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Aspects of the Renormalisation Group for gauge theories and gravity

by

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Abstract

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This thesis is devoted to exploring various aspects of the renormalisation group for gauge theories and gravity. After introducing the necessary concepts, we study a manifestly gauge invariant and background independent ERG flow equation for gravity and Yang-Mills theories, which seeks to bypass the usual problems regarding gauge-fixing and ghosts by implementing a geometric version of the higher covariant derivative regularisation scheme. In the process we develop the machinery needed to study it rigorously and show that, under rather loose assumptions, the regularisation is at odds with gauge invariance. Next we investigate scalar field theories within the Local Potential Approximation (LPA). We show that in the large field limit the eigenoperator equation can be treated as a Sturm-Liouville (SL) problem, which in turn allows us to recast it as a Schrödinger type equation, and compute the scaling dimension of the corresponding eigenoperators using Wentzel-Kramers-Brillouin (WKB) analysis. We find that the scaling dimension for the $O(N)$ case is twice that for a single scalar field, and moreover that both results are universal and independent of the details of the regularisation used. Finally, we consider off-shell perturbative renormalisation of pure quantum gravity. We show that at each new loop order the divergences that do not vanish on-shell are constructed from only the total metric, whilst those that vanish on-shell are renormalised by canonical transformations involving the quantum fields. Purely background metric divergences do not separately appear and the background metric does not get renormalised. We verify these assertions by computing leading off-shell divergences to two loops, exploiting off-shell BRST invariance and the renormalisation group equations. Furthermore, although some divergences can be absorbed by field redefinitions, we explain why this does not lead to finite beta functions for the corresponding field.

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Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:
    - Vlad-Mihai Mandric and Tim R. Morris. Properties of a proposed background independent exact renormalization group. *Phys. Rev. D*, 107(6):065012, 2023, 2210.00492 [1]
    - Vlad-Mihai Mandric, Tim R. Morris, and Dalius Stulga. Universal scaling dimensions for highly irrelevant operators in the local potential approximation. *Phys. Rev. D*, 108(10):105003, 2023, 2306.14643 [2]
    - Vlad-Mihai Mandric, Tim R. Morris, and Dalius Stulga. Off-shell divergences in quantum gravity. *JHEP*, 11:149, 2023, 2308.07382 [3]

Signed:..... Date:.....

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Last, but by no means least, I would like to thank my parents for their unconditional love and support. This thesis is dedicated to them.

Pentru Mama și Tata

Chapter 1

Introduction

Physics underwent a profound transformation in the twentieth century which resulted in the birth of General Relativity (GR) and the Standard Model (SM) of particles. This was possible through a series of revisions of old scientific beliefs (*i.e.* paradigm shifts). Thus, special relativity (SR), and GR afterwards, emerged after abandoning the view that space and time are independent from one another. Similarly, leaving behind a classical deterministic world, physicists uncovered the quantum probabilistic nature of the universe, which in turn enabled them to construct the SM.

One can truly term the physics of the twentieth century as revolutionary. In some sense, much of the modern developments in fundamental physics can be regarded as footnotes to what has been achieved in the last century, and this thesis makes no exception. This is not to say that modern theoretical physics is irrelevant or unworthy of attention. It is just a way of highlighting that we still live in the aftermath of the paradigm shifts that relativity and quantum theory brought about.

The SM of particles is a quantum field theory (QFT) which describes three of the four known fundamental forces (*i.e.* strong, weak and electromagnetic) present in nature. It is a very successful theory which, despite physicists' efforts, still defies to this date all the experimental attempts to prove it wrong. However, it is not the end of the story. There are still phenomena which the Standard Model fails to account for: dark matter and dark energy, neutrino masses *etc.* . In addition to these, there exist unresolved anomalies which await more refined research, *e.g.* the anomalous magnetic moment of the muon [4,5]. Furthermore, the SM does not take into account gravity, which is nonetheless accurately described by GR on large scales. The latter has been thoroughly tested (gravitational waves [6], perihelion of Mercury *etc.*) and explored [7]. However, similar to the SM, it is not perfect. It comes short of being a truly fundamental theory. For example, the Penrose-Hawking theorems show that spacetime singularities are present both inside black holes [8], and at the beginning of time (*i.e.* Big Bang) [9]. Their presence highlights one of the limitations of classical GR.

This suggests the need to go beyond and search for a more comprehensive framework. One of the possible paths, which motivates most of the work undergone in this thesis, is to unify the principles which lie at the core of GR and the SM into a quantum theory of gravity, which one would expect to succeed where the previous two failed. In addition to this, there are other approaches currently pursued, such as String Theory [10] or Loop Quantum Gravity (LQG) [11]. Although very interesting, their aim is to construct a different and more fundamental framework. However, this lies beyond the scope of this thesis, in which we take a more conservative stance and explore the possibility of quantising gravity. In other words, we treat on equal footing gauge theories and gravity, and hence consider gravity as a field theory. This shares many similarities with the Asymptotic Safety (AS) approach to gravity [12–16].

If we naively proceed about treating gravity as a simple gauge theory, we soon realise that it is more complicated than that [17]. We inevitably encounter infinities which cannot be cast away (*i.e.* renormalised) in the usual way. It is well established by now that the Einstein-Hilbert action is not (perturbatively) renormalisable on its own at two loops [18–20], or even at the one-loop level if one takes into account a cosmological constant [21], adds matter fields [22,23], or introduces gauge fields [24]. However, it does not follow from this that gravity cannot be quantised in the usual way. It just proves to be more subtle in some sense, and that diffeomorphism invariance is somewhat more stubborn than the usual gauge invariance one encounters in the SM.

The difficulty arises from the requirement to bridge the gap between the large scale structure of the universe, as successfully accounted for in GR, and the small scale, where particle physics emerges. The way out, we believe, is provided by the Renormalisation Group (RG), a framework based on the simple observation that the same physics can look different at different physical scales. And for this reason, this thesis is dedicated to the study of various implementations of the RG group ideas, whilst always keeping in the back of our heads the original goal that we ultimately want to find a consistent way of making gravity quantum.

In the following sections we give the necessary background for the subsequent chapters. We begin with a review of the RG ideas in section 1.1. In the following section we introduce the theory space on which all possible effective actions lie and highlight the connection between fixed points and renormalisability. In section 1.3 we present the effective average action, the IR regulated generator of one-particle irreducible (1PI) diagrams, and its corresponding flow equation. Finally, we conclude this chapter outlining the structure for the rest of this thesis.

1.1 The Renormalisation Group

The scale at which we observe a physical phenomenon will ultimately determine how we describe it. Physics stays the same, but some variables prove to be more appropriate than others for describing a physical system at a particular viewing scale. And this very fact, as mentioned before, is encoded within the RG framework.

For example, to understand ocean waves or the flow of water through pipes, one does not have to understand how water molecules interact with one another or how hydrogen and oxygen atoms behave within them. A suitable description of water related phenomena at the macroscopic level can be constructed using the language of fluid mechanics (*i.e.* classical) without ever referring to any quantum effects at all. Similar examples can be given in the context of field theories, and gauge theories in particular, which represent one of the main interests of this thesis. To name just one, quantum chromodynamics (QCD) in the ultraviolet (UV) or, equivalently, high energy/short distance regime, is best described in terms of quarks and gluons which are weakly coupled as a consequence of asymptotic freedom [25, 26]. In contrast to this, in the infrared (IR) or, equivalently, low energy/large distance regime, QCD is best understood in terms of bound states (*i.e.* hadrons) and their corresponding interactions, the reason behind this being that quarks and gluons become strongly coupled, and thus undergo confinement [27].

In other words, the RG approach can be viewed as a procedure for systematically dealing with problems happening on many length scales. In general, from a microscopic description of a physical system one should be able to track how its macroscopic behaviour unfolds using the RG framework. However, this is not as simple a task, and one often requires to make approximations to obtain a feasible result, or any result at all. But before delving into the mathematical details, it is instructive to build a more intuitive understanding of the renormalisation group. One particularly useful analogy is that with a microscope of varying magnification [28, 29]. This proceeds as follows. If we consider a physical system in position space, in which fields are labelled in terms of their spacetime indices (*i.e.* $\phi = \phi(x)$), then, the RG procedure amounts to replacing the degrees of freedom (d.o.f) within a small part of the system by an effective one such that physics remains unchanged. Equivalently, we can think of this as 'zooming out' using a microscope, but in a rather abstract way and at the level of the d.o.f. This is as far as we can take this analogy however. If we stretch it too far, it will snap, and it will lose its usefulness.

Central to the RG approach is the effective action S_Λ, a scale dependent functional which encodes the relevant degrees of freedom of a physical system and the strength of the corresponding interactions at some viewing scale Λ. We can develop a better understanding of this concept if we shift to momentum space, where fields are labelled in terms of their corresponding momenta (*i.e.* $\phi = \phi(p)$). In this language, the above philosophy of averaging over local patches translates to integrating out high energy

modes between some original scale Λ_0, which is called the bare scale, and some effective scale Λ. Carrying out this procedure, the original action describing our physical system, S_{Λ_0}, will generally get modified, and hence one obtains an effective action S_Λ. This procedure can in turn be iterated, and thus one ends up with a series of effective actions:

$$S_{\Lambda_0} \to S_{\Lambda_1} \to S_{\Lambda_2} \to \cdots , \tag{1.1.1}$$

where $\Lambda_0 > \Lambda_1 > \Lambda_2 > \cdots$. If each of the above steps are performed over infinitesimal momentum shells (*i.e.* $\Lambda_i - \Lambda_{i+1} = \delta\Lambda$ for all $i \in \mathbb{N}$), then (1.1.1) becomes a partial differential equation,

$$\Lambda \partial_\Lambda S_\Lambda = \cdots , \tag{1.1.2}$$

where we have used the shorthand $\partial_\Lambda \equiv \frac{\partial}{\partial \Lambda}$. The above equation is called the *RG flow equation* and it is the mathematical formulation of the continuous RG. This means that, whenever the RG approach is used to study continuous systems, one can write that down as a flow equation, whose basic structure is given by (1.1.2) above.

Up to this point, we have outlined the philosophy behind the renormalisation group, and, in the process, we have also presented its qualitative features such that one could develop an intuitive understanding without having to bring into play technical details. We will cover these in the subsequent sections of this chapter. However, there is still a crucial question left unanswered: When is the RG approach useful? (or, shorter, why bother?) To answer this, we will follow the excellent discussion of ref. [28], which in turn builds on the one in ref. [30]. We first give a more general explanation, and then further expand on how this applies to quantum field theories, which lie at the heart of this thesis.

In short, the deciding factor is the correlation length ξ of the system or, more precisely, the number of degrees of freedom within a correlation length. Now, for any physical system, ξ represents the typical size of the regions in which the degrees of freedom act collectively as a single unit (*i.e.* are correlated). Thus, one expects a small subsystem of characteristic size ξ to exhibit similar properties to the whole system. In other words, the d.o.f within a correlation length will dictate the behaviour and properties of the bulk (*e.g.* for the analysis of crystalline solids it is sufficient to consider only the Brillouin zone for most physical purposes). If, for a certain physical system, this number is small, and thus only a few d.o.f are relevant within a correlation length, then RG methods can not be used to further simplify the problem.

On the other hand, if the opposite holds, and one has to deal with a large density of d.o.f per correlation length, then the renormalisation group proves to be crucial for understanding and uncovering the underlying physics. The reason behind this boils down to two key ideas: interactions are assumed to be local of range $O(L_0)$ with $L_0 \ll \xi$, and the coarse graining procedure can be iterated. We can understand this as follows. Let us assume that we have a small subsystem of size $O(\xi)$. As before, understanding

what happens within a correlation length will ultimately translate to understanding the properties of the whole (larger) system. Now, we can break the subsystem into smaller local patches of characteristic size L_0, which will contain a finite, ideally small, number of relevant d.o.f. The strategy of sampling the subsystem in this way is similar to what we have done before, but, in the current case, given that $L_0 \ll \xi$, there is no hope of deducing the properties of the bulk directly. Nonetheless, one can still follow the coarse graining recipe. If we now average over patches slightly larger than the original one, say of characteristic size $2L_0$, then we end up with an effective action in which the range of the interactions increased (it is now $O(2L_0)$), and there are fewer d.o.f present (since we averaged over). This is an improvement, but still not enough for our purposes. However, the crucial observation is that this can be repeated as many times as required until the range of the interactions within the effective action equals the correlation length. And, in this way, provided that we figure out how to coarse grain over neighbouring patches, which in general is not easy, we will be able to deduce the properties of the whole system.

Extending this analysis to quantum field theories is not straightforward because they present difficulties of their own. Any patch, regardless of its size, will exhibit an infinite number of degrees of freedom, which at first seems to be at odds with the arguments outlined above. The way out of this conundrum is provided by the point-like nature of QFT interactions (*i.e.* $L_0 \to 0$). Although an infinite number of d.o.f can certainly not be reconciled with a finite range of interaction, sensible answers can in principle be obtained if this holds no more. In momentum space, this translates to integrating out the high energy modes within an infinitesimal momentum shell, and then iterating this process (*i.e.* integrating shell by shell), rather than considering all the modes in one go.

In the remainder of this section, as mentioned above, we will focus on the more technical aspects of the renormalisation group. Historically, one can trace the origin of the RG methods back to the seminal work of Kadanoff [31], where the idea of coarse graining over d.o.f was first used to study the properties of the Ising model. We first recall in the next subsection how this works. Many of the key ideas, although formulated in the discrete case, will continue to hold for continuous systems as well. Then, in subsection 1.1.2, we move on to explore the RG approach to field theories as envisaged by Wilson. In the process, we explain how to construct flow equations, and as a first example derive Polchinski's flow equation. We end this section with a brief discussion on renormalisability in subsection 1.1.3.

1.1.1 Kadanoff blocking

To be able to make concrete statements regarding the renormalisation group we consider the two-dimensional Ising model. This is a lattice of spins which can be oriented either up or down, and exhibit only nearest-neighbour interactions, see Fig. 1.1.1 below. The characteristic length, and implicitly the cutoff scale, is set by the lattice spacing (*i.e.*

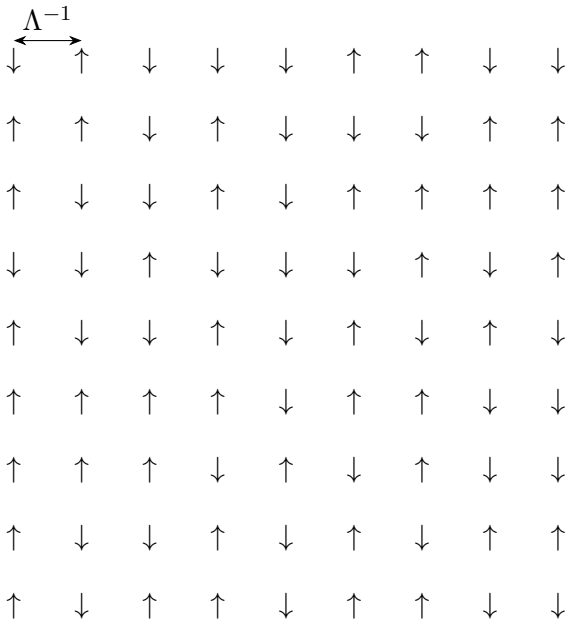

FIGURE 1.1.1: 2D Ising model

the distance between neighbouring spins). As outlined above, the regime in which the RG proves to be indispensable is when the density of d.o.f per correlation length is large. Given the periodicity of the lattice, this simply translates to a large, potentially divergent, correlation length. In other words, we want to apply the RG transformations when the system is brought near criticality, where it undergoes a continuous, second order phase transition. We proceed as follows.

We first divide the system into blocks of adjacent spins. Then, for each of the individual blocks, we average over the spins within and replace them by a single *blocked* spin. In this way, we end up with a system described in terms of fewer d.o.f (*i.e.* the blocked spins), which now lie further apart from one another. In this context, the coarse graining bears the name of *Kadanoff blocking* or, simply, *blocking*.

Now, the power of the RG procedure, as emphasised in the previous section, lies in the fact that this procedure can be iterated. This, however, poses another problem. We have two descriptions of the same physical system, *i.e.* the original one and the one in terms of the blocked variables, and we want to compare the two. Unfortunately, as emphasised above, the blocking procedure modifies the lattice spacing, and hence there is no common ground left for us to draw any conclusions. Nonetheless, this can be solved by performing a second complementary step, which amounts to rescaling the lattice spacing back to its original size after coarse graining.

The two steps outlined above make up the celebrated *block-spin* renormalisation originally introduced by Kadanoff [31]. And, in fact, any RG transformation has a similar structure: a coarse graining followed by a rescaling.

At this stage it is natural to ask ourselves what have we achieved by doing this. First of all, the original set of spins has been replaced by a smaller set of blocked spins. Since the latter have been obtained by averaging over the former, we inevitably end up modifying the strength of the corresponding spin-spin interactions. Moreover, despite the fact the the original spins interact with their adjacent neighbours only, the blocked spins can in principle exhibit all possible interactions (*i.e.* nearest neighbour interactions, next-to-nearest neighbour interactions and so on). At first sight, this seems at odds with the assumed locality which makes the RG procedure possible to begin with. However, the long range interactions generated are suppressed provided that the blocking procedure is performed in a sensible way, and thus locality survives. For the ease of exposure, we defer a discussion of the various notions of locality and their corresponding implementation to Chapter 2, where we present them in the context of gauge theories.

Furthermore, from the discussion above, one can easily see that the system described in terms of blocked spins looks rather different than the original one. However, despite all these rather technical differences, physics remains the same, or more precisely all macroscopic observables (*i.e.* measured at length scales much larger than the cutoff scale Λ) are left invariant under a RG transformation. And, in fact, this still holds regardless of how the RG transformation is performed, as long as it is performed in a sensible manner. This property that IR physics does not depend on the exact microscopic details bears the name of *universality*. What is more, this property can be used to classify physical systems, and thus those systems whose microscopic description differs, but exhibit a similar behaviour in the IR, fall into the same *universality class*.

1.1.2 The Functional Renormalisation group

The modern understanding of the renormalisation group is due to the pioneering work of Kenneth Wilson [32]. As emphasised before, in contrast with the RG picture developed by Kadanoff, where the cutoff is modified in discrete steps, the framework envisaged by Wilson can successfully account for continuous (infinitesimal) changes in the cutoff, and thus proves to be suitable for investigating field theories. Consequently, it comes as no surprise that the renormalisation group is commonly referred as the Continuous Renormalisation Group (continuous RG) or the Wilsonian Renormalisation Group (WRG). Furthermore, for reasons which will become evident throughout this subsection, it also bears the name of Exact Renormalisation Group (ERG) or Functional Renormalisation Group (FRG) [33]. We use all these terms interchangeably in this chapter and in the remainder of the thesis.

