2 + x^2\right)\left(1 - \frac{\rho}{2}\right), \qquad X^1 = t\left(1 + \frac{\rho}{2}\right), \tag{2.34}$$

$$X^a = \frac{t}{z} x^a \left(1 - \frac{\rho}{2}\right), \qquad X^{d+1} = \frac{t}{2\sqrt{2}z}\left(2 - z^2 - x^2\right)\left(1 - \frac{\rho}{2}\right).$$

To summarise, an ambient space is flat if and only if $g_{(0)}$ is conformally flat and has vanishing stress tensor VEV. It is therefore in such case that the ambient space is locally diffeomorphic to the embedding space. Whenever $g_{(0)}$ is non-conformally flat or there is a non-vanishing stress tensor VEV, the ambient space represents a generalisation of the embedding space geometry.

$d = 2$

Consider now a generic $g_{(0)}$ in $d = 2$. Following similar arguments as for the $d \geq 3$ case, one can show that the Weyl tensors of both the ambient space and of its ALAdS sections are vanishing, entailing that for $d = 2$ any ambient space is necessarily flat (and thus locally diffeomorphic to the embedding space). Solving the Ricci-flatness equations, the ambient geometry can be shown to take the form [23, 76]

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \left[g_{(0)ij} + 2L_{ij}\rho + L_i{}^k L_{kj}\rho^2\right] dx^i dx^j \,, \tag{2.35}$$

where L_{ij} encodes the 2-dimensional stress tensor VEV. It is partially constrained by the requirements

$$L^i{}_i = \frac{1}{2}R \,, \qquad \nabla^j L_{ij} = \frac{1}{2}\nabla_i R \,. \tag{2.36}$$

2.3.2 Conformally Einstein $g_{(0)}$

In this section we present properties of the ambient spaces associated to a conformally Einstein $g_{(0)}$. As customary, by Einstein manifold we mean a space where the Ricci tensor is proportional to the metric,

$$R_{ij} = 2\lambda(d-1)g_{(0)ij}, \tag{2.37}$$

for some constant λ. The case $\lambda = \frac{1}{2L^2}$ includes $\mathbb{S}^d$, $\lambda = 0$ the plane, while $\lambda = -\frac{1}{2L^2}$ the hyperboloid $\mathbb{H}^d$. The d-dimensional Schouten tensor for such class of metrics is $P_{ij} = \lambda g_{(0)ij}$.

For any $d \geq 3$ solving Ricci-flatness one can show [23] that the ambient geometry takes the form,

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \left[(1+\lambda\rho)^2 g_{(0)ij} + g_{(d)ij}\rho^{\frac{d}{2}} + O\left(\rho^{\frac{d}{2}+1}\right)\right] dx^i dx^j \,. \tag{2.38}$$

An interesting feature of this expansion is that the logarithmic piece $h_{(d)ij}$ (referred to as the *obstruction tensor* in the mathematical literature) appearing in the general expansion (2.4) is automatically vanishing for Einstein $g_{(0)}$ in any dimension d. We anticipate that these properties extend to the whole class of ambient spaces associated to a *conformally* Einstein $g_{(0)}$ through the observations of Section 2.4.

2.3.3 Self-dual ambient spaces

Another class of interesting ambient spaces consist in self-dual and anti-self-dual solutions. A representative of these geometries will appear in Chapter 6.

We focus on $d = 3$ and it is convenient to take the perspective of each 4-dimensional ALAdS slice rather than of the related ambient space. We denote the conformal compactification of the ALAdS sections by $\overline{g} = dr^2 + g_r$, with $g_r = g_{ij}(x,r)dx^i dx^j$ as in equation (2.8). We also define the three-dimensional volume form induced by g_r onto the boundary $r = 0$ as

$$\mu_{ijk} = \overline{\mu}_{0ijk} = \sqrt{\det g_r}\, \varepsilon_{ijk} \,. \tag{2.39}$$

The conformal compactification $\overline{g}$ determines a Hodge operator $*$ on the bulk 2-forms such that $(*)^2 = 1$ and coinciding with the one induced by g^+ (since the Hodge operator is Weyl-invariant). Given then the 4-dimensional Weyl tensor $W_{\alpha\beta\gamma\delta}$ of the ALAdS slice[5], we can define its dual as

$$(*W)_{\alpha\beta\gamma\delta} = \frac{1}{2}\, \overline{\mu}_{\alpha\beta}{}^{\rho\sigma} W_{\rho\sigma\gamma\delta} \,. \tag{2.40}$$

We will say that $W^{+}_{\alpha\beta\gamma\delta}$ is (anti)self-dual if

$$W_{\alpha\beta\gamma\delta} = \pm (*W)_{\alpha\beta\gamma\delta} \,. \tag{2.41}$$

As a consequence, we can split any Weyl tensor according to the eigenvalues ± 1 of the Hodge operator,

$$W_{\alpha\beta\gamma\delta} = W^{(+)}_{\alpha\beta\gamma\delta} + W^{(-)}_{\alpha\beta\gamma\delta} \,, \tag{2.42}$$

effectively decomposing the Weyl bundle in two sub-bundles

$$\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^- . \tag{2.43}$$

Self-dual Weyl tensors have vanishing projection onto $\mathcal{W}^-$ and vice versa for the anti-self-dual case. One interesting observation is that in case of (anti)self-duality, any of the two sub-bundles with eigenvalues ± 1 is isomorphic to the space of symmetric traceless 2-tensors on the d-dimensional boundary, as it is easy to show in the Fefferman-Graham gauge [23]. In particular, the independent degrees of freedom of the bulk Weyl tensor in the (anti)self-dual case can be conveniently gathered in the components W_{rirj}.

As a consequence, the independent equations for (anti)self-duality read

$$W_{rirj} = \pm (*W)_{rirj} \,, \tag{2.44}$$

These components read

$$W_{rirj} = -\frac{1}{4r^2}\, \mathrm{tf}\left[g''_{ij} + 2R_{ij} - \frac{1}{2}\, \mathrm{tr}[g'] g'_{ij} \right] , \tag{2.45a}$$

$$(*W)_{rirj} = \frac{1}{2r^2} \mu_{(i}{}^{kl}\, \nabla_l g'_{j)k} , \tag{2.45b}$$

[5]Above we denoted it by $W^{+}_{\alpha\beta\gamma\delta}$; here we remove the + to avoid cluttering.

where traces, R_{ij} and ∇_l are computed with $g_{ij}(x,r)$ at fixed r. Primes indicate derivatives in r and tf stands for the trace-free part. Therefore the (anti)self-duality conditions read

$$\mathrm{tf}\left[g''_{ij} + 2R_{ij} - \frac{1}{2}\,\mathrm{tr}[g']g'_{ij}\right] \pm 2\mu_{(i}{}^{kl}\,\nabla_l g'_{j)k} = 0. \tag{2.46}$$

One can solve the self-duality condition order by order in small r and find that in $d=3$ an ALAdS metric (and hence an ambient space) is (anti)self-dual if and only if

$$g_{(3)ij} = \pm 2C_{ij}, \tag{2.47}$$

where we have repackaged the 3-dimensional Cotton tensor C_{ijk} into a 2-tensor according to

$$C_{ij} = \mu_i{}^{kl}C_{jkl}\,. \tag{2.48}$$

The condition (2.47) is consistent with the general properties of $g_{(3)}$ as a consequence of the properties of the Cotton tensor,

- $C_{[ijk]} = 0 \;\longrightarrow\; g^{ij}C_{ij} = 0,$
- $g^{ij}C_{ijk} = 0 \;\longrightarrow\; C_{ij}$ is symmetric,
- $\nabla^i C_{ijk} = 0 \;\longrightarrow\; \nabla^i C_{ij} = 0.$

Therefore through (2.47) (anti)self-duality of the ambient space (or equivalently (anti)self-duality of its ALAdS sections) constrains the stress tensor VEV in terms of the metric $g_{(0)}$.

2.4 Weyl invariance and the Weyl connection

In the previous sections we discussed how the ambient construction reduces to the embedding space for CFTs on conformally flat d-dimensional backgrounds in the vacuum state. Correlators must be invariant under conformal transformations, which are conveniently realised as symmetries in the embedding space. As detailed in Chapter 3, the same happens in the ambient space. In particular, conformal Killing vectors on $g_{(0)}$ are lifted to near-lightcone isometries on the ambient space. Such a feature is already present in standard holography as one relates asymptotic symmetries in the bulk to conformal transformations on the boundary. This property of the ambient space can be thought of as inherited from the ALAdS realization (2.8), where asymptotic symmetries on the ALAdS slices are to be understood as near-nullcone isometries on the ambient space.

Given a conformal symmetry transformation, the corresponding Ward identity in the CFT constrains ambient correlators in the same way as embedding correlators. For each such isometry K in $d+2$ dimensions, ambient correlators F of quasi-primary operators must satisfy

$$\sum_{i=1}^{n} \mathcal{L}_K^{(i)} F(X_1, X_2 \dots X_n) = 0\,, \tag{2.49}$$

where $\mathcal{L}^{(i)}$ is the Lie derivative operator acting on the i-th insertion point and where F is a tensor on the ambient space, in general with different tensorial transformation properties for each insertion.

Since we are interested in CFT backgrounds and states that may break all near-lightcone isometries K that so usefully constrain correlators through (2.49), how then is the ambient space formalism useful? The answer is *Weyl covariance*, which represents the universal kinematical constraint on correlators. For a CFT on a generic background $g_{(0)}$ it reads as in (1.9).[6]

Assume we have an ambient space $\tilde{g}$ of the form (2.1) constructed from the CFT background metric $g_{(0)ij}$; we wish to construct another one compatible with the metric $\hat{g}_{(0)ij} = e^{2\Omega(x)} g_{(0)ij}$. It turns out these two ambient spaces are locally diffeomorphic, so that in a new set of coordinates $(\hat{t}, \hat{x}^i, \hat{\rho})$ the ambient metric $\tilde{g}$ reads

$$\tilde{g} = 2\hat{\rho} d\hat{t}^2 + 2\hat{t} d\hat{t} d\hat{\rho} + \hat{t}^2 \hat{g}_{ij}(\hat{x}, \hat{\rho}) d\hat{x}^i d\hat{x}^j\,, \tag{2.50}$$

which induces the metric $\hat{g}_{(0)ij}(\hat{x})$ when taking $\hat{\rho} = 0$, $\hat{t} = 1$. Formally (2.50) is an ambient space constructed from the metric $\hat{g}_{(0)ij}$. One can interpret this fact as the statement that an ambient space is canonically related not only to $g_{(0)ij}$ but to the whole conformal class of metrics $[g_{(0)}]$, all equivalent to $g_{(0)}$ modulo a Weyl transformation.[7]

The coordinate transformation from (t, x^i, ρ) to $(\hat{t}, \hat{x}^i, \hat{\rho})$ can be easily found by working perturbatively in ρ.[8] Algorithmically, one imposes order by order that the background metric induced at $\hat{\rho} = 0$, $\hat{t} = 1$ is the Weyl-rescaled $\hat{g}_{(0)ij}$, as well as that the ambient gauge is preserved (i.e. $\hat{t}$ and $\hat{\rho}$ are Gaussian null coordinates). For what follows we are interested only in the first few orders,

$$\hat{t} = e^{-\Omega(x)} t \left[1 - \frac{1}{2}\Omega^i \Omega_i \rho + O(\rho^2)\right]\,, \tag{2.51a}$$

$$\hat{x}^i = x^i + \Omega^i \rho + O(\rho^2)\,, \tag{2.51b}$$

$$\hat{\rho} = e^{2\Omega(x)} \rho + O(\rho^2)\,, \tag{2.51c}$$

with $\Omega_i = \partial_i \Omega$ and where indices are raised and lowered using $g_{(0)ij}$.

[6] In this context we do not discuss the contributions from Weyl anomalies that can be arise in even dimensions d.

[7] This parallels the case of $(d+1)$-dimensional ALAdS spaces, where Weyl transformations are induced onto the boundary by a special class of bulk diffeomorphisms (see *e.g.* [65, 68]).

[8] We proceed in this way to keep the discussion as general as possible. Given an exact ambient solution to all orders in ρ, this diffeomorphism may be found in closed form.

As anticipated on the nullcone $\rho = 0$ this diffeomorphism reduces to a local rescaling of the coordinate t. This is the analogue of the local rescaling of the projective section in the embedding space in (1.18) which leads to a Weyl-rescaled background. This agrees with the intuition that t measures the engineering dimensions of ambient quantities. In particular, the scalar invariant $\widetilde{X}_{ij}$ defined in equation (2.25) for conformally flat backgrounds is manifestly homogeneous in t in both insertions $\widetilde{X}_{ij} \propto t_i t_j$, hence transforms homogeneously under Weyl transformations with dimension -2. Analogously, X_{12} is a dimension -2 invariant in the embedding space.

The fact that Weyl transformations are induced by ambient diffeomorphisms represents the key property of the ambient space and it has been the main motivation for its use in conformal geometry, allowing one to find and classify Weyl-invariant objects on arbitrary d-dimensional manifolds [22, 23, 77–86]. This aspect is obscured in the ALAdS_{d+1} realization and in the standard holographic setup. There one focuses on a single hyperbolic slice of radius s, effectively quotienting by the action of the homothety $T = s\partial_s$.

Our goal is to use the ambient space to study correlators, meant as multi-local tensorial objects living on the ambient nullcone. To impose their Weyl-covariance, one has to study the precise action of the diffeomorphisms (2.51) on ambient tensors when restricted to the nullcone [23, 77, 80].

Let us focus on vector fields on the ambient space for simplicity. It will be straightforward to generalise the discussion to any ambient tensor. When we restrict an ambient vector field to the nullcone, its components will only depend on t and x^i. There turns out to be a privileged class of ambient vectors whose components can be written in the form

$$V^M = \left(v^0(x), \frac{v^i(x)}{t}, \frac{v^\rho(x)}{t} \right) \tag{2.52}$$

once restricted to the CFT background.[9] Here t should be thought of as a formal parameter $t = 1$ which keeps track of the weight under Weyl transformations of each component. In conformal geometry, the vector (2.52) is known as a (weight 0) *tractor*.

More specifically, under the ambient diffeomorphisms (2.51) the components of any such vector restricted to $g_{(0)}$ transform according to

$$\widehat{V}^M = \frac{\partial \widehat{\widetilde{X}}^M}{\partial \widetilde{X}^N} V^N \bigg|_{\rho=0} = \begin{pmatrix} \Omega^{-1} \left[v^0 - \Upsilon_i v^i - \frac{1}{2} \Upsilon_i \Upsilon^i v^\rho \right] \\ \frac{1}{t} \left[v^i + \Upsilon^i v^\rho \right] \\ \frac{\Omega^2}{t} v^\rho \end{pmatrix}, \tag{2.53}$$

where $\Upsilon_i = \Omega^{-1} \Omega_i$. The resulting components $\widehat{V}^M$ have the same weight in t as the initial vector (2.52) and thus this transformation preserves the class of tractor fields. We can

[9] By doing this we are effectively constructing a $(d+2)$-dimensional vector bundle on the d-dimensional background.

rewrite this action in terms of a linear transformation on the $(d+2)$-dimensional space of such vector fields,

$$U(\Omega)^M_{\ \ N} = \begin{pmatrix} \Omega^{-1} & -\Upsilon_n \Omega^{-1} & -\frac{1}{2}\Upsilon_i \Upsilon^i \Omega^{-1} \\ 0 & \delta^m_n \Omega^{-1} & \Upsilon^m \Omega^{-1} \\ 0 & 0 & \Omega \end{pmatrix}, \tag{2.54}$$

parametrised by a function $\Omega(x)$ on the CFT background.

One can analogously define weight w tractors as the restriction to a d-dimensional nullcone section of ambient vectors with an additional overall homogeneity of t^w with respect to (2.52). Weighted tractors can be simply thought of as the restriction of ambient vectors $V = V^M \partial_M$ with homogeneity $w-1$ in t to the section $t=1$ of the nullcone. They transform as

$$\widehat{V}^M = \Omega^w \, U(\Omega)^M_{\ \ N} V^N \tag{2.55}$$

under a Weyl transformation.

If we further inspect the components of the ambient connection, the action of the ambient covariant derivative along the x^i directions (once restricted to the d-dimensional section) $\mathcal{D}_k \equiv \widetilde{\nabla}\Big|_{\substack{\rho=0\\t=1}}$ can be split as

$$\mathcal{D}_k = \nabla_k + \mathcal{A}_k, \qquad \text{with} \qquad (\mathcal{A}_k)^M_{\ \ N} = \begin{pmatrix} 0 & -P_{kn} & 0 \\ P_k^{\ m} & 0 & \delta^m_k \\ 0 & -g_{kn} & 0 \end{pmatrix}, \tag{2.56}$$

where the first piece is simply the covariant derivative compatible with the background metric $g_{(0)}$, under which v^0 and v^ρ are scalars. Thus the ambient connection acts on a tractor (of any weight) as

$$\mathcal{D}_k V^M = \partial_k V^M + \delta^M_m \Gamma^m_{kn} v^n + (\mathcal{A}_k)^M_{\ \ N} V^N. \tag{2.57}$$

The additional piece $\mathcal{A}_k$ that the ambient connection induces onto the CFT background is what makes $\mathcal{D}_k$ covariant under Weyl transformations when acting on tractors. In particular one can check that $\mathcal{D}_k$ commutes with Weyl transformations $U(\Omega)$, i.e. $\mathcal{D}_k V^M$ transforms in the same way as V^M,

$$\widehat{\mathcal{D}}_k \, U(\Omega) = U(\Omega) \, \mathcal{D}_k \,, \tag{2.58}$$

where $\widehat{\mathcal{D}}_k$ indicates the covariant derivative compatible with the ambient metric (2.50). This shows that the ambient connection canonically induces a Weyl connection on the boundary. Finding Weyl covariant objects in d dimensions (such as CFT correlators) boils down to the study of multiplets under Weyl transformations given by the matrix $U(\Omega)$. It is in this sense that Weyl transformations are linearly realised on the ambient nullcone, similarly to what happens for conformal symmetries in the embedding space. This is the perspective adopted in the so-called *tractor calculus* [80, 87]. In Section 4.5 we discuss the implications for the computation of spinning correlators in general backgrounds and states.

2.5 GJMS conformal operators

In this section we review a class of differential operators that can be defined using the ambient space. Their importance resides in the fact that they can be used on objects of a certain scaling dimension to produce new objects of a different scaling dimension. It is in this sense that they can be thought of as a first example of weight-shifting operators, which will be discussed more in detail in Section 4.5.

The original paper [78] constructs a family of differential operators P_{2k} of order $2k$ which are conformally covariant and act on scalars $\phi_{(0)}(x)$ of conformal weight $w = k - \frac{d}{2}$. They takes the form

$$P_{2k} = \Box^k + \dots , \tag{2.59}$$

where $\Box = \nabla_i \nabla^i$ is the Laplacian associated to $g_{(0)}$ and where the dots denote corrections involving the d-dimensional curvature. The resulting scalar $P_{2k}\phi$ has conformal weight $w - 2k = -w - d$ in t. These operators can thus be thought of as conformal Laplacians in d dimensions.

Let us now turn to their construction using the ambient space. We first fix an ambient space $\tilde{g}\left[g_{(0)}, g_{(d)}\right]$ associated to a d-dimensional metric $g_{(0)}$ and stress tensor VEV $g_{(d)}$. By definition, a field $\phi_{(0)}(x)$ is a conformal density of weight w in d dimensions if under conformal rescaling of the metric

$$\hat{g}_{(0)ij} = \Omega^2 g_{(0)ij} \qquad \Rightarrow \qquad \hat{\phi}_{(0)} = \Omega^w \phi_{(0)}. \tag{2.60}$$

In CFT terms, the conformal weight w is therefore minus the scaling dimension Δ. From the form of the coordinate transformation (2.51) inducing a d-dimensional Weyl transformation, one concludes the canonical nullcone extension of such a conformal density field is the homogeneous function

$$f(t, x) = t^w \phi_{(0)}(x). \tag{2.61}$$

We now would like to solve the massless Klein-Gordon equation

$$\Box_{\tilde{g}} f(t, x, \rho) = 0 \tag{2.62}$$

on the ambient space $\tilde{g}\left[g_{(0)}, g_{(d)}\right]$ with Dirichlet boundary condition at $\rho = 0$

$$f(t, x, \rho = 0) = t^w \phi_{(0)}(x)\,. \tag{2.63}$$

This boundary condition does not fully characterise the second order problem, and we expect an integration function to appear in the solution.

By parametrizing the conformal weight as $w = k - \frac{d}{2}$, $k = 1, 2 \dots$, for a straight ambient metric in normal form we can rewrite the ambient box operator as

$$\frac{\Box_{\tilde{g}} f(t, x, \rho)}{t^{w-2}} = -2\rho\phi'' + \left(2k - 2 - \rho g^{ij} g'_{ij}\right)\phi' + \Box\phi + \frac{1}{2}\left(k - \frac{d}{2}\right) g^{ij} g'_{ij}\phi \tag{2.64}$$

where as usual primes denote derivatives in ρ and we defined $f(t,x,\rho) = t^w \phi(x,\rho)$. The solution to equation (2.62) has then a ρ expansion of the form [78]

$$\phi(x,\rho) = \phi_{(0)}(x) + \rho\phi_{(1)}(x) + \cdots + \rho^k \left[\phi_{(k)}(x) + \psi_{(k)}(x)\log\rho\right] + O\left(\rho^{k+1}\right), \qquad (2.65)$$

where $\psi_{(k)}(x)$ as well as all the coefficients $\phi_{(m)}$ up to $m = k-1$ are determined by $\phi_{(0)}$. The conformally covariant operator P_{2k} is then defined through the equality

$$P_{2k}\phi_{(0)} = c_k \psi_{(k)}\,, \qquad (2.66)$$

where c_k is an arbitrary constant. The logarithmic term in (2.65) is present only for integer k and thus the differential operators P_{2k} can only be defined for integer k.

As an example, in the case of $k = 1$ the expansion for ϕ takes the form

$$\phi(x,\rho) = \phi_{(0)} + \rho\left(\phi_{(1)} + \psi_{(1)}\log\rho\right) + \ldots. \qquad (2.67)$$

Evaluating (2.64) at $\rho = 0$, one finds an equation for $\psi_{(1)}$, which leads to

$$\psi_{(1)} = \frac{1}{2}\left[\Box\phi_{(0)} - \frac{n-2}{4(n-1)}R\,\phi_{(0)}\right]. \qquad (2.68)$$

One does not find any equation for $\phi_{(1)}$, which constitutes the other piece of boundary data to be specified.

It is possible to find the following recursive relation for the coefficient $\phi_{(\ell+1)}$,

$$2(\ell+1-k)\phi_{(\ell+1)} = \partial_\rho^{(\ell)}\left[\Box\phi - g^{ij}g'_{ij}\left(\rho\phi' - \frac{w}{2}\phi\right)\right]\Big|_{\rho=0}, \qquad (2.69)$$

which holds as long as $\ell < k$. From this, one can show that $\phi_{(\ell+1)}$ involves all the coefficients of the ambient metric expansion $g^{(m)}$ with $m \leq \ell$, in addition to the trace of $g_{(\ell)}$. This means that an equation for $\psi_{(k)}$ appears after differentiating (2.64) $k-1$ times and it involves $\mathrm{tr}[g_{(k)}]$, on top of all the lower order coefficients $g_{(m)}$ with $m < k$.

Notice that $\phi_{(1)}$ contains $\Box\phi_{(0)}$, and similarly $\phi_{(2)} \sim \Box\phi_{(1)} \sim \Box^2\phi_{(0)}$, so that $\phi_{(\ell)}$ will have leading term $\Box^\ell\phi_{(0)}$ and $\psi^{(k)} \sim \Box^k\phi^{(0)} + \ldots$.

For specific classes of metrics $g_{(0)}$, equation (2.64) simplifies considerably and one can find the operators P_{2k} in a closed form. For flat boundaries, $g_{ij}(x,\rho) = \delta_{ij}$ are constants and $P_{2k} = \Box^k$. For conformally Einstein $g_{(0)}$ such that $R_{ij} = 2\lambda(d-1)g_{(0)ij}$, and vanishing $g_{(d)}$ they take the following closed form,

$$P_{2k} = \prod_{j=1}^{k}(-\Box + 2\lambda c_j), \qquad (2.70)$$

with $c_j = \left(\frac{d}{2} + j - 1\right)\left(\frac{d}{2} - j\right)$.

A connection can be made between these results and holography switching from the ambient coordinates (t, x, ρ) to the ALAdS slicing picture in equation (2.8). In these coordinates (s, r, x), the ambient box operator factorizes as

$$\Box_{\tilde{g}} f = -\partial_s^2 f + \frac{g_+^{\mu\nu}}{s^2}\,\partial_\mu \partial_\nu f + \widetilde{\Gamma}^M_{00}\,\partial_M f - \frac{g_+^{\mu\nu}}{s^2}\,\widetilde{\Gamma}^M_{\mu\nu}\,\partial_M f, \tag{2.71}$$

where $\widetilde{\Gamma}^M_{00} = 0$ and

$$\widetilde{\Gamma}^M_{\mu\nu} = \delta^M_\sigma \left(\Gamma^+\right)^\sigma_{\mu\nu} + s g^+_{\mu\nu} \delta^M_0 . \tag{2.72}$$

Then

$$\Box_{\tilde{g}} f = -\partial_s^2 f - \frac{d+1}{s}\partial_s f + \frac{1}{s^2}\Box^+ f\,, \tag{2.73}$$

where $\Box^+$ indicates the box operator on an ALAdS slice. Under the coordinate transformation (2.7), $f(t, x, \rho) = t^w \phi(x, \rho) = \frac{s^w}{r^w}\phi(x, r)$. By defining $F(x, r) = \frac{\phi(x,r)}{r^w}$, one finds that

$$\Box_{\tilde{g}} f = s^{w-2}\left[\Box^+ - w(w+d)\right] F(x, r). \tag{2.74}$$

Therefore the equation $\Box_{\tilde{g}} f = 0$ for the ambient extension of a conformal density field of weight w corresponds to the bulk equations of a massive scalar field $F(x, r)$ with $M^2 = w(w+d)$. This can be interpreted as a generalisation of the calculation performed for example in [40] on Minkowski space, where the dynamics of a massless scalar in four dimensions is decomposed in terms of an infinite number of massive scalar modes on the Euclidean AdS sections of Minkowski. In this section we have restricted for simplicity to one such mode by fixing a specific conformal weight w.

Via this map, the properties of the expansion (2.65) can be easily related to those of the holographic scalar computation in Subsection 1.3.3 through the identifications of $\phi_{(0)}$ and $\phi_{(k)}$ as the holographic source and VEV for the dual scalar operator, as well as by identifying $w = \Delta - d$. Holographically, it is then the coefficient $\psi_{(k)}$ related to the scalar Weyl anomaly to define the conformal powers of the boundary Laplacian P_{2k}.

Chapter 3

Ambient isometries

As anticipated in Section 2.4, in this chapter we show that any conformal Killing vector of the CFT background $g_{(0)}$ can be lifted to a near-nullcone isometry on the ambient space. This is analogous to what happens in the embedding space formalism, where d-dimensional conformal transformations are lifted to Lorentz transformations in $(d+2)$-dimensional Minkowski spacetime. We first summarise the results, and in the sections below we provide the details of the computation as well as several examples.

Consider a conformal transformation generated in d dimensions by $E_j^{(0)}(x)$ satisfying

$$\nabla_i E_j^{(0)} + \nabla_j E_i^{(0)} = 2\psi\, g_{(0)ij}(x), \tag{3.1}$$

with conformal factor $\psi(x) = \frac{1}{d}\nabla_l E^l_{(0)}$. One can extend it to the ambient space (at least close enough to the nullcone) as an isometry K with components

$$K(t,\rho,x) = -t\psi(x)\,\partial_t + 2\rho\,\psi(x)\,\partial_\rho + E^i(\rho,x)\,\partial_i \tag{3.2}$$

Here we denote

$$E^j(\rho,x) = E^j_{(0)}(x) + (\partial_i\psi)\int_0^\rho d\rho' g^{ij}(x,\rho'), \tag{3.3}$$

where the integral of the inverse metric expansion yields for the first few orders

$$E^j(\rho,x) = E^j_{(0)}(x) + (\partial_i\psi)\left[g^{ij}_{(0)}\,\rho - P^{ij}\rho^2 + o\left(\rho^2\right)\right]. \tag{3.4}$$

3.1 Derivation of the ambient isometries and their relation with conformal symmetries

We would like to study the solutions to the Killing equation on a given ambient space $\tilde{g}$ with fixed free data $\left\{g_{(0)}, g_{(d)}\right\}$. Working in the ambient coordinates $\widetilde{X}^M = (t, x^i, \rho)$, a Killing vector

$$K = K^0\partial_t + K^\rho\partial_\rho + K^i\partial_i \tag{3.5}$$

satisfies

$$\widetilde{\nabla}_M K_N + \widetilde{\nabla}_N K_M = 0, \tag{3.6}$$

which can be expanded as

$$\begin{aligned}
\partial_0 K_0 &= 0, &(3.7)\\
\partial_\rho K_\rho &= 0, &(3.8)\\
\partial_0 K_j + \partial_j K_0 &= \frac{2}{t} K_j, &(3.9)\\
\partial_0 K_\rho + \partial_\rho K_0 &= \frac{2}{t} K_\rho, &(3.10)\\
\partial_i K_\rho + \partial_\rho K_i &= g^{ml} g'_{il} K_m. &(3.11)\\
\nabla_i K_j + \nabla_j K_i &= -t g'_{ij} K_0 + 2\left(-g_{ij} + \rho g'_{ij}\right) K_\rho, &(3.12)
\end{aligned}$$

The first two equations imply that the Killing vector components depend on the coordinates according to $K_0(x,\rho),\ K_i(t,x,\rho),\ K_\rho(t,x)$. Equations (3.9) and (3.10) allow one to fix the t and ρ dependence of K_M. In particular, from (3.10) one finds that

$$\begin{aligned}
K_0(x,\rho) &= A(x)\rho + B(x), &(3.13)\\
K_\rho(t,x) &= t\left[t a(x) + A(x)\right], &(3.14)
\end{aligned}$$

for some integration functions a, A and B. From (3.9) then it follows that

$$K_j(t,x,\rho) = t\left[\rho \partial_j A(x) + \partial_j B(x) + t E_j(x,\rho)\right], \tag{3.15}$$

for some undetermined vector $E_j(x,\rho)$.

If we plug these functions into (3.11), we find

$$t\left[\partial_i a(x) + \partial_\rho E_i(x,\rho) - g^{ml} g'_{il} E_m(x,\rho)\right] + 2\partial_i A(x) - g^{ml} g'_{il}\left(\rho \partial_m A(x) + \partial_m B(x)\right) = 0, \tag{3.16}$$

which can be split into two equations,

$$\begin{aligned}
\partial_\rho E_i(x,\rho) - g^{ml} g'_{il} E_m(x,\rho) + \partial_i a(x) &= 0 &(3.17)\\
\partial_i A(x) - \frac{1}{2} g^{ml} g'_{il}\left(\rho \partial_m A(x) + \partial_m B(x)\right) &= 0. &(3.18)
\end{aligned}$$

Similarly, from equation (3.12) one finds

$$\begin{aligned}
\nabla_i E_j(x,\rho) + \nabla_j E_i(x,\rho) &= 2(-g_{ij} + \rho g'_{ij}) a(x), &(3.19)\\
\nabla_i \partial_j \left[\rho A(x) + B(x)\right] &= -\frac{1}{2} g'_{ij}\left[\rho A(x) + B(x)\right] + (-g_{ij} + \rho g'_{ij}) A(x). &(3.20)
\end{aligned}$$

The functions a, A and B do not depend on ρ and we can find them by considering the equations above at $\rho = 0$. Here the components K_M reduce to

$$K_0(x,0) = B(x), \tag{3.21a}$$
$$K_j(t,x,0) = t\left[\partial_j B(x) + tE_j^{(0)}(x)\right], \tag{3.21b}$$
$$K_\rho(t,x) = t\left[ta(x) + A(x)\right], \tag{3.21c}$$

where $E_j^{(0)}(x) = E_j(x, \rho = 0)$. Equations (3.17)-(3.20) at $\rho = 0$ read

$$E_i^{(1)}(x) - g^m_{(1)i}E_m^{(0)}(x) + \partial_i a(x) = 0 \tag{3.22}$$
$$\partial_i A(x) - \frac{1}{2}g^m_{(1)i}\partial_m B(x) = 0. \tag{3.23}$$
$$\nabla_i E_j^{(0)}(x) + \nabla_j E_i^{(0)}(x) + 2g_{(0)ij}a(x) = 0, \tag{3.24}$$
$$\nabla_i\partial_j B(x) + \frac{1}{2}g_{(1)ij}B(x) + g_{(0)ij}A(x) = 0, \tag{3.25}$$

where $g_{(1)ij} = 2P_{ij}$, $\mathrm{tr}[g_{(1)}] = \frac{R}{d-1}$ and indices are raised and lowered with the metric $g_{(0)}$.

Let us focus on (3.22) and (3.24) first. The latter has precisely the form of the conformal Killing equation for $E_i^{(0)}(x)$ on $g_{(0)}$, and in particular its trace implies that at any ρ

$$a(x) = -\frac{1}{d}\nabla_i E^i_{(0)}(x), \tag{3.26}$$

so that (3.24) can be rewritten as equation (3.1). One can also rewrite equation (3.17) as

$$\partial_\rho E_i(x,\rho) - g^{ml}g'_{il}E_m(x,\rho) - \frac{1}{d}\partial_i\left(\nabla_l E^l_{(0)}\right) = 0, \tag{3.27}$$

which can be integrated yielding (3.3) and (3.3). Therefore, given a conformal Killing vector $E_j^{(0)}(x)$ on $g_{(0)}$ satisfying (3.1), one can use (3.2) and (3.3) to define its ambient extension K^M, which is an isometry. The scalars $A(x)$ and $B(x)$ appearing in the general solution of the ambient Killing vector equations described above are instead independent and ultimately related to ambient translational invariance, as we show in the following sections.

3.2 The $B(x)$ ambient isometries

We now turn to (3.23) and (3.25). From the trace part of (3.25),

$$A(x) = -\frac{1}{d}\left[\Box + \frac{1}{2}\,\mathrm{tr}[g_{(1)}]\right]B(x) = -\frac{1}{d}\left[\Box + \frac{R}{2(d-1)}\right]B(x). \tag{3.28}$$

Equations (3.23) and (3.25) respectively can thus be rewritten as

$$\left[2\partial_i\Box + dg^{m}_{(1)i}\partial_m + \mathrm{tr}[g_{(1)}]\partial_i + \partial_i(\mathrm{tr}\, g_{(1)})\right] B(x) = 0 \tag{3.29a}$$

$$\left[\nabla_i\partial_j - \frac{1}{d}g_{(0)ij}\Box + \frac{1}{2}\left(g_{(1)ij} - \frac{1}{d}\,\mathrm{tr}[g_{(1)}]g_{(0)ij}\right)\right] B(x) = 0. \tag{3.29b}$$

These are two equations that $B(x)$ must satisfy at any ρ in order for K_M to be a Killing vector on the ambient space. We can simplify them noting that by the second Bianchi identity and the commutator of two covariant derivatives

$$\nabla^j R_{ij} = \frac{1}{2}\nabla_i R, \qquad\qquad [\nabla_j, \nabla_i]\, V^k = R^k{}_{mji}V^m, \tag{3.30}$$

the covariant derivative ∇^j of (3.29b) reads

$$\partial_i\Box B + \frac{d}{d-2}\left[R_{im} - \frac{R}{d(d-1)}g_{(0)im}\right]\partial^m B + \frac{d-1}{2}(\nabla_i R)B = 0 \tag{3.31}$$

By subtracting it from (3.29a), we obtain

$$(\nabla_i R)\, B(x) = 0. \tag{3.32}$$

Therefore, the equations (3.29) are satisfied in two cases, wither $\nabla_i R = 0$ or $B = 0$. If $\nabla_i R = 0$, then a new class of ambient Killing vectors parametrized by B is turned on besides those related to conformal symmetries $E^{(0)}_j$ on $g_{(0)}$. Their components read

$$K^0(x) = -\frac{1}{d}\left[\Box + \frac{R}{2(d-1)}\right] B, \tag{3.33a}$$

$$K^\rho(t, \rho, x) = \frac{1}{td}\left[\rho\,\Box + \frac{R}{2(d-1)}\,\rho + d\right] B, \tag{3.33b}$$

$$K^i(t, \rho, x) = \frac{g^{ij}(x, \rho)}{t}\left[\delta^m_j + \rho P^m{}_j\right]\partial_m B. \tag{3.33c}$$

For a conformally flat $g_{(0)}$, these isometries are simply the $d + 2$ translations of the embedding space in disguise, as we check in Section 3.3. More generally, the fact that the d-dimensional manifold must have constant curvature for the $B(x)$ transformations to be present means that locally $g_{(0)}$ is either a sphere, Euclidean space or a hyperboloid if we assume it is a complete manifold. In all these cases the ambient space near the nullcone is locally Minkowski and hence there we expect translations to be isometries. We check this explicitly in the case of Euclidean AdS $g_{(0)}$ in Section 3.4. In less trivial cases fewer such isometries may exist but nevertheless one can prove in full generality that they form a self-commuting sub-algebra as we discuss in Section 3.4. In practice these additional symmetries are useful to find an adapted set of coordinates to rewrite the ambient space as Minkowski if $d+2$ such isometries exist. Note that generally when a stress tensor VEV $g_{(d)}$ is present in the ambient expansion this additional class of $B(x)$ ambient isometries does not exist, even for $g_{(0)}$ with constant sectional curvature $\nabla_i R = 0$.

3.3 Ambient isometries for flat $g_{(0)}$

We now discuss the ambient isometries in the case of flat $g_{(0)ij} = \delta_{ij}$. Here $R = 0$ and as we discussed in Section 1.1, the generator of conformal transformations in d dimensions is

$$E_{(0)} = \left[a^i + \omega^i{}_j x^j + \lambda x^i + b^i x^2 - 2b^k x_k x^i\right] \partial_i. \tag{3.34}$$

Since in this case

$$\psi(x) = \frac{1}{d}\nabla \cdot E_{(0)} = \lambda - 2b \cdot x, \tag{3.35}$$

$$E^j(x,\rho) = E^j_{(0)} - 2\rho b^j, \tag{3.36}$$

its ambient extension has components

$$K^0 = -t\,(\lambda - 2b\cdot x)\,, \tag{3.37a}$$

$$K^i = E^i_{(0)} - 2\rho b^i, \tag{3.37b}$$

$$K^\rho = 2\rho\,(\lambda - 2b\cdot x)\,. \tag{3.37c}$$

The equations (3.29) for B are simply

$$\partial^2\partial_i B = 0, \tag{3.38}$$

$$\left(\partial_i\partial_j - \frac{1}{d}\partial^2\delta_{ij}\right) B = 0. \tag{3.39}$$

The most general form of B is then is

$$B(x) = C + T_i x^i + \alpha x^i x_i\,, \tag{3.40}$$

where C, T_i and α are $d+2$ integration constants.

Through (3.33), the corresponding ambient isometries read

$$K(t,x,\rho) = -2\alpha\partial_t + \frac{1}{t}\left[T^i + 2\alpha x^i\right]\partial_i + \frac{1}{t}\left[C + T^i x_i + \alpha(x^2 + 2\rho)\right]\partial_\rho\,. \tag{3.41}$$

We can thus identify the following three classes of ambient isometries related to $B(x)$,

$$K_{(C)} = \frac{1}{t}\partial_\rho, \tag{3.42a}$$

$$K_{(i)} = \frac{1}{t}\left(\partial_i + x_i\partial_\rho\right), \tag{3.42b}$$

$$K_{(\alpha)} = -2\partial_t + \frac{2x^i}{t}\partial_i + \frac{1}{t}\left(x^2 + 2\rho\right)\partial_\rho. \tag{3.42c}$$

Using the change to Cartesian coordinates (2.17) and defining $X^{\pm} = X^0 \pm X^{d+1}$, these vectors read on Minkowski

$$K_{(C)} = -\partial_0 + \partial_{d+1}, \tag{3.43a}$$
$$K_{(i)} = \partial_i, \tag{3.43b}$$
$$K_{(\alpha)} = -\partial_0 - \partial_{d+1}. \tag{3.43c}$$

As anticipated, we conclude that on flat ambient spaces $B(x)$ transformations correspond to the $d+2$ translations.

If we turn on both $B(x)$ and $E_j^{(0)}$, the most general ambient Killing vector is

$$K^0 = -2\alpha - t\left(\lambda - 2b\cdot x\right), \tag{3.44a}$$
$$K^i = \frac{1}{t}\left(T^i + 2\alpha x^i\right) + E^i_{(0)} - 2\rho b^i, \tag{3.44b}$$
$$K^\infty = \frac{1}{t}\left[C + T^i x_i + \alpha x^2\right] + 2\rho\left(\lambda - 2b\cdot x + \frac{\alpha}{t}\right), \tag{3.44c}$$

with $E_j^{(0)}$ given by (3.34). One can check that their commutators reproduce the Poincaré algebra $\mathrm{ISO}(1, d+1)$.

3.4 Ambient isometries for $g_{(0)}$ with $\nabla_i R = 0$

In this section we consider ambient spaces where $g_{(0)}$ has constant sectional curvature, $\nabla_i R = 0$. Parametrising the Ricci scalar as $R = 2\lambda d(d-1)$ and restricting to $d > 2$, this entails that $R_{ij} = 2\lambda(d-1)g_{ij} + H_{ij}$, with H_{ij} traceless symmetric and satisfying $\nabla^j H_{ij} = 0$ due to Bianchi identities. Accordingly,

$$P_{ij} = \lambda g_{ij} + \frac{H_{ij}}{d-2}, \qquad g_{(1)ij} = 2P_{ij}, \qquad \mathrm{tr}\left[g_{(1)}\right] = \frac{R}{d-1} = 2d\lambda. \tag{3.45}$$

The class of isometries parametrized by $B(x)$ in this case are of the form

$$K_0(x,\rho) = \left[1 - \frac{\rho}{d}\left(\Box + d\lambda\right)\right] B(x), \tag{3.46a}$$
$$K_j(t,x,\rho) = t\left[\rho\left(\frac{H^i{}_j}{d-2} + \lambda\delta^i_j\right) + \delta^i_j\right]\partial_i B(x), \tag{3.46b}$$
$$K_\rho(t,x) = -\frac{t}{d}\left(\Box + d\lambda\right) B(x), \tag{3.46c}$$

with $B(x)$ satisfying (3.29b), i.e.

$$\left[\nabla_i\partial_j - \frac{1}{d}g_{ij}\Box + \frac{H_{ij}}{d-2}\right] B = 0, \tag{3.47}$$

which can be checked to imply (3.29a) once derived in ∇^j.

The case of Einstein $g_{(0)}$ (with $H_{ij} = 0$) is more tractable. The ambient isometries parametrised by $B(x)$ become

$$K_0(x,\rho) = \left[1 - \frac{\rho}{d}\left(\Box + d\lambda\right)\right] B(x), \tag{3.48a}$$

$$K_j(t,x,\rho) = t\left(1+\lambda\rho\right)\partial_j B(x), \tag{3.48b}$$

$$K_\rho(t,x) = -\frac{t}{d}\left(\Box + d\lambda\right) B(x), \tag{3.48c}$$

with $B(x)$ satisfying (3.29b), i.e.

$$\nabla_i \partial_j B = \frac{1}{d} g_{ij} \Box B. \tag{3.49}$$

Given $\nabla_i R = 0$, the latter implies (3.29a), that is

$$\left(\Box + 2\lambda\right)\partial_i B(x) = 0, \tag{3.50}$$

or equivalently

$$\partial_i\left(\Box + 2\lambda d\right) B(x) = 0. \tag{3.51}$$

If we fix two generic Killing vectors $K^M(B_1)$ and $K^N(B_2)$ (where B_1 and B_2 indicate two different choices of the transformations parameters contained in B), one can check that they commute for Einstein boundaries by virtue of equations (3.49) and (3.51). Thus, the B isometries constitute a Cartan subalgebra of the ambient isometries. This is compatible with them being translations on the ambient space.

