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CONVERGENCE OF THE SOLUTION SETS FOR SET OPTIMIZATION PROBLEMS

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Abstract. In this paper, we present the stability analysis of solution sets for set optimization problems with respect to the set order relation defined by means of Minkowski difference. We introduce the concepts of weak / weak# locally Lipschitz continuity and the concepts of *ml*-quasiconnectedness and strictly *ml*-quasiconnectedness for set-valued mappings. By using these concepts, we study the Painlevé–Kuratowski convergence of the solution sets for perturbed set optimization problems. Several examples are given to illustrate our results.

Keywords. Locally lipschitz continuity; Painlevé–Kuratowski convergence; Set optimization problems; Set relations.

1. Introduction

The set-valued optimization problems, where the objective function is a set-valued map, have been extensively studied during the last decades because of their applications in different areas of science, social sciences, management, medical sciences, etc.; see, e.g., [1] and the references therein. The stability of the solution sets under certain perturbations (with respect to the feasible region and the objective set-valued mapping) is one of the most important and attractive topics in optimization. It was initiated by Attouch and Riahi [2], where the convergence of the solution sets for optimization problems were studied. The Painlevé–Kuratowski convergence of the solution sets for vector optimization problems has been studied by several authors; see, e.g., [3, 4, 5, 6, 7, 8, 9] and the references therein.

In the recent years, the stability analysis of the solution sets for set optimization problems has been studied by several authors. Gutiérrez *et al.* [10] investigated external and internal stability of the solutions of set optimization problems in the image space by using the set convergence notions. Xu and Li [11] established the semicontinuity of minimal solution mappings and weak minimal solution mappings to a parametric set optimization problem by using the converse *u*-property of objective mappings. By virtue of the level mappings, Han and Huang [12, 13] discussed the upper semicontinuity, lower semicontinuity and convexity of solution mappings of the parametric set optimization problems. Han [14] further established the Hölder

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continuity of the nonlinear scalarizing function, and applied it to obtain the Lipschitz continuity of strongly approximate solution mappings of the parametric set optimization problems. Khoshkhabar-amiranloo [15] discussed the upper semicontinuity and lower semicontinuity and compactness of the minimal solutions of parametric set optimization problems. Under appropriate assumptions of compactness and strict quasiconvexity, Karuna and Lalitha [16] established external and internal stability in terms of the Hausdorff convergence and Kuratowski–Painlevé convergence of a sequence of solution sets of perturbed set optimization problems to the solution set of the original set optimization problem both in the image space and the feasible space. Han *et al.* [17] derived the Painlevé–Kuratowski upper / lower convergence of the approximate solution sets for set optimization problems, where the objective mapping is continuous and convex. Very recently, Anh *et al.* [18] obtained the external and internal stability results of the solutions of the perturbed set optimization problems with set less order relation in the image space.

Note that the convexity of the objective mapping plays an important role in establishing the stability of the solution sets for set optimization problems; see, e.g., [12, 13, 14, 15, 16, 17, 19] and the references therein. Therefore, it is important and interesting to study the stability of the solution sets for set optimization problems without the convexity of the objective mapping. The first aim of this paper is to make an attempt in this direction.

Recently, Karaman *et al.* [20] introduced set order relations on the family of sets based on Minkowski difference. In comparison to Kuroiwa's set order relations, these set order relations are partial ordered on the family of bounded sets, and hence provide a new approach to study the set optimization problems. Recently, the set optimization problems were investigated and studied in [20, 21, 22] by using Karaman's set order relations. Khushboo and Lalitha [22] studied the relationship among different kinds of solution sets of set optimization problems defined by means of Kuroiwa's set order relations or Karaman's set order relations. They also investigated that the solution sets of a set optimization problem defined by different kinds of set order relations are different. Therefore, it is interesting and important to investigate the set optimization problems by using the Karaman's set order relations.

Motivated by [23], we first introduce the concepts of *ml*-quasiconnectedness and strictly *ml*-quasiconnectedness for set-valued mappings, and then establish the Painlevé–Kuratowski convergence of the solution sets for set optimization problems.

Rest of the paper is organized as follows. In Section 2, we first present some necessary notations, concepts, and results which will be used throughout the paper. Then we introduce the concept of weak locally Lipschitz continuity, weak$^{\#}$ locally Lipschitz continuity *ml*-quasiconnectedness, and strictly *ml*-quasiconnectedness for set-valued mappings. In Section 3, by using set order relations given in [20], we introduce the concept of convergence for the sequence of set-valued mappings and obtain convergence results, which will be used in discussing the Painlevé–Kuratowski convergence of the solution sets for set optimization problems. In Section 4, we establish the Painlevé–Kuratowski convergence of the sets of *ml*-minimal solutions and weak *ml*-minimal solutions for set optimization problems with respect to the perturbations of the feasible set and the objective mapping by using *ml*-quasiconnectedness and strictly *ml*-quasiconnectedness. The results of this paper are different and even not comparable with the corresponding results in [23].

2. PRELIMINARIES

Throughout the paper, unless otherwise specified, we adopt the following notations and terminology.

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be real normed vector spaces. We denote an open ball centered at y and radius $r > 0$ in Y by $\mathbb{B}_Y(y,r)$, the closed unit ball centered at origin in X (respectively, Y) by $\mathbb{B}_X$ (respectively, $\mathbb{B}_Y$). Let $\mathscr{P}(Y)$ (respectively, $\mathscr{B}(Y)$) denote the family of all nonempty subsets (respectively, nonempty, proper bounded subsets) of Y. For a subset A of Y, we denoted by int$A$, cl$A$, and ∂A the topological interior, the topological closure and the topological boundary of the set A, respectively. A set A in Y is said to be a cone if for all $x \in A$ and all $\lambda \geq 0$, we have $\lambda x \in A$. Throughout the paper, $\mathbf{0}$ represents the zero vector of the corresponding vector space. It can be easily seen that if K is a compact subset of Y, then, for any $\hat{k} \in Y$, $\hat{k} + K$ is also compact.

Let $A, B \in \mathscr{P}(Y)$.

(a) The algebraic sum (respectively, difference) of A and B is defined as

$$A + B := \{a + b : a \in A, b \in B\} \text{ (respectively, } A - B := \{a - b : a \in A, b \in B\});$$

(b) The Minkowski or Pontryagin difference of A and B is defined as

$$A \dot{-} B := \{y \in Y : y + B \subseteq A\} = \bigcap_{b \in B} (A - b);$$

(c) For an arbitrary $\lambda \in \mathbb{R}$, $\lambda A := \{\lambda a : a \in A\}$.

Note that $A \dot{-} a = A - a$ for all $A \in \mathscr{P}(Y)$ and $a \in Y$. For further detail on Minkowski difference, we refer to [24].

Lemma 2.1. [23] *Let A and B be nonempty subsets of Y. If $0 < \lambda < \delta$, B is convex, and $A + \delta\mathbb{B}_Y \subseteq B + \lambda\mathbb{B}_Y$, then $A \subseteq \text{int}B$.*

Definition 2.1. [25, 26] Let $A, B \in \mathscr{P}(Y)$ and C be a convex cone. The lower set less relation, denoted by $\preceq_C^l$, is defined as

$$A \preceq_C^l B :\Leftrightarrow B \subseteq A + C. \tag{2.1}$$

In addition, if int$C \neq \emptyset$, then the weak lower set less relation, denoted by $\prec_C^l$, is defined by $A \prec_C^l B :\Leftrightarrow B \subseteq A + \text{int}C$.

Note that the set order relation $\preceq_C^l$ is pre-order on $\mathscr{P}(Y)$, that is, reflexive and transitive on $\mathscr{P}(Y)$.

Recently, Karaman *et al.* [20] introduced the following order relations on the family of sets by using the Minkowski difference.

Definition 2.2. [20] Let $A, B, K \in \mathscr{P}(Y)$. The m-lower set less relation, denoted by $\preceq_K^{ml}$, is defined as

$$A \preceq_K^{ml} B :\Leftrightarrow (A \dot{-} B) \cap (-K) \neq \emptyset. \tag{2.2}$$

In addition, if int$K \neq \emptyset$, then the weak m-lower set less relation, denoted by $\prec_K^{ml}$, is defined by $A \prec_K^{ml} B :\Leftrightarrow (A \dot{-} B) \cap (-\text{int}K) \neq \emptyset$.

Karaman *et al.* [20] studied several properties of order relation $\preceq_K^{ml}$. Some of them are gathered in the following result.

Proposition 2.1. [20] *Let $K \in \mathscr{P}(Y)$. The following assertions hold.*

(a) *The relation $\preceq_K^{ml}$ is compatible with addition.*
(b) *The relation $\preceq_K^{ml}$ is compatible with scalar multiplication if and only if K is a cone.*
(c) *The relation $\preceq_K^{ml}$ is reflexive if and only if $\mathbf{0} \in K$.*
(d) *The relations $\preceq_K^{ml}$ is transitive if and only if K is a convex cone.*

It is shown in [20, Corollary 3] that if K is a convex cone, then $\preceq_K^{ml}$ is a pre order relation on $\mathscr{P}(Y)$. Mereover, if K is pointed, then $\preceq_K^{ml}$ is a partial order relation on $\mathscr{B}(Y)$; see, [20, Corollary 4].

