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SOME TOPOLOGICAL PROPERTIES OF SOLUTION SETS IN PARTIALLY ORDERED SET OPTIMIZATION

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Abstract. In this paper, we study some topological properties, in particular, arcwise connectedness and connectedness of solution sets in set optimization, where the acting space is equipped with partial set order relations. We obtain continuity, generalized convexity, and natural quasi arcwise connectedness of the nonlinear scalarization function and use them to study some topological properties and convergence of efficient and weak efficient solution sets in partially ordered set optimization. We also employ derived results to vector-valued game theory with uncertainty.

Keywords. Arcwise connectedness; Game theory; Generalized convexity; Nonlinear scalarization; Set optimization.

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1. Introduction

Set optimization has gained increasing attention due to several applications in mathematical finance, control theory, game theory, welfare economics, engineering and medical sciences. For more details, see [1] and the references therein. Set optimization problem is based on comparison of values of the set-valued objective map by means of set order relations. Several kinds of set order relations are available in the literature; see, e.g., [2, 3, 4]. Recently, Karaman et al. [5] introduced set order relations on the family of sets involving the Minkowski difference. In comparison to other existing set order relations, these set order relations are partial order relations on the family of bounded sets and hence provide a new approach to study set optimization problems. Recently, set optimization with respect to partial set order relations has been studied and investigated in [6, 7, 8, 9] and the references therein.

Nonlinear scalarization functions are essential tools to study set optimization problems in terms of associated scalar optimization problems. By using nonlinear scalarization functions, we can study optimality conditions, Ekeland's variational principle, topological properties, the existence results, and so on; see [7, 9, 10, 11, 12] and the references therein. It is well known that the continuity and convexity of the nonlinear scalarizing function play an important role in establishing existence and stability results. Therefore, it is important and interesting to study

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the continuity and convexity of the nonlinear scalarizing function introduced by Karaman et al. [5].

Connectedness and arcwise connectedness are the most fundamental topological properties of solution sets in set optimization problems. Also, the convergence of the solution sets in set optimization is one of the most important aspect. Very recently, Han [13, 14] established the connectedness and arcwise connecedness of the optimal solutions of set optimization problems using both linear as well as nonlinear scalarization functions. Khushboo and Lalitha [9] studied the lower and upper convergence of the optimal solution sets of a sequence of perturbed scalarized problems for the unified set optimization problem. Ansari et al. [6] studied the Painlevé-Kuratowski convergence of the solution sets for perturbed set optimization problems by using the locally Lipschitz continuity and the concept of *ml*-quasiconnectedness and strictly *ml*-quasiconnectedness for set-valued map. Several authors studied the convergence of solution sets of set optimization problems; see, e.g., [15, 16, 17] and the references therein. However, in set optimization, the study of connectedness, arcwise connectedness, and convergence on the ground of scalarization techniques is not much advanced and is still in initial stages.

Inspired by the work in [6, 11, 12, 14, 18, 19, 20], we consider the set optimization problem equipped with partial set order relations. We drive the continuity, generalized convexity, and arcwise connectedness of the nonlinear scalarization function for sets introduced by Karaman [5]. We characterize the set of strict efficient and weak efficient solutions as the union of optimal solutions of a family of scalar optimization problem scalarized by the nonlinear function. We further derive the connectedness of efficient, strict efficient and weak efficient solution sets of partially ordered set optimization problems using the objective set-valued maps to be strictly nearly convexlike and natural arcwise quasi connectedness. We also derive the convergence of the strict efficient and weak efficient solution sets. Finally, we derive the connectedness and arcwise connectedness of solution sets to vector-valued game theory with uncertainty as an application.

The rest of the paper is organized as follows. In Section 2, we recall some basic notions and definitions which will be used in the sequel. In Section 3, we study the continuity of the nonlinear scalarization function for sets considered in [5]. In Section 4, we study the generalized convexity and the natural arcwise quasi connectedness of the nonlinear scalarization function for sets. In Section 5, we establish the connectedness and arcwise connectedness of the efficient, strict efficient and weak efficient solution sets using the strictly nearly convexlikeness and the natural arcwise quasi connectedness of the set-valued maps. In Section 6, we study the convergence of the strict efficient and weak efficient solution sets by using nonlinear scalarization function. Section 7 deals with an application to vector-valued game theory with uncertainty. Last section concludes the paper. Section 8 ends this paper.

2. PRELIMINARIES

Throughout the paper, unless otherwise specified, we assume that X and Y are real normed vector spaces and $\mathbf{0}$ denotes the zero vector in Y. We denote the family of nonempty proper subsets of Y and the family of nonempty proper bounded subsets of Y by $\mathscr{P}(Y)$ and $\mathscr{B}(Y)$, respectively. For a set $A \in \mathscr{P}(Y)$, we denote by int$A$, cl$A$ and A^c, the interior, the closure and the complement of A, respectively.

For $A,B \in \mathscr{P}(Y)$, the algebraic sum of A and B is defined by

$$A+B := \{a+b : a \in A, b \in B\} = \bigcup_{b\in B}(A+b),$$

and the Minkowski or Pontryagin difference of A and B, considered in [21], is defined as

$$A \dot{-} B := \{y \in Y : y+B \subseteq A\} = \bigcap_{b\in B}(A-b).$$

We now recall some basic properties of Minkowski difference.

Proposition 2.1. [5] *Let $A,B \in \mathscr{P}(Y)$ and $\alpha \in Y$. The following assertions hold.*

(a) $(\alpha+A)\dot{-}B = \alpha+(A\dot{-}B)$.
(b) $A\dot{-}(\alpha+B) = -\alpha+(A\dot{-}B)$.
(c) *If A is closed, then $A\dot{-}B$ is also closed.*
(d) *If A is bounded, then $A\dot{-}A = \{\mathbf{0}\}$.*

We now recall the following ordering relations $\preceq_K^{ml}$ and $\prec_K^{ml}$ on $\mathscr{P}(Y)$, introduced by Karaman et al. [5]. For $A,B,K \in \mathscr{P}(Y)$,

$$A \preceq_K^{ml} B :\Leftrightarrow (A\dot{-}B)\cap(-K) \neq \emptyset,$$

and

$$A \prec_K^{ml} B :\Leftrightarrow (A\dot{-}B)\cap(-\mathrm{int}K) \neq \emptyset.$$

Karaman et al. [5, Corollary 4] proved that the relation $\preceq_K^{ml}$ is partial order relation on $\mathscr{B}(Y)$ provided that K is a pointed convex cone in Y with $\mathbf{0} \in K$. The relation $\preceq_K^{ml}$ is compatible with addition see [5, Proposition 7(i)]. Moreover, the relation $\preceq_K^{ml}$ is compatible with scalar multiplication if and only if K is a cone, see [5, Proposition 7(ii)].

We now recall a set order relation, namely, weak l-set order relation, proposed by Kuroiwa [4]. For $A,B \in \mathscr{P}(Y)$ and proper convex cone K in Y with nonempty interior,

$$A \prec_K^{l} B :\Leftrightarrow B \subseteq A+\mathrm{int}K.$$

It can be observed from [5, Proposition 9] that $A \prec_K^{ml} B \Rightarrow A \prec_K^{l} B$.

We now consider the following set optimization problem

$$\begin{aligned} &\text{Minimize } F(x)\\ &\text{subject to } x \in S, \end{aligned} \tag{P}$$

where $\emptyset \neq S \subseteq X$ and $F : X \rightrightarrows Y$ is a set-valued map with $F(x) \neq \emptyset$ for all $x \in X$. Let $F(S) = \bigcup_{x\in S} F(x)$.

To define the notions of efficient solutions of the problem (P) with respect to $\preceq_K^{ml}$ and $\prec_K^{ml}$, we assume that $F(x) \in \mathscr{B}(Y)$ for all $x \in X$ and K is a closed convex pointed cone with nonempty interior. We now recall some notions of efficient solutions of the problem (P).

Definition 2.1. [5, Definition 7] A point $\bar{x} \in S$ said to be

(a) an ml-efficient solution of (P) if there does not exist any $x \in X$ such that $F(x) \preceq_K^{ml} F(\bar{x})$ and $F(x) \neq F(\bar{x})$, that is, either $F(x) \not\preceq_K^{ml} F(\bar{x})$ or $F(x) = F(\bar{x})$, for any $x \in X$;
(b) a weak ml-efficient solution to (P) if there does not exist any $x \in X$ such that $F(x) \prec_K^{ml} F(\bar{x})$.