3.4.1 AdS$_d$.

Consider the example of a Euclidean AdS$_d$ in Poincaré coordinates as the $g_{(0)}$ metric,

$$g(z,x) = \frac{1}{z^2}\left(dz^2 + dx^a dx^a\right), \qquad a = 2\ldots d, \tag{3.52}$$

which is Einstein with $\lambda = -\frac{1}{2}$. Assuming no stress tensor VEV, the subsequent ambient metric is

$$\tilde{g} = 2\rho dt^2 + 2t dt d\rho + t^2\left(1 - \frac{\rho}{2}\right)^2 g(z,x). \tag{3.53}$$

From equation (3.51) we conclude that

$$(\Box - d)B(z,x) = \alpha d, \tag{3.54}$$

for some constant α, so that we can rewrite (3.49) as

$$\nabla_i \partial_j B = g_{ij}(B + \alpha). \tag{3.55}$$

From the $i = j = z$ component we find

$$B(z,x) = \frac{S(x)}{z} + zT(x) - \alpha, \tag{3.56}$$

while the remaining equations constrain T to be a constant, as well as imposing that

$$\begin{aligned} \partial_a \partial_b S &= 0 \qquad \text{for } a \neq b, && (3.57a)\\ \partial_a^2 S &= 2T, && (3.57b) \end{aligned}$$

leading to $S(x) = \beta + v^a x^a + T x^2$. Therefore, $B(z,x)$ contains again $d+2$ free constant parameters. It reads

$$B(z,x^a) = \frac{\beta}{z} + \frac{v^a x^a}{z} + T\left(z + \frac{x^2}{z}\right) - \alpha. \tag{3.58}$$

The corresponding transformations on the ambient space are

$$\begin{aligned} K_{(\beta)} &= -\frac{1}{2z}\partial_t - \frac{1}{t\left(1-\frac{\rho}{2}\right)}\partial_z + \frac{1}{zt}\left(1+\frac{\rho}{2}\right)\partial_\rho, && (3.59a)\\ K_{(a)} &= -\frac{1}{2z}x^a\partial_t - \frac{x^a}{t\left(1-\frac{\rho}{2}\right)}\partial_z + \frac{z}{t\left(1-\frac{\rho}{2}\right)}\partial_a + \frac{1}{zt}\left(1+\frac{\rho}{2}\right)x^a\partial_\rho, && (3.59b)\\ K_{(T)} &= -\frac{1}{2z}(z^2+x^2)\partial_t + \frac{z^2-x^2}{t\left(1-\frac{\rho}{2}\right)}\partial_z + \frac{2z}{t\left(1-\frac{\rho}{2}\right)}x^b\partial_b + \frac{z^2+x^2}{zt}\left(1+\frac{\rho}{2}\right)\partial_\rho, && (3.59c)\\ K_{(\alpha)} &= -\frac{1}{2}\partial_t - \frac{1}{t}\left(1-\frac{\rho}{2}\right)\partial_\rho. && (3.59d) \end{aligned}$$

One can check explicitly that commute with each other. Through the coordinate transformation (2.34) one can indeed show these are the translations on Minkowski space.

Turning to the ambient isometries inducing conformal transformations on $g_{(0)}$, recall that being AdS conformally flat, a conformal vector on the flat space parametrized by (z, x^a) such as (3.34) is also a conformal vector on AdS. Thus in this case

$$E^{(0)} = \xi^z\partial_z + \xi^a\partial_a, \tag{3.60}$$

with

$$\begin{aligned} \xi^z &= a^z + \omega^{za}x_a + \lambda z + (x^2 - z^2)b^z - 2zb^a x_a, && (3.61a)\\ \xi^a &= a^a + \omega^{ab}x_b + \lambda x^a + (x^2+z^2)b^a - 2(b^z z + b^c x_c)x^a, && (3.61b) \end{aligned}$$

with associated conformal factor

$$\psi = -\frac{1}{z}\left[a^z - \omega^{za}x_a + (z^2+x^2)b^z\right]. \tag{3.62}$$

so that from (3.3) one finds

$$E^j(x,\rho) = \xi^j + g^{ij}(x)\frac{\rho}{1-\frac{\rho}{2}}\partial_i\psi. \tag{3.63}$$

The components of this class of ambient isometries thus read,

$$K^0(t,x,\rho) = \frac{t}{z}\left[a^z - \omega^{za}x_a + (z^2+x^2)b^z\right], \tag{3.64a}$$

$$K^z(x,\rho) = \xi^z + \frac{\rho}{1-\frac{\rho}{2}}\left[a^z - \omega^{za}x_a + (x^2-z^2)b^z\right], \tag{3.64b}$$

$$K^a(x,\rho) = \xi^a + \frac{z\rho}{1-\frac{\rho}{2}}\left[\omega^{za} - 2x^a b^z\right], \tag{3.64c}$$

$$K^\rho(x,\rho) = -\frac{2\rho}{z}\left[a^z - \omega^{za}x_a + (z^2+x^2)b^z\right]. \tag{3.64d}$$

3.5 Concluding remarks

In the context of the ALAdS realization it is well known that conformal symmetries are mapped to asymptotic symmetries in the bulk and a statement analogous to (3.2) can be found for instance in [70]. This relation between conformal transformations on $g_{(0)}$ and ambient isometries can thus be seen as inherited from the ALAdS slicing (2.15). Note that by definition any ambient space is also endowed with a conformal Killing vector, the homothety $T = t\partial_t$ under which $\mathcal{L}_T\tilde{g} = 2\tilde{g}$.

CFT correlators on a given non-conformally flat metric $g_{(0)}$ satisfy Ward Identities associated to the residual conformal symmetries of the background. From the ambient perspective, such Ward Identities take the form (2.49) and thus require to know the generators of such symmetries on $g_{(0)}$ as well as their ambient extension. However it may not be a trivial task to identify them for complex enough geometries. For this reason, in Appendix A we illustrate useful techniques to determine the conformal Killing vectors for several classes of manifolds $g_{(0)}$.

We now wish to make few comment about the use that we can make of the ambient isometries (3.2) and (3.33). First of all, they help find adapted coordinates to describe the ambient geometry. They also play an important role when solving the geodesic equations on a given ambient space since they provide first integrals of motion that allow one to automatically reduce part of the geodesic equations to first order ODEs (see Appendix B for more details). As we will see in Chapter 4, solving the geodesic equations on a given ambient space is a crucial step for the ambient formalism. Finally, these ambient isometries may further constrain the form of ambient correlators when considering more general states requiring additional ingredients other than the ambient metric and covariant derivatives of the curvature, and may enter themselves as ingredients for ambient building blocks. We refer the reader to Chapter 4 for more comments on these issues.

Let us finally redirect the reader to Chapter 7 for the discussion of an interesting connection between ambient isometries and the asymptotic symmetries of flat spacetimes as addressed in the context of flat holography.

Chapter 4

Ambient correlators

Given a CFT in a state defined by the VEVs $\{\langle O_i \rangle\}$ and on the metric background $g_{(0)}$, we must find a prescription to associate a specific ambient space to it. As discussed in Section 2.1 the data $g_{(0)}$ are not enough to specify the ambient metric, since one must also provide additional near-nullcone data $g_{(d)ij}$. Once this data is specified the construction proceeds by fulfilling the Ricci-flatness condition. It is natural to associate $g_{(d)ij}$ with the VEV of the stress-energy tensor in some way.

In order to provide a concrete proposal, we lean on AdS/CFT. As reviewed in Subsection 1.3.2, according to AdS/CFT any hyperbolic slice of an ambient space in the form (2.8) encodes the dynamics of a CFT on the background $g_{(0)}$ and in a precise state. We propose to associate a CFT in the state $\{\langle O_i \rangle\}$ and background $g_{(0)}$ to the ambient space constructed with the corresponding ALAdS slices according to AdS/CFT. Other states where additional VEVs are turned on would require an extension of the ambient space to accommodate for such additional data. In this case one should include other matter fields and a modification of the Ricci-flatness condition.

This proposal does not necessarily mean that the resulting ambient-space analysis only applies to holographic CFTs. Recall that the embedding space solves the kinematics of CFTs in the vacuum state on conformally flat backgrounds, and its hyperbolic slices are pure (A)dS spaces. Nonetheless, we know that using the embedding space we can solve the symmetry constraints on correlators not only for theories which are strictly-speaking holographically dual to pure AdS, but also for free or weakly coupled CFTs in the vacuum state. The ambient space will be treated in a similar way: although we use the AdS/CFT dictionary to construct it, we expect it to allow one to solve the kinematical constraints of any CFT in that background and state, also non-holographic ones. We will explicitly see this in the example of thermal CFTs discussed in Chapter 5.

Let us now construct Weyl-covariant building blocks that can appear in correlators of a CFT on the metric background $g_{(0)}$ with a given stress tensor VEV $\langle T_{ij} \rangle$, using the corresponding ambient space we prescribed. The focus will be on scalar n-point functions as a first test of the formalism. The case of flat ambient space correlators described in

Section 2.2 will guide our steps. There, for scalar n-point functions the only available building block is X_{ij}. Since the ambient space accounts for setups with less symmetry we expect a larger number of allowed invariants than in the embedding space. After assembling these Weyl-covariant building blocks into correlators on the ambient space, the CFT correlators are obtained by taking the projection onto a section of the nullcone. We assume the section to be at $t = 1$; through a Weyl transformation it is easy to move to a conformally-related section. In Section 4.5 we describe how to generalise this discussion to spinning correlators.

4.1 The ingredients

We have to identify the $(d + 2)$-dimensional objects that are suitable ingredients for constructing Weyl-covariant building blocks.The objects at hand are the ambient space metric and covariant derivative. Whilst in some sense these objects survive in the embedding space limit as the Minkowski metric and partial derivative, the ambient Riemann tensor $\widetilde{R}_{ABCD}$ does not. Thus, $\widetilde{R}_{ABCD}$ and its ambient covariant derivatives form natural ingredients that embody departures from embedding space results.

For correlation functions the other important ingredient is the homothetic vector, T. As discussed in Section 2.2, T provides the ambient space generalisation of the embedding space insertion points X_i for correlation functions. For n-point functions we have multiple distinct insertion points and need to parallel transport all relevant quantities to the same point, so that everything lives in the same tangent space and contractions can be made. Typically the geodesics along which we transport leave the ambient nullcone and explore the bulk of ambient space. This means that transported quantities get affected by the non-trivial $(d+1)$-dimensional ALAdS geometry. The ambient curvature itself contains information about the state. Explicitly,

$$\text{even } d: \qquad (\widetilde{\nabla}_\rho)^{\frac{d}{2}-2}\widetilde{R}_{\rho ij\rho} = \frac{t^2}{2}\left(\frac{d}{2}\right)!\; g_{(d)ij} + F[g_{(0)}] + O(\rho)\,, \tag{4.1}$$

$$\text{odd } d: \qquad (\widetilde{\nabla}_\rho)^{\frac{d+1}{2}-2}\widetilde{R}_{\rho ij\rho} = \frac{t^2}{2\sqrt{\pi}}\,\Gamma\left(\frac{d}{2}+1\right)\frac{g_{(d)ij}}{\sqrt{\rho}} + O(\rho^0)\,, \tag{4.2}$$

where $F[g_{(0)}]$ is a local functional of $g_{(0)}$, while $g_{(d)ij}$ is related to the state in the prescription outlined above. This is one of the main ways the CFT state enters the building blocks that we are constructing.

Note that in more general settings where other operators take non-vanishing VEVs these must be added to the legitimate ingredients. If residual conformal Killing vectors are present, the corresponding ambient isometries and their parallel transport may also enter the list of ingredients necessary to construct a complete set of invariants.

Finally, we close this subsection by listing some useful properties of the ambient Riemann tensor, $\widetilde{R}_{ABCD}$. The Weyl, Cotton and Bach tensors can be obtained as the restriction of the ambient curvature to the d-dimensional background [23][1],

$$\widetilde{R}_{ijkl}\Big|_{\rho=0,t=1} = W_{ijkl}\,, \qquad \widetilde{R}_{\rho jkl}\Big|_{\rho=0,t=1} = C_{jkl}\,, \qquad \widetilde{R}_{\rho jk\rho}\Big|_{\rho=0,t=1} = -\frac{B_{ij}}{d-4}\,. \tag{4.3}$$

Working perturbatively at small ρ one can obtain expressions in closed form for the components of the ambient Riemann tensor. For conformally flat $g_{(0)}$,

$$\widetilde{R}_{\rho jk\rho} = \frac{d}{4}\left(\frac{d}{2}-1\right) g_{(d)jk}\,\rho^{\frac{d}{2}-2}t^2 + O(\rho^{\frac{d}{2}-1})\,, \tag{4.4}$$

$$\widetilde{R}_{\rho jkl} = \frac{d}{4}\left[\nabla_l g_{(d)jk} - \nabla_k g_{(d)jl}\right]\rho^{\frac{d}{2}-1}t^2 + O(\rho^{\frac{d}{2}})\,, \tag{4.5}$$

$$\widetilde{R}_{ijkl} = \frac{d}{4}\left(g_{(0)il}g_{(d)jk} + g_{(0)jk}g_{(d)il} - g_{(0)ik}g_{(d)jl} - g_{(0)jl}g_{(d)ik}\right)\rho^{\frac{d}{2}-1}t^2 + O(\rho^{\frac{d}{2}})\,, \tag{4.6}$$

while for generic $g_{(0)}$ in $d=3$ the components take the same form as above except for

$$\widetilde{R}_{\rho jkl} = \left[\nabla_l P_{jk} - \nabla_k P_{jl}\right]t^2 + \frac{d}{4}\left[\nabla_l g_{(d)jk} - \nabla_k g_{(d)jl}\right]\rho^{\frac{d}{2}-1}t^2 + O(\rho)\,. \tag{4.7}$$

where P_{ij} is the boundary Schouten tensor, (2.6). We will make use of these expressions later when studying CFTs at finite temperature and on squashed sphere backgrounds.

4.2 The building blocks

We now construct building blocks on the ambient space that can enter CFT correlators based on Weyl covariance. Following the previous discussion, using parallel transport we must combine the local quantities

$$T, \quad \tilde{g}, \quad (\widetilde{\nabla})^k\widetilde{\mathrm{Riem}} \qquad (k = 0, 1\ldots), \tag{4.8}$$

evaluated at the different insertion points.[2] We first focus on scalar invariants, turning to invariants with spin in Section 4.5.

The simplest scalar invariant whose expression we are missing on a generic ambient space is $\widetilde{X}_{ij}$, the ambient space analogue of the square-distance between insertions. We construct it just as prescribed in the flat case in Section 2.2: we parallel transport $T_i = T(\widetilde{X}_i)$ to $\widetilde{X}_j$ yielding $\hat{T}_i$, which we then contract with $T_j = T(\widetilde{X}_j)$ at $\widetilde{X}_j$. In Appendix B we discuss in detail how to find geodesics between two points lying on a section of the

[1] Here we are assuming that $d \neq 4$. If $d = 4$, as we will remark later, the stress tensor VEV contained in $g_{(d)ij}$ enters the $\widetilde{R}_{\rho jk\rho}\Big|_{\rho=0,t=1}$ components, and thus these cannot be written only in terms of the boundary metric $g^{(0)}$. As a consequence, the expression in (4.3) is no longer valid in $d = 4$.

[2] Note that gradients of T do not need to be considered since $\widetilde{\nabla}_A T_B = \tilde{g}_{AB}$.

ambient nullcone, how to perform the parallel transport and finally obtain $\hat{T}_i$. The key result is that

$$\widetilde{X}_{ij} = -2\,\hat{T}_i \cdot T_j = \ell(\widetilde{X}_i, \widetilde{X}_j)^2, \tag{4.9}$$

where $\ell(\widetilde{X}_i, \widetilde{X}_j)$ is the geodesic distance between the two insertion points on the ambient space. This generalises the result we found earlier for the flat background, (2.25). Note that it does not matter which insertion we parallel transport, the result is symmetric under $i \leftrightarrow j$.

Note that the invariant (4.9) relies on the existence of an ambient geodesic between the two insertions. It is conceivable that in some cases no such geodesic exists, in which case we lose this building block for constructing correlators. It is also possible that there is more than one geodesic, in which case there will be an enhancement in the number of available invariants to build correlators.[3] However, under mild assumptions given any two points on the ambient nullcone there is one and only one geodesic connecting them [88,89].

In addition to $\widetilde{X}_{ij}$ we can construct new bi-local scalar invariants by directly using the ambient curvature and its covariant derivatives. Assembling these ingredients one immediately discovers that not all such invariants are independent, due to a number of identities: the contractions of T with Riemann are trivial [23],

$$T_j^D \widetilde{R}_{ABCD} = 0, \tag{4.10}$$

and contractions with gradients of Riemann are redundant since,

$$T_j^D \widetilde{R}_{ABCD,M_1\ldots M_r} = -\sum_{s=1}^{r} \widetilde{R}_{ABCM_s,M_1\ldots \widehat{M}_s\ldots M_r}, \tag{4.11a}$$

$$\begin{aligned} T_j^P \widetilde{R}_{ABCD,M_1\ldots M_s P M_{s+1}\ldots M_r} = &-(s+2)\widetilde{R}_{ABCD,M_1\ldots M_r} \\ &- \sum_{t=s+1}^{r} \widetilde{R}_{ABCD,M_1\ldots M_s M_t M_{s+1}\ldots \widehat{M}_t\ldots M_r}, \end{aligned} \tag{4.11b}$$

where commas denote covariant derivatives and hatted indices are understood as removed. These properties, along with Ricci-flatness $\widetilde{R}_{AB} = 0$, reduce the number of independent scalar invariants.

Based on these observations, in what follows we restrict our attention to the following set of scalar invariants constructed at X_j, the *weighted curvature invariants*:

$$W_{ij}^{(k,n)} \sim \mathrm{contr}\left[\hat{T}_i \otimes \ldots \otimes (\widetilde{\nabla})^{r_1}\mathrm{Riem} \otimes \ldots \otimes (\widetilde{\nabla})^{r_k}\mathrm{Riem}\right], \tag{4.12}$$

[3]This latter possibility occurs for states described by thermal AdS spaces, where there is an infinite number of geodesics connecting any two nullcone points, enumerated by the number of windings around the thermal circle. Indeed, a sum over such contributions is required to reproduce the corresponding thermal correlator, as we show later in Section 5.5.3.

where contr indicates the full contraction of all the indices using the ambient metric at X_j. They are diffeomorphism invariants in $d+2$ dimensions, while displaying a precise weight under Weyl transformations (hence being Weyl covariant quantities). Since $\hat{T}_i$ are obtained by parallel transport, one can build a distinct set of invariants for each corresponding geodesic.

We have labelled the $W_{ij}^{(k,n)}$ by the number of Riemann's they contain, k. This is a good label once we fix some redundancies. The first redundancy is associated to use of the identity $\widetilde{\nabla}^D \widetilde{R}_{ABCD} = 0$ which follows from the second Bianchi identity and Ricci-flatness. Because of this, we require that none of the covariant derivatives within each factor $(\widetilde{\nabla})^r\mathrm{Riem}$ in (4.12) are contracted with the Riemann tensor itself, regardless of the ordering. This is because by repeated commutation of the covariant derivatives one can eventually reach a form where $\widetilde{\nabla}^D \widetilde{R}_{ABCD} = 0$ can be applied to one term; all remaining terms then take the form of other terms appearing in (4.12) with higher k. The second redundancy is the remaining ordering ambiguity of the covariant derivatives within each factor $(\widetilde{\nabla})^r\mathrm{Riem}$, which we fix by symmetrisation, as a matter of convention. The remaining label n enumerates all possible invariants with that k.

As a note of caution, the weighted curvature invariants (4.12) do not necessarily include all possible invariants. For example, we have not considered covariant derivatives of $\hat{T}_i$ at $\widetilde{X}_j$, nor do we consider parallel transport of the ambient curvature and its covariant derivatives from $\widetilde{X}_i$ to $\widetilde{X}_j$. In what follows we assume that (4.12) constitute a basis without including such contributions. Evidence in support of these assumptions is brought by the results presented in the explicit examples in Chapters 5 and 6 where we show that the invariants of the form (4.12) under these assumptions constitute a basis.

Let us discuss invariants with $k = 0, 1, 2$. There are no nontrivial $k = 0$ weighted curvature invariants, since without Riemanns in (4.12) there are only contractions of $\hat{T}_i$ which are zero; there is just the identity. There are also no $k = 1$ weighted curvature invariants, and a proof of this result proceeds as follows. At most two of the indices of Riemann can be contracted with $\hat{T}_i$ due to the antisymmetry of Riemann indices. Thus at least two of the four indices of the ambient Riemann are to be contracted with either the inverse metric or covariant derivatives. Any contraction with an inverse metric yields zero by Ricci-flatness. Any contraction with covariant derivatives is a term that is not a member of the $k = 1$ set of invariants, according to the definition given above. Later, in the examples discussed in Chapters 5 and 6 we provide explicit examples of $k = 2$ building blocks, which play an important role in constructing ambient 2-point functions.

As explained in Section 2.4 the engineering dimension, Δ, of an ambient scalar is minus its overall weight in t. It can be easily computed with the same rules used in the embedding space[4], by viewing T^M and $\widetilde{\nabla}_M$ as dimension -1 and 1 quantities re-

[4]For ease of comparison with the mathematical literature, we observe this is not the perspective typically adopted in conformal geometry. There the metric $\tilde{g}$ and the Riemann (meant as tensors) both have dimensions –2 following from their homogeneity in t, while T and the ambient derivative $\widetilde{\nabla}_M$ have

spectively. The Riemann tensor contains two derivatives of the metric and we conclude it has dimension 2. For weighted curvature invariant (4.12) with k Riemann tensors, r covariant derivatives and ℓ $\hat{T}_i$ vectors,

$$\Delta = 2k + r - \ell. \tag{4.13}$$

Note that $r+\ell$ must be even in order to be able to build a scalar with an integral number of inverse metrics. From (4.13) this entails that all such invariants have even dimensions.

If an invariant of the form (4.12) has $\Delta \neq 0$ we can easily construct a $\Delta = 0$ invariant from it by multiplying by an appropriate power of $\widetilde{X}_{ij}$. However, a useful class of $\Delta = 0$ invariants are those of the form (4.12) with $2k + r = \ell$. Due to the symmetries of the Riemann tensor their structure is completely fixed and one can list them in full generality. If we define the partial contraction

$$\mathcal{R}^{(\hat{r})}_{AC} = \hat{T}_i^{M_1} \dots \hat{T}_i^{M_{\hat{r}}}\, \hat{T}_i^U \hat{T}_i^V\, \widetilde{\nabla}_{M_1} \dots \widetilde{\nabla}_{M_{\hat{r}}} \widetilde{R}_{AUCV}\,, \tag{4.14}$$

any $\Delta = 0$ curvature scalar constructed out of k Riemann's and r derivatives can be written as a linear combination of chains of the form

$$\mathcal{R}^{(r_1)\ M_2}_{\ \ \ \ M_1}\, \mathcal{R}^{(r_2)\ M_3}_{\ \ \ \ M_2} \dots \mathcal{R}^{(r_k)\ M_1}_{\ \ \ \ M_k}, \tag{4.15}$$

where each such chain is constrained to have $\sum_i r_i = r$. We will utilise invariants from this class in Sections 5 and 6.

A caveat to be aware of concerns the limit of expressions of the form (4.12) to a section of the nullcone $\rho = 0$, $t = 1$. In particular this involves the behaviour of the ambient Riemann tensor when approaching the nullcone, some examples of which are given in (4.1)-(4.2) and (4.4)-(4.7). From the metric expansion one can show that in even d only non-negative integer powers of ρ appear in components of the ambient Riemann tensor, while in odd d fractional powers of ρ appear when a non-vanishing $g_{(d)}$ is present. For odd d the RHS of equation (4.2) diverges for $\rho \to 0$. By taking more derivatives such divergences become stronger. For the purpose of constructing ambient invariants this means that scalars constructed using curvature terms $(\widetilde{\nabla})^r \widetilde{R}$ with high enough r may be singular in odd d when restricted to the boundary. Such terms must either be discarded, or combined into linear combinations to cancel such infinities. Despite these apparent complications for odd d, we were able to find a complete basis of curvature invariants for the $d = 3$ example of a CFT on a squashed 3-sphere in Chapter 6.

Analogously to the embedding formalism, correlators in $d + 2$ dimensions must be invariant under the (near-nullcone) isometries encoding d-dimensional conformal symmetries. Geodesics and geodesic transport preserve the symmetries of the geometry

dimension zero. Their components have of course different weight, and this is what one considers in the embedding formalism instead. For example, the vector $T = t\partial_t$ has weight zero in t, while its components $T^M = t\delta_0^M$ clearly have dimension -1. In practice either perspectives lead to the same answer (4.13), hence in the main discussion we stick to the component-based picture, which is rather unnatural from the perspective of conformal geometry but very common in the QFT literature.

and hence ambient building blocks constructed out of the ambient metric and covariant derivatives of the curvature automatically satisfy the constraints imposed by the near-nullcone symmetries. One can explicitly see this for instance in the invariants constructed in Section 5.3 in the case of thermal CFTs – they are invariant under the residual symmetries of the CFT. To conclude, the prescription for the ambient building blocks discussed here automatically implements the residual conformal Ward Identities, leaving Weyl covariance as the only non-trivial kinematic constraint to be imposed.

4.3 Scalar 2-point functions

In the previous sections we constructed a class of ambient invariants – namely, $\widetilde{X}_{ij}$ (4.9) and $W_{ij}^{(k,n)}$ (4.12) – that enter CFT correlators on general backgrounds and states based on Weyl covariance. We now propose a general form of ambient scalar 2-point functions that arranges those invariants so as to exhibit the required properties,

$$\langle O(\widetilde{X}_1)O(\widetilde{X}_2)\rangle = \frac{C_\Delta}{(\widetilde{X}_{12})^\Delta}\lim_{\substack{\rho\to 0\\ t\to 1}}\left[1+\sum_{k=2}^{\infty}\mathcal{I}_2^{(k)}\right], \tag{4.16}$$

where

$$\mathcal{I}_2^{(k)} = \sum_n c_n \widetilde{X}_{12}^{\Delta_n/2} W_{12}^{(k,n)}, \tag{4.17}$$

and Δ_n denotes the dimension of $W_{ij}^{(k,n)}$ given by (4.13). The constant coefficients c_n are determined by the dynamics of the CFT. The sum over k in (4.16) starts from terms of order $O(\widetilde{\text{Riem}})^2$ since in Subsection 4.2 we proved that $\mathcal{I}_2^{(1)}=0$, while $\mathcal{I}_2^{(0)}$ is just the identity, already accounted for as the first term in (4.16). The overall scaling dimension is -2Δ, as required by Weyl covariance. The correlator is analytic in curvatures and continuously connected to the flat space limit in which $\widetilde{X}_{12}\to X_{12}$ and $\mathcal{I}_2^{(k)}\to 0$, where we recover the embedding space 2-point function (2.26) with the same constant C_Δ.

As discussed in Section 4.2 there may be more than one geodesic path connecting the two insertion points. Parallel transporting along each of them can generate independent invariants and thus an implicit sum over all the ambient geodesics connecting $\widetilde{X}_1$ and $\widetilde{X}_2$ is understood in the RHS of (4.16).

Let us now discuss which states we expect to be able to describe using (4.16). For a CFT in any background $g_{(0)}$ and state, at short distances the background becomes approximately flat and as such we should have a convergent OPE of the form (1.14). We can use it to reduce a 2-point function of a scalar operator O of scaling dimension Δ to a sum of 1-point functions of exchanged operators,

$$\langle O(x_1)O(x_2)\rangle \simeq \frac{1}{|x_{12}|^{2\Delta}}\sum_{\phi\in O\times O} h_\phi(x_i,\partial_i)\,\langle\phi(x_2)\rangle\ , \tag{4.18}$$

where the $\simeq$ is understood as an equality modulo contact terms. Since $\widetilde{\mathrm{Riem}} \sim \langle T \rangle$ (see (4.1) and (4.2)) schematically we have that $\mathcal{I}_2^{(k)} \sim \langle T \rangle^k$. Therefore we expect (4.16) to account for the multi-stress tensor contributions in (4.18), at least for large-N theories where multi-stress tensor 1-point functions factorise. However, we conjecture that our curvature invariants provide a basis for multi-stress tensor contributions also for theories which are not at large-N. Operators other than the multi-stress tensors contributing to the RHS of (4.18) must be captured using other classes of ambient invariants, and we comment on this issue in Section 4.5. We stress that the multi-stress tensors are universal contributions in any CFT correlator, and this is what the ambient geometry captures through (4.16).

For holographic CFTs the ambient 2-point function (4.16) has an additional interpretation, providing multi-stress tensor corrections to the well-known geodesic approximation of 2-point functions in the context of AdS/CFT [90, 91]. In Appendix B we discuss how the presence of the homothetic vector T on the ambient space fully fixes the component of a particle trajectory along that direction. As we show in Appendix C if we focus on geodesics connecting points on the ambient nullcone, the remaining $d+1$ equations for the unknown components of the geodesic path turn out to be the geodesic equations on the ALAdS$_{d+1}$ section associated to that ambient space in a non-affine parametrisation. In this picture, the endpoints of the geodesic are boundary points on the conformal compactification of ALAdS$_{d+1}$. In Appendix C we further prove that the square-geodesic distance on the ambient space $\widetilde{X}_{12}$ is related to the (renormalised) geodesic distance on the associated ALAdS$_{d+1}$ space. Through (4.9) we can write their relation as

$$\frac{1}{(\widetilde{X}_{12})^{\Delta}} = r^{-2\Delta} e^{-\Delta L_{AdS}}\Big|_{r=0} \tag{4.19}$$

for an arbitrary real Δ, where r is the Fefferman-Graham coordinate on the ALAdS space as in the metric (2.7). Here L_{AdS} indicates the (divergent) length of the corresponding geodesic on the ALAdS$_{d+1}$ section. The RHS of (4.19) coincides with the geodesic approximation for a scalar 2-point function of an operator of dimension Δ in the context of AdS/CFT. It can be argued to follow from the saddle-point approximation of the first-quantised path integral for a massive particle and consequently its validity is restricted to the large-Δ regime. We can thus interpret the ambient curvature invariants in (4.16) as encoding the quantum corrections from the multi-stress tensor contributions at finite Δ beyond the semi-classical approximation provided by $(\widetilde{X}_{12})^{-\Delta}$.

Given that $\mathcal{I}^{(1)} = 0$, the coefficient in the ambient expansion at order $O(\widetilde{\mathrm{Riem}})$ predicted by the geodesic approximation is exact. Therefore a universal prediction from the ambient formalism is that correlators in the geodesic approximation are exact up to order $O(\widetilde{\mathrm{Riem}})^2$ corrections if no other operator with scaling dimension $\Delta < 2d$ acquires a VEV.

4.4 Scalar higher-point functions

Similar expressions to (4.16) can be written for scalar higher-point functions. In the ambient formalism scalar 3-point functions read

$$\langle O_1 O_2 O_3\rangle = \frac{C_{123}}{(\widetilde{X}_{12})^{\alpha_{123}}(\widetilde{X}_{13})^{\alpha_{132}}(\widetilde{X}_{23})^{\alpha_{231}}} \lim_{\substack{\rho\to 0\\ t\to 1}} \left[1+\sum_{k=2}^{\infty}\mathcal{I}_3^{(k)}\right], \tag{4.20}$$

where the α_{ijk} coefficients are the same as those defined in (1.23) to ensure the correct scaling properties. To recover the expression on the embedding space in the flat limit, C_{123} must be the same as in (1.23). Here $\mathcal{I}_3^{(k)}$ denote linear combinations of weight-0 curvature invariants containing k ambient Riemanns and constructed with the pairwise parallel transport of tensors from the three insertions $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3$. The fact that bi-local invariants provide a basis for 3- and higher-point functions (with no need to resort to n-local invariants) is justified by the following remarks. First, this is what happens in embedding space correlators with arbitrary spin and with an arbitrary number of insertions. Second and more fundamental, as stressed above the OPE is expected to converge at short enough distances in general backgrounds and states, and OPE contractions are pairwise.

The linear combinations $\mathcal{I}_3^{(k)}$ are thus products of bi-local invariants from the three insertion points and as such they can be decomposed in terms of the 2-point linear combinations $\mathcal{I}_2^{(m)}$ with generic coefficients. From $\mathcal{I}_2^{(1)} = 0$ it follows that $\mathcal{I}_3^{(1)} = 0$; furthermore one can check explicitly that

$$\mathcal{I}_3^{(2)}(\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3) = P_3^{(2)}(\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3)\,, \tag{4.21}$$

$$\mathcal{I}_3^{(3)}(\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3) = P_3^{(3)}(\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3)\,, \tag{4.22}$$

where we defined

$$P_3^{(k)}(\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3) = \mathcal{I}_2^{(k)}(\widetilde{X}_1,\widetilde{X}_2)+\mathcal{I}_2^{(k)}(\widetilde{X}_1,\widetilde{X}_3)+\mathcal{I}_2^{(k)}(\widetilde{X}_2,\widetilde{X}_3), \tag{4.23}$$

and where each $\mathcal{I}_2^{(k)}$ is thought of as containing generic different constant coefficients. Turning to fourth order invariants, the most general linear combination is of the form

$$\mathcal{I}_3^{(4)}(123) = P_3^{(4)}(123)+\mathcal{I}_2^{(2)}(12)\,\mathcal{I}_2^{(2)}(13)+\mathcal{I}_2^{(2)}(12)\,\mathcal{I}_2^{(2)}(23)+\mathcal{I}_2^{(2)}(13)\,\mathcal{I}_2^{(2)}(23)\,, \tag{4.24}$$

where we adopted a convention where one denotes X_ℓ by ℓ. The last three terms can be rewritten as $[P_3^{(2)}(123)]^2$. The latter also includes $[\mathcal{I}_2^{(2)}(12)]^2$, $[\mathcal{I}_2^{(2)}(13)]^2$ and $[\mathcal{I}_2^{(2)}(23)]^2$. Note that these three terms are already contained in $P_3^{(4)}(123)$ and their appearance in $[P_3^{(2)}(123)]^2$ can be reabsorbed by shifting the corresponding arbitrary coefficients in $P_3^{(4)}(123)$. All in all we can rewrite the linear combination of fourth order 3-point invariants as

$$\mathcal{I}_3^{(4)}(123) = P_3^{(4)}(123)+\Big(P_3^{(2)}(123)\Big)^2\,. \tag{4.25}$$

Studying higher orders one finds the following recursive relation between order k and order $k-2$ invariants,

$$\mathcal{I}_3^{(k)}(123) = P_3^{(k)}(123) + P_3^{(2)}(123) P_3^{(k-2)}(123)\,. \tag{4.26}$$

Using these recursion relations one finds the expression for the general linear combination of curvature invariants of order k in terms of the bi-locals $\mathcal{I}_2^{(k)}(\widetilde{X}_i, \widetilde{X}_j)$ involved in 2-point functions,

$$k \text{ even:} \qquad \mathcal{I}_3^{(k)}(\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3) = \sum_{\ell=1}^{k/2} \left(P_3^{(2)}\right)^{\frac{k}{2}-\ell} P_3^{(2\ell)}\,, \tag{4.27}$$

$$k \text{ odd:} \qquad \mathcal{I}_3^{(k)}(\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3) = \sum_{\ell=3/2}^{k/2} \left(P_3^{(2)}\right)^{\frac{k}{2}-\ell} P_3^{(2\ell)}\,, \tag{4.28}$$

where the sum in the odd case is over half-odd ℓ. These expressions are manifestly symmetric (modulo the different coefficients in the linear combinations) under permutations of the insertion points $\widetilde{X}_i$ and are of the appropriate order in the ambient Riemann. The full 3-point function is not invariant under permutations of the three insertion points for different scaling dimensions Δ_i. However this different behaviour under Weyl transformations is accounted for by the overall factor in front of (4.20).

Let us now turn to scalar n-point functions. Based on our assumptions and on a consistent reduction to (1.24) in the flat limit, their form on the ambient space is fixed to

$$\langle O_1(\widetilde{X}_1) \dots O_n(\widetilde{X}_n)\rangle = \left(\prod_{i<j} \widetilde{X}_{ij}^{\alpha_{ij}}\right) \lim_{\substack{\rho\to 0\\ t\to 1}} \left[f\,(u) + \sum_{k=2}^{\infty} \mathcal{I}_n^{(k)} \right], \tag{4.29}$$

where the cross-ratios u are now in terms of the ambient geodesic distances

$$u_{[pqrs]} = \frac{\widetilde{X}_{pr}\widetilde{X}_{qs}}{\widetilde{X}_{pq}\widetilde{X}_{rs}}\,, \tag{4.30}$$

and f is the same function of the cross-ratios present in the corresponding correlator for the same CFT in vacuum on flat space.

Using combinatorial arguments similar to those for 3-point functions one can straightforwardly generalise (4.27)-(4.28) to any n,

$$k \text{ even:} \qquad \mathcal{I}_n^{(k)}(\widetilde{X}_1 \dots \widetilde{X}_n) = \sum_{\ell=1}^{k/2} \left(P_n^{(2)}\right)^{\frac{k}{2}-\ell} P_n^{(2\ell)}\,, \tag{4.31}$$

$$k \text{ odd:} \qquad \mathcal{I}_n^{(k)}(\widetilde{X}_1 \dots \widetilde{X}_n) = \sum_{\ell=3/2}^{k/2} \left(P_n^{(2)}\right)^{\frac{k}{2}-\ell} P_n^{(2\ell)}\,. \tag{4.32}$$

Here we defined

$$P_n^{(k)}(\widetilde{X}_1 \ldots \widetilde{X}_n) = \sum_{(Y,Z)\in C_2^n(\widetilde{X}_1 \ldots \widetilde{X}_n)} \mathcal{I}_2^{(k)}(Y,Z)\,, \tag{4.33}$$

where the sum is over the $\binom{n}{2}$ pairwise combinations (Y,Z) of the points $(\widetilde{X}_1 \ldots \widetilde{X}_n)$. This definition reduces to (4.23) for $n=3$.

These combinatorial relations show that knowing the ambient curvature invariants that enter the scalar 2-point function up to a certain order k allows one to straightforwardly write the form of generic ambient scalar n-point functions to the same order k. In particular, $\mathcal{I}_2^{(1)} = 0$ implies $\mathcal{I}_n^{(1)} = 0$ for any n. This entails that the universal validity of the geodesic approximation up to $O(\widetilde{\mathrm{Riem}}^2)$ corrections extends to any scalar n-point function in any CFT on generic backgrounds and states, as long as no operator with scaling dimension lower or equal than d has a non-vanishing VEV. Here by geodesic approximation of a generic n-point function we mean the first term $\left(\prod \widetilde{X}_{ij}^{\alpha_{ij}}\right) f(u)$ in (4.29).

4.5 Open directions: correlators with spin and more general states

We would like now to provide some comments on how to generalise the scalar correlators discussed above to those with spin. As before, the embedding space correlators are generalised to ambient correlators by adopting the homothetic vector T^M in lieu of the position on Minkowski X^M, and the ambient metric $\tilde{g}_{MN}$ instead of η_{MN}. On top of this, one considers generically infinite sum of ambient curvature invariants which vanish in the flat limit.

The building blocks to be used in this case must have free indices on the ambient space, meaning that curvature invariants will be of the same form as (4.12), where contractions are understood as partial contractions so as to end up with the appropriate spin. Such free ambient indices transform as generic ambient tensors (i.e. weighted tractor tensors when restricted to the nullcone) under Weyl transformations through the matrix ${U^M}_N(\Omega)$. We thus conjecture that a set of ambient curvature invariants with such partial contractions form a basis for multi-stress tensor contributions to n-point functions of general spin.

As is customary in the embedding space [20], for practical purposes it is convenient to reduce the problem of classifying ambient spinning structures to finding scalar structures by considering ambient polarisation vectors $Z_{(i)}^M$, one at each insertion point, and treating them as additional local ingredients to be used to construct scalar bi-local invariants besides (4.8). The number of $Z_{(i)}$'s that such invariants must contain is fixed by the spin of the inserted operators. To retrieve the spinning expression on the ambient space it is sufficient to use appropriate differential operators acting on the $Z_{(i)}$'s.

However there exists a possible alternative path for spinning ambient building blocks. The relationship described in Section 2.4 between tractor and ambient connections allows one to generalise the so-called weight- and spin-shifting operators on the embedding space introduced in [57] to the ambient space. These differential operators act on tensor structures modifying their scaling dimension and spin, and by leveraging the many results of tractor calculus one is able to generalise them to the ambient space.

More precisely, given such operators on the embedding space it is sufficient to perform the map

$$X^M \to T^M, \qquad \partial_M \to \widetilde{\nabla}_M, \qquad \eta_{MN} \to \tilde{g}_{MN}\,, \tag{4.34}$$

giving local weight- and spin-shifting operators on the ambient space. Operators obtained in this way satisfy all the required properties of a weight- or spin-shifting operator as put forward in the flat space case [57]. Note that to use these operators requires pairwise contractions to obtain bi-local differential operators acting on two distinct insertions, and on the ambient space this involves parallel transport of differential operators.

We should point out that the use of these differential operators on the embedding space is subject to issues regarding the completeness of the resulting tensor structures. The same issues will also arise for their ambient space counterparts. Nevertheless, ambient weight- and spin-shifting operators surely appears as an interesting way to study correlators with spin. We defer more detailed discussions of these issues and generalisations to future work.

As mentioned at the beginning of this chapter, according to our prescription it would be necessary to couple the ambient metric to matter fields in $(d+2)$ dimensions in case of more general states where operators other than the multi-stress tensors acquire a non-vanishing VEV. In such setups the ambient equations take the form of Einstein's equations sourced by matter fields. The scaling dimensions of the corresponding CFT operators specify their required weight in t similarly to the probe ambient scalar example presented in Section 2.5. The homothetic symmetry T as well as the ambient nullcone structure are thus preserved. This ensures that Weyl covariance is still canonically encoded in the ambient geometry, although no longer Ricci-flat. In this case the ALAdS sections encode the matter sources and VEVs that define the CFT state as per AdS/CFT. The matter fields in $(d+2)$ dimensions become additional ambient ingredients that must be used to construct invariants. The framework we discussed in this chapter is thus conceptually unmodified for more general states, and it would be interesting to explicitly compute ambient correlators for such cases.

Chapter 5

Finite temperature CFTs

In this section we apply the ambient space formalism to finite temperature CFTs, and our goal is to show that the the ambient 2-point function (4.16) accounts for the multi-stress tensor contributions as conjectured in the previous chapter. To this aim, after a review of the properties of Euclidean CFTs at finite temperature in Section 5.1, we set up the ambient space suitable for this problem, we perturbatively solve the ambient geodesics and construct the appropriate ambient curvature invariants in Sections 5.2 and 5.3. We then construct the ambient scalar 2-point function and we indeed confirm that it captures the multi-stress tensor blocks in the OPE limit in Section 5.4. As a corollary, this also confirms that the ambient invariants represent a kinematic basis for generic CFTs, and not only for holographic ones.