Proposition 2.2. [20] *Let $A, B \in \mathscr{P}(Y)$. Then $A \preceq_K^{ml} B$ implies $A \preceq_K^{l} B$.*

Lemma 2.2. [22] *If K is a closed convex pointed cone with nonempty interior, then for $A, B \in \mathscr{P}(Y)$, $A \prec_K^{ml} B \Rightarrow A \prec_K^{l} B$.*

Definition 2.3. A subset M of Y is said to be free-disposal with respect to the set $E \subseteq Y$ if $M + E = M$.

Example 2.1. Let $Y = \mathbb{R}^2$, $M := \{(x,y) \in Y : 1 \leq x < \infty, 1 \leq y < \infty\}$, and $E := \{(x,y) \in Y : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. It is easy to see that $M + E = M$, that is, M is free-disposal with respect to E.

Bao *et al.* [27] considered E to be a cone in Y to define the free-disposal set. By using the definition of the free-disposal set, we can easily prove the following lemma.

Lemma 2.3. *If K is a convex cone with nonempty interior, then, for any $A, B \in \mathscr{P}(Y)$ such that A and B are free-disposal with respect to K, then $A \prec_K^{l} B \Rightarrow A \preceq_K^{ml} B$.*

Recall that a subset A of a topological space is said to be arcwise connected [28] if, for every pair of points $x, y \in A$, there exists a continuous function $\phi : [0,1] \to A$ such that $\phi(0) = x$ and $\phi(1) = y$.

Let $F : X \rightrightarrows Y$ be a set-valued mapping, and let $S \subseteq \mathrm{Dom}(F) := \{x \in X : F(x) \neq \emptyset\}$ be a nonempty subset of X. Consider the following set optimization problem:

$$\begin{aligned} &\text{Minimize } F(x), \\ &\text{subject to } x \in S. \end{aligned} \tag{SOP}$$

We recall the notion of the optimal solutions of the problem (SOP) with respect to the set order relation $\preceq_C^{l}$.

Definition 2.4. [25] An element $\hat{x} \in S$ is called

(a) an l-minimal solution of the problem (SOP) if
$$F(x) \preceq_C^{l} F(\hat{x}) \text{ for some } x \in S \quad \Rightarrow \quad F(\hat{x}) \preceq_C^{l} F(x);$$
(b) a weak l-minimal solution of (SOP) if there does not exist any $x \in S$ such that $F(x) \prec_C^{l} F(\hat{x})$.

The set of all l-minimal solutions and the set all weak l-minimal solutions of the problem (SOP) are denoted by $E_{l-F}(S)$ and $W_{l-F}(S)$, respectively.

We now recall the notion of the optimal solutions of the problem (SOP) with respect to the relation $\preceq_K^{ml}$.

From now onwards, we assume that $K := C$ is a closed, convex, and pointed cone with $\mathrm{int} C \neq \emptyset$. For defining the solution concepts, we consider that $F(x) \in \mathscr{B}(Y)$, for all $x \in S$.

Definition 2.5. [20] An element $\hat{x} \in S$ is said to be

(a) an ml-minimal solution of problem (SOP) if there does not exist any $x \in S$ such that $F(x) \preceq_K^{ml} F(\hat{x})$ and $F(x) \neq F(\hat{x})$, i.e., either $F(x) \not\preceq_K^{ml} F(\hat{x})$ or $F(x) = F(\hat{x})$, for any $x \in S$;

(b) a weak ml-minimal solution of (SOP) if there does not exist any $x \in S$ such that $F(x) \prec_K^{ml} F(\hat{x})$.

The set of ml-minimal (respectively, weak ml-minimal) solutions of problem (SOP) is denoted by $E_{ml-F}(S)$ (respectively, $W_{ml-F}(S)$). It is clear that $E_{ml-F}(S) \subseteq W_{ml-F}(S)$. However, Khushboo and Lalitha [22] showed that the reverse inclusion may not hold.

Remark 2.1. It was shown in [22] that there is no relation between $E_{l-F}(S)$ and $E_{ml-F}(S)$. However, if $F(x)$ is free-disposal with respect to K for all $x \in S$, then $E_{ml-F}(S) = E_{l-F}(S)$.

Indeed, let $\hat{x} \in E_{ml-F}(S)$. If $F(x) \preceq_K^{l} F(\hat{x})$ for some $x \in S$, i.e., $F(\hat{x}) \subseteq F(x) + K = F(x)$. Then $\mathbf{0} + F(\hat{x}) \subseteq F(x)$, which implies that $F(x) \preceq_K^{ml} F(\hat{x}) \Rightarrow F(\hat{x}) \preceq_K^{ml} F(x)$ for some $x \in S$. Now there exists $\hat{k} \in K$ such that $-\hat{k} + F(x) \subseteq F(\hat{x}) \Rightarrow F(x) \subseteq F(\hat{x}) + \hat{k} \subseteq F(\hat{x}) + K \Rightarrow F(\hat{x}) \preceq_K^{l} F(x)$ for some $x \in S$. Therefore $\hat{x} \in E_{l-F}(S)$. Conversely, let $\hat{x} \in E_{l-F}(S)$. If $F(x) \preceq_K^{ml} F(\hat{x})$ for some $x \in S$, i.e., there exists $\tilde{k} \in K$ such that $-\tilde{k} + F(\hat{x}) \subseteq F(x) \Rightarrow F(\hat{x}) \subseteq F(x) + \tilde{k} \subseteq F(x) + K \Rightarrow F(x) \preceq_K^{l} F(\hat{x}) \Rightarrow F(\hat{x}) \preceq_K^{l} F(x) \Rightarrow F(x) \subseteq F(\hat{x}) + K = F(\hat{x})$. Then, $\mathbf{0} + F(x) \subseteq F(\hat{x})$, this implies that $F(\hat{x}) \preceq_K^{ml} F(x)$. Therefore, $\hat{x} \in E_{ml-F}(S)$.

By using Lemma 2.2, we have the following result.

Lemma 2.4. [22] *If K is a closed convex pointed cone with nonempty interior, then*

$$W_{l-F}(S) \subseteq W_{ml-F}(S).$$

We recall the concept of the Painlevé-Kuratowski set convergence (see, e.g., [29]). Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X. Define

$$\mathrm{Ls}A_n := \left\{ x \in X \,:\, x = \lim_{k\to+\infty} x_{n_k}, x_{n_k} \in A_{n_k}, \{x_{n_k}\}_{n_k\in\mathbb{N}} \text{ is a subsequence of } \{x_n\}_{n\in\mathbb{N}} \right\},$$

$$\mathrm{Li}A_n := \left\{ x \in X \,:\, x = \lim_{k\to+\infty} x_n, x_n \in A_n \text{ for sufficiently large } n \right\}.$$

The set $\mathrm{Ls}A_n$ (respectively, $\mathrm{Li}A_n$) is called upper (respectively, lower) limit of the sequence $\{A_n\}_{n\in\mathbb{N}}$. We say that the sequence $\{A_n\}_{n\in\mathbb{N}}$ converges in the sense of Painlevé-Kuratowski to a set A if

$$\mathrm{Ls}A_n \subseteq A \subseteq \mathrm{Li}A_n. \tag{2.3}$$

We denote the Painlevé-Kuratowski convergence by $A_n \xrightarrow{PK} A$. It can be easily seen that $\mathrm{Ls}A_n \subseteq \mathrm{Ls}B_n$ if $A_n \subseteq B_n$ and all $n \in \mathbb{N}$.

Definition 2.6. [30] Let X and Y be Hausdorff topological spaces. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(a) upper semicontinuous (u.s.c.) at $\hat{u} \in X$ if for any open set V in Y with $F(\hat{u}) \subseteq V$, there exists a neighbourhood $U(\hat{u})$ of $\hat{u}$ such that

$$F(u) \subseteq V, \quad \forall u \in U(\hat{u});$$

(b) lower semicontinuous (l.s.c.) at $\hat{u} \in X$ if for any open set V in Y with $F(\hat{u}) \cap V \neq \emptyset$, there exists a neighbourhood $U(\hat{u})$ of $\hat{u}$ such that

$$F(u) \cap V \neq \emptyset, \quad \forall u \in U(\hat{u}).$$

We say that F is u.s.c. (respectively, l.s.c.) on X if it is u.s.c. (respectively, l.s.c.) at each point of X. The set-valued mapping F is said to be continuous on X if it is u.s.c. as well as l.s.c. on X.