Before proceeding further with the explicit construction of the ERG procedure, for clarity, it is worth fixing the notation and outlining the different conventions that we use here and in the rest of the thesis. Thus, for position and momentum space integrals

we introduce the shorthand

$$\int d^d x \equiv \int_x , \qquad \int \frac{d^d p}{(2\pi)^d} \equiv \int_p , \tag{1.1.3}$$

respectively, where d is the number of dimensions we are working in. We will frequently switch between position and momentum space representations and this will help us keep everything clean and concise. For similar reasons we will adopt the following convention:

$$\delta(p) \equiv (2\pi)^d \delta^{(d)}(p) , \tag{1.1.4}$$

where $\delta^{(d)}(p)$ is the standard d-dimensional Dirac delta function. Our convention for Fourier transforms is then

$$\phi(p) = \int_x \phi(x)\, e^{-ipx} . \tag{1.1.5}$$

We use the compact DeWitt notation in which repeated indices stand for both a summation over discrete indices and an integration over spacetime. For expressions involving a single component field $\phi(x)$, we adopt the following shorthand together with its explicit representations in both position and momentum space, respectively:

$$J \cdot \phi \equiv \int_x J(x)\phi(x) = \int_p J(-p)\phi(p) . \tag{1.1.6}$$

In addition to this, for two functions $f(x)$ and $g(y)$, and a momentum kernel $K(p^2/\Lambda^2)$, we introduce the shorthand:

$$f \cdot K \cdot g \equiv \int_{x,y} f(x)K(x,y)g(y) = \int_x f(x) K\left(-\partial^2/\Lambda^2\right) g(x) , \tag{1.1.7}$$

where $K(x,y)$ is given by:

$$K(x,y) = K\left(-\partial^2/\Lambda^2\right)\delta(x-y) = \int_p K\left(p^2/\Lambda^2\right)\, e^{ip(x-y)} . \tag{1.1.8}$$

Note again that Λ is the effective cutoff scale. Last, but not least, one more requirement that we impose is Euclidean signature. This allows us to construct a well-defined blocking procedure. More precisely, it ensures that patches which are light-like separated can not be blocked together, and hence that we do not average over arbitrary spatial separation.

All this being said, we can now return to our original task of applying the RG ideas to study continuous systems. The central element of the Wilsonian framework is the distinction between high and low energy modes. For this reason, we choose to work mostly in momentum space, and not in position space as we did in the previous section. In addition to this, for simplicity, we consider single component scalar field theory. All the arguments presented here are general and can be translated straightforwardly to more complicated systems. We will explicitly see how this happens for gauge theories

(and gravity) in chapter 2, multi-component scalar field theory in Chapter 3 and gravity in Chapter 4.

The discussion in the remaining of this and the next section is based on ref. [34]. We start with what is generally regarded as the main object of interest in a quantum field theory, the partition function $Z[J]$. For a single scalar field $\phi(x)$, it takes the following form:

$$Z[J] = \int \mathcal{D}\phi \, e^{-S^{\text{tot}}_{\Lambda_0}[\phi] + J\cdot\phi} \,, \tag{1.1.9}$$

where we also include a source term for the scalar field in the usual way. Note that the above equation holds up to some constant prefactor, which ensures proper normalisation. Here and in general we drop these prefactors since their presence or absence does not interfere with physics (*i.e.* all Green functions remain unchanged). Equivalently, we can just set $Z[J=0]=1$. Also note that the integrand above is defined at some original scale Λ_0, which we call the *bare* scale.

Now, as emphasised at the start of this section, the strength of the ERG lies in the fact that the coarse graining procedure can be iterated. Thus, we first integrate over all the modes between the starting bare scale Λ_0 and a lower effective scale Λ, and then repeat this process until we integrate over all the d.o.f. The net effect of any RG step is to replace the original bare action $S^{\text{tot}}_{\Lambda_0}$ by an effective action S^{tot}_{Λ}. The latter is in general more complicated as a consequence of the fact that coarse graining, similarly to the blocking procedure described in the previous subsection, modifies the strength of the original interactions, and alongside gives rise to new ones. However, the crucial point is that all the underlying physics stays the same. This, together with the fact that both Λ_0 and Λ are put in by hand, and hence carry no physics, implies that the partition function has to be scale invariant:

$$\Lambda\partial_\Lambda Z[J] = 0 \,. \tag{1.1.10}$$

We can make more concrete statements about this if we first split ϕ into high and low energy modes, respectively, *i.e.*

$$\phi = \phi_H + \phi_L \,, \tag{1.1.11}$$

with ϕ_H carrying momenta $|p| > \Lambda$, and ϕ_L momenta $|p| \leq \Lambda$. We can then use this to recast the partition function (1.1.9) as follows:

$$Z[J] = \int \mathcal{D}\phi_H \mathcal{D}\phi_L \, e^{-S^{\text{tot}}_{\Lambda_0}[\phi_H+\phi_L] + J\cdot(\phi_H+\phi_L)} \tag{1.1.12}$$

$$= \int \mathcal{D}\phi_L \, e^{J\cdot\phi_L} \int \mathcal{D}\phi_H e^{-S^{\text{tot}}_{\Lambda_0}[\phi_H+\phi_L] + J\cdot\phi_H} \,. \tag{1.1.13}$$

Performing the second integral above over the high energy modes, and taking into account (1.1.10), we arrive at the following identity:

$$Z[J] = \int \mathcal{D}\phi_L \, e^{-S_\Lambda^{\rm tot}[\phi_L]+J\cdot\phi_L} \,, \tag{1.1.14}$$

where the effective action $S_\Lambda^{\rm tot}[\phi_L]$ is given by:

$$S_\Lambda^{\rm tot}[\phi_L] = -\ln \int \mathcal{D}\phi_H \, e^{-S_{\Lambda_0}^{\rm tot}[\phi_H+\phi_L]+J\cdot\phi_H} \,. \tag{1.1.15}$$

This means that the information pertaining to the high energy modes is encoded within the effective action $S_\Lambda^{\rm tot}$. Once all the modes have been integrated out, we can make physical predictions. Or, in other words, observables are to be computed from the effective action $S_\Lambda^{\rm tot}$ in the limit $\Lambda \to 0$. Figure 1.1.2 below summarises these ideas.

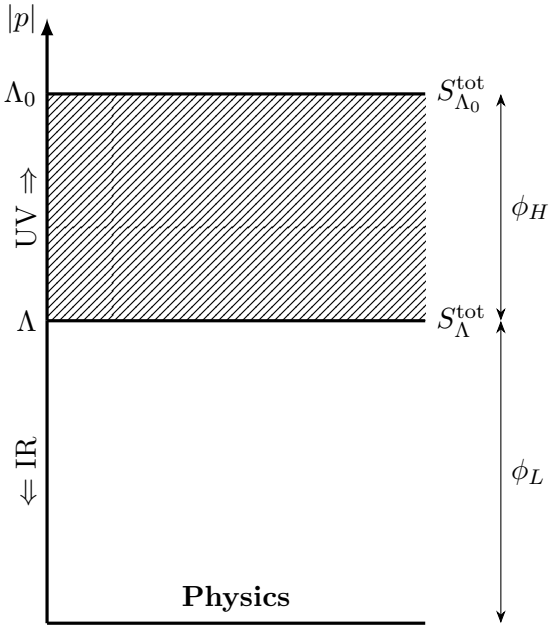

FIGURE 1.1.2: The effective action $S_\Lambda^{\rm tot}$ encodes the effects of integrating out the high energy modes ϕ_H with momenta between the original bare scale Λ_0 and Λ.

Kadanoff's approach reviewed in the previous subsection involved a coarse graining procedure followed by a rescaling. For the continuous RG case, we have seen how the former can be implemented, but we have swept under the rug the latter. In direct analogy to block-spin renormalisation, we should restore the effective scale Λ to its original value Λ_0. The easy way to do this is by simply rendering all the fields and couplings dimensionless using Λ. In other words, one just needs to divide all the quantities by Λ raised to the appropriate power (*i.e.* the scaling dimension). We assume this to be the case everywhere in this chapter.

Up to this point, we have established that the effective action is arguably the most important object within ERG. Understanding its behaviour and properties, ultimately

translates to understanding the physical properties of the system it describes. As highlighted above, we want to be able to track how it evolves when the effective scale Λ changes since, eventually, we will need to do this all the way down into deep IR (*i.e.* $\Lambda \to 0$) to compute the complete set of Green functions. To achieve this, we will recast the partition function (1.1.9) as a flow equation for the effective action. We proceed as follows.

We first rewrite the bare action $S^{\text{tot}}_{\Lambda_0}$ as a sum between a kinetic term and an interaction term, respectively, *i.e.*

$$S^{\text{tot}}_{\Lambda_0} = \frac{1}{2}\phi \cdot p^2 \cdot \phi + S_{\Lambda_0}[\phi]\,. \tag{1.1.16}$$

With this modification, the partition function (1.1.9) becomes:

$$Z[J] = \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi\cdot p^2\cdot\phi - S_{\Lambda_0}[\phi] + J\cdot\phi}\,. \tag{1.1.17}$$

We now introduce the effective cutoff scale using two smooth cutoff profiles $C_{UV}(p,\Lambda)$ and $C_{IR}(p,\Lambda)$, respectively, which add up to unity, *i.e.*

$$C_{UV}(p,\Lambda) + C_{IR}(p,\Lambda) = 1\,. \tag{1.1.18}$$

Furthermore, we impose $C_{UV}(p,\Lambda)$ to act as an UV cutoff, *i.e.*

$$C_{UV}(0,\Lambda) = 1\,, \quad C_{UV}(p,\Lambda) \to 0 \text{ (sufficiently fast) as } |p| \to \infty\,. \tag{1.1.19}$$

This, together with the relation (1.1.18), constrains $C_{IR}(p,\Lambda)$ to behave as follows:

$$C_{IR}(0,\Lambda) = 0\,, \quad C_{UV}(p,\Lambda) \to 1 \text{ as } |p| \to \infty\,. \tag{1.1.20}$$

Using these two profiles we can define regularised propagators:

$$\Delta_{UV} \equiv \frac{C_{UV}(p,\Lambda)}{p^2}\,, \quad \Delta_{IR} \equiv \frac{C_{IR}(p,\Lambda)}{p^2}\,, \tag{1.1.21}$$

which are related with the original propagators Δ in the following way:

$$\Delta \equiv \frac{1}{p^2} = \Delta_{UV} + \Delta_{IR}\,. \tag{1.1.22}$$

Using the regularised propagators in (1.1.21) together with the split (1.1.11), we can recast (1.1.17) in the following way:

$$Z[J] = \int \mathcal{D}\phi_L \mathcal{D}\phi_H \ \exp\Big\{ -\frac{1}{2}\phi_L \cdot \Delta^{-1}_{UV} \cdot \phi_L - \frac{1}{2}\phi_H \cdot \Delta^{-1}_{IR} \cdot \phi_H \\ - S_{\Lambda_0}[\phi_H + \phi_L] + J \cdot (\phi_H + \phi_L) \Big\}\,. \tag{1.1.23}$$

This can be easily checked. We first write $\phi_H = \phi - \phi_L$ in the above expression, then perform the shift $\phi_L \to \phi_L + (\Delta_{UV}/\Delta)\cdot\phi$, compute the integral over ϕ_L, which at this stage is just a gaussian integral, and finally discard the resulting constant term to arrive back at (1.1.17). On a side note, in the light of the structure of the expression above, we can still interpret ϕ_H and ϕ_L as high energy and low energy modes, respectively. However, given that the effective scale is introduced using the smooth cutoff profiles (1.1.19) and (1.1.20), and not as a sharp cutoff as before, the boundary between the two sets of modes can not be drawn exactly at the scale Λ anymore. Thus, one needs to take this interpretation with a pinch of salt.

The next step is to integrate out the high energy modes to get the effective action. For this reason, we rewrite the identity (1.1.23) as follows:

$$Z[J] = \int \mathcal{D}\phi_L \ \exp\left\{-\frac{1}{2}\phi_L\cdot\Delta_{UV}^{-1}\cdot\phi_L\right\} Z_\Lambda[J,\phi_L]\,, \tag{1.1.24}$$

where $Z_\Lambda[J,\phi_L]$ is given by:

$$Z_\Lambda[J,\phi_L] = \int \mathcal{D}\phi_H \ \exp\left\{-\frac{1}{2}\phi_H\cdot\Delta_{IR}^{-1}\cdot\phi_H - S_{\Lambda_0}[\phi_H+\phi_L] + J\cdot(\phi_H+\phi_L)\right\}. \tag{1.1.25}$$

If we now write $\phi_H = \phi - \phi_L$, the above identity becomes:

$$\begin{aligned} Z_\Lambda[J,\phi_L] = \exp&\left\{-\frac{1}{2}\phi_L\cdot\Delta_{IR}^{-1}\cdot\phi_L\right\} \\ &\times\int \mathcal{D}\phi \ \exp\left\{-\frac{1}{2}\phi\cdot\Delta_{IR}^{-1}\cdot\phi + \phi\cdot\left(J+\Delta_{IR}^{-1}\cdot\phi_L\right) - S_{\Lambda_0}[\phi]\right\}. \end{aligned} \tag{1.1.26}$$

We can perform the gaussian integral over ϕ provided that we pull the bare action term in front of the integral beforehand, and hence we obtain [35]:

$$\begin{aligned} Z_\Lambda[J,\phi_L] = \exp&\left\{\frac{1}{2}J\cdot\Delta_{IR}^{-1}\cdot J + J\cdot\phi_L\right\} \\ &\times\exp\left\{-\frac{1}{2}\left(J+\Delta_{IR}^{-1}\cdot\phi_L\right)\cdot\Delta_{IR}\cdot\left(J+\Delta_{IR}^{-1}\cdot\phi_L\right)\right\} \\ &\times\exp\left\{-S_{\Lambda_0}\left[\frac{\delta}{\delta J}\right]\right\}\exp\left\{-\frac{1}{2}\left(J+\Delta_{IR}^{-1}\cdot\phi_L\right)\cdot\Delta_{IR}\cdot\left(J+\Delta_{IR}^{-1}\cdot\phi_L\right)\right\}, \end{aligned} \tag{1.1.27}$$

where the first two lines are just the first line of (1.1.26) rewritten in a more convenient way. Also, note that in the last line above the functional derivative will pull from the rightmost exponential either $\phi_L + J\cdot\Delta_{IR}$, or Δ_{IR}, after differentiation. This means that there exists some functional $S_\Lambda\left[\phi_L + J\cdot\Delta_{IR}\right]$, such that the following identity holds:

$$Z_\Lambda[J,\phi_L] = \exp\left\{\frac{1}{2}J\cdot\Delta_{IR}\cdot J + J\cdot\phi_L - S_\Lambda\left[\phi_L + J\cdot\Delta_{IR}\right]\right\}. \tag{1.1.28}$$

At this point it is difficult to tell precisely what the meaning of S_Λ really is. However, we can still make progress if we slightly alter our computations. In the equation (1.1.23) we have the freedom to source the low energy modes only. Doing so, and following the same path as above, instead of (1.1.28) we arrive at the following:

$$Z_\Lambda[J,\phi_L] = \exp\left\{-S_\Lambda[\phi_L] + J\cdot\phi_L\right\}\,, \tag{1.1.29}$$

which is the same as setting $J\cdot\Delta_{IR}=0$ in (1.1.28). From this and (1.1.24) it follows that the full partition function $Z[J]$ is given by:

$$Z[J] = \int \mathcal{D}\phi_L \exp\left\{-\frac{1}{2}\phi_L\cdot\Delta_{UV}^{-1}\cdot\phi_L - S_\Lambda[\phi_L] + J\cdot\phi_L\right\}\,. \tag{1.1.30}$$

Here we see that the restriction we previously imposed on J pays off. The above equation allows us to identify S_Λ as the interaction part of the full effective action S_Λ^{tot}, provided that we choose to write the latter similarly to (1.1.16), *i.e.*

$$S_\Lambda^{\text{tot}}[\phi_L] = \frac{1}{2}\phi_L\cdot\Delta_{UV}^{-1}\cdot\phi_L + S_\Lambda[\phi_L]\,. \tag{1.1.31}$$

It is worth emphasising that, although we write the effective action as a sum between a (regularised) kinetic term and an interaction term, the truth is slightly fuzzier. The effective action is in general more complicated than the original bare action it originates from. The coarse graining procedure generates new interactions which can in fact spoil the bilinear kinetic term. However, if we do not exclude the possibility that the effective action might exhibit interactions at the bilinear level already, then we can always write down a relation similar to (1.1.31) above.

The FRG flow equation can be obtained by first differentiating (1.1.25) with respect to Λ:

$$\frac{\partial}{\partial\Lambda}Z_\Lambda[J,\phi_L] = -\frac{1}{2}\left(\frac{\delta}{\delta J}-\phi_L\right)\cdot\left(\frac{\partial}{\partial\Lambda}\Delta_{IR}^{-1}\right)\cdot\left(\frac{\delta}{\delta J}-\phi_L\right)Z_\Lambda[J,\phi_L]\,, \tag{1.1.32}$$

and, then, substituting in (1.1.28) to obtain:

$$\frac{\partial}{\partial\Lambda}S_\Lambda[\phi] = \frac{1}{2}\frac{\delta S_\Lambda}{\delta\phi}\cdot\frac{\partial\Delta_{UV}}{\partial\Lambda}\cdot\frac{\delta S_\Lambda}{\delta\phi} - \frac{1}{2}\frac{\delta}{\delta\phi}\cdot\frac{\partial\Delta_{UV}}{\partial\Lambda}\cdot\frac{\delta S_\Lambda}{\delta\phi}\,. \tag{1.1.33}$$

This is the ERG flow equation for the effective action, known as Polchinski's equation [36]. To understand its structure better we can write the effective action in terms of its vertices and represent it graphically as in Fig. 1.1.3 below. We can depict the two terms as the tree level and one-loop contributions, respectively, and one could be tempted to interpret them as the classical and quantum parts, respectively. However, this is misleading since it is the effective action S_Λ present in both terms which encompasses all the quantum corrections.

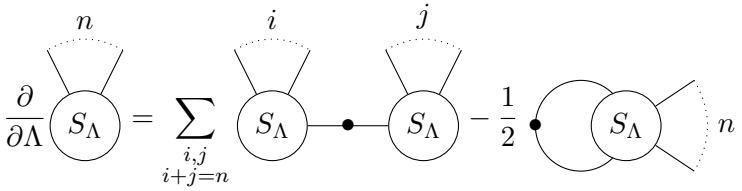

FIGURE 1.1.3: Diagrammatic representation of Polchniski's flow equation

An important feature of the flow equation above is that it is an exact reformulation of the partition function (hence the term ERG) as the flow equation of some functional, which turns out to be the effective action which lies at the heart of the renormalisation group (hence the term FRG) [37]. Although exact, in practice one can not solve it exactly and has to rely on approximations in the hope of gaining some insight. One such class of approximations are truncations, which amount to enforcing that the effective action contains only a finite number of operators. One such truncation is the Local Potential Approximation (LPA), which we use in Chapter 3 to investigate scalar field theories. Another widely employed method for solving FRG flow equations is perturbation theory, which generally requires the existence of a small parameter which one can potentially expand into. We follow this path to investigate the quantum structure of gravity in Chapter 4. One disadvantage of this approach is that it is computationally very expensive, and thus very difficult to track higher order contributions. Moreover, there are non-perturbative aspects which cannot be accounted for in perturbation theory, and hence one cannot rely exclusively on the latter.