To test the ambient predictions beyond the OPE regime, we also perform novel holographic computations for the scalar 2-point function on the planar AdS black hole. In Subsection 5.5.1 we describe a perturbative computation at large inverse temperature yielding the 2-point function to first order for generic d and Δ and to arbitrarily high order for odd $d+2\Delta$, and they confirm that the ambient invariants account for the multi-stress tensor contributions. To test this statement in a fully non-perturbative regime, in Subsection 5.5.2 we present a numerical holographic computation on the same AdS black hole bulk. Its results signal the appearance of different operators contributing to the 2-point function, the so-called double-twist spectrum. For generic Δ these operators do not mix with the multi-stress tensors, supporting the conclusions of the ambient formalism at the non-perturbative level.

In Subsection 5.5.3 we use the insights from these holographic computations to provide new understanding on the non-perturbative nature of double-twist operators and on the analytic structure of thermal scalar 2-point functions. First, we show how the double-twist spectrum is produced by summing the multi-stress tensor blocks over thermal images for a large class of correlators. In particular, we provide explicit expressions for the double-twist OPE coefficients as functions of the multi-stress tensor OPE coefficients. From the asymptotic behaviour at large order of the perturbative holographic

2-point function for $d = 4$ and $\Delta = 3/2$ we are also able to extract the radius of convergence of the thermal OPE in this case, and its value is less than β as originally conjectured in [25]. We also discuss possible ways to extend the ambient space formalism to capture such double-twist contributions in correlators. We conclude the chapter with the simpler case of $d = 2$ thermal CFTs, where the ambient space is locally Riemann-flat, all curvature invariants vanish and the ambient formalism accounts for the full 2-point function.

5.1 Thermal CFTs and the ambient setup

Euclidean thermal CFTs in d dimensions on flat space live on the thermal cylinder $S^1_\beta \times \mathbb{R}^{d-1}$ where β is the inverse temperature. We parametrise this background with coordinates $x^i = (\tau, x^a)$ where $0 \leq \tau < \beta$. This geometry breaks conformal invariance because of the length scale β; the only global symmetries remaining are translations along the τ and x^a directions, as well as rotations on $\mathbb{R}^{d-1}$. We restrict our analysis to states which respect these spacetime symmetries and do not spontaneously break them further.

These residual symmetries fully constrain 1-point functions. By translational symmetry they are non-vanishing only for primary operators, and rotational symmetry implies they must be constant tensors of the form

$$\langle O^{i_1 \dots i_J} \rangle^{(\beta)}_\Delta = \frac{b_O}{\beta^\Delta} \left(e^{i_1} \dots e^{i_J} - \text{traces} \right) , \tag{5.1}$$

where e^i is the unit vector along τ [25]. In particular, for the stress tensor VEV we have

$$\langle T_{ij} \rangle^{(\beta)} = \frac{c_{(d)}}{\beta^d} \, \text{diag} \left(1 - d, 1, \dots, 1 \right) , \tag{5.2}$$

which is traceless, as expected on the thermal cylinder. From now on (unless stated otherwise) we restrict our attention to $d = 4$; we will comment later on the generalisation to any d. Given the state specified by (5.2), the ambient space to be used has the AdS planar black hole as ALAdS$_5$ slices. The six-dimensional ambient geometry relevant for this problem then reads

$$\tilde{g} = -ds^2 + \frac{s^2}{z^2} \left[\frac{dz^2}{1 - \frac{z^4}{z_H^4}} + \left(1 - \frac{z^4}{z_H^4} \right) d\tau^2 + \delta_{ab} dx^a dx^b \right] , \tag{5.3}$$

with $a, b = 1, 2, 3$, horizon scale $z_H = \pi/\beta$ and compact time direction $0 \leq \tau < \beta$. This choice of AdS bulk metric corresponds to $c_{(4)} = 2\pi^4$ in (5.2); it is straightforward to rescale the temperature to attain a generic $c_{(4)}$. Finally note that (5.3) is not in the usual Fefferman-Graham ambient gauge (2.8), which can be reached with the transformation

$$z = \frac{r}{\sqrt{1 + \frac{r^4}{4z_H^4}}}\,. \tag{5.4}$$

Our aim is to find the expression for scalar 2-point functions in such a thermal CFT using the ambient space formalism. This translates into finding the ambient building blocks that account for the multi-stress tensor contributions. Following the prescription in (4.16) we set up the problem so as to identify these invariants order by order in the ambient Riemann. In this specific case, since β is the only scale in the CFT, we have

$$\widetilde{\text{Riem}} \sim \beta^{-4}. \tag{5.5}$$

Thus the Riemann expansion in (4.16) can be viewed either as an expansion in small temperature, or as an expansion in small distance between insertions. The former allows us to use β power-counting to organise the number of Riemann tensors in (4.16). The latter allows us to make contact with the thermal OPE, presented in Section 5.3.

As a first step we find the relevant ambient invariants up to second order in the Riemann tensor.

5.2 Ambient geodesics and geodesic transport

To implement the ambient formalism, we must identify the geodesics between the two insertion points on the ambient nullcone and compute the corresponding geodesic distance. As we showed in Section 4.2 this yields the invariant $\widetilde{X}_{12}$. Adopting the ambient parametrisation $\widetilde{X}^M = (t, z, \tau, x^a)$ and using the residual rotational and translational symmetries of the problem, we can move the two insertions to lie at $\widetilde{X}_1 = (t_1, 0, 0, 0, 0, 0)$ and $\widetilde{X}_2 = (t_2, 0, \tau_f, x_f, 0, 0)$.

The strategy to solve the geodesic equations is the following. Because of the presence of the homothetic vector $T = s\partial_s$, the expression for the trajectory along $s = r\,t$ is automatically fixed up to an integration constant (the square geodesic length C) as from equation (B.10). The second order geodesic equation for s then becomes a first order equation involving z, τ and x_1 and their derivatives. One can get rid of $\dot{\tau}$ and $\dot{x_1}$ using the equations for the integrals of motion related to translations along τ and x_1,

$$\dot{\tau} = \frac{A_1 z^2}{\lambda(1-\lambda)\left(1 - \frac{z^4}{z_H^4}\right)}, \qquad \dot{x_1} = \frac{A_2 z^2}{\lambda(1-\lambda)}, \tag{5.6}$$

with A_1, A_2 constants of motion. The geodesic equation for s thus becomes a non-linear first order equation in z only,

$$4\lambda^2(1-\lambda)^2\dot{z}^2 - \frac{4A_2^2 z^8}{z_H^4} + \frac{z^6}{z_H^4} + 4\left(A_1^2 + A_2^2\right) z^4 - z^2 = 0. \tag{5.7}$$

The three equations (5.6) and (5.7) are the only independent equations left.

We are interested in computing ambient correlators, which are expressed as expansions in terms of the ambient Riemann. Given (5.5), it is sufficient to solve the geodesic equations perturbatively, considering the distance between the insertions as small compared to the inverse temperature. Denoting the distance between the insertions on the thermal cylinder by $|x|=\sqrt{\tau_f^2+x_f^2}$, this corresponds to the regime $|x|/\beta \ll 1$. We solve the equations by expanding the trajectory z, τ, x_1 and the integration constants C, A_1, A_2 as,

$$z(\lambda)=\sum_{k=0}^{\infty}\frac{z^{(k)}(\lambda)}{z_H^{4k}}, \qquad A_i=\sum_{k=0}^{\infty}\frac{A_i^{(k)}}{z_H^{4k}}, \tag{5.8}$$

and analogously for τ, x_1 and C. This is a consistent expansion since in this perturbative scheme we intend to capture the corrections to geodesics on $(d+2)$-dimensional Minkowski provided by the non-trivial geometry on the ALAdS slices, where only powers of z_H^4 appear. We start by solving equation (5.7) in $z(\lambda)$ order by order. By subsequently feeding the $z^{(k)}(\lambda)$'s into (5.6) one finds the coefficients in the expansion of τ and x_1.

At each perturbative order, the solution just obtained contains six integration constants, that is $A_1^{(k)}, A_2^{(k)}, C^{(k)}$ as well as the three following from the integration of the first order equations (5.6)-(5.7). These can be fixed order by order imposing the boundary conditions[1]

$$\tau(0)=0\,, \qquad \tau(1)=\tau_f\,, \tag{5.9a}$$

$$x_1(0)=0\,, \qquad x_1(1)=x_f\,, \tag{5.9b}$$

$$\lim_{\lambda\to 0}\frac{s(\lambda)}{z(\lambda)}=t_1\,, \qquad \lim_{\lambda\to 1}\frac{s(\lambda)}{z(\lambda)}=t_2\,. \tag{5.9c}$$

The leading order of both the trajectory and the integration constants coincide with the corresponding Minkowski expressions shown in Subsection 2.2. Following this integration scheme and renaming $\tau_f \to \tau$ and $x_f \to x$, to second order in the perturbative parameter the invariant $\widetilde{X}_{12}$ reads,

$$\widetilde{X}_{12}=t_1t_2|x|^2\left[1+\frac{|x|^2(x^2-3\tau^2)}{120z_H^4}-\frac{|x|^4(91\tau^4-98\tau^2x^2+19x^4)}{201600z_H^8}+O\left(\frac{|x|^{12}}{z_H^{12}}\right)\right]. \tag{5.10}$$

One can straightforwardly proceed to arbitrarily high order. Through the relation between the ambient and AdS geodesic lengths (4.19) this result matches the geodesic distance on the AdS planar black hole found in [92, 93].

[1] Note that z differs from the Fefferman-Graham coordinate r by $O\left(z_H^{-4}\right)$ corrections. However, close to the boundary $\lambda \to 0, 1$ the behaviour in λ of $z(\lambda)$ and $r(\lambda)$ is the same and this ensures we can use (5.9c) as boundary conditions.

5.3 The ambient 2-point function

After finding the geodesic trajectories and $\widetilde{X}_{12}$ we turn to the curvature invariants. As a first step we are interested in writing the ambient 2-point function (4.16) up to second order in the ambient Riemann. The homothetic vector T can be parallel transported along the perturbative geodesics we are considering order by order in z_H^{-4} taking the form

$$\hat{T} = \hat{T}^{(0)} + \sum_{n=1}^{\infty} \frac{\hat{T}^{(n)}}{z_H^{4n}} \,, \tag{5.11}$$

where $\hat{T}^{(0)}$ is the homothetic vector (2.24) transported on the flat ambient space. Since $\mathcal{I}_2^{(1)} = 0$, it is sufficient to use $\hat{T}^{(0)}$ for invariants up to second order in the Riemann because higher order $\hat{T}^{(n)}$'s contribute at order $O(\widetilde{\text{Riem}})^3$ in contractions of the form (4.12).

We now turn to determining a basis of ambient invariants quadratic in the curvature. In principle one could pick them to be of any scaling dimension and then multiply them by the appropriate power of $\widetilde{X}_{12}$. As we discussed in Section 4.2, invariants with vanishing scaling dimension are particularly rigid in their structure and thus easy to completely classify. Their general form is given in equation (4.15) and if we restrict to $k = 2$ curvature tensors, one can show that in the present setup there are only three independent such invariants of order $O(z_H^{-8})$. One possible choice is

$$e_0 = \mathcal{R}^{(0)}_{AC}\,\mathcal{R}^{(0)AC} = \frac{3}{4}\frac{|x|^8}{z_H^8} + O\left(\frac{|x|}{z_H}\right)^{12} , \tag{5.12a}$$

$$e_1 = \mathcal{R}^{(1)}_{AC}\,\mathcal{R}^{(0)AC} = -\frac{|x|^6}{z_H^8}(3\tau^2 + 7x^2) + O\left(\frac{|x|}{z_H}\right)^{12} , \tag{5.12b}$$

$$e_2 = \mathcal{R}^{(1)}_{AC}\,\mathcal{R}^{(1)AC} = 4\frac{|x|^4}{z_H^8}(3\tau^4 + 16\tau^2x^2 + 17x^4) + O\left(\frac{|x|}{z_H}\right)^{12} , \tag{5.12c}$$

where the subscripts refer to the number of covariant derivatives required to construct them, with the tensors $\mathcal{R}^{(r)}$ defined in (4.14). In these expressions we have already taken the limit from generic ambient points to the CFT background on the nullcone. As we detail in Appendix D any curvature invariant quadratic in the ambient Riemann can be obtained as a linear combination of the form

$$\mathcal{I}_2^{(2)} = c_0e_0 + c_1e_1 + c_2e_2 \,. \tag{5.13}$$

Putting all together, we assemble the ambient scalar 2-point function for operators of scaling dimension Δ as prescribed by equation (4.16),

$$\langle O(\tau,x)O(0)\rangle^{(\beta)}_{d=4,\Delta} = \frac{C_\Delta}{|x|^{2\Delta}}\Bigg[1 - \frac{\Delta\,(x^2-3\tau^2)|x|^2}{120\pi^{-4}\beta^4} + \frac{|x|^4}{\pi^{-8}\beta^8}\Bigg[\frac{3}{4}\left(c_0 + \frac{\Delta(63\Delta+170)}{30240}\right)|x|^4$$
$$- \left(c_1 + \frac{\Delta(14\Delta+39)}{25200}\right)|x|^2(3\tau^2+7x^2) + 4\left(c_2 + \frac{\Delta(7\Delta+20)}{201600}\right)(3\tau^4+16\tau^2x^2+17x^4)\Bigg]$$
$$+ O\left(\frac{|x|}{\beta}\right)^{12}\Bigg]. \tag{5.14}$$

Here the constants c_i are to be fixed by the dynamics of the specific thermal CFT, and they quantify the quantum corrections to the semi-classical geodesic approximation as discussed in Section 4.3.

5.4 Matching with the thermal OPE

As reviewed in Section 4.3, the OPE is expected to converge for CFTs on generic backgrounds and states for short enough distances. It can thus be used to reduce 2-point functions to a sum over 1-point functions as in equation (4.18). Specialising to thermal CFTs on flat space and given the form (5.1) of 1-point functions, it was argued in [25] that for a distance between insertions shorter than the thermal radius $|x|< \beta$ one can expand a scalar correlation function of two operators of dimension Δ as

$$\langle O(\tau,x)O(0)\rangle^{(\beta)}_{d,\Delta} = \sum_{\phi\in O\times O} \frac{a_\phi}{\beta^{\Delta_\phi}}\, C^{(\nu)}_J(q)|x|^{\Delta_\phi-2\Delta}\,, \tag{5.15}$$

where J and Δ_ϕ are the spin and scaling dimension of the exchanged operator ϕ, and we defined $\nu = \frac{d}{2}-1$ and $a_\phi = \frac{f_{OO\phi}b_\phi}{c_\phi}\frac{J}{2^J(\nu)_J}$. Here c_ϕ and $f_{O\phi\phi}$ are the 2- and 3-point function coefficients on flat space, while $C^{(\nu)}_J(q)$ are Gegenbauer polynomials of the dimensionless ratio $q=\tau/|x|$.

The products $C^{(\nu)}_J(q)|x|^{\Delta_\phi-2\Delta}$ can be thought of as thermal conformal blocks. The fact that in thermal CFTs 1- and 2-point functions contain non-trivial dynamical data through the coefficients a_ϕ mirrors the freedom in the coefficients c_i appearing in the ambient 2-point function (5.14). In this subsection we would like to make this connection more precise by relating the coefficients a_ϕ with the c_i.

As anticipated in Section 4.3 we expect the ambient curvature invariants to account for the multi-stress tensor contributions $:T^n:$. These operators are defined as the $n+1$ symmetrised traceless partial contractions of tensor products of n stress tensors, with scaling dimensions nd in d dimensions and even spins ranging from $J=0$ to $J=2n$.

Their contribution takes the form

$$\langle O(\tau,x)O(0)\rangle^{(\beta)}_{d,\Delta} \supset \sum_{n=0}^{\infty}\sum_{\substack{J=0\\ J\,\mathrm{even}}}^{2n} a^{(T)}_{n,J}\, C^{(\nu)}_J(q)\frac{|x|^{nd-2\Delta}}{\beta^{nd}}. \tag{5.16}$$

Comparing (5.14) with (5.16) to second order in the stress tensor yields the following dictionary re-expressing the thermal OPE coefficients in terms of the ambient free coefficients for any Δ in $d=4$,

$$a^{(T)}_{0,0} = C_\Delta, \qquad a^{(T)}_{1,0} = 0, \qquad a^{(T)}_{1,2} = \frac{\Delta}{120}C_\Delta, \tag{5.17}$$

$$a^{(T)}_{2,0} = \left(\frac{3c_0}{4} - 6c_1 + 52c_2 + \frac{\Delta(7\Delta+18)}{201600}\right)C_\Delta, \tag{5.18}$$

$$a^{(T)}_{2,2} = \left(c_1 - 15c_2 + \frac{\Delta(7\Delta+12)}{201600}\right)C_\Delta, \tag{5.19}$$

$$a^{(T)}_{2,4} = \left(c_2 + \frac{\Delta(7\Delta+20)}{201600}\right)C_\Delta. \tag{5.20}$$

Note once more that the ambient prediction at first order in the stress tensor is fully fixed by the geodesic distance factor $(\widetilde{X}_{12})^{-\Delta}$ as a consequence of $\mathcal{I}^{(1)}_2 = 0$.

The relations (5.17)-(5.20) entail that to this order, ambient curvature invariants and thermal conformal blocks are two equivalent bases to describe multi-stress tensor contributions. This can be made more precise by mapping the thermal conformal blocks to the basis of curvature invariants $\{e_0, e_1, e_2\}$. After taking the large-N limit in the CFT the multi-stress tensor VEVs factorise, $\langle :T^n:\rangle \sim \langle T\rangle^n$. Denoting the stress tensor VEV (5.2) by T_{ij} to avoid cluttering, in terms of T_{ij} the double-stress tensor VEVs with $J=0,2,4$ read,

$$\langle T^2\rangle = T^{kl}T_{kl}, \tag{5.21a}$$

$$\langle T^2\rangle_{ij} = T_{ik}T^k{}_j - \frac{1}{4}T^{kl}T_{kl}\delta_{ij}, \tag{5.21b}$$

$$\langle T^2\rangle_{ijkl} = \Sigma_{ijkl} - \frac{3}{4}\delta_{(ij}\Sigma_{kl)m}{}^m + \frac{1}{16}\Sigma^m{}_m{}^n{}_n\,\delta_{(ij}\delta_{kl)}, \tag{5.21c}$$

where we defined $\Sigma_{ijkl} = T_{(ij}T_{kl)}$. In terms of these the second order curvature invariants can be written as

$$64\,e_0 = \langle T^2\rangle\,|x|^8, \tag{5.22a}$$

$$8\,e_1 = \langle T^2\rangle_{ij}\,x^ix^j|x|^6 - \langle T^2\rangle\,|x|^8, \tag{5.22b}$$

$$4\,e_2 = \langle T^2\rangle_{ijkl}\,x^ix^jx^kx^l|x|^4 - \frac{15}{2}\langle T^2\rangle_{ij}\,x^ix^j|x|^6 + \frac{13}{3}\langle T^2\rangle\,|x|^8. \tag{5.22c}$$

Thus the thermal conformal blocks at order $n = 2$ in the large-N limit are simply proportional to trace modifications of the ambient invariants e_i. In Appendix D we describe how to extend these conclusions to any order in the ambient Riemann and to other dimensions d. In particular we argue that the dimensionless invariants (4.15) constructed as chains of tensors $\mathcal{R}^{(r)}$ form a basis for the contribution of generic multi-stress tensor operators $:T^n:$ for thermal CFTs in even dimensions d.[2]

At finite N, typically more operators take non-trivial VEVs and contribute in correlators, meaning that usually additional ambient invariants are required. However conformal blocks retain their form independently of the regime the theory is in, since they follow from kinematics and not from dynamics. In the thermal case this means that the thermal conformal blocks describing the multi-stress tensor contributions in (5.16) are the same Gegenbauer polynomials at any N. We have shown that multi-stress tensor conformal blocks are equivalent to the basis of ambient curvature invariants of the form (4.15) at large N. We now conclude that this equivalence extends trivially to finite N: the ambient curvature invariants provide a basis for multi-stress tensor contributions in any thermal CFT.

This represents strong evidence for the conjectured validity of the ambient formalism as a tool to solve the kinematics of generic CFTs. In this case it was possible to compare the ambient prediction with OPE computations and we found perfect agreement even for non-holographic CFTs. In Chapter 6 we treat CFTs on squashed spheres, where no OPE result is available and the ambient formalism produces genuinely new predictions.

5.5 Matching with a holographic correlator

In the previous section we showed that ambient 2-point functions account for the multi-stress tensor contributions to correlators in thermal CFTs (holographic or otherwise). We did this by comparing with the thermal OPE. In this section we check this statement through a holographic computation, without relying on the thermal OPE. To this aim we will study holographic correlators in the state dual to the Euclidean AdS$_{d+1}$ planar black hole with metric

$$ds^2 = \frac{1}{z^2}\left[\frac{dz^2}{1-\frac{z^d}{z_H^d}} + \left(1-\frac{z^d}{z_H^d}\right)d\tau^2 + \delta_{ab}dx^a dx^b\right]. \tag{5.23}$$

The dual CFT is in the same background and state as the previous subsections, with inverse temperature $\beta = 4\pi z_H/d$ and stress tensor expectation value (5.2). This problem

[2] As we discussed in Section 4.2, in odd d divergences appear in the ambient Riemann in the limit $\rho \to 0$. This implies that some of the weight-0 scalars (4.15) may diverge and other invariants must be used in addition to them. We will see this explicitly in Chapter 6.

involves solving the free scalar equation

$$[-\Box_{d+1} + \Delta(\Delta - d)]\, \Phi(z, \tau, x) = 0 \tag{5.24}$$

on the fixed background (5.23), subject to Dirichlet conditions at the boundary $z \to 0$ and regularity conditions in the bulk interior $z \to \infty$. Here Δ is the scaling dimension of the operator whose 2-point function we wish to compute.

Translational symmetry along the boundary directions and periodicity along τ allow one to expand the scalar fields in Fourier modes,

$$\Phi(z, \tau, x) = \sum_{m \in \mathbb{Z}} \int d^{d-1}\mathbf{k}\, e^{i(\omega_m \tau + \mathbf{k}\cdot\mathbf{x})} B(z, \omega_m, \mathbf{k}), \tag{5.25}$$

where $\omega_m = 2\pi m/\beta$ are the Matsubara frequencies. Defining $k = \sqrt{\omega_m^2 + \mathbf{k}^2}$ and after rescaling the radial coordinate as $r = kz$, $r_H = kz_H$ and redefining $r_H = \epsilon^{-\frac{1}{d}}$, equation (5.24) reads

$$\begin{aligned} &r\left(\epsilon r^d - 1\right)\left(rB''(r)\left(\epsilon r^d - 1\right) + B'(r)\left(\epsilon r^d + d - 1\right)\right) \\ &\quad + B(r)\left(\left(\Delta(\Delta - d) + r^2\right)\left(\epsilon r^d - 1\right) - \frac{\omega_m^2}{k^2}\epsilon r^{d+2}\right) = 0. \end{aligned} \tag{5.26}$$

where we left the dependence of B on ω_m and $\mathbf{k}$ implicit.

This equation is of Heun type. We will first solve it perturbatively in the limit of short boundary distance between the insertions (or equivalently at high momenta) with respect to the thermal radius, $\epsilon = (kz_H)^{-d} \ll 1$. In this regime we are able to compare with the expansion in the curvature of ambient correlators (and also double-check results from the thermal OPE). We will then turn to a fully-non-perturbative numerical computation to further check the ambient correlator, as well as to study effects that may elude the perturbative analysis.

5.5.1 Perturbative 2-point function

We set up the perturbative problem by expanding at $\epsilon \ll 1$ corresponding to large momenta $k \gg (z_H)^{-1}$,

$$B(r) = \sum_{n=0}^{\infty} b_n(r)\epsilon^n\,. \tag{5.27}$$

The equations for the first few orders read

$$\mathcal{D}\, b_0(r) = 0, \tag{5.28a}$$

$$\mathcal{D}\, b_1(r) = r^{d-2}\left[b_0(r)\left(\Delta(\Delta - d) + \left(\eta^2 + 1\right) r^2\right) + drb_0'(r)\right], \tag{5.28b}$$

$$\begin{aligned} \mathcal{D}\, b_2(r) = r^{d-2}&\left[b_0(r) r^d\left(\Delta(\Delta - d) + \left(2\eta^2 + 1\right) r^2\right) + b_1(r)\left(\Delta(\Delta - d) + \left(\eta^2 + 1\right) r^2\right)\right. \\ &\left. + dr\left(r^d b_0'(r) + b_1'(r)\right)\right], \end{aligned} \tag{5.28c}$$

where we defined $\eta = \omega_m/k$ and the differential operator

$$\mathcal{D} = \partial_r^2 + \frac{1-d}{r}\partial_r - \frac{\Delta(\Delta-d)+k^2}{r^2}. \tag{5.29}$$

The perturbative equations at a generic order n reads

$$\mathcal{D}\, b_n(r) = \sum_{\ell=1}^{n} r^{\ell d-1}\left[d b'_{n-\ell}(r) + r\left(\Delta(\Delta-d) + (1+\ell\eta^2)r^2\right) b_{n-\ell}(r)\right]. \tag{5.30}$$

The solution to the leading order equation (5.28a) corresponds to a free scalar wavefunction on pure Euclidean AdS$_{d+1}$. Defining $\kappa = \Delta - \frac{d}{2}$, if we assume κ is not an integer[3], a possible choice for the basis of the solutions space is in terms of modified Bessel functions of the first kind,

$$u_1(r) = \sqrt{\frac{\pi}{2}}\, r^{\frac{d}{2}}\, I_{-\kappa}(r)\,, \qquad u_2(r) = \sqrt{\frac{\pi}{2}}\, r^{\frac{d}{2}}\, I_{\kappa}(r)\,. \tag{5.31}$$

Imposing regularity in the interior $r \to \infty$ fixes the leading order solution to a modified Bessel function of the second kind,

$$b_0 = u_2 - u_1 = -\sqrt{\frac{2}{\pi}}\cos\left(\frac{2\kappa-1}{2}\pi\right) K_\kappa(r), \tag{5.32}$$

recovering the expected solution on pure AdS (see e.g. [67]).

Solving the first order equation (5.28b) is more involved, and we refer the reader to Appendix E for a detailed computation. The holographic correlator to first order in ϵ in momentum space results in

$$\begin{aligned}\langle OO\rangle^{(\beta)}_{d,\Delta}(\omega_m,\mathbf{k}) = -\frac{2^{d-2\Delta}\Gamma\left(\frac{d}{2}-\Delta+1\right)}{\Gamma\left(-\frac{d}{2}+\Delta+1\right)} k^{2\Delta-d}\Bigg[1+ \\ \frac{\pi^{3/2+d}(-1)^{d+1}\cot\left(\frac{\pi d}{2}\right)\Gamma\left(-\frac{d}{2}-\frac{1}{2}\right)\csc^2(\pi\Delta)\sin\left(\frac{1}{2}\pi(d-2\Delta)\right)(k^2-d\omega_m^2)}{4\Gamma\left(1-\frac{d}{2}\right)\Gamma(-\Delta)\Gamma(\Delta-d)\,k^{d+2}\beta^d} + O(\epsilon^2)\Bigg].\end{aligned} \tag{5.33}$$

Note that for even d and integer Δ or for odd d and half-integer Δ with $\Delta > \frac{d}{2}$ one has poles in the scaling dimension. For example in $d=4$ and $\Delta=3$ the leading terms are

$$\langle OO\rangle^{(\beta)}_{d=4,\Delta=3} = \frac{k^2}{4(\Delta-3)} + \frac{k^2}{4}\left(\log\frac{k^2}{4} + 2\gamma - 1 + \frac{4}{5}\left(1-4\omega_m^2\right)\epsilon\right) + O\left(\Delta-3\right) + O(\epsilon^2). \tag{5.34}$$

[3]For such values of Δ the CFT correlators have short-distance singularities leading to conformal anomalies [94]. On the bulk side a different choice of basis for the solution space must be made since m is half-odd implying $u_1 = u_2$. As reviewed in Subsection 1.3.1, logarithmic terms also appear in the Fefferman-Graham near-boundary expansion [66] and in this case the present analysis must be modified.

These divergent contributions constitute contact terms in position space and we discard them.

Solving the second order equation (5.28c) and higher is particularly involved for general d and Δ. A simplification happens when $2\Delta + d$ is integer (i.e. κ is half-odd). In this case the homogeneous solutions can be written in terms of products of polynomials and exponentials since

$$\begin{aligned} I_\kappa(r) &= \sqrt{\frac{2}{\pi}}\, i^{\kappa-\frac{3}{2}} r^\kappa \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\kappa-\frac{1}{2}} \frac{\sinh r}{r}, \\ I_{-\kappa}(r) &= \sqrt{\frac{2}{\pi}}\, i^{\kappa-\frac{3}{2}} r^{\kappa+1} \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\kappa+\frac{1}{2}} \frac{\cosh r}{r}. \end{aligned} \tag{5.35}$$

This observation allows one to find a case-by-case solution to arbitrarily high order in the inverse temperature. Considering for simplicity $\frac{d}{2} \le \Delta \le d$, the generic form of such momentum space correlators is

$$\langle OO\rangle^{(\beta)}_{d,\Delta}(\omega_n, \mathbf{k}) = \frac{1}{k^{d-2\Delta}} \sum_{q=0}^{\infty} \frac{\pi^{qd}}{k^{q(d+2)}\beta^{qd}} \sum_{j=0}^{q} \alpha_j^{(q)}\, \mathbf{k}^{2q-2j} \omega_n^{2j}, \tag{5.36}$$

where the coefficients $\alpha_j^{(q)}$ are in terms of d and Δ. Up to second order it reads explicitly

$$\langle OO\rangle^{(\beta)}_{d,\Delta}(\omega_m, \mathbf{k}) = \frac{\alpha_0^{(0)}}{k^{d-2\Delta}} + \frac{\alpha_0^{(1)}\mathbf{k}^2 + \alpha_1^{(1)}\omega_m^2}{\pi^{-d}\, k^{2d-2\Delta+2}\beta^d} + \frac{\alpha_0^{(2)}\mathbf{k}^4 + \alpha_1^{(2)}\mathbf{k}^2\omega_m^2 + \alpha_2^{(2)}\omega_m^4}{\pi^{-2d}\, k^{3d-2\Delta+4}\beta^{2d}} + O\left(k\beta\right)^{-3d}. \tag{5.37}$$

The coefficients $\alpha_0^{(0)}, \alpha_0^{(1)}$ and $\alpha_1^{(1)}$ for generic d and Δ can be extracted from (5.33); in particular, $\alpha_1^{(1)} = (1-d)\alpha_0^{(1)}$. As an example, for $d = 4$ and $\Delta = \frac{3}{2}$ the first few coefficients take the values

$$\alpha_0^{(0)} = -1, \qquad \alpha_0^{(1)} = -\frac{3}{16}, \qquad \alpha_1^{(1)} = \frac{9}{16}, \tag{5.38}$$

$$\alpha_0^{(2)} = -\frac{2637}{512}, \qquad \alpha_1^{(2)} = \frac{11511}{256}, \qquad \alpha_2^{(2)} = -\frac{10773}{512}. \tag{5.39}$$

Transforming momentum space correlators of the form (5.37) back to position space is subtle since the Fourier transform should be performed over all real momenta and all Matsubara frequencies, while the expression for the correlators we have found is only valid at large frequencies $\omega_m \gg 1/\beta$. To explicitly perform the Fourier transform one should thus resum the perturbative expansion to assess the full dependence on k and ω_m. One should assume the sum (5.36) is convergent, although there are hints that it contains non-perturbative effects in the Lorentzian case [95,96]. As we discuss in Subsection 5.5.3, such non-perturbative parts do not affect the multi-stress tensor blocks we wish to compute, hence for us this subtlety will not play any role.

Furthermore, it is particularly hard to directly compute the compact Fourier transform (5.25) of expansions of the form (5.36). However, note that as a consequence of the Poisson summation formula the compact Fourier transform over the Matsubara frequencies can be recast as a sum over images of the non-compact Fourier transform,

$$\begin{aligned}
\langle O(\tau,x)O(0)\rangle^{(\beta)}_{d,\Delta} &= \sum_{m\in\mathbb{Z}} \int d^{d-1}\mathbf{k}\, e^{i\omega_m\tau+i\mathbf{k}\cdot\mathbf{x}} \langle OO\rangle_{d,\Delta}(\omega_m,\mathbf{k}) \\
&= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} d\omega \int d^{d-1}\mathbf{k}\, e^{i\omega(\tau+m\beta)+i\mathbf{k}\cdot\mathbf{x}} \langle OO\rangle_{d,\Delta}(\omega,\mathbf{k}) \\
&\equiv \sum_{m\in\mathbb{Z}} \langle O(\tau+m\beta,x)O(0)\rangle^{(T)}_{d,\Delta} \,.
\end{aligned} \tag{5.40}$$

Therefore we proceed as follows. We compute the non-compact Fourier transform of the momentum space correlator (5.37) order by order in the perturbative expansion to obtain the non-compact correlator $\langle O(\tau,x)O(0)\rangle^{(T)}_{d,\Delta}$. As we motivate in Subsection 5.5.2, the sum over images does not affect the multi-stress tensor contributions in position space as long as κ is not an integer. Thus knowing the non-compact correlator is sufficient to match the multi-stress tensor contributions with the predictions from the ambient space.[4]

Let us now perform the non-compact Fourier transform of the perturbative terms in the expansion (5.37). The Fourier transform of a spherically symmetric distribution in momentum space reduces to a Hankel transform,

$$\begin{aligned}
F(x) &= \int d^d p f(|p|) e^{ip\cdot x} = \frac{(2\pi)^{\frac{d}{2}}}{|x|^{\frac{d}{2}-1}} \int_0^\infty dp f(p) J_{\frac{d}{2}-1}(|x|p) p^{\frac{d}{2}} \\
&= \frac{(2\pi)^{\frac{d}{2}}}{|x|^{\frac{d}{2}-1}} H_{\frac{d}{2}-1}\left[p^{\frac{d}{2}-1} f(p)\right](x)\,.
\end{aligned} \tag{5.41}$$

The non-compact transform of each order in (5.37) can thus be rewritten as a linear combination of derivatives of Hankel transforms. Defining the integral

$$I_\gamma(\tau,x) \equiv \int d^d p |p|^\gamma e^{ip\cdot x} = \frac{(2\pi)^{\frac{d}{2}}}{|x|^{\frac{d}{2}-1}} H_{\frac{d}{2}-1}\left[p^{\gamma+\frac{d}{2}-1}\right](x) = \frac{\pi^{d/2} 2^{\gamma+d} \Gamma\left(\frac{d+\gamma}{2}\right)}{\Gamma\left(-\frac{\gamma}{2}\right)} \frac{1}{|x|^{\gamma+d}}\,, \tag{5.42}$$

[4] In Subsection 5.5.2 we further comment on these points.

one can rewrite the first few orders of the correlator in position space as

$$\langle O(\tau,x)O(0)\rangle^{[0]}_{d,\Delta} = \alpha^{(0)}_0\, I_{2\Delta-d}\,, \tag{5.43}$$

$$\langle O(\tau,x)O(0)\rangle^{[d]}_{d,\Delta} = \alpha^{(1)}_0\left[I_{2\Delta-2d} - d\left(\frac{\partial_\tau}{i}\right)^2 I_{2\Delta-2d-2}\right], \tag{5.44}$$

$$\begin{aligned}\langle O(\tau,x)O(0)\rangle^{[2d]}_{d,\Delta} &= \alpha^{(2)}_0\, I_{2\Delta-3d} + (\alpha^{(2)}_1 - 2\alpha^{(2)}_0)\left(\frac{\partial_\tau}{i}\right)^2 I_{2\Delta-3d-2}\\ &\quad + (\alpha^{(2)}_2 - \alpha^{(2)}_1 + \alpha^{(2)}_0)\left(\frac{\partial_\tau}{i}\right)^4 I_{2\Delta-3d-4}.\end{aligned} \tag{5.45}$$

These relations can be straightforwardly obtained to arbitrarily high order.

Using these expressions on (5.33) one finds the position space correlator at general d and Δ to first order. Normalising the operators so that the leading order constant is normalised to 1, the non-compact 2-point function reads

$$\langle O(\tau,x)O(0)\rangle^{(T)}_{d,\Delta} = \frac{1}{|x|^{2\Delta}}\left[1+\tilde{\lambda}_1\left(x^2-(d-1)\tau^2\right)\frac{|x|^{d-2}}{\beta^d}\right] + O\left(\frac{|x|}{\beta}\right)^{2d}, \tag{5.46}$$

with

$$\tilde{\lambda}_1 = \left(\frac{4\pi}{d}\right)^d \frac{\sqrt{\pi}(-1)^{d+1}\Delta\,\Gamma\left(-\frac{d}{2}-\frac{1}{2}\right)\sin(\pi(d-\Delta))}{2^{d+2}\Gamma\left(1-\frac{d}{2}\right)\tan\left(\frac{\pi d}{2}\right)\sin(\pi\Delta)}. \tag{5.47}$$

One can check that this expression matches both the geodesic approximation and the ambient correlator (5.14) to first order in β^{-d} for $d=4$. This result hence substantiates the universality of the geodesic approximation for the stress tensor contribution as predicted by the ambient space formalism.

Furthermore, using (5.43)-(5.45) on (5.37) one is able to check for odd $2\Delta+d$ that the higher orders in $|x|/\beta$ of such position space correlators can be decomposed in terms of the ambient curvature invariants. For instance, for $d=4$ and $\Delta=\frac{3}{2}$ the correlator up to second order reads

$$\begin{aligned}\langle O(\tau,x)O(0)\rangle^{(T)}_{d=4,\Delta=\frac{3}{2}} = \frac{1}{|x|^3}\Bigg[1 &- \pi^4\frac{|x|^2(x^2-3\tau^2)}{80\,\beta^4}\\ &-\pi^8\frac{|x|^4(479\tau^4-1162\tau^2x^2+199x^4)}{268800\,\beta^8} + O\left(\frac{|x|^{12}}{\beta^{12}}\right)\Bigg],\end{aligned} \tag{5.48}$$

which fixes the coefficients in (5.14) to

$$c_0 = -\frac{53}{1575}, \qquad c_1 = -\frac{11}{1120}, \qquad c_2 = -\frac{11}{16800}\,. \tag{5.49}$$

This brings further evidence that the ambient curvature invariants form a basis for the multi-stress tensor spectrum and it confirms the ambient prediction (5.14). Via the relations between ambient and thermal OPE coefficients (5.17)-(5.20), this also represents a non-trivial check of the expansion in terms of thermal conformal blocks (5.16) in a non-trivial thermal state, in particular beyond the large-Δ regime studied in [92, 93] and to arbitrarily high order in $|x|/\beta$.

5.5.2 Non-perturbative 2-point function

In this subsection we discuss the non-perturbative effects in $|x|/\beta \to 0$ entering the thermal holographic correlator on the planar black hole background. In momentum space such contributions can be studied along the lines of [96, 97], at least perturbatively in the instanton number. We are however interested in the correlator in position space and for this purpose we resort to a numerical calculation, fully non-perturbative in the boundary temperature.

Since we are not working perturbatively in $|x|/\beta$, to compute the position space 2-point function we must solve (5.24) on the Euclidean cigar geometry with period β. The boundary conditions are a delta-function source at $\tau = |\vec{x}| = 0$ and we demand regularity in the interior. With the Euclidean time circle τ, the holographic radial direction z, and noting a rotational symmetry in the spatial boundary directions x^i, this leaves a 3d PDE problem. Without loss of generality we set $z_H = 1$ so that $\beta = \pi$. Next, we make the following coordinate changes,

$$z = 1 - \rho^2, \qquad \tau = \frac{1}{2}\phi, \qquad |\vec{x}| = \frac{R}{1 - R^2}. \tag{5.50}$$

In these coordinates we have $\rho \in [0, 1]$ where $\rho = 0$ is the tip of the Euclidean cigar geometry and $\rho = 1$ is the conformal boundary, $\phi = (0, 2\pi]$ is the angle around the thermal circle, and $R \in [0, 1)$ where $R = 0$ is the origin of spatial coordinates on the boundary and $R = 1$ is the compactification of spatial infinity.

The principal numerical challenge is handling the delta function source at the origin on the boundary. We subtract a function from Φ with the correct singularity structure, i.e. we define a new field Ψ via,

$$\Phi = \Psi + \tilde{G}_{AdS} \tag{5.51}$$

where $\tilde{G}_{AdS}$ is an analytically known function containing the correct source behaviour. A candidate function is the vacuum AdS bulk-boundary propagator,

$$\frac{z^{\Delta}}{(\tau^2 + r^2 + z^2)^{\Delta}} = \frac{(1 - \rho^2)^{\Delta}}{\left(\frac{\phi^2}{4} + (1 - \rho^2)^2 + \frac{R^2}{(1 - R^2)^2}\right)^{\Delta}}. \tag{5.52}$$

however this is not periodic in ϕ. To address this we make the replacement

$$\phi^2 \to \frac{2}{3}(7 - \cos(\phi)) \sin\left(\frac{\phi}{2}\right)^2. \tag{5.53}$$

The resulting function $\tilde{G}_{AdS}$ is then periodic $\phi \sim \phi + 2\pi$, contains no additional singularities, and is regular in the interior. Hence to find the 2-point function we now need to solve,

$$(\Box - \Delta(\Delta - d))\,\Psi = -\,(\Box - \Delta(\Delta - d))\,\tilde{G}_{AdS}. \tag{5.54}$$

where Ψ obeys a Dirichlet zero boundary condition at the conformal boundary, and is also regular in the interior.

We work with $\Delta = 5/2$, so that the near boundary behaviour of Ψ is,

$$\Psi = a(\tau, r) z^{\frac{3}{2}} + b(\tau, r) z^{\frac{5}{2}} + \ldots = a(\tau, r)(1-\rho^2)^{\frac{3}{2}} + b(\tau, r)(1-\rho^2)^{\frac{5}{2}} + \ldots \tag{5.55}$$

To this order the z expansion is equivalent to the Fefferman-Graham expansion. Note that

$$\Phi = a(\tau, r) z^{\frac{3}{2}} + \left(b(\tau, r) + \frac{1152\sqrt{6}}{(15 + 24r^2 - 16\cos(2\tau) + \cos(4\tau))^{\frac{5}{2}}} \right) z^{\frac{5}{2}} + \ldots \tag{5.56}$$

We define

$$\Psi = (1-\rho^2)^{\frac{3}{2}} H \tag{5.57}$$

enforce $a = 0$ through a Dirichlet boundary condition $H_{\rho=1} = 0$, and read off b from the solution as $b = \partial_\rho H|_{\rho=1}$. The two point function is then given by the data b, corrected by the subtracted function,

$$\langle O(0,0) O(\tau, r)\rangle = b(\tau, r) + \frac{1152\sqrt{6}}{(15 + 24r^2 - 16\cos(2\tau) + \cos(4\tau))^{\frac{5}{2}}}. \tag{5.58}$$

For the rest of the problem we enforce tip of the cigar regularity with $\partial_\rho H|_{\rho=0} = 0$, origin regularity on the boundary with $\partial_R H|_{R=0} = 0$, and at spatial infinity on the boundary the response to the delta should vanish, so we also set $H|_{R=1} = 0$.

The PDE is discritised using a grid of N_ρ, N_ϕ, N_R points in the ρ, ϕ, R directions respectively. We utilize Chebyschev collocation in ρ with second-order finite difference methods for ϕ and R. This discritisation of (5.54) give rise to a linear problem

$$MH = S \tag{5.59}$$

where M is a matrix of size $(N_\rho N_\phi N_R)^2$ and S is a vector of size $N_\rho N_\phi N_R$. We then solve for H, read off $b = \partial_\rho H|_{\rho=1}$ and compute $\langle O(0,0)O(\tau, r)\rangle$ using (5.58).

The results at $d = 4$, $\Delta = 5/2$ are shown in figure 5.1. In particular, the behaviour of the 2-point function in the limit $x \to 0$ is consistent with the prediction of the ambient formalism (5.14) and with the perturbative holographic value (5.46).