Definition 2.7. [31] Let S be a nonempty subset of X. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(a) locally Lipschitz continuous at $\hat{u} \in S$ if there exist a constant $L > 0$ and a neighbourhood $U(\hat{u})$ of $\hat{u}$ in X such that

$$F(u_1) \subseteq F(u_2) + L\|u_2 - u_1\|_X \mathbb{B}_Y, \quad \forall u_1, u_2 \in U(\hat{u}) \cap S;$$

(b) locally K-Lipschitz continuous at $\hat{u} \in S$ if there exist a constant $L > 0$ and a neighbourhood $U(\hat{u})$ of $\hat{u}$ in X such that

$$F(u_1) \subseteq F(u_2) + L\|u_2 - u_1\|_X \mathbb{B}_Y + K, \quad \forall u_1, u_2 \in U(\hat{u}) \cap S.$$

Remark 2.2. If we replace K by $-K$ in the Definition 2.7 (b), then we say that F is locally $(-K)$-Lipschitz continous at $\hat{u} \in S$. Clearly, if F is locally Lipschitz continuous at $\hat{u} \in S$, then it is locally K-Lipschitz continous at $\hat{u}$. But the converse is not true as the following example shows.

Example 2.2. [31] Let $S = X = \mathbb{R}$, $Y = \mathbb{R}^2$, and $K = \mathbb{R}^2_+ := [0, +\infty) \times [0, +\infty)$. Define a set-valued mapping $F : S \rightrightarrows Y$ by

$$F(x) = \begin{cases} \mathbb{R}^2_+, & \text{if } x = 0, \\ [0,1] \times [0,1], & \text{if } x \neq 0. \end{cases}$$

Then F is locally K-Lipschitz continuous at 0, but it is not locally Lipschitz continuous at 0.

Remark 2.3. If $F(x)$ is free-disposal with respect to K for all $x \in S$, then Definition 2.7 (a) and (b) coincide.

The following examples show that the concept of locally Lipschitz continuity is different from the concept of continuity / l.s.c. / u.s.c. of a set-valued mapping.

Example 2.3. [31] Let $X = \mathbb{R}$, $S = \{x \in \mathbb{R} : x > 0\}$, $Y = \mathbb{R}^2$, and $K = \mathbb{R}^2_+ := [0, +\infty) \times [0, +\infty)$. Define a set-valued mapping $F : S \rightrightarrows Y$ by

$$F(x) = (\ln x, \ln x) + \mathbb{R}^2_+, \quad \forall x \in S.$$

Then it is easy to see that F is continuous on S, while it is not locally K-Lipschitz continous on S, and hence not locally Lipschitz on S.

Example 2.4. Let $S = X = \mathbb{R}$ and $Y = \mathbb{R}$. Define a set-valued mapping $F : S \rightrightarrows Y$ by

$$F(x) = \begin{cases} [-1,1], & \text{if } x = 0, \\ \{0\}, & \text{if } x \neq 0. \end{cases}$$

Then one can easily see that F is locally Lipschitz at 0, but it is not l.s.c. at 0.

Example 2.5. Let $S = X = \mathbb{R}$ and $Y = \mathbb{R}$. Define a set-valued mapping $F : S \rightrightarrows Y$ by

$$F(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ [-1,1], & \text{if } x \neq 0. \end{cases}$$

Then it is easy to see that F is locally Lipschitz at 0, but it is not u.s.c. at 0.

Definition 2.8. Let S be a nonempty subset of X. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(a) weak locally Lipschitz continuous at $\hat{u} \in S$ if there exist a constant $L > 0$ and a neighbourhood $U(\hat{u})$ of $\hat{u}$ in X such that

$$F(u_1) \cap \{F(u_2) + L\|u_2 - u_1\|_X \mathbb{B}_Y\} \neq \emptyset, \quad \forall u_1, u_2 \in U(\hat{u}) \cap S;$$

(b) weak locally K-Lipschitz continuous at $\hat{u} \in S$ if there exist a constant $L > 0$ and a neighbourhood $U(\hat{u})$ of $\hat{u}$ in X such that

$$F(u_1) \cap \{F(u_2) + L\|u_2 - u_1\|_X \mathbb{B}_Y + K\} \neq \emptyset, \quad \forall u_1, u_2 \in U(\hat{u}) \cap S.$$

Clearly, every locally Lipschitz continous set-valued mapping is weak locally Lipschitz continous.

Definition 2.9. Let S be a nonempty subset of X. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(a) weak$^{\#}$ locally Lipschitz continuous at $\hat{u} \in S$ if there exist a constant $L > 0$ and a neighbourhood $U(\hat{u})$ of $\hat{u}$ in X such that

$$F(u_1) \cap \{z + L\|u_2 - u_1\|_X \mathbb{B}_Y\} \neq \emptyset, \quad \forall u_1, u_2 \in U(\hat{u}) \cap S,\ \forall z \in F(u_2);$$

(b) weak$^{\#}$ locally K-Lipschitz continuous at $\hat{u} \in S$ if there exist a constant $L > 0$ and a neighbourhood $U(\hat{u})$ of $\hat{u}$ in X such that

$$F(u_1) \cap \{z + L\|u_2 - u_1\|_X \mathbb{B}_Y + K\} \neq \emptyset, \quad \forall u_1, u_2 \in U(\hat{u}) \cap S,\ \forall z \in F(u_2).$$

It is clear that a weak$^{\#}$ locally Lipschitz continuous set-valued mapping is weak locally Lipschitz continuous.

Definition 2.10. Let S be a nonempty convex subset of X. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(a) naturally quasi ml-convex on S if, for any $x_1, x_2 \in S$ and for any $t \in [0,1]$, there exists $\lambda \in [0,1]$ such that

$$F(tx_1 + (1-t)x_2) \preceq_K^{ml} \lambda F(x_1) + (1-\lambda)F(x_2);$$

(b) strictly naturally quasi ml-convex on S if, for any $x_1, x_2 \in S$ with $x_1 \neq x_2$ and for any $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$F(tx_1 + (1-t)x_2) \prec_K^{ml} \lambda F(x_1) + (1-\lambda)F(x_2).$$

Example 2.6. Let $X = \mathbb{R}$, $S = [0,1]$, $Y = \mathbb{R}^2$, and $K := \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be a set-valued mapping defined as $F(x) = [1-x, 2-x] \times [1-x, 2-x]$ for all $x \in S$. Then F is naturally quasi ml-convex on S.

Inspired by the Definition 2.5 in [23], we introduce the concepts of ml-quasiconnectedness and strictly ml-quasiconnectedness for set-valued mappings.

Definition 2.11. Let S be an arcwise connected subset of X. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(a) ml-quasiconnected on S if for any $A \in \mathscr{P}(Y)$ and for any $u,v \in S$ with $F(u) \preceq_K^{ml} A$ and $F(v) \preceq_K^{ml} A$, there exists a continuous path $\gamma : [0,1] \to S$ with $\gamma(0) = u$ and $\gamma(1) = v$ such that

$$F\big(\gamma(t)\big) \preceq_K^{ml} A, \quad \forall t \in [0,1];$$

(b) strictly ml-quasiconnected on S if for any $A \in \mathscr{P}(Y)$ and for any $u,v \in S$ with $u \neq v$, $F(u) \preceq_K^{ml} A$ and $F(v) \preceq_K^{ml} A$, there exists a continuous path $\gamma : [0,1] \to S$ with $\gamma(0) = u$ and $\gamma(1) = v$ such that

$$F\big(\gamma(t)\big) \prec_K^{ml} A, \quad \forall t \in (0,1).$$

Example 2.7. Let $X = \mathbb{R}$, $S = [0,1]$, $Y = \mathbb{R}$, and $K = \mathbb{R}_+$. A set-valued mapping $F : X \rightrightarrows Y$ defined by $F(x) = [0, 1-x^2]$ for all $x \in S$, is ml-quasiconnected on S with $\gamma(t) := (1-t)u + tv$ for any $u,v \in S$.

Example 2.8. Let $X = \mathbb{R}^2$, $S = \{(0,0)\} \cup (0,3) \times (0,3)$, $Y = \mathbb{R}^2$, and $K := \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be a set-valued mapping defined as $F(x,y) = (x^2y^2, x^2y^2) + \mathbb{B}_Y$, for all $(x,y) \in \mathbb{R}^2$. Then F is strictly ml-quasiconnected on S with $\gamma(t) := (1-t)u + tv$ for any $u,v \in S$.

It is clear that if $F : X \rightrightarrows Y$ is naturally quasi ml-convex (respectively, strictly naturally quasi ml-convex) on S, then it is ml-quasiconnected (respectively, strictly ml-quasiconnected) by using the fact that $K + \text{int}K \subseteq \text{int}K$.

Definition 2.12. Let S be a nonempty subset of X and $F : X \rightrightarrows Y$ be a set-valued mapping. The set $Q_{ml}(x,F,S)$ defined by

$$Q_{ml}(x,F,S) = \{u \in S : F(u) \preceq_K^{ml} F(x)\},$$

is known as level set of F.

Remark 2.4. Clearly, for any $x \in S$, $E_{ml-F}\big(Q_{ml}(x,F,S)\big) \subseteq E_{ml-F}(S)$.