1.1.3 Renormalisability

One other crucial insight due to the pioneering work of Wilson outlined above is related to renormalisability. Prior to the development of Wilsonian RG, the idea of renormalisability was intimately linked with perturbation theory and relied on the existence of a small parameter, which one could use as a starting point for a loop expansion of the action. However, as is well known, if we attempt to compute loop integrals, we eventually encounter infinities. To handle them, we need to introduce in one way or another a cutoff Λ_0. This is just a mathematical trick commonly used to deal with improper integrals, which enables one to isolate infinities (*i.e.* regularise) and rewrite them in terms of the cutoff. In doing so, all the quantities of interest pick a unphysical Λ_0 dependence, which ultimately has to be removed since the cutoff carries no physical meaning.

Perturbative renormalisation amounts to the possibility of safely removing the cutoff without altering the physics. Additionally, one needs to make sure that all physical obervables remain finite and carry no Λ_0 dependence anymore. This is equivalent to saying that all physical quantities are *renormalised*. The most common way to achieve

this is by introducing a finite number of counterterms to cancel divergences at all orders in the loop expansion (*i.e. renormalised perturbation theory*). Another way to proceed about this, more demanding computationally and less intuitive, is by rewriting bare quantities in terms of renormalised ones (*i.e. bare perturbation theory*).

Wilson's approach offers a more comprehensive perspective. It allows us to define renormalisability without referring to perturbation theory or to any small coupling whatsoever. The essential ingredient for this to be possible amounts to a shift in paradigm. Contrary to the view employed before that the cutoff scale Λ_0 is not physically meaningful, in Wilsonian RG the cutoff carries some physical meaning. All theories are to be viewed as effective theories, hence, not valid at all scales, but valid up to some scale set by the cutoff Λ_0. In other words, the latter comes into game to humble us and show the limitation of our knowledge. However, as emphasised before, even if our understanding fails at the cutoff, physics does not, and thus one has to venture past the boundary set by Λ_0. And it is for this reason that we seek QFTs in which we are able to take the *continuum limit* (*i.e.* $\Lambda_0 \to \infty$) safely, without encountering any divergences. The possibility to do so is one of the many incarnations of *non-perturbative renormalisability*. We will see in the next section how this is realised at the technical level, when discussing the fixed point (FP) structure of the theory space.

One of the advantages of the framework envisaged by Wilson is that it enables us to treat both perturbative and non-perturbative renormalisation on equal footing. From the definition of the latter given above, one can easily see that the former is just a realisation of the latter under some specific conditions. We mentioned before that these two approaches can reach different conclusions for the same physical system, and thus relying solely on perturbative RG can be misleading. One such example is given by quantum electrodyanmics (QED). This theory is perturbatively renormalisable in the sense that we can get rid of divergences using a finite number of counterterms order by order in a loop expansion. This is valid only at low energies when the coupling is small, and thus we can treat it perturbatively. However, at high energies, QED becomes strongly coupled, and the perturbative series developed in the low energy regime develop a Landau pole [38–40]. This means that QED is not truly renormalisable since its continuum limit, from the point of view of perturbation theory, it is not well-defined. More generally, this shows that the action of taking the continuum limit $\Lambda \to \infty$ in perturbation theory can be misleading. For this reason, from here onward, unless otherwise stated, we will only refer to non-perturbative renormalisation in the Wilsonian sense as renormalisation since this is equivalent to a well-defined continuum limit.

1.2 Theory space and Fixed Points

We have outlined in the previous section the ideas which represent the backbone of the RG approach to physical systems. It all boils down to understanding the behaviour of the effective action S_Λ. This means that we need to understand the solution space of the FRG flow equation (1.1.33) (or of any other equivalent FRG flow equation).

Any effective action can be parameterised as follows:

$$S_\Lambda[\phi] = \sum_{i=1}^{\infty} g_i(\Lambda) O_i[\phi] \,, \tag{1.2.1}$$

where the scale dependence is carried by the dimensionless couplings g_i, whilst their conjugate operators O_i carry the dependence on the dimensionless field ϕ. Note that the sum above is infinite. The reason behind this is that, no matter how simple we choose the bare action to be, the effective action will exhibit all possible interactions allowed by symmetry constraints and field content. Of course, not all the couplings will be independent of one another, but we still have to account for all of them. Since the above identity holds for any effective action S_Λ, we can interpret the operators O_i as the basis of an infinite dimensional space, which we will call the *theory space*, and the couplings g_i as the corresponding coordinates in this basis. Thus, any action can be represented as a point within theory space.

From this it follows straightforwardly that a RG transformation corresponds to a trajectory within theory space, which starts at the point where the original bare action lies, and ends at the point corresponding to the coarse grained action. This means that any QFT should be identified with a full RG trajectory. We should emphasise that it is the full (infinite) collection of effective actions at various degrees of coarse graining that defines a physical theory, and not just one particular action. Also, note that RG trajectories are conventionally regarded as flowing from UV to IR since this is the natural direction dictated by coarse graining.

All RG trajectories are generated by the corresponding RG flow equations. In geometric terms, they are the integral curves generated by the vector field induced within theory space by the flow equation. Generally, we can schematically write the latter as follows:

$$\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda = \mathcal{F}\left[S_\Lambda, \frac{\delta S_\Lambda}{\delta \phi}, \frac{\delta^2 S_\Lambda}{\delta \phi \delta \phi}, \cdots \right] , \tag{1.2.2}$$

where $\mathcal{F}$ stands for some functional dependence on the effective action and its functional derivatives [37, 41]. One such example is Polchinski's equation (1.1.33) derived above.

We ultimately want to study renormalisability, which, as outlined in the previous subsection, amounts to scale invariance. This motivates us to first look at the fixed point structure of the RG flow. *Fixed points* (FPs) are stationary points of (1.2.2), which

means that the following identity holds:

$$\Lambda \frac{\partial}{\partial \Lambda} S_* = 0\,, \tag{1.2.3}$$

where S_* is the FP action. All theories defined at a FP are trivially renormalisable. This is true because all the couplings and fields in the effective action being dimensionless, the above equation implies full scale independence, and hence renormalisability. Moreover, it represents a *massless continuum limit*. The presence of any dimensionful parameter (*i.e.* like Λ_0 or the mass) would break scale independence, and thus would be at odds with (1.2.3). One such fixed point is the *Gaussian fixed point* (GFP), which exists in any QFT. Also, it is achieved in the limit that all couplings vanish, and for this reason, although almost never explicitly stated, renormalisability with respect to the GFP is equivalent to perturbative renormalisability. Moreover, any theory defined at the GFP is *trivial* (*i.e.* non-interacting).

All bare actions which flow towards the same FP under an RG flow lie on the *critical manifold*. We can get more precise about this. In the vicinity of a FP we can write the effective action S_Λ as follows:

$$S_\Lambda[\phi] = S_*[\phi] + \sum_i \alpha_i \left(\frac{\mu}{\Lambda}\right)^{\lambda_i} \mathcal{O}_i[\phi]\,. \tag{1.2.4}$$

Here μ is the usual arbitrary mass scale, α_i are positive dimensionless constants and λ_i are the RG eigenvalues of the eigenoperators $\mathcal{O}_i$. Depending on the sign of the eigenvalues, the operators can be classified into relevant ($\lambda_i > 0$), marginal ($\lambda_i = 0$) or irrelevant ($\lambda_i < 0$). A direct consequence of this is that perturbations in any of the irrelevant directions will inevitably sink back into the FP. Moreover, it follows that the irrelevant and marginally irrelevant eigenoperators form the basis of the critical manifold.

To see how this unfolds, we can rewrite (1.2.4) as follows:

$$S_\Lambda[\phi] = S_*[\phi] + \sum_{\lambda_i \leq 0} \alpha_i \left(\frac{\mu}{\Lambda}\right)^{\lambda_i} \mathcal{O}_i^{\mathrm{irr}}[\phi] + \sum_{\lambda_i \geq 0} \alpha_i \left(\frac{\mu}{\Lambda}\right)^{\lambda_i} \mathcal{O}_i^{\mathrm{rel}}[\phi]\,. \tag{1.2.5}$$

The second term on the right-hand side contains all the irrelevant and marginally irrelevant operators, whilst the third one all the relevant and marginally relevant operators. As outlined above, the former describes perturbations within the critical manifold. The remaining set of operators quantify perturbations off the critical manifold. And such a trajectory which corresponds to perturbing along one of the relevant directions (or marginally relevant) is called *renormalised trajectory* (RT).

All effective actions which lie on the renormalised trajectory are called *perfect actions*. One can easily see that any theory, whose effective actions are of this kind, must be

renormalisable since their UV behaviour is controlled by the FP the RT emanates from. Moreover, such theories are interacting. Or, better said, given that the fixed point supports renormalised trajectories, from a free theory in the UV we uncover an interacting theory as we flow into the IR.

At this point it is worth pausing to look at the big picture so far. Renormalisability is intimately linked with the existence of fixed points. All theories defined at a fixed point are trivially renormalisable and represent massless non-interacting theories. Any fixed point has a critical manifold spanned by the irrelevant (and marginally irrelevant) eigenperturbations, and thus the FP acts as a sink for all the actions which lie on this manifold. If the FP supports relevant directions, then we can use these so-called renormalised trajectories to construct renormalisable interacting theories. The only question left unanswered is what happens when we start with a bare action which neither lies on the critical manifold, nor on a renormalised trajectory.

Let us suppose that we start with such an (bare) action which sits slightly off the critical manifold. Also, for the ease of exposure, we shall assume that there exists only one relevant direction. The flow will first head towards the fixed point since the irrelevant operators (and marginally irrelevant ones) dominate in the early stages of the flow. At some point, the relevant direction becomes the dominant one, and correspondingly the flow will head in the same direction as the renormalised trajectory. However, this does not have a well-defined continuum limit since there is no FP to control the UV behaviour.

This problem can be solved if we now tune the original action back onto the critical manifold. In this limit, the flow splits, and we end up with an initial path which sinks back into the FP and a second one which goes out of the FP along the RT. To be able to do so, we have to account for the diverging couplings. This can be done if, parallel to bringing the bare action back onto the critical manifold, we express all the quantities of interest in terms of renormalised couplings $g_i(\Lambda)$ as follows:

$$g_i(\Lambda) \sim \alpha_i \left(\frac{\mu}{\Lambda}\right)^{\lambda_i} \qquad \text{as } \Lambda \to \infty \,. \tag{1.2.6}$$

This ensures that the RT is properly parameterised (*i.e.* in terms of renormalised quantities) and that the deep IR limit of the flow corresponds to the effective action of a non-trivial renormalisable QFT.

There are a few things worth highlighting before proceeding further. First of all, note that the FRG framework allows us to search for interacting renormalisable theories without specifying a bare action or a bare scale. As presented above, for any flow the important part is the one which coincides with a renormalised trajectory. The existence of the latter is equivalent to the existence of a non-trivial continuum limit. Furthermore, since the RT can be fully described using renormalised quantities, one can directly work

in the continuum (*i.e.* in terms of renormalised quantities). This represents one of the big advantages of the continuous RG framework.

Another important conclusion that can be drawn is that the effective action S_Λ can be recast in *self-similar* form, *i.e.*

$$S_\Lambda[\phi] = S[\phi]\Big(g_1(\Lambda), g_2(\Lambda), \cdots, g_n(\Lambda)\Big)\,. \tag{1.2.7}$$

This means that all the scale dependence is carried by the (renormalised) couplings, and that the form of the action itself does not change. Moreover, one can show starting from the discussion above that self-similarity is simply another incarnation of (non-perturbative) renormalisability [34].

1.3 The effective average action

We turn our attention now to a reformulation of the RG ideas, in particular, to the flow equation of the *effective average action* Γ_Λ, one of the most commonly used flow equations in practical computations. In the rest of this subsection we will outline how one can derive it, and in the process we will point out its main advantages. Prior to that, note that Γ_Λ is the IR regulated generator of one-particle irreducible (1PI) diagrams. This may come as a surprise given that we have emphasised in the previous sections the need for UV finiteness. Nonetheless, one can get an intuition for this as follows. We start by integrating out high-energy modes between two scales, Λ_0 and Λ, respectively, with $\Lambda_0 > \Lambda$. This means that Λ acts as an IR cutoff for the integrated (high-energy) modes, or equivalently as an UV cutoff for the remaining (low-energy) modes [34]. The latter perspective is the one used in Wilsonian RG and discussed at large in the previous sections. However, adopting the former stance (*i.e.* Λ as an IR cutoff), will allow us to construct a flow equation for Γ_Λ which successfully interpolates between UV and IR, and is free of divergences.

For the ease of exposure, we continue to work within single component scalar field theory. The starting point in deriving the flow equation for the effective average action $\Gamma_\Lambda[\varphi]$ is the usual partition function (1.1.9) supplemented by a source term. However, for the present purpose, we choose to write it as follows:

$$Z[J] = \int^{\Lambda_0} \mathcal{D}\phi\, e^{-S[\phi]+J\cdot\phi}\,, \tag{1.3.1}$$

where $S[\phi]$ is the action. Also, similarly to (1.1.9) before, note that the above integral is subjected to some overall UV cutoff Λ_0, which is kept here as a superscript. Our aim is to introduce a scale dependence into (1.3.1), and, at the same time, some IR

regularisation for the high-energy modes. Hence, we choose to modify (1.3.1) as follows:

$$Z_\Lambda[J] = \int^{\Lambda_0} \mathcal{D}\phi\, e^{-S[\phi]-\Delta S_\Lambda[\phi]+J\cdot\phi}\,, \tag{1.3.2}$$

where the extra regulator term introduced, $\Delta S_\Lambda[\phi]$, takes the following form:

$$\Delta S_\Lambda[\phi] \equiv \frac{1}{2}\phi\cdot R_\Lambda\cdot\phi \tag{1.3.3}$$

$$= \frac{1}{2}\int_p \phi(-p) R_\Lambda\left(p^2\right)\phi(p)\,. \tag{1.3.4}$$

This has the structure of a momentum-dependent mass-like term. The reason behind this choice is quite simple. We want to selectively modify the propagators such that only high-energy degrees of freedom are integrated over, whilst low-energy modes are suppressed in (1.3.2). This can be achieved if we impose on the IR cutoff operator R_Λ, which is a function of the laplacian (*e.g.* $R_\Lambda = R_\Lambda\left(-\partial^2\right)$ for scalar field theory), the following requirements:

1. IR regularisation (*i.e.* screen modes with $p^2 \ll \Lambda^2$):

$$\lim_{p^2/\Lambda^2\to 0} R_\Lambda(p^2) > 0\,. \tag{1.3.5}$$

2. Regulator does not modify UV modes (*i.e.* modes with $p^2 \gg \Lambda^2$ are integrated over in (1.3.2) in the usual way):

$$\lim_{p^2/\Lambda^2\to \infty} R_\Lambda(p^2) = 0\,. \tag{1.3.6}$$

Also, note that the second property above ensures that the regulator term vanishes in the limit $\Lambda \to 0$, and thus the original partition function is recovered:

$$\lim_{\Lambda\to 0} Z_\Lambda[J] = Z[J]\,. \tag{1.3.7}$$

Two frequently used regulators are the optimised cutoff [42–44]:

$$R_\Lambda(p^2) = (\Lambda^2 - p^2)\Theta(\Lambda^2 - p^2)\,, \tag{1.3.8}$$

where Θ is the Heaviside step function, and the exponential cutoff:

$$R_\Lambda(p^2) = \frac{\frac{p^2}{\Lambda^2}}{e^{\frac{p^2}{\Lambda^2}} - 1}\,, \tag{1.3.9}$$

respectively. The former is often used for analytical computations, whereas the latter is more commonly used for numerical computations.

Prior to introducing any IR regularisation, the Wilsonian effective action $W[J]$, the generator of connected Green functions, is related to the partition function in the following way:

$$Z[J] = e^{W[J]}\,. \tag{1.3.10}$$

This, in turn, can be used to construct the (quantum) effective action $\Gamma[\varphi]$, the generator of 1PI diagrams, via a Legendre transform, *i.e.*

$$\Gamma[\varphi] = W[J] - J\cdot\varphi\,, \tag{1.3.11}$$

where the classical field $\varphi(x)$ is given by:

$$\varphi(x) \equiv \phi^c(x) = \langle\phi(x)\rangle = \frac{\delta W[J]}{J(x)}\,. \tag{1.3.12}$$

Now, with the IR regularisation in place, the equation (1.3.10) remains virtually the same, *i.e.*

$$W_\Lambda^{\Lambda_0}[J] \equiv \ln Z_\Lambda[J]\,, \tag{1.3.13}$$

where $W_\Lambda^{\Lambda_0}[J]$ is the IR regulated generator of connected Green functions. However, the original definition of the quantum effective action has to be modified to account for the presence of the IR regulator ΔS_Λ. Thus, (1.3.11) above takes the form of a modified Legendre transformation:

$$\Gamma_\Lambda^{\Lambda_0}[\varphi] \equiv J\cdot\phi - W_\Lambda^{\Lambda_0}[J] - \Delta S_\Lambda[\phi]\,, \tag{1.3.14}$$

where the *classical field* $\varphi(x)$ is again the expectation value of the original field $\phi(x)$, *i.e.*

$$\varphi(x) \equiv \langle\phi(x)\rangle_J = \frac{\delta W_\Lambda^{\Lambda_0}[J]}{\delta J(x)}\,. \tag{1.3.15}$$

If we now take the derivative of (1.3.14) with respect to Λ and use (1.3.2), together with (1.3.13) and (1.3.15), we arrive at the following flow equation:

$$\partial_\Lambda\Gamma_\Lambda^{\Lambda_0}[\varphi] = \frac{1}{2}\mathrm{Tr}^{\Lambda_0}\left[\left(\frac{\delta^2\Gamma_\Lambda^{\Lambda_0}[\varphi]}{\delta\varphi\delta\varphi} + R_\Lambda\right)^{-1}\partial_\Lambda R_\Lambda\right]\,, \tag{1.3.16}$$

where the functional trace runs over momenta, and is subject to the same overall UV cutoff Λ_0 as the path integral (1.3.2) it was derived from. However, the true advantage of the above differential equation lies in its structure. The presence of the cutoff operator R_Λ in the first term on the RHS ensures that the full (quantum) propagator, *i.e.*

$$\frac{1}{\left(\Gamma_\Lambda^{\Lambda_0}\right)^{(2)} + R_\Lambda}\,, \tag{1.3.17}$$

where the Hessian take the following form:

$$\left(\Gamma_{\Lambda}^{\Lambda_0}\right)^{(2)} \equiv \frac{\delta^2 \Gamma_{\Lambda}^{\Lambda_0}}{\delta\varphi\delta\varphi}\,, \tag{1.3.18}$$

is well-behaved in the IR. After all, this is the original motivation for introducing the regulator term $\Delta S_\Lambda[\phi]$ in (1.3.2) to begin with. However, as it turns out, there is more to this than that. Due to (1.3.5) and (1.3.6), the second term on the RHS of (1.3.16) (*i.e.* the Λ derivative of R_Λ) will be sharply peaked around $p^2 \approx \Lambda^2$, and hence only these momenta will be integrated over in the flow equation above. This means that one can safely remove the original (bare) scale in (1.3.16) by taking the limit $\Lambda_0 \to \infty$. This yields the following:

$$\partial_\Lambda \Gamma_\Lambda[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_\Lambda[\varphi]}{\delta\varphi\delta\varphi} + R_\Lambda\right)^{-1}\partial_\Lambda R_\Lambda\right]\,. \tag{1.3.19}$$

The above equation is the so-called effective average action flow equation [35, 45, 46]. The regulator ensures that it is both UV and IR finite, and thus one can search for solutions directly in the continuum. Another property of this flow equation is that, by construction, it interpolates between UV (*i.e.* $\Gamma_{\Lambda=\Lambda_0} = S$) and IR (*i.e.* $\Gamma_{\Lambda\to 0} = \Gamma$). Similarly to the Polchinski equation (1.1.33), to recover physics, one has to integrate over all the modes (*i.e.* $\Lambda \to 0$) to recover the standard quantum effective action.