(c) a strict ml-efficient solution to (P) if there does not exist any $x \in X \setminus \{\bar{x}\}$ such that $F(x) \preceq_K^{ml} F(\bar{x})$.

We recall a weaker notion of efficient solution with respect to $\prec_K^l$ form the book by Khan et al. [1]. A point $\bar{x} \in S$ said to be a weak l-efficient solution of (P) if there does not exist any $x \in X$ such that $F(x) \prec_K^l F(\bar{x})$.

We denote the set of ml-efficient, weak ml-efficient, strict ml-efficient, and weak l-efficient solutions of (P) by $ml - \text{Eff}(F,K)$, $ml - \text{WEff}(F,K)$, $ml - \text{SEff}(F,K)$, and $l - \text{WEff}(F,K)$, respectively. From the above definitions, it is clear that $ml - \text{SEff}(F,K) \subseteq ml - \text{Eff}(F,K) \subseteq ml - \text{WEff}(F,K)$.

Remark 2.1. From [9, Theorem 2.2], we observe that

$$l - \text{WEff}(F,K) \subseteq ml - \text{WEff}(F,K).$$

However, the reverse inclusion may fail to hold; see [9, pp. 6].

A nonempty subset S of X is said to be an arcwise connected set if for any $z_1, z_2 \in S$ there exists a continuous map $\varphi_{z_1,z_2} : [0,1] \to S$ such that $\varphi_{z_1,z_2}(0) = z_1$ and $\varphi_{z_1,z_2}(1) = z_2$, see [22, Section 6.2]. Also, it is well-known that an arcwise connected set is a connected set.

To characterize ml-efficient and weak ml-efficient solutions, Karaman et al. [5] introduced the scalarization function $I_k^{ml}(\cdot,\cdot) : \mathscr{P}(Y) \times \mathscr{P}(Y) \rightrightarrows \mathbb{R} \cup \{\pm\infty\} := \overline{\mathbb{R}}$ defined by

$$I_k^{ml}(A,B) = \inf\{t \in \mathbb{R} : A \preceq_K^{ml} tk + B\},$$

for $A, B \in \mathscr{P}(Y)$ and $k \in \text{int}K$.

Karaman et al. [5] studied the following properties of the function I_k^{ml} to establish scalarizations.

Proposition 2.2. [5] *Let $A_1, A_2, A, B \in \mathscr{P}(Y)$ and $t \in \mathbb{R}$. The function $I_k^{ml}(\cdot,\cdot) : \mathscr{P}(Y) \times \mathscr{P}(Y) \to \overline{\mathbb{R}}$ has the following properties:*

(a) $I_k^{ml}(A,B) = +\infty$ *if and only if* $A \dot{-} B = \emptyset$.

(b) *If B is bounded, then* $I_k^{ml}(A,B) > -\infty$.

(c) *If A is bounded, then* $I_k^{ml}(A,A) = 0$.

(d) *If $A \dot{-} B$ is compact and K is closed, then*

$$I_k^{ml}(A,B) \le t \quad \Leftrightarrow \quad A \preceq_K^{ml} tk + B.$$

(e) $I_k^{ml}(A,B) < t$ *if and only if* $A \prec_K^{ml} tk + B$.

(f) $I_k^{ml}(\cdot,B)$ *is ml-increasing on* $\mathscr{P}(Y)$, *that is, if*

$$A_1 \preceq_K^{ml} A_2 \quad \Rightarrow \quad I_k^{ml}(A_1,B) \le I_k^{ml}(A_2,B).$$

(g) $I_k^{ml}(\cdot,B)$ *is strictly ml-increasing on compact sets in* $\mathscr{P}(Y)$, *that is, for all compact sets* $A_1, A_2 \in \mathscr{P}(Y)$ *if*

$$A_1 \prec_K^{ml} A_2 \quad \Rightarrow \quad I_k^{ml}(A_1,B) < I_k^{ml}(A_2,B).$$

Remark 2.2. It may be noted that Proposition 2.2(d) can be proved on similar lines of [5, Proposition 9 (i)].

3. Continuity of Nonlinear Scalarization Function

This section deals with the continuity of the nonlinear scalarization function I_k^{ml} which plays a crucial role to establish arcwise connectedness and convergence of the solution sets.

We next recall the upper and lower semi-continuity notions for set-valued maps from the book by Khan et al. [1].

Definition 3.1. [1, Definition 3.1.1(a)-(c)] Let $\bar{x} \in X$. A map $F : X \rightrightarrows Y$ is said to be

(a) upper semi-continuous at $\bar{x}$ iff, for any open set $F(\bar{x}) \subseteq V$, there exists a neighborhood U of $\bar{x}$ such that $F(x) \subseteq V$, for all $x \in U \cap X$.
(b) lower semi-continuous at $\bar{x}$ iff, for any open set V such that $F(\bar{x}) \cap V \neq \emptyset$, there exists a neighborhood U of $\bar{x}$ such that $F(x) \cap V \neq \emptyset$, for all $x \in U \cap X$.
(c) continuous at $\bar{x}$ if F is both upper semi-continuous and lower semi-continuous at $\bar{x}$.

We say that F is upper semi-continuous and lower semi-continuous on X if it is upper semi-continuous and lower semi-continuous at each point $x \in X$, respectively. We say that F is continuous on X if it is both upper semi-continuous and lower semi-continuous on X.

We recall sequential characterizations of upper semi-continuity and lower semi-continuity from [1].

Lemma 3.1. [1, Proposition 3.1.6(iv), Definition 3.1.7, Proposition 3.1.9] *Let $F : X \rightrightarrows Y$ and $\bar{x} \in X$.*

(i) *F is lower semi-continuous at $\bar{x} \in X$ iff, for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to \bar{x}$ and $\bar{y} \in F(\bar{x})$ there exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq Y$ converging to $\bar{y}$ such that $y_n \in F(x_n)$ for sufficiently large values of n.*
(ii) *F is compact at $\bar{x}$ iff, for any sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \to \bar{x}$ and $(y_n)_{n\in\mathbb{N}}$ with $y_n \in F(x_n)$, there exist a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ and $\bar{y} \in F(\bar{x})$ such that $y_{n_k} \to \bar{y}$.*
(iii) *If F is compact at $\bar{x}$, then F is upper semi-continuous at $\bar{x}$.*
(iv) *F is compact at $\bar{x}$ iff F is upper semi-continuous at $\bar{x}$ and $F(\bar{x})$ is compact.*

The following lemma is required to prove the connectedness of efficient and weak efficient solution sets.

Lemma 3.2. [1, Proposition 3.1.8] *Let $\emptyset \neq S \subseteq X$ and $F : S \rightrightarrows Y$ be upper semi-continuous and $F(x) \neq \emptyset$ for every $x \in S$.*

(i) *If S is compact and $F(x)$ is compact for each $x \in S$, then $F(S)$ is compact.*
(ii) *If S is connected and $F(x)$ is connected for each $x \in S$, then $F(S)$ is connected.*

The following lemma is required to prove the connectedness and arcwise connectedness of efficient and weak efficient solution sets.

Lemma 3.3. *Let $\emptyset \neq S \subseteq X$ and $F : S \rightrightarrows Y$ be continuous map and $F(x) \neq \emptyset$ for every $x \in S$.*

(i) [23, Theorem 4.5.1] *If S is connected and $F(x)$ is connected for each $x \in S$, then $F(S)$ is arcwise connected.*
(ii) [23, Theorem 4.5.2] *If S is arcwise connected and $F(x)$ is arcwise connected for each $x \in S$, then $F(S)$ is connected.*

We now proceed to prove the continuity of the function I_k^{ml}. Let Λ_1 and Λ_2 be two real normed vector spaces. Let $A : \Lambda_1 \rightrightarrows Y$ and $B : \Lambda_2 \rightrightarrows Y$ be two set-valued maps. Define $\Theta_k : \Lambda_1 \times \Lambda_2 \to \overline{\mathbb{R}}$

by

$$\Theta_k(\lambda,\mu) := I_k^{ml}(A(\lambda),B(\mu)) := \inf\{t \in \mathbb{R} : A(\lambda) \preceq_K^{ml} tk + B(\mu)\}, \tag{3.1}$$

for all $(\lambda,\mu) \in \Lambda_1 \times \Lambda_2$.