Lemma 2.5. [23] *Let A be a nonempty subset of Y and $g : [0,1] \to Y$ be continuous. If $g(0) \in A$ and $g(1) \notin A$, then there exists $t_0 \in [0,1]$ such that $g(t_0) \in \partial A$.*

Lemma 2.6. *Let S be a nonempty and closed subset of X. Let $F : X \rightrightarrows Y$ be locally Lipschitz continuous on S with closed values. Then $Q_{ml}(x,F,S)$ is closed.*

Proof. Let $\{y_n\}$ be a sequence in $Q_{ml}(x,F,S)$ such that $y_n \to \hat{y}$. Then $\hat{y} \in S$ as S is closed, and $F(y_n) \preceq_K^{ml} F(x)$. Therefore,

$$(F(y_n) \dot{-} F(x)) \cap (-K) \neq \emptyset, \tag{2.4}$$

that is, there exists $\hat{k} \in K$ such that

$$-\hat{k} + F(x) \subseteq F(y_n). \tag{2.5}$$

We claim that $(F(\hat{y}) \dot{-} F(x)) \cap (-K) \neq \emptyset$, that is, there exists $\tilde{k} \in K$ such that $-\tilde{k} + F(x) \subseteq F(\hat{y})$. Assume to the contrary that there exists $v_0 \in F(x)$ such that $-\tilde{k} + v_0 \notin F(\hat{y})$. Since F is closed valued, there exists $\delta > 0$ such that $((-\tilde{k} + v_0) + \delta\mathbb{B}_Y) \cap F(\hat{y}) = \emptyset$, and so

$$-\tilde{k} + v_0 \notin F(\hat{y}) + \delta\mathbb{B}_Y. \tag{2.6}$$

Since F is locally Lipschitz continuous at $\hat{y}$, there exist a constant $L > 0$ and a neighbourhood $U(\hat{y})$ of $\hat{y}$ such that

$$F(y) \subseteq F(\hat{y}) + L\|y - \hat{y}\|_X \mathbb{B}_Y, \quad \forall y \in U(\hat{y}) \cap S. \tag{2.7}$$

As $y_n \to \hat{y}$, there exists $\mathcal{N} \in \mathbb{N}$ such that $y_n \in U(\hat{y})$ for all $n \geq \mathcal{N}$. Therefore,

$$F(y_n) \subseteq F(\hat{y}) + L\|y_n - \hat{y}\|_X \mathbb{B}_Y, \quad \forall y_n \in U(\hat{y}) \cap S. \tag{2.8}$$

Since $v_0 \in F(x)$, from (2.5) and (2.8), we obtain

$$-\hat{k} + v_0 \in -\hat{k} + F(x) \subseteq F(y_n) \subseteq F(\hat{y}) + L\|y_n - \hat{y}\|_X \mathbb{B}_Y,$$

that is,

$$-\hat{k} + v_0 \in F(\hat{y}) + L\|y_n - \hat{y}\|_X \mathbb{B}_Y.$$

Since $\mathbb{B}_Y$ is a closed ball around $\mathbf{0}$ and $y_n \to \hat{y}$ as $n \to \infty$, $L\|y_n - \hat{y}\|_X \mathbb{B}_Y$ will reduce to the origin $\mathbf{0}$ as $n \to \infty$. Then, we have $-\hat{k} + v_0 \in F(\hat{y}) + \delta \mathbb{B}_Y$, which contradicts (2.6). Therefore, $-\tilde{k} + F(x) \subseteq F(\hat{y})$, and so $(F(\hat{y}) \dot{-} F(x)) \cap (-K) \neq \emptyset$. This implies that $F(\hat{y}) \preceq_K^{ml} F(x)$. This together with $\hat{y} \in S$ implies that $\hat{y} \in Q_{ml}(x, F, S)$, and hence $Q_{ml}(x, F, S)$ is closed. □

We next show that the set of all *ml*-minimal elements and the set of all weak *ml*-minimal elements are equal if F is strictly *ml*-quasiconnected on S.

Lemma 2.7. *If S is an arcwise connected subset of X and $F : X \rightrightarrows Y$ is strictly ml-quasiconnected on S with nonempty values, then $W_{ml-F}(S) = E_{ml-F}(S)$.*

Proof. Obviously, $E_{ml-F}(S) \subseteq W_{ml-F}(S)$. To prove reverse inclusion, we assume that $\hat{x} \in W_{ml-F}(S)$ but $\hat{x} \notin E_{ml-F}(S)$. Then there exists $\bar{x} \in X$ such that $F(\bar{x}) \preceq_K^{ml} F(\hat{x})$ and $F(\bar{x}) \neq F(\hat{x})$. Since $F(\hat{x}) \preceq_K^{ml} F(\hat{x})$ and F is strictly *ml*-quasiconnected, there exists a continuous path $\gamma : [0,1] \to S$ with $\gamma(0) = \bar{x}$ and $\gamma(0) = \hat{x}$ such that

$$F(\gamma(t)) \prec_K^{ml} F(\hat{x}), \quad \forall t \in (0,1),$$

which contradicts that $\hat{x} \in W_{ml-F}(S)$. Therefore, $\hat{x} \in E_{ml-F}(S)$. □

Lemma 2.8. *If S is arcwise connected subset of X, $F : X \rightrightarrows Y$ is strictly ml-quasiconnected on S with nonempty values and $\hat{x} \in E_{ml-F}(S)$, then $Q_{ml}(\hat{x}, F, S) = \{\hat{x}\}$.*

Proof. Clearly, $\hat{x} \in Q_{ml}(\hat{x}, F, S)$. Assume that there exists $\bar{x} \in Q_{ml}(\hat{x}, F, S)$ such that $\bar{x} \neq \hat{x}$. Since F is strictly *ml*-quasiconnected on S, there exists a continuous path $\gamma : [0,1] \to S$ with $\gamma(0) = \bar{x}$ and $\gamma(0) = \hat{x}$ such that

$$F(\gamma(t)) \prec_K^{ml} F(\hat{x}), \quad \forall t \in (0,1).$$

This implies that $\hat{x} \notin W_{ml-F}(S)$, which contradicts that $\hat{x} \in E_{ml-F}(S) \subseteq W_{ml-F}(S)$. □

3. Convergence of the Sequences of Set-Valued Mappings

From now on, we shall use $\mathbb{N}_0$ instead of $\mathbb{N} \cup \{0\}$.

Definition 3.1. For each $n \in \mathbb{N}_0$, let $F_n : X \rightrightarrows Y$ be a set-valued mapping. Let A be a nonempty subset of X with $A \subseteq \mathrm{Dom}(F_n)$ for all $n \in \mathbb{N}_0$. We say that

(a) $F_n \xrightarrow{Cml_1} F_0$ with respect to A if, for any neighbourhood V of $\mathbf{0} \in Y$, there exists $N \in \mathbb{N}$ such that

$$F_n(x) + V \preceq_K^{ml} F_0(x), \quad \forall x \in A \text{ and } \forall n \geq N;$$

(b) $F_n \xrightarrow{Cml_2} F_0$ with respect to A if, for any neighbourhood V of $\mathbf{0} \in Y$, there exists $N \in \mathbb{N}$ such that

$$F_0(x) + V \preceq_K^{ml} F_n(x), \quad \forall x \in A \text{ and } \forall n \geq N.$$

We say that $F_n \xrightarrow{Cml} F_0$ with respect to A if $F_n \xrightarrow{Cml_1} F_0$ with respect to A as well as $F_n \xrightarrow{Cml_2} F_0$ with respect to A.

Assumption 3.1. (a) For each $n \in \mathbb{N}_0$, let $F_n : X \rightrightarrows Y$ be a set-valued mapping.
(b) Let A be a nonempty subset of X with $A \subseteq \mathrm{Dom}(F_n)$ for all $n \in \mathbb{N}_0$.
(c) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in A that converges to $\hat{x} \in X$.
(d) Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in A that converges to $\hat{y} \in X$.
(e) Let F_0 be locally Lipschitz continuous at $\hat{x}$.

We now establish the following results which will be used to study the Painlevé–Kuratowski convergence of solution set of (SOP).