One last idea worth highlighting is that both Polchinski's equation (1.1.33) and the effective average action flow equation (1.3.19) are equivalent. With the following change of variables [34]:

$$S_\Lambda[\phi] = \Gamma_\Lambda[\varphi] + \frac{1}{2}(\varphi-\phi)\cdot\Delta_{IR}^{-1}\cdot(\varphi-\phi)\,, \tag{1.3.20}$$

one can recover (1.3.19) from (1.1.33). Both these equations are quite versatile, and picking one over the other is usually motivated by the precise details of the system being analysed.

1.4 Thesis outline

Each of the following three chapters investigates different aspects of the renormalisation group.

Chapter 2 is based on the work done in [1]. Here, we consider the possibility to formulate a manifestly gauge invariant flow equation. This has the advantage that it avoids gauge-fixing and the problems thereof, which makes it a suitable candidate for non-perturbative studies of gauge theories. In the process, we develop the machinery needed to study the proposal of ref. [47] rigorously, and show that the regularisation is at odds with gauge invariance.

In chapter 3 we present the work done in [2], in which we investigate asymptotically the scaling dimensions of operators in the LPA. We show that, within this approximation, it is possible to compute them analytically for scalar field theory and $O(N)$ invariant scalar field theory. Moreover, we show that the results are universal (*i.e.* independent of the choice of cutoff functions), subject only to some general weak constraints.

Chapter 4 presents the work done in [3], in which we use off-shell BRST methods together within the background field method [48–50] to study the perturbative renormalisability of pure gravity. We show that at each new loop order, the divergences that do not vanish on-shell are constructed from only the total metric, whilst those that vanish on-shell are renormalised by canonical transformations involving the quantum fields.

Finally, in the last chapter we summarise the results of each individual chapter and draw some concluding remarks regarding their significance in the context of the renormalisation group approach to gravity and gauge theories.

Chapter 2

Manifestly gauge invariant ERGs

2.1 Introduction

As emphasised above, understanding the Wilsonian renormalization group (RG) structure of quantum gravity is surely of importance, see *e.g.* [14, 51–59], and central to this is the role of diffeomorphism invariance. In this chapter we explore the possibility to generalise the exact RG [30, 60] to gravity, in such a way that it is manifestly diffeomorphism invariant. This would allow computations to be done whilst keeping exact diffeomorphism invariance at every stage (*i.e.* without gauge fixing), and, moreover, these computations would be background independent (*i.e.* performed without first choosing the spacetime manifold and background metric).

At the classical level this can be done [61]. However, in order to compute quantum corrections, extra ultraviolet regularisation has to be incorporated into the exact RG so that the integration is properly cut off in some diffeomorphism invariant way at the effective cutoff scale Λ. For the simpler case of gauge invariance in $SU(N)$ Yang-Mills theory this problem was solved by incorporating gauge invariant PV (Pauli-Villars) fields arising from a spontaneously broken $SU(N|N)$ gauge theory [28, 62–83]. It is not clear whether one can generalise such a scheme to gravity however, in particular, it is not clear what should play the role of $SU(N|N)$, although a kind of supergravity has been suggested in [84].

Recently a different approach to the problem of regularisation has been pursued in ref. [47]. Explicit PV fields are avoided and instead replaced by functional determinants which, if constructed as squares of simpler determinants, can be shown to work in the framework of standard perturbation theory [85–87]. Furthermore a geometric approach is followed where the determinants can be regarded as defining a regularised volume element on the orbit space of the gauge theory [87]. The flow equation is then formulated in a way that is manifestly invariant under field redefinitions, and applies equally well to both gravity and gauge theory [47].

Although this new proposal has elegant features, it is not immediately evident that the regularisation is successfully implemented in the flow equation once we delve into the details of the relevant Feynman diagrams, first at one loop and then also at higher loops. In the rest of this chapter we put the proposal to the test by constructing explicit expressions for the relevant vertices, and then carefully analyse their UV behaviour, as a function of loop momentum, in the form that they appear in quantum corrections. We focus on massless Yang-Mills since, if the regularisation fails in this case, then it most certainly fails for quantum gravity (given the latter's poor UV behaviour). Working with Yang-Mills also means we can take over methods used for these investigations in the earlier successful construction [28, 62–83]. As we will see, the proposal of ref. [47] unfortunately fails to fully regularise already at one loop, but in a rather subtle way, which in particular invalidates powerful techniques previously used to extract universal information [71, 88].

This chapter is organised as follows. In sec. 2.2 we review the proposal. In sec. 2.3 we sketch why the regularisation can fail, and in sec. 2.4 we build the machinery needed to rigorously test the structure of the renormalisation scheme at the perturbative level. Following that, in sec. 2.5 we confirm that the higher point classical vertices incorporate the assumed regularisation. Then, in sec. 2.6, we apply the techniques extensively to an analysis of the simplest one-loop correction, namely that for the effective action two-point vertex. We show that the regularisation is sufficient for the momentum independent part, giving a vanishing result as it should by gauge invariance, only if the Λ derivative of quadratically divergent constant part is discarded. The part that is second order in momentum ought to give the one-loop beta function, if properly regulated. However we show that the result cannot both be completely regularised and transverse. It can be taken to be transverse only if the Λ derivative of a linearly divergent part is discarded. Since this holds for all choices of covariantisation and cutoff profiles, it points to some inherent limitations in the structure of the proposed flow equation. In sec. 2.7 we summarise and draw our conclusions.

2.2 A proposal for a background independent exact RG

2.2.1 Basic ingredients for regularisation

The flow equation proposed in ref. [47] incorporates a geometric approach to the quantisation of gauge theories [89, 90]. As we will see, a detailed understanding of the UV properties can only be reached by working with explicit expressions for the vertices. In the case of Yang-Mills theories, the flow equations then take their simplest form if we regard the gauge fields as valued in the Lie algebra, *i.e.* contracted into the generators [62, 63]. Also, since we work in Euclidean signature, we will keep all gauge group

indices as superscripts and Lorentz indices as subscripts for convenience. Thus, the DeWitt compact notation and its explicit representation take here the following form:

$$\phi^a J_a \equiv \sum_a \int_x A^a_\mu(x) J^a_\mu(x)\,. \tag{2.2.1}$$

We work with the gauge group SU(N). We use DeWitt Latin indices from the start of the alphabet to label gauge fields, thus ϕ^a. As already mentioned, when we need more explicit expressions it will be convenient to regard the gauge fields $A_\mu(x)$ as contracted into the generators:

$$\phi^a \equiv A_\mu(x) = A^a_\mu(x) T^a\,. \tag{2.2.2}$$

With appropriate definitions for the vertex functions in either language, the expressions will of course be equal, and the two representations are, thus, equivalent. The generators T^a are taken to be hermitian, in the fundamental representation, and orthonormalised as

$$tr(T^a T^b) = \frac{1}{2}\delta^{ab}\,. \tag{2.2.3}$$

A second set of DeWitt Greek indices from the start of the alphabet is used to label gauge parameters; the map from the two languages is thus:

$$\epsilon^\alpha \equiv \omega(x) = \omega^a(x) T^a\,. \tag{2.2.4}$$

If we adopt the geometric approach of [89, 90] to the quantisation of gauge theories, we regard the fields ϕ^a as coordinates on an infinite dimensional 'manifold' Φ, the space of all possible field configurations. This has the structure of a fibre bundle, and the fibres are the gauge orbits $\mathcal{G}$. However, all the physics happens on the quotient space $\Phi/\mathcal{G}$, where each point belongs to a unique equivalence class $\{\phi\}$, which encompasses all the possible field configurations related by gauge transformations. The generators of the gauge transformation are of the following form:

$$K^a_\alpha[\phi] \equiv D_\mu \delta(x-y)\,, \tag{2.2.5}$$

where the covariant derivative is given by $D_\mu := \partial_\mu - iA_\mu(x)$, and A_μ is understood to act by commutation. This means that gauge transformations can be written equivalently as

$$\delta\phi^a = K^a_\alpha[\phi]\,\epsilon^\alpha\,, \tag{2.2.6}$$

$$\delta A_\mu(x) = D_\mu \cdot \omega := [D_\mu, \omega]\,. \tag{2.2.7}$$

The field strength is given by $F_{\mu\nu} := i[D_\mu, D_\nu]$. We write the SU($N$) Yang-Mills action in the following way:

$$I[\phi] \equiv \frac{1}{4g^2}\int_x \left(F^a_{\mu\nu}\right)^2 = \frac{1}{2g^2}\ \mathrm{tr}\int_x F^2_{\mu\nu}\,. \tag{2.2.8}$$

In momentum space this becomes:

$$\begin{aligned} I[\phi] \equiv{}& \frac{1}{2g^2}\int_p I_{\mu_1\mu_2}(p)\ \mathrm{tr}\Big(A_{\mu_1}(p)A_{\mu_2}(-p)\Big) + \\ &+ \frac{1}{3g^2}\int_{p_1\ldots p_3} I_{\mu_1\ldots\mu_3}(p_1\ldots p_3)\ \mathrm{tr}\Big(A_{\mu_1}(p_1)\ldots A_{\mu_3}(p_3)\Big)\,\delta(p_1+p_2+p_3) + \\ &+ \frac{1}{4g^2}\int_{p_1\ldots p_4} I_{\mu_1\ldots\mu_4}(p_1\ldots p_4)\ \mathrm{tr}\Big(A_{\mu_1}(p_1)\ldots A_{\mu_4}(p_4)\Big)\,\delta(p_1+\cdots+p_4)\,, \end{aligned} \tag{2.2.9}$$

where the vertex functions $I^{(2)}, I^{(3)}$ and $I^{(4)}$, are given by:

$$I_{\mu_1\mu_2}(p) = 2\,\Box_{\mu_1\mu_2}(p) := 2(p^2\delta_{\mu_1\mu_2} - p_{\mu_1}p_{\mu_2})\,, \tag{2.2.10}$$

$$I_{\mu_1\ldots\mu_3}(p_1\ldots p_3) = 2(p_{3\mu_2}\delta_{\mu_1\mu_3} - p_{3\mu_1}\delta_{\mu_2\mu_3}) + \text{cycles}\,, \tag{2.2.11}$$

$$I_{\mu_1\ldots\mu_4}(p_1\ldots p_4) = (\delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3} - \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}) + \text{cycles}\,. \tag{2.2.12}$$

We note that 'cycles' stands for all cyclic permutations of momenta $p_{1\mu_1} \to p_{2\mu_2} \to \ldots \to p_{1\mu_1}$. The exact preservation of gauge invariance at all stages during the flow, together with the choice to rescale the coupling in front of the integral in (2.2.8), ensures that A_μ cannot run. The reason behind this is that any wavefunction renormalisation by $Z \neq 1$, $A^R_\mu = Z^{-1/2}A_\mu$, would break gauge invariance: $\delta A^R_\mu = Z^{-1/2}\partial_\mu\omega - i[A^R_\mu, \omega]$ [49, 62, 63]. Therefore the coupling $g = g(\Lambda)$ is the only quantity that runs.

We write functional derivatives as follows:

$$S_{,a} = \frac{\delta S}{\delta\phi^a} \equiv \frac{\delta S}{\delta A_\mu(x)}\,, \tag{2.2.13}$$

where

$$\frac{\delta}{\delta A_\mu(x)} := 2\,T^a\,\frac{\delta}{\delta A_\mu(x)^a}\,. \tag{2.2.14}$$

The properties of the latter can be understood as follows [63, 64]. For convenience here we temporarily write the gauge fields $A_\mu(x)$ simply as A, suppressing spacetime dependence and Lorentz indices. Given a well-behaved gauge invariant function $f(A)$ such that $\delta f(A) = \mathrm{tr}(\delta A\, X)$ for some X, we can exploit the $SU(N)$ completeness relation to see that:

$$\frac{\delta f}{\delta A} = X - \frac{1}{N}\mathrm{tr}X\,, \tag{2.2.15}$$

effectively isolating X. This property will be crucial in our endeavour later on. Following the same reasoning, we have two more useful relations:

$$\mathrm{tr}\, Y \frac{\delta f}{\delta A} = \mathrm{tr} YX - \frac{1}{N}\mathrm{tr} Y \mathrm{tr} X\,, \tag{2.2.16}$$

and,

$$\mathrm{tr} \frac{\delta}{\delta A} W = \mathrm{tr} Y \mathrm{tr} Z - \frac{1}{N}\mathrm{tr} YZ\,, \tag{2.2.17}$$

for some $W = T^a W^a$ such that $\delta W = Y\,\delta A\, Z$. Thus, the functional derivatives of $I[\phi]$ are (in position space):

$$\mathcal{E}_a = I_{,a} \equiv \frac{1}{g^2} D_\alpha F_{\mu\alpha}\,, \tag{2.2.18}$$

$$I_{,ab} \equiv \frac{1}{g^2}\Big(iF_{\mu\nu} + D_\nu D_\mu - \delta_{\mu\nu} D^2\Big)\delta(x-y)\,. \tag{2.2.19}$$

Ultra-local metrics (*i.e.* strictly local) are introduced on Φ and $\mathcal{G}$, respectively, such that their corresponding line element (*e.g.* $\delta\phi^a \gamma_{ab} \delta\phi^b$) is made dimensionless by using the appropriate power of the effective scale Λ [47]. They allow us to raise or lower DeWitt indices, *e.g.* $\phi^a = \gamma^{ab}\phi_b$. At this abstract level the formulation can be developed in a way that applies equally well to gravity, and indeed ref. [47] treats gravity as a special case. For gravity however, γ_{ab} and $\eta^{\alpha\beta}$ necessarily depend on ϕ (which in this case is the spacetime metric). Then, to keep equations covariant on Φ, requires the use of a connection Γ^c_{ab} in function space and covariant derivatives, *e.g.* replacing $I_{,ab}$ above with $I_{;ab} = I_{,ab} - \Gamma^c_{ab} I_{,c}$. Since we will treat only Yang-Mills theory in this chapter, where one can take the metrics to be ϕ independent:

$$\gamma_{ab} = \frac{\Lambda^2}{g^2}\delta_{\mu\nu}\delta(x-y)\,, \qquad \gamma^{ab} = \frac{g^2}{\Lambda^2}\delta_{\mu\nu}\delta(x-y)\,, \tag{2.2.20}$$

$$\eta_{\alpha\beta} = \frac{\Lambda^4}{g^2}\delta(x-y)\,, \qquad \eta^{\alpha\beta} = \frac{g^2}{\Lambda^4}\delta(x-y)\,, \tag{2.2.21}$$

we do not need this extra complication, and thus we work only in the flat function space limit.

Again, following [47], we will introduce higher covariant derivatives into the effective action S via functions of Laplace-like differential operators, which we list below:

$$\Delta^a{}_b = \gamma^{ac} I_{,cb} + K^a_\alpha \eta^{\alpha\beta} K^c_\beta \gamma_{cb} \quad \equiv \quad \Delta_{\mu\nu} = \Lambda^{-2}\left(2iF_{\mu\nu} - \delta_{\mu\nu} D^2\right)\delta(x-y)\,, \tag{2.2.22}$$

$$\left(\Delta_\parallel\right)^\alpha{}_\beta = \eta^{\alpha\delta} K^a_\delta \gamma_{ab} K^b_\beta \quad \equiv \quad \Delta_\parallel = -\Lambda^{-2} D^2 \delta(x-y)\,. \tag{2.2.23}$$

Gauge invariant quantities have all DeWitt indices contracted (*i.e.* behave as scalars on Φ and $\mathcal{G}$). One gauge invariant object that plays a fundamental role in the regularisation

is the action A :

$$A = I + \frac{1}{2} I_{,a} \left[X(\Delta)\right]^{ab} I_{,b} \,. \tag{2.2.24}$$

It is *quasi-local* (*i.e.* has an all orders expansion in momenta) [63, 64]. Here, the kernel $X(z)$ is given by:

$$X(z) = \frac{1 - c(z)}{z c(z)} \,, \tag{2.2.25}$$

where $c(z)$ is a smooth UV cutoff profile such that $c(0) = 1$, and $c(p^2/\Lambda^2) \to 0$ as $p^2/\Lambda^2 \to \infty$.

The anomalous dimension η is given by:

$$\eta := -g^2 \Lambda \partial_\Lambda \left(\frac{1}{g^2}\right) = \frac{2}{g} \Lambda \partial_\Lambda g \,. \tag{2.2.26}$$

Given that $\eta \propto \hbar$ (*i.e.* actually $\hbar g^2$), we can write down a loop expansion for it:

$$\eta = \eta_1 g^2 + \eta_2 g^4 + \cdots \,, \tag{2.2.27}$$

and thus for $\beta := \Lambda \partial_\Lambda g$ as well ($\eta_i = 2\beta_i$):

$$\beta = \beta_1 g^3 + \beta_2 g^5 + \cdots \,. \tag{2.2.28}$$

It will prove useful to analyse expressions using the following projection operators:

$$\Pi_T{}^a{}_b := \delta^a{}_b - \Pi_L{}^a{}_b \qquad \equiv \qquad \Pi^T_{\mu\nu} := \delta_{\mu\nu} - \Pi^L_{\mu\nu} \,, \tag{2.2.29}$$

$$\Pi_L{}^a{}_b := K^a_\alpha \left(\Delta_\parallel^{-1}\right)^{\alpha\beta} K^c_\beta \gamma_{bc} \qquad \equiv \qquad \Pi^L_{\mu\nu} := D_\mu \Big(D^2\Big)^{-1} D_\nu \,, \tag{2.2.30}$$

although they are not directly involved in the regularisation scheme because of their non-local nature.