Throughout the section, we assume that $A(\lambda), B(\mu) \in \mathscr{B}(Y)$ and $A(\lambda) \dot{-} B(\mu) \neq \emptyset$, for all $(\lambda,\mu) \in \Lambda_1 \times \Lambda_2$. From Proposition 2.2 (a) and (b), we have

$$-\infty < \Theta_k(\lambda,\mu) < +\infty \quad \text{for all } (\lambda,\mu) \in \Lambda_1 \times \Lambda_2.$$

We next define a set-valued map $\Phi : \Lambda_1 \times \Lambda_2 \rightrightarrows Y$ by

$$\Phi(\lambda,\mu) \ := \ A(\lambda) \dot{-} B(\mu) = \{y \in Y : y + B(\mu) \subseteq A(\lambda)\},$$

for all $(\lambda,\mu) \in \Lambda_1 \times \Lambda_2$.

We next show that the map Φ is compact and lower semi-continuous which will be used to prove the continuity of the set-valued map Θ_k.

Lemma 3.4. *If A is upper semi-continuous and compact valued on Λ_1 and B is lower semi-continuous on Λ_2, then Φ is compact on $\Lambda_1 \times \Lambda_2$.*

Proof. Let $\{(\lambda_n,\mu_n)\}_{n\in\mathbb{N}} \subseteq \Lambda_1 \times \Lambda_2$ be such that $(\lambda_n,\mu_n) \to (\lambda_0,\mu_0)$ and $z_n \in \Phi(\lambda_n,\mu_n)$. We have to show that there exists a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $z_{n_k} \to z_0 \in \Phi(\lambda_0,\mu_0)$, that is, $z_0 + B(\mu_0) \subseteq A(\lambda_0)$.

Let $v_0 \in B(\mu_0)$. As B is lower semi-continuous at v_0, by Lemma 3.1(i) there exists $v_n \in B(\mu_n)$ such that $v_n \to v_0$. Since $z_n \in \Phi(\lambda_n,\mu_n)$ we have $z_n + B(\mu_n) \subseteq A(\lambda_n)$ which implies that

$$w_n = z_n + v_n \in A(\lambda_n), \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Using upper semi-continuity of A and Lemma 3.1(iv), we get a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ such that $w_{n_k} \to w_0$ and $w_0 \in A(\lambda_0)$. From (3.2), it follows that $z_{n_k} = w_{n_k} - v_{n_k} \to w_0 - v_0$ as $k \to \infty$. Hence there exists a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $z_{n_k} \to z_0 = w_0 - v_0$. This implies that $z_0 + v_0 \in A(\lambda_0)$. Since $v_0 \in B(\mu_0)$ is arbitrary, it follows that $z_0 + B(\mu_0) \subseteq A(\lambda_0)$. □

Lemma 3.5. *If A is upper semi-continuous and compact valued on Λ_1 and B is lower semi-continuous on Λ_2, then Φ is upper semi-continuous on $\Lambda_1 \times \Lambda_2$.*

Proof. The proof follows from Lemma 3.4 and Lemma 3.1(iii). □

Lemma 3.6. *If A is lower semi-continuous on Λ_1 and B is upper semi-continuous and compact valued on Λ_2, then Φ is lower semi-continuous on $\Lambda_1 \times \Lambda_2$.*

Proof. Let $\{(\lambda_n,\mu_n)\}_{n\in\mathbb{N}} \subseteq \Lambda_1 \times \Lambda_2$ be such that $(\lambda_n,\mu_n) \to (\lambda_0,\mu_0)$ and $z_0 \in \Phi(\lambda_0,\mu_0)$. As $z_0 \in \Phi(\lambda_0,\mu_0)$ it follows that

$$z_0 + B(\mu_0) \subseteq A(\lambda_0). \tag{3.3}$$

Let $w_0 \in B(\mu_0)$ and $w_n \in B(\mu_n)$. By Lemma 3.1(iv), there exists a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ of $\{w_n\}_{n\in\mathbb{N}}$ such that $w_{n_k} \to w_0$. It follows from (3.3) that $z_0 + w_0 \in A(\lambda_0)$. As A is lower semi-continuous at λ_0, it follows that there exists a sequence $s_n \in A(\lambda_n)$ such that $s_n \to z_0 + w_0$. Let $z_n = s_n - w_n$. Then, $z_n + w_n = s_n \in A(\lambda_n)$ and $z_n \to z_0$. Since $w_n \in B(\mu_n)$ is arbitrary, it follows that $z_n + B(\mu_n) \subseteq A(\lambda_n)$, that is, $z_n \in \Phi(\lambda_n,\mu_n)$. □

We now present the continuity of the function Θ_k.

Theorem 3.1. *Let $\Phi : \Lambda_1 \times \Lambda_2 \rightrightarrows Y$ be a continuous function on $\Lambda_1 \times \Lambda_2$. Then the function $\Theta_k(\cdot,\cdot)$ is a continuous on $\Lambda_1 \times \Lambda_2$.*

Proof. Let $\varepsilon > 0$ be any real number and $(\lambda_0, \mu_0) \in \Lambda_1 \times \Lambda_2$. Let $\Theta_k(\lambda_0, \mu_0) = \bar{t}$. Then $A(\lambda_0) \not\preceq_K^{ml} (\bar{t}-\varepsilon)k + B(\mu_0)$, that is, $[A(\lambda_0) \dot{-} ((\bar{t}-\varepsilon)k + B(\mu_0))] \cap (-K) = \emptyset$, which implies that $[A(\lambda_0) \dot{-} ((\bar{t}-\varepsilon)k + B(\mu_0))] \subseteq (-K)^c$. By using Proposition 2.1 (b), it follows that

$$-(\bar{t}-\varepsilon)k + [A(\lambda_0) \dot{-} B(\mu_0)] \subseteq (-K)^c,$$

that is,

$$\Phi(\lambda_0, \mu_0) = A(\lambda_0) \dot{-} B(\mu_0) \subseteq (-K)^c + (\bar{t}-\varepsilon)k.$$

As Φ is continuous on $\Lambda_1 \times \Lambda_2$, there exists a neighbourhood $U^1_{\lambda_0} \times U^1_{\mu_0}$ of (λ_0, μ_0) such that $\Phi(\lambda, \mu) \subseteq (-K)^c + (\bar{t}-\varepsilon)k$ for all $(\lambda, \mu) \in U^1_{\lambda_0} \times U^1_{\mu_0}$, which further implies that $A(\lambda) \not\preceq_K^{ml} (\bar{t}-\varepsilon)k + B(\mu)$ for all $(\lambda, \mu) \in U^1_{\lambda_0} \times U^1_{\mu_0}$. Hence,

$$\Theta_k(\lambda, \mu) > \bar{t} - \varepsilon, \quad \text{for all } (\lambda, \mu) \in U^1_{\lambda_0} \times U^1_{\mu_0}. \tag{3.4}$$

From Proposition 2.2(d), we have $A(\lambda_0) \preceq_K^{ml} B(\mu_0) + \bar{t}k$, that is, there exists $-k' \in -K$ such that

$$-k' + \bar{t}k + B(\mu_0) \subseteq A(\lambda_0).$$

For any $\varepsilon > 0$, it follows that $(-k' - \frac{\varepsilon}{2}k) + \frac{\varepsilon}{2}k + \bar{t}k + B(\mu_0) \subseteq A(\lambda_0)$, which implies that $A(\lambda_0) \prec_K^{ml} (\bar{t} + \frac{1}{2}\varepsilon)k + B(\mu_0)$, that is, $[A(\lambda_0) \dot{-} ((\bar{t} + \frac{1}{2}\varepsilon)k + B(\mu_0))] \cap (-\text{int}K) \neq \emptyset$. Using Proposition 2.1 (b), we have that $[-(\bar{t} + \frac{1}{2}\varepsilon)k + \Phi(\lambda_0, \mu_0)] \cap (-\text{int}K) \neq \emptyset$. As Φ is lower semi-continuous at (λ_0, μ_0), it follows that there exists a neighbourhood $U^2_{\lambda_0} \times U^2_{\mu_0}$ of (λ_0, μ_0) such that

$$[-(\bar{t} + \frac{1}{2}\varepsilon)k + \Phi(\lambda, \mu)] \cap (-\text{int}K) \neq \emptyset, \quad \text{for all } (\lambda, \mu) \in U^2_{\lambda_0} \times U^2_{\mu_0}.$$