Theorem 3.1. *Assume that the Assumption 3.1 (a)-(c) and (e) holds. Further assume that $F_0(\hat{x})$ is closed and $F_n \xrightarrow{Cml_1} F_0$ with respect to A. Then, for any neighbourhood V of $\mathbf{0} \in Y$, there exist $N \in \mathbb{N}$ such that*

$$F_n(x_n) + V \preceq_K^{ml} F_0(\hat{x}), \quad \forall n \geq N.$$

Proof. It is well-known that, for any neighbourhood V of $\mathbf{0} \in Y$, there exists a neighbourhood $\mathscr{N}$ of $\mathbf{0} \in Y$ such that

$$\mathscr{N} + \mathscr{N} \subseteq V. \tag{3.1}$$

Since F_0 is locally Lipschitz at $\hat{x}$, there exist a constant $L > 0$ and a neighbourhood $U(\hat{x})$ of $\hat{x}$ such that

$$F_0(\hat{x}) \subseteq F_0(x) + L\|x - \hat{x}\|_X \mathbb{B}_Y, \quad \forall x \in U(\hat{x}) \cap S. \tag{3.2}$$

As $x_n \to \hat{x}$, there exists $N_1 \in \mathbb{N}$ such that $x_n \in U(\hat{x})$ for all $n \geq N_1$. This together with (3.2) implies that

$$F_0(\hat{x}) \subseteq F_0(x_n) + L\|x_n - \hat{x}\|_X \mathbb{B}_Y, \quad \forall n \geq N_1. \tag{3.3}$$

Also, since $F_n \xrightarrow{Cml_1} F_0$ with respect to A, there exists $N_2 \in \mathbb{N}$ such that

$$F_n(x) + \mathscr{N} \preceq_K^{ml} F_0(x), \quad \forall n \geq N_2,$$

which implies that

$$\big((F_n(x) + \mathscr{N}) \dot{-} F_0(x)\big) \cap (-K) \neq \emptyset, \quad \forall n \geq N_2.$$

Therefore, there exists $k_1 \in K$ such that

$$-k_1 + F_0(x) \subseteq F_n(x) + \mathscr{N}, \quad \forall n \geq N_2. \tag{3.4}$$

Let $N = \max\{N_1, N_2\}$. Then we conclude from $x_n \in A$ and (3.4) that

$$-k_1 + F_0(x_n) \subseteq F_n(x_n) + \mathscr{N}, \quad \forall n \geq N. \tag{3.5}$$

From (3.1), (3.3) and (3.5), we obtain

$$-k_1 + F_0(\hat{x}) \subseteq F_n(x_n) + L\|x_n - \hat{x}\|_X \mathbb{B}_Y + \mathscr{N}, \quad \forall n \geq N. \tag{3.6}$$

Since $\mathbb{B}_Y$ is a closed ball around $\mathbf{0}$ and $x_n \to \hat{x}$ as $n \to \infty$, the radius of the closed ball $L\|x_n - \hat{x}\|_X \mathbb{B}_Y$ will approach to zero as $n \to \infty$. Therefore, from (3.6), we have

$$-k_1 + F_0(\hat{x}) \subseteq F_n(x_n) + L\|x_n - \hat{x}\|_X \mathbb{B}_Y + \mathscr{N} \subseteq F_n(x_n) + \mathscr{N} + \mathscr{N},$$

that is,

$$-k_1+F_0(\hat{x})\subseteq F_n(x_n)+V.$$

Therefore,

$$((F_n(x_n)+V)\dot{-}F_0(\hat{x}))\cap(-K)\neq\emptyset,$$

and hence,

$$F_n(x_n)+V\preceq_K^{ml}F_0(\hat{x})\quad\forall n\geq N.$$

This completes the proof. □

Theorem 3.2. *Assume that the Assumption 3.1 (a)-(c) and (e) holds, and* $F_n\xrightarrow{Cml_2}F_0$ *with respect to* A. *Let* G *be a nonempty subset of* Y *and assume that there exists* $\varepsilon_n>0$ *with* $\varepsilon_n\to 0$ *such that* $F_n(x_n)+\varepsilon_n\mathbb{B}_Y\preceq_K^{ml}G$. *Then* $F_0(\hat{x})\preceq_K^{ml}G$.

Proof. Assume to the contrary that $F_0(\hat{x})\not\preceq_K^{ml}G$, that is, $-k+G\not\subseteq F_0(\hat{x})$ for all $k\in K$. Then there exists $\hat{g}\in G$ such that $-k+\hat{g}\notin F_0(\hat{x})$. Since $F_0(\hat{x})$ is closed, there exists $\delta>0$ such that $(-k+\hat{g}+\delta\mathbb{B}_Y)\cap F_0(\hat{x})=\emptyset$, and so

$$-k+\hat{g}\notin F_0(\hat{x})+\delta\mathbb{B}_Y. \tag{3.7}$$

Since F_0 is locally Lipschitz continuous at $\hat{x}$, there exist a constant $L>0$ and a neighbourhood $U(\hat{x})$ of $\hat{x}$ such that

$$F_0(x)\subseteq F_0(\hat{x})+L\|x-\hat{x}\|_X\mathbb{B}_Y,\quad\forall x\in U(\hat{x})\cap A. \tag{3.8}$$

Since $F_n\xrightarrow{Cml_2}F_0$ with respect to A, there exists $N_1\in\mathbb{N}$ such that

$$F_0(x)+\frac{1}{3}\delta\mathbb{B}_Y\preceq_K^{ml}F_n(x),\quad\forall x\in A\text{ and }\forall n\geq N_1. \tag{3.9}$$

Then there exists $k_1\in K$ such that

$$-k_1+F_n(x)\subseteq F_0(x)+\frac{1}{3}\delta\mathbb{B}_Y,\quad\forall x\in A\text{ and }\forall n\geq N_1. \tag{3.10}$$

Since $x_n\to\hat{x}$ and $\varepsilon_n\to 0$, there are $N_2,N_3\in\mathbb{N}$ such that $x_n\in U(\hat{x})$ for all $n\geq N_2$ and $\varepsilon_n\in\left(0,\frac{1}{3}\delta\right)$ for all $n\geq N_3$. It follows form (3.8) that

$$F_0(x_n)\subseteq F_0(\hat{x})+L\|x_n-\hat{x}\|_X\mathbb{B}_Y,\quad\forall n\geq N_2. \tag{3.11}$$

By using (3.10), we obtain

$$-k_1+F_n(x_n)\subseteq F_0(x_n)+\frac{1}{3}\delta\mathbb{B}_Y,\quad\forall n\geq N_1. \tag{3.12}$$

By hypothesis $F_n(x_n)+\varepsilon_n\mathbb{B}_Y\preceq_K^{ml}G$, we have that there exists $k_2\in K$ such that

$$-k_2+G\subseteq F_n(x_n)+\varepsilon_n\mathbb{B}_Y. \tag{3.13}$$

For any $N\geq\max\{N_1,N_2,N_3\}$, we conclude from (3.11), (3.12), and (3.13) that

$$-k_1-k_2+G\subseteq F_0(\hat{x})+L\|x_n-\hat{x}\|_X\mathbb{B}_Y+\frac{1}{3}\delta\mathbb{B}_Y+\varepsilon_n\mathbb{B}_Y, \tag{3.14}$$

and therefore,

$$-k+G\subseteq F_0(\hat{x})+L\|x_n-\hat{x}\|_X\mathbb{B}_Y+\frac{1}{3}\delta\mathbb{B}_Y+\varepsilon_n\mathbb{B}_Y,\quad\text{where }k=k_1+k_2. \tag{3.15}$$

Since $\mathbb{B}_Y$ is a closed ball around $\mathbf{0}$ and $x_n \to \hat{x}$ as $n \to \infty$, the radius of the closed ball $L\|x_n - \hat{x}\|_X \mathbb{B}_Y$ will approach to zero as $n \to \infty$. Therefore, from (3.15), we have

$$-k + G \subseteq F_0(\hat{x}) + \delta \mathbb{B}_Y, \tag{3.16}$$

which contradicts (3.7). Therefore, $F_0(\hat{x}) \preceq_K^{ml} G$. □

Theorem 3.3. *Assume that the Assumption 3.1 (a)-(e) holds. Further assume that*

(i) *$F_n(y_n)$ are convex subsets of Y,*
(ii) *$F_0(\hat{x})$ is compact and $F_0(\hat{y}) \prec_K^{ml} F_0(\hat{x})$,*
(iii) *F_0 is* weak$^{\#}$ *locally $(-K)$-Lipschitz at $\hat{y}$,*
(iv) *$F_n \xrightarrow{Cml} F_0$ with respect to A.*

Then there exists $N \in \mathbb{N}$ such that

$$F_n(y_n) \prec_K^{ml} F_n(x_n), \quad \forall n \geq N.$$

Proof. Since $F_0(\hat{y}) \prec_K^{ml} F_0(\hat{x})$, there exists $\hat{k} \in \text{int}K$ such that $-\hat{k} + F_0(\hat{x}) \subseteq F_0(\hat{y})$. Then, for any $r > 0$, $-\hat{k} + F_0(\hat{x}) \subseteq F_0(\hat{y}) \subseteq \bigcup_{z \in F_0(\hat{y})} \mathbb{B}_Y(z, r)$. Since $F_0(\hat{x})$ is compact, there is a finite set $\{z_1, z_2, \ldots, z_m\} \subseteq F_0(\hat{y})$ such that

$$-\hat{k} + F_0(\hat{x}) \subseteq \bigcup_{i=1}^{m} \mathbb{B}_Y(z_i, r).$$

From the fact that $-\hat{k} + F_0(\hat{x})$ is compact, we see that there exists $\delta > 0$ such that

$$-\hat{k} + F_0(\hat{x}) + 5\delta \mathbb{B}_Y \subseteq \bigcup_{i=1}^{m} \mathbb{B}_Y(z_i, r). \tag{3.17}$$