2.2.2 The main idea

Before delving into details, it is instructive to outline the main steps undertaken in ref. [47] to construct a flow equation for the effective action S. Essentially this is a three step procedure, where the first two steps represent a geometric reformulation of Slavnov's higher derivative regularisation scheme [85, 86], while in the last step the freedom one has to design an exact RG [62, 91–93] is exploited to avoid fixing the gauge, and to manifestly preserve gauge invariance at all stages during the flow.

First of all we need to incorporate higher covariant derivatives, S_{UV}, into the effective action S, to improve the UV behaviour by modifying the propagators:[1]

$$S = I + S_{UV} + \mathcal{S}\,, \tag{2.2.31}$$

where $\mathcal{S}$ stands for all the other interactions in the effective action that are generated by the flow. However higher covariant derivatives do not regularise all one-loop divergences, and therefore further regularisation is required [85, 86].

The second step addresses this issue by introducing PV regularisation, which however is introduced directly through functional determinants in such a way that these act as quasi-local metrics for Φ, and $\mathcal{G}$; they are called $G_{ab}[\phi]$ and $H_{\alpha\beta}[\phi]$ respectively. Now, for gauge theories, the partition function $\mathcal{Z}$ (*cf.* (1.1.9)) takes in general the following form:

$$\mathcal{Z} = \int_{\Phi/\mathcal{G}} d\{\phi\}\, M[\{\phi\}]\, e^{-S[\{\phi\}]}\,, \tag{2.2.32}$$

where $M[\{\phi\}]$ is the volume element on the quotient space $\Phi/\mathcal{G}$. Factoring the integration measure using the functionals mentioned above,

$$d\phi\sqrt{det\, G[\phi]} = d\{\phi\}M[\{\phi\}]\, d\xi\sqrt{det\, H[\phi]}\,, \tag{2.2.33}$$

we thus rewrite the partition function:

$$\mathcal{Z} \propto \int_\Phi d\phi\, \frac{\sqrt{det\, G[\phi]}}{\sqrt{det\, H[\phi]}}\, e^{-S[\phi]} = \int_\Phi d\phi\, e^{-S[\phi]+\frac{1}{2}\mathrm{Tr}\ln G_{ab}[\phi]-\frac{1}{2}\mathrm{Tr}\ln H_{\alpha\beta}[\phi]} \tag{2.2.34}$$

(up to some actually-infinite constant that, as mentioned before, we are free to discard without altering the physics).

The idea then is that G is chosen to cancel one-loop UV divergences arising from S. In the IR, G and H are both order Λ^2, corresponding to PV masses of order the cutoff. G thus has a longitudinal part which will contribute divergences, but H can be used to cancel them. Finally, by choosing G and H each such that they factor into other operators that are less divergent, we can avoid so-called 'overlapping' divergences, ones that correspond at one loop to having external PV legs [86, 94, 95]. Thus, at this stage, PV regularisation will require the following:

(PV1) At high momentum the Hessian for fluctuations matches that of G in the transverse space:

$$\Pi_T{}^c{}_a\, G_{cd}\, \Pi_T{}^d{}_b \sim S_{,ab}\,. \tag{2.2.35}$$

[1] As it stands, this equation is almost devoid of meaning since $\mathcal{S}$ could modify or even cancel terms in S_{UV}. At the classical level, we give it meaning by insisting that $\mathcal{S}^0$ contains only three-point and higher vertices: $\mathcal{S}^{0(n<3)} = 0$, which, in turn, fixes the on-shell two-point classical effective action. This choice will allow us to compute classical vertices in terms of quantities already known [71].

(PV2) Similarly, H matches the longitudinal part of G:

$$K^a_\alpha G_{ab} K^b_\beta \sim H_{\alpha\beta}\,. \tag{2.2.36}$$

(PV3) $G_{ab}[\phi]$ and $H_{\alpha\beta}[\phi]$ must be chosen so that they factor into operators that are less divergent.

Slavnov's original scheme does not avoid fixing the gauge [85, 86]. In ref. [47] gauge fixing is completely avoided by exploiting the freedom [62, 63, 88, 91–93, 96] one has to design an exact RG [30, 60] flow for the effective action S. This leads to the final flow equation:

$$\Lambda D_\Lambda S = \frac{1}{2}\mathrm{Tr}\big[G^{-1}\Lambda D_\Lambda G - H^{-1}\Lambda D_\Lambda H\big]\,, \tag{2.2.37}$$

where $\Lambda D_\Lambda := \Lambda\partial_\Lambda + \mathcal{L}_\Psi$ is the total RG derivative, $\mathcal{L}_\Psi$ being a Lie derivative on Φ associated to the change of variables $\phi^a \mapsto \phi^a - \delta\Lambda\Psi^a$. The total RG derivative acts in the following way:

$$\Lambda D_\Lambda S = \Lambda\partial_\Lambda S + \Psi^a S_{,a}\,, \tag{2.2.38}$$

$$\Lambda D_\Lambda H_{\alpha\beta} = \Lambda\partial_\Lambda H_{\alpha\beta} + \Psi^a H_{\alpha\beta,a}\,, \tag{2.2.39}$$

$$\Lambda D_\Lambda G_{ab} = \Lambda\partial_\Lambda G_{ab} + \Psi^c G_{ab,c} + \Psi^c{}_{,a} G_{bc} + \Psi^c{}_{,b} G_{ac}\,. \tag{2.2.40}$$

The strategy is that, altogether, the higher covariant derivatives and PV field determinants should provide a regularised 'kinetic' $I + S_{UV} \in S$ part, while a careful choice of Ψ^a is meant to ensure that the exact RG then generates only an $\mathcal{S} \in S$ part that is free of divergences.

2.2.3 Regularisation structure

The method proposed in ref. [47] to construct the regularisation scheme, *i.e.* PV operators and higher covariant derivatives, proceeds as follows. First of all, we need to construct $G_{ab}[\phi]$ and $H_{\alpha\beta}[\phi]$ by taking into account the requirements outlined in section 2.2.2. To satisfy (PV3) they are set to

$$G_{ab} = \big(C^{-1}\big)^c{}_a\,\gamma_{cd}\,\big(C^{-1}\big)^d{}_b\,, \tag{2.2.41}$$

$$H_{\alpha\beta} = B^\gamma{}_\alpha\, b_{\gamma\delta}\, B^\delta{}_\beta\,, \tag{2.2.42}$$

where we require $B^\alpha{}_\beta$, $b^\alpha{}_\beta$, and $\big(C^{-1}\big)^a{}_b$ to be constructed such that $G_{ab} \to \gamma_{ab}$ and $H_{\alpha\beta} \to \eta_{\alpha\beta}$ as $\Lambda \to \infty$. This latter requirement ensures that in the continuum limit the quasi-local and ultra-local metrics coincide, and hence encode the same physics [82]. Furthermore, the condition (PV1) implies that $C^2 S^{(2)} \sim 1$, which essentially means that all divergences coming from C^{-1} vertices should be regulated by contributions in

the effective action involving the effective propagator. The remaining condition (PV2) together with (2.2.41), and (2.2.42) relates the PV operators to one another in the high momentum limit:

$$\left(C^{-1}\right)^{a}{}_{b}K^{b}_{\alpha} \sim K^{a}_{\beta}B^{\beta}{}_{\alpha}\,, \tag{2.2.43}$$

$$b_{\alpha\beta} \sim \left(\Delta_{\|}\right)_{\alpha\beta} = K^{a}_{\alpha}\gamma_{ab}K^{b}_{\beta}\,. \tag{2.2.44}$$

The second part of the regularisation scheme amounts to introducing higher covariant derivatives into the effective action S. Given that G is bi-linear in C^{-1} we will keep the same structure for S_{UV} [86],

$$S_{UV} = \frac{1}{2}A_{,a}\gamma^{ab}A_{,b}\,, \tag{2.2.45}$$

and thus write the effective action S as

$$S = I + \mathcal{S} + \frac{1}{2}A_{,a}\gamma^{ab}A_{,b}\,. \tag{2.2.46}$$

Here it is instructive to pause and make some comments on the choice of using A and on the strategy for analysis of the regularisation and flow equation. If we functionally differentiate (2.2.24) twice we get:

$$A_{,ab} = \left(\Delta\cdot c^{-1}(\Delta)\right)_{ad}\Pi_{T}{}^{d}{}_{b} + O(\mathcal{E})\,. \tag{2.2.47}$$

Following ref. [47], it is useful to write equations such as this, where the equality is given up to terms that vanish on the equations of motion $\mathcal{E}$, *cf.* (2.2.18). Since $\mathcal{E}$ vanishes if we set $A_{\mu} = 0$, we can read off from the above that the first term provides the effective inverse propagator that would arise from using A as the action. We see that it is transverse and that it would lead to an effective propagator $\sim c_q/q^2$ which is regularised by a cutoff function in the usual way. The first term above is also solely responsible for the one-loop contribution that A would provide (through the functional determinant of $A_{,ab}$) when the result is evaluated on shell. From (2.2.47) we can also infer the large momentum behaviour of the vertices $A^{(n)}$. Clearly $A^{(2)} \sim q^2 c_q^{-1}$. In fact, for any integer n, $A^{(n)} \sim c_q^{-1}$ up to some multiplicative power of momentum which can be ignored in comparison if the cutoff is chosen to be strong enough, for example $c(x) = e^{-x}$ or $c(x) = (1+x)^{-m}$ for some large m. To write this we will use the following shorthand $A^{(n)} \approx c_q^{-1}$. These properties will be more precisely defined and confirmed in sec. 2.4.2.6. Until then, in the spirit of ref. [47], we count only powers of cutoffs to analyse the behaviour of different expressions in the large momentum limit. For example, looking at (2.2.46) we see that

$$S_{,ab} \sim A^{(2)}A^{(2)} \approx c_q^{-2}\,, \tag{2.2.48}$$

which means that the (PV1) condition can now be recast as:

$$C^{-1} \approx c_q^{-1} \,. \tag{2.2.49}$$

To properly implement regularisation, (PV1), (PV2) and (PV3) ought to be fulfilled or, equivalently, (2.2.43), (2.2.44) and (2.2.49). A particular choice which satisfies this is the following :

$$\big(C^{-1}\big)^a{}_b = \delta^a{}_b + \gamma^{ac} A_{,cb} + K^a_\alpha Y^{\alpha\beta} K^c_\beta \gamma_{cb} \,, \tag{2.2.50}$$

$$B^\alpha{}_\beta = \delta^\alpha{}_\beta + Y^{\alpha\gamma}\big(\Delta_\parallel\big)_{\gamma\beta} \,, \tag{2.2.51}$$

$$b^\alpha{}_\beta = t^\alpha{}_\beta + \big(\Delta_\parallel\big)^\alpha{}_\beta \,, \tag{2.2.52}$$

where $Y = Y\big(\Delta_\parallel\big)$ is a quasi-local function such that $Y(z) \to \infty$ as $z \to \infty$ at the same rate as $1/c(z)$ and $Y(0)$ is finite, while $t = t\big(\Delta_\parallel\big)$ is a quasi-local function such that $t(z) \to 0$ as $z \to \infty$ and $t(0) = 1$.

2.2.4 Flow equation

For the flow equation to have fixed points it has to be non-linear, which implies that the blocking functional $\Psi[\phi]$ must itself depend on S. Generalising [28, 62–83, 88, 96, 97], which are themselves generalisations of the Polchinski equation [36], the blocking functional is written in the following way:

$$\Psi^a = -\frac{1}{2}\mathcal{K}^{ab}\Sigma_{,b} + \psi^a \,, \tag{2.2.53}$$

where

$$\Sigma := S - \hat{S} = \mathcal{S} - \frac{1}{2} I_{,a} X^{ab} I_{,b} \,, \tag{2.2.54}$$

and the "seed" action $\hat{S}$ is given by:

$$\hat{S} := A + S_{UV} = A + \frac{1}{2} A_{,a} \gamma^{ab} A_{,b} \,. \tag{2.2.55}$$

We will call the quasi-local functionals $\mathcal{K}^{ab}$ and ψ^a the *exact RG kernels*. We will try to fine-tune them to remove the unwanted divergences still present in the flow equation. In addition to this, it will prove useful to fix the classical two-point function [63, 64, 71]. As such, we require

$$S_0 = I + \frac{1}{2} A_{,a} \gamma^{ab} A_{,b} + \mathcal{S}_0 \,, \tag{2.2.56}$$

where $\mathcal{S}_0 = O\big(\mathcal{E}^3\big)$, to be a solution of the classical flow equation

$$\Lambda D_\Lambda S_0 = 0 \,. \tag{2.2.57}$$

This fixes the classical two-point function to be

$$S_{0,ab} = I_{,ab} + A_{,ma}\gamma^{mn}A_{,nb} + O(\mathcal{E})\,. \tag{2.2.58}$$

The only thing left now is to fix the exact RG kernels. The main constraint comes from the left-hand side of (2.2.37), *i.e.* the classical flow equation. One needs to carefully choose the kernels such that the effective interactions generated during the flow (*i.e.* $\mathcal{S}$) do not destroy the regularisation. In other words we need to ensure that $\mathcal{S}$ diverges more weakly than $S_{UV} \approx c_q^{-2}$. Since we are unable to write a closed form expression for $\mathcal{S}$ (we can only determine a finite number of higher order contributions to it by repeated iterations), following ref. [47], we suppose that its maximum divergence rate is given by:

$$\mathcal{S}^{(n)} \sim A^{(n)} \approx c_q^{-1}\,, \tag{2.2.59}$$

or, equivalently, using the definition (2.2.54) of Σ, that

$$\Sigma^{(n)} \approx c_q^{-1}\,. \tag{2.2.60}$$

The strategy now is to fine-tune the exact RG kernels such that they cancel all the terms which diverge faster than c_q^{-1} in the flow equation of $\mathcal{S}^{(n)}$, and hence prove *a posteriori* that the assumption (2.2.59) is consistent. If we unwrap the left-hand side of (2.2.37) we get:

$$\begin{aligned}\Lambda D_\Lambda S = \dot{A}_{,a}\gamma^{ab}A_{,b} + \frac{1}{2}A_{,a}\dot{\gamma}^{ab}A_{,b} + A_{,a}\big(C^{-1}\big)^a{}_b\psi^b - \frac{1}{2}A_{,c}\big(C^{-1}\big)^c{}_a\mathcal{K}^{ab}\Sigma_{,b} \\ + \dot{A} + \dot{\Sigma} - \frac{1}{2}\Sigma_{,a}\mathcal{K}^{ab}\Sigma_{,b} + \Sigma_{,a}\psi^a\,,\end{aligned} \tag{2.2.61}$$

where, for any functional F, we write $\dot{F} = \Lambda\partial_\Lambda F$. To arrive at this expression we have used the identity:

$$\hat{S}_{,b} = A_{,a}\big(C^{-1}\big)^a{}_b\,, \tag{2.2.62}$$

which follows from (2.2.50) and the gauge invariance of A (*i.e.* $K^a_\alpha A_{,a} = 0$). The first two terms of (2.2.61) are bi-linear in $A_{,a}$ and hence diverge as $\approx c_q^{-2}$. To eliminate them we use the third term, setting

$$\psi^a = -C^a{}_b\Big(\gamma^{bc}\dot{A}_{,c} + \frac{1}{2}\dot{\gamma}^{bc}A_{,c}\Big)\,. \tag{2.2.63}$$

Now, looking at the remaining terms, we see that the potentially dangerous ones are the fourth term, which, ignoring $\mathcal{K}^{ab}$ for the moment, diverges as $\approx c_q^{-3}$, and similarly the next-to-last term, which diverges as $\approx c_q^{-2}$. Cancelling the C^{-1} factor in the former by writing

$$\mathcal{K}^{ab} = C^a{}_c[\kappa(\Delta)]^{cb}\,, \tag{2.2.64}$$

the kernel κ^{ab} is, then, fixed uniquely by substituting S_0 from (2.2.56) into (2.2.61), and requiring cancellation, *i.e.* requiring the classical flow (2.2.57) at $O(\mathcal{E}^2)$. The result is:

$$\kappa^a{}_b = \left(2\frac{c(\Delta) - 2\Delta c'(\Delta)}{c^2(\Delta) + \Delta}\right)^a{}_b \tag{2.2.65}$$

(see sec. IIIB and app. A of ref. [47]). This means that $\kappa^{ab} \approx c_q$ and hence ref. [47] concluded that all the potentially offending terms in (2.2.61) are now brought under control. This part of the flow equation (the classical part) now simplifies to:

$$\Lambda D_\Lambda S = -\frac{1}{2}(\Sigma_{,a} + \hat{S}_{,a})C^a{}_c\kappa^{cb}\Sigma_{,b} + \dot{A} + \dot{\Sigma} + \Sigma_{,a}\psi^a\,, \tag{2.2.66}$$

and indeed a direct substitution of the above estimates suggests that now no terms diverge faster than c_q^{-1}. Moreover, although more involved, one can similarly check that the exact RG kernels (2.2.63), (2.2.64) and (2.2.65) ensure that the RHS of (2.2.37) is also apparently properly regularised [47].

2.3 Regularisation failure: a sketch

The problem with the analysis reviewed in the previous section is more easily grasped if we convert the analysis into the language of Feynman graphs. The reason this is helpful, is because we can then derive closed formulae for all the component vertices and propagator-like terms of these graphs, and then rigorously characterise their asymptotic behaviour in the limit that certain momenta diverge. In particular, UV divergences arise in quantum corrections from the limit in which loop momenta $q \to \infty$ while external momenta are held fixed. The components in the flow equation (2.2.66) do not necessarily behave in this limit with the assumed powers of c_q because their UV behaviour depends on the details of how q is routed through the vertices.

By focusing in sec. 2.6 on the part of the one-loop action that should provide the beta function, we show that regularisation fails here in a somewhat subtle way. There are parts that are UV divergent, but can be set to zero once treated as Λ independent. However, gauge transformations still map the one-loop beta function contribution into those improperly regularised parts. In fact, we show that the contribution cannot then both be gauge invariant and properly regularised. Actually, this latter conclusion holds for all choices of cutoff functions and some generalisations of the construction. Although the flow equation is constructed to be covariant by using functions of the differential operators Δ and $\Delta_\parallel$, other choices of covariantisation are possible that still reduce to the same functions when the gauge field A_μ is set to zero (for example by allowing the functions to depend separately on $F_{\mu\nu}$). Our derivation is independent of these details.