It follows that $A(\lambda) \prec_K^{ml} (\bar{t} + \frac{1}{2}\varepsilon)k + B(\mu)$ for all $(\lambda, \mu) \in U^2_{\lambda_0} \times U^2_{\mu_0}$. Hence,

$$\Theta_k(\lambda, \mu) \leq \bar{t} + \frac{1}{2}\varepsilon < \bar{t} + \varepsilon, \quad \text{for all } (\lambda, \mu) \in U^2_{\lambda_0} \times U^2_{\mu_0}. \tag{3.5}$$

Let $U_{\lambda_0} = U^1_{\lambda_0} \cap U^2_{\lambda_0}$ and $U_{\mu_0} = U^1_{\mu_0} \cap U^2_{\mu_0}$. Then it follows from (3.4) and (3.5) that $\bar{t} - \varepsilon < \Theta_k(\lambda, \mu) < \bar{t} + \varepsilon$ for all $(\lambda, \mu) \in U_{\lambda_0} \times U_{\mu_0}$, which implies that $|\Theta_k(\lambda, \mu) - \Theta_k(\lambda_0, \mu_0)| < \varepsilon$ for all $(\lambda, \mu) \in U_{\lambda_0} \times U_{\mu_0}$. Hence, Θ_k is continuous at (λ_0, μ_0). Since $(\lambda_0, \mu_0) \in \Lambda_1 \times \Lambda_2$ is arbitrary, therefore, Θ_k is continuous on $\Lambda_1 \times \Lambda_2$. □

The following a direct consequence of Lemma 3.5, Lemma 3.6 and Theorem 3.1.

Corollary 3.1. *If A and B are continuous on Λ_1 and Λ_2, respectively, with nonempty compact values, then the function $\Theta_k(\cdot,\cdot)$ is continuous on $\Lambda_1 \times \Lambda_2$.*

4. Generalized Convexity and Natural Arcwise Quasi Connectedness of Nonlinear Scalarization Function

In this section, we discuss the generalized convexity and the natural arcwise quasi connectedness of the nonlinear scalarization function I_k^{ml}. We also establish the convexity of I_k^{ml} over $\mathscr{P}(Y)$.

Corresponding to the set-valued map $F : X \rightrightarrows Y$, we redefine the function Θ_k given in (3.1). Let $\Theta_k : X \times X \to \overline{\mathbb{R}}$ be defined by $\Theta_k(x,u) := I_k^{ml}(F(x), F(u))$ for all $x, u \in X$. From now onwards, we assume that $F(x) \dot{-} F(u) \neq \emptyset$ for all $x, u \in X$. From Proposition 2.2 (a) and (b), we have $-\infty < \Theta_k(x,u) < +\infty$ for all $x, u \in X$. We next prove the convexity of the function $I_k^{ml}(\cdot, B)$, for $B \in \mathscr{P}(Y)$.

Proposition 4.1. *Let $B \in \mathscr{P}(Y)$. Then $I_k^{ml}(\cdot, B)$ is a convex function on $\mathscr{P}(Y)$.*

Proof. Let $\mu \in [0,1]$ and $A_1, A_2 \in \mathscr{P}(Y)$. For any $t > 0$, we have

$$A_1 \preceq_K^{ml} (I_k^{ml}(A_1, B) + t)k + B \text{ and } A_2 \preceq_K^{ml} (I_k^{ml}(A_2, B) + t)k + B.$$

Using the compatibility of the relation $\preceq_K^{ml}$ with respect to addition and scalar multiplication, we have

$$\mu A_1 + (1-\mu)A_2 \preceq_K^{ml} \mu(I_k^{ml}(A_1, B) + t)k + \mu B + (1-\mu)(I_k^{ml}(A_2, B) + t)k + (1-\mu)B,$$

that is, there exists $k^{'} \in K$ such that

$$-k^{'} + \mu(I_k^{ml}(A_1, B) + t)k + (1-\mu)(I_k^{ml}(A_2, B) + t)k + \mu B + (1-\mu)B \subseteq \mu A_1 + (1-\mu)A_2,$$

which gives

$$-k^{'} + \mu(I_k^{ml}(A_1, B) + t)k + (1-\mu)(I_k^{ml}(A_2, B) + t)k + B \subseteq \mu A_1 + (1-\mu)A_2.$$

Hence,

$$\mu A_1 + (1-\mu)A_2 \preceq_K^{ml} (\mu I_k^{ml}(A_1, B) + (1-\mu)I_k^{ml}(A_2, B) + t)k + B,$$

that is,

$$I_k^{ml}(\mu A_1 + (1-\mu)A_2, B) \leq \mu I_k^{ml}(A_1, B) + (1-\mu)I_k^{ml}(A_2, B).$$

□

Mastroeni and Rapcsák [24, Definition 4.1(ii)] introduced a notion of nearly cone-convexlike set-valued maps. We define similar notions with respect to the relations $\preceq_K^{ml}$ and $\prec_K^{ml}$.

Definition 4.1. Let $S \subseteq X$ and $F : S \rightrightarrows Y$ be a set-valued map. The map F is said to be

(a) nearly *ml*-*K*-convexlike on S if, for all $x_1, x_2 \in S$, there exist $z \in S$ and $\mu \in (0,1)$ such that $F(z) \preceq_K^{ml} (1-\mu)F(x_1) + \mu F(x_2)$.

(b) strictly nearly *ml*-*K*-convexlike on S if, for all $x_1, x_2 \in S$, $x_1 \neq x_2$, there exist $z \in S$ and $\mu \in (0,1)$ such that $F(z) \prec_K^{ml} (1-\mu)F(x_1) + \mu F(x_2)$.

If $F : S \to \mathbb{R}$, then we refer to the notions as nearly $\mathbb{R}_+$-convexlike and strictly nearly $\mathbb{R}_+$-convexlike on S, respectively. We next establish convexlikeness of the function Θ_k in the first variable provided that the set-valued map F is convexlike on S.

Theorem 4.1. *Let $u \in S$.*

(i) *If F is nearly ml-K-convexlike on S, then the function $\Theta_k(\cdot, u)$ is nearly $\mathbb{R}_+$-convexlike on S.*

(ii) *If $F(x)$ is compact for all $x \in S$ and F is strictly nearly ml-K-convexlike on S, then the function $\Theta_k(\cdot, u)$ is strictly nearly $\mathbb{R}_+$-convexlike on S.*

Proof.

(i) Let F be nearly *ml*-*K*-convexlike on S. Then, for all $x_1, x_2 \in S$, there exists $z \in S$ and $\mu \in (0,1)$ such that $F(z) \preceq_K^{ml} (1-\mu)F(x_1) + \mu F(x_2)$. Using Proposition 2.2 (f) and Proposition 4.1, we obtain

$$\begin{aligned}
\Theta_k(z, u) = I_k^{ml}(F(z), F(u)) &\leq I_k^{ml}((1-\mu)F(x_1) + \mu F(x_2), F(u)) \\
&\leq (1-\mu)I_k^{ml}(F(x_1), F(u)) + \mu I_k^{ml}(F(x_2), F(u)) \\
&= (1-\mu)\Theta_k(x_1, u) + \mu\Theta_k(x_2, u).
\end{aligned}$$

(ii) This follows similar lines by using Proposition 2.2 (g) and Proposition 4.1. □

Motivated by the notion of natural quasi K-convexity of set-valued maps given in [19, Definition 2.2], we define the notion of natural arcwise quasi K-connectedness for set-valued maps in terms of the relations $\preceq_K^{ml}$ and $\prec_K^{ml}$.

Definition 4.2. Let $S \subseteq X$ be a nonempty arcwise connected set with respect to arc φ and $F : S \rightrightarrows Y$ be a set-valued map. The map F is said to be

(a) natural arcwise ml-quasi K-connected on S if, for any $x_1, x_2 \in S$ and for any $t \in [0,1]$, there exists $\mu \in [0,1]$ such that $F(\varphi_{x_1,x_2}(t)) \preceq_K^{ml} (1-\mu)F(x_1) + \mu F(x_2)$.