Since F_0 is locally Lipschitz at $\hat{x}$, there exist a constant $L > 0$ and a neighbourhood $U(\hat{x})$ of $\hat{x}$ such that

$$F_0(x) \subseteq F_0(\hat{x}) + L\|x - \hat{x}\|_X \mathbb{B}_Y, \quad \forall x \in U(\hat{x}). \tag{3.18}$$

Since F_0 is weak$^{\#}$ locally $(-K)$-Lipschitz continuous at $\hat{y}$, there exist a constant $L_1 > 0$ and a neighbourhood $U(\hat{y})$ of $\hat{y}$ such that

$$F_0(y) \cap \{z + L_1\|y - \hat{y}\|_X \mathbb{B}_Y - K\} \neq \emptyset, \quad \forall y \in U(\hat{y}),\ \forall z \in F_0(\hat{y}). \tag{3.19}$$

As $F_n \xrightarrow{Cml_1} F_0$ with respect to A, there exists $N_1 \in \mathbb{N}$ such that

$$F_n(x) + \delta \mathbb{B}_Y \preceq_K^{ml} F_0(x), \quad \forall x \in A \text{ and } \forall n \geq N_1. \tag{3.20}$$

Then there exists $k_1 \in K$ such that

$$-k_1 + F_0(x) \subseteq F_n(x) + \delta \mathbb{B}_Y, \quad \forall x \in A \text{ and } \forall n \geq N_1. \tag{3.21}$$

Since $F_n \xrightarrow{Cml_2} F_0$ with respect to A, there exists $N_2 \in \mathbb{N}$ such that

$$F_0(x) + \delta \mathbb{B}_Y \preceq_K^{ml} F_n(x), \quad \forall x \in A \text{ and } \forall n \geq N_2. \tag{3.22}$$

Then there exists $k_2 \in K$ such that

$$-k_2 + F_n(x) \subseteq F_0(x) + \delta \mathbb{B}_Y, \quad \forall x \in A \text{ and } \forall n \geq N_2. \tag{3.23}$$

As $x_n \to \hat{x}$ and $y_n \to \hat{y}$, there are $N_3, N_4 \in \mathbb{N}$ such that $x_n \in U(\hat{x})$ for any $n \geq N_3$ and $y_n \in U(\hat{y})$ for any $n \geq N_4$. Let $\widetilde{N} = \max\{N_1, N_2, N_3, N_4\}$. Then, for any $n \geq \widetilde{N}$, it follows form (3.18) that

$$F_0(x_n) + 4\delta \mathbb{B}_Y \subseteq F_0(\hat{x}) + L\|x_n - \hat{x}\|_X \mathbb{B}_Y + 4\delta \mathbb{B}_Y. \tag{3.24}$$

Since $\mathbb{B}_Y$ is a closed ball around $\mathbf{0}$ and $x_n \to \hat{x}$ as $n \to \infty$, the radius of the closed ball $L\|x_n - \hat{x}\|_X \mathbb{B}_Y$ will approach to zero as $n \to \infty$. Therefore, it follows from (3.24) and (3.17), that

$$-\hat{k} + F_0(x_n) + 4\delta \mathbb{B}_Y \subseteq -\hat{k} + F_0(\hat{x}) + 5\delta \mathbb{B}_Y \subseteq \bigcup_{i=1}^{m} \mathbb{B}_Y(z_i, r) \tag{3.25}$$

$$= \{z_1, z_2, \ldots, z_m\} + \mathbb{B}_Y(\mathbf{0}, r), \tag{3.26}$$

and therefore,

$$-\hat{k} + F_0(x_n) + 4\delta \mathbb{B}_Y \subseteq \{z_1, z_2, \ldots, z_m\} + \mathbb{B}_Y(\mathbf{0}, r). \tag{3.27}$$

For any $u \in -\hat{k} + F_0(x_n) + 4\delta \mathbb{B}_Y$, we conclude from (3.27) that there is $i_0 \in \{1, 2, \ldots, m\}$ and $\bar{b} \in \mathbb{B}_Y(\mathbf{0}, r)$ such that

$$u = z_{i_0} + \bar{b}. \tag{3.28}$$

From (3.19), we have $F_0(y_n) \cap \{z + L_1\|y_n - \hat{y}\|_X \mathbb{B}_Y - K\} \neq \emptyset$, $\forall y_n \in U(\hat{y})$, and $\forall z \in F_0(\hat{y})$. Since $\mathbb{B}_Y$ is a closed ball around $\mathbf{0}$ and $y_n \to \hat{y}$ as $n \to \infty$, the radius of the closed ball $L\|y_n - \hat{y}\|_X \mathbb{B}_Y$ will approach to the zero as $n \to \infty$. Therefore, $F_0(y_n) \cap \{z + \delta \mathbb{B}_Y - K\} \neq \emptyset$, $\forall y_n \in U(\hat{y})$, and $\forall z \in F_0(\hat{y})$. Then there exist $s \in F_0(y_n)$, $b_0 \in \mathbb{B}_Y$ and $\tilde{k} \in K$ such that $s = z_{i_0} + \delta b_0 - \tilde{k}$, and so $z_{i_0} = s - \delta b_0 + \tilde{k}$. Combining this with (3.28), we get

$$u = z_{i_0} + \bar{b} = s - \delta b_0 + \tilde{k} + \bar{b},$$

that is,

$$-\tilde{k} + u = s - \delta b_0 + \bar{b} \in F_0(y_n) + \delta \mathbb{B}_Y + \mathbb{B}_Y(\mathbf{0}, r). \tag{3.29}$$

Since $u \in -\hat{k} + F_0(x_n) + 4\delta \mathbb{B}_Y$, we have $-\tilde{k} + u \in -\tilde{k} - \hat{k} + F_0(x_n) + 4\delta \mathbb{B}_Y$. From this and (3.29), we have

$$-\tilde{k} - \hat{k} + F_0(x_n) + 4\delta \mathbb{B}_Y \subseteq F_0(y_n) + \delta \mathbb{B}_Y + \mathbb{B}_Y(\mathbf{0}, r) \subseteq F_0(y_n) + \delta \mathbb{B}_Y + r\mathbb{B}_Y. \tag{3.30}$$

Let $\tilde{k} + \hat{k} = \bar{k}$. Since $\tilde{k} \in K$ and $\hat{k} \in \text{int}K$, by using the fact that $K + \text{int}K \subseteq \text{int}K$, we have, $\bar{k} \in \text{int}K$. Now

$$-\bar{k} + F_0(x_n) + 4\delta \mathbb{B}_Y \subseteq F_0(y_n) + \delta \mathbb{B}_Y + r\mathbb{B}_Y. \tag{3.31}$$

From (3.21) and (3.23), we conclude that, for any $n \geq \widetilde{N}$,

$$-k_1 + F_0(y_n) \subseteq F_n(y_n) + \delta \mathbb{B}_Y, \tag{3.32}$$

and

$$-k_2 + F_n(x_n) \subseteq F_0(x_n) + \delta \mathbb{B}_Y. \tag{3.33}$$

From (3.31), (3.32), and (3.33), and using again the fact that $K + \text{int}K \subseteq \text{int}K$, we obtain

$$-k_0 + 3\delta \mathbb{B}_Y + F_n(x_n) \subseteq F_n(y_n) + (2\delta + r)\mathbb{B}_Y, \quad \text{where } k_0 = k_1 + k_2 + \bar{k}. \tag{3.34}$$

Let $r \to 0$. Then $2\delta + r \to 2\delta < 3\delta$, and hence, by using Lemma 2.1, we get

$$-k_0 + F_n(x_n) \subseteq \text{int}F_0(y_n) \subseteq F_n(y_n). \tag{3.35}$$

This implies that

$$F_n(y_n) \prec_K^{ml} F_n(x_n).$$

This completes the proof. □

4. Convergence for Solution Sets

In this section, we establish the convergence of the set of all ml-minimal solutions and set of all weak ml-minimal solutions for the family of perturbed set optimization problems obtained by perturbing objective function as well as feasible region in set optimization problem (SOP).

The proof of all the results of this section lies on the proof of the corresponding results in [23]. However, the results are different and even not comparable with the corresponding results in [23]. We include the proof of all the results for the sake of completeness of the paper.

Assume that $A \subseteq \mathrm{Dom}(F_n)$ for all $n \in \mathbb{N}_0$ and S is a nonempty subset of $A \subseteq X$. For each $n \in \mathbb{N}$, let $\{S_n\}$ be a sequence of nonempty subsets of X such that $\bigcup_{n\in\mathbb{N}} S_n \subseteq A$. The family of perturbed constrained set optimization problems ($\mathrm{SOP_n}$), $n \in \mathbb{N}_0$, is given by

$$\begin{aligned}&\text{Minimize } F_n(x),\\ &\text{subject to } x \in S_n,\end{aligned} \tag{$\mathrm{SOP_n}$}$$

where F_n is given in Assumption 3.1 (a).