In fact the first hint of regularisation failure seems to come from how the regularisation is constructed in the first place. Ref. [47] argues that in the high momentum limit the leading divergent part of the propagator is its transverse part. The argument goes as follows. Starting from the gauge invariance of S (under a gauge transformation $\phi^a \to \phi^a + \delta\phi^a$),

$$K^a_\alpha S_{,a} = 0\,, \tag{2.3.1}$$

and differentiating it once, we obtain:

$$K^a_\alpha S_{,ab} = -K^a_{\alpha,b} S_{,a}\,. \tag{2.3.2}$$

Regarding this expression as a differential operator one can think of the large momentum as going from α to b. Then, this equation tells us that the behaviour at large momentum is given by $K^a_{\alpha,b}$, and since K^a_α is a first order differential operator, the divergence is only with one power of momentum. However, as an expression about vertices, more generally we see that we cannot exclude the possibility that the loop momentum threads instead through $S_{,a}$ whose high momentum behaviour can behave as badly as does $S_{,ab}$.

2.4 Perturbative expansion

In summary, the robustness of the regularisation scheme outlined in sec. 2.2 needs to be thoroughly tested by analysing carefully the high momentum behaviour of n-point vertices at each order in the loop expansion. In what follows we will lay out the machinery needed to do this properly, before returning to these issues in sec. 2.6.

2.4.1 Loop expansion

The effective action S, and in fact any gauge invariant action appearing in this analysis, has a weak coupling expansion (*i.e.* loop expansion) in g^2 (actually in $\hbar g^2$):

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots\,. \tag{2.4.1}$$

This means that purely classical actions such as S_0, but also *e.g.* A, S_{UV} and $\hat{S}$, carry only a single $1/g^2$ prefactor.

The form of the ultralocal metrics γ_{ab} and $\eta_{\alpha\beta}$ ensure that the factors of g embedded in the kernels combine with actions to preserve this property. Therefore we simplify matters from now on by ignoring the factors of g entirely and trusting that they can be put back in the form (2.4.1) above. (Actually, one can also simplify matters by recognising that the powers of Λ appearing through these metrics are as required to make the equations dimensionally correct, and thus essentially ignore these metrics.) Note that

in the ensuing we will also place the loop index as a superscript when convenient (so write $S^0 \equiv S_0$ *etc.*).

Similarly to (2.4.1) we can write down a loop expansion for the blocking functional Ψ^a:

$$\Psi^a = \Psi^a_0 + g^2\Psi^a_1 + \cdots \tag{2.4.2}$$

$$= \Big[-\frac{1}{2}\mathcal{K}^{ab}\Sigma^0{}_{,b} + \psi^a_0 \Big] + g^2\Big[-\frac{1}{2}\mathcal{K}^{ab}\Sigma^1{}_{,b} + \psi^a_1 \Big] + \cdots . \tag{2.4.3}$$

This can be further refined. If we explicitly compute the RG time derivatives in (2.2.63) and use the commutativity of RG time and functional derivatives, we can recast ψ^a in the following way:

$$\psi^a = C^{ab}\theta_{,b}\,, \tag{2.4.4}$$

where we have introduced the gauge invariant action θ, given by

$$\theta := \frac{\eta}{2}A + I + I_{,m}[W(\Delta)]^{mn}I_{,n}\,, \tag{2.4.5}$$

with $W(z) := \frac{3}{2}X(z) + zX'(z)$, which thus has a loop expansion:

$$\theta^0 = I + I_{,m}W^{mn}I_{,n}\,, \tag{2.4.6}$$

$$\theta^n = \frac{\eta_n}{2}A\,. \tag{2.4.7}$$

This enables us to rewrite (2.4.3) as

$$\Psi^a = \Big[-\frac{1}{2}\mathcal{K}^{ab}\Sigma^0{}_{,b} + C^{ab}\theta^0{}_{,b} \Big] + g^2\Big[-\frac{1}{2}\mathcal{K}^{ab}\Sigma^1{}_{,b} + C^{ab}\theta^1{}_{,b} \Big] + \cdots . \tag{2.4.8}$$

If we substitute (2.4.1) and (2.4.8) into the flow equation (2.2.37), we will obtain its loopwise expansion. Thus we find that the flow of the effective action at the classical level is given by:

$$\Lambda\partial_\Lambda S^0 = \frac{1}{2}S^0{}_{,a}\mathcal{K}^{ab}\Sigma^0{}_{,b} - S^0{}_{,a}C^{ab}\theta^0{}_{,b}\,, \tag{2.4.9}$$

at the one-loop level by:

$$\begin{aligned}\Lambda\partial_\Lambda S^1 = {}& \eta_1 S^0 + \frac{1}{2}S^1{}_{,a}\mathcal{K}^{ab}\Sigma^0{}_{,b} + \frac{1}{2}S^0{}_{,a}\mathcal{K}^{ab}\Sigma^1{}_{,b} - S^1{}_{,a}C^{ab}\theta^0{}_{,b} - S^0{}_{,a}C^{ab}\theta^1{}_{,b} \\ & + \mathrm{Tr}\Bigg[C^a{}_c\Big[\Lambda\partial_\Lambda(C^{-1})^c{}_b\Big] + C^a{}_c(C^{-1})^c{}_{b,d}\,\Psi^d_0 + \frac{\delta\Psi^a_0}{\delta\phi^b} + \delta^a{}_b - (B^{-1})^\alpha{}_\gamma\big[\Lambda\partial_\Lambda B^\gamma{}_\beta\big] \\ & \qquad -(B^{-1})^\alpha{}_\gamma B^\gamma{}_{\beta,a}\Psi^a_0 - \frac{1}{2}(b^{-1})^\alpha{}_\gamma\big[\Lambda\partial_\Lambda b^\gamma{}_\beta\big] - \frac{1}{2}(b^{-1})^\alpha{}_\gamma b^\gamma{}_{\beta,a}\Psi^a_0 - 2\,\delta^\alpha{}_\beta \Bigg]\,,\end{aligned} \tag{2.4.10}$$

at the two-loop level by:

$$\Lambda\partial_\Lambda S^2 = \eta_2 S^0 + \frac{1}{2}S^2{}_{,a}\mathcal{K}^{ab}\Sigma^0{}_{,b} + \frac{1}{2}S^1{}_{,a}\mathcal{K}^{ab}\Sigma^1{}_{,b} + \frac{1}{2}S^0{}_{,a}\mathcal{K}^{ab}\Sigma^2{}_{,b} - S^2{}_{,a}C^{ab}\theta^0{}_{,b} - S^1{}_{,a}C^{ab}\theta^1{}_{,b} - S^0{}_{,a}C^{ab}\theta^2{}_{,b} + \mathrm{Tr}\Bigg[C^a{}_c\,(C^{-1})^c{}_{b,d}\Psi_1^d + \frac{\delta\Psi_1^a}{\delta\phi^b} - \frac{\eta_1}{2}\delta^a{}_b + \frac{\eta_1}{2}\delta^\alpha{}_\beta - (B^{-1})^\alpha{}_\gamma B^\gamma{}_{\beta,a}\Psi_1^a - \frac{1}{2}(b^{-1})^\alpha{}_\gamma b^\gamma{}_{\beta,a}\Psi_1^a\Bigg], \tag{2.4.11}$$

etc. It will prove useful in the subsequent analysis to rewrite (2.4.10) in a different way:

$$\Lambda\partial_\Lambda S^1 = \eta_1 S^0 + \frac{1}{2}S^1{}_{,a}\mathcal{K}^{ab}\Sigma^0{}_{,b} + \frac{1}{2}S^0{}_{,a}\mathcal{K}^{ab}\Sigma^1{}_{,b} - S^1{}_{,a}C^{ab}\theta^0{}_{,b} - S^0{}_{,a}C^{ab}\theta^1{}_{,b} + \mathrm{Tr}\Bigg[\Lambda\partial_\Lambda\big[\ln\big(C^{-1}\big)\big]^a{}_b + \Psi_0^d\frac{\delta}{\delta\phi^d}\big[\ln\big(C^{-1}\big)\big]^a{}_b + \frac{\delta\Psi_0^a}{\delta\phi^b} - \Lambda\partial_\Lambda\big[\ln B^\alpha{}_\beta\big] - \Psi_0^a\frac{\delta}{\delta\psi^a}\big[\ln B^\alpha{}_\beta\big] - \frac{1}{2}\Lambda\partial_\Lambda\big[\ln b^\alpha{}_\beta\big] - \frac{1}{2}\Psi_0^a\frac{\delta}{\delta\psi^a}\big[\ln b^\alpha{}_\beta\big]\Bigg]. \tag{2.4.12}$$

Here note that we have also discarded the two vacuum contributions $\delta^a{}_b$ and $\delta^\alpha{}_\beta$.

2.4.2 Vertex expansion

In the above equations we have actions $(S^i, \Sigma^i, \theta^i)$ and kernels (*e.g.* $\mathcal{K}^{ab}, C^{ab}, B^\gamma_\beta$ *etc.*). To analyse their behaviour precisely we break them down into vertices. The above equations then tell us how these vertices fit together to determine effective action vertices at each loop order. In this subsection we sketch the form of these vertices and their properties [63, 64].

2.4.2.1 Action vertex properties

Any gauge invariant action has an expansion in traces and products of traces [63, 64], which, in position space, takes the following form:

$$S = \sum_{n=2}^{\infty}\frac{1}{n}\int_{x_1\ldots x_n} S_{\mu_1\ldots\mu_n}(x_1\ldots x_n)\mathrm{tr}\Big(A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\Big) + \frac{1}{2!}\sum_{n,m=2}^{\infty}\frac{1}{nm}\int_{x_1\ldots x_n}\int_{y_1\ldots y_m} S_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_m}(x_1\ldots x_n; y_1\ldots y_m) \mathrm{tr}\Big(A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\Big)\mathrm{tr}\Big(A_{\nu_1}(y_1)\ldots A_{\nu_m}(y_m)\Big) + \cdots. \tag{2.4.13}$$

From this we can see that single trace vertices are defined as cyclically symmetric in all indices,

$$S_{\mu_1\dots\mu_n}(x_1\dots x_n) = S_{\mu_n\mu_1\dots\mu_{n-1}}(x_n, x_1\dots x_{n-1})\,, \tag{2.4.14}$$

whereas the remaining vertices are separately cyclically symmetric for each string. However, the latter have been included for completeness only and are of no interest to us in the present chapter. Therefore, from now on, we will consider only the single trace vertices. The fact that the action S is real, together with the hermiticity of the generators T^a, implies that $S^*_{\mu_1\dots\mu_n}(x_1\dots x_n) = S_{\mu_n\dots\mu_1}(x_n\dots x_1)$. This is an expression of charge conjugation invariance. To exploit translation invariance we work in momentum space where it implies momentum conservation:

$$S_{\mu_1\dots\mu_n}(p_1\dots p_n)\,\delta\Big(\sum_i p_i\Big) = \int_{x_1\dots x_n} S_{\mu_1\dots\mu_n}(x_1\dots x_n)\,e^{-i\sum_i p_i\cdot x_i}\,. \tag{2.4.15}$$

By convention, we take all momenta pointing in towards the vertex. Note that the $S_{\mu_1\dots\mu_n}(p_1\dots p_n)$ are well-defined only when momentum is conserved. For the two-point vertex we write more simply $S_{\mu\nu}(p) = S_{\mu\nu}(p,-p)$. Lorentz invariance implies that:

$$S_{\mu_1\dots\mu_n}(p_1,\dots,p_n) = (-1)^n S_{\mu_1\dots\mu_n}(-p_1,\dots,-p_n)\,. \tag{2.4.16}$$

In other words, vertices with an even/odd number of legs are even/odd under a change of sign in all its momentum arguments. Combined with charge conjugation invariance we thus also have that they are even/odd under reversal of their arguments

$$S_{\mu_n\dots\mu_1}(p_n,\dots,p_1) = (-1)^n S_{\mu_1\dots\mu_n}(p_1,\dots,p_n)\,. \tag{2.4.17}$$

As an example we note that these symmetries tell us that a three-point action vertex is totally antisymmetric in its arguments $(p_{1\mu_1}, p_{2\mu_2}, p_{3\mu_3})$.

The flow equation (2.2.37) manifestly preserves gauge invariance, namely (2.2.7), and this allows us to write a set of 'trivial' Ward identities for the effective action S that relates the longitudinal part of n-point vertices to differences of $(n-1)$-point vertices:

$$p_{1\mu_1} S_{\mu_1\dots\mu_n}(p_1\dots p_n) = S_{\mu_2\dots\mu_n}(p_1{+}p_2, p_3\dots p_n) - S_{\mu_2\dots\mu_n}(p_2\dots p_{n-1}, p_1{+}p_n)\,. \tag{2.4.18}$$

Since there are no $(n<2)$-point vertices, the two-point vertex is transverse:

$$p_\mu S_{\mu\nu}(p) = 0\,. \tag{2.4.19}$$

In the limit of small contracted momentum p_1, (2.4.18) yields differential Ward identities *e.g.*

$$S_{\mu\nu\lambda}(0,p,-p) = \partial^p_\mu S_{\nu\lambda}(p)\,. \tag{2.4.20}$$

where we have introduced the shorthand $\partial^p_\mu = \frac{\partial}{\partial p_\mu}$. This follows straightforwardly starting from (2.4.18) after Taylor expanding its right-hand side:

$$\epsilon_\mu S_{\mu\nu\lambda}(\epsilon, p, -p-\epsilon) = S_{\nu\lambda}(p+\epsilon) - S_{\nu\lambda}(p) \tag{2.4.21}$$

$$= \epsilon_\mu \partial^p_\mu S_{\nu\lambda}(p) + O(\epsilon^2)\,, \tag{2.4.22}$$

recovering (2.4.20) from $O(\epsilon)$. We can proceed similarly about the four-point vertex to get:

$$S_{\mu\nu\lambda\sigma}(0,0,p,-p) + S_{\nu\mu\lambda\sigma}(0,0,p,-p) = \partial^p_\mu \partial^p_\nu S_{\lambda\sigma}(p)\,. \tag{2.4.23}$$

The $I^{(n)}$ vertices already introduced in eqs. (2.2.10)–(2.2.12) satisfy the above properties.

2.4.2.2 Kernel vertex properties

The regularisation and the flow equation are built up from constructs where actions are joined together by kernels to form new compound actions. Generically for two actions, say $R[A]$ and $S[A]$, the construct takes the form $R_{,a}K^{ab}S_{,b}$ and involves some kernel $K^{ab}[A]$. Since everything is constructed gauge covariantly, these kernels are themselves necessarily functionals of A_μ. Although the kernels we need are constructed in a specific way, as reviewed in sec. 2.2.3, they are special cases of a more general form of kernel namely any such that provides a *covariantisation* of some momentum kernel $K^{ab}[A = 0] \equiv K_{\mu\nu}(p)$, via gauge fields A_μ that act by commutation. (The latter property is assured here to the 2-point level that we will require them, in essence by constructing the kernels using covariant derivatives D_μ, see sec. 2.4.2.5 and app. 2.A.) Such kernels have vertices whose properties generalise those for an action and ensure that the vertices of the compound action continue to satisfy the correct properties.

Expressions involving such a covariantised kernel $K_{\mu\nu}$ can be equivalently written in the following way:

$$R_{,a}K^{ab}S_{,b} \equiv \frac{1}{\Lambda^2}\frac{\delta R}{\delta A_\mu}\cdot K_{\mu\nu}\cdot\frac{\delta S}{\delta A_\nu} \tag{2.4.24}$$

$$\equiv \frac{1}{2\Lambda^2}\mathrm{tr}\int_x\int_y \frac{\delta R}{\delta A_\mu(x)}K_{\mu\nu}(x,y)\cdot\frac{\delta S}{\delta A_\nu(y)}\,, \tag{2.4.25}$$

where we suppress the functional dependence on the gauge field. Taylor expanding with respect to the gauge field in $K^{ab}[A]$ gives a series expansion in nested commutators [63, 64, 71]:

$$R_{,a}K^{ab}S_{,b} \equiv \frac{1}{2\Lambda^2}\sum_{n=0}\int_x\int_y\int_{x_1\cdots x_n} K_{\mu_1\ldots\mu_n;\mu\nu}(x_1\ldots x_n;x,y)$$
$$\mathrm{tr}\Big[\frac{\delta R}{\delta A_\mu(x)}A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\cdot\frac{\delta S}{\delta A_\nu(y)}\Big]\,. \tag{2.4.26}$$

Now, expanding the commutators, we obtain:

$$R_{,a}K^{ab}S_{,b} \equiv \frac{1}{2\Lambda^2}\sum_{m,n=0}\int_x\int_y\int_{x_1\cdots x_n,y_1\cdots y_m} K_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_m;\mu\nu}(x_1\ldots x_n;y_1\ldots y_m;x,y)$$
$$\mathrm{tr}\left[\frac{\delta R}{\delta A_\mu(x)}A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\frac{\delta S}{\delta A_\nu(y)}A_{\nu_1}(y_1)\cdots A_{\nu_m}(y_m)\right]. \quad (2.4.27)$$

In momentum space, kernel vertices take the following form:

$$K_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_m;\mu\nu}(p_1\ldots p_n;q_1\ldots q_m;p,q)\,\delta\left(\sum p_i+\sum q_i+p+q\right) =$$
$$= \int_x\int_y\int_{x_1\cdots x_n,y_1\cdots y_m} K_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_m;\mu\nu}(x_1\ldots x_n;y_1\ldots y_m;x,y)$$
$$\times\exp\left[-i\left(p\cdot x+q\cdot y+\sum_i p_i\cdot x_i+\sum_j q_j\cdot y_j\right)\right]. \quad (2.4.28)$$

The two sets of vertices in (2.4.26) and (2.4.27) are related through 'interleave' identities (also called 'coincident line' identities [63, 64, 71]) which here just express the fact that covariantisation is via commutation:

$$K_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_m;\mu\nu}(p_1\ldots p_n;q_1\ldots q_m;p,q) =$$
$$= (-1)^m\sum_{\text{interleaves}} K_{\alpha_1\ldots\alpha_{n+m};\mu\nu}(k_1\ldots k_{m+n};p,q)\,. \quad (2.4.29)$$

In the above the sum runs over all the possible arrangements of the combined sequence of $p_1^{\mu_1}\ldots p_n^{\mu_n}$ and $q_1^{\nu_1}\ldots q_m^{\nu_m}$, in which the p momenta remain ordered with one another, whereas the q momenta order is reversed (via the so-called shuffle product $\sum k_1^{\alpha_1}\ldots k_{m+n}^{\alpha_{m+n}} = p_1^{\mu_1}\ldots p_n^{\mu_n}\sqcup\!\sqcup\, q_m^{\nu_m}\ldots q_1^{\nu_1}$).