In Figure 5.2 we show the results for $d = 4$, $\Delta = 3/2$. The behaviour of the first subleading term in $\tau \to 0$ differs from that expected for the single stress tensor block,

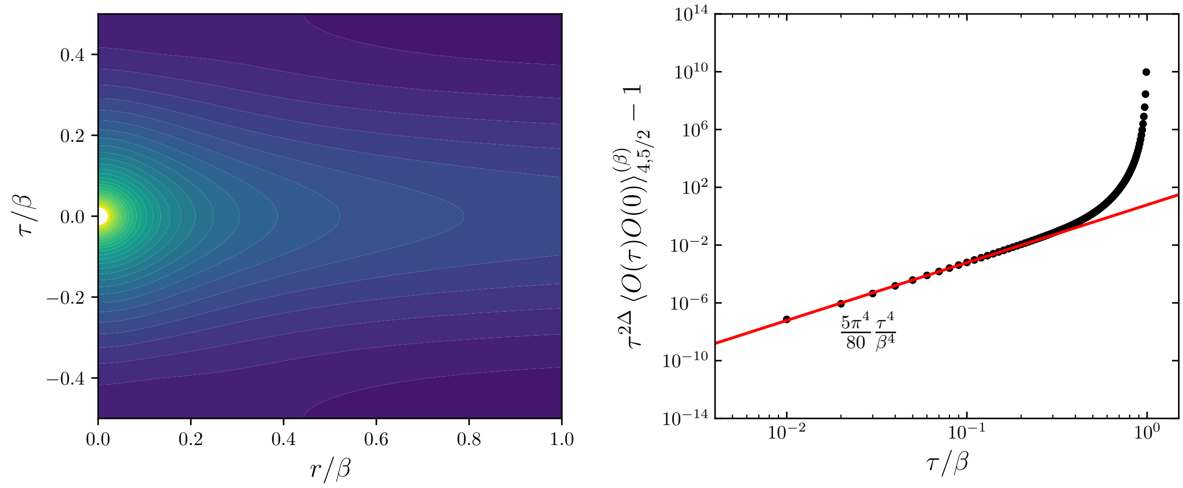

Figure 5.1: Non-perturbative thermal 2-point function for $d = 4$, $\Delta = 5/2$ from holography. **Left:**. Contour plot of the 2-point function over the full range of the thermal circle. **Right:** Showing a log-log plot to illustrate the leading behaviour at $x^i = 0$ near $\tau = 0$ (black dots). The power-law behaviour is consistent with the analytically derived stress tensor contribution (red line).

and it is compatible with the exchange of an operator of dimension $\Delta = 3$. This suggests the appearance of the operator $:OO:$ belonging to the so-called double-twist spectrum. These are operators of the schematic form $:O\Box^n\partial_{i_1}\ldots\partial_{i_J}O:$, symmetric and traceless in the J indices. They are primaries with scaling dimensions $\Delta_{p,J} = 2\Delta + 2p + J$ and even spin J. Given their dimensions and tensorial properties, double-twist operators appear in the thermal OPE (5.15) with blocks of the form,

$$\langle O(\tau,x)O(0)\rangle^{\beta}_{d,\Delta} \supset \frac{1}{\beta^{2\Delta}}\sum_{p=0}^{\infty}\sum_{\substack{J=0\\ J\text{even}}}^{\infty} a^{(OO)}_{p,J}\, C^{(\nu)}_J(q)\left(\frac{|x|}{\beta}\right)^{2p+J}. \tag{5.60}$$

In the limit $x \to 0$ the $n = 0, J = 0$ block precisely reproduces the scaling in τ displayed in Figure 5.2, and our non-perturbative computation thus makes a prediction for the dynamical OPE coefficient

$$a^{(OO)}_{0,0} \simeq 1.1\,. \tag{5.61}$$

We discuss the appearance of the double-twist spectrum and its non-perturbative nature at length in Subsection 5.5.3.

In Figure 5.2 we also show the second-subleading behaviour which we recognise as the stress tensor block. Also in this case the value of its coefficient is compatible with the ambient result (5.46), thus confirming the ambient prediction about the exactness

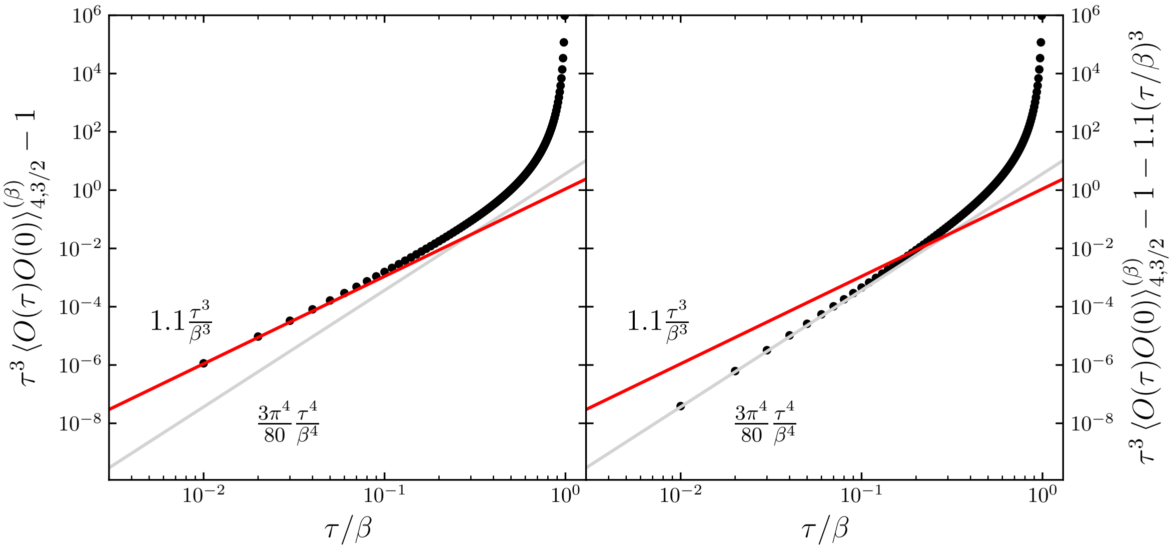

Figure 5.2: Non-perturbative thermal 2-point function for $d = 4$, $\Delta = 3/2$ from holography. Showing a log-log plot to illustrate the leading behaviour at $x^i = 0$ near $\tau = 0$ (black dots). **Left:** With the leading conformal behaviour subtracted, the remaining power-law at short distances is consistent with the leading term in the double-twist spectrum, $a^{(OO)}_{0,0}$ (red line). **Right:** Making a further subtraction to remove the leading double-twist contribution reveals the analytically derived stress tensor contribution (grey line).

of the stress tensor coefficient at the non-perturbative level. The numerical value of the coefficient also matches the perturbative analytic correlator (5.46), supporting the claim that multi-stress tensors do not receive non-perturbative corrections (as long as κ is not integer).

5.5.3 Obtaining the double-twist spectrum

In position space the presence of double-twist operators in the holographic scalar 2-point function on the planar black hole has been argued in [25, 98–102]. For non-integer κ they exhibit no mixing with the multi-stress tensors and hence they do not modify the conclusions concerning the first-order exactness of the geodesic approximation as well as the completeness of the basis for multi-stress tensors provided by the ambient curvature invariants. Let us now make few observations on the non-perturbative nature of double-twist operators, so as to clarify why they are absent in perturbative correlators such as (5.46) and (5.48).

In Euclidean signature there is evidence from [103] that double-twist contributions cannot appear as perturbative terms in an expansion of a momentum space 2-point function of the form (5.36).[5] This statement is supported by our analytic perturbative results in momentum space, where only the multi-stress tensor spectrum is present.

For Lorentzian thermal CFTs it has been argued in [96] that the double twist spectrum arises by Fourier transforming non-perturbative contributions present in the momentum space 2-point function [95]. Similar non-perturbative pieces $\sim e^{-\beta\omega}$ in the momentum space holographic correlator may be present also in Euclidean signature as they are not captured by the perturbative treatment of Subsection 5.5.1. They may yield double-twist contributions in the non-compact correlator once transformed to position space.

To gain some intuition, consider the simple case of thermal AdS$_{d+1}$ in the bulk. The scalar 2-point function takes the form of a sum over images (5.40) of the correlator computed on Euclidean AdS (which consists in the sole identity block),

$$\langle O(\tau,x)O(0)\rangle_{\Delta}^{(\beta)} = \sum_{m\in\mathbb{Z}} \frac{1}{[(\tau+m\beta)^2+x^2]^{\Delta}}\,. \tag{5.62}$$

Note that this sum over images is intrinsically non-perturbative in $|x|/\beta \to 0$, as β is kept finite while the correlator on the non-compact bulk must be evaluated at parametrically large Euclidean time $\tau+m\beta$. In this case the sum over images can be carried out explicitly and gives rise to the double-twist spectrum.

Following this observation we now show how the double-twist spectrum arises by a sum over images of the non-compact correlator containing multi-stress tensor contributions as prescribed by (5.40). Let us assume that the two insertions are separated along τ only. Following the discussion in Subsection 5.5.1, the non-compact correlator contains only multi-stress tensor blocks, taking the form,[6]

$$\langle O(\tau)O(0)\rangle_{d,\Delta}^{(T)} = \sum_{n=0}^{\infty} \frac{a_n^{(T)}}{\beta^{2\Delta}} \left|\frac{\tau}{\beta}\right|^{nd-2\Delta}\,, \tag{5.63}$$

since in the limit $x\to 0$ the sum of the multi-stress tensor contributions of different spin at a given order d reduces to a power of τ times a collective constant $a_n^{(T)}$. As an OPE this expression is valid in some interval $0<|\tau|<\tau^*$.

In order to perform the image sum of (5.63) it is convenient to analytically continue the complex τ plane. Because of the absolute value in (5.63), we first focus on the case

[5] Note that the discussion in [103] is based on Fourier transforming the thermal conformal blocks in (5.15) to momentum space. This is a questionable procedure since the Fourier transform is performed over the full thermal cylinder $S^1\times\mathbb{R}^{d-1}$, while the thermal OPE is convergent only for $|x|<\beta$. In addition to this, the thermal OPE (being based on the flat space OPE) does not capture the contact terms in the 2-point function. In the current state it thus appears to us far from obvious that the analysis of [103] can be trusted in full generality.

[6] We comment later on how this discussion generalises when additional operators enter the OPE limit of the non-compact correlator.

$\tau > 0$ where,

$$\langle O(\tau)O(0)\rangle^{(+)}_{d,\Delta} = \sum_{n=0}^{\infty} \frac{a_n^{(T)}}{\beta^{2\Delta}} \left(\frac{\tau}{\beta}\right)^{nd-2\Delta} . \tag{5.64}$$

We then subsequently continue to $\tau \in \mathbb{C}$ so that (5.64) is valid in an annulus $0 < |\tau| < \tau^*$, with τ^* corresponding to the smallest radius at which there will be singularities in the complex τ plane. We then attempt to analytically continue beyond τ^*, to a function which we denote $G_+(\tau)$. We assume that there are no singularities of $G_+(\tau)$ lying on the positive real axis, that is, the only singularity of the non-compact correlator is the one at coincident points.

To extend the range of validity of the sum (5.64) we take G_+ to be composed of singular and non-singular parts,

$$G_+(\tau) = \sum_{\ell} W_\ell(\tau) + \sum_{n=0}^{\infty} \frac{\tilde{a}_n^{(T)}}{\beta^{2\Delta}} \left(\frac{\tau}{\beta}\right)^{nd-2\Delta} , \tag{5.65}$$

where the first sum includes all poles and branch points whose positions are governed by the parameters y_ℓ,

$$W_\ell(\tau) = \frac{1}{\tau^{2\Delta}} \frac{A_{(\ell)}}{\Big((\tau/\beta)^d - y_\ell\Big)^{\mu_{(\ell)}}} , \tag{5.66}$$

with non-negative real $\mu_{(\ell)}$. The second sum in (5.65) has an infinite radius of convergence.

With $G_+(\tau)$ known, the sum over images (5.40) is given by

$$\langle O(\tau)O(0)\rangle^{(\beta)}_{d,\Delta} = G_+(\tau) + \sum_{m=1}^{\infty} \left[G_+(\tau+m) + G_+(-\tau+m)\right] . \tag{5.67}$$

In Appendix F we give a detailed account of how to perform these sums over images. The resulting thermal correlator arising from the non-compact correlator (5.65) reads

$$\langle O(\tau)O(0)\rangle^{(\beta)}_{d,\Delta} = \sum_{n=0}^{\infty} \frac{a_n^{(T)}}{\beta^{2\Delta}} \left|\frac{\tau}{\beta}\right|^{nd-2\Delta} + \frac{1}{\beta^{2\Delta}} \sum_{p=0}^{\infty} \left[a^{(OO)}_{\mathrm{reg},\,p} + \sum_{\ell} a^{(OO)}_{(\ell)\,p}\right] \frac{\tau^{2p}}{\beta^{2p}} , \tag{5.68}$$

where we defined the coefficients

$$a^{(OO)}_{\mathrm{reg},\,p} = 2\sum_{n=0}^{\infty} \frac{\Gamma(2p+2\Delta-nd)}{(2p)!\,\Gamma(2\Delta-nd)}\, \zeta(2p+2\Delta-nd)\, \tilde{a}_n^{(T)} , \tag{5.69}$$

$$\begin{aligned} a^{(OO)}_{(\ell)\,p} = \frac{2A_{(\ell)}}{(2p)!} \sum_{j=0}^{\infty} \binom{\mu_{(\ell)}+j-1}{j} \Bigg[\\ (y_\ell)^j \left(d(\mu_{(\ell)}+j)+2\Delta\right)_{2p} \zeta(2p+d(\mu_{(\ell)}+j)+2\Delta, M^*_{(\ell)}) \\ + (-1)^j (-\tau_\ell)^{-\mu_{(\ell)}-j} \,(2\Delta-dj)_{2p} \Big(\zeta(2p+2\Delta-dj) - \zeta(2p+2\Delta-dj, M^*_{(\ell)})\Big)\Bigg] , \end{aligned} \tag{5.70}$$

as well as $M^*_{(\ell)} = \lceil |y_\ell|^{1/d} \rceil$. The first sum in (5.68) contains the multi-stress tensor spectrum (5.63), left untouched by the sum over images. This result justifies directly matching the ambient correlator with the non-compact 2-point function $\langle O(\tau, x)O(0)\rangle^{(T)}_{d,\Delta}$. Furthermore, through the second sum this expression (5.68) provides a prediction for the double-twist coefficients, taking as an input the multi-stress tensor coefficients $a_n^{(T)}$ and the singularities of the analytically continued non-compact correlator (in particular, their positions, orders and the factors $A_{(\ell)}$). Although this computation was carried out in the limit $x \to 0$, these same techniques can be applied in the case of non-vanishing x, as well as for theories where space-like directions are compact.

As mentioned, non-perturbative effects $\sim e^{-\beta\omega}$ in momentum space may yield additional regular and singular contributions besides the multi-stress tensor operators in the non-compact correlator (5.63). In that case this computation can be repeated without obstructions. The form of the double-twist coefficients changes accordingly, while the structure of (5.68) is preserved. This suggests that under sum over images any operator entering the OPE limit of the non-compact correlator contributes to the double-twist coefficients in an analogous way to multi-stress tensors. We can conclude that the double-twist spectrum in the holographic Euclidean thermal 2-point function on the planar black hole arises from the sum over images of the non-compact position space correlator, and it may receive further contributions from possible non-perturbative pieces in momentum space, whose existence was not probed in our perturbative computation.

Let us now examine how these results apply to the holographic thermal 2-point function on the planar black hole. Using the techniques of Subsection 5.5.1, for $\Delta = \frac{3}{2}$ and $d = 4$ we computed the momentum space correlator (5.36) up to order $O\left(k\beta\right)^{-360}$, from which we extracted the first 91 coefficients $a_n^{(T)}$ in (5.63). The asymptotic growth at large n is captured by $a_n^{(T)} \approx (-1)^n 4^n$, meaning that the radius of convergence of this non-compact correlator is $1/\sqrt{2} \simeq 0.707$ and that the four closest singularities to the origin on the complex τ-plane are the four roots of $1 + 4\tau^4 = 0$. We were however unable to robustly subtract this singular behaviour from the series obtained to order $O\left(|x|/\beta\right)^{-360}$, so as to attain an infinite radius of convergence for the remainder as in equation (5.65), and subsequently use (5.68) to make predictions on the double-twist coefficients.

The physical interpretation for these singularities in the complex τ-plane is not currently clear. Similar singularities have appeared in the large$-\Delta$ eikonal approximation of holographic correlators on Lorentzian AdS black holes in [95], and they signal the presence of a singularity beyond the horizon. Our computations are carried out at finite Δ and they may provide important information about the signature of a black hole singularity in the dual correlators. We also note that this provide the first fully-fledged explicit example where the radius of convergence of the thermal OPE in this setup is less than β, as conjectured in the analysis of [25].

For our purposes, an interesting question is how to account for these double-twist contributions using ambient invariants. For CFTs in thermal states dual to thermal AdS and BTZ, the ambient space is a quotient of $(d+2)$-dimensional Minkowski space (being each ALAdS$_{d+1}$ slice simply a quotient of Euclidean AdS). One can check that the double-twist spectrum arises automatically from the sum over the distinct geodesics that connect the same two nullcone points, which is implicit in the prescription (4.16). In particular, the ambient correlator is equal to a sum of terms $\widetilde{X}_{12}^{-\Delta}$, each evaluated on one among the infinite distinct ambient geodesics that wrap the thermal circle, each characterised by a different winding.

However, as we detail in Appendix G there is a unique geodesic on the AdS planar black hole that connects two given boundary points, and hence there is only one ambient geodesic that connects a given pair of points on the nullcone. This entails that there are no periodic ambient geodesics to sum over, suggesting that double-twist contributions must be described by a novel class of invariants on the ambient space. We leave this interesting question to future work.

5.6 The $d = 2$ case and the BTZ black hole

As a simpler example we would like to study thermal correlators in $d = 2$ Euclidean CFTs using the ambient formalism. Following the literature, we parametrise the thermal cylinder with coordinates $x^i = (\tau, \phi)$ with $0 \leq \tau < \beta$ and consider for now a non-compact ϕ. The states we are interested in are characterised by a stress tensor VEV of the form (5.2), which we write in $d = 2$ as

$$\langle T_{ij}\rangle\, dx^i dx^j = \frac{\pi}{4G\,\beta^2}\left(d\tau^2 + d\phi^2\right). \tag{5.71}$$

The ambient space one must use is then a foliation of Euclidean BTZ black holes,

$$\tilde{g} = -ds^2 + s^2\left[\frac{dr^2}{r^2 - r_H^2} + \left(r^2 - r_H^2\right) d\tau^2 + r^2 d\phi^2\right], \tag{5.72}$$

with $r_H = 2\pi/\beta$ and $r > r_H$. The radial r coordinate is related to the ambient ρ coordinate by

$$r = \frac{1 - 2\rho^2}{2\sqrt{-2\rho}}\, r_H\,. \tag{5.73}$$

Let us now study scalar 2-point functions in such thermal state using this ambient space. Leveraging translational symmetries and turning to the ambient gauge $\widetilde{X} = (t, \rho, \tau, \phi)$ as in (2.1), we place the insertion points at

$$\widetilde{X}_1 = (t_1, 0, 0, 0)\,, \qquad \widetilde{X}_2 = (t_2, 0, \tau, \phi)\,. \tag{5.74}$$

Through the coordinate transformation

$$X^0 = \frac{s\,r}{r_H}\cosh(r_H\phi), \qquad X^1 = \frac{s\,r}{r_H}\sinh(r_H\phi)\,, \tag{5.75}$$

$$X^2 = s\sqrt{\frac{r^2}{r_H^2}-1}\,\cos(r_H\tau) \qquad X^3 = s\sqrt{\frac{r^2}{r_H^2}-1}\,\sin(r_H\tau)\,, \tag{5.76}$$

one can show the ambient metric (5.72) describes the geometry of Minkowski space on the whole region in the causal future of the origin $X^M = 0$. As we discussed in Section 2.2, any 4-dimensional ambient space is locally diffeomorphic to Minkowski space. This entails that all ambient curvature invariants are identically vanishing, and the only non-trivial building block is the geodesic distance square $\widetilde{X}_{12}$.

Geodesics on this geometry are simply straight lines on Minkowski. The boundary conditions (5.74) fix the integration constants, yielding

$$X^0(\lambda) = \frac{1}{2}\Big[t_0 - t_0\lambda + t_1\lambda\cosh(r_H\phi)\Big]\,, \qquad X^1(\lambda) = \frac{1}{2}t_1\lambda\sinh(r_H\phi)\,, \tag{5.77}$$

$$X^2(\lambda) = \frac{1}{2}\Big[t_0 - t_0\lambda + t_1\lambda\cos(r_H\tau)\Big]\,, \qquad X^3(\lambda) = \frac{1}{2}t_1\lambda\sin(r_H\tau)\,. \tag{5.78}$$

We thus obtain the invariant

$$\widetilde{X}_{12} = \frac{t_0 t_1}{2}\Big[\cosh(r_H\phi) - \cos(r_H\tau)\Big]. \tag{5.79}$$

Contrarily to Thermal AdS, there is only one geodesic connecting any pair of insertion points on the thermal cylinder for non-compact ϕ, regardless of the periodicity in τ. This is analogous to what happens with the higher dimensional black brane as discussed in Subsection 5.5.3 and Appendix G. The resulting ambient 2-point function is therefore

$$\langle O(\tau,\phi)O(0)\rangle^{(\beta)}_{d=2,\Delta} = \frac{1}{\beta^{2\Delta}}\frac{C_\Delta}{[\cosh\frac{2\pi\phi}{\beta} - \cos\frac{2\pi\tau}{\beta}]^\Delta}\,. \tag{5.80}$$

Expanding this correlator in the OPE limit, only negative even powers of β appear, describing the multi-stress tensor spectrum.

The non-singular BTZ black hole geometry is however periodic in ϕ with period 2π, and on the corresponding ambient space one has an infinite number of geodesics. Their form is the same as in equations (5.77)-(5.78) with $\phi \to \phi + 2\pi m$, where $m \in \mathbb{Z}$ parametrises the winding around the ϕ circle. There is an invariant analogue to (5.79) for each such geodesic, yielding a correlator of the form,

$$\langle O(\tau,\phi)O(0)\rangle^{(\beta)}_{d=2,\Delta} = \frac{1}{\beta^{2\Delta}}\sum_{m=-\infty}^{\infty}\frac{C_\Delta}{[\cosh\frac{2\pi(\phi+2\pi m)}{\beta} - \cos\frac{2\pi\tau}{\beta}]^\Delta}\,. \tag{5.81}$$

This expression matches the corresponding holographic result [104, 105], and represents another successful test of the ambient formalism.

Chapter 6

CFTs on squashed spheres

Squashed spheres are a class of non-conformally flat and non-Einstein manifolds and represent an interesting case of study to make predictions using the ambient space formalism. Previous works on CFTs on squashed spheres include [26–33].

Let us summarise the content of this chapter. In Section 6.1 we review the geometry of squashed spheres and their symmetries, and in Section 6.2 we study how such symmetries constrain the form of CFT 1- and 2-point functions on these metric backgrounds. In Section 6.3 we set up the appropriate class of ambient spaces to be used in this case, and in Section 6.4 we solve the geodesic equations on such ambient spaces. In Section 6.5 we build the relevant ambient curvature invariants and we assemble them into an ambient scalar 2-point function, allowing us to point out a mismatch with the Ansatz for the holographic correlator on squashed spheres considered in [27]. We conclude this chapter with interesting open questions concerning the classification of observables of CFTs on squashed spheres and the potential of the ambient space formalism for this class of theories.

6.1 Geometry

In odd $d = 2k+1$ their geometry can be conveniently written in terms as a Hopf fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^d_\alpha \longrightarrow \mathbb{CP}^k$ with metric

$$ds^2 = g_{\mathbb{CP}^k} + \frac{d+1}{1+\alpha}\left(d\psi + \frac{A_{\mathbb{CP}^k}}{d+1}\right)^2 , \tag{6.1}$$

with $g_{\mathbb{CP}^k}$ normalized such that $R^{\mathbb{CP}^k}_{ab} = g^{\mathbb{CP}^k}_{ab}$, and where $A_{\mathbb{CP}^k}$ is related to the Kähler form J on $\mathbb{CP}^k$ by $J = dA_{\mathbb{CP}^k}$. The real parameter α defines the squashing. For $\alpha = 0$ we recover the round sphere $\mathbb{S}^d$, while in the limiting case $\alpha \to \infty$ we obtain the cylinder $\mathbb{R} \times \mathbb{S}^{2k}$.

The complex projective space $\mathbb{CP}^k$ has $SU(k+1) = SU\left(\frac{d+1}{2}\right)$ as isometry group. Thus for generic α, squashed spheres have $\left(\frac{d+1}{2}\right)^2$ independent isometries generating $SU\left(\frac{d+1}{2}\right) \times U(1)$. In the round sphere case $\alpha = 0$, one has an enhancement of the isometry group to $\mathrm{SO}(d+1)$, i.e. $\frac{d(d+1)}{2}$ isometries.

Fixing to $d = 3$ for concreteness, the geometry reads

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 + \frac{1}{1+\alpha}\left(d\psi + \cos\theta d\varphi\right)^2 , \tag{6.2}$$

where $0 \leq \theta < \pi$, $0 \leq \varphi < 2\pi$, $0 \leq \psi < 4\pi$ are the Euler angles. [1] As anticipated, these spaces are not Einstein but they are close to being Einstein in the sense that one can recast their Ricci tensor as

$$R_{ij}(\theta) = \frac{R}{3} g_{ij}(\theta) + H_{ij}(\theta), \tag{6.3}$$

where H_{ij} is traceless, so that they have constant curvature $R = \frac{3+4\alpha}{2(1+\alpha)}$. Their Cotton tensor is non-vanishing, hence they are non-conformally flat. From the results in Subsection 2.3.1, this means that regardless of the CFT state, their ambient space locally is not flat space.

For generic α, the squashing breaks $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ down to $\mathrm{SU}(2)_L \times \mathrm{U}(1)_R$. Thus $d = 3$ squashed spheres are endowed with only four out of the six isometries of round spheres. These isometries can be written as

$$K_1 = -\sin\varphi\partial_\theta + \frac{\cos\varphi}{\sin\theta}\partial_\psi - \cot\theta\cos\varphi\partial_\varphi, \tag{6.4}$$

$$K_2 = \cos\varphi\partial_\theta + \frac{\sin\varphi}{\sin\theta}\partial_\psi - \cot\theta\sin\varphi\partial_\varphi, \tag{6.5}$$

$$K_3 = \partial_\varphi, \tag{6.6}$$

$$K_4 = \partial_\psi, \tag{6.7}$$

where K_4 generates the residual $U(1)_R$ symmetry. No additional conformal Killing vector is present for a generic squashing α.

6.2 Ward Identities and correlators

The Ward Identities associated to the vectors $K_1 \ldots K_4$ fix scalar 1-point functions of quasi-primary operators to constants, while a dependence on θ is allowed for 1-point functions of operators with spin. Defining the invariant 1-form

$$\zeta = d\psi + \cos\theta d\varphi, \qquad \mathcal{L}_{K_i}\zeta = 0\,, \tag{6.8}$$

[1] Performing the limit $\alpha \to 0$ on the metric (6.2) yields a degenerate geometry. We can attain a non-degenerate metric by first unwrapping the fibred $\mathbb{S}^1$ so that $0 \leq \psi < \infty$, and subsequently rescaling it as $\widetilde{\psi} = \frac{\psi}{\sqrt{1+\alpha}}$.

a generic spin-1 1-point function can be written as

$$\langle O_i(\theta)\rangle_\alpha = u_1\,\zeta_i\,, \tag{6.9}$$

and a generic spin-2 1-point function can be written as

$$\langle O_{ij}(\theta)\rangle_\alpha = u_2\,\zeta_i\,\zeta_j + u_2^{(\mathrm{tr})} g_{(0)ij}\,, \tag{6.10}$$

where the constants u_2 and $u_2^{(\mathrm{tr})}$ are fixed by dynamics. Note that no antisymmetric part is allowed. $\langle O_{ij}(\theta)\rangle_\alpha$ has a non-trivial dependence on θ, which entails that an infinite tower of descendants can be constructed acting with three-dimensional covariant derivatives on the squashed sphere.

The form of scalar 2-point functions of quasi-primary operators is partially fixed by the Ward Identities

$$\left[\mathcal{L}_{K_i}(\theta_1,\varphi_1,\psi_1) + \mathcal{L}_{K_i}(\theta_2,\varphi_2,\psi_2)\right]\langle O_1 O_2\rangle_\alpha = 0\,. \tag{6.11}$$

Leveraging rotational symmetry along ψ and φ, we can move the first insertion to lie at $\psi_1 = \varphi_1 = 0$ and the second insertion to be at $\psi_2 = \psi$, $\varphi_2 = \varphi$. Adopting the basis of cross-ratios from [27], scalar 2-point correlators must be of the form

$$\langle O_1(\theta_1,0,0)\,O_2(\theta_2,\psi,\varphi)\rangle_\alpha = F(v_1,v_2), \tag{6.12}$$

where

$$v_1 = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\psi+\varphi}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\psi-\varphi}{2}\,, \tag{6.13a}$$

$$v_2 = \frac{1}{2}\left(1 + \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\varphi\right). \tag{6.13b}$$

Although it will not be pursued in the following sections, an alternative interesting perspective is to consider the squashing as a metric variation and use conformal perturbation theory [30]. Writing the squashed sphere metric as a power expansion in small α,

$$g_{(0)} = g + \alpha\, h + \ldots, \tag{6.14}$$

where g is the metric on the round $\mathbb{S}^3$, we can interpret a small squashing as the deformation of the boundary action by

$$S[g_{(0)},\phi] = S[g,\phi] + \alpha\,\delta S[g,\phi] + \ldots = S[g,\phi] - \frac{\alpha}{2}\int d^3x\sqrt{g}h^{\mu\nu}T_{\mu\nu} + \ldots. \tag{6.15}$$

Assuming the CFT partition function and observables are analytic in α, the 2-point functions take the following structure,

$$\begin{aligned}\langle O(x_1)O(x_2)\rangle_{g_{(0)}} &= \int D\phi\, O(x_1)O(x_2)e^{-S[g_{(0)},\phi]} \\ &= \langle O(x_1)O(x_2)\rangle_g - \frac{\alpha}{2}\int d^3x\sqrt{g}\,\langle \widetilde{T}(x)O(x_1)O(x_2)\rangle_g + \ldots,\end{aligned} \tag{6.16}$$

with $\widetilde{T}(x) = h^{\mu\nu}(x)T_{\mu\nu}(x)$ dimension 3 operator. This turns a 2-point function on a squashed sphere into a sum of higher point functions on a round sphere involving additional stress tensors, averaged on the round sphere.

6.3 The ambient setup

We intend to study CFTs on squashed spheres in states with a non-vanishing stress tensor VEV. To describe the multi-stress tensor contributions of their correlators using the ambient formalism we have to identify suitable bulks with a squashed 3-sphere as a boundary and a non-vanishing holographic stress tensor VEV. Four-dimensional AdS Taub-NUT and -bolt spaces allow one to study a wide class of such states. Their metric reads [106]

$$ds^2 = \frac{dr^2}{V(r)} + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + 4n^2 V(r)\left(d\psi + \cos\theta d\varphi\right)^2, \tag{6.17}$$

where we defined

$$V(r) = \frac{r^2 + n^2 - 2mr + (r^4 - 6n^2r^2 - 3n^4)}{r^2 - n^2}. \tag{6.18}$$

The nut parameter is related to the squashing of the boundary by $n = (2\sqrt{\alpha+1})^{-1}$, and the boundary is reached for $r \to \infty$. The holographic stress tensor 1-point function in such geometries is parametrised by the mass parameter m as

$$u_2 = -\frac{3}{8\pi}\frac{m}{1+\alpha}, \qquad u_2^{(\mathrm{tr})} = \frac{m}{8\pi}, \tag{6.19}$$

and as such they can be used to describe any state where

$$\frac{u_2}{u_2^{(\mathrm{tr})}} = -\frac{3}{1+\alpha}. \tag{6.20}$$

As an illustrative example, the CFT state that we consider is characterised by the stress tensor VEV associated to the self-dual AdS Taub-NUT geometry with no conical singularities. The ambient space we need is then (2.8) with metric (6.17) as $(d+1)-$dimensional hyperbolic slices, and with the choice of the mass parameter $m = \frac{\alpha}{2(1+\alpha)^{3/2}}$. For later convenience, we write explicitly the stress tensor VEV $\langle T_{ij}\rangle_\alpha = \frac{3}{16\pi}g_{(3)ij}$, with

$$\begin{aligned} g_{(3)ij}dx^i dx^j &= \frac{\alpha}{3(\alpha+1)^{3/2}}\left[d\theta^2 - \frac{2d\psi^2}{1+\alpha} - \frac{4\cos\theta}{1+\alpha}d\psi d\varphi - \frac{(\alpha+3)\cos 2\theta - \alpha + 1}{2(\alpha+1)}d\varphi^2\right] \\ &= \frac{\alpha}{3}\left[d\theta^2 - 2d\psi^2 - 4\cos\theta\, d\psi d\varphi - \frac{3\cos 2\theta + 1}{2}d\varphi^2\right] + O(\alpha^2), \end{aligned} \tag{6.21}$$

We work perturbatively in small α. To avoid cluttering in the expressions below we fix $\theta_1 = 0$, rename $\theta_2 = \theta$ and define $\chi = (\varphi + \psi)/2$. The two insertion points on the ambient space are thus $\widetilde{X}_1 = (s_1, r_1, 0, 0, 0)$ and $\widetilde{X}_2 = (s_2, r_2, \theta, \psi, \varphi)$, where the limit to the lightcone $s_i, r_i \to \infty$ with fixed $s_i/r_i = t_i = 1$ is understood.

6.4 Geodesics

Let us first solve the geodesic equations on this geometry between $\widetilde{X}_1$ and $\widetilde{X}_2$ so as to obtain the invariant $\widetilde{X}_{12}$. In this case it is convenient to compute the (divergent) geodesic length L_{AdS} on a fixed hyperbolic slice and then use the relation (4.19) to find the finite ambient invariant $\widetilde{X}_{12}$.

We hence consider the 4-dimensional self-dual AdS Taub-NUT metric (6.17) with $m = \frac{\alpha}{2(1+\alpha)^{3/2}}$. We would like to study geodesics on this background with endpoints on the boundary $r \to \infty$ at the generic points $x_1 = (\theta_1, 0, 0)$ and $x_2 = (\theta_2, \psi, \varphi)$ corresponding to the values of the affine parameter $\lambda = 0$ and $\lambda = 1$ respectively. For simplicity we restrict to $\theta_1 = \theta_2 = 0$ and fix the dynamics along θ.

The boundary isometries (6.4)-(6.7) are also bulk isometries and one can use them to partially integrate the bulk geodesic equations. From the integrals of motion related to translational symmetries (6.6)-(6.7) along φ and ψ one obtains the first-order equations

$$\dot{\varphi} = \frac{A_\varphi}{n^2 - r^2}\,, \tag{6.22}$$

$$\dot{\psi} = -\frac{A_\psi (n + r)}{4n^2(n - r)\left(-3n^2 + 2nr + r^2 + 1\right)} - \frac{A_\varphi}{n^2 - r^2}\,, \tag{6.23}$$

where A_ψ and A_φ are the constants of motions. Using equations (6.22)-(6.23) the 4-velocity constraint in the bulk $\dot{x}_\mu \dot{x}^\mu = L^2_{AdS}$ can be expanded as

$$\begin{aligned} &4n^2(n + r(\lambda))\dot{r}(\lambda)^2 + nA_\psi^2 + 4\left(1 - 3n^2\right) n^3 L^2_{\text{AdS}} + \\ &r(\lambda)\left(A_\psi^2 - 4n^2 L^2_{\text{AdS}} r(\lambda)(n + r(\lambda)) + 4n^2\left(5n^2 - 1\right) L^2_{\text{AdS}}\right) = 0\,. \end{aligned} \tag{6.24}$$

Instead of solving directly this equation in $r(\lambda)$, we find it more convenient to use it to simplify the form of the second-order radial equation by removing the $\dot{r}(\lambda)^2$ term. The radial equation that we are going to solve then reads

$$(n + r(\lambda))^2 \ddot{r}(\lambda) - L^2_{\text{AdS}}\left(n - 4n^3 + r(\lambda)(n + r(\lambda))^2\right) = 0\,. \tag{6.25}$$

The strategy is to first solve this radial equation. To regulate the divergence in the geodesic distance as one approaches the boundary we use boundary conditions $r(0) = r(1) = R$ with a radial regulator R to be eventually set to infinity.

One then plugs the solution $r(\lambda)$ into the angular equations and solves them subject to the Dirichlet boundary conditions at x_1 and x_2. This fully determines the trajectory.

Finally, substituting the onshell $r(\lambda)$ and A_ψ into (6.24) allows one to find the value of L_{AdS} in terms of the boundary points x_1 and x_2.

Note that A_φ does not appear in (6.24), meaning that to obtain the geodesic distance L_{AdS} it is sufficient to solve the reduced ODE

$$\dot{\chi}(\lambda) + \frac{A_\psi(n + r(\lambda))}{8n^2(n - r(\lambda))\left(-3n^2 + 2nr(\lambda) + r(\lambda)^2 + 1\right)} = 0\,, \tag{6.26}$$

in terms of $\chi(\lambda) = \frac{\psi(\lambda)+\varphi(\lambda)}{2}$ with boundary conditions $\chi(0) = 0$ and $\chi(1) = \chi$.

The explicit solution to (6.25) can be found in terms of inverse elliptic functions. However we are interested in an $\alpha \to 0$ expansion of the geodesic distance, which corresponds to an $n \to \frac{1}{2}$ expansion in the current parametrisation. We thus expand the unknown functions and integration constants as

$$r(\lambda) = \sum_{k=0}^{\infty}\left(n - \frac{1}{2}\right)^k r_k(\lambda)\,, \qquad \chi(\lambda) = \sum_{k=0}^{\infty}\left(n - \frac{1}{2}\right)^k \chi_k(\lambda)\,, \tag{6.27}$$

$$A_\psi = \sum_{k=0}^{\infty}\left(n - \frac{1}{2}\right)^k A_\psi^{(k)}\,, \qquad L_{\mathrm{AdS}} = \sum_{k=0}^{\infty}\left(n - \frac{1}{2}\right)^k L_{\mathrm{AdS}}^{(k)}\,. \tag{6.28}$$

At leading order $k = 0$ the bulk is simply global Euclidean AdS_4 and the boundary is a round sphere. Following this integration scheme, one finds

$$r_0(\lambda) = \frac{(8 - 8\cos\chi)^{-\lambda}\left[64^\lambda R^{2\lambda+1}(1 - \cos\chi)^{2\lambda} + 8R^{3-2\lambda}(1 - \cos\chi)\right]}{1 + 8R^2(1 - \cos\chi)}\,, \tag{6.29}$$

$$\chi_0(\lambda) = -\frac{\chi \arctan\left[\frac{8\sin\chi(\cos\chi-1)\left(R^{4\lambda}(8-8\cos\chi)^{2\lambda}-1\right)}{R^{4\lambda-2}(8-8\cos\chi)^{2\lambda}(8R^2(\cos\chi-1)\cos\chi-1)-8(\cos\chi-1)((8R^2-1)\cos\chi-8R^2)}\right]}{\arctan\left[\frac{\sin\chi(-32R^4(\cos 2\chi-4\cos\chi)-96R^4+1)}{\cos\chi\left(32R^4(-4\cos\chi+\cos 2\chi+1)+(1-8R^2)^2\right)+16R^2}\right]}\,, \tag{6.30}$$

$$L_{\mathrm{AdS}}^{(0)} = \log\left[8R^2(1 - \cos\chi)\right]\,, \tag{6.31}$$

$$A_\psi^{(0)} = -\frac{\sqrt{32R^4\cos 2\chi - 16R^2\cos\chi - 32R^4 + 16R^2 + 1}}{4\left(8R^2\cos\chi - 8R^2 - 1\right)} \times \tag{6.32}$$

$$\operatorname{arctanh}\left[\frac{\left(-8R^2\cos\chi + 8R^2 - 1\right)\sqrt{32R^4\cos 2\chi - 16R^2\cos\chi - 32R^4 + 16R^2 + 1}}{64R^4(\cos\chi - 1)\cos\chi + 1}\right]^{-1}.$$

The solution at first order is rather lengthier and we avoid displaying it here. The first order geodesic distance at leading order in $R \to \infty$ takes however a particularly compact form,

$$L_{\mathrm{AdS}}^{(1)} = \left[4(\pi - \chi)\sin^3\frac{\chi}{2} + \cos\frac{\chi}{2} + 3\cos\frac{3\chi}{2}\right]\sec^3\frac{\chi}{2}\,. \tag{6.33}$$

Knowing $L^{(0)}_{\text{AdS}}$ and $L^{(1)}_{\text{AdS}}$, one can compute the invariant $\widetilde{X}_{12}$ to first order in α through (4.19) setting $R \to \infty$. It reads

$$\widetilde{X}_{12} = 8(1-v_1)\left[1+\alpha\left(\frac{1-3v_1}{1+v_1}+(\arccos v_1-\pi)\left(\frac{1-v_1}{1+v_1}\right)^{\frac{3}{2}}\right)+O(\alpha)^2\right]. \qquad (6.34)$$

In the limit $\theta_1 \to \theta_2$ we are considering here the cross ratios reduce to $v_1 = \cos\chi$ and $v_2 = 1$. The fact that v_2 trivialises means that only v_1 can appear in invariants contributing to correlators such as (6.34) in this limit.