Lemma 4.1. *Let X be a finite-dimensional space, and for each $n \in \mathbb{N}$, $x_n \in S_n$ such that $x_n \to x \in X$. Assume that*

(i) *F_0 is locally Lipschitz continuous on S with closed values;*
(ii) *$S_n \xrightarrow{PK} S$ and $F_n \xrightarrow{Cml} F_0$ with respect to A;*
(iii) *S_n is arcwise connected, and F_n is ml-quasiconnected on S_n;*
(iv) *$Q_{ml}(x, F_0, S)$ is nonempty and bounded.*

Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$Q_{ml}(x_n, F_n, S_n) \subseteq Q_{ml}(x, F_0, S) + \varepsilon \mathbb{B}_X, \quad \forall n \geq N.$$

Proof. Assume to the contrary that there exists $\varepsilon_0 > 0$ such that, for any $N \in \mathbb{N}$, there exists $m_N \geq N$ satisfying

$$Q_{ml}(x_{m_N}, F_{m_N}, S_{m_N}) \nsubseteq Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X.$$

Without loss of generality, we may assume that

$$Q_{ml}(x_n, F_n, S_n) \nsubseteq Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X, \quad \forall n \in \mathbb{N}.$$

Then there exists $y_n \in Q_{ml}(x_n, F_n, S_n)$ such that

$$y_n \notin Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X. \tag{4.1}$$

By view of condition (iv), we can choose $\bar{x} \in Q_{ml}(x, F_0, S)$. Thus $\bar{x} \in S$. Since $S_n \xrightarrow{PK} S$, there exists $\bar{x}_n \in S_n$ such that $\bar{x}_n \to \bar{x}$. As $\bar{x} \in Q_{ml}(x, F_0, S)$, we have $F_0(\bar{x}) \preceq_K^{ml} F_0(x)$, and so there exists $k \in K$ such that

$$-k + F_0(x) \subseteq F_0(\bar{x}). \tag{4.2}$$

Let $e \in \mathrm{int}K$. Then $K - \frac{1}{n}e$ is a neighbourhood of $\mathbf{0} \in Y$. Since F_0 is locally Lipschitz continuous at $\bar{x}$, by Theorem 3.1, there exists $m_N \geq N$ such that

$$F_0(\bar{x}) \subseteq F_{m_N}(\bar{x}_{m_N}) + K + K - \frac{1}{n}e \subseteq F_{m_N}(\bar{x}_{m_N}) + K - \frac{1}{n}e. \tag{4.3}$$

Without loss of generality, we may assume that

$$F_0(\bar{x}) \subseteq F_n(\bar{x}_n) + K - \frac{1}{n}e, \quad \forall n \in \mathbb{N}. \tag{4.4}$$

It follows from (4.2) and (4.4) that

$$-k + F_0(x) \subseteq F_n(\bar{x}_n) + K - \frac{1}{n}e. \tag{4.5}$$

Since F_0 is locally Lipschitz ontinuous at x and $K - \frac{1}{n}e$ is a neighbourhood of $\mathbf{0} \in Y$, it follows from Theorem 3.1 that there exists $t_N \geq N$ such that $F_{t_N}(x_{t_N}) + \left(K - \frac{1}{n}e\right) \preceq_K^{ml} F_0(x)$. Without loss of generality, we may assume that

$$F_n(x_n) + \left(K - \frac{1}{n}e\right) \preceq_K^{ml} F_0(x), \quad \forall n \in \mathbb{N}. \tag{4.6}$$

Then there exists $k_1^{'} \in K$ such that

$$-k_1^{'} + F_0(x) \subseteq F_n(x_n) + K - \frac{1}{n}e, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

Since $y_n \in Q_{ml}(x_n, F_n, S_n)$, we have $F_n(y_n) \preceq_K^{ml} F_n(x_n)$, and so there exists $k_2^{'} \in K$ such that

$$-k_2^{'} + F_n(x_n) \subseteq F_n(y_n). \tag{4.8}$$

From (4.7) and (4.8), we have

$$-k_3 + F_0(x) \subseteq F_n(y_n) + K - \frac{1}{n}e, \quad \text{where } k_3 = k_1^{'} + k_2^{'}. \tag{4.9}$$

In view of (4.5) and (4.9), we have

$$F_n(\bar{x}_n) \preceq_K^{ml} F_0(x) + \frac{1}{n}e, \tag{4.10}$$

and

$$F_n(y_n) \preceq_K^{ml} F_0(x) + \frac{1}{n}e. \tag{4.11}$$

Since F_n is ml-quasiconnected on S_n, there exists a continuous path $\gamma_n : [0,1] \to S_n$ with $\gamma_n(0) = \bar{x}_n$ and $\gamma_n(1) = y_n$ such that

$$F_n(\gamma_n(t)) \preceq_K^{ml} F_0(x) + \frac{1}{n}e, \quad \forall t \in [0,1]. \tag{4.12}$$

Since $\bar{x}_n \to \bar{x} \in Q_{ml}(x, F_0, S)$, we have $\bar{x}_n \in Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X$ for large n enough. This together with (4.1) and Lemma 2.5 implies that there exists $t_n \in [0,1]$ such that

$$\beta_n := \gamma_n(t_n) \in \partial[Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X].$$

Since $Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X$ is bounded and X is a finite dimensional space, it is easy to see that $\partial[Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X]$ is compact. So we assume that

$$\beta_n \to \beta \in \partial[Q_{ml}(x, F_0, S) + \varepsilon_0 \mathbb{B}_X]. \tag{4.13}$$

From (4.12), there exists $k_4 \in K$ such that

$$-k_4 + F_0(x) \subseteq F_n(\beta_n) - \frac{1}{n}e. \tag{4.14}$$

Letting $\lambda_n = \frac{\|e\|}{n}$, we conclude from (4.14) that

$$-k_4 + F_0(x) \subseteq F_n(\beta_n) + \lambda_n \mathbb{B}_Y, \tag{4.15}$$

and therefore,

$$F_n(\beta_n) + \lambda_n \mathbb{B}_Y \preceq_K^{ml} F_0(x). \tag{4.16}$$

Since $F_0(\beta)$ is closed, combining this with (4.16) and Theorem 3.2, we obtain

$$F_0(\beta) \preceq_K^{ml} F_0(x). \tag{4.17}$$

Since $\beta_n \in S_n, S_n \xrightarrow{PK} S$ and $\beta_n \to \beta$, we have $\beta \in S$. This together with (4.17) implies that $\beta \in Q_{ml}(x, F_0, S)$, which contradicts (4.13). This completes the proof. □

Remark 4.1. In [23], F_0 was considered to be K-continuous with compact values, while in Lemma 4.1, F_0 is locally Lipschitz continous with closed values. Therefore, with regard to the continuity, Lemma 4.1 is not comparable with the Lemma 4.1 in [23], while in regard to the values for the mapping F_0, Lemma 4.1 is more general than the Lemma 4.1 in [23]. Similar comparison between Lemma 4.1 and Lemma 3.1 in [17] can also be seen.

Theorem 4.1. *Assume that the following conditions hold:*

(i) *F_0 is locally Lipschitz continuous on S with compact values;*
(ii) *F_0 is weak# locally $(-K)$-Lipschitz continuous on S;*
(iii) *$S_n \xrightarrow{PK} S$ and $F_n \xrightarrow{Cml} F_0$ with respect to A;*
(iv) *For any $x \in S_n, F_n(x)$ is convex.*

Then $\mathrm{Ls}W_{ml-F_n}(S_n) \subseteq W_{ml-F_0}(S)$. *Moreover, if S is arcwise connected and F_0 is strictly ml-quasiconnected on S, then* $\mathrm{Ls}E_{ml-F_n}(S_n) \subseteq E_{ml-F_0}(S)$.