The case where $m=n=0$ (thus leaving a double semi-colon) is of course just the momentum kernel again:

$$K_{;;\mu\nu}(;;p,-p) = K_{\mu\nu}(p)\,. \quad (2.4.30)$$

We write the $m=0$ case more simply as:

$$K_{\mu_1\ldots\mu_n,;\mu\nu}(p_1\ldots p_n;;p,q) = K_{\mu_1\ldots\mu_n;\mu\nu}(p_1\ldots p_n;p,q)\,, \quad (2.4.31)$$

then from (2.4.29) the $n=0$ case is given by

$$K_{,\nu_1\ldots\nu_m;\mu\nu}(;q_1\ldots q_m;p,q) = (-1)^m K_{\nu_m\ldots\nu_1;\mu\nu}(q_m\ldots q_1;p,q)\,. \quad (2.4.32)$$

A useful special case of (2.4.29) is:

$$K_{\mu,\nu;\alpha\beta}(p;q;r,s) = -K_{\mu\nu;\alpha\beta}(p,q;r,s) - K_{\nu\mu;\alpha\beta}(q,p;r,s)\,. \quad (2.4.33)$$

Kernel vertices obey Ward identities closely similar to (2.4.18) [63,64]. Furthermore the contracted momentum can reach the end of the string of momenta, where it attaches to the outer momentum:

$$\begin{aligned} p_{1\mu_1} K_{\mu_1\dots\mu_n,\nu_1\dots\nu_m;\mu\nu}(p_1\dots p_n;q_1\dots q_m;p,q) = \\ = K_{\mu_2\dots\mu_n,\nu_1\dots\nu_m;\mu\nu}(p_1+p_2,p_3\dots p_n;q_1\dots q_m;p,q) \\ - K_{\mu_2\dots\mu_n,\nu_1\dots\nu_m;\mu\nu}(p_2\dots p_n;q_1\dots q_m;p_1+p,q)\,. \end{aligned} \tag{2.4.34}$$

Again one can derive differential Ward identities (*cf.* (2.4.20), (2.4.23)). We list below some of the more useful ones:

$$K_{\lambda;\mu\nu}(0;-p,p) = \partial^p_\lambda K_{\mu\nu}(p)\,, \tag{2.4.35}$$

$$K_{\lambda\sigma;\mu\nu}(-p,0;0,p) = \partial^p_\sigma K_{\lambda;\mu\nu}(-p;0,p)\,. \tag{2.4.36}$$

2.4.2.3 Two-point action vertices

Combining these definitions with those of sec. 2.2 one can derive the expressions for the vertices that are needed and work through the classical and quantum corrections systematically. Thus for example, from the definition (2.2.24) of the quasi-local action A, combined with that of X, (2.2.25), and the fundamental two-point vertex $I^{(2)}$ in (2.2.10), or alternatively directly from (2.2.47), one gets its two-point vertex:

$$A_{\mu_1\mu_2}(p) = 2\,\Box_{\mu_1\mu_2}(p) F_A(p)\,, \qquad F_A(p) := \frac{1}{c_p}\,. \tag{2.4.37}$$

Armed with this one finds from (2.2.55) the two-point vertex for the seed action:

$$\hat{S}_{\mu_1\mu_2}(p) = 2\,\Box_{\mu_1\mu_2}(p) F_{\hat{S}}(p)\,, \qquad F_{\hat{S}}(p) := \frac{\tilde{p}^2 + c_p}{c_p^2}\,, \tag{2.4.38}$$

and from (2.2.56) the classical effective action S^0 two-point vertex:

$$S^0_{\mu_1\mu_2}(p) = 2\,\Box_{\mu_1\mu_2}(p) F(p)\,, \qquad F(p) := 1 + \frac{\tilde{p}^2}{c_p^2}\,. \tag{2.4.39}$$

From the definition (2.2.54) of Σ and X one gets immediately its classical two-point vertex:

$$\Sigma^0_{\mu_1\mu_2}(p) = 2\,\Box_{\mu_1\mu_2}(p) F_\Sigma(p)\,, \qquad F_\Sigma(p) := 1 - \frac{1}{c_p}\,. \tag{2.4.40}$$

Finally from its definition (2.4.5) we get the classical two-point vertex for the action θ:

$$\theta^0_{\mu_1\mu_2}(p) = 2\,\Box_{\mu_1\mu_2}(p) F_\theta(p)\,, \qquad F_\theta(p) := \frac{c_p - 2\tilde{p}^2 c'_p}{c_p^2}\,. \tag{2.4.41}$$

2.4.2.4 Zero-point kernel functions

The zero-point vertices for the *simple* (*i.e.* non-compound) kernels X, Y, B, b, t, κ and W, follow straightforwardly from their definitions in (2.2.25), (2.2.51), (2.2.52), (2.2.65) and (2.4.5), by replacing their argument (Δ or $\Delta_\parallel$ as appropriate) by $\tilde{p}^2$, for example:

$$\kappa(p) = 2\,\frac{c_p - 2\tilde{p}^2 c_p'}{c_p^2 + \tilde{p}^2}\,. \tag{2.4.42}$$

(The zero-point X kernel was in effect already used to derive the two-point action vertices above.) Those for the *compound* kernels follow almost as straightforwardly. Thus the zero point vertex for $\left(C^{-1}\right)^a{}_b$ follows from its definition (2.2.50) since the generator of gauge transformations (2.2.5) collapses to a partial derivative in this case. Writing

$$C^{-1}_{\mu\nu}(p) = \frac{1}{C_T(p)}\frac{\Box_{\mu\nu}(p)}{p^2} + \frac{1}{C_L(p)}\frac{p_\mu p_\nu}{p^2}\,, \tag{2.4.43}$$

we have that its longitudinal and transverse functions are given by:

$$C_T(p) := \frac{c_p}{c_p + \tilde{p}^2}\,, \qquad C_L(p) := \frac{1}{1 + \tilde{p}^2 Y_p}\,. \tag{2.4.44}$$

Note that despite appearances in (2.4.43), the kernel has a Taylor expansion in p_μ (is quasilocal) because Y is quasilocal and c_p is normalised to $c(0) = 1$ (*cf.* below (2.2.25)). Inverting gives us the zero-point vertex for $C^a{}_b$:

$$C_{\mu\nu}(p) = C_T(p)\frac{\Box_{\mu\nu}(p)}{p^2} + C_L(p)\frac{p_\mu p_\nu}{p^2}\,, \tag{2.4.45}$$

which is also quasi-local for the same reasons. Then that for $\mathcal{K}$ follows from (2.2.64):

$$\mathcal{K}_{\mu\nu}(p) = \mathcal{K}_T(p)\frac{\Box_{\mu\nu}(p)}{p^2} + \mathcal{K}_L(p)\frac{p_\mu p_\nu}{p^2}\,, \qquad \mathcal{K}_I(p) := \kappa(p) C_I(p) \quad (I = T, L)\,. \tag{2.4.46}$$

2.4.2.5 Higher-point vertices

The n-point vertices for the simple kernels can be computed by Taylor expanding in Δ or $\Delta_\parallel$ as appropriate. Keeping only n instances of A_{μ_i} as in (2.4.26), the momentum dependence can then be resummed to give the explicit formula. The vertices for those that are simply functions of $\Delta_\parallel = -D^2/\Lambda^2$ – namely Y, B, b and t – have vertices that

have already been computed this way in sec. 5 of ref. [64]. We just quote the results:

$$Y_\mu(p;r,s) = (r-s)_\mu \frac{Y_r - Y_s}{p\cdot(r-s)}\,, \tag{2.4.47}$$

$$Y_{\mu\nu}(p,q;r,s) = \delta_{\mu\nu}\frac{Y_s - Y_r}{s^2-r^2} - (p+2r)_\mu(q+2s)_\nu\,\Xi_Y(p,q;r,s)\,, \tag{2.4.48}$$

$$\Xi_Y(p,q;r,s) = \frac{Y_{s+q}}{q\cdot(q+2s)\,p\cdot(p+2r)} + \frac{1}{s^2-r^2}\left[\frac{Y_r}{p\cdot(p+2r)} - \frac{Y_s}{q\cdot(q+2s)}\right]\,, \tag{2.4.49}$$

where we have introduced the shorthand Ξ_Y, and where for the others one simply replaces Y by B, b or t as appropriate. We note that these equations need care at special momenta where denominators vanish. For example the correct equation for $Y_{\mu\nu}(p,-p;r,-r)$ follows from $\lim_{\epsilon\to 0} Y_{\mu\nu}(p,-p-\epsilon;r+\epsilon,-r)$. Again we refer to sec. 5 of ref. [64] for details. The vertices for the other simple kernels X, κ and W follow from using a similar strategy. We only have to take into account the presence of $F_{\mu\nu}$ when expanding in a series of powers of $\Delta_{\mu\nu}$, *cf.* (2.2.22). Thus we find:

$$X_{\lambda;\mu\nu}(p;r,s) = \left\{(r-s)_\lambda\delta_{\mu\nu} + 4\delta_{\lambda[\mu}p_{\nu]}\right\}\frac{X_r - X_s}{p\cdot(r-s)}\,, \tag{2.4.50}$$

$$\begin{aligned} X_{\lambda\sigma;\mu\nu}(p,q;r,s) = {}& \left(\delta_{\mu\nu}\delta_{\lambda\sigma} + 4\delta_{\lambda[\mu}\delta_{\nu]\sigma}\right)\frac{X_s - X_r}{s^2-r^2} \\ & + \Big(16\delta_{\lambda[\mu}p_{\epsilon]}\delta_{\sigma[\epsilon}q_{\nu]} - 4\delta_{\lambda[\mu}p_{\nu]}(q+2s)_\sigma + 4(p+2r)_\lambda\delta_{\sigma[\mu}q_{\nu]} \\ & \quad - \delta_{\mu\nu}(p+2r)_\lambda(q+2s)_\sigma\Big)\Xi_X(p,q;r,s)\,. \end{aligned} \tag{2.4.51}$$

Here $T_{[\mu\nu]} := \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$. As above, those for κ and W follow simply by replacing the name.

With the above building blocks and techniques it is a tedious but straightforward exercise to arrive at explicit expressions for compound kernel and action n-point vertices. For example from (2.2.24), one finds

$$\begin{aligned} A_{\mu\nu\lambda}(p,q,-p-q) = {}& I_{\mu\nu\lambda}(p,q,-p-q) \\ & + \frac{1}{2\Lambda^2}\Big[I_{\alpha\mu\nu}(-p-q,p,q)X_{p+q}I_{\alpha\lambda}(p+q) \\ & \quad + I_{\alpha\nu\lambda}(p,q,-p-q)X_pI_{\alpha\mu}(p) \\ & \quad + I_{\alpha\lambda\mu}(q,-p-q,p)X_qI_{\alpha\nu}(q) \\ & \quad + I_{\alpha\mu}(p)X_{\nu;\alpha\beta}(q;p,-p-q)I_{\beta\lambda}(p+q) \\ & \quad + I_{\alpha\nu}(q)X_{\lambda;\alpha\beta}(-p-q;q,p)I_{\beta\mu}(p) \\ & \quad + I_{\alpha\lambda}(p+q)X_{\mu;\alpha\beta}(p;-p-q,q)I_{\beta\nu}(q)\Big]\,. \end{aligned} \tag{2.4.52}$$

Using (2.2.50), this is sufficient to construct the one-point vertex for the compound kernel $\left(C^{-1}\right)^{a}{}_{b}$:

$$C^{-1}_{\mu;\alpha\beta}(p;r,s) = \frac{1}{2\Lambda^2}A_{\mu\beta\alpha}(p,s,r) + \frac{1}{\Lambda^2}\left\{\delta_{\mu\alpha}s_\beta Y_s - r_\alpha\delta_{\mu\beta}Y_r - r_\alpha s_\beta Y_\mu(p;r,s)\right\}\,, \quad (2.4.53)$$

and from this we also have the C one-point vertex:

$$C_{\mu;\alpha\beta}(p;r,s) = -C_{\alpha\gamma}(r)C^{-1}_{\mu;\gamma\delta}(p;r,s)C_{\delta\beta}(s)\,. \quad (2.4.54)$$

We give further details on the construction of the compound kernels in app. 2.A where we also prove that up to the two-point level the compound kernels obey the interleave identities (2.4.29). Note that, as a consequence of their symmetric definition as operators, and the interleave identities, all the kernel one-point vertices so far treated are antisymmetric under $r_\alpha \leftrightarrow s_\beta$, for example:

$$C^{-1}_{\mu;\alpha\beta}(p;r,s) = C^{-1}_{,\mu;\beta\alpha}(;p;s,r) = -C^{-1}_{\mu;\beta\alpha}(p;s,r)\,. \quad (2.4.55)$$

The definition (2.2.64), $\mathcal{K}^{ab} = C^{a}{}_{c}\kappa^{cb}$, is not symmetric, and thus its one-point vertex does not obey this identity. However, using (2.4.54), and (2.4.50) with $X \mapsto \kappa$, we get an explicit expression for it also:

$$\mathcal{K}_{\mu;\alpha\beta}(p;r,s) = C_{\mu;\alpha\beta}(p;r,s)\,\kappa(s) + C_{\alpha\gamma}(r)\,\kappa_{\mu;\gamma\beta}(p;r,s)\,. \quad (2.4.56)$$

We should distinguish this from the vertex made from the hermitian conjugate $\kappa^{ac}C_{c}{}^{b}$:

$$\overleftarrow{\mathcal{K}}_{\mu;\alpha\beta}(p;r,s) = \kappa(r)\,C_{\mu;\alpha\beta}(p;r,s) + \kappa_{\mu;\alpha\gamma}(p;r,s)\,C_{\gamma\beta}(s)\,, \quad (2.4.57)$$

and the symmetrised combination

$$\overleftrightarrow{\mathcal{K}}_{\mu;\alpha\beta}(p;r,s) = \frac{1}{2}\left[\mathcal{K}_{\mu;\alpha\beta}(p;r,s) + \overleftarrow{\mathcal{K}}_{\mu;\alpha\beta}(p;r,s)\right]\,, \quad (2.4.58)$$

coming from $\frac{1}{2}(C^{a}{}_{c}\kappa^{cb} + \kappa^{ac}C_{c}{}^{b})$, since these versions can also appear. As a consequence of the interleave identities, $\overleftrightarrow{\mathcal{K}}_{\mu;\alpha\beta}(p;r,s)$ obeys (2.4.55), whilst the directed versions satisfy

$$\mathcal{K}_{\mu;\alpha\beta}(p;r,s) = -\overleftarrow{\mathcal{K}}_{\mu;\beta\alpha}(p;s,r)\,. \quad (2.4.59)$$

2.4.2.6 Large momentum behaviour

As we will see in sec. 2.6, these vertices play a closely similar role to those of Feynman rules. In particular in one-loop diagrams, three-point vertices will carry a loop momentum q and external momentum p and thus have arguments p, q, and $-p-q$. Therefore the large momentum behaviour that actually determines whether the flow equation is properly regularised is one where $q \to \infty$ while keeping p fixed. As before, it is sufficient

to characterise the large momentum behaviour in terms of the power of c_q, since the proposed regularisation structure works only if the quantum corrections are regularised overall by some negative power of c_q.

In fact the three-point action vertices[2] $S_{\mu\nu\lambda}(p, q, -p-q)$ diverge at least as rapidly in terms of c_q, as their corresponding two-point action vertices $S_{\mu\lambda}(q)$. This follows from the Ward identity:

$$q^\nu S_{\mu\nu\lambda}(p, q, -p-q) = S_{\mu\lambda}(p) - S_{\mu\lambda}(p+q)\,. \tag{2.4.60}$$

Similarly the degree of divergence of a one-point kernel is set by the zero-point kernel. If q goes from end to end, the one-point kernel behaves with at least the same power of c_q as K_q:

$$p^\lambda K_{\lambda;\mu\nu}(p; q, -p-q) = K_{\mu\nu}(q) - K_{\mu\nu}(q+p)\,. \tag{2.4.61}$$

But if q passes through the side, the large q behaviour depends on whether K_q diverges or decays:

$$q^\lambda K_{\lambda;\mu\nu}(q; p, -p-q) = K_{\mu\nu}(p) - K_{\mu\nu}(q+p)\,. \tag{2.4.62}$$

If K_q diverges, then the one-point kernel diverges at least as rapidly, whilst if K_q decays then $K_{\lambda;\mu\nu}(q; p, -p-q)$ cannot decay, because the right hand side of the Ward identity has $K_{\mu\nu}(p)$ which is independent of q.

However, Ward identities only set a lower bound on the power of c_q, because there remains the possibility that vertices have a worse behaving transverse part. We now pin down the precise behaviour as a power of c_q, *i.e.* both longitudinal and transverse parts.[3]

We start by analysing $A^{(3)}$. This is displayed in (2.4.52). The fundamental vertices $I^{(n)}$ in (2.4.52) only provide some power of q at worst, *cf.* (2.2.10) – (2.2.12), so $A^{(3)}$s large q behaviour is set by its kernels. From (2.2.25) we have $X_q \approx X_{p+q} \approx c_q^{-1}$, and from (2.4.50) we see that all the X one-point vertices in (2.4.52) also diverge in this way. Two terms in $A^{(3)}$ do not diverge with the cutoff, namely the pure I term (*i.e.* the first term) and the term containing X_p on the second line, but the others dominate, so overall $A^{(3)} \approx c_q^{-1}$, as previously assumed, and matching its two-point vertex (2.2.47) as per the minimum required by the Ward identity.

Since from (2.4.6) the classical action θ_0 has the same formula as (2.4.52), only with the replacement $X \mapsto W$, we see immediately that its three-point vertex also $\approx c_q^{-1}$. In fact, given that the quantum correction is $\propto A$, this holds for the full $\theta^{(3)}$.

[2] Recall that three-point action vertices are totally antisymmetric so the order of the arguments is irrelevant.

[3] For a detailed analysis of the large momentum behaviour of arbitrary n higher-point (simple) kernel vertices see sec. 5 of ref. [64].

We can borrow the same formula also for the seed action $\hat{S}$. From (2.2.55) we replace $I \mapsto A$ and $X \mapsto 1$ on the right hand side of (2.4.52) (and so delete the three terms involving the X one-point vertex). Since we have already established that $A_{\mu\nu\lambda}(p,q,-p-q) \approx c_q^{-1}$, we see that indeed $\hat{S}_{\mu\nu\lambda}(p,q,-p-q) \approx c_q^{-2}$ again as previously assumed and enforced by the Ward identity.

It is not hard to see that these estimates hold also for the higher point $A^{(n)}$, $\theta^{(n)}$ and $\hat{S}^{(n)}$ action vertices, where again two of their momentum arguments diverge as $\sim \pm q$ with the others held fixed. For the four-point example, their explicit formulae can be read off from the right hand side of (2.6.11). The same is true for higher point kernel vertices where the zero-point kernel diverges at large momentum. However we will only use the estimates for the three-point vertices throughout this chapter.