6.5 Invariants and ambient 2-point functions

Using the techniques developed in Appendix B one can straightforwardly compute the parallel transported vector $\hat{T}_1$ from $\widetilde{X}_1$ to $\widetilde{X}_2$. In particular, to order $O(\alpha^0)$ the hyperbolic sections are global AdS$_4$ spaces and the ambient geometry is simply Minkowski spacetime. Thus, to find $\hat{T}_1$ to this order in α it is sufficient to take the Euler vector in Minkowski $X^M\partial_M$ evaluated at the point $\widetilde{X}_1$, and make a transformation $X^M \to \widetilde{X}^M = (s, r, \theta, \psi, \varphi)$ to the ambient coordinates. This change of coordinates reads

$$X_0 = t\left(1-\frac{\rho}{2}\right), \qquad (6.35a)$$

$$X_1 = t\left(1+\frac{\rho}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi-\psi}{2}\right), \qquad (6.35b)$$

$$X_2 = t\left(1+\frac{\rho}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi-\psi}{2}\right), \qquad (6.35c)$$

$$X_3 = t\left(1+\frac{\rho}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi+\phi}{2}\right), \qquad (6.35d)$$

$$X_4 = t\left(1+\frac{\rho}{2}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\psi+\phi}{2}\right), \qquad (6.35e)$$

where the ambient t and ρ are related to the AdS Taub-NUT radial coordinate (at zero-th order in α) by

$$t = \frac{s}{4\left(\sqrt{r^2-\frac{1}{4}}+r\right)}, \qquad (6.36)$$

$$\rho = -8\left(\sqrt{r^2-\frac{1}{4}}+r\right)^2. \qquad (6.37)$$

The resulting transported $\hat{T}_1$ is consequently

$$\begin{aligned}\hat{T}_1 = &\frac{\cos\frac{\theta}{2}\cos\chi+1}{2r}\partial_s + \frac{\cos\frac{\theta}{2}\cos\chi-1}{64r}\partial_r \\ &- 2\sin\frac{\theta}{2}\cos\chi\partial_\theta - \sec\frac{\theta}{2}\sin\chi\partial_\psi - \sec\frac{\theta}{2}\sin\chi\partial_\varphi + O(\alpha).\end{aligned} \tag{6.38}$$

From equations (4.4), (4.6) and (4.7) the non-vanishing components of the ambient Riemann read

$$\begin{aligned}\widetilde{R}_{rirj} &= -\frac{3t^2}{2r^5}g_{(3)ij} + O(\alpha^2), \\ \widetilde{R}_{rijk} &= \frac{t^2}{r^3}\left(\nabla_k g_{(3)ij} - \nabla_j g_{(3)ik}\right) + O(\alpha^2), \\ \widetilde{R}_{ijkl} &= \frac{3t^2}{2r}\left[g_{(0)ik}g_{(3)jl} + g_{(0)jl}g_{(3)ik} - (l \leftrightarrow k)\right] + O(\alpha)^2.\end{aligned} \tag{6.39}$$

These ingredients can be assembled to form the ambient curvature invariants that enter scalar correlators. As we showed in Section 4.2 the ambient formalism predicts that the single-stress tensor contributions to scalar correlators are fully fixed by the leading geodesic distance $(\widetilde{X}_{12})^{-\Delta}$. Since $\langle T_{ij}\rangle_\alpha = O(\alpha)$, the leading curvature invariants are of order $O(\alpha)^2$. As expected due to the infinite tower of non-trivial descendants of $:T^2:$, one can construct an infinite number of independent ambient curvature invariants at this order. However, in a short distance expansion the dominant contributions can be identified as the three invariants accounting for the three independent $\sim \langle :T^2: \rangle$ contributions, while the others include a higher and higher number of covariant derivatives and are thus subleading. Focusing on the dominant ones, a suitable basis is provided by the three curvature invariants

$$(\nabla\widetilde{\text{Riem}})^2 = \frac{42\alpha^2}{t^6} + O(\alpha)^3, \tag{6.40}$$

$$\widetilde{\mathcal{R}}^{(1)}_{AC}\,\widetilde{\mathcal{R}}^{(1)AC} = 18\alpha^2\sin^2\frac{\theta}{2}\left[3-\cos\theta-2\cos^2\frac{\theta}{2}\cos2\chi\right]^2 + O(\alpha)^3, \tag{6.41}$$

$$\widetilde{\mathcal{R}}^{(0)}_{AC}\,\widetilde{\mathcal{R}}^{(2)AC} = \frac{3}{2}\alpha^2\left[-3+5\cos\theta-2\cos^2\frac{\theta}{2}\cos2\chi\right]\left[3-\cos\theta-2\cos^2\frac{\theta}{2}\cos2\chi\right]^2 + O(\alpha)^3, \tag{6.42}$$

where the tensors $\mathcal{R}^{(r)}$ are defined in (4.14). Using the expressions for the ambient Riemann components in (6.39) one can explicitly check that these invariants do not contain derivatives of the stress tensor VEV, ensuring they describe the independent $:T^2:$ blocks.

Note that weight-0 invariants of the form (4.14) constructed as chains of tensors $\mathcal{R}^{(r)}$ are not sufficient to account for all the three $:T^2:$ blocks, as opposed to the finite temperature example in Chapter 5 where they can be used as a basis for *any* multi-stress tensor contribution $:T^n:$ as we showed. The difference resides in the fact that in the present case the dimension of the CFT background is odd. As discussed in the end of Section 4.2 this causes a number of ambient invariants to be either divergent or vanishing in the limit to the nullcone, as it is the case for instance for $\widetilde{\mathcal{R}}^{(0)}_{AC}\widetilde{\mathcal{R}}^{(0)AC}$ here.

Following the prescription in Section 4.3, the scalar 2-point function in this background and state takes the form

$$\langle O(X_1)O(X_2)\rangle_\alpha = \frac{C_\Delta}{(\widetilde{X}_{12})^\Delta}\Bigg[1+\frac{3}{2}\alpha^2\Bigg[28c_1-\Big(2\cos^2\frac{\theta}{2}\cos 2\chi+\cos\theta-3\Big)^2\times$$
$$\Bigg(6c_2(\cos\theta-1)+c_3\Big(2\cos^2\frac{\theta}{2}\cos 2\chi-5\cos\theta+3\Big)\Bigg)+\dots\Bigg]+O(\alpha^3)\Bigg], \tag{6.43}$$

where c_1, c_2 and c_3 are theory-dependent constants, while the dots denote subleading terms in the short distance limit. These contributions can be constructed in a similar way using invariants containing more ambient covariant derivatives, while multi-stress tensor blocks can be accounted for with invariants of higher order in the ambient Riemann.

Given the form (6.34) of the invariant $\widetilde{X}_{12}$, in the case of $\theta = 0$ we are able to explicitly write the form of the scalar 2-point function to first order in α,

$$\langle O(\widetilde{X}_1)O(\widetilde{X}_2)\rangle_\alpha = \frac{C_\Delta 8^{-\Delta}}{(1-v_1)^\Delta}\Bigg[1-\Delta\left(\frac{1-3v_1}{1+v_1}+(\arccos v_1-\pi)\left(\frac{1-v_1}{1+v_1}\right)^{\frac{3}{2}}\right)\alpha+$$
$$+O(\alpha)^2\Bigg]. \tag{6.44}$$

In [27] the same holographic scalar 2-point function was computed for $\Delta = 1$ by making an Ansatz on the form of the bulk propagator. In the limit $\theta_1 = \theta_2 = 0$, the expression provided in [27] reduces to the correlator on the round sphere

$$\langle O(\widetilde{X}_1)O(\widetilde{X}_2)\rangle_\alpha^{[Ansatz]} = \frac{2}{1-v_1}, \tag{6.45}$$

to all orders in α. Comparing it with (6.44) we note a clash with the predictions of the ambient space formalism. We observe that both (6.44) and (6.45) satisfy the correct kinematic constraints. We discuss some of the possible routes to discriminate between them in the next section.

6.6 Outlooks

There are several interesting directions one can take to further develop what presented in this chapter.

First of all, there is currently no available general treatment of 1-point functions of arbitrary spin on squashed spheres. In Section 6.2 we initiated the study of the allowed tensor structures for spin up to two using the invariant 1-form ζ. As an immediate consequence, a generalisation to arbitrary spin would allow one to study the OPE limit of two point functions on squashed spheres and the corresponding conformal blocks, similarly to what pursued in [25] for thermal 2-point functions.

It would also be interesting to develop the approach making use of conformal perturbation theory described at the end of Section 6.2. So far such perspective has only been adopted for the computation of the free energy corrections due to squashing in [30], and it appears especially promising to study the dynamics encoded in correlators on squashed spheres.

Note that both the OPE limit and conformal perturbation theory would be able to resolve the tension between the ambient 2-point function (6.44) and the proposed holographic correlator (6.45). Alternatively, to this aim it may be interesting to perform a numerical integration of the scalar propagator on the four-dimensional self-dual AdS Taub-NUT spacetime and extract the corresponding 2-point function. To our knowledge, this has never been carried out.

Regarding ambient correlators for theories on squashed spheres, an open direction concerns listing the ambient curvature invariants accounting for an arbitrary number of stress tensors and their descendants. In addition, from the discussion in Section 4.4 knowing the 2-point ambient blocks would straightforwardly allow one to study higher-point functions on squashed spheres, which are completely unexplored as of now. Similarly, any information from the ambient formalism related to spinning correlators would prove extremely useful in characterising these theories. We hope to return to these open questions in the future.

Chapter 7

Flat holography, the Beig-Schmidt gauge and the ambient space

As we discussed in the previous chapters, the ambient space is a Ricci-flat spacetime which can be locally characterised by the metric (2.1). Geometric quantities and fields on this $(d+2)$-dimensional geometry are related to QFT observables of a d-dimensional theory. From this perspective the ambient space has a manifest holographic flavour, and one may wonder whether it is able to provide a better understanding of holography for asymptotically flat spacetimes. Studying this connection will be the goal of this chapter.

Conventions. Unless specified otherwise, in this chapter as well as in Chapter 8 we adopt the most common indices notations in the flat holography literature. We indicate four-dimensional directions with lowercase Greek letters; three-dimensional indices are lowercase Latin letters; two-dimensional indices are denoted by capital Latin letters.

Before delving into flat holography, it is convenient to discuss the kind of spacetimes flat holography intends to address and their asymptotic structure. We fix $d+2 = 4$ spacetime dimensions. The conformal compactification of Minkowski spacetime is depicted in Figure 7.1. We consider the Minkowski metric in the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2\,, \tag{7.1}$$

and define the null coordinates $v = t + r$, $u = t - r$. The lines denoted by $\mathscr{I}^-$ and $\mathscr{I}^+$ indicate past and future null infinity, where null trajectories start and end. Different points on $\mathscr{I}^+$ ($\mathscr{I}^-$) are identified by $r \to \infty$ and $u = const$ ($v = const$). The locus i^0 represents spatial infinity, reached for $r \to \infty$ at constant finite t. It can be approached from $\mathscr{I}^+$ ($\mathscr{I}^-$) in the limit $u \to -\infty$ ($v \to \infty$). The loci $i^\pm$ are called past and future infinity and they are reached for $t \to \pm\infty$ and finite r; timelike trajectories start at i^- and end at i^+. For future convenience, we indicate the locus resulting from the limit on $\mathscr{I}^+$ to $u \to -\infty$ as $\mathscr{I}^+_-$, and similarly the limit on $\mathscr{I}^-$ to $v \to \infty$ as $\mathscr{I}^-_+$.

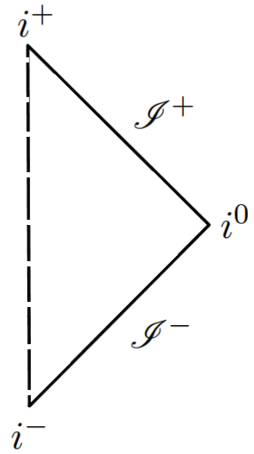

Figure 7.1: This figure displays the conformal compactification of Minkowski spacetime.

The class of spacetimes whose asymptotic structure is of this form are called *asymptotically Minkowski* (or *asymptotically flat*) *spacetimes*. We will see what the form of their metric is near $\mathscr{I}$ (Section 7.1) and near i^0 (Section 7.3). The key difference with Minkowski spacetime that will play a role in Chapter 8 is that i^0 is in general a singular locus in asymptotically Minkowski spacetimes. In particular, it becomes singular as soon as the spacetime exhibits a non-vanishing ADM mass, and in this case the result of a limit to spatial infinity depends on the direction of the limit itself [107–110].

Asymptotically locally flat spacetimes are a further generalisation, and their asymptotic structure may be significantly different. In particular, also $\mathscr{I}^-$ and $\mathscr{I}^+$ may exhibit singularities: either of them may not be a connected manifold and they may not extend up to $i^\pm$ and i^0. Conceptually they are considerably harder to describe holographically, to the same extent as ALAdS spacetimes where bulk matter back-reacts on the boundary structure (for instance, this is the case for matter corresponding to irrelevant deformations in the boundary theory [111–116]).

In the next section we review the Bondi gauge, a convenient set of coordinates to describe physics in a neighbourhood of null infinity. We also review the study of asymptotic symmetries at null infinity and their charges. In Section 7.2 we discuss current proposals of flat holography and the role these asymptotic symmetries play. We then turn to future, past and spatial infinity by discussing the properties of the Beig-Schmidt gauge in Section 7.3. Finally, in Section 7.4 we analyse the connections between the Beig-Schmidt gauge and the ambient space, and what they entail for flat holography.

7.1 The Bondi gauge and the BMS asymptotic symmetry group

A suitable chart to describe the gravitational field at null infinity is the Bondi gauge. Focusing for now on $\mathscr{I}^+$, one can define the so-called *retarded Bondi coordinates* (u, r, x^A) [117–119]. This is a gauge that one can choose in space-time with general asymptotics and it is adapted to a family of outgoing null surfaces $u = const$. The angular coordinates x^A, $A = 1, 2$ describe a space which is conformal to $\mathbb{S}^2$ [120], and we impose x^A to be constant along these null rays. Such conditions imply the metric conditions $g^{uu} = 0$ and $g^{uA} = 0$. Finally, the radial coordinate r varies along these light rays and it is chosen so that

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0, \tag{7.2}$$

implying that

$$\det g_{AB} = r^4 F(u, x) \tag{7.3}$$

for some function $F(u, x)$. Under these conditions, r is effectively the areal coordinate. Overall, this means that a general metric in the Bondi gauge has the form

$$ds^2 = g_{uu}\, du^2 + 2\, g_{ur}\, du\, dr + 2\, g_{uA}\, du\, dx^A + g_{AB}\, dx^A\, dx^B\,, \tag{7.4}$$

A similar gauge called *advanced Bondi coordinates* (v, r, x^a) can be adopted near $\mathscr{I}^-$. Following the conventions of Strominger [121], the Bondi gauge at $\mathscr{I}^-$ is formally obtained by applying the transformation $u \mapsto -v$ to (7.4) and all subsequent equations.

Assuming the spacetime is asymptotically Minkowski and that the large-r expansion does not contain logarithmic terms[1], Einstein's equations $R_{\mu\nu} = 0$ are solved by

$$g_{uu} = -1 + \frac{2m}{r} + \frac{\phi}{r^2} + O(r^{-3})\,, \tag{7.5a}$$

$$g_{ur} = -1 + \frac{1}{16r^2}\, C_{AB}C^{AB} + O(r^{-3})\,, \tag{7.5b}$$

$$g_{uA} = \frac{1}{2}\nabla^B C_{AB} + \frac{2}{3r}\left(N_A + u\,\partial_A m - \frac{3}{32}\partial_A\left(C_{BC}C^{BC}\right)\right) + O(r^{-2})\,, \tag{7.5c}$$

$$g_{AB} = r^2\,\gamma_{AB} + r\, C_{AB} + \frac{1}{4}\gamma_{AB}\, C_{CD}C^{CD} + O(r^{-1})\,. \tag{7.5d}$$

Here we are considering a spherical section of $\mathscr{I}$, with metric γ_{AB}. The quantities m, N_A and C_{AB} are called the Bondi mass aspect, the angular momentum aspect and the

[1]The most general solution of Einstein equations in this setting contains terms of the form $r^i \log r^j$ even if the initial characteristic data do not [122, 123]. Setting them to zero thus changes the solution space. Here we point out that the assumption of smooth null infinity excludes infalling matter from past timelike infinity [124–128].

shear tensor, respectively[2]. The shear tensor satisfies $\gamma^{AB}C_{AB} = 0$ as part of the gauge conditions. Einstein's equations also imply the evolution equations

$$\partial_u m = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}\nabla_A\nabla_B N^{AB} - 4\pi \lim_{r\to\infty}(r^2\, T_{uu})\,, \tag{7.6a}$$

$$\begin{aligned}\partial_u N_A = {}& -u\,\partial_A\partial_u m + \frac{1}{4}\partial_A\left(N_{BC}C^{BC}\right) - \frac{1}{4}\nabla_B\left(C^{BC}N_{CA}\right) + \frac{1}{2}C_{AB}\nabla_C N^{BC} \\ & - \frac{1}{4}\nabla_B\left(\nabla^B\nabla^C C_{AC} - \nabla_A\nabla_C C^{BC}\right) - 8\pi \lim_{r\to\infty}(r^2\, T_{uA})\,,\end{aligned} \tag{7.6b}$$

where $T_{\mu\nu}$ is the matter stress tensor and $N_{AB} \equiv \partial_u C_{AB}$ is the News tensor.

The asymptotic symmetries of this class of spacetimes must preserve both the Bondi gauge and the asymptotic Minkowski structure that we are assuming. If we denote the generator of such asymptotic transformations by ξ, preserving the Bondi gauge corresponds to requiring

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad g^{AB}\mathcal{L}_\xi g_{AB} = 0, \tag{7.7}$$

while preserving the asymptotically Minkowski structure imposes the following falloff for the remaining components of the metric variation,

$$\mathcal{L}_\xi g_{uu} = O\left(\frac{1}{r}\right), \qquad \mathcal{L}_\xi g_{uA} = O(1), \tag{7.8a}$$

$$\mathcal{L}_\xi g_{ur} = O\left(\frac{1}{r^2}\right), \qquad \mathcal{L}_\xi g_{AB} = O(r). \tag{7.8b}$$

As usual for asymptotic symmetries, these are milder conditions than those for proper isometries. For this reason the Killing vector fields of a given spacetime are always a subset of the asymptotic symmetries. Notice also that in this formulation asymptotic symmetries depend on which falloff conditions one demands. With stricter ones, it is also possible to remove some of them (though this may result in the loss of legitimate physical content).

[2]The definition of the angular momentum aspect varies in the literature. The conventions adopted by Flanagan–Nichols [129] or by Barnich–Troessaert and Compère–Fiorucci–Ruzziconi [130–132] are related to the one used here respectively by

$$\begin{aligned} N_A^{\mathrm{FN}} &= N_A + u\,\partial_A m\,, \\ N_A^{\mathrm{BT}} &= N_A + u\,\partial_A m - \frac{3}{32}\partial_A(C_{BC}C^{BC}) - \frac{1}{4}C_{AB}\nabla_C C^{BC}\,. \end{aligned}$$

The quantity ϕ is needed for completeness of the map between Bondi and Beig-Schmidt gauges at the order we work in Chapter 8. The only information we actually use is that is behaves like $\phi = u\,\phi^{-1} + \phi^0 + o(u^0)$. The reader can find its explicit expression in Appendix I.

One can easily solve the equations in the asymptotic transformations ξ. Equations (7.7) fix the dependence on r of its components, while (7.8) are meant to determine the u and x^A dependence. The most generic transformation then reads

$$\xi^u = T(x) + \frac{u}{2}\Psi(x), \tag{7.9a}$$

$$\xi^A = Y^A(x) + (\partial_B \xi^u) \int_r^\infty dr' \, g^{AB}, \tag{7.9b}$$

$$\xi^r = -\frac{r}{2}\left[D_C \xi^C + Y^A D_A \xi^u\right]. \tag{7.9c}$$

It is parametrized by two functions. $Y^A(x)$ is a conformal vector on $\mathbb{S}^2$ with $\Psi = D_A Y^A$ as its conformal factor, while $T(x)$ is an arbitrary function on $\mathbb{S}^2$.

The function $T(x)$ can be expanded in terms of spherical harmonics or as a power series in stereographic coordinates, and it is the generator of the infinite set of transformations called *supertranslations*. They generate the Abelian algebra $\mathfrak{st}_4$. Asymptotically, they act as

$$\xi_T = T(x)\partial_u - \frac{1}{r}\gamma^{AB}\partial_A T(x)\partial_B + \frac{1}{2}\left[D_B \partial^B T(x)\right]\partial_r + \ldots \tag{7.10}$$

where D_b denotes the covariant derivative on $\mathbb{S}^2$ and the dots must be removed if the spacetime is exactly four-dimensional Minkowski since the subleading corrections in the metric components vanish identically.

As for the generators $Y^A(x)$, if we restrict to the global conformal transformations forming the Moebius subgroup $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \simeq \mathrm{SO}(1,3)$, one obtains the asymptotic *global BMS group* for four-dimensional asymptotically Minkowski spacetimes, with algebra

$$(\text{global})\mathfrak{bms}_4 \simeq \mathfrak{st}_4 \times \mathfrak{so}(1,3). \tag{7.11}$$

Otherwise, one can include all local conformal transformations on $\mathbb{S}^2$, consisting in two copies of the Witt algebra [130, 133, 134] corresponding to transformations called *superrotations*. The *local BMS group* has thus associated algebra

$$(\text{local})\mathfrak{bms}_4 \simeq \mathfrak{st}_4 \times \mathfrak{w}_L \times \mathfrak{w}_R. \tag{7.12}$$

With the aim of finding an explicit realisation of these vector fields, let us consider the stereographic coordinates

$$z = e^{i\phi}\cot\frac{\theta}{2}, \qquad \bar{z} = e^{-i\phi}\cot\frac{\theta}{2}. \tag{7.13}$$

We can expand $Y^z = Y(z)$ and $Y^{\bar{z}} = Y(\bar{z})$ in terms of the generators

$$\ell_n = -z^{n+1}\partial_z, \qquad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{7.14}$$

Similarly, we can expand the supertranslations generator $T(x)$ as a series

$$T(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} a^{mn} T_{mn} \tag{7.15}$$

with constant a^{mn} and

$$T_{mn} = \frac{2}{1+z\bar{z}}\, z^m \bar{z}^n. \tag{7.16}$$

The asymptotic algebra reads

$$[\ell_m, \ell_n] = (m-n)\,\ell_{m+n}, \tag{7.17a}$$
$$\left[\bar{\ell}_m, \bar{\ell}_n\right] = (m-n)\,\bar{\ell}_{m+n}, \tag{7.17b}$$
$$[\ell_k, T_{mn}] = \left(\frac{k+1}{2} - m\right) T_{m+k,n}, \tag{7.17c}$$
$$\left[\bar{\ell}_k, T_{mn}\right] = \left(\frac{k+1}{2} - n\right) T_{m,n+k}, \tag{7.17d}$$
$$[T_{kj}, T_{mn}] = 0. \tag{7.17e}$$

Thus the superrotation generators $\{\ell_k, \bar{\ell}_k\}$ form two copies of Witt algebras. In the case of Minkowski spacetime, the Poincaré algebra is recovered as an isometry subalgebra on the full spacetime formed by $\ell_{-1,0,1}$, $\bar{\ell}_{-1,0,1}$ and $T_{00}, T_{01}, T_{10}, T_{11}$.

Let us now turn our attention to a generic even number of dimensions $d+2 \geq 6$. In this case the Bondi expansion has a similar form to (7.5) [123] and considering the same asymptotic conditions as in equations (7.7) and (7.8), the generator for the asymptotic transformations reads [135]

$$\xi^u = T(x) + \frac{u}{d}\Psi(x), \tag{7.18a}$$
$$\xi^a = Y^A(x) + (\partial_A \xi^u) \int_r^\infty dr'\, g^{AB}, \tag{7.18b}$$
$$\xi^r = -\frac{r}{d}\left[D_C \xi^C + Y^A D_A \xi^u\right], \tag{7.18c}$$

with $Y^A(x)$ conformal vector on $\mathbb{S}^d$. Thus in dimensions higher than four, the asymptotic symmetry algebra appears to include supertranslations and simple rotations [120], since there conformal vectors Y^A form the finite algebra $\mathfrak{so}(1, d+1)$.[3] In higher dimensions one can only define the global BMS algebra,

$$(\text{global})\mathfrak{bms}_d \simeq \mathfrak{st}_d \times \mathfrak{so}(1, d+1). \tag{7.19}$$

[3]Note that in [136] a version of superrotations that is amenable to generalisation to any number of dimensions was proposed. There Y^A is required to be a vector generating smooth diffeomorphisms on $\mathbb{S}^d$.

Returning to $d+2=4$ spacetime dimensions, it is possible to associate finite charges to them (see [130,131,137]). These charges are labelled by the quantities (T, Y^A) defined over the celestial sphere $\mathbb{S}^2$, where T is a function parametrising supertranslations while Y^A is a conformal Killing vector parametrising Lorentz rotations and boosts. We give the following parametrisation where different values of the constants reproduce the various proposals for global BMS charges in Bondi gauge,

$$Q_{(\alpha,\beta)}[T, Y^A] = \frac{1}{8\pi G}\int_{\mathbb{S}^2} d\Omega \left(2T\, m + Y^A \hat{N}_A\right) , \tag{7.20}$$

with

$$\hat{N}_A \equiv N_A - \frac{\alpha}{16}\partial_A(C_{BC}C^{BC}) - \frac{\alpha}{4}C_{AB}\nabla_C C^{BC} + u\,\frac{\beta}{4}\nabla^B\left(\nabla_B\nabla^C C_{AC} - \nabla_A\nabla^C C_{BC}\right) . \tag{7.21}$$

This parametrisation slightly extends the one defined in [138] in that we include β in order to properly account for the recently constructed charges in [131, 132, 139]. The specific values of α and β in each renormalisation scheme will not be important.

The charges (7.20) are non-trivial for generic asymptotically Minkowski spacetimes, and thus the global BMS group represents an infinite-dimensional family of *physical* transformations. Let us now delve into the meaning of such statement. We restrict for simplicity to Minkowski spacetime, $m = C_{AB} = N_A = 0$. In the stereographic parametrisation (7.13) of $\mathbb{S}^2$, performing a supertranslation with parameter $f(z,\bar{z})$ induces a non-vanishing shear [140],

$$C_{zz} = -2\nabla_z^2 f(z,\bar{z})\,. \tag{7.22}$$

This can be shown to be precisely the effect of the passage of a train of gravitational waves between two asymptotic observers. If two asymptotic observers are at a certain distance in Minkowski, the passage of gravitational waves modifies the proper distance between them. Therefore the spacetime has memory of the passage of gravitational waves, and such memory can be seen as the action of BMS transformations on the spacetime. The values of the Poincaré charges are however left unchanged, therefore different values of supertranslation charges parametrise degenerate gravitational vacua with different supertranslation charges. The scalar function f entering the shear tensor as in (7.22) plays therefore the role of a supertranslation Goldstone mode, parametrising the vacua of the corresponding superselection sector.

A similar connection can be made between local superrotations [141, 142], another classes of memory effects (the *spin memory*), and the corresponding appearance of superrotation Goldstone mode in the metric.

7.2 Towards flat holography

Taking inspiration from AdS/CFT, as discussed in Section 1.3 we expect that if a holographic description of asymptotically flat spacetimes exists, bulk asymptotic isometries must be realised as conformal symmetries in the boundary dual theory. In particular, if flat holography is realised as an equivalence between observables similar to (1.29), bulk and boundary observables must be invariant under the same transformations, appropriately realised.

This is the main motivation for the proposal called *celestial holography* [34–38]. In that approach, scattering amplitudes in four-dimensional Minkowski spacetime are related to correlators on a $\mathbb{S}^2$. As described in the previous sections, the asymptotic symmetries of asymptotically Minkowski spacetimes include superrotations, which are parametrised by conformal transformations on $\mathbb{S}^2$ (either global or local depending on the considered version of the BMS group). Restricting to massless m-to-n scattering processes, the asymptotic states define m insertion points on a 2-sphere $\mathbb{S}^-$ which is a section of $\mathscr{I}^-$. Similarly, they define n insertion points on a 2-sphere $\mathbb{S}^+$ which is a section of $\mathscr{I}^+$. As proposed by celestial holography, such spheres are however antipodally related, in such a way that the bulk m-to-n scattering amplitude corresponds to a $(m+n)$-point function on a single 2-sphere, called the celestial sphere. Taking advantage of the known results and properties of the S-matrix on Minkowski space, one is then able to rewrite the four-dimensional scattering amplitudes as putative dual conformal correlators. This bottom-up approach hence allows one to investigate the putative holographic theory dual to weakly-coupled physics on flat spacetimes. Note that the BMS symmetry group also involves supertranslations, and the dual CFTs on the celestial sphere must realise these additional symmetries too.

This procedure however involves subtleties. Consider an amplitude $\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle$, where $\langle\text{out}|$ and $|\text{in}\rangle$ denote the outgoing and ingoing states respectively, while $\mathcal{S}$ indicates the scattering matrix. Note that $\langle\text{out}|$ and $|\text{in}\rangle$ are representations of two distinct symmetry groups BMS($\mathscr{I}^-$) and BMS($\mathscr{I}^+$) defined on cross-sections of $\mathscr{I}^-$ and $\mathscr{I}^+$ respectively. However, holographically we expect only one dual theory with a single associated phase space. The proposed solution in celestial holography is then to assume antipodal matching conditions for the bulk data in a vicinity of spatial infinity so as to make the value of the BMS($\mathscr{I}^-$) charges match the value of the corresponding BMS($\mathscr{I}^+$) charges at i^0. This effectively diagonalises the BMS($\mathscr{I}^+$) $\times$BMS($\mathscr{I}^-$) naive boundary symmetry group to a single BMS group, with a single set of independent BMS charges. There is evidence that this is a consistent procedure at least in Minkowski spacetime, since it allows one to show that the celebrated graviton soft theorems imply Ward Identities associated to BMS symmetries for the celestial correlators in two dimensions [121, 143].

In the case of gravitational scattering on asymptotically Minkowski spacetimes, these antipodal conditions on the bulk matter read

$$m(x^E)\big|_{\mathscr{I}^+_-} = m(\Upsilon x^E)\big|_{\mathscr{I}^-_+}\,, \qquad C_{AB}(x^E)\big|_{\mathscr{I}^+_-} = -\,C_{AB}(\Upsilon x^E)\big|_{\mathscr{I}^-_+}\,, \tag{7.23}$$

$$N_A(x^E)\big|_{\mathscr{I}^+_-} = -\,N_A(\Upsilon x^E)\big|_{\mathscr{I}^-_+}\,, \tag{7.24}$$

where by Υx^A we denote the antipodally-related point to x^A: if we cover the sphere $\mathbb{S}^2$ with angular coordinates $x^A = (\theta\,,\varphi)$, we define it by $\Upsilon(\theta\,,\varphi) = (\pi - \theta\,,\varphi + \pi)$. Although (7.23) and (7.24) are taken in celestial holography as an additional assumption on the bulk gravitational scattering processes, in Chapter 8 we will prove these antipodal matching conditions on the bulk data by analysing the dynamics of the gravitational field near spatial infinity.

Being a codimension-2 type of holography, the celestial holography program appears as a promising direction to connect the ambient space to flat holography. At the core of the celestial picture lie several key ideas from another approach to flat holography very much inspired by AdS/CFT and first put on paper by de Boer and Solodukhin [40–43], which we now review.

For their proposal, they restrict to four-dimensional Minkowski spacetime. As we known from Section 1.2, one can slice the region with $X^2 < 0$ in terms of Euclidean AdS_3 spaces, with metric there of the form,

$$\eta = -ds^2 + \frac{s^2}{r^2}\left[dr^2 + \delta_{AB}dx^A dx^B\right], \tag{7.25}$$

where $s > 0$ and for simplicity we consider the Poincaré patch. Analogously, the region of Minkowski with $X^2 > 0$ can be foliated with dS_3 slices, with metric

$$\eta = ds^2 + \frac{s^2}{r^2}\left[-dr^2 + \delta_{AB}dx^A dx^B\right]. \tag{7.26}$$

Their key idea is to perform a non-compact dimensional reduction of a given theory on Minkowski space along the s direction onto a single representative three-dimensional hyperbolic space, separately in the regions $X^2 > 0$ and $X^2 < 0$. In principle this allows one to describe the dynamics on Minkowski space as physics on a lower dimensional (A)dS space. The intention is then to use AdS/CFT to relate this theory on $(A)dS_3$ to a CFT_2, which would likely correspond to the celestial CFT in celestial holography.

Let us discuss more in detail this proposal and its open issues through an explicit example, a massless scalar on Minkowski spacetime [40]. Focusing for now on the $X^2 < 0$ region in the coordinates of (7.25) and denoting the Euclidean AdS_3 metric as g^+, the bulk equation of motion

$$\Box_\eta \Phi(s, r, x) = 0 \tag{7.27}$$

takes the form of a Sturm-Liouville problem, provided the appropriate boundary conditions which we will discuss later. The solution hence takes the form

$$\Phi(s, r, x) = \int d\Delta\left[s^{-\Delta}\phi^+_\Delta(r, x) + s^{\Delta-d}\phi^-_\Delta(r, x)\right], \tag{7.28}$$

where the modes $\phi^\pm_\Delta$ must satisfy the equation of a massive scalar on AdS_3,

$$(\Box^+ - \Delta(\Delta - d))\phi^\pm_\Delta(r, x) = 0\,. \tag{7.29}$$

The integral over Δ is formal since the set of Δ to sum over depends on the boundary conditions for the problem. Several possible choices are physically motivated, however the requirements of propagating modes as well as of regularity on the Minkowski lightcone at $s = 0$ restrict Δ to belong to the principal series $\Delta_{\pm} = 1 \pm i\lambda$, with real λ.

Therefore a massless scalar field with support on the region $X^2 < 0$ of four-dimensional Minkowski space is equivalent to an infinite set of non-interacting massive modes on Euclidean AdS_3. Using AdS/CFT, one is tempted to relate such modes to a family of scalar dual operators on a putative CFT_2. The AdS/CFT dictionary as currently established and as described in Subsection 1.3.3 involves only positive real scaling dimensions Δ. For complex Δ's, the CFT interpretation becomes more obscure since there is no clear distinction between normalisable and non-normalisable modes. Furthermore, to extract CFT observables one has to renormalise the divergences at the level of the bulk onshell action. This is again complicated by the presence of complex scaling dimensions and the corresponding counterterms are consequently not known. In addition to these divergences, other divergences appear in the onshell action in the limit $s \to \infty$, and they are completely different in nature and closer to the type of divergences that are addressed in gravity on asymptotically flat spacetimes. The decomposition on dS_3 in the region $X^2 > 0$ can be treated analogously, and a further issue of this picture concerns how to join the descriptions in terms of modes on either AdS_3 or dS_3 for Minkowski fields with supports on both $X^2 < 0$ and $X^2 > 0$ regions.

This approach by dimensional reduction exhibits several connections with the ambient space setup, which naturally involves hyperbolic slicings of the form (2.8) and (2.15). Indeed, the computation in Section 2.5 can be viewed as a generalisation of the dimensional reduction approach by de Boer and Solodukhin beyond four-dimensional Minkowski. In particular, there the spacetime dimensions are $d + 2$, the spacetime is generically Ricci-flat, and consequently the codimension-1 slices are $\mathrm{AL(A)dS}_{d+1}$ (as opposed to pure $\mathrm{(A)dS}_{d+1}$) spaces.

To make this connection with the ambient space more explicit and with the aim of extending the celestial holography framework as well as this dimensional reduction approach beyond four-dimensional Minkowski spacetimes, in the next section we introduce an interesting gauge for asymptotically Minkowski spacetimes, the Beig-Schmidt gauge.

7.3 The Beig-Schmidt gauge

Originally studied as a convenient way to describe the dynamics of the gravitational field on asymptotically Minkowski spacetimes near spatial infinity i^0 (for which we use it in Chapter 8), the Beig–Schmidt gauge [39] also proves useful for connecting the ambient space to flat holography. A metric in the Beig-Schmidt gauge takes the form

$$ds^2 = N^2 \, d\rho^2 + H_{ab} \, (N^a \, d\rho + dx^a) \, (N^b \, d\rho + dx^b) \,, \tag{7.30}$$

where spatial infinity is approached in the limit $\rho \to \infty$. As this limit is taken, the metric behaves as

$$N = 1 + \frac{\sigma}{\rho}\,, \tag{7.31a}$$

$$H_{ab}N^b = o(\rho^{-1})\,, \tag{7.31b}$$

$$H_{ab} = \rho^2 \left(h_{ab} + \rho^{-1} f_{ab} + \frac{\log \rho}{\rho^2}\, i_{ab} + \rho^{-2} j_{ab} + o(\rho^{-2}) \right)\,. \tag{7.31c}$$

These falloff properties of the metric components are ensured also in the case of an asymptotically Minkowski spacetime sourced by a matter stress tensor, as long as

$$T_{\rho\rho} = o(\rho^{-4})\,, \qquad T_{\rho a} = o(\rho^{-3})\,, \qquad T_{ab} = o(\rho^{-2})\,. \tag{7.32}$$

Einstein's equations at leading order imply

$$R[h]_{ab} = 2h_{ab}\,. \tag{7.33}$$

We take h_{ab} to be globally the three-dimensional de Sitter space $\mathcal{H}$. We stick to the boundary conditions in [144], where h_{ab} is not allowed to fluctuate. We treat h_{ab} as a genuine metric on $\mathcal{H}$ and indicate the corresponding covariant derivative as D_a. All three-dimensional indices $a, b, c, \ldots$ are raised and lowered with this metric. As it will be discussed in Section 7.4, an analogous Beig-Schmidt gauge can be set up in a vicinity of i^+ or i^- and in that case h_{ab} is required to be ALAdS$_3$, as opposed to ALdS$_3$ as in the present case.

The leading non-vanishing terms of the electric and magnetic parts of the Weyl tensor are respectively

$$E_{ab} = -\left(D_a D_b + h_{ab}\right)\sigma\,, \qquad B_{ab} = \frac{1}{2}\epsilon_a{}^{cd} D_c k_{db}\,, \tag{7.34}$$

where

$$k_{ab} \equiv f_{ab} + 2\sigma h_{ab}\,. \tag{7.35}$$

The fields σ and k_{ab} play the role of potentials for these two pieces of the Weyl tensor. In order to allow for a well-posed action principle, the trace of k_{ab} must vanish as part of the boundary conditions [144],

$$k^a_a = h^{ab} k_{ab} = 0\,. \tag{7.36}$$

Given these definitions and boundary conditions, Einstein's equations reduce to dynamical equations on the three-dimensional de Sitter hyperboloid $\mathcal{H}$, which can be solved order by order in the ρ^{-1} expansion. The leading order fields σ, k_{ab} and i_{ab} satisfy homogeneous partial differential equations, and act as sources for the subleading field j_{ab}. More specifically, the homogeneous equations satisfied by the leading fields are given by

$$(D^2 + 3)\sigma = 0\,, \qquad (D^2 - 3)k_{ab} = 0\,, \qquad D^a k_{ab} = 0\,, \tag{7.37}$$

$$(D^2-2)i_{ab}=0\,, \qquad i^a_a=D^a i_{ab}=0\,. \tag{7.38}$$

The inhomogeneous equations satisfied by the subleading field j_{ab} is given by

$$\left(D^2-2\right)j_{ab}=2i_{ab}+S_{ab}\,, \tag{7.39}$$

subject to the constraints

$$j^a_a=12\sigma^2+D_a\sigma\,D^a\sigma+\frac{1}{4}k^{ab}k_{ab}+k^{ab}D_aD_b\sigma\,, \tag{7.40a}$$

$$D^b j_{ba}=\frac{1}{2}k^c_b\,D^b k_{ca}+D_a\left(8\sigma^2+D_a\sigma\,D^a\sigma-\frac{1}{8}k^{cd}k_{cd}+k^{cd}D_cD_d\sigma\right)\,. \tag{7.40b}$$

The constraints (7.40) determine the trace and divergence of j_{ab} in terms of the first order data σ and k_{ab}. The remaining equation (7.39) should be understood as an equation for the remaining undetermined degrees of freedom carried by j_{ab}. This evolution equation is sourced by a term S_{ab} quadratic in σ and k_{ab},

$$S_{ab}=\mathrm{NL}_{ab}(\sigma,\sigma)+\mathrm{NL}_{ab}(\sigma,k)+\mathrm{NL}_{ab}(k,k)\,. \tag{7.41}$$

We refer to Appendix C of [144] for explicit expression of (7.41) which is lengthy and not particularly illuminating.

The asymptotic symmetries at spatial infinity with thee boundary conditions can be shown to be Poincaré transformations, supertranslations and logarithmic translations [144]. We discuss the role of superrotations in this gauge in Section 7.4. The action of logarithmic translations is

$$\rho\to\rho+\log\rho\,H(x^a)+o(\rho^0)\,, \tag{7.42}$$

$$x^a\to x^a+\frac{1+\log\rho}{\rho}\,D^aH(x^b)+o(\rho^{-1})\,, \tag{7.43}$$

where $H(x^a)$ is a function on the hyperboloid satisfying the condition $(D_aD_b+h_{ab})H=0$. Such transformations do not change k_{ab}, while they modify $\sigma\to\sigma+H$.

Supertranslations act as

$$\rho\to\rho+\omega(x^a)+o(\rho^0)\,, \tag{7.44}$$

$$x^a\to x^a+\frac{1}{\rho}\,D^a\omega(x^b)+o(\rho^{-1})\,, \tag{7.45}$$

inducing the variation

$$k_{ab}\to k_{ab}+2(D_aD_b+h_{ab})\omega\,, \tag{7.46}$$

while σ is left unchanged. Here ω is a function on the hyperboloid satisfying

$$(D_aD^a+3)\omega=0\,. \tag{7.47}$$

Translations are the transformations which further satisfy

$$(D_a D_b + h_{ab})\omega = 0\,, \tag{7.48}$$

and they thus lead to no variation in the metric. More details regarding the asymptotic symmetries in the Beig-Schmidt gauge and the associated finite charges [144] will be provided in Chapter 8.

7.4 Relations with the ambient space

Conventions. In this section we reinstate the ambient notation for indices adopted in Chapters 1 to 6. We denote four-dimensional indices by capital Latin letters, three-dimensional indices by lowercase Greek letters, and two-dimensional indices by lowercase Latin letters.

As we mentioned in Section 7.3, for four-dimensional asymptotically Minkowski spacetimes the vicinity of i^+ can also be described in terms of Beig-Schmidt gauge. The corresponding metric expansion reads

$$\tilde{g} = -\left(1 + \frac{\sigma(x)}{s}\right)^2 ds^2 + s^2\left[g^+_{\alpha\beta}(x) + \frac{f_{\alpha\beta}(x)}{s} + O(s)^{-2}\right] dx^\alpha dx^\beta\,, \tag{7.49}$$

where s describes the geodesic distance from future infinity, reached for $s \to \infty$. Considering a Ricci-flat spacetime, the condition $\widetilde{R}_{MN} = 0$ can be solved order by order at large s, from which it follows that $g^+_{\alpha\beta}$ must be the metric of a 3-dimensional ALAdS space (as opposed to the Beig-Schmidt expansion near spatial infinity, where the corresponding 2-tensor is ALdS).

Performing the transformation

$$s \to s - \sigma(x)\,\log s + o(s^0)\,, \tag{7.50}$$

$$x^\alpha \to x^\alpha - \frac{1 + \log s}{s}\,D^\alpha\sigma(x) + o(s^{-1})\,, \tag{7.51}$$

one can transfer the degrees of freedom of σ into an additional logarithmic term with coefficient $\tilde{f}_{\alpha\beta}(x)$, yielding the following metric in normal coordinates,

$$\tilde{g} = -ds^2 + s^2\left[g^+_{\alpha\beta}(x) + \frac{f_{\alpha\beta}(x)}{s} + \frac{\tilde{f}_{\alpha\beta}(x)}{s}\log s + O(s)^{-2}\right] dx^\alpha dx^\beta\,. \tag{7.52}$$

This transformation is nothing but a logarithmic supertranslation [145], that is a logarithmic translation of the form (7.42) where the function $H(x)$ that parametrises it does not satisfy $(D_\alpha D_\beta + g^+_{\alpha\beta})H = 0$.

If one now imposes $T = s\partial_s$ to be a homothety of the 4-dimensional spacetime such that $\mathcal{L}_T \tilde{g} = 2\tilde{g}$, one restricts to spacetimes where $f_{\alpha\beta}$, $\tilde{f}_{\alpha\beta}$ as well as all the higher order data in the expansion is vanishing. The resulting geometry is that of an ambient space in hyperbolic slicing (2.8). Similarly, if one starts from a Beig-Schmidt expansion near spatial infinity as the one described in Section 7.3, the resulting metric after imposing the homothety T is the ALdS slicing (2.15) of the ambient space.

In this perspective, the ambient space is a generalisation of Minkowski space which maintains an analogue of the Euler vector $X^M \partial_M$, while a general geometry in the Beig-Schmidt gauge allows one to describe more general Ricci-flat spacetimes where this homothety is absent. In this latter case, the nullcone structure of ambient spaces is broken.