Proof. Let $\hat{x} \in \mathrm{Ls}W_{ml-F_n}(S_n)$. Since $W_{ml-F_n}(S_n) \subseteq S_n$, we have $\mathrm{Ls}W_{ml-F_n}(S_n) \subseteq \mathrm{Ls}S_n$. By (2.3) and assumption (iii), we have $\mathrm{Ls}S_n \subseteq S$. Therefore, $\hat{x} \in S$. We claim that $\hat{x} \in W_{ml-F_0}(S)$. Assume contrary that $\hat{x} \notin W_{ml-F_0}(S)$. Then there exists $\hat{y} \in S$ such that

$$F_0(\hat{y}) \prec_K^{ml} F_0(\hat{x}). \tag{4.18}$$

Since $\hat{y} \in S \subseteq \mathrm{Li}S_n$, there exists $y_n \in S_n$ such that $y_n \to \hat{y}$. In particular, every subsequence $\{y_{n_k}\}$ of $\{y_n\}$ will converges to $\hat{y}$. On the other hand, since $\hat{x} \in \mathrm{Ls}W_{ml-F_n}(S_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in W_{ml-F_{n_k}}(S_{n_k})$ such that $x_{n_k} \to \hat{x}$. Then by applying Theorem 3.3 and using (4.18), there exists $k_0 \in \mathbb{N}$ such that

$$F_{n_k}(y_{n_k}) \prec_K^{ml} F_{n_k}(x_{n_k}), \quad \forall k \geq k_0, \tag{4.19}$$

that is, $x_{n_k} \notin W_{ml-F_{n_k}}(S_{n_k})$, which contradicts that $x_{n_k} \in W_{ml-F_{n_k}}(S_{n_k})$. Therefore, $\hat{x} \in W_{ml-F_0}(S)$. We now show that $\mathrm{Ls}E_{ml-F_n}(S_n) \subseteq E_{ml-F_0}(S)$. By Lemma 2.7, we have $W_{ml-F_0}(S) = E_{ml-F_0}(S)$. Since $E_{ml-F_n}(S_n) \subseteq W_{ml-F_n}(S_n)$, we have $\mathrm{Ls}E_{ml-F_n}(S_n) \subseteq \mathrm{Ls}W_{ml-F_n}(S_n)$, therefore,

$$\mathrm{Ls}E_{ml-F_n}(S_n) \subseteq \mathrm{Ls}W_{ml-F_n}(S_n) \subseteq W_{ml-F_0}(S) = E_{ml-F_0}(S),$$

and the proof is complete. □

Remark 4.2. If $F_n = F$ for all $n \in \mathbb{N}$, then Theorem 4.1 can be written as "if F is locally Lipschitz continuous on S with compact values, F is weak$^{\#}$ locally $(-K)$-Lipschitz continuous on S, $S_n \xrightarrow{PK} S$ and $F(x)$ is convex for any $x \in S_n$, then $\mathrm{Ls}W_{ml-F}(S_n) \subseteq W_{ml-F}(S)$." On comparing Theorem 4.1 with Theorem 3.1 in [17], we have established Theorem 4.1 for weak m-lower set less realtion $(\prec_C^{ml})$, however in Theorem 3.1 in [17], the weak lower set less relation $(\prec_C^{l})$ is considered. By Theorem 2.4, we can see that the solution set $W_{ml-F}(S)$ is larger than the solution set $W_{l-F}(S)$. Hence, we have studied Theorem 4.1 for a large set. In Theorem 3.1 of [17], the continuity of F on S is assumed, however, we need only locally Lipschitz continuity and weak$^{\#}$ locally $(-K)$-Lipschitz continuity on S with the assumption that $F(x)$ is convex for all $x \in S_n$.

Theorem 4.2. *Let X be a finite-dimensional space and let S_n and S be arcwise connected. Assume that the following conditions hold:*

(i) *F_0 is locally Lipschitz continuous and strictly ml-quasiconnected on S with compact values;*
(ii) *F_0 is* weak$^{\#}$ *locally $(-K)$-Lipschitz continuous on S;*
(iii) *$S_n \xrightarrow{PK} S$ and $F_n \xrightarrow{Cml} F_0$ with respect to A;*
(iv) *S_n is closed, and F_n is ml-quasiconnected and locally Lipschitz continuous on S_n with convex and closed values;*
(v) *$E_{ml-F_n}\big(Q_{ml}(x_n,F_n,S_n)\big) \neq \emptyset$.*

Then,

$$E_{ml-F_n}(S_n) \xrightarrow{PK} E_{ml-F_0}(S) \quad \text{and} \quad W_{ml-F_n}(S_n) \xrightarrow{PK} W_{ml-F_0}(S).$$

Proof. From Theorem 4.1, we have $\mathrm{Ls}W_{ml-F_n}(S_n) \subseteq W_{ml-F_0}(S)$ and $\mathrm{Ls}E_{ml-F_n}(S_n) \subseteq E_{ml-F_0}(S)$. To show $E_{ml-F_0}(S) \subseteq \mathrm{Li}E_{ml-F_n}(S_n)$, we let $\hat{x} \in E_{ml-F_0}(S)$. Since $E_{ml-F_0}(S) \subseteq S$. By (2.3) and assumption (iii), we have $S \subseteq \mathrm{Li}S_n$, and therefore, $\hat{x} \in \mathrm{Li}S_n$. Then, there exists $x_n \in S_n$ such that $x_n \to \hat{x}$. By Lemma 2.8, we have $Q_{ml}(\hat{x},F_0,S) = \{\hat{x}\}$. Clearly, $Q_{ml}(\hat{x},F_0,S)$ is nonempty and bounded. For any $\varepsilon > 0$, by Lemma 4.1, there exists $N \in \mathbb{N}$ such that

$$Q_{ml}(x_n,F_n,S_n) \subseteq Q_{ml}(\hat{x},F_0,S) + \varepsilon\mathbb{B}_X = \{\hat{x}\} + \varepsilon\mathbb{B}_X, \quad \forall n \geq N. \tag{4.20}$$

Since F_n is locally Lipschitz continuous on S_n with closed values, it follows from Lemma 2.6 that $Q_{ml}(x_n,F_n,S_n)$ is closed. In view of (4.20), we obtain that $Q_{ml}(x_n,F_n,S_n)$ is bounded. Since X is a finite-dimensional space and $Q_{ml}(x_n,F_n,S_n)$ is closed, we have that $Q_{ml}(x_n,F_n,S_n)$ is compact. Clearly, $x_n \in Q_{ml}(x_n,F_n,S_n)$, and so $Q_{ml}(x_n,F_n,S_n)$ is nonempty. By assumption (v), we have $E_{ml-F_n}(Q_{ml}(x_n,F_n,S_n)) \neq \emptyset$. Let $\beta_n \in E_{ml-F_n}(Q_{ml}(x_n,F_n,S_n))$. Then by Remark 2.4, we have

$$\beta_n \in E_{ml-F_n}\big(Q_{ml}(x_n,F_n,S_n)\big) \subseteq E_{ml-F_n}(S_n). \tag{4.21}$$

From (4.20), we have that

$$\begin{aligned}\beta_n &\in E_{ml-F_n}\big(Q_{ml}(x_n,F_n,S_n)\big) \subseteq Q_{ml}(x_n,F_n,S_n)\\ &\subseteq Q_{ml}(\hat{x},F_0,S) + \varepsilon\mathbb{B}_X = \{\hat{x}\} + \varepsilon\mathbb{B}_X, \quad \forall n \geq N,\end{aligned}$$

which implies that $\beta_n \to \hat{x}$. Therefore, by (4.21), $\hat{x} \in \mathrm{Li}E_{ml-F_n}(S_n)$, and hence, $E_{ml-F_0}(S) \subseteq \mathrm{Li}E_{ml-F_n}(S_n)$.

Finally, we prove that $W_{ml-F_0}(S) \subseteq \text{Li}W_{ml-F_n}(S_n)$. By Lemma 2.7, we have $W_{ml-F_0}(S) = E_{ml-F_0}(S)$. Since $E_{ml-F_n}(S_n) \subseteq W_{ml-F_n}(S_n)$, we have $\text{Li}E_{ml-F_n}(S_n) \subseteq \text{Li}W_{ml-F_n}(S_n)$, and therefore,

$$W_{ml-F_0}(S) = E_{ml-F_0}(S) \subseteq \text{Li}E_{ml-F_n}(S_n) \subseteq \text{Li}W_{ml-F_n}(S_n).$$

This completes the proof. □

Remark 4.3. Since there is no relationship between $E_{ml-F}(S)$ and $E_{ml-F}(S)$, and the locally Lipschitz continuity is different from continuity / l.s.c. / u.s.c. for set-valued mappings, Theorem 4.2 is new and not comparable with the Theorem 4.2 in [23].

Remark 4.4. In Theorem 3.1, 4.1 and 4.2 in [10], Theorem 3.1, 4.1 and 4.5 in [16] and Theorems 4.1 and 4.3 in [18], some stability results for the minimal solutions and for the weak minimal solutions were carried out in the image space. While, in Theorem 4.2, we have studied stability results for the minimal solutions and weak minimal solutions in the decision space.

The following example illustrates Theorem 4.2.

Example 4.1. In Example 2.8, consider $F_n : X \rightrightarrows Y$ defined by

$$F_n(x,y) = \left(x^2y^2 + \frac{1}{n}, x^2y^2 + \frac{1}{n}\right) + \mathbb{B}_Y, \quad \forall (x,y) \in \mathbb{R}^2,$$

$S_n = \{(0,0)\} \cup [1/n, 3) \times [1/n, 3)$ and $A = \mathbb{R}^2_+$. Then it can be easily seen that $\bigcup_{n\in\mathbb{N}} S_n \subseteq A$ and $S \subseteq A$. Thus, after some calculations, we can see that all the conditions of Theorem 4.2 are fulfilled. Hence by Theorem 4.2, we conclude that $E_{ml-F_n}(S_n) \xrightarrow{PK} E_{ml-F_0}(S)$ and $W_{ml-F_n}(S_n) \xrightarrow{PK} W_{ml-F_0}(S)$.

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