(b) strictly natural arcwise ml-quasi K-connected on S if, for any $x_1, x_2 \in S$ with $x_1 \neq x_2$ and for any $t \in (0,1)$, there exists $\mu \in [0,1]$ such that $F(\varphi_{x_1,x_2}(t)) \prec_K^{ml} (1-\mu)F(x_1) + \mu F(x_2)$.

We refer the above notions as natural arcwise quasi $\mathbb{R}_+$-connected and strictly natural arcwise quasi $\mathbb{R}_+$-connected on S, respectively, provided $F : S \to \mathbb{R}$.

We now establish natural arcwise quasi $\mathbb{R}_+$connectedness of the function Θ_k on S.

Theorem 4.2. *Let S be arcwise connected and $u \in S$.*

(i) *If F is natural arcwise ml-quasi K-connected on S, then the function $\Theta_k(\cdot, u)$ is natural arcwise quasi $\mathbb{R}_+$-connected function on S;*

(ii) *If $F(x)$ is compact for all $x \in S$ and F is strictly natural arcwise ml-quasi K-connected on S, then the function $\Theta_k(\cdot, u)$ is strictly natural arcwise quasi $\mathbb{R}_+$-connected function on S.*

Proof. The proof follows on similar lines of Theorem 4.1. □

5. Connectedness and Arcwise Connectedness of Solution Sets

This section deals with the connectedness and arcwise connectedness of ml-efficient, weak ml-efficient, and strict ml-efficient solution sets in partially ordered set optimization problems.

We now consider the following parametric scalar optimization problem associated with $x \in S$

$$\begin{aligned} &\text{Minimize } I_k^{ml}(F(u), F(x)) \\ &\text{subject to } u \in S. \end{aligned} \tag{P(x)}$$

We define the set-valued maps $\Gamma : S \rightrightarrows S$ and $\Upsilon : S \rightrightarrows S$ by

$$\begin{aligned} \Gamma(x) &:= \{w \in S : I_k^{ml}(F(w), F(x)) \leq I_k^{ml}(F(u), F(x)),\ \forall u \in S\}, \\ &:= \{w \in S : \Theta_k(w, x) \leq \Theta_k(u, x),\ \forall u \in S\}, \end{aligned}$$

and

$$\begin{aligned} \Upsilon(x) &:= \{w \in S : I_k^{ml}(F(w), F(x)) < I_k^{ml}(F(u), F(x)),\ \forall u \in S\}, \\ &:= \{w \in S : \Theta_k(w, x) < \Theta_k(u, x),\ \forall u \in S\}, \end{aligned}$$

respectively.

In the next theorem, we characterize the set of weak ml-efficient solutions in terms of the image of the set-valued map Γ.

Theorem 5.1. *Assume that $F(x)$ is compact for all $x \in S$. Then $ml - \mathrm{WEff}(F, K) = \Gamma(S)$.*

Proof. Let $\bar{x} \in ml-\mathrm{WEff}(F,K)$, which implies that $F(x) \not\prec_K^{ml} F(\bar{x})$ for any $x \in S$. From Proposition 2.2 (e), it follows that $I_k^{ml}(F(x),F(\bar{x})) \geq 0, \forall x \in S$. Since $F(x)$ is bounded for each $x \in S$, by Proposition 2.2 (c), we have $I_k^{ml}(F(\bar{x}),F(\bar{x})) = 0$. Thus

$$I_k^{ml}(F(x),F(\bar{x})) \geq 0 = I_k^{ml}(F(\bar{x}),F(\bar{x})),\ \forall x \in S,$$

that is, $\bar{x} \in \Gamma(\bar{x}) \subseteq \Gamma(S)$. Let $w \in \Gamma(S)$. Then there exists $\bar{x} \in S$ such that $w \in \Gamma(\bar{x})$, which implies that $\Theta_k(w,\bar{x}) \leq \Theta_k(u,\bar{x})$ for all $u \in S$, that is,

$$I_k^{ml}(F(w),F(\bar{x})) \leq I_k^{ml}(F(u),F(\bar{x})), \forall u \in S. \tag{5.1}$$

Assume that $w \notin ml-\mathrm{WEff}(F,K)$, that is, there exists $w_0 \in S$ such that $F(w_0) \prec_K^{ml} F(w)$. Using Proposition 2.2 (g), we obtain $I_k^{ml}(F(w_0),F(\bar{x})) < I_k^{ml}(F(w),F(\bar{x}))$, which contradicts (5.1). Hence $w \in ml-\mathrm{WEff}(F,K)$ and $\Gamma(S) \subseteq ml-\mathrm{WEff}(F,K)$. □

In the following theorem, we characterize the set of strict ml-efficient solutions in terms of the image of the set-valued map Υ.

Theorem 5.2. $ml-\mathrm{SEff}(F,K) = \Upsilon(S)$.

Proof. Let $\bar{x} \in ml-\mathrm{SEff}(F,K)$. Then $F(x) \not\preceq_K^{ml} F(\bar{x})$ for all $x \in S\backslash\{\bar{x}\}$, which implies that $I_k^{ml}(F(x),F(\bar{x})) > 0$ for all $x \in S\backslash\{\bar{x}\}$. From Proposition 2.1(d), we further have $I_k^{ml}(F(x),F(\bar{x})) > 0 = I_k^{ml}(F(\bar{x}),F(\bar{x}))$ for all $x \in S\backslash\{\bar{x}\}$, that is, $\bar{x} \in \Upsilon(\bar{x}) \subseteq \Upsilon(S)$. Let $w \in \Upsilon(S)$. Then there exists $\bar{x} \in S$ such that $w \in \Upsilon(\bar{x})$ and so $\Theta_k(w,\bar{x}) < \Theta_k(u,\bar{x}), \forall u \in S$, that is,

$$I_k^{ml}(F(w),F(\bar{x})) < I_k^{ml}(F(u),F(\bar{x})), \forall u \in S. \tag{5.2}$$

Assume that $w \notin ml-\mathrm{SEff}(F,K)$. Then there exists $w_0 \in S\backslash\{w\}$ such that $F(w_0) \preceq_K^{ml} F(w)$. Using Proposition 2.2 (f), we have

$$I_k^{ml}(F(w_0),F(\bar{x})) \leq I_k^{ml}(F(w),F(\bar{x})),$$

which contradicts (5.2). Hence $w \in ml-\mathrm{SEff}(F,K)$ and so $\Upsilon(S) \subseteq ml-\mathrm{SEff}(F,K)$. □

We next show that the set of ml-efficient, weak ml-efficient, and strict ml-efficient solutions of problem (P) are equal provided that F is strictly natural arcwise ml-quasi K-connected.

Proposition 5.1. *Assume that*

(a) *S is an arcwise connected set with respect to arc φ;*
(b) *F is strictly natural arcwise ml-quasi K-connected on S.*

Then $ml-\mathrm{Eff}(F,K) = ml-\mathrm{WEff}(F,K) = ml-\mathrm{SEff}(F,K)$.

Proof. It is suffices to prove that $ml-\mathrm{WEff}(F,K) \subseteq ml-\mathrm{Eff}(F,K)$.

Let $\bar{x} \in ml-\mathrm{WEff}(F,K)$. If $\bar{x} \notin ml-\mathrm{Eff}(F,K)$, then there exists $\hat{x} \in S$ such that

$$F(\hat{x}) \preceq_K^{ml} F(\bar{x}) \text{ and } F(\hat{x}) \neq F(\bar{x}). \tag{5.3}$$

Clearly, $\hat{x} \neq \bar{x}$. Since F is strictly natural arcwise ml-quasi K-connected on S, then, for any $t \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\varphi_{\bar{x},\hat{x}}(t)) \prec_K^{ml} \mu F(\bar{x}) + (1-\mu)F(\hat{x}). \tag{5.4}$$

By (5.3) and (5.4), together with transitivity of the relation $\preceq_K^{ml}$, and the fact that $F(\bar{x}) \subseteq \mu F(\bar{x}) + (1-\mu)F(\bar{x})$, we have $F(\varphi_{\bar{x},\hat{x}}(t)) \prec_K^{ml} F(\bar{x})$, which contradicts the fact that $\bar{x} \notin \mathrm{WEff}(F,K)$. The proof of second equality follows on the similar lines. □

5.1. **Arcwise connectedness of solution sets using arcwise quasi-connected set-valued maps.** In this subsection, we derive the arcwise connectedness of ml-efficient, strict ml-efficient and weak ml-efficient solution sets of the problem (P) using arcwise quasi-connected set-valued maps.