In view of generalising the approaches to flat holography described in Section 7.2 to spacetimes other than four-dimensional Minkowski, it is interesting to discuss where the BMS Goldstone modes appear in the ambient and Beig-Schmidt metrics. Considering the local BMS group (7.12), one can show that the superrotation mode is encoded in the $g_{(d)ij}$ term appearing in the hyperbolic metric $g^{+}_{\alpha\beta}$ related to the dual holographic stress tensor according to AdS/CFT [43]. [4] The supertranslation Goldstone mode is encoded in $f_{\alpha\beta}$ [50, 144]. Thus, enforcing the presence of the homothety $T = s\partial_s$ requires $f_{\alpha\beta}$ to vanish and fully fixes the supertranslation mode. If we then consider flat spacetimes which are locally ambient spaces, we expect to be able to describe the soft physics related to Poincaré transformations and superrotations, while less clearly so for supertranslations.

In particular from the algebra (7.17) it follows that Poincaré transformations together with superrotations do not form a closed subalgebra of local BMS transformations (7.12). This entails that if one fixes supertranslations as one does on ambient spaces, the maximal consistent algebra of near-lightcone transformations is simply $\mathrm{ISO}(1,3)$. Analogous arguments yield a near-lightcone algebra $\mathrm{ISO}(1, d+1)$ in general $d+2$ spacetime dimensions. This hence seems to indicate that one has to give up the global homothety T and resort to the more general geometries of the Beig-Schmidt class in order to describe the soft physics of gravity in flat spacetimes.

At these stage, several questions remain open. It is not yet clear how to use the ambient and Beig-Schmidt geometries to generalise the proposal by [40, 41] of flat holography as an uplift of AdS/CFT to generic asymptotically Minkowski spacetimes in an arbitrary number of dimensions. It is also unclear how this framework precisely relates to celestial holography, and how celestial holography can be extended to non-trivial spacetimes where the gravitational scattering problem is not a priori well-defined.

In the next chapter we will provide useful insight to address all these questions, by relating the gravitational data appearing in Bondi gauge (where celestial holography is formulated) to the gravitational data present in Beig-Schmidt gauge (which can serve

[4] In terms of the ambient isometries presented in Chapter 3, such local superrotations are associated to a vector $E^{(0)}$ which generates local conformal transformations on the two-dimensional sphere $g_{(0)}$.

as a potential extension of the dimensional reduction proposal for flat holography to generic spacetimes). This will allow us to prove the antipodal matching conditions (7.23)-(7.24) for a general class of asymptotically Minkowski spacetimes. As mentioned, these conditions are assumed in celestial holography to match the BMS charges on $\mathscr{I}^-$ to those on $\mathscr{I}^+$ and define a well-posed gravitational scattering problem. With these results we are therefore able to provide a sound foundation to the celestial picture.

Chapter 8

The antipodal matching conditions in General Relativity

8.1 The gravitational scattering problem and celestial holography

In order for the program of celestial holography to reach its full potential, we should carefully set it up in a way which appropriately incorporates the nonlinear nature of General Relativity. Generic asymptotically flat spacetimes significantly differ from Minkowski space, especially with regards to their structure at spatial infinity i^0. As we discussed in Chapter 7, physical fields are generically not single-valued at i^0, such that continuity cannot be invoked in order to relate their behavior from past null infinity $\mathscr{I}^-$ to future null infinity $\mathscr{I}^+$. These considerations should play an important role in celestial holography. Indeed the newly discovered connections between asymptotic symmetries in General Relativity and soft graviton theorems crucially rely on *i*) the definition of a single BMS group acting simultaneously on both $\mathscr{I}^+$ and $\mathscr{I}^-$ via antipodal identifications of the symmetry generators and asymptotic fields, and on *ii*) the conservation of BMS charges from $\mathscr{I}^-_+$ to $\mathscr{I}^+_-$ across i^0. The validity of these two conditions in the nonlinear theory should be tightly connected with the behavior of the gravitational field in a neighborhood of i^0.

In this chapter we wish to shed further light on the matching of BMS charges across spatial infinity i^0 and on the corresponding antipodal matching conditions *in the context relevant to the gravitational scattering problem*. The class of spacetimes typically considered in that context are a peeling version of those studied by Christodoulou and Klainerman (CK), which constitute a set of asymptotically flat geometries non-linearly close to Minkowski space[1] [146]. Although this class of spacetimes satisfies conditions i) and ii) as defined above, they certainly do not contain all configurations of interest. In

[1] The initial data sets considered by CK are characterised by a spherically symmetric mass parameter

particular, they do not account for spacetimes with nonzero supertranslation charges at i^0 [147]. Thus a nontrivial matching of the charges requires one to consider a broader class of spacetime asymptotics. We will however not investigate how and whether these asymptotics result from the evolution of mathematically well-defined initial data sets. See the work of Mohamed and Valiente Kroon along these lines in the case of spin-1 and spin-2 fields [148].

A key result of our approach is the mapping of scattering data at $\mathscr{I}$ to gravitational data in a neighborhood of i^0. Our treatment is entirely coordinate-based and relies on the Bondi–Sachs description of the gravitational field near $\mathscr{I}$ [149–151] and on the Beig–Schmidt description of the gravitational field near i^0 [39, 145]. It therefore differs from the recent work of Prabhu and Shehzad [152, 153] who studied the matching of charges within the Ashtekar–Hansen formalism set up to treat i^0 and $\mathscr{I}$ simultaneously [107,108]. Rather we proceed by performing an asymptotic coordinate transformation between Bondi and Beig–Schmidt gauges in order to obtain an explicit map relating the respective asymptotic data.

The descriptions of the gravitational field at $\mathscr{I}$ or i^0 have distinctive features, and relating them is therefore of high interest. On the one hand, spatial infinity is the locus where the variational principle is well-defined, and charges are both integrable and conserved. The most general phase space in Beig-Schmidt gauge was analysed by Compère and Dehouck (CD) [144] (see also [154–156]). Their charges satisfy all the desirable properties[2] and give a faithful representation of the BMS algebra without central extension. Their treatment also extends previous constructions [107, 157, 158, 160–162] in that they account for nonzero leading electric and magnetic Weyl tensor E_{ab} and B_{ab}, and do not impose parity conditions on the corresponding potentials σ and k_{ab}. On the other hand, charges computed at $\mathscr{I}$ are neither integrable nor conserved, a fact closely related to the leakage of symplectic flux through $\mathscr{I}$ in the form of gravitational radiation.

Inspired by the celestial holography literature, we assume the scattering data at $\mathscr{I}$ to admit a polynomial expansion in negative powers of the radial and retarded time coordinates r and u, and no radiation in the limit to i^0. This is the notion of gravitational scattering we consider here. Under these assumptions we find that the scattering data maps onto a restricted subset of the CD phase space. In particular both σ and k_{ab} turn

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and result in spacetimes which do not satisfy the peeling property.

[2]Note that the renormalization procedure that CD propose involves a Mann-Marolf-type counterterm [157, 158]. This prescription is well-known to partially break bulk diffeomorphisms, in the sense that in $d = 4$ it requires additional boundary conditions on the gravitational fields depending on the choice of the regulating surfaces in order for the variational principle to be well-defined. With our boundary conditions, this issue does not play any role. It is however not clear whether in principle it is possible to define alternative schemes that fully preserve covariance at spatial infinity. Another long-known problematic feature of renormalization at spatial infinity is the non-locality of counterterms, as first stressed in [159]. From a holographic point of view, this may be due to either some form of incompleteness of the current perspectives or a fundamental property of theories dual to flat spaces. It would be thus interesting to assess more in depth these two points.

out to satisfy specific parity conditions, although the well-posedness of Einstein's equations near spatial infinity does not require them a priori. We also demonstrate that the resulting magnetic Weyl tensor B_{ab} vanishes, a property previously assumed by Prabhu and Shehzad when considering the matching of Lorentz charges [153]. As far as we know, $B_{ab} = 0$ had only been established for spacetimes that are axisymmetric or stationary [163]. We confirm that it is in fact a feature of the gravitational scattering problem (as considered here).

The explicit map between Bondi and Beig–Schmidt data allows us to derive the antipodal matching relations together with the conservation of the BMS charges across spatial infinity. This in particular implies that only the diagonal subgroup of BMS($\mathscr{I}^+$) $\times$BMS($\mathscr{I}^-$) is a symmetry of the entire spacetime asymptotic structure, as it was originally assumed by Strominger in his seminal work [121].

Various analyses of the requirements under which $\mathscr{I}^+$ and $\mathscr{I}^-$ and their symmetries can be matched across spatial infinity can be found in the literature. Perhaps the first step in this direction was taken by Herberthson and Ludvigsen in demonstrating the antipodal matching of the Bondi mass aspect [164]. More recently Troessaert derived Strominger's original antipodal matching condition relating the supertranslation symmetry parameters of BMS($\mathscr{I}^+$) and BMS($\mathscr{I}^-$) [121, 165]. In subsequent work Henneaux and Troessaert studied a set of parity conditions in the Hamiltonian formulation of gravity that allows for a canonical realisation of BMS symmetries at i^0 and argued that such a phase space supports Strominger's antipodal matching condition [166–168]. The aforementioned analysis of Prabhu and Shezad is instead framed within the Ashtekar–Hansen formalism and focuses on the matching of the charges themselves [152, 153]. Our analysis builds upon this literature by giving the complete map of asymptotic data and charges between $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$.

From our analysis it also follows that the various proposals of BMS charges in Bondi gauge found in the literature [129–132, 138, 139] all match with the conserved charges at spatial infinity. This is a consequence of the fact that the terms by which they differ vanish in the limit to i^0 under our working assumptions.

In addition, our work highlights the restrictions on the global spacetime asymptotic structure resulting from a choice of data at $\mathscr{I}$. This turns into a signpost indicating the limits of validity of the standard setup, as well as a pathway to envision extensions towards a more general holographic framework. Indeed we have not restricted our analysis to solutions that are close to Minkowski space in the sense of CK. For these reasons, we believe that our approach is naturally suited to study the interplay between null and spatial infinity and to explore phenomenologically relevant processes beyond perturbative quantum gravity.

This chapter is organized as follows. In Section 8.2 we introduce assumptions regarding the behavior of the fields in their limit to $\mathscr{I}^-_+$ and $\mathscr{I}^+_-$. In Section 8.3 we describe the onshell late-time behaviour of fields entering the Beig–Schmidt metric presented in Section 7.3. In Section 8.4 we present the map from the Bondi gauge to the Beig–Schmidt gauge, whose details are given in Appendix I. In Section 8.5 we use this map together with Einstein's equations to derive the antipodal matching relations of the Bondi mass aspect, angular momentum aspect and shear tensor. In Section 8.6 we provide the matching of BMS charges between null and spatial infinity.

Conventions. Here we use different conventions to those used in Chapters 1 to 6. Four-dimensional indices are labelled by lowercase Greek letters, while three-dimensional indices are denoted with lowercase Latin letters. h_{ab} is the metric of the three-dimensional de Sitter spacetime $\mathcal{H}$ and D_a is its compatible covariant derivative. The metric on the celestial sphere $\mathbb{S}^2$ is denoted with γ_{AB} and ∇_A is the covariant derivative, capital Roman indices label the coordinates on this manifold. As in Section 7.2, covering the sphere $\mathbb{S}^2$ with angular coordinates $x^A = (\theta\,, \varphi)$, we define the *antipodal map* $\Upsilon(\theta\,, \varphi) = (\pi - \theta\,, \varphi + \pi)$. A tensor T on $\mathbb{S}^2$ is of *odd parity* under Υ if $\Upsilon^* T = -T$. In terms of components of a vector field for example, this means that T_φ and T_θ are odd and even functions on the sphere, respectively. Even parity under Υ is similarly defined. Covering $\mathcal{H}$ with global coordinates $x^a = (\tau\,, \theta\,, \varphi)$, we also define the *$\mathcal{H}$-antipodal map* $\Upsilon_{\mathcal{H}}(\tau\,, \theta\,, \varphi) = (-\tau\,, \pi - \theta\,, \varphi + \pi)$ and analogous considerations on parity properties apply.

8.2 Specifying the data at null infinity

To proceed with the analysis of the matching between quantities defined at past null infinity $\mathscr{I}^-$ and at future null infinity $\mathscr{I}^+$ (in their limit to spatial infinity i^0), we assume that the Bondi gauge is well-suited to describe the region $\mathscr{I}^+_-$ in the limit $u \to -\infty$. We start by imposing the following falloff conditions,

$$m = m^0 + u^{-1}\, m^1 + o(u^{-1})\,, \tag{8.1a}$$
$$N_A = N_A^0 + o(u^0)\,, \tag{8.1b}$$
$$C_{AB} = C_{AB}^0 + u^{-1}\, C_{AB}^1 + o(u^{-1})\,, \tag{8.1c}$$

together with the falloff rate of the matter stress tensor near $\mathscr{I}^+_-$,

$$\lim_{r\to\infty} r^2\, T_{uu} = o(u^{-2})\,, \qquad \lim_{r\to\infty} r^2\, T_{uA} = o(u^{-1})\,. \tag{8.2}$$

These fall-offs correspond to those typically underlying the proofs of the relation between soft theorems and asymptotic symmetries. We further discuss the scope of these conditions later in this section and in Section 8.7. The evolution equations (7.6) yield constraints[3] on the coefficients of the expansions (8.1), as they directly imply

$$\nabla^B\left(\nabla_B\nabla^C C^0_{AC}-\nabla_A\nabla^C C^0_{BC}\right)=0\,, \tag{8.3}$$

and

$$m^1=\frac{1}{4}\nabla^A\nabla^B C^1_{AB}\,, \tag{8.4a}$$

$$\partial_A m^1=\frac{1}{4}\nabla^B\left(\nabla_B\nabla^C C^1_{AC}-\nabla_A\nabla^C C^1_{BC}\right)\,. \tag{8.4b}$$

The meaning of these constraints becomes manifest once we decompose C^0_{AB} and C^1_{AB} in electric and magnetic parts,

$$C^i_{AB}=-2\nabla_A\nabla_B C^i+\gamma_{AB}\nabla^2 C^i+\epsilon_{C(A}\nabla_{B)}\nabla^C\Psi^i\,,\qquad i=0,1\,, \tag{8.5}$$

where C^i is the corresponding electric scalar potential and Ψ^i is the corresponding magnetic pseudo-scalar potential. Note that the $l=0,1$ spherical harmonics in C^i and Ψ^i do not contribute to (8.5). We can check that the two differential operators appearing in (8.3)-(8.4) respectively project out the electric or magnetic modes,

$$\nabla^A\nabla^B C^i_{AB}=-\nabla^2(\nabla^2+2)C^i\,, \tag{8.6a}$$

$$\nabla^B\left(\nabla_B\nabla^C C^i_{AC}-\nabla_A\nabla^C C^i_{BC}\right)=-\epsilon_{AB}\nabla^B\nabla^2(\nabla^2+2)\Psi^i\,. \tag{8.6b}$$

Hence the constraint (8.3) requires $\nabla^2(\nabla^2+2)\Psi^0=0$ which eliminates spherical harmonics with $l>1$ in Ψ^0. Since C^1 and Ψ^1 are necessarily independent, the constraints (8.4) can only be satisfied provided $m^1=\nabla^2(\nabla^2+2)C^1=\nabla^2(\nabla^2+2)\Psi^1=0$ which similarly eliminate all spherical harmonics with $l>1$ in C^1 and Ψ^1. In summary, the Ansatz (8.1) together with the evolution equations (7.6) imply

$$C^0_{AB}=-2\nabla_A\nabla_B C+\gamma_{AB}\nabla^2 C\,,\qquad m^1=C^1_{AB}=0\,, \tag{8.7}$$

where the electric potential $C\equiv C^0$ is known as the supertranslation Goldstone mode. In particular we conclude that the News tensor satisfies the stronger falloff $N_{AB}=o(u^{-2})$. This stronger falloff is in fact required for finiteness of the BMS charge fluxes along $\mathscr{I}$ [139], and enters the assumptions for deriving the subleading soft graviton theorem [169]. Notice that the falloffs (8.1) are also such that the BMS charges studied in Section 8.6 are finite in the limit $u\to-\infty$, while allowing for overleading terms would result in divergent charges. The expansions (8.1) can also be considered as inspired by those resulting from the well-posed Cauchy problem studied by Christodoulou and Klainerman (CK), although here m^0 is not restricted to be a constant and N_{AB} falls off faster than the $O(u^{-\frac{3}{2}})$ obtained by CK [146].

[3] If considered alone, the expansion of C_{AB} (8.1c) would generically yield $O(u)$ and $O(\ln u)$ terms in N_A by integration of the corresponding evolution equation. Since we do not allow for such terms, the evolution equation rather gives constraints solved by (8.7).

8.3 Late-time expansions of the Beig-Schmidt fields

The connection between quantities at $\mathscr{I}^-_+$ and $\mathscr{I}^+_-$ will involve the dynamics of the gravitational field near spatial infinity i^0. As we reviewed in Section 7.3, a convenient way to describe this dynamics is to adopt the Beig–Schmidt gauge [39]. We now study the behavior of the fields $\sigma\,, k_{ab}$ and j_{ab} in the limits to infinite past and future on the hyperboloid $\mathcal{H}$, which we can expect to connect to past and future null infinity $\mathscr{I}^\pm$ respectively. It will not be required to discuss i_{ab} further since we will find in Section 8.4 that no such term is needed in order to account for the Bondi data appropriate to the gravitational scattering problem. Covering $\mathcal{H}$ with coordinates (τ, x^A) and metric

$$ds^2_{\mathcal{H}} = -\,d\tau^2 + \cosh^2\tau\,\gamma_{AB}\,dx^A\,dx^B\,, \tag{8.8}$$

the loci of interest correspond to the limits $\tau \to \pm\infty$. For simplicity we will only describe the late-time limit and expand the fields $\sigma\,, k_{ab}$ and j_{ab} in the small parameter $e^{-\tau}$, but a similar expansion in the early-time limit obviously holds. Such expansions are completely analogous to the Fefferman–Graham expansions in AdS space described in Subsection 1.3.1. We give the details of these computations in Appendix H and collect the relevant results here.

Leading fields. The large-τ expansion of the electric potential σ and magnetic potential k_{ab} are found to be

$$\sigma(\tau,x) = e^{\tau}\,\sigma^{(-1)} + e^{-\tau}\sigma^{(1)} + e^{-3\tau}\tau\,\tilde{\sigma} + e^{-3\tau}\,\sigma^{(3)} + \dots\,, \tag{8.9}$$

and

$$k_{\tau\tau} = e^{-3\tau}\tau\,\tilde{k}_{\tau\tau} + e^{-3\tau}\,k^{(3)}_{\tau\tau} + \dots\,, \tag{8.10a}$$

$$k_{\tau A} = e^{-\tau}\tau\,\tilde{k}_{\tau A} + e^{-\tau}\,k^{(1)}_{\tau A} + \dots\,, \tag{8.10b}$$

$$k_{AB} = e^{\tau}\tau\,\tilde{k}_{AB} + e^{\tau}\,k^{(-1)}_{AB} + \dots\,. \tag{8.10c}$$

The equations of motion (7.37) are quadratic differential equations, and as such they generically admit two independent sets of solutions with distinct asymptotic behaviors. The two independent solutions for the electric potential σ are characterized respectively by $\sigma^{(-1)}$ and $\sigma^{(3)}$, while all the other functions appearing in the τ-expansion (8.9) can be fully determined in terms of these data. A similar structure applies to the components of k_{ab}, where the two sets of independent solutions are characterized by the first two functions on the sphere appearing in each of the τ-expansions (8.10).

Subleading field. The analysis of the large-τ behavior of j_{ab} is significantly more delicate due to the appearance of terms quadratic in σ and k_{ab} on the right-hand side of

(7.39)-(7.40). These terms are the manifestation of the nonlinear nature of Einstein's equations, and their careful treatment is precisely what will allow us to prove the antipodal matching condition of the angular momentum aspect without imposing dramatic restrictions on the Bondi data. We can summarise the situation in the following way. The solutions to (7.39)-(7.40) are given by the superposition of a particular solution that depends on pre-determined source terms such as S_{ab}, and a combination of homogeneous solutions. The asymptotic behavior of the homogeneous solutions is easily worked out,

$$j_{\tau\tau} = e^{-2\tau} j^{(2)}_{\tau\tau} + e^{-4\tau} j^{(4)}_{\tau\tau} + \dots\,, \tag{8.11a}$$

$$j_{\tau A} = j^{(0)}_{\tau A} + e^{-2\tau} j^{(2)}_{\tau A} + \dots\,, \tag{8.11b}$$

$$j_{AB} = e^{2\tau} j^{(-2)}_{AB} + j^{(0)}_{AB} + \dots\,, \tag{8.11c}$$

while the behavior of the particular solution strongly depends on the form of σ and k_{ab}. A key result of the analysis to be presented in Section 8.4 is that the Bondi data maps onto a *subset* of the allowed Beig–Schmidt data, in such a way that the large-τ behavior of the particular solution is subleading compared to that of the homogeneous solutions. Thus (8.11) holds true for the full solution of Einstein's equations provided such a solution can be mapped onto the Bondi phase space. On the other hand, a generic solution of the Beig–Schmidt equations (7.39)-(7.40) which is not connected to the Bondi phase space would see its leading asymptotics (8.11) modified due to weaker falloffs of the source terms quadratic in σ and k_{ab}. This clean separation between large-τ asymptotics of the homogeneous and particular solutions is a property of the Bondi phase space, not one of the larger Beig–Schmidt phase space. As we will further show in Section 8.4, the angular momentum aspect sits in $j^{(2)}_{\tau A}$ and assessing that this term is controlled by homogeneous solutions appears crucial to the derivation of the corresponding antipodal matching condition in Section 8.5.

8.4 From Bondi to Beig-Schmidt

At leading order in r and ρ respectively, the Bondi and Beig-Schmidt metrics are simply that of Minkowski space written in two different coordinate systems. The coordinate transformation between these two is explicitly given by

$$u = -\rho\, e^{-\tau}, \tag{8.12a}$$

$$r = \rho \cosh \tau. \tag{8.12b}$$

Obviously there exists an analogous coordinate transformation to the advanced Bondi gauge describing the neighborhood of $\mathscr{I}^-$,

$$v = \rho\, e^{\tau}, \tag{8.13a}$$

$$r = \rho \cosh \tau. \tag{8.13b}$$

We want to find a map between data of asymptotically flat gravity at null and spatial infinity. We will proceed by explicit coordinate transformation from the Bondi gauge to the Beig–Schmidt gauge. However each of these two asymptotic expansions are valid in different regions of spacetime and one can only hope to relate them where these expansions overlap.[4] This happens in the regime $r \to \infty\,, u \to -\infty\,(v \to \infty)$ or equivalently in the limit $\rho\,, \tau \to \infty\,(\tau \to -\infty)$, which can intuitively be thought of as the neighborhood of $\mathscr{I}^+_-\,(\mathscr{I}^-_+)$. To be more precise, we will start from Bondi metrics written as a double asymptotic expansion in $r \gg |u| \gg 1$, which we will map to Beig–Schmidt metrics written as a double asymptotic expansion in $\rho \gg e^\tau \gg 1$. Since

$$\frac{u}{r} = O(e^{-2\tau})\,, \tag{8.14}$$

terms that are subleading in r but overleading in u will contribute at the same order in ρ but to subleading order in $e^{-\tau}$. The explicit details of this transformation are relegated to Appendix I.

A first important observation is that the logarithmic term i_{ab} is not generated by this mapping of the Bondi data onto the Beig–Schmidt data. For the electric potential, the map yields

$$\sigma^{(-1)} = \sigma^{(1)} = \tilde{\sigma} = 0\,, \qquad \sigma^{(3)} = 2m^0\,, \tag{8.15}$$

while for the magnetic potential, we find

$$\tilde{k}_{\tau A} = \tilde{k}_{AB} = 0\,, \qquad k^{(1)}_{\tau A} = 2\nabla^B C^0_{AB}\,, \qquad k^{(-1)}_{AB} = \frac{1}{2}\,C^0_{AB}\,. \tag{8.16}$$

Note that this is enough to also determine $\tilde{k}_{\tau\tau}$ from the the constraints $k^a_a = D^a k_{ab} = 0$, and the only undetermined data is therefore $k^{(3)}_{\tau\tau}$. We show in Appendix J that (8.16) necessarily implies that k_{ab} takes the form

$$k_{ab} = -\,(D_a D_b + h_{ab})\,\Phi\,, \qquad (D^2 + 3)\,\Phi = 0\,, \tag{8.17}$$

where Φ is the Goldstone mode of supertranslations at spatial infinity [173]. Just like the electric potential, Φ is fully characterised by its leading asymptotic data $\Phi^{(-1)}$ and $\Phi^{(3)}$. In Appendix J we confirm the identification $\Phi^{(-1)} = C$ with the supertranslation mode (8.7) previously made in [173]. The remaining degree of freedom $\Phi^{(3)}$ then corresponds to the undetermined data $k^{(3)}_{\tau\tau}$. It is known since the work of Troessaert that $\Phi^{(3)}$ is in fact pure gauge [165], and we can therefore consider $\Phi^{(3)} = k^{(3)}_{\tau\tau} = 0$ without loss of generality. Thus k_{ab} is fully determined by the supertranslation mode C. Another direct

[4] Alternatively one could start from the Friedrich gauge which naturally covers a finite portion of $\mathscr{I}$ at least [170–172]. For this to also be the case in Beig-Schmidt coordinates would require to keep the quantity $u = -\rho e^{-\tau}$ tunable, a feature which we have to give up once we assume the large-ρ expansion (7.31). The limit $u \to -\infty$ is however compatible with the latter asymptotic expansion and turns out to be sufficient for our purpose of mapping data from $\mathscr{I}^+_-$ to i^0.

consequence of the restricted form (8.17) is the vanishing of the leading magnetic Weyl tensor (7.34),

$$B_{ab} = 0\,. \tag{8.18}$$

This result has often been assumed in the literature [107, 108, 174], and played a crucial role in the previous matching of Lorentz charges by Prabhu and Shehzad [153] (it allows to single out a Lorentz group within the BMS group). We just showed that (8.18) actually follows from the particular Bondi phase space described in Section 8.2. Because of (8.18), the solution space does not include Taub-NUT solutions which are relevant when discussing dual gravitational charges and their implications on soft graviton theorems [175–177]. Note that such configurations are allowed by the boundary conditions employed in the Hamiltonian formulation of [167].

Similarly, we find that the leading asymptotic data allowed by the homogeneous solutions for the subleading field j_{ab} actually vanishes,

$$j^{(2)}_{\tau\tau} = j^{(0)}_{\tau A} = j^{(-2)}_{AB} = 0\,, \tag{8.19}$$

while the subleading asymptotic data is given by

$$\begin{aligned}
j^{(4)}_{\tau\tau} &= 4\nabla_A C^0_{BC}\nabla^A C^{BC}_0 - 4\nabla_E C^0_{AB}\nabla^A C^{EB}_0 + 64\phi^0\,, && (8.20\text{a})\\
j^{(2)}_{\tau A} &= 4N^0_A + C^0_{AB}\nabla_C C^{BC}_0\,, && (8.20\text{b})\\
j^{(0)}_{AB} &= \frac{1}{8} C^0_{CD} C^{CD}_0\,\gamma_{AB}\,. && (8.20\text{c})
\end{aligned}$$

At this point we can compute the trace and divergence of j_{ab} and verify that they do satisfy the constraints (7.40) resulting from Einstein's equations in Beig–Schmidt gauge. Up to the available orders in τ, we indeed find perfect agreement between the direct computation from (8.20) and the evaluation of (7.40) requiring only knowledge of the leading data (8.15)-(8.16), namely

$$\begin{aligned}
j^a_a &= e^{-2\tau}\, C^0_{AB} C^{AB}_0 + O(e^{-4\tau})\,, && (8.21\text{a})\\
D^b j_{b\tau} &= -e^{-2\tau}\, C^0_{AB} C^{AB}_0 + O(e^{-4\tau})\,, && (8.21\text{b})\\
D^b j_{bA} &= \frac{1}{2} e^{-2\tau}\,\partial_A\left(C^0_{BC} C^{BC}_0\right) + O(e^{-4\tau})\,. && (8.21\text{c})
\end{aligned}$$

We can now come back to the discussion started at the end of Section 7.3 regarding the clean separation observed between homogeneous and particular solutions of j_{ab}. By explicit coordinate transformation between Bondi and Beig–Schmidt gauges, we just obtained a large-τ behavior for j_{ab} which coincides with that of the homogeneous solutions to (7.39)-(7.40) and described in (8.11). Consistency of our findings with the Beig–Schmidt dynamics therefore requires that the particular solution of j_{ab} determined by the source terms in (7.39) be subleading in τ, a fact which we have verified in Appendix H by direct evaluation of the source terms. Thus the quantities (8.20) are entirely governed by homogeneous solutions to (7.39)-(7.40), which will prove crucial to the derivation of the antipodal matching condition of the angular momentum aspect N_A.

The map between data at $\mathscr{I}^-_+$ in retarded Bondi gauge and data in the limit $\tau \to -\infty$ in Beig–Schmidt gauge is worked out in a similar way. The resulting identifications have the same functional form as above, with a few minus signs differences. The rule of thumb is that any τ index yields a relative minus sign. In particular, we find

$$\sigma(\tau, x^A) = 2m^0|_{\mathscr{I}^-_+}\, e^{3\tau} + O(e^{5\tau})\,, \tag{8.22a}$$

$$k_{AB}(\tau, x^A) = \frac{1}{2} C^0_{AB}|_{\mathscr{I}^-_+}\, e^{-\tau} + O(e^{\tau})\,, \tag{8.22b}$$

$$j_{\tau A}(\tau, x^A) = -\left(4N^0_A + C^0_{AB}\nabla_C C^{BC}_0\right)|_{\mathscr{I}^-_+}\, e^{2\tau} + O(e^{4\tau})\,. \tag{8.22c}$$

The antipodal matching conditions, to be studied in the next section, give relations between quantities defined in the two limits $\tau \to \infty$ and $\tau \to -\infty$.

8.5 Derivation of the antipodal matching relations

We are now ready to give a complete derivation of the antipodal matching conditions used by Strominger and crucial in establishing an equivalence between soft graviton theorems and Ward identities associated with BMS symmetries [121, 143, 178]. In terms of the antipodal map Υ defined at the end of Section 8.1, these antipodal matching relations read

$$\Upsilon^*\, m|_{\mathscr{I}^+_-} = m|_{\mathscr{I}^-_+}\,, \tag{8.23a}$$

$$\Upsilon^*\, C_{AB}|_{\mathscr{I}^+_-} = -\, C_{AB}|_{\mathscr{I}^-_+}\,, \tag{8.23b}$$

$$\Upsilon^*\, N_A|_{\mathscr{I}^+_-} = -N_A|_{\mathscr{I}^-_+}\,. \tag{8.23c}$$

In the previous section we mapped the leading Bondi data onto a subset of the leading Beig–Schmidt data. A key observation is that the latter is fully governed by homogeneous solutions of the Beig–Schmidt equations. In this section we show that the antipodal matching relations follow from the parity properties of these homogeneous solutions under the $\mathcal{H}$-antipodal map $\Upsilon_{\mathcal{H}}$.

As commented on in Section 8.1, various works already exist on this topic [152, 153, 165, 168]. As summarised in [168], in the Hamiltonian framework conditions have been given at spacelike infinity in order to recover BMS symmetries and argue in favour of the antipodal matching among past and future null infinities. Earlier, some steps towards the derivation of (8.23) were taken by Troessaert [165] by mapping the Bondi-Sachs gauge to the Beig-Schmidt gauge at leading order. After showing that the Lie algebras associated with global BMS symmetries ($\mathscr{I}$) and Spi-symmetries (i^0) are isomorphic, he argued that the charge density and symmetry parameter associated with Spi-supertranslations both satisfy antipodal relations. However his analysis was restricted to linearized gravity. Recently Prabhu [152] and Prabhu and Shehzad [153] tackled the antipodal matching of both supertranslation and angular momentum charges using the formalism of Ashtekar and Hansen [107], formally without restricting to the linear theory. Specifically, a number of assumptions were required to achieve the angular momentum matching, among them the vanishing of the leading magnetic Weyl tensor B_{ab} was assumed and the inhomogeneous terms were neglected in the subleading equation of motion defining the angular momentum data [153]. In this approach, furthermore, the matching is somewhat indirect because the charges at $\mathscr{I}^+$ and i^0 are independent and linked through the observation that the asymptotic limit of certain bulk spacetime quantity is the same as the limit toward i^0 ($\mathscr{I}^+$) of the charge defined on $\mathscr{I}^+$ (i^0).

The analysis that we present here completes these results in various important ways. First, the connection we make between $\mathscr{I}$ and i^0 goes beyond that of matching Lie algebras associated with asymptotic symmetries, since we have provided in Section 8.4 a precise dictionary between Bondi data and Beig–Schmidt data. This allows us to unambiguously identify where the initial data of the shear C_{AB}, mass aspect m and angular momentum aspect N_A sits in the Beig–Schmidt gauge, and to proceed with the study of their parity properties and matching of the asymptotic charges at $\mathscr{I}$ and i^0.

Such analysis does not require any further assumption in the bulk, except those made on the structure near null infinity. For example, as seen in the previous section, the leading magnetic Weyl tensor at spacelike infinity B_{ab} vanishes as a consequence of these. In the current section, the key point we will use in the derivation of the antipodal matching (8.23) also follows from the structure at null infinity. The relations (8.23) in fact stem from parity properties of the *homogeneous solutions* for σ, k_{ab} and j_{ab} under the $\mathcal{H}$-antipodal map $\Upsilon_{\mathcal{H}}$. While these were crucially used in [152, 153, 165] as well, here we do not need to discard source terms quadratic in σ and k_{ab} in the equations (7.39)-(7.40) determining j_{ab}. Rather, the dictionary of Section 8.4 between Bondi and Beig–Schmidt data shows that such source terms do not affect the large-τ behavior of j_{ab}, and hence the angular momentum aspect N_A is still fully governed by homogeneous solutions of j_{ab}. The derivation of the antipodal relations (8.23) based on parity properties of homogeneous solutions to the Beig–Schmidt equations then proceeds unobstructed.

Harmonic and Legendre functions. The dependence on the sphere coordinates x^A will be treated by decomposition into scalar spherical harmonics $Y_l^m(x^A)$, satisfying

$$\nabla^2 Y_l^m = -l(l+1) Y_l^m\,, \qquad \Upsilon^* \, Y_l^m = (-1)^l \, Y_l^m\,. \tag{8.24}$$

Beig–Schmidt equations then generically reduce to Legendre equations of the form

$$\left[(1-s^2)\partial_s^2 - 2s\partial_s + l(l+1) - \frac{n^2}{1-s^2}\right] F(s) = S(s)\,, \quad s \equiv \tanh\tau \in (-1,1)\,, \tag{8.25}$$

with $S(s)$ a generic source term. The homogeneous solutions are associated Legendre functions on the cut [179],

$$P_l^n(s)\,,\; Q_l^n(s)\,, \qquad (n \geq 0)\,, \tag{8.26}$$

satisfying the parity properties

$$P_l^n(-s) = (-1)^{l+n} P_l^n(s)\,, \qquad Q_l^n(-s) = (-1)^{l+n+1} Q_l^n(s)\,. \tag{8.27}$$

Of importance to us will be their asymptotic behavior in the limit $s \to \pm 1$, or equivalently in the limit $\tau \to \pm\infty$. For $l \geq n$, we have

$$P_l^n(s) = O\left((1-s)^{n/2}\right) = O(e^{-n\tau})\,, \qquad Q_l^n(s) = O\left((1-s)^{-n/2}\right) = O(e^{n\tau})\,, \tag{8.28}$$

while solutions with $l < n$ have a separate asymptotic behavior. For $n = 1$, we have

$$P_0^1(s)\,, Q_0^1(s) = O(1/\sqrt{1-s}) = O(e^{\tau})\,, \tag{8.29}$$

while for $n = 2$, we have

$$P_0^2(s)\,, P_1^2(s)\,, Q_0^2(s)\,, Q_1^2(s) = O\left(1/(1-s)\right) = O(e^{2\tau})\,. \tag{8.30}$$

We can now proceed to the derivation of the antipodal relations (8.23).

Mass aspect. The initial value of the Bondi mass aspect at $\mathscr{I}^+_-$ is carried by

$$\sigma^{(3)} = 2m|_{\mathscr{I}^+_-}\,, \tag{8.31}$$

where the electric potential σ solves the homogeneous equation

$$\left[-\partial_\tau^2 - 2\tanh\tau\,\partial_\tau + \cosh^{-2}\tau\,\nabla^2 + 3\right]\sigma = 0\,. \tag{8.32}$$

We introduce the variable $s = \tanh\tau \in (-1, 1)$ and decompose σ in spherical harmonics,

$$\sigma(s, x^A) = \sqrt{1-s^2}\sum_{l,m} \sigma_{lm}(s) Y_l^m(x^A)\,. \tag{8.33}$$

The coefficients then satisfy the Legendre differential equation

$$\left[(1-s^2)\partial_s^2 - 2s\partial_s + l(l+1) - \frac{4}{1-s^2}\right]\sigma_{lm}(s) = 0\,, \tag{8.34}$$

whose solutions are the Legendre functions

$$P_l^2(s)\,, Q_l^2(s)\,. \tag{8.35}$$

Taking into account the prefactor $\sqrt{1-s^2}$ in (8.33), we can confirm that independent solutions behave either as $O(e^{\tau})$ or $O(e^{-3\tau})$ in agreement with (8.9). In Section 8.4 we found that the mode $O(e^{\tau})$ is absent, by explicit mapping of the Bondi data onto the Beig-Schmidt data. Therefore we conclude that the relevant general solution for the electric potential takes the form

$$\sigma(s, x^A) = \sqrt{1-s^2} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} \, P_l^2(s) Y_l^m(x^A) \,. \tag{8.36}$$

In particular, it is parity-even under the $\mathcal{H}$-antipodal map,

$$\Upsilon_{\mathcal{H}}^* \sigma = \sigma \,, \tag{8.37}$$

in agreement with earlier discussions on the existence of a regular null infinity [152, 164, 165, 173]. Making use of (8.22), this also directly yields the antipodal relation for the Bondi mass aspect,

$$\Upsilon^* \, m|_{\mathscr{I}^+_-} = m|_{\mathscr{I}^-_+} \,. \tag{8.38}$$

Shear tensor. The initial value of the shear tensor at $\mathscr{I}^+_-$ is encoded in the leading nontrivial component of the magnetic potential,

$$2\, k_{AB}^{(-1)} = C_{AB}|_{\mathscr{I}^+_-} = -2\nabla_A \nabla_B C + \gamma_{AB} \nabla^2 C \,. \tag{8.39}$$

We know from Appendix J that the relevant solutions for k_{ab} take the form

$$k_{ab} = -\left(D_a D_b + h_{ab}\right) \Phi \,, \qquad \left(D^2 + 3\right) \Phi = 0 \,, \tag{8.40}$$

where the supertranslation mode C is identified with the leading large-τ behavior of Φ,

$$\Phi(\tau, x^A) = e^{\tau}\, C(x^A) + O(e^{-\tau}) \,. \tag{8.41}$$

Thus it is the scalar potential Φ that carries the relevant information. Its evolution equation is identical to that of σ, although this time there is no restriction on its asymptotic behavior. Given (8.41) we are interested in the set of solutions scaling like $O(e^{\tau})$,

$$\Phi(s, x^A) = \sqrt{1-s^2} \left(\sum_{l=0,1} \sum_{m=-l}^{l} a_{lm} \, P_l^2(s) + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm} \, Q_l^2(s) \right) Y_l^m(x^A) \,. \tag{8.42}$$

This solution *almost* has definite odd parity under $\Upsilon_{\mathcal{H}}$, which is however spoiled by the a_{lm} with $l = 0, 1$. But these do not contribute to the shear (8.39) since the four lowest spherical harmonics Y_0^m, Y_1^m are precisely annihilated by the differential operator $-2\nabla_A \nabla_B + \gamma_{AB} \nabla^2$. Moreover it can be seen from (J.6) that none of the data specifying

k_{ab} is actually sensitive to the a_{lm}, such that we can as well set them to zero. This implies that the k_{ab} satisfies the parity property

$$\Upsilon^*_{\mathcal{H}}\, k_{ab} = -k_{ab}\,. \tag{8.43}$$

Strictly speaking this requires the solutions of Φ scaling like $O(e^{-3\tau})$ to be absent, and this can always be achieved without loss of generality since these solutions can be removed by pure gauge transformations [165]. It is sometimes useful to express these relations in terms of $k_{\tau\tau}\,, k_{\tau A}$ and k_{AB} viewed as time-dependent scalar, vector and tensor fields on $\mathbb{S}^2$, respectively. These read

$$\Upsilon^*\, k_{\tau\tau}(-\tau) = -k_{\tau\tau}(\tau)\,, \qquad \Upsilon^*\, k_{\tau A}(-\tau) = k_{\tau A}(\tau)\,, \qquad \Upsilon^*\, k_{AB}(-\tau) = -k_{AB}(\tau)\,. \tag{8.44}$$

Using (8.22), this yields in particular the antipodal relation of the shear tensor,

$$\Upsilon^*\, C_{AB}|_{\mathscr{I}^+_-} = -\, C_{AB}|_{\mathscr{I}^-_+}\,. \tag{8.45}$$

Angular momentum aspect. The initial value of the angular momentum aspect at $\mathscr{I}^+_-$ is carried by the leading data of the field j_{ab}, namely

$$j^{(2)}_{\tau A} = 4N_A + C_{AB}\nabla_C C^{BC}|_{\mathscr{I}^+_-}\,. \tag{8.46}$$

In order to discuss the relevant solutions, we make use of Helmholtz decomposition

$$j_{\tau A} = \nabla_A \Psi_1 + \epsilon_{AB}\nabla^B \Psi_2\,. \tag{8.47}$$

In Appendix H we show that the leading order $\Psi_1 = O(e^{-2\tau})$ is fully determined in terms of $j_{\tau\tau}$ through

$$\nabla^2 \Psi_1 = \cosh^2\tau\,(\partial_\tau + 3\tanh\tau)\, j_{\tau\tau}\,, \tag{8.48}$$

while the leading order $\Psi_2 = O(e^{-2\tau})$ is associated with homogeneous solutions that satisfy

$$\left[-\partial_\tau^2 - 2\tanh\tau\,\partial_\tau + \cosh^{-2}\tau\,\nabla^2\right]\Psi_2 = 0\,. \tag{8.49}$$

The determination of Ψ_1 therefore follows the determination of $j_{\tau\tau}$. Expanding the latter into spherical harmonics,

$$j_{\tau\tau}(s, x^A) = (1-s^2)^{3/2}\sum_{l,m} j_{lm}(s) Y_l^m(x^A)\,, \tag{8.50}$$

the coefficients $j_{lm}(s)$ corresponding to the homogeneous solutions of (H.8a) can be shown to satisfy the Legendre equation (8.25) with $n = 1$. Therefore the solutions with allowed asymptotic behavior $O(e^{-4\tau})$ are of the form

$$j_{\tau\tau}(s, x^A) = (1-s^2)^{3/2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} j_{lm}\, P_l^1(s)\, Y_l^m(x^A)\,, \tag{8.51}$$

which are parity-odd under the $\mathcal{H}$-antipodal map. Thus the right-hand side of (8.48) is parity-even and behaves as $O(e^{-2\tau})$. Hence we conclude that the leading term in Ψ_1 is parity-even, and similarly for the corresponding vector field $j^{\parallel}_{\tau A} \equiv \nabla_A \Psi_1$,

$$\Upsilon^* \, j^{\parallel}_{\tau A}(-\tau) = j^{\parallel}_{\tau A}(\tau) \,. \tag{8.52}$$

The determination of Ψ_2 is straightforward. Expanding into spherical harmonics,

$$\Psi_2(s, x^A) = \sqrt{1-s^2} \sum_{l,m} \Psi_{2,lm}(s) \, Y_l^m(x^A) \,, \tag{8.53}$$

the coefficients $\Psi_{2,lm}(s)$ then satisfy the Legendre equation (8.25) with $n = 1$. The solutions with asymptotic behavior $O(e^{-2\tau})$ are of the form

$$\Psi_2(s, x^A) = \sqrt{1-s^2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} \, P_l^1(s) \, Y_l^m(x^A) \,, \tag{8.54}$$

which are parity-odd under $\Upsilon_{\mathcal{H}}$ following the parity properties of spherical harmonics and Legendre functions. The corresponding solutions of $j^{\perp}_{\tau A} \equiv \epsilon_{AB} \nabla^B \Psi_2$ viewed as a time-dependent vector field on the sphere $\mathbb{S}^2$, then satisfy

$$\Upsilon^* \, j^{\perp}_{\tau A}(-\tau) = j^{\perp}_{\tau A}(\tau) \,. \tag{8.55}$$

Let us illustrate this with an example of direct relevance to the Kerr solution. Since $P_1^1(s) = \sqrt{1-s^2}$ and $Y_1^0(\theta, \varphi) = \cos\theta$, the mode with $(l, m) = (1, 0)$ is given by

$$\Psi_2 = a_{10} \, (1 - s^2) \cos\theta \,, \tag{8.56}$$

and correspondingly

$$j^{\perp}_{\tau\varphi} = a_{10} \, (1 - s^2) \sin^2\theta \,, \qquad j^{\perp}_{\tau\theta} = 0 \,. \tag{8.57}$$

The above nonzero component is parity-even under $\Upsilon_{\mathcal{H}}$ in agreement with (8.55) and the conventions spelled out at the end of the introduction. Note that the angular momentum aspect of the Kerr solution precisely takes the form (8.57), in which case one can explicitly identify a_{10} with the angular velocity of the Kerr solution [160].

Putting (8.52) and (8.55) together and making use of (8.22), we thus obtain the antipodal relation of the angular momentum aspect,

$$\Upsilon^* \, N_A|_{\mathscr{I}^+_-} = -N_A|_{\mathscr{I}^-_+} \,. \tag{8.58}$$

8.6 Matching of BMS charges from $\mathscr{I}$ to i^0

As advertised in the Section 8.1 we are able to explicitly match various proposals of global BMS charges in Bondi gauge (which we presented as $Q_{(\alpha,\beta)}[T, Y^A]$ in Section 7.1) to the charges that were directly constructed in Beig–Schmidt gauge by Compère and Dehouck (CD) at spatial infinity [144]. Conservation of the CD charges then directly implies conservation of the corresponding BMS charges across i^0.

BMS charges at $\mathscr{I}^+_-$. If we evaluate the Bondi charges $Q_{(\alpha,\beta)}[T, Y^A]$ in the limit to $\mathscr{I}^+_-$ $(u \to -\infty)$ using the expansions (8.1), they result in

$$\lim_{u\to-\infty} Q_{(\alpha,\beta)}[T] = \frac{1}{4\pi G}\int d\Omega\, T\, m^0\,, \tag{8.59}$$

$$\lim_{u\to-\infty} Q_{(\alpha,\beta)}[Y^A] = \frac{1}{8\pi G}\int d\Omega\, Y^A N^0_A\,. \tag{8.60}$$

We note that the charges in this limit are independent of the parameters $(\alpha\,,\beta)$. In particular the term controlled by α is proportional to

$$\int d\Omega\, Y^A\left[\partial_A(C^0_{BC}C_0^{BC}) + 4C^0_{AB}\nabla_C C_0^{BC}\right] = 0\,, \tag{8.61}$$

which integrates to zero on account of the various identities satisfied by C^0_{AB} and Y^A.

We show below that the expressions (8.59)-(8.60) indeed coincide with the conserved CD charges. Since the latter are conserved, they can be evaluated on any spacelike cut of $\mathcal{H}$ with topology of the sphere $\mathbb{S}^2$. We will restrict to constant-τ cuts, and subsequently take the limit $\tau \to \infty$ such that we can easily write them in the terms of the leading Bondi data m^0, C^0_{AB} and N^0_A.

Supertranslation charges. The supertranslation charges at spatial infinity are given by [155]

$$Q_{\mathrm{CD}}[\omega] = \frac{\cosh^2\tau}{4\pi G}\oint d\Omega\,\left(\omega\partial_\tau\sigma - \sigma\partial_\tau\omega\right)\,, \tag{8.62}$$

where the symmetry parameter $\omega(x^a)$ satisfies the constraint

$$\left(D^2+3\right)\omega = 0\,, \tag{8.63}$$

and therefore admits the large-τ expansion

$$\omega(\tau, x^A) = e^\tau\,\bar\omega(x^A) + O(e^{-\tau})\,. \tag{8.64}$$

Under supertranslation, the magnetic potential k_{ab} and supertranslation mode Φ defined in (8.17) transform as

$$\delta_\omega k_{ab} = 2\left(D_aD_b + h_{ab}\right)\omega\,, \qquad \delta_\omega\Phi = -2\omega\,. \tag{8.65}$$

Using (H.2) and (J.8), we can further identify $\bar\omega$ as the symmetry parameter of supertranslations at $\mathscr{I}$,

$$\bar\omega = -\frac{1}{2}T\,, \qquad \delta_T C = T\,. \tag{8.66}$$

By identical arguments as those used in Section 8.5, we also infer the antipodal matching of the supertranslation parameter,[5]

$$\Upsilon^* T|_{\mathscr{I}^+_-} = -T|_{\mathscr{I}^-_+} \,. \tag{8.68}$$

This is the condition used by Strominger to single out the diagonal subgroup of BMS($\mathscr{I}^+$) $\times$ BMS($\mathscr{I}^-$) as the symmetry group of the gravitational $\mathcal{S}$-matrix [121]. We evaluate these charges in the limit $\tau \to \infty$ and use the dictionary of Section 8.4 to express them in terms of Bondi data,

$$Q_{\text{CD}}[\omega] = -\frac{1}{4\pi G} \oint d\Omega\, \bar{\omega}\, \sigma^{(3)} = -\frac{1}{2\pi G} \oint d\Omega\, \bar{\omega}\, m^0 = \frac{1}{4\pi G} \oint d\Omega\, T\, m^0 \,. \tag{8.69}$$

This indeed agrees with (8.59).

Lorentz charges. The Lorentz charges are given by [155]

$$Q_{\text{CD}}[\xi^a] = \frac{\cosh^2 \tau}{8\pi G} \int d\Omega\, \xi^a \delta^b_\tau \left[-j_{ab} + \frac{1}{2} i_{ab} + \frac{1}{2} k^c_a k_{cb} + h_{ab} F \right] , \tag{8.70}$$

with

$$F \equiv 8\sigma^2 + D^c \sigma D_c \sigma - \frac{1}{8} k^{cd} k_{cd} + k^{cd} D_c D_d \sigma \,. \tag{8.71}$$

The symmetry parameters ξ^a are the Killing vector fields of the hyperboloid $\mathcal{H}$, and indeed the isometry group of three-dimensional de Sitter space is isomorphic to the Lorentz group SO(1,3). They satisfy

$$D_a \xi_b + D_b \xi_a = 0 \,, \tag{8.72}$$

which in the (τ, x^A) coordinate system reads

$$0 = \partial_\tau \xi^\tau \,, \tag{8.73a}$$

$$0 = \partial_\tau \xi_A - \partial_A \xi^\tau - 2 \tanh \tau\, \xi_A \,, \tag{8.73b}$$

$$0 = \nabla_A \xi_B + \nabla_B \xi_A + 2 \cosh \tau\, \sinh \tau\, \gamma_{AB}\, \xi^\tau \,. \tag{8.73c}$$

The solutions to these equations are given by

$$\xi^\tau = b(x^A) \,, \tag{8.74a}$$

$$\xi^A = \tilde{\xi}^A + \tanh \tau\, \partial^A b \,, \qquad \partial_\tau \tilde{\xi}^A = 0 \,, \tag{8.74b}$$

[5] The conventions adopted here are such that the supertranslation mode C transforms in the same way at $\mathscr{I}^\pm$,

$$\delta_T C = T|_{\mathscr{I}^+} \,, \qquad \delta_T C = T|_{\mathscr{I}^-} \,, \tag{8.67}$$

and therefore differ by a relative minus sign from those adopted by Strominger [121].

with the constraints

$$0 = \nabla_A \tilde{\xi}_B + \nabla_B \tilde{\xi}_A \,, \tag{8.75a}$$
$$0 = (\nabla_A \nabla_B + \nabla_B \nabla_A + 2\gamma_{AB})\, b \,. \tag{8.75b}$$

The first constraint implies that $\tilde{\xi}^A$ is a Killing vector field on the sphere $\mathbb{S}^2$ that parametrise rotations. The second constraint implies that b is a linear combination of the three spherical harmonics $Y^m_{l=1}$ parametrising boosts. $\tilde{\xi}^A$ and b have even and odd parities under the antipodal map Υ, respectively, such that ξ^a is even under the $\mathcal{H}$-antipodal map $\Upsilon_{\mathcal{H}}$. In the limit $\tau \to \infty$, we can define the vector fields on the sphere

$$Y^A \equiv - \lim_{\tau\to\infty} \xi^A = -(\tilde{\xi}^A + \partial^A b) \,, \qquad b = \frac{1}{2}\nabla_A Y^A \,. \tag{8.76}$$

Using the above relations, one finds that they satisfy the conformal Killing equation on the sphere,

$$\nabla_A Y_B + \nabla_B Y_A = \gamma_{AB}\, \nabla_C Y^C \,. \tag{8.77}$$

Consistently, the group of conformal isometries of $\mathbb{S}^2$ is isomorphic to the Lorentz group SO(1,3). The even parity of ξ^a under $\Upsilon_{\mathcal{H}}$ in particular implies the antipodal matching relation

$$\Upsilon^* \, Y^A|_{\mathscr{I}^+_-} = Y^A|_{\mathscr{I}^-_+} \,. \tag{8.78}$$

We can then evaluate the Lorentz charges (8.70) in the limit $\tau \to \infty$ using the expansions of Section 8.3 and their expressions in terns of Bondi data given in Section 8.4. One obtains

$$Q_{\mathrm{CD}}[\xi^a] = -\frac{1}{32\pi G}\int d\Omega \left[Y^A \left(-j^{(2)}_{\tau A} + 2k^{(-1)}_{AB} k^{(1)B}{}_{\tau} \right) - \nabla_A Y^A \, k^{(-1)}_{BC} k^{(-1)BC} \right] \tag{8.79a}$$
$$= \frac{1}{8\pi G}\int d\Omega \, Y^A \left[N^0_A - \frac{1}{16}\partial_A (C^0_{BC} C_0^{BC}) - \frac{1}{4} C^0_{AB} \nabla_C C_0^{BC} \right] \tag{8.79b}$$
$$= \frac{1}{8\pi G}\int d\Omega \, Y^A N^0_A \,, \tag{8.79c}$$

where we made use of (8.61) in the last equality. This exactly coincides with (8.60).

8.7 Concluding remarks

In this chapter we derived the antipodal matching relations (8.23) used in proving the equivalence between the soft graviton theorems and BMS charge conservation across spatial infinity [121,143,178]. We also explicitly demonstrated that the various proposals for global BMS charges at $\mathscr{I}$ precisely match the conserved charges at spatial infinity. To derive these results we made a few assumptions in Section 8.2 regarding the gravitational phase space at $\mathscr{I}$ in Bondi gauge, in line with what is usually considered appropriate to the gravitational scattering problem (at least within the celestial holography program). We then provided a precise map between gravitational data in Bondi gauge and Beig–Schmidt gauge, allowing us to address the dynamical evolution taking place between $\mathscr{I}^-_+$ and $\mathscr{I}^+_-$. The Bondi data maps onto a restricted subset of the Beig–Schmidt data, yielding in particular a vanishing leading magnetic Weyl tensor B_{ab} at i^0. This justifies to some extent the assumption made by Prabhu and Shehzad who provided a different derivation of charge matching between i^0 and $\mathscr{I}$ [153]. We also confirmed that the electric and magnetic potentials σ and k_{ab} satisfy specific parity properties on the hyperboloid $\mathcal{H}$ (similar parity properties naturally arise in the Hamiltonian framework [166–168]).

Several comments are in order along with a cursory overview of future directions.

General u-behavior at $\mathscr{I}$. We made specific assumptions regarding the falloff rate of the matter stress tensor (8.2) and other gravitational quantities (8.1) in the limit to $\mathscr{I}^+_-$. While it has been recognised long ago that m^0 should not be restricted to be spherically symmetric – as opposed to what characterise CK spacetimes [146] – in order to not trivialize BMS charges [152], the conditions on the asymptotic behaviour of the shear/news tensor are much subtler.

We start from the assumption that the shear is expanded in a Taylor series in u^{-1}. The picture we present is in line with the minimal requirements for recovering the subleading soft graviton theorem. The condition $N_{AB} = o(u^{-\gamma})$ is usually taken in order to guarantee that the associated BMS fluxes are finite on all of $\mathscr{I}$: greater than $\gamma = 1$ for the leading soft theorem [121], $\gamma = 2$ for the subleading soft theorem [169], and $\gamma = 3$ for the sub-subleading soft theorem [180]. However, it is important to stress that our work does not exclude other possibilities that have been given both in the context of mathematical general relativity and soft theorems, as exemplified below.

For example, Christodoulou–Klainerman proof of the non-linear stability of Minkowski space implies $N_{AB} = O(u^{-3/2})$ [146], while both different stability proofs [181, 182] and the work of Prabhu and Shezad result in less stringent falloff behaviours [152, 153]. We can compare such various proposal with our working hypotheses by noticing that the crucial argument given at the end of Section 8.2 still sets to zero C^{α}_{AB} in $C_{AB} = C^0_{AB} + u^{-\alpha}C^{\alpha}_{AB} + \ldots$ for any $\alpha \in (0, 1)$. Furthermore, it is clear from our map that Bondi data with non-integer α are mapped to phase spaces at spacelike infinity that differ from the standard Beig-Schmidt phase space because of necessarily non-integer powers of ρ and τ needed in the map.

Similarly, the vanishing of the $O(u^{-1})$ term in the shear C_{AB} can be contrasted with the appearance of an analogous term in linearised gravity in conjunction with logarithmic corrections to the subleading soft graviton theorem [183, 184]. According to (7.6b), such piece would yield a $O(\log u)$ term in the angular momentum aspect, which is excluded in our non-polyhomogeneous falloff conditions (8.1).

Connected to this, but not only tied to it, the assumption that the large u expansion is polynomial both for C_{AB} and the other metric functions that we consider, as well as for all r-subleading terms of the metric, does not hold in general [185, 186]. While polyhomogeneous asymptotic expansions in r are believed to be a generic feature of physically relevant asymptotically flat systems (of which CK spacetimes are an example) [122, 123, 187], although contrasting results exist (see [188] and references therein), the polyhomogeneous behaviour in u is less understood. The potential interconnection of the two sources of polyhomogeneity has been briefly recognised in [185]. It is clear from what presented in this chapter that more general Beig–Schmidt data than those we have considered might result in such configurations at null infinity.

Further investigation of all these points are clearly desired. As a guiding principle, one could hope to deal with the issue of flux divergences in the u-integrals not by imposing ad hoc conditions, however well motivated, but by developing a suitable renormalization scheme. This is currently missing.

Independently of the issue of u-renormalization, a typical question within mathematical general relativity is that concerning the existence of physically relevant configurations that satisfy given conditions on the asymptotic structure or, somewhat equivalently, the existence of well-defined Cauchy data that evolve to a given asymptotic structure. The works on the non-linear stability of Minkowski spacetime are pivotal examples. Recently, Mohamed and Valiente Kroon studied the interplay between initial data sets of spin-1 and spin-2 fields and matching of the corresponding asymptotic charges across spatial infinity [148]. In some sense, the philosophy of this latter work is reverse to ours, as they assess which subset of asymptotic initial data gives rise to finite charges at the corners of $\mathscr{I}$, while we prescribe conditions at $\mathscr{I}$ such that charges are finite and reconstruct the corresponding data at spatial infinity, which turns out to have restricted parity properties.

Sub-subleading antipodal matching. The antipodal matching relations provided in (8.23) do not suffice to derive the sub-subleading soft graviton theorem [180, 189–191]. For this one needs an extra antipodal matching condition on one of the subleading Bondi fields sitting at order $O(r^{-1})$ in the angular components of the metric [180]. An obvious continuation of this work would be to work out the dictionary between Bondi and Beig–Schmidt data to lower orders in r and u such as to access the relevant field.

Extensions of BMS and corresponding phase space. We restricted our attention to the standard Bondi phase space in which the metric on the sphere $\mathbb{S}^2$ is the smooth unit round sphere metric. This constraint should be relaxed in order to allow for extensions of the BMS group at $\mathscr{I}$ that also include *superrotations* [133, 134, 136, 190–192]. However it is not known whether spatial infinity i^0 also admits such extensions of the BMS group, and if it exists, the corresponding extended phase space in Beig–Schmidt gauge is yet to be uncovered. This is the reason we did not discuss the matching of superrotation charges between $\mathscr{I}$ and i^0 in Section 8.6; none has been defined at i^0 as of yet.

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So long, and thanks for all the metrics!

Appendix A

Tools to find conformal vectors

In this appendix we provide a framework and useful formulas to find conformal Killing vectors on several families of manifolds. It should be thought of as complementary material to what discussed in Chapter 3, as remarked in Section 3.5.

A.1 Conformal vectors on product spaces

Let us consider a product manifold $M \times N$ of dimension $d = m + n$, parametrized by $x^A = (x^a, x^i)$ and with metric

$$g = g_{AB} dx^A dx^B = g_{ab}(x^a) dx^a dx^b + g_{ij}(x^i) dx^i dx^j. \tag{A.1}$$

Only the components Γ^c_{ab} and Γ^k_{ij} of the connection are non vanishing. The conformal Killing equations read

$$\nabla_A K_B + \nabla_B K_A = 2\psi(x^A) g_{AB}, \tag{A.2}$$

with

$$\psi(x^A) = \frac{1}{d} \nabla_E K^E = \frac{1}{d} \left(\nabla_a K^a + \nabla_i K^i \right). \tag{A.3}$$

Given the block-diagonal form (A.1) of the metric, these equations reduce to

$$\nabla_a K_b + \nabla_b K_a = 2\psi(x^A) g_{ab}, \tag{A.4}$$

$$\nabla_a K_j + \nabla_j K_a = 0 \tag{A.5}$$

$$\nabla_i K_j + \nabla_j K_i = 2\psi(x^A) g_{ij}, \tag{A.6}$$

where $K_a(x^a, x^i)$, $K_i(x^a, x^i)$. From the trace of the first and third equations, we understand that

$$\psi(x^A) = \frac{1}{d} \nabla_E K^E = \frac{1}{m} \nabla_a K^a = \frac{1}{n} \nabla_i K^i. \tag{A.7}$$

Thus, once fixed a conformal factor $\psi(x^A)$, we can solve (A.4) and (A.6) for the dependence of K_a, K_i on the respective coordinates x^a, x^i. Equations (A.5) and (A.7) can then be used to find the dependence on the coordinates of the orthogonal submanifold.

It is possible to proceed further by dividing the derivatives $\nabla_A K_B$ in trace, symmetric traceless and antisymmetric parts as

$$\nabla_B K_A = \psi(x^A) g_{AB} + H_{AB}(x^A) + F_{AB}(x^A), \tag{A.8}$$

$$\nabla_b K_a = \psi(x^A) g_{ab} + H_{ab}(x^A) + F_{ab}(x^A), \tag{A.9}$$

$$\nabla_j K_i = \psi(x^A) g_{ij} + H_{ij}(x^A) + F_{ij}(x^A), \tag{A.10}$$

where H_{AB} is traceless symmetric and F is antisymmetric. By plugging them into (A.4) and (A.6) one finds $H_{AB} = 0$. Furthermore, by deriving (A.5) in ∇_b with (A.9) one has

$$\nabla_b \nabla_a K_j + g_{ab} \nabla_j \psi(x^A) + \nabla_j F_{ab}(x^A) = 0. \tag{A.11}$$

Since K_j is a scalar according to the connection of g_{ab}, one can extract the symmetric and antisymmetric parts, resulting in the equations

$$\nabla_a \partial_b K_j + g_{ab} \nabla_j \psi(x^A) = 0, \tag{A.12}$$

$$\nabla_j F_{ab} = 0. \tag{A.13}$$

Similar equations can be obtained for K_j and F_{ij}. The second equation implies that $F_{ab} = F_{ab}(x^a)$ and $F_{ij} = F_{ij}(x^i)$. We recall that a *conformal gradient vector* of g_{ab} is one of the form $K_a = \nabla_a \Omega(x^a)$, satisfying

$$\nabla_a \nabla_b \Omega = \frac{1}{m} \left(\Box_M \Omega\right) g_{ab}, \tag{A.14}$$

where we denote the Laplacians of the two submanifolds by $\Box_M$ and $\Box_N$. Note that this is precisely the content of equations (A.12); from its trace we find

$$\nabla_j \psi(x^A) = -\frac{1}{m} \Box_M K_j, \tag{A.15}$$

$$\nabla_a \psi(x^A) = -\frac{1}{n} \Box_N K_a. \tag{A.16}$$

We can hence rewrite (A.12) as

$$\nabla_a \partial_b K_j = g_{ab} \frac{1}{m} \Box_M K_j, \tag{A.17}$$

$$\nabla_i \partial_j K_a = g_{ij} \frac{1}{n} \Box_N K_a, \tag{A.18}$$

so that $\nabla_a K_j$ must be a gradient vector according to g_{ab} and analogously $\nabla_i K_a$ for g_{ab}. From the commutator in (3.30) one can also show that

$$(m-1)\nabla_i \nabla_a \psi(x^A) = R_{ab} \nabla^b K_i, \tag{A.19}$$

$$(n-1)\nabla_i \nabla_a \psi(x^A) = R_{ij} \nabla^j K_a, \tag{A.20}$$

which also imply (given (A.5))

$$\frac{R_{ab}}{m-1} \nabla^b K_i = -\frac{R_{ij}}{n-1} \nabla_a K^j. \tag{A.21}$$

These relations are useful to constrain the dependence of K_i on x^a and K_b on x^j (especially if the submanifolds have special properties, such as being Einstein or 1-dimensional).

What found so far can be summarised as follows:

- The components of $K_A = (K_a, K_i)$ must satisfy the conformal equations (A.4) and (A.6) separately on the submanifolds, together with the matching condition (A.7) and (A.5), which determines the dependence of a conformal vector on one submanifold on the coordinates of the other submanifold.
- With the decomposition in trace, symmetric traceless and antisymmetric parts of $\nabla_a K_b$ and $\nabla_i K_j$, from the equations above we obtain

$$\nabla_b K_a = \psi(x^A) g_{ab} + F_{ab}(x^a), \tag{A.22}$$

$$\nabla_j K_i = \psi(x^A) g_{ij} + F_{ij}(x^i), \tag{A.23}$$

  so that only the traceful part contains the coordinates of the other submanifold.
- From (A.5) also the relations (A.15)-(A.18) follow, indicating that the dependence of K_a on x^i is such that $\nabla_j K_a$ is a gradient conformal vector on N, and similarly K_j on M. Note that, once we fix a conformal factor $\psi(x^A)$, we automatically know the dependence of K_j on x^a (K_a on x^i) from (A.15)-(A.16), since their solution is formally the massless scalar propagator on M (N) with source $-m\nabla_j\psi$ ($-n\nabla_a\psi$).

We can now consider more specific classes of conformal vectors and specialize this discussion further.

Killing vectors. They represent proper isometries on $M \times N$, i.e. $\psi = 0$. Then equations (A.22)-(A.23) reduce to

$$\nabla_b K_a = F_{ab}(x^a), \qquad \nabla_j K_i = F_{ij}(x^i), \tag{A.24}$$

i.e the covariant derivatives do not contain the coordinates of the other submanifold. This means that K_a and K_j must be solutions to the distinct Killing equations on the respective submanifolds. Thus, sums of Killing vectors of the single subspaces will be Killing vectors of the product space, but the most general isometric K_a, K_i have non-trivial dependences on all the coordinates (x^a, x^i) determined by (A.15)-(A.16) and (A.19)-(A.20), that is

$$\Box_M K_j = 0, \qquad \Box_N K_a = 0, \tag{A.25a}$$

$$R_{ab}\nabla^b K_i = 0, \qquad R_{ij}\nabla^j K_a = 0. \tag{A.25b}$$

Homothetic vectors. They act as dilations, $\nabla_A\psi = 0$, i.e. $\psi = const$. For the same arguments as for Killing vectors, we need to solve (A.24), along with (A.25) so as to determine the dependence on the orthogonal directions. The components K_a and K_j must also be solutions of

$$\nabla_a K_b + \nabla_b K_a = 2\psi g_{ab}, \tag{A.26}$$

$$\nabla_i K_j + \nabla_j K_i = 2\psi g_{ij}. \tag{A.27}$$

This means that given two homothetic vectors $K_a(x^a)$ on M and $K_i(x^i)$ on N both with conformal factor ψ, then we can extend them to $K_a(x^a, x^i)$ and $K_i(x^a, x^i)$ on $M \times N$ with (A.25). The resulting $K_A = (K_a, K_j)$ is then a homothetic vector on $M \times N$ with conformal factor ψ.

Special vectors. These have conformal factor satisfying $\nabla_A\nabla_B\psi = 0$, which is equivalent to

$$\partial_A\partial_B\psi = \Gamma^C_{AB}\partial_C\psi. \tag{A.28}$$

In this case, (A.25b) also holds.

Generic conformal vectors. From (A.19)-(A.20) one obtains

$$\nabla_a\nabla_b\psi = \left(\frac{\Box_M\psi}{m}\right) g_{ab}, \tag{A.29a}$$

$$\nabla_i\nabla_j\psi = \left(\frac{\Box_N\psi}{n}\right) g_{ij}, \tag{A.29b}$$

along with

$$\frac{\Box_M\psi}{m} = \frac{\Box_N\psi}{n}. \tag{A.30}$$

We recognize here the equations for the gradient vectors $\nabla_b\psi$ and $\nabla_i\psi$. They can be seen as a generalization of the above conditions $\psi = 0$, $\nabla_A\psi = 0$ and $\nabla_A\nabla_B\psi = 0$. Understanding which conformal transformations ψ are present on a product manifold amounts to solving equations (A.29).

A.2 Conformal vectors on conformally-related spaces

In this section we would like to prove that under a Weyl transformation, a conformal Killing vector of the original manifold remains a conformal Killing vector for the final conformally-related manifold.

Let us first recall few well-known facts about conformal isometries. A d-dimensional space can have at most $\frac{d(d+1)}{2}$ independent Killing vectors and the maximal number is attained for spaces where all the curvature scalars (constructed with the Riemann tensor) are constant. The maximal number of conformal vectors in $d \geq 3$ is $\frac{(d+1)(d+2)}{2}$ (for $d = 2$, it is infinite). This is the case when a space is conformally flat.

Let us consider a manifold with metric g and conformal vector ξ_a, satisfying

$$\nabla_a \xi_b + \nabla_b \xi_a = \frac{2}{n} \left(\nabla_c \xi^c \right) g_{ab}. \tag{A.31}$$

In terms of a Weyl-rescaled metric $\hat{g} = e^{2\omega} g$, this equation reads

$$\hat{\nabla}_a \xi_b + \hat{\nabla}_b \xi_a + \xi_a \partial_b (2\omega) + \xi_b \partial_a (2\omega) = \frac{2}{n} \left(\hat{\nabla}_c \xi^c \right) \hat{g}_{ab} + \frac{2}{n} \xi_c \left(\partial^c 2\omega \right) \hat{g}_{ab}, \tag{A.32}$$

which is precisely the conformal equation for $\hat{\xi}_a = e^{2\omega} \xi_a$. We can conclude that given g with conformal vector ξ_a, after a Weyl transformation $\hat{\xi}_a = e^{2\omega} \xi_a$ (i.e. $\hat{\xi}^a = \xi^a$) is a conformal vector for $\hat{g} = e^{2\omega} g$, as one could expect.

Notice instead that under a Weyl transformation a Killing vector will be mapped to a conformal vector in general, since (A.32) with $\hat{\nabla}_c \xi^c = 0$ simply reads

$$\hat{\nabla}_a \xi_b + \hat{\nabla}_b \xi_a + \xi_a \partial_b (2\omega) + \xi_b \partial_a (2\omega) = \frac{2}{n} \xi_c \left(\partial^c 2\omega \right) \hat{g}_{ab}, \tag{A.33}$$

which is again the equation for a conformal vector $\hat{\xi}_a = e^{2\omega} \xi_a$ for $\hat{g}$. It will be an isometry of $\hat{g}$ if $\omega = const$, that is if the Weyl transformation is simply a global rescaling of the metric g.

A.3 An equation for the conformal factor

Let us obtain an equation that any conformal factor ψ on a given metric g must satisfy. We start from the conformal Killing equation

$$\nabla_i \xi_j + \nabla_j \xi_i = 2\psi(x) g_{ij}, \tag{A.34}$$

where $\psi(x) = \frac{1}{d} \nabla \cdot \xi$. By acting with ∇^i, one finds

$$\Box \xi_j + (d-2) \nabla_j \psi + R_{mj} \xi^m = 0, \tag{A.35}$$

and performing now the derivative ∇^j,

$$\nabla^j \Box \xi_j + (d-2)\Box\psi + \nabla^j \left(R_{mj} \xi^m \right) = 0. \tag{A.36}$$

From this equation thus follows that

$$(d-1)\Box\psi + R_{mj} \nabla^m \xi^j + \frac{1}{2} \xi^m \nabla_m R = 0. \tag{A.37}$$

For Einstein manifolds with $R = 2\lambda d(d-1)$ and $R_{ij} = 2\lambda(d-1)g_{ij}$, this equation reduces to

$$\left(\Box + 2\lambda d\right) \psi(x) = 0. \tag{A.38}$$

Appendix B

Ambient geodesics and parallel transport of T

In this appendix we provide more details and results concerning the solution of the ambient geodesic equations and the parallel transport of the homothety T along such geodesics between two arbitrary points $\widetilde{X}_0$ and $\widetilde{X}_1$ on the ambient nullcone, that is $\widetilde{X}_0 = (t_0, 0, x_0^i)$ and $\widetilde{X}_1 = (t_1, 0, x_1^i)$.

Let us first focus on the geodesic problem. We indicate the geodesic trajectories with $\widetilde{X}^M(\lambda)$, where $0 \leq \lambda \leq 1$. The boundary conditions are $\widetilde{X}(0) = \widetilde{X}_0$ and $\widetilde{X}(1) = \widetilde{X}_1$. In this affine parametrisation the velocity is normalised according to $\dot{\widetilde{X}}^M \dot{\widetilde{X}}^N \tilde{g}_{MN} = C$. Here C is a constant fixed by the boundary conditions and its sign is related to the causal nature of the ambient trajectory. Its norm is the square of the geodesic length between the two points,

$$\ell(\widetilde{X}_0, \widetilde{X}_1) = \int_0^1 d\lambda \sqrt{\left|\tilde{g}_{MN} \dot{\widetilde{X}}^M \dot{\widetilde{X}}^N\right|} = \sqrt{|C|}. \tag{B.1}$$

The ambient geodesic equations

$$\ddot{\widetilde{X}}^M(\lambda) + \widetilde{\Gamma}^M_{AB}(\lambda) \dot{\widetilde{X}}^A(\lambda) \dot{\widetilde{X}}^B(\lambda) = 0, \tag{B.2}$$

can be expanded as

$$\ddot{t} - \frac{1}{2} t g'_{ij} \dot{x}^i \dot{x}^j = 0, \tag{B.3}$$

$$\ddot{\rho} + \frac{2}{t} \dot{t} \dot{\rho} - \left(g_{ij} - \rho g'_{ij}\right) \dot{x}^i \dot{x}^j = 0, \tag{B.4}$$

$$\ddot{x}^k + \frac{2}{t} \dot{x}^k \dot{t} + \Gamma^k_{ij} \dot{x}^i \dot{x}^j + g^{kl} g'_{il} \dot{\rho} \dot{x}^i = 0, \tag{B.5}$$

where the Christoffel symbols in (B.5) are computed using $g_{ij}(x, \rho)$ at fixed ρ. Here the prime denotes a derivative in ρ, while the dot stands for a derivative in λ. The velocity normalization condition reads

$$2\rho\,\dot{t}^2 + 2\,t\,\dot{t}\,\dot{\rho} + t^2\,g_{ij}(x,\rho)\dot{x}^i\dot{x}^j = C. \tag{B.6}$$

These equations cannot be integrated in full generality. As customary some of them can be reduced to first order ODEs using the ambient isometries $K^{(i)}_M$ described in Chapter 3, if any is present. They lead to integrals of motion whose value is fixed by the boundary conditions of the problem,

$$Q_i = K^{(i)}_M(\lambda)\dot{\widetilde{X}}^M(\lambda)\,. \tag{B.7}$$

The geodesic equations can however be partially solved on general grounds. From (B.3) and (B.6) we can extract $g_{ij}\dot{x}^i\dot{x}^j$ and $g'_{ij}\dot{x}^i\dot{x}^j$ as functions of ρ and t. Plugging them into (B.4) one finds

$$\ddot{\rho} + 4\frac{\dot{t}}{t}\dot{\rho} - \frac{C}{t^2} + 2\rho\frac{\dot{t}^2}{t^2} + 2\rho\frac{\ddot{t}}{t} = 0\,, \tag{B.8}$$

which can be easily integrated. Imposing the boundary conditions specified above, the solution reads

$$\rho(\lambda) = -C\,\frac{\lambda(1-\lambda)}{2t(\lambda)^2}\,. \tag{B.9}$$

Observe that the sign of C tells the sign of ρ, that is related to which region of the ambient space the geodesic is moving through, either the region with ALAdS foliation or the one with ALdS slices. Note also that equation (B.9) can be rewritten in terms of the coordinates (2.7) as

$$s(\lambda) = t\sqrt{-2\rho^2} = \sqrt{C\lambda(1-\lambda)}, \tag{B.10}$$

so that the trajectory along s is completely specified once we fix C. This relation turns out to be sufficient to find the explicit expression of the ambient invariant $\widetilde{X}_{ij}$.

Let us now move to the parallel transport of the homothetic vector T from $\widetilde{X}_0$ to $\widetilde{X}_1$, defined by the equations

$$\dot{\widetilde{X}}^M(\lambda)\,\widetilde{\nabla}_M T^A(\lambda) = 0\,, \tag{B.11}$$

which can be expanded as

$$\partial_\lambda T^0 - \frac{t}{2}g'_{ij}\dot{x}^iT^j = 0, \tag{B.12}$$

$$\partial_\lambda T^\rho + \frac{\dot{t}}{t}T^\rho + \frac{\dot{\rho}}{t}T^0 + (-g_{ij} + \rho g'_{ij})\dot{x}^iT^j = 0, \tag{B.13}$$

$$\partial_\lambda T^l + \frac{1}{t}\dot{x}^lT^0 + \frac{1}{t}\dot{t}T^l + \frac{1}{2}g^{lm}g'_{jm}\left(\dot{x}^jT^\rho + \dot{\rho}T^j\right) + \Gamma^l_{ij}\dot{x}^iT^j = 0. \tag{B.14}$$

One of them can be automatically integrated in view of the fact that the norm of T must stay constant along a geodesic, and in this case the constant is zero (as its norm at $\widetilde{X}_0$ vanishes). Therefore $T^M(\lambda)T_M(\lambda) = 0$ entails

$$2\rho T^0(\lambda)^2 + 2tT^0(\lambda)T^\rho(\lambda) + t^2 g_{ij}(x,\rho)T^i(\lambda)T^j(\lambda) = 0. \tag{B.15}$$

Being the norms of $\dot{\widetilde{X}}$ and T constant along the geodesic, one can also show that the angle between them stays constant, $\dot{\widetilde{X}}^M T^N \tilde{g}_{MN} = W$. Explicitly,

$$2\rho\dot{t}T^0 + t\dot{t}T^\rho + t\dot{\rho}T^0 + t^2 g_{ij}(x,\rho)\dot{x}^i T^j = -\frac{C}{2}. \tag{B.16}$$

where the boundary conditions and equation (B.9) fix $W = -\frac{C}{2}$.

Observe that from (B.12) we can obtain $g'_{ij}\dot{x}^i T^j$ in terms of T^0 and T^ρ, while from (B.16) one finds $g_{ij}(x,\rho)\dot{x}^i T^j$. If we plug them into (B.13), the resulting equation can be integrated in terms of $T^\rho + \frac{2\rho}{t}T^0$ yielding

$$T^\rho = -\frac{2\rho}{t}T^0 - \frac{C\lambda}{2t^2}\,. \tag{B.17}$$

As anticipated these general features of the solutions are enough to compute the invariant $\widetilde{X}_{ij}$. Since the homothetic vector at $\widetilde{X}_1$ has components $T^M_{(1)} = (t_1, 0, 0)$, using the ambient metric at $\widetilde{X}_1$ one has

$$\widetilde{X}_{01} = -2\,\hat{T}^M_{(0)}T^N_{(1)}\,\tilde{g}^{(1)}_{MN} = -2\,t_1^2\,\hat{T}^\rho_{(0)}. \tag{B.18}$$

Hence, evaluating (B.17) at $\lambda = 1$ we can conclude that $\widetilde{X}_{ij} = C = \ell(\widetilde{X}_i, \widetilde{X}_j)^2$ as claimed in the main discussion.

This entails that solving parallel transport is not required to construct the invariant $\widetilde{X}_{ij}$ (even though it generally is necessary when constructing other ambient invariants). One has to simply solve geodesic equations and extract C from the norm of the velocity $\dot{\widetilde{X}}^M\dot{\widetilde{X}}^N\tilde{g}_{MN} = C$ at any point λ.

Appendix C

Relation between ambient and AdS geodesics

Consider an ambient space of the form (2.8). We intend to relate the geodesic distance between two nullcone points $\widetilde{X}_0 = (t_0, 0, x_0^i)$ and $\widetilde{X}_1 = (t_1, 0, x_1^i)$ on the ambient space with the geodesic distance L_{AdS} between those same points x_0^i and x_1^i at the boundary of a single Euclidean ALAdS slice. In particular we wish to show that

$$\frac{1}{(\widetilde{X}_{12})^{\Delta}} = r^{-2\Delta} e^{-\Delta L_{AdS}}\big|_{r=0}\,. \tag{C.1}$$

We assume $t_0, t_1 > 0$ meaning that we are interested in spacelike geodesics on the ambient space. The same result can be attained with completely analogous computations in the cases of null and timelike ambient geodesics. We choose a slightly different parametrization consisting in a rescaling of the one used in Appendix B,

$$\tilde{g}_{AB}\dot{\widetilde{X}}^A\dot{\widetilde{X}}^B = 1. \tag{C.2}$$

We take advantage of the parent description

$$S_A = \frac{1}{2}\int_\gamma d\lambda \left[\frac{1}{e}\tilde{g}_{AB}\dot{\widetilde{X}}^A\dot{\widetilde{X}}^B + e\right], \tag{C.3}$$

where one can put the einbein onshell compatibly with the parametrization above by setting

$$e = \sqrt{\tilde{g}_{AB}\dot{\widetilde{X}}^A\dot{\widetilde{X}}^B} = 1. \tag{C.4}$$

The ambient action is then equal to

$$S_A = \int_0^L d\lambda \sqrt{\tilde{g}_{AB}\dot{\widetilde{X}}^A\dot{\widetilde{X}}^B} = L, \tag{C.5}$$

i.e. is simply the ambient geodesic length L, with $L^2 = C = \widetilde{X}_{12}$.

As shown in Appendix B the presence of the homothetic vector $T = s\partial_s$ allows one to automatically integrate the geodesic equation along s. In the current parametrization this entails $s(\lambda) = \sqrt{\lambda(L-\lambda)}$. Regulating the integrals by taking the domain $\lambda \in (\varepsilon, L-\varepsilon)$ we can set $s(\lambda)$ onshell in (C.3) (after setting $e = 1$) and rewrite S_A as,

$$S_A = L + \frac{L}{4}\log\frac{\varepsilon}{L} + \frac{1}{2}\int_{\varepsilon}^{L-\varepsilon} d\lambda\, \lambda(L-\lambda) g^{+}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}, \tag{C.6}$$

where as customary $g^{+}_{\mu\nu}$ is the metric on a ALAdS slice. By rewriting the integral using the new parametrization

$$p(\lambda) = \frac{1}{L}\log\frac{\lambda}{L-\lambda}, \tag{C.7}$$

one obtains the constraint

$$\frac{L}{4}\log\frac{\varepsilon}{L} + \frac{1}{2}\int_{\frac{1}{L}\log\frac{\varepsilon}{L}}^{-\frac{1}{L}\log\frac{\varepsilon}{L}} dp\; g^{+}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0\,. \tag{C.8}$$

Normalizing the velocity on the Euclidean ALAdS slice as

$$g^{+}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = q^2, \tag{C.9}$$

for some constant q we have to determine, the parent action for a trajectory on such $(d+1)$ dimensional slice reads

$$L_{AdS} = S_{EAdS} = \frac{1}{2q}\int_{\frac{1}{L}\log\frac{\varepsilon}{L}}^{-\frac{1}{L}\log\frac{\varepsilon}{L}} dp\; g^{+}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - \frac{q}{L}\log\frac{\varepsilon}{L}. \tag{C.10}$$

One also has

$$L_{AdS} = \int_{\frac{1}{L}\log\frac{\varepsilon}{L}}^{-\frac{1}{L}\log\frac{\varepsilon}{L}} dp\; \sqrt{g^{+}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} = -\frac{q}{L}\log\frac{\varepsilon}{L}. \tag{C.11}$$

We can then use the constraint from the ambient space to fix q. To leading order in ε one finds $q = L/2$, entailing

$$L_{AdS} = -\log\frac{\varepsilon}{L}. \tag{C.12}$$

Finally, note that in the current parametrization of geodesics,

$$r(\lambda) = \frac{s(\lambda)}{t(\lambda)} = \frac{1}{t(\lambda)}\sqrt{\lambda(L-\lambda)}. \tag{C.13}$$

From the boundary conditions, close to $\lambda = 0$ one has $t(\lambda) = t_0 + O(\lambda) = 1 + O(\lambda)$. This leads to $r(\lambda) = \sqrt{L\lambda} + O(\lambda)$ (where an analogous statement holds near $\lambda = 1$), meaning that we can express the regulator in terms of the radial component of the trajectory $r(\lambda)$ as $\varepsilon = r^2/L$ and obtain

$$L_{AdS} = -\log \frac{r^2}{L^2}, \tag{C.14}$$

which precisely reproduces the expected relation (4.19) between the geodesic approximation on ALAdS and ambient spaces.

Appendix D

Details on the curvature invariants at finite temperature

Let us now discuss the completeness of the basis for weighted curvature invariants as provided by the weight-0 scalars (4.15) in the finite temperature case presented in Chapter 5. Compatibly with equations (4.4)-(4.6), the only non-vanishing components of the ambient Riemann are

$$\widetilde{R}_{\rho jk\rho} = \frac{d}{4}\left(\frac{d}{2}-1\right) g_{(d)jk}\, \rho^{\frac{d}{2}-2} t^2, \tag{D.1a}$$

$$\widetilde{R}_{ijkl} = \frac{d}{4}\left[\delta_{il} g_{(d)jk} + \delta_{jk} g_{(d)il} - \delta_{ik} g_{(d)jl} - \delta_{jl} g_{(d)ik}\right] \rho^{\frac{d}{2}-1} t^2. \tag{D.1b}$$

The subleading orders in ρ in equations (4.4)-(4.6) are proportional to d-dimensional covariant derivatives acting on $g_{(0)}$ and $g_{(d)}$. In this case these are constant tensors and hence the expansion in ρ of the ambient Riemann truncates at the leading order.