We prove that Γ is arcwise connected-valued map.

Theorem 5.3. *Assume that*

(a) *S is an arcwise connected set with respect to arc φ;*
(b) *F is natural arcwise ml-quasi K-connected on S.*

Then, $\Gamma(x)$ is arcwise connected set with respect to arc φ, for all $x \in S$.

Proof. Let $t \in (0,1)$ and $x_1, x_2 \in \Gamma(x), x_1 \neq x_2$. Then, for any $\bar{x} \in S$,

$$\Theta_k(x_1,x) \leq \Theta_k(\bar{x},x) \text{ and } \Theta_k(x_2,x) \leq \Theta_k(\bar{x},x). \tag{5.5}$$

From Theorem 4.2 (i), it follows that the function $\Theta_k(\cdot,x)$ is natural arcwise quasi $\mathbb{R}_+$-connected function on S. Then, for any $t \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\Theta_k(\varphi_{x_1,x_2}(t),x) \leq (1-\mu)\Theta_k(x_1,x) + \mu\Theta_k(x_2,x).$$

Using (5.5), it follows that $\varphi_{x_1,x_2}(t) \in \Gamma(x)$ for any $t \in (0,1)$. Clearly, by (a) we have $\varphi_{x_1,x_2}(0) = x_1$ and $\varphi_{x_1,x_2}(1) = x_2$. Hence $\Gamma(x)$ is an arcwise connected set with respect to arc φ. $\square$

We now establish the arcwise connectedness and connectedness of the sets of ml-efficient, weak ml-efficient, and strict ml-efficient solutions.

Theorem 5.4. *Assume that*

(a) *S is compact and arcwise connected set with respect to arc φ;*
(b) *F is continuous on S;*
(c) *F is natural arcwise ml-quasi K-connected on S;*
(d) *$F(x)$ is compact for all $x \in S$.*

Then $ml-\mathrm{WEff}(F,K)$ is a connected set.

Moreover, if we replace the condition (c) by the following condition

(c$'$) *F is strictly natural arcwise ml-quasi K-connected on S;*

then $ml-\mathrm{Eff}(F,K)$ and $ml-\mathrm{SEff}(F,K)$ are connected sets.

Proof. We divide the proof into the following two steps:
Step 1: $\Gamma(x)$ is nonempty.

Let $x \in S$. Using Theorem 3.1 it follows that $\Theta_k(\cdot,x)$ is continuous on S. Hence there exists $w \in S$ such that $\Theta_k(w,x) = \min_{u \in S} \Theta_k(u,x)$, which implies that $\Gamma(x) \neq \emptyset$.
Step 2: Γ is upper semi-continuous on S.

Assume on the contrary that Γ is not upper semi-continuous at $x_0 \in S$. Then there exist an open set U containing $\Gamma(x_0)$, a sequence $\{x_n\}$ in S converging to x_0 and $w_n \in \Gamma(x_n)$ such that $w_n \notin U$ for all n. Since S is compact so without loss of generality we assume that $w_n \to w_0$ for some $w_0 \in S$. Since $w_n \in \Gamma(x_n)$ we have $\Theta_k(w_n,x_n) \leq \Theta_k(u,x_n)$ for all $u \in S$. By Theorem 3.1, it follows that $\Theta_k(w_0,x_0) \leq \Theta_k(u,x_0)$ for all $u \in S$, that is, $w_0 \in \Gamma(x_0)$. As $\Gamma(x_0) \subseteq U$ we have $w_0 \in U$. Thus we arrive at a contradiction as $w_n \to w_0$ but $w_n \notin U$ for all n. From Theorem 5.3 we have $\Gamma(x)$ is arcwise connected for each $x \in S$ and hence connected set. Further, using

Lemma 3.2 (ii) and Step 2, we obtain that $\Gamma(S)$ is an connected set. In view of Theorem 5.1, we have $ml-\mathrm{WEff}(F,K)=\Gamma(S)$ which implies that $ml-\mathrm{WEff}(F,K)$ is a connected set. Using Proposition 5.1, we obtain that $ml-\mathrm{Eff}(F,K)$ and $ml-\mathrm{SEff}(F,K)$ are connected sets. □

Remark 5.1. Han [14] studied the connectedness of the approximate solution sets in set optimization problems for lower set order relations. But our results involve different set order relations and the convexity of the constraint set S is weaken by arcwise connectedness. Also, we have defined and used new notions of natural arcwise ml-quasi K-connected set-valued maps. However, our results are not comparable to [14].

5.2. **Connectedness and arcwise connectedness of solution sets using nearly convexlike set-valued maps.** In this subsection, we discuss the connectedness and arcwise connectedness of ml-efficient, strict ml-efficient and weak ml-efficient solution sets of the problem (P) using nearly K-convexlike set-valued map.

We show that the set of ml-efficient, strict ml-efficient and weak ml-efficient solutions of the problem (P) coincide if F is strictly nearly ml-K-convexlike on S.

Proposition 5.2. *Let F be strictly nearly ml-K-convexlike on S with nonempty compact values. Then*

$$ml-\mathrm{Eff}(F,K)=ml-\mathrm{WEff}(F,K)=ml-\mathrm{SEff}(F,K).$$

Proof. From Proposition 5.1, one can obtain desired conclusion immediately. □

We next establish the connectedness and arcwise connectedness of the solutions sets of the problem (P).

Theorem 5.5. *Assume that*

(a) *S is a compact and connected (respectively, arcwise connected) set;*
(b) *F is continuous on S;*
(c) *F is strictly nearly ml-K-convexlike on S;*
(d) *$F(x)$ is compact for all $x\in S$.*

Then $ml-\mathrm{WEff}(F,K)$ is a connected (respectively, arcwise connected) set. Moreover, $ml-\mathrm{Eff}(F,K)$ and $ml-\mathrm{SEff}(F,K)$ are connected (respectively, arcwise connected) sets.

Proof. We divide the proof into the following two steps:
Step 1: $\Gamma(x)$ is a singleton set for all $x\in S$.

Assume to the contrary that there exist $z_1,z_2\in\Gamma(x)$ such that $z_1\neq z_2$. By Theorem 4.1 (ii) it follows that $\Theta_k(\cdot,x)$ is strictly nearly $\mathbb{R}_+$-convexlike on S. Hence there exists $z_3\in S$ such that

$$\Theta_k(z_3,x)<\frac{1}{2}\Theta_k(z_1,x)+\frac{1}{2}\Theta_k(z_2,x)=\Theta_k(z_1,x)\leq\Theta_k(z,x),\ \forall z\in S.$$

Hence $z_3\in\Gamma(x)$ which contradicts the fact that $\Theta_k(\cdot,x)$ achieves minimum at z_1.
Step 2: Γ is continuous on S.

Assume on the contrary that Γ is not continuous at $x_0\in S$. Then there exist an open set U containing $\Gamma(x_0)$, a sequence $\{x_n\}$ in S converging to x_0 and $w_n\in\Gamma(x_n)$ such that $w_n\notin U$ for all n. Since S is compact so without loss of generality we assume that $w_n\to w_0$ for some $w_0\in S$. Since $w_n\in\Gamma(x_n)$ we have $\Theta_k(w_n,x_n)\leq\Theta_k(u,x_n)$ for all $u\in S$. By Theorem 3.1, it follows that $\Theta_k(w_0,x_0)\leq\Theta_k(u,x_0)$ for all $u\in S$, that is, $w_0\in\Gamma(x_0)$. As $\Gamma(x_0)$ is singleton and $\Gamma(x_0)\subseteq U$ we have $w_0\in U$. Thus we arrive at a contradiction as $w_n\to w_0$ but $w_n\notin U$ for all n. By the hypothesis, S is a connected (respectively, arcwise connected) set. By the first step of Theorem 5.4, it follows that $\Gamma(x)$ is nonempty for each $x \in S$. Since $\Gamma(x)$ is a singleton set for all $x \in S$ and the map Γ is continuous on S, it is connected (respectively, arcwise connected) for each $x \in S$. From Theorem 5.1, we have $ml - \mathrm{WEff}(F,K) = \Gamma(S)$. Hence, by Lemma 3.3(i) (respectively, Lemma 3.3(ii)), it follows that $ml - \mathrm{WEff}(F,K)$ is a connected (respectively, arcwise connected) set. It follows from Proposition 5.2 that $ml - \mathrm{Eff}(F,K)$ and $ml - \mathrm{SEff}(F,K)$ are also connected (respectively, arcwise connected) sets. □

Remark 5.2. Anh et al. [18] studied the connectedness and arcwise connectedness of the efficient solution sets in vector optimization problem. Recently, Sharma and Lalitha [20] studied the connectedness of the efficient and weak efficient solution sets in generalized semi infinite set optimization. From Theorem 5.5, we can observe that we do not need any convexity assumption on the constraint set S to establish the connectedness and arcwise connectedness of the solution sets of set optimization problems. Therefore, Theorem 5.5 extends [18, Theorem 4.1] from vector-valued maps to set-valued maps. Also, we use different the set order relations, used in [20].