Because of the homogeneity in t of the Riemann and of the fact that the geometry does not depend on the boundary directions x^i, the action of the ambient covariant derivative on the Riemann decomposes as a derivative along ρ plus terms which are proportional to the Riemann. Schematically,

$$\widetilde{\nabla}_M \widetilde{\mathrm{Riem}} = \delta^{\rho}_{M}\, \partial_\rho \widetilde{\mathrm{Riem}} + \widetilde{\mathrm{Riem}}, \tag{D.2}$$

and the same holds for higher order derivatives. Focusing on $d = 4$, this means that the only independent tensorial structures containing the stress tensor VEV $g_{(d)ij}$ are $\delta_{ij} g_{(d)jk}$ (and symmetrisations) and $g_{(d)ij}$ itself, and they can be extracted from $\widetilde{\mathrm{Riem}}|_{\rho=0}$ and $\widetilde{\nabla}\widetilde{\mathrm{Riem}}|_{\rho=0}$. Higher order derivatives of the Riemann simply yield different linear combinations of those two structures. This entails that $\mathcal{R}^{(0)}$ and $\mathcal{R}^{(1)}$ are the only independent objects that one needs in order to construct the weight-0 invariants (4.15).

As a consequence, any order two weight-0 invariant can be written as a linear combination of the scalars e_0, e_1,e_2 defined in (5.12).

At a given order β^{-nd} the weight-0 invariants are to be constructed as a chain of n $\mathcal{R}$'s,

$$\mathcal{R}^{(r_1)\,M_2}_{M_1}\,\mathcal{R}^{(r_2)\,M_3}_{M_2}\ldots\mathcal{R}^{(r_n)\,M_1}_{M_n}\,, \tag{D.3}$$

where for each of them one has two possible choices, $r_i = 0, 1$. This means that they provide at most $2n$ different invariants, modulo cyclic permutations. Our aim is to reproduce the whole set of multi-stress tensor blocks entering the thermal OPE at order β^{-nd}. Each of them is proportional to a Gegenbauer polynomial $C_J^{(1)}$, with even $J = 0\ldots 2n$, hence at this order one needs $n+1$ independent invariants.

At order $n = 2$ this entails that the three independent scalars e_0, e_1,e_2 in (5.12) form a basis of ambient invariants. At a generic order n, based on the counting above the weight-0 curvature scalars are in principle able to form an over-complete basis. We checked explicitly that they generate a basis of $n+1$ invariants in $d = 4$ up to $n = 6$, and one may check it to arbitrarily high order n.

With similar arguments based on the action (D.2) of ambient covariant derivatives, this discussion can be easily extended to any even $d \geq 4$, where now the two independent objects are $\mathcal{R}^{(d/2-2)}$ and $\mathcal{R}^{(d/2-1)}$ and any weight-0 invariant is built as a chain of them.

Appendix E

Perturbative thermal holographic correlator for general d and Δ

In this appendix we intend to solve the first order equation (5.28b). We will consider here the case of generic d and Δ with non-integer κ. Using the leading order solution, the onshell source on the RHS of (5.28b) reads

$$S(r) = -\sqrt{\frac{2}{\pi}} r^{\frac{3d}{2}-2} \cos\left(\frac{2\kappa-1}{2}\pi\right) \left[K_\kappa(r)\left(\Delta^2 + \left(\eta^2+1\right) r^2\right) - dr K_{\kappa+1}(r)\right], \tag{E.1}$$

while the Wronskian of the homogeneous solutions is

$$W(u_1, u_2) = \cos\left(\frac{2\kappa-1}{2}\pi\right) r^{d-1}. \tag{E.2}$$

The first order solution will then be of the form

$$b_1(r) = (A(r) + a_1)u_1 + (B(r) + a_2)u_2\,, \tag{E.3}$$

where a_1 and a_2 are integration constants while

$$\begin{aligned} A(r) &= -\int_0^r dr'\, \frac{u_2(r')S(r')}{W(r')} && \text{(E.4a)} \\ &= -d\,\mathcal{I}^{(1)}(d,\kappa,1) + \Delta^2\,\mathcal{I}^{(1)}(d-1,\kappa,0) + (1+\eta^2)\,\mathcal{I}^{(1)}(d+1,\kappa,0)\,, \\ B(r) &= \int_0^r dr'\, \frac{u_1(r')S(r')}{W(r')} && \text{(E.4b)} \\ &= d\,\mathcal{I}^{(2)}(d,\kappa,1) - \Delta^2\,\mathcal{I}^{(2)}(d-1,\kappa,0) - (1+\eta^2)\,\mathcal{I}^{(2)}(d+1,\kappa,0)\,. \end{aligned}$$

Here we defined the following class of integrals involving two Bessel functions,

$$\mathcal{I}^{(\ell)}(\alpha,\kappa,\delta) = \int_0^r dr'\, r'^{\alpha}\, I_{(-1)^{\ell+1}(\kappa)}(r')\, K_{\kappa+\delta}(r')\,. \tag{E.5}$$

The explicit expressions for these integrals are

$$\mathcal{I}^{(1)}(\alpha,\kappa,0) = \frac{r^{\alpha+1}}{2\kappa(\alpha+1)}\,{}_2F_3\left(\frac{1}{2},\frac{\alpha+1}{2};1-\kappa,\kappa+1,\frac{\alpha+3}{2};r^2\right) \tag{E.6a}$$

$$+\frac{2^{-2\kappa-1}\Gamma\left(-\kappa\right)r^{\alpha+2\kappa+1}}{(\alpha+2\kappa+1)\Gamma\left(\kappa+1\right)}\,{}_2F_3\left(\kappa+\frac{3}{2},\frac{\alpha}{2}+\kappa+1;\kappa+2,2\kappa+2,\frac{\alpha}{2}+\kappa+2;r^2\right),$$

$$\mathcal{I}^{(1)}(\alpha,\kappa,1) = \frac{r^{\alpha}}{2\alpha}+\frac{r^{\alpha}}{2\alpha}\,{}_2F_3\left(\frac{1}{2},\frac{\alpha}{2};-\kappa,\kappa+1,\frac{\alpha}{2}+1;r^2\right) \tag{E.6b}$$

$$+\frac{\pi 2^{-2(\kappa+1)}\sec\left(\frac{2\kappa-1}{2}\pi\right)r^{\alpha+2\kappa+2}}{(\alpha+2\kappa+2)\Gamma(\kappa+1)\,\Gamma(\kappa+2)}\,{}_2F_3\left(\kappa+\frac{3}{2},\frac{\alpha}{2}+\kappa+1;\kappa+2,2\kappa+2,\frac{\alpha}{2}+\kappa+2;r^2\right),$$

$$\mathcal{I}^{(2)}(\alpha,\kappa,\delta) = \frac{\pi 2^{-\delta-1}\sec\left(\left(\delta+\kappa-\frac{1}{2}\right)\pi\right)r^{\alpha-\delta}}{\Gamma\left(1-\kappa\right)}\Bigg[-\frac{2^{2\delta+2\kappa}r^{-2\kappa+1}}{(-\alpha+\delta+2\kappa-1)\Gamma\left(1-\delta-\kappa\right)}\times \tag{E.6c}$$

$${}_3F_4\Bigg(-\kappa+\frac{1-\delta}{2},-\kappa-\frac{\delta}{2},-\kappa+\frac{\alpha-\delta+1}{2};1-\kappa,-2\kappa+1-\delta,1-\delta-\kappa,\frac{3}{2}-\kappa+\frac{\alpha-\delta}{2};r^2\Bigg)$$

$$-\frac{r^{2\delta+1}\,{}_3F_4\left(\frac{\delta+1}{2},\frac{\delta}{2}+1,\frac{\alpha+\delta+1}{2};1-\kappa,\frac{\alpha+\delta+3}{2},\delta+1,\kappa+\delta+1;r^2\right)}{(\alpha+\delta+1)\,\Gamma\left(\kappa+\delta+1\right)}\Bigg],$$

We have fixed the source at the leading $O(\epsilon^0)$ order, hence the order corresponding to the source in the near-boundary Fefferman-Graham expansion of the function b_1 must vanish. Integrating over $r'\in(0,r)$ as in equation (E.4), this is automatically true for $a_1=0$. The remaining integration constant a_2 is fixed by imposing regularity in the bulk interior $r\to\infty$. By studying the large–r behaviour of b_1 this fixes

$$a_2 = \frac{\pi^{3/2}\left(d\eta^2-1\right)\cot\left(\frac{\pi d}{2}\right)\Gamma\left(-\frac{d}{2}-\frac{1}{2}\right)\csc(\pi\Delta)\sin\left(\frac{1}{2}\pi(d-2\Delta)\right)\csc(\pi(d-\Delta))}{4\Gamma\left(1-\frac{d}{2}\right)\Gamma(-\Delta)\Gamma(\Delta-d)}\,. \tag{E.7}$$

The resulting holographic correlator to first order in ϵ is in equation (5.33). Note that after the appearance of these computations and results in [24], a similar approach for the holographic correlator was taken in [193], whose results match (5.33) and (5.46).

Appendix F

Computation of the double-twist coefficients from the multi-stress tensor spectrum

In this appendix we provide details on how to perform the sum over images (5.67) to obtain the thermal correlator (5.68).

First we consider the analytic part of G_+ in (5.65). Setting for the moment $\beta = 1$, we thus have to evaluate the sum

$$S_\gamma(\tau) = \sum_{m=1}^{\infty} |m + \tau|^\gamma \,, \tag{F.1}$$

with $\gamma = -2\Delta + nd$. As it is this sum converges for $\gamma < -1$, however we can analytically continue it using the Hurwitz ζ function, in terms of which it reads

$$S_\gamma(\tau) = \zeta(-\gamma, 1 + \tau) \,. \tag{F.2}$$

This expression is finite on the whole complex γ-plane except for a simple pole at $\gamma = -1$. To avoid it, it is sufficient to pick $\Delta \neq \mathtt{n}\,\frac{d}{2} + \frac{1}{2}$ for all non-negative integer $\mathtt{n}$.

Using the expansion of the Hurwitz ζ for $\gamma \neq -1$ and $|\tau| < 1$,

$$\zeta(-\gamma, 1 - \tau) = \sum_{p=0}^{\infty} \frac{\Gamma(p - \gamma)}{p!\ \Gamma(-\gamma)}\, \zeta(p - \gamma)\, \tau^p \,, \tag{F.3}$$

the sum over images of the analytic part then reads,

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\tilde{a}_n^{(T)}}{\beta^{2\Delta}} \left(\frac{\tau}{\beta}\right)^{nd-2\Delta} = \frac{1}{\beta^\Delta} \sum_{p=0}^{\infty} Q^{(OO)}_{\mathrm{reg},\, p} \frac{\tau^p}{\beta^p} \,, \tag{F.4}$$

where we define the coefficients

$$Q^{(OO)}_{\mathrm{reg},\,p} = \sum_{n=0}^{\infty} \frac{\Gamma(p+2\Delta-nd)}{p!\;\Gamma(2\Delta-nd)}\,\zeta(p+2\Delta-nd)\,\tilde{a}^{(T)}_n\,, \tag{F.5}$$

Turning to the singular part of G_+ in (5.65), for the moment we consider terms with poles of arbitrary positive integer order $\mu_{(\ell)}$,

$$W_\ell(\tau) = \frac{1}{|\tau|^{2\Delta}} \frac{A_{(\ell)}}{\Big(|\tau/\beta|^d - y_\ell\Big)^{\mu_{(\ell)}}}\,. \tag{F.6}$$

Such a singular term describes $d - \sin\left(\frac{\pi}{2}d\right)$ poles in the complex τ-plane lying on the circle of radius $|y_\ell|$. Its form is dictated by the dependence on τ of the multi-stress tensor series (5.63).

The sum over images that we wish to evaluate is then

$$\sum_{m=1}^{\infty} W_\ell(\tau+m) = \sum_{m=-\infty}^{\infty} \frac{1}{(\tau+m)^{2\Delta}} \frac{A_{(\ell)}}{((\tau+m)^d - y_\ell)^{\mu_{(\ell)}}}\,, \tag{F.7}$$

Due to the singularities, to ensure convergence one has to split this sum based on whether $\tau+m$ is either inside or outside the circle of radius $|y_\ell|^{1/d}$, with τ arbitrarily small. We thus define $M^*_{(\ell)} = \left\lceil |y_\ell|^{1/d} \right\rceil$ and split the sum as

$$\sum_{m=1}^{\infty} W_\ell(\tau+m) = Z_0(\tau) + Z_+(\tau)\,, \tag{F.8}$$

with

$$Z_0(\tau) = \sum_{m=1}^{M^*_{(\ell)}-1} W_\ell(\tau+m)\,, \qquad Z_+(\tau) = \sum_{m=M^*_{(\ell)}}^{\infty} W_\ell(\tau+m)\,. \tag{F.9}$$

Starting with Z_+, the range of m in the sum guarantees that the we can expand each summand as

$$\frac{1}{[(\tau+m)^d - y_\ell]^{\mu_{(\ell)}}} = \sum_{j=0}^{\infty} \binom{\mu_{(\ell)}+j-1}{j} (y_\ell)^j\,(\tau+m)^{-d(\mu_{(\ell)}+j)}\,. \tag{F.10}$$

We can then rewrite

$$Z_+(\tau) = A_{(\ell)} \sum_{m=M^*_{(\ell)}}^{\infty} \sum_{j=0}^{\infty} \binom{\mu_{(\ell)}+j-1}{j} (y_\ell)^j\,(\tau+m)^{-d(\mu_{(\ell)}+j)-2\Delta}\,, \tag{F.11}$$

$$= A_{(\ell)} \sum_{j=0}^{\infty} \binom{\mu_{(\ell)}+j-1}{j} (y_\ell)^j\,\zeta\Big(d(\mu_{(\ell)}+j)+2\Delta, \tau+M^*_{(\ell)}\Big)\,, \tag{F.12}$$

which converges for $\Delta \neq \mathtt{n}\frac{d}{2}+\mu_{(\ell)}$ with integer $\mathtt{n}$. Using the expansion in τ of the Hurwitz ζ,

$$\zeta(y, \tau + M^*_{(\ell)}) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} (y)_p \, \zeta(p+y, M^*_{(\ell)}) \, \tau^p \,, \tag{F.13}$$

where $(a)_b$ indicates the Pochhammer symbol, the sum over images in the region outside the singularity takes the form

$$\begin{aligned} Z_+ = A_{(\ell)} \sum_{p=0}^{\infty} \Bigg[\frac{(-1)^p}{p!} \sum_{j=0}^{\infty} \binom{\mu_{(\ell)}+j-1}{j} (y_\ell)^j \left(d(\mu_{(\ell)}+j)+2\Delta\right)_p \\ \times\, \zeta\Big(p + d(\mu_{(\ell)}+j)+2\Delta, M^*_{(\ell)}\Big) \Bigg] \tau^p \,. \end{aligned} \tag{F.14}$$

The finite sum Z_0 can be expanded in powers of τ in an analogous way, with the only caveat that given $|\tau + m| < M^*_{(\ell)}$ one must expand

$$\frac{1}{[(\tau+m)^d - y_\ell]^{\mu_{(\ell)}}} = \sum_{j=0}^{\infty} (-1)^j \binom{\mu_{(\ell)}+j-1}{j} (\tau+m)^{dj} (-y_\ell)^{-\mu_{(\ell)}-j} \,, \tag{F.15}$$

so as to ensure convergence. Reinstating the appropriate scalings in β, the sum over images of each singular piece hence takes the form,

$$\sum_{m=1}^{\infty} W_\ell(\tau + m\beta) = \frac{1}{\beta^{2\Delta}} \sum_{p=0}^{\infty} Q^{(OO)}_{(\ell)\,p} \, \frac{\tau^p}{\beta^p} \,, \tag{F.16}$$

where we defined the coefficients

$$\begin{aligned} Q^{(OO)}_{(\ell)\,p} = \frac{(-1)^p A_{(\ell)}}{p!} \sum_{j=0}^{\infty} \binom{\mu_{(\ell)}+j-1}{j} \Bigg[\\ (y_\ell)^j \left(d(\mu_{(\ell)}+j)+2\Delta\right)_p \zeta(p + d(\mu_{(\ell)}+j) + 2\Delta, M^*_{(\ell)}) \\ + (-1)^j (-\tau_\ell)^{-\mu_{(\ell)}-j} \, (2\Delta - dj)_p \Big(\zeta(p+2\Delta-dj) - \zeta(p+2\Delta-dj, M^*_{(\ell)}) \Big) \Bigg] \,, \end{aligned} \tag{F.17}$$

This expression has been obtained assuming integer order $\mu_{(\ell)}$. However one can check that it can be analytically continued for all real positive $\mu_{(\ell)}$, thus allowing for branching points in the complex τ-plane.

Overall, summing these contributions according to (5.67), only even powers of τ survive and one finds the thermal correlator (5.68).

Appendix G

Non-perturbative geodesics on the planar black hole

In Section 5.5 we focused on correlators with close insertions $|x|/\beta \ll 1$ on the planar AdS black hole. From the ambient perspective this coincides with considering perturbatively short geodesics. These can be obtained as perturbations in $|x|^d/\beta^d$ of geodesics on Minkowski spacetime as shown in Section 5.2. By making this choice we do not expect to be able to account for the double-twist spectrum in the corresponding correlator. The motivation for this is that as discussed in Subsection 5.5.2, their appearance is related to non-perturbative effects in $|x|/\beta \to 0$ and as such one has to take into account the global properties of the background (in this case the periodicity of the τ direction) to fully describe them.

As mentioned in Subsection 5.5.2 a possibility to describe the double-twist spectrum in terms of the ambient formalism is that long geodesics exist on the ambient space (5.3). By long geodesics here we mean ambient geodesics that connect nullcone points that are close to each other, and whose length is not perturbatively short in $|x|/\beta \to 0$. More explicitly, the length of such geodesics would scale like β instead of $|x|$, as instead it happens for perturbatively short geodesics (see equation (5.10)). This would be the case for geodesics that wrap the thermal circle multiple times for example. If this class of ambient geodesics existed, the double-twist contributions in the ambient correlator would likely emerge from the sum over geodesics of the ambient curvature invariants, paralleling what happens in the holographic correlator (5.40) where they arise from the sum over images of the multi-stress tensor spectrum. This is however not the case and we do not find any such geodesic as we detail below.

One can study long geodesics on the planar black hole by finding exact solutions to the geodesic equations (5.6)-(5.7) with boundary conditions (5.9). As already mentioned in the main discussion, generic solutions to (5.7) are in terms of inverse elliptic functions and thus not easily tractable. Nonetheless, if one restricts to trajectories moving only along the τ direction (i.e. setting $x(\lambda) = 0$ as an initial condition, meaning that $A_2 = 0$ in

(5.6)), the equation for $z(\lambda)$ is explicitly solvable for any A_1. Defining the dimensionless parameter $A = \frac{\sqrt{2}}{\pi}\beta A_1$, one finds

$$z(\lambda) = \frac{\sqrt{2}}{\pi}\sqrt{\frac{(1-\lambda)\lambda}{2A^2(1-\lambda)\lambda + \sqrt{1+A^4}(1-2(1-\lambda)\lambda)}}\,\beta\,, \tag{G.1}$$

where we set the endpoints to lie on the same slice of the ambient nullcone, $t_0 = t_f = 1$. Through (5.6) this leads to

$$\tau(\lambda) = \frac{\tau_f}{2} + \frac{\beta}{2\pi}\left(\arctan\left[Y_-\left(\lambda - \frac{1}{2}\right)\right] + \operatorname{arctanh}\left[Y_+\left(\lambda - \frac{1}{2}\right)\right]\right)\,, \tag{G.2}$$

where

$$Y_\pm = \sqrt{2}\,\frac{1 \pm (A^2 - \sqrt{A^4+1})}{A}. \tag{G.3}$$

In expression (G.2) we are still to impose the condition $\tau(1) = \tau_f$, which fixes the value of the integration constant A. Renaming $\tau_f \to \tau$ and considering trajectories with $A > 0$, the relation that A must satisfy is,

$$\frac{\tau}{\beta} = -\left(\frac{1}{4} + \frac{i}{4}\right)\left[\arctan\left(\sqrt[4]{-1}A\right) - \operatorname{arctanh}\left(\sqrt[4]{-1}A\right)\right]. \tag{G.4}$$

This equation is transcendental and cannot be inverted analytically to obtain an expression for $A(\tau)$. Nonetheless, we can extract interesting information from this class of orbits.[1]

For all $A > 0$ these trajectories represent physical solutions. In particular $\tau(\lambda)$ is real and $0 < z(\lambda) < z_H$ for any $0 < \lambda < 1$. We can interpret (G.4) as indicating which point τ on the thermal circle is reached by the geodesic as a function of the integration constant A. In Figure G.1 we plot $\tau(A)$. We see that the furthest τ one is able to reach is half of the circle, $\tau = \beta/2$, corresponding to $A = 0$. An analogous behavior is found if one repeats the analysis for $A < 0$.

The ambient geodesic distance square spanned by this class of trajectories is

$$\widetilde{X}_{12}(\tau) = \frac{2}{\pi^2}\frac{\beta^2}{\sqrt{1+A(\tau)^4}}. \tag{G.5}$$

One can test the small-τ behaviour of this geodesic distance to exclude the presence of long geodesics, which would correspond to $\widetilde{X}_{12} \sim \beta^2$ in $\tau/\beta \to 0$, as opposed to short geodesics whose scaling is $\widetilde{X}_{12} \sim \tau^2$. By inverting the relation (G.4) in a series at small

[1]Note that these geodesics were also found in [92].

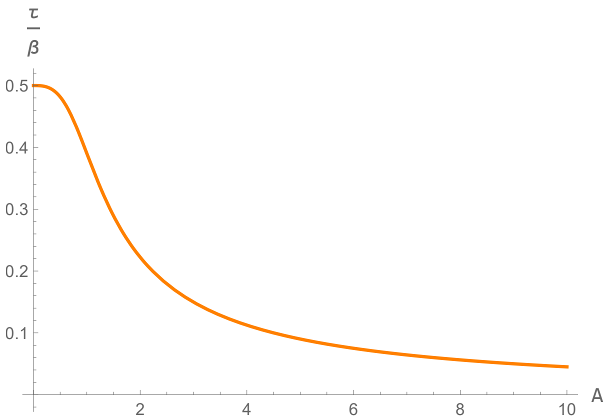

Figure G.1: This plot shows the behaviour of $\frac{\tau}{\beta}(A)$ for positive A as fixed by the boundary condition (G.4) at $\lambda = 1$.

τ/β one can check that this class of exact ambient geodesics reduces to the perturbative geodesics of Section 5.2 for close insertions, meaning in particular that

$$\widetilde{X}_{12}(\tau) = \tau^2 \left(1 + O(\tau^2)\right) \ . \tag{G.6}$$

Therefore, perturbatively close insertions always correspond to perturbatively short geodesics, contrarily to what happens for instance in Thermal AdS. To further confirm this picture, we performed a numerical scan allowing for non-trivial dynamics along the x direction. Also in this more general case no long geodesic was found. This suggests that the double twist spectrum arises in ambient correlators in a different way than a sum over long ambient geodesics connecting the same pair of nullcone points, hinting at the existence of a new class of genuine ambient invariants.

Appendix H

Details on the late-time expansions of the Beig-Schmidt fields

In this appendix we work out the behavior of the Beig–Schmidt fields $\sigma\,, k_{ab}$ and j_{ab} in the asymptotic past and future of the de Sitter hyperboloid $\mathcal{H}$, corresponding to the limits $\tau \to \pm\infty$ in the global coordinate system (τ, x^A).

Electric potential. The equation of motion (7.37) of σ takes the form

$$\left(-\partial_\tau^2 - 2\tanh\tau\,\partial_\tau + \cosh^{-2}\tau\,\nabla^2 + 3\right)\sigma = 0\,. \tag{H.1}$$

The corresponding asymptotic solution is found to be

$$\sigma(\tau, x) = e^{\tau}\,\sigma^{(-1)} + e^{-\tau}\sigma^{(1)} + e^{-3\tau}\tau\,\tilde{\sigma} + e^{-3\tau}\,\sigma^{(3)} + \dots\,, \tag{H.2}$$

where $\sigma^{(-1)}$ and $\sigma^{(3)}$ are free functions that specify a solution to the quadratic differential equation (H.1), while all other functions can be fully determined,

$$\sigma^{(1)} = -\left(\nabla^2 + 1\right)\sigma^{(-1)}\,, \qquad \tilde{\sigma} = \nabla^2\left(\nabla^2 + 2\right)\sigma^{(-1)}\,, \qquad \dots \tag{H.3}$$

Mathematical interlude. In order to streamline the analysis of k_{ab} and j_{ab} to come momentarily, we consider the generic inhomogeneous differential equation for a symmetric tensor t_{ab},

$$(D^2 - \alpha)\,t_{ab} = S_{ab}\,, \qquad \alpha \in \mathbb{R}\,, \tag{H.4}$$

in the case where not only the source S_{ab} but also the trace t^a_a and divergence $D^a t_{ab}$ are non-dynamical and pre-determined quantities. This situation will apply to both k_{ab} and j_{ab}. The trace and divergence of t_{ab} can be written

$$t^a_a = -t_{\tau\tau} + h^{AB}t_{AB},\,, \tag{H.5a}$$

$$D^a t_{a\tau} = -\left(\partial_\tau + 2\tanh\tau\right)t_{\tau\tau} - \tanh\tau\,h^{AB}t_{AB} + h^{AB}\nabla_A t_{B\tau}\,, \tag{H.5b}$$

$$D^b t_{bA} = -\left(\partial_\tau + 2\tanh\tau\right)t_{A\tau} + h^{BC}\nabla_B t_{CA}\,, \tag{H.5c}$$

or equivalently

$$h^{AB}t_{AB} = t_{\tau\tau} + t^a_a\,, \tag{H.6a}$$
$$h^{AB}\nabla_A t_{B\tau} = (\partial_\tau + 3\tanh\tau)\, t_{\tau\tau} + D^b t_{b\tau} + \tanh\tau\, t^a_a\,, \tag{H.6b}$$
$$h^{BC}\nabla_B t_{CA} = (\partial_\tau + 2\tanh\tau)\, t_{A\tau} + D^b t_{bA}\,. \tag{H.6c}$$

These equations allow to eliminate the combinations on the left-hand side in terms of $t_{\tau a}$ and the non-dynamical quantities t^a_a and $D^a t_{ab}$. Introducing the differential operators

$$D_1 \equiv -\partial_\tau^2 - 6\tanh\tau\,\partial_\tau - 6\tanh^2\tau + \cosh^{-2}\tau\,\nabla^2\,, \tag{H.7a}$$
$$D_2 \equiv -\partial_\tau^2 - 2\tanh\tau\,\partial_\tau + 2\tanh^2\tau + \cosh^{-2}\tau\,(1+\nabla^2)\,, \tag{H.7b}$$
$$D_3 \equiv -\partial_\tau^2 + 2\tanh\tau\,\partial_\tau + \cosh^{-2}\tau\,\nabla^2 + 2\,, \tag{H.7c}$$

and making use of (H.6), the equations of motion (H.4) become

$$(D_1 - \alpha)\, t_{\tau\tau} = S_{\tau\tau} + 4\tanh\tau\, D^b t_{b\tau} + 2\tanh^2\tau\, t^a_a\,, \tag{H.8a}$$
$$(D_2 - \alpha)\, t_{\tau A} = S_{\tau A} + 2\tanh\tau\,\partial_A t_{\tau\tau} + 2\tanh\tau\, D^b t_{bA}\,, \tag{H.8b}$$
$$(D_3 - \alpha)\, t_{AB} = S_{AB} + 2\tanh\tau\left(\nabla_A t_{B\tau} + \nabla_B t_{A\tau}\right)\,. \tag{H.8c}$$

They can be solved each one in turn. Indeed (H.8a) is a simple inhomogeneous ordinary differential equation governing the dynamics of the component $t_{\tau\tau}$, with all quantities on the right-hand side being pre-determined functions. Once the solution for $t_{\tau\tau}$ has been found, one can go on and solve (H.8b) in order to determine $t_{\tau A}$, and then similarly solve (H.8c) in order to determine t_{AB} .

Magnetic potential. The magnetic potential k_{ab} satisfies the equation of motion (H.4) with $\alpha = 3$ and vanishing source term, trace and divergence. Solving (H.8) asymptotically we find

$$k_{\tau\tau} = e^{-3\tau}\tau\,\tilde{k}_{\tau\tau} + e^{-3\tau}\,k^{(3)}_{\tau\tau} + \dots\,, \tag{H.9a}$$
$$k_{\tau A} = e^{-\tau}\tau\,\tilde{k}_{\tau A} + e^{-\tau}\,k^{(1)}_{\tau A} + \dots\,, \tag{H.9b}$$
$$k_{AB} = e^{\tau}\tau\,\tilde{k}_{AB} + e^{\tau}\,k^{(-1)}_{AB} + \dots\,, \tag{H.9c}$$

where all functions appearing are free data which serve to specify a particular solution, while all subleading terms can be determined order by order. Note that these asymptotics are those of the homogeneous solutions, while the dependencies on the terms on the right-hand side of (H.8) appear at subleading order in these expansions.

Subleading field. The subleading field j_{ab} satisfies the equation of motion (H.4) with $\alpha = 2$ and pre-determined but nonzero source, trace and divergence given in (7.39)-(7.40). Generic solutions involve superpositions of homogeneous solutions and particular solutions depending on the pre-determined quantities appearing on the right-hand side of (H.8). The asymptotic behavior of the homogeneous solutions is straightforward to determine,

$$j_{\tau\tau} = e^{-2\tau} j_{\tau\tau}^{(2)} + e^{-4\tau} j_{\tau\tau}^{(4)} + \dots\,, \tag{H.10a}$$

$$j_{\tau A} = j_{\tau A}^{(0)} + e^{-2\tau} j_{\tau A}^{(2)} + \dots\,, \tag{H.10b}$$

$$j_{AB} = e^{2\tau} j_{AB}^{(-2)} + j_{AB}^{(0)} + \dots\,. \tag{H.10c}$$

On the other hand, the asymptotic behavior of the particular solutions strongly depends on the asymptotic behavior of the first order fields σ and k_{ab}. Using the data determined from the Bondi phase space and given in Section 8.4, we explicitly evaluate the right-hand side of (H.8),

$$S_{\tau\tau} + 4\tanh\tau\, D^b j_{b\tau} + 2\tanh^2\tau\, j^a_a = O(e^{-6\tau})\,, \tag{H.11a}$$

$$S_{\tau A} + 2\tanh\tau\, \partial_A j_{\tau\tau} + 2\tanh\tau\, D^b j_{bA} = O(e^{-4\tau})\,, \tag{H.11b}$$

$$S_{AB} + 2\tanh\tau\, (\nabla_A j_{B\tau} + \nabla_B j_{A\tau}) = O(e^{-2\tau})\,. \tag{H.11c}$$

This means that the particular solutions of j_{ab} are subleading in τ compared to the homogeneous solutions. Thus (8.11) indeed describes the asymptotic behavior of a generic solution which can be mapped to the Bondi phase space. Note that some of the subleading Bondi data which we have not explicitly considered in this work would in principle contribute to (H.11a) at order $O(e^{-4\tau})$, but for consistency such terms must cancel out.

Note that the trace and divergence constraints (H.6) restrict the number of independent dynamical degrees of freedom. For $j_{\tau A}$ this is most easily seen by considering the Helmholtz decomposition (8.47), in which case the divergence constraint (H.6) fully determines Ψ_1 in terms of other dynamical fields,

$$\nabla^2\Psi_1 = \cosh^2\tau\, [(\partial_\tau + 3\tanh\tau) j_{\tau\tau} + D^a j_{\tau a} + \tanh\tau\, j^a_a]\,. \tag{H.12}$$

By careful analysis of (7.40) one can show that the combination $D^a j_{\tau a} + \tanh\tau\, j^a_a$ decays faster than $O(e^{-4\tau})$, and therefore does not contribute to the order $\Psi_1 = O(e^{-2\tau})$ of interest in Section 8.5. On the other hand Ψ_2 is an independent mode whose dynamics follows from (H.8b),

$$\left[-\partial_\tau^2 - 2\tanh\tau\, \partial_\tau + \cosh^{-2}\tau\, \nabla^2\right] \Psi_2 = O(e^{-4\tau})\,. \tag{H.13}$$

The order $\Psi_2 = O(e^{-2\tau})$ of interest in Section 8.5 is controlled by the homogeneous solutions to the above equation.

Appendix I

Details of the map between Bondi and Beig–Schmidt data

In this appendix we present the details of the doubly-asymptotic coordinate transformation used to map the Bondi data of Section 8.2 onto the Beig-Schmidt data described in Section 7.3. The resulting map is summarised in Section 8.4.

As explained in the main discussion, the idea is to map the second order Bondi metric (7.4) to the Beig-Schmidt gauge, to second order in $1/\rho$ and to leading order in τ. We consider an appropriate Ansatz for the transformation between Bondi coordinates (u, r, x^A) and Beig–Schmidt coordinates (ρ, τ, y^A),

$$u = -\rho\, e^{-\tau} + \alpha(\tau, y^A) + \frac{A(\tau, y^A)}{\rho} + \dots\,, \tag{I.1a}$$

$$r = \rho \cosh\tau + \beta(\tau, y^A) + \frac{B(\tau, y^A)}{\rho} + \dots\,, \tag{I.1b}$$

$$x^A = y^A + \frac{p^A(\tau, y^A)}{\rho} + \frac{q^A(\tau, y^A)}{\rho^2} + \dots\,, \tag{I.1c}$$

where $\alpha, A, \kappa, B, p^A$ and q^A are arbitrary functions on the hyperboloid that are to be determined order by order in ρ by enforcing the Beig-Schmidt gauge. At leading order this transformation coincides with (8.12), i.e. it relates Minkowski space written in retarded coordinates and in hyperbolic foliation. Note also that although the sphere coordinates x^A and y^A differ by terms which vanish in the limit $\rho \to \infty$, they can be swapped at will once the mapping of fields at a given order in ρ has been determined. This allows us to write down the dictionary in a way that is manifestly covariant on the (celestial) sphere $\mathbb{S}^2$.

The Beig–Schmidt gauge conditions to be imposed at the relevant order in ρ consist in

$$g_{\rho\rho} = 1 + \frac{2\sigma}{\rho} + \frac{\sigma^2}{\rho^2} + o(\rho^{-2})\,, \qquad g_{\rho a} = o(\rho^{-1})\,. \tag{I.2}$$

At first order in ρ, this relates σ to the Bondi mass aspect according to

$$\sigma\,(\tau,y) = 2m^0 e^{-3\tau} + \ldots\,, \tag{I.3}$$

while to leading order in τ one finds for the coordinate transformation

$$\alpha(\tau,y) = 8m^0\left(\tau - \frac{1}{3}\right)e^{-4\tau} + \ldots\,, \tag{I.4a}$$

$$\beta(\tau,y) = 8m^0\left(\tau - \frac{1}{3}\right)e^{-2\tau} + \ldots\,, \tag{I.4b}$$

$$p^A\,(\tau,y) = -2\,\nabla_B C_0^{AB}\,e^{-3\tau} + \ldots\,. \tag{I.4c}$$

This entails

$$k_{\tau\tau} = \frac{8}{3}\,m^0\,(24\tau - 17)\,e^{-3\tau} + \ldots\,, \tag{I.5a}$$

$$k_{\tau A} = 2\,\nabla^B C^0_{AB}\,e^{-\tau} + \ldots\,, \tag{I.5b}$$

$$k_{AB} = \frac{1}{2}\,C^0_{AB}\,e^{\tau} + \ldots\,. \tag{I.5c}$$

In a similar way, the functions in the second order transformation are found to be

$$A(\tau,y) = \left(\nabla_E C^0_{AB}\nabla^A C_0^{EB} - \nabla_E C^0_{AB}\nabla^E C_0^{AB} + 4\phi\right)e^{-5\tau} + \ldots\,, \tag{I.6a}$$

$$B(\tau,y) = -\frac{1}{8}C^0_{AB}C_0^{AB}e^{-\tau} + \ldots\,, \tag{I.6b}$$

$$q^A(\tau,y) = 2\gamma^{AB}\left(-\frac{4}{3}N^0_B + C_0^{EF}\nabla_B C^0_{EF} - C_0^{EF}\nabla_E C^0_{BF}\right)e^{-4\tau} + \ldots\,. \tag{I.6c}$$

From the resulting Beig-Schmidt metric we can read off the leading data of j_{ab},

$$j_{\tau\tau} = -4\left(\nabla_E C^0_{AB}\nabla^A C_0^{EB} - \nabla_E C^0_{AB}\nabla^E C_0^{AB} - 16\,\phi^0\right)e^{-4\tau} + \ldots\,, \tag{I.7a}$$

$$j_{\tau A} = \left(4N^0_A + C^0_{AB}\nabla_C C_0^{BC}\right)e^{-2\tau} + \ldots\,, \tag{I.7b}$$

$$j_{AB} = \frac{1}{8}C^0_{EF}C_0^{EF}\gamma_{AB} + \left(-\frac{4}{3}\nabla_A N^0_B + 8\left(\tau - \frac{1}{3}\right)m^0 C^0_{AB} + U^0_{AB}\right)e^{-2\tau} + \ldots\,, \tag{I.7c}$$

where U_{AB} is defined as the following tracefree tensor on the 2-sphere,

$$\begin{aligned} U_{AB} = &-(\nabla^E\nabla^F C_{EF})C_{AB} - \nabla_E C^{EF}\nabla_F C_{AB} + \nabla_E C_{FA}\nabla_B C^{EF} \\ &+ C^{EF}\nabla_E\nabla_F C_{AB} - C^{EF}\nabla_E\nabla_A C_{FB} - \text{trace}\,, \end{aligned} \tag{I.8}$$

and ϕ^0 is the order $O(u^0)$ piece in the second order term ϕ appearing in (7.5a),

$$\phi = \frac{1}{2}\left(R_{(2)} - \nabla^A g_{(1)uA} + \nabla^2 g_{(2)ur}\right) \equiv u\,\phi^{-1} + \phi^0 + o(u^0)\,. \tag{I.9}$$

The coefficients $g_{(1)uA}$ and $g_{(2)ur}$ are the orders $O(r^{-1})$ and $O(r^{-2})$ of the corresponding metric components (7.5c) and (7.5b), and $R_{(2)} = \gamma^{AB} R_{(2)AB} - C^{AB} R_{(1)AB} + \bar{g}^{AB}_{(0)} R_{(0)AB}$ is the order $O(r^{-2})$ in the asymptotic expansion of the Ricci scalar associated with g_{AB} ($\bar{g}^{AB}_{(0)}$ is the inverse of the order $O(r^0)$ in g_{AB}). One can show that $R_{(2)}$ only depends on γ_{AB} and C_{AB} since $g_{(0)AB}$ does not have a C_{AB}-independent trace-free part.

Appendix J

Vanishing of the leading magnetic Weyl tensor

In this appendix we give the form of k_{ab} in case where the leading magnetic Weyl tensor B_{ab} vanishes, and we work out the corresponding large-τ expansion. This allows us to demonstrate that the Beig–Schmidt data (8.16) obtained by explicit coordinate transformation from Bondi gauge is that associated with a vanishing B_{ab}. We also confirm the identification between the supertranslation Goldstone mode at null infinity and the supertranslation Goldstone mode at spatial infinity that was previously made in [173].

The vanishing of the leading magnetic part of the Weyl tensor B_{ab} defined in (7.34) is equivalent to the condition

$$D_{[a}k_{b]c} = 0\,. \tag{J.1}$$

On the three-dimensional hyperboloid $\mathcal{H}$ a symmetric traceless tensor satisfying the above condition can be written in terms of a scalar potential Φ [107, 194],

$$k_{ab} = -\left(D_a D_b + h_{ab}\right)\Phi\,, \qquad (D^2+3)\,\Phi = 0\,. \tag{J.2}$$

The scalar field Φ is the Goldstone mode of supertranslations at spatial infinity [173]. Proceeding exactly as we did with the electric potential σ in Appendix H, we find its large-τ expansion to be

$$\Phi(\tau,x) = e^{\tau}\,\Phi^{(-1)} + e^{-\tau}\Phi^{(1)} + e^{-3\tau}\tau\,\tilde{\Phi} + e^{-3\tau}\,\Phi^{(3)} + \dots\,, \tag{J.3}$$

with

$$\Phi^{(1)} = -\left(\nabla^2+1\right)\Phi^{(-1)}\,, \qquad \tilde{\Phi} = \nabla^2\left(\nabla^2+2\right)\Phi^{(-1)}\,, \qquad \dots \tag{J.4}$$

Now we express (J.2) in global coordinates (τ, x^A),

$$k_{\tau\tau} = \left(-\partial_\tau^2+1\right)\Phi\,, \tag{J.5a}$$

$$k_{\tau A} = \left(-\partial_\tau + \tanh\tau\right)\partial_A\Phi\,, \tag{J.5b}$$

$$k_{AB} = \left(\gamma_{AB}\,\cosh^2\tau\,(\tanh\tau\,\partial_\tau - 1) - \nabla_A\nabla_B\right)\Phi\,, \tag{J.5c}$$

and plug in the asymptotic expansion (J.3) of the scalar potential Φ. Doing so we find that the leading order terms for the various components of k_{ab} are

$$\tilde{k}_{\tau\tau} = -8\,\tilde{\Phi} = 8\,\nabla^A\nabla^B C^{\Phi}_{AB}\,, \tag{J.6a}$$
$$k^{(3)}_{\tau\tau} = -8\,\Phi^{(3)}\,, \tag{J.6b}$$
$$\tilde{k}_{\tau A} = 0\,, \tag{J.6c}$$
$$k^{(1)}_{\tau A} = -2\,\partial_A(\Phi^{(-1)} - \Phi^{(1)}) = 2\,\nabla^B C^{\Phi}_{AB}\,, \tag{J.6d}$$
$$\tilde{k}_{AB} = 0\,, \tag{J.6e}$$
$$k^{(-1)}_{AB} = -\nabla_A\nabla_B\Phi^{(-1)} - \frac{1}{2}\gamma_{AB}\,(\Phi^{(-1)} + \Phi^{(1)}) = \frac{1}{2}C^{\Phi}_{AB}\,, \tag{J.6f}$$

where we have suggestively defined

$$C^{\Phi}_{AB} \equiv -2\,\nabla_A\nabla_B\Phi^{(-1)} + \gamma_{AB}\nabla^2\Phi^{(-1)}\,. \tag{J.7}$$

We observe that (J.6) perfectly agree with (8.16) provided that we make the identification

$$\Phi^{(-1)} = C\,, \tag{J.8}$$

which in particular also implies $C^{\Phi}_{AB} = C^0_{AB}$. Building on the work of Troessaert who pointed out that BMS supertranslations are isomorphic to supertranslations at spatial infinity [165], (J.8) had been recently argued to hold since both members transform in the same way [173]. Since we now have the map (8.16) between the large-τ behavior of k_{ab} and the supertranslation mode C, we are able to confirm the identification (J.8) without relying on their transformation properties, while at the same time proving a consistency check of our findings. Finally, note that it is known that the subleading mode $\Phi^{(3)}$ can always be removed by a pure gauge transformation [165], and we can therefore set it to zero without loss of generality.

The data $\tilde{k}_{\tau\tau}\,, k^{(3)}_{\tau\tau}\,, \tilde{k}_{\tau A}\,, k^{(1)}_{\tau A}\,, \tilde{k}_{AB}\,, k^{(-1)}_{AB}$ fully specifies a solution for k_{ab}. Since the data (8.16) obtained in the main text can be written in the form (J.6), we conclude that the full solution of k_{ab} also admits the form (J.2). This further implies the vanishing of the leading magnetic Weyl tensor whenever the solution can be mapped onto the Bondi phase space defined in Section 8.2,

$$B_{ab} = 0\,. \tag{J.9}$$

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