6. Convergence of Solution Sets via Nonlinear Scalarization

This section deals with the convergence of strict efficient and weak efficient solution sets by means of scalarizations. In particular, we derive the lower and upper Painlevé–Kuratowski set convergence of the sequence of efficient and weak efficient solution sets, respectively of parametric scalarized optimization problems. The main tools used are scalarization results and the continuity of the map associated with parametric scalar problem.

We recall the notion of Painlevé–Kuratowski convergence for a sequence of sets in X from [1]. Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence in X. Consider

$$\mathrm{Ls}(\Omega_n) := \{x \in X \ : \ \exists\, x_{n_k} \to x \text{ with } x_{n_k} \in \Omega_{n_k}\},$$

$$\mathrm{Li}(\Omega_n) := \{x \in X \ : \ \exists\, x_n \to x \text{ with } x_n \in \Omega_n, \text{ for } n \text{ sufficiently large}\}.$$

The set $\mathrm{Ls}(\Omega_n)$ (respectively, $\mathrm{Li}(\Omega_n)$) is called upper (respectively, lower) limit of the sequence $\{\Omega_n\}_{n\in\mathbb{N}}$. We say that the sequence $\{\Omega_n\}_{n\in\mathbb{N}} \subseteq X$ converges to the set $\Omega \subseteq X$ in the *Painlevé–Kuratowski* sense, denoted by $\Omega_n \xrightarrow{\mathrm{K}} \Omega$ if $\mathrm{Ls}(\Omega_n) \subseteq \Omega \subseteq \mathrm{Li}(\Omega_n)$.

The inclusion $\mathrm{Ls}(\Omega_n) \subseteq \Omega$ (respectively, $\Omega \subseteq \mathrm{Li}(\Omega_n)$) is referred to as *upper* (respectively, *lower*) part of *Painlevé–Kuratowski convergence*, denoted by $\Omega_n \xrightharpoonup{\mathrm{K}} \Omega$ (resp. $\Omega_n \xrightharpoondown{\mathrm{K}} \Omega$). Clearly, $\mathrm{Li}(\Omega_n) \subseteq \mathrm{Ls}(\Omega_n)$.

From [25], we recall that a sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ of subsets of X upper converges to a set $\Omega \subseteq X$ in the Hausdorff sense if $\rho(\Omega_n,\Omega) \to 0$, where $\rho(\Omega_n,\Omega) := \sup_{x\in\Omega_n} d(x,\Omega)$. We denote this convergence by $\Omega_n \xrightharpoonup{\mathrm{H}} \Omega$. If Ω is a closed set and $\Omega_n \xrightharpoonup{\mathrm{H}} \Omega$, then $\Omega_n \xrightharpoonup{\mathrm{K}} \Omega$, see [17, Corollary 2.1].

We now recall the following lemma, which will be used in the sequel.

Lemma 6.1. [15, Lemma 3.3] *Let $\Omega_n \xrightharpoonup{\mathrm{H}} \Omega$, where Ω is a nonempty compact set in X and $\{\Omega_n\}_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of X. Then, for any sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in \Omega_n$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \to x$ and $x \in \Omega$.*

We consider the following family of parametric scalar optimization problems by perturbing the feasible set of the scalar problem (P(x)). We consider the problem (P$_n(x)$) for each $n \in \mathbb{N}$, where $S_n \subseteq X$ is a nonempty set and $x \in S_n$, as follows

$$\begin{aligned} &\text{Minimize } I_k^{ml}(F(u), F(x)) \\ &\text{subject to } u \in S_n. \end{aligned} \tag{P$_n(x)$}$$

We denote the set of minimal solutions of (P$_n(x)$) by $\Gamma_n(x)$.

We now establish the lower part of convergence of the set of minimal solutions of the perturbed problem (P$_n(x)$) to the set of minimal solutions of (P(x)).

Theorem 6.1. *Assume that*

(a) $S_n \xrightarrow{\mathrm{K}} S$,
(b) *F is continuous and compact-valued on X,*
(c) $S_n \xrightarrow{\mathrm{H}} S$ *and S is compact.*

Then $\Gamma_n(S_n) \xrightarrow{\mathrm{K}} \Upsilon(S)$.

Proof. Let $w \in \Upsilon(S)$ which implies that $w \in \Upsilon(x)$ for some $x \in S$, that is,

$$I_k^{ml}(F(w), F(x)) < I_k^{ml}(F(u), F(x)), \tag{6.1}$$

for all $u \in S$. Since $S_n \xrightarrow{\mathrm{K}} S$, then there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ in S_n such that $w_n \to w$ and $x_n \to x$. We claim that for sufficiently large n, $I_k^{ml}(F(w_n), F(x_n)) \leq I_k^{ml}(F(u), F(x_n))$ for all $u \in S_n$. Assume to the contrary that there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and $\hat{w}_{n_k} \in S_{n_k}$ such that

$$I_k^{ml}(F(\hat{w}_{n_k}), F(x_{n_k})) < I_k^{ml}(F(w_{n_k}), F(x_{n_k})). \tag{6.2}$$

Since $S_n \xrightarrow{\mathrm{H}} S$ and S is compact, therefore by Lemma 6.1 there exists a subsequence $\{\hat{w}_{n_{k_l}}\}_{l\in\mathbb{N}}$ of $\{\hat{w}_{n_k}\}_{k\in\mathbb{N}}$ and $\hat{w} \in S$ such that $\hat{w}_{n_{k_l}} \to \hat{w}$. Taking limit along the subsequences in (6.2) and using Theorem 3.1 we have $I_k^{ml}(F(\hat{w}), F(x)) \leq I_k^{ml}(F(w), F(x))$, which contradicts (6.1). □

We next prove the upper part of convergence of the set of minimal solutions of the perturbed problem (P$_n(x)$) to the set of minimal solutions of (P(x)).

Theorem 6.2. *Assume that the conditions* (a)-(c) *of Theorem 6.1 hold. Then*

$$\Gamma_n(S_n) \xrightarrow{\mathrm{K}} \Gamma(S).$$

Proof. Let $w \in \mathrm{Ls}(\Gamma_n(S_n))$. Then, there exist a subsequence $\{w_{n_k}\}_{k\in\mathbb{N}}$ with $w_{n_k} \in \Gamma_{n_k}(S_{n_k})$ such that $w_{n_k} \to w$. Let $w_{n_k} \in \Gamma_{n_k}(x_{n_k})$ for some $x_{n_k} \in S_{n_k}$. Since $x_{n_k} \in S_{n_k}$ and $S_{n_k} \xrightarrow{\mathrm{H}} S$, it follows by Lemma 6.1 that there exists a subsequence $\{x_{n_{k_l}}\}_{l\in\mathbb{N}}$ of $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $x \in S$ such that $x_{n_{k_l}} \to x$. It is sufficient to show that $w \in \Gamma(x)$. Let $\hat{x} \in S$. Since $S_n \xrightarrow{\mathrm{K}} S$, there exists a sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ with $\hat{x}_n \in S_n$ such that $\hat{x}_n \to \hat{x}$. As $w_{n_{k_l}} \in \Gamma_{n_{k_l}}(x_{n_{k_l}})$, we have

$$I_k^{ml}(F(w_{n_{k_l}}), F(x_{n_{k_l}})) \leq I_k^{ml}(F(\hat{x}_{n_{k_l}}), F(x_{n_{k_l}})).$$

By Theorem 3.1 it follows that $I_k^{ml}(F(w), F(x)) \leq I_k^{ml}(F(\hat{x}), F(x))$ and hence, $w \in \Gamma(x)$. □

By using Theorem 5.2, Theorem 6.1, Theorem 6.2, Proposition 5.1, and Proposition 5.2, we obtain the convergence to sets of weak efficient solutions and strict efficient solutions.

Corollary 6.1. *Assume that the conditions* (a)-(c) *of Theorem* 6.1 *hold and F is strictly nearly ml-K-convexlike (or, strictly natural arcwise ml-quasi K-connected) on X. Then*

(a) $\Gamma_n(S_n) \overset{\mathrm{K}}{\to} ml - \mathrm{WEff}(F,K)$.
(b) $\Gamma_n(S_n) \overset{\mathrm{K}}{\to} ml - \mathrm{SEff}(F,K)$.

Remark 6.1. Khushboo et al. [12] and Ansari et al. [6] studied similar results for solution sets in set optimization problems with different set order relations. However, our results are not comparable to them.

7. Application to Game Theory with Uncertainty

This section deals with an application of the derived results to uncertain game theory with vector-valued objective map. In particular, we establish the arcwise connectedness and connectedness of *ml*-efficient, weak *ml*-efficient and strict *ml*-efficient solutions for uncertain vector-valued games with uncertain parameter.

Consider the game $\mathscr{G} := (N, \{\Omega_i\}, \{\xi_i\}, \mathscr{U})_{i\in N}$, where $N := \{1,2,\cdots,n\}$ is a finite player set, Ω_i (a nonempty subset of real normed space X) is the set of strategies of the ith player for each $i \in N$ and setting $\Omega := \Pi_{i\in N}\Omega_i$, $\xi_i : \Omega \times \mathscr{U} \to Y$ is the playoff function of the ith player for each $i \in N$ and the uncertainty set $\mathscr{U}$. Set $\Omega_{-i} := \Pi_{j\in N\setminus\{i\}}\Omega_j$. For each $i \in N$, we define

$$\omega_{-i} := \{\omega_1,\cdots,\omega_{i-1},\omega_{i+1},\cdots,\omega_n\} \in \Omega_{-i}, \quad \forall\, \omega = (\omega_1,\cdots,\omega_n) \in \Omega.$$

For each $i \in N$, $\bar{\omega}_i \in \Omega_i$, we define

$$(\bar{\omega}_i, \omega_{-i}) := \{\omega_1,\cdots,\omega_{i-1},\bar{\omega}_i,\omega_{i+1},\cdots,\omega_n\} \in \Omega.$$

The image of the uncertainty set $\mathscr{U}$ and all $\omega \in \Omega$ under ξ_i is the set

$$\xi_i(\omega,\mathscr{U}) := \{\xi_i(\omega,u) : u \in \mathscr{U}\}$$

which represent all possible realizations of the vector-valued loss function when decision w_i is assumed. These sets will be compared by using set order relations $\preceq_K^{ml}$ and $\prec_K^{ml}$ in order to obtain the notions of robust Nash equilibria.

Motivated by the notions of robust Nash equilibria from [26, Definition 3.1], we define the following notions of robust Nash equilibria for vector-valued games with uncertain parameter.

Definition 7.1. An element $\hat{\omega} = (\hat{\omega}_1,\hat{\omega}_2,\cdots,\hat{\omega}_n) \in \Omega$ is said to be

(a) a robust Nash equilibrium for the game $\mathscr{G}$ if and only if for any $i \in N$, there is no $\omega_i \in \Omega_i$ such that $f_i(\omega_i,\hat{\omega}_{-i},\mathscr{U}) \preceq_K^{ml} f_i(\hat{\omega},\mathscr{U})$ and $f_i(\omega_i,\hat{\omega}_{-i},\mathscr{U}) \neq f_i(\hat{\omega},\mathscr{U})$, that is, either $f_i(\omega_i,\hat{\omega}_{-i},\mathscr{U}) \npreceq_K^{ml} f_i(\hat{\omega},\mathscr{U})$ or $f_i(\omega_i,\hat{\omega}_{-i},\mathscr{U}) = f_i(\hat{\omega},\mathscr{U})$, for any $\omega_i \in \Omega_i$;
(b) a weak robust Nash equilibrium for the game $\mathscr{G}$ if and only if for any $i \in N$, there is no $\omega_i \in \Omega_i$ such that $\xi_i(\omega_i,\hat{\omega}_{-i},\mathscr{U}) \prec_K^{ml} \xi_i(\hat{\omega},\mathscr{U})$.
(c) a strict robust Nash equilibrium for the game $\mathscr{G}$ if and only if for any $i \in N$, there is no $\omega_i \in \Omega_i$ such that $\xi_i(\omega_i,\hat{\omega}_{-i},\mathscr{U}) \preceq_K^{ml} \xi_i(\hat{\omega},\mathscr{U})$.

We denote the set of robust Nash equilibrium, weak robust Nash equilibrium and strict robust Nash equilibrium for the game $\mathscr{G}$ by $ml - \mathrm{REff}(F,K)$, $ml - \mathrm{RWEff}(F,K)$ and $ml - \mathrm{RSEff}(F,K)$, respectively.

An element $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_n) \in \Omega$ is a robust Nash or weak robust Nash equilibrium if and only if $\hat{\omega}_i \in \Omega_i$ is a efficient or a weak efficient solution, respectively, of the following set optimization problem

$$\begin{aligned} &\text{Minimize } \xi_i(\omega_i, \hat{\omega}_{-i}, \mathscr{U}) \\ &\text{subject to } \omega_i \in \Omega_i, \end{aligned} \tag{P_i}$$

for each $i \in N$.

We now establish the arcwise connectedness of the vector-valued game $\mathscr{G}$ with uncertainty using strictly natural arcwise *ml*-quasi K-connected set-valued maps.

Theorem 7.1. *Assume that*

(a) *for each $i \in N$, Ω_i is nonempty, compact and arcwise connected subset of X;*
(b) $w \mapsto \xi_i(\omega, \mathscr{U})$ *is continuous on Ω;*
(c) $w \mapsto \xi_i(x, \mathscr{U})$ *is natural arcwise ml-quasi K-connected set-valued map on Ω;*
(d) $\xi_i(\omega, \mathscr{U})$ *is compact for all $\omega \in \Omega$.*

Then the set of weak robust Nash equilibrium elements ($ml - \mathrm{RWEff}(F,K)$) is a connected set.

Moreover, if we replace condition (c) by the following condition

(c$'$) $w \mapsto \xi_i(x, \mathscr{U})$ *is strictly natural arcwise ml-quasi K-connected set-valued map on Ω;*

then the set of robust Nash equilibrium elements ($ml - \mathrm{REff}(F,K)$) and the set of strict robust Nash equilibrium elements ($ml - \mathrm{RSEff}(F,K)$) are connected sets.

We now establish the connectedness and arcwise connectedness of the vector-valued game $\mathscr{G}$ with uncertainty using strictly nearly *ml*-K-convexlike maps.

Theorem 7.2. *Assume that*

(a) *for each $i \in N$, Ω_i is nonempty, compact and connected (respectively, arcwise connected) subset of X;*
(e) $\omega \mapsto \xi_i(\omega, \mathscr{U})$ *is continuous on Ω;*
(f) $\omega \mapsto \xi_i(\omega, \mathscr{U})$ *is strictly nearly ml-K-convexlike on Ω;*
(g) $\xi_i(\omega, \mathscr{U})$ *is compact for all $x \in \Omega$.*

Then the set of weak robust Nash equilibrium elements ($ml - \mathrm{RWEff}(F,K)$) is connected (respectively, arcwise connected) set. Moreover, the set of robust Nash equilibrium elements ($ml - \mathrm{REff}(F,K)$) and the set of strict robust Nash equilibrium elements ($ml - \mathrm{RSEff}(F,K)$) are connected (respectively, arcwise connected) sets.

8. Conclusions

In this paper, we investigated the continuity and convexity of the nonlinear scalarization function introduced by Karaman et al. [5]. Furthermore, we studied the topological properties, namely, connectedness, arcwise connectedness and convergence of efficient, strict efficient, and weak efficient solution sets of partially ordered set optimization problems by means of scalarization. We applied our results to vector-valued games with uncertainty. It will be interesting to establish these topological properties under weaker assumptions and for other kinds of set order relations.

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