At one loop we need also the classical effective action three-point vertex and this in turn depends on the kernels $\mathcal{K}$ and C and their one-point vertices (see eqn. (2.6.10)). From (2.4.45) we clearly have $C_{\alpha\beta}(q) \approx c_q$ (recall $Y_q \approx c_q^{-1}$). Clearly from (2.4.53), the C^{-1} one-point vertex $\approx c_q^{-1}$ whatever the arrangement of the arguments p, q, and $-p-q$, this being the minimum imposed by the Ward identity. But from (2.4.54) the arrangement of the arguments matters for C:

$$C_{\mu;\alpha\beta}(p;q,-p-q) \approx C_q \approx c_q\,, \tag{2.4.63}$$

$$C_{\nu;\alpha\beta}(q;p,-p-q) \approx 1\,, \tag{2.4.64}$$

again precisely as predicted by the Ward identities. One can check using the explicit formulae from the previous section that the behaviour predicted by the Ward identities, holds for all the kernel vertices whose zero-point kernels decay at large momentum, namely t, κ, C and $\mathcal{K}$. In particular we have also

$$\mathcal{K}_{\mu;\alpha\beta}(p;q,-p-q) \approx \mathcal{K}_q \approx c_q^2\,, \tag{2.4.65}$$

$$\mathcal{K}_{\nu;\alpha\beta}(q;p,-p-q) \approx 1\,. \tag{2.4.66}$$

2.5 Regularisation of higher point classical vertices

As we saw in sec. 2.2, the space-time trace terms are designed to implement PV regularisation of the $S_{UV} \approx c_q^{-2}$ part. We will see in sec. 2.6.4 that there are subtleties with this however. The remaining regularisation works provided $\mathcal{S}_0^{(n\geq 3)}$ corrections diverge slower than c_q^{-2}. This was the key property set out in eqns. (2.2.56) and (2.2.59), where it was assumed that in fact $\mathcal{S}_0^{(n)} \approx c_q^{-1}$, or equivalently $\Sigma_0^{(n)} \approx c_q^{-1}$. For the three-point we can now state this more precisely as:

$$\Sigma^0_{\beta\nu\alpha}(q+p,-p,-q) \approx c_q^{-1}\,. \tag{2.5.1}$$

In this section we confirm in detail that this estimate holds true. It is then clear that the estimates are correct for all the higher point $\Sigma_0^{(n)}$ vertices also.

In order to compute $\Sigma_0^{(3)}$ we need the classical action three-point vertex. The latter follows from the flow equation $\Lambda D_\Lambda S^0 = 0$. The analysis is simpler if instead we recast it directly as a flow for Σ_0. Taking the classical limit (*i.e.* $O(1/g^2)$ in the loop expansion) of (2.2.66), and using the identity (2.2.62), we have

$$\dot{\Sigma}^0 = -\dot{A} + \frac{1}{2}\Sigma^0_{,a}\mathcal{K}^{ab}\Sigma^0_{,b} + \frac{1}{2}A_{,a}\kappa^{ab}\Sigma^0_{,b} - \Sigma^0_{,a}C^{ab}\theta^0_{,b}\,. \tag{2.5.2}$$

Expanding to the three-point level:

$$\begin{aligned}
\dot{\Sigma}^0_{\mu\nu\lambda}(p,q,-p-q) = &- \dot{A}^0_{\mu\nu\lambda}(p,q,-p-q) \\
&+ \frac{1}{2\Lambda^2}\Sigma^0_{\mu\alpha}(p)\mathcal{K}_T(p)\Sigma^0_{\alpha\nu\lambda}(p,q,-p-q) \\
&+ \frac{1}{2\Lambda^2}\Sigma^0_{\mu\alpha}(p)\overleftrightarrow{\mathcal{K}}_{\nu;\alpha\beta}(q;p,-p-q)\Sigma^0_{\beta\lambda}(p+q) \\
&+ \frac{1}{4\Lambda^2}\Sigma^0_{\alpha\mu}(p)\kappa(p)A_{\alpha\nu\lambda}(p,q,-p-q) \\
&+ \frac{1}{4\Lambda^2}\Sigma^0_{\mu\nu\alpha}(p,q,-p-q)\kappa(p+q)A_{\alpha\lambda}(p+q) \\
&+ \frac{1}{4\Lambda^2}\Sigma^0_{\mu\alpha}(p)\kappa_{\nu;\alpha\beta}(q;p,-p-q)A_{\beta\lambda}(p+q) \\
&- \frac{1}{4\Lambda^2}\Sigma^0_{\mu\alpha}(p)\kappa_{\lambda;\alpha\beta}(-p-q;p,q)A_{\beta\nu}(q) \\
&- \frac{1}{2\Lambda^2}\Sigma^0_{\alpha\mu}(p)C_T(p)\theta^0_{\alpha\nu\lambda}(p,q,-p-q) \\
&- \frac{1}{2\Lambda^2}\Sigma^0_{\mu\nu\alpha}(p,q,-p-q)C_T(p+q)\theta^0_{\beta\lambda}(p+q) \\
&- \frac{1}{2\Lambda^2}\Sigma^0_{\mu\alpha}(p)C_{\nu;\alpha\beta}(q;p,-p-q)\theta^0_{\beta\lambda}(p+q) \\
&+ \frac{1}{2\Lambda^2}\Sigma^0_{\mu\alpha}(p)C_{\lambda;\alpha\beta}(-p-q;p,q)\theta^0_{\beta\nu}(q) \\
&+ \text{cycles}\,.
\end{aligned} \tag{2.5.3}$$

Here we have used properties of the kernels. First of all, if we contract a zero-point kernel into a two-point vertex, *e.g.* as on second line, then only its transverse part contributes because of the transversality of the two-point function. In addition to this, we used that $\mathcal{K}^a{}_b$, $C^a{}_b$ and $\kappa^a{}_b$, kernel one-point vertices obey interleave identities (2.4.29).

Those terms that depend on $\Sigma_0^{(3)}$ in (2.5.3) we take to the left of the equation. Their multiplying factors collect into

$$Z_{\mu\nu}(p_i) = \mathcal{K}_T(p_i)\,\Sigma^0_{\mu\nu}(p_i) \tag{2.5.4}$$

for each external momentum p_i, as follows from the identity

$$C_T(p)\,\theta^0_{\mu\nu}(p) = \frac{1}{2}\mathcal{K}_T(p)\,S^0_{\mu\nu}(p) \tag{2.5.5}$$

(this can be shown from their explicit form given in secs. 2.4.2.3, 2.4.2.4) and $\kappa(p)A_{\mu\nu}(p) = \mathcal{K}_T(p)\hat{S}_{\mu\nu}(p)$, which is the identity (2.2.62) at two-point level. Using the results of the previous section it is straightforward to verify explicitly that the remaining terms, which we call $T_{\mu\nu\lambda}(p,q,-p-q)$, diverge as $T_{\mu\nu\lambda}(p,q,-p-q) \approx c_q^{-1}$. Indeed, given that in (2.5.2) the action vertices diverge as c_q^{-1}, the only way to produce a contribution that diverges faster than this would be to have the high momentum flow, q, go through both action factors. But this requires q to flow through the kernel from end to end, thus reducing the divergence by at least a factor of c_q again (c_q for κ and C, while $\mathcal{K}$ would provide c_q^2). In this way we verify $\Sigma_0^{(n)} \approx c_q^{-1}$ for all n, *i.e.* (2.2.60) at the classical level.

The flow at the three-point level can be written now as

$$\begin{aligned}&\Lambda\partial_\Lambda\Sigma^0_{\mu\nu\lambda}(p,q,-p-q)\\ &-\frac{1}{4\Lambda^2}\Big[Z_{\mu\alpha}(p)\Sigma^0_{\alpha\nu\lambda}(p,q,-p-q)+Z_{\nu\alpha}(q)\Sigma^0_{\alpha\lambda\mu}(q,-p-q,p)+Z_{\lambda\alpha}(p+q)\Sigma^0_{\alpha\mu\nu}(-p-q,p,q)\Big]\\ &\qquad\qquad = T_{\mu\nu\lambda}(p,q,-p-q)\,. \quad (2.5.6)\end{aligned}$$

This is in a form where it can also be integrated with respect to Λ. $T_{\mu\nu\lambda}(p,q,-p-q)$ is a known function which can be constructed explicitly from the vertices discussed in the previous sections. The terms in square brackets on the left hand side of (2.5.6) provide the integrating factor for the differential equation. Regarding $Z_{\mu\nu}(p)$ as a matrix $Z(p)$, we define

$$\zeta_{\mu\nu}(p) = \left[\exp\int_\Lambda^\infty \frac{d\Lambda_1}{4\Lambda_1^3}\,Z(p)\right]_{\mu\nu} = \delta_{\mu\nu} + \frac{\Box_{\mu\nu}(p)}{p^2}\left(-1+\exp\int_\Lambda^\infty \frac{d\Lambda_1}{2\Lambda_1}\,\tilde{p}^2\mathcal{K}_T(p)F_\Sigma(p)\right), \quad (2.5.7)$$

where for the second equality we use (2.5.4) and (2.4.40). The integrated $\Sigma_0^{(3)}$ is then given by

$$\begin{aligned}\Sigma^0_{\mu\nu\lambda}(p,q,-p-q) &= -\zeta^{-1}_{\mu\alpha}(p)\,\zeta^{-1}_{\nu\beta}(q)\,\zeta^{-1}_{\lambda\gamma}(-p-q)\\ &\quad\times\int_\Lambda^\infty\frac{d\Lambda_1}{\Lambda_1}\,\zeta_{\alpha\alpha'}(p)\,\zeta_{\beta\beta'}(q)\,\zeta_{\gamma\gamma'}(-p-q)\,T_{\alpha'\beta'\gamma'}(p,q,-p-q)\,, \quad (2.5.8)\end{aligned}$$

where $\zeta^{-1}_{\mu\nu}(p)$ is of course given by the same expression as (2.5.7) except for a minus sign in the exponential, and it is understood that all terms under a Λ_1-integral are evaluated at cutoff scale Λ_1. Note that the integration constant vanishes because, by gauge invariance and dimensions, Σ_0 vanishes in the limit $\Lambda \to \infty$.[4] Expanding the exponential in (2.5.7) we see that all the corrections that arise from these integrating factors are also transverse. In this sense the full vertex involves in fact an exponentiation of the equations of motion. It does not affect the divergence with c_q however, since $Z(q)$ vanishes in the large q limit. The integrated version of the $n \geq 4$-point vertices can be explicitly written down in a similar way.

[4] As can be seen by Taylor expanding (2.4.40) for example, Σ_0 has a minimum of four space-time derivatives and thus has an overall $1/\Lambda^2$ factor.

2.6 One loop beta function

To see why it is the large q behaviour of the above vertices that is important, and to provide a test of the formalism, we focus on the simplest quantum correction: the one-loop contribution to the effective action two-point vertex. Since gauge invariance is exactly preserved this computation should, if the flow equation is regularised correctly, also yield the one-loop beta function (i.e. $\beta_1 = \eta_1/2$). To extract it, we need to define the coupling constant $g(\Lambda)$ beyond classical level. We do this by imposing a convenient *renormalization condition*:

$$S = \frac{1}{2g^2(\Lambda)}\mathrm{tr}\int_x F_{\mu\nu}^2 + O(\partial^3)\,, \tag{2.6.1}$$

which means that the full two-point function of S at $O(p^2)$ is given by:

$$S_{\mu_1\mu_2}(p)\big|_{O(p^2)} = 2\,\Box_{\mu_1\mu_2}(p)\,. \tag{2.6.2}$$

Extracting the $O(p^2)$ part of (2.4.39) we have that:

$$S^0_{\mu_1\mu_2}(p)\big|_{O(p^2)} = 2\,\Box_{\mu_1\mu_2}(p) = S_{\mu_1\mu_2}(p)\big|_{O(p^2)}\,. \tag{2.6.3}$$

This means that the renormalization condition at $O(p^2)$ is already saturated at tree level, and, thus, all higher order loop contributions must vanish,

$$S^n_{\mu_1\mu_2}(p)\big|_{O(p^2)} = 0 \qquad\qquad ,\forall n \geq 1\,. \tag{2.6.4}$$

Since $\Sigma^0 = S^0 - \hat{S}$ and $\Sigma^n = S^n$, we also obtain that:

$$\Sigma^n_{\mu_1\mu_2}(p)\big|_{O(p^2)} = 0 \qquad\qquad ,\ \forall n \geq 1\,. \tag{2.6.5}$$

The remaining action θ behaves differently with respect to S and Σ. Its loopwise expansion was already given in (2.4.7) and from there we see that at $O(p^2)$ we have a non-vanishing higher loop contribution from its two-point vertex:

$$\theta^n_{\mu_1\mu_2}(p)\big|_{O(p^2)} = \eta_n \Box_{\mu_1\mu_2}(p). \tag{2.6.6}$$

We computed explicitly all the relevant classical action two-point vertex functions in sec. 2.4.2.3.

At this point we are able to recast (2.4.12) as an algebraic equation for β_1. If we look at the flow equation of $S^1_{\mu\nu}(p)$ and restrict it to $O(p^2)$, we obtain:

$$-4\beta_1\Box_{\mu\nu}(p) = \mathrm{Tr}\Big\{\Lambda\partial_\Lambda\Big[\ln\big(C^{-1}\big)^a{}_b\Big] - \Lambda\partial_\Lambda\big[\ln B^\alpha{}_\beta\big] - \frac{1}{2}\Lambda\partial_\Lambda\big[\ln b^\alpha{}_\beta\big] + \Psi_0^d\frac{\delta}{\delta\phi^d}\Big[\ln\big(C^{-1}\big)^a{}_b\Big] - \Psi_0^a\frac{\delta}{\delta\phi^a}\big[\ln B^\alpha{}_\beta\big] - \frac{1}{2}\Psi_0^a\frac{\delta}{\delta\phi^a}\big[\ln b^\alpha{}_\beta\big] + \frac{\delta\Psi_0^a}{\delta\phi^b}\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^2)}}. \tag{2.6.7}$$

Here we have used the requirement imposed on us by the renormalization condition (2.6.1) and the fact that $\Sigma^0_{\mu\nu}(p)$ is $O(p^4)$, whereas $\theta^0_{\mu\nu}(p)$ and $S^0_{\mu\nu}(p)$ are $O(p^2)$. The only task we are left with is to evaluate the right-hand side of (2.6.7), and then β_1 can be extracted from its coefficient.

2.6.1 Classical flow equations

The classical flow equation (2.4.9) can equivalently be written as follows:

$$\Lambda\partial_\Lambda S^0 = \frac{1}{2\Lambda^2}\frac{\delta S^0}{\delta A_\alpha}\cdot\mathcal{K}_{\alpha\beta}\cdot\frac{\delta\Sigma^0}{\delta A_\beta} - \frac{1}{\Lambda^2}\frac{\delta S^0}{\delta A_\alpha}\cdot C_{\alpha\beta}\cdot\frac{\delta\theta^0}{\delta A_\beta}. \tag{2.6.8}$$

The right-hand side of the above expression has two terms with similar structure, and thus, to write the flow equation for some n-point vertex, one just needs to compute the contributions coming from one term as the contributions coming from the second one follow from the former by a mere relabelling. We write below the flow equations for those vertices that are relevant for the computation of β_1.

The flow equation for $S^0_{\mu_1\mu_2}(p)$ takes the following form:

$$\Lambda\partial_\Lambda S^0_{\mu_1\mu_2}(p) = \frac{1}{4\Lambda^2}S^0_{\alpha\mu_1}(p)\mathcal{K}_T(p)\Sigma^0_{\alpha\mu_2}(p) - \frac{1}{2\Lambda^2}S^0_{\alpha\mu_1}(p)C_T(p)\theta^0_{\alpha\mu_2}(p) + (\mu_1\leftrightarrow\mu_2). \tag{2.6.9}$$

By gauge invariance and dimensions, $S^0_{\mu\nu}(p)$ (and in fact any two-point function) must have a structure similar to (2.4.39). If we substitute this into the above flow equation and solve for some function $F(p)$, we recover the same result as stated in (2.4.39). This is an extra check that the flow equation (2.6.9) is derived in a consistent manner.

The flow equation for $S^0_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$ is given by:

$$\begin{aligned}\Lambda\partial_\Lambda S^0_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = \frac{1}{4\Lambda^2}\Big[&S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_2;\alpha\beta}(p_2;p_1,p_3)\Sigma^0_{\beta\mu_3}(p_3)\\ &- S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_3;\alpha\beta}(p_3;p_1,p_2)\Sigma^0_{\beta\mu_2}(p_2)\\ &+ S^0_{\alpha\mu_1\mu_2}(p_3,p_1,p_2)\mathcal{K}_T(p_3)\Sigma^0_{\alpha\mu_3}(p_3)\\ &+ S^0_{\alpha\mu_1}(p_1)\mathcal{K}_T(p_1)\Sigma^0_{\alpha\mu_2\mu_3}(p_1,p_2,p_3)\Big]\\ -\frac{1}{2\Lambda^2}\Big[\mathcal{K}\mapsto C\;;\;\Sigma\mapsto\theta\Big] &+ \text{cycles}\,,\end{aligned} \tag{2.6.10}$$

where the terms inside brackets from the last line follow from those written explicitly inside brackets above them after the relabelling indicated. One can proceed similarly to derive the flow equation for $S^0_{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4)$:

$$\begin{aligned}&\Lambda\partial_\Lambda S^0_{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) =\\ &\quad= \frac{1}{4\Lambda^2}\Big[S^0_{\alpha\mu_1\mu_2\mu_3}(p_4,p_1,p_2,p_3)\mathcal{K}_T(p_4)\Sigma^0_{\alpha\mu_4}(p_4)\\ &\qquad+ S^0_{\alpha\mu_1\mu_2}(-p_1-p_2,p_1,p_2)\mathcal{K}_{\alpha\beta}(p_1+p_2)\Sigma^0_{\beta\mu_3\mu_4}(-p_3-p_4,p_3,p_4)\\ &\qquad+ S^0_{\alpha\mu_1}(p_1)\mathcal{K}_T(p_1)\Sigma^0_{\alpha\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4)\\ &\qquad+ S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_2;\alpha\beta}(p_2;p_1,p_3+p_4)\Sigma^0_{\beta\mu_3\mu_4}(-p_3-p_4,p_3,p_4)\\ &\qquad- S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_4;\alpha\beta}(p_4;p_1,p_2+p_3)\Sigma^0_{\beta\mu_2\mu_3}(-p_2-p_3,p_2,p_3)\\ &\qquad+ S^0_{\alpha\mu_1\mu_2}(-p_1-p_2,p_1,p_2)\mathcal{K}_{\mu_3;\alpha\beta}(p_3;p_1+p_2,p_4)\Sigma^0_{\beta\mu_4}(p_4)\\ &\qquad- S^0_{\alpha\mu_1\mu_2}(-p_1-p_2,p_1,p_2)\mathcal{K}_{\mu_4;\alpha\beta}(p_4;p_1+p_2,p_3)\Sigma^0_{\beta\mu_3}(p_3)\\ &\qquad+ S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_2\mu_3;\alpha\beta}(p_2,p_3;p_1,p_4)\Sigma^0_{\beta\mu_4}(p_4)\\ &\qquad- S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_2\mu_4;\alpha\beta}(p_2,p_4;p_1,p_3)\Sigma^0_{\beta\mu_3}(p_3)\\ &\qquad- S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_4\mu_2;\alpha\beta}(p_4,p_2;p_1,p_3)\Sigma^0_{\beta\mu_3}(p_3)\\ &\qquad+ S^0_{\alpha\mu_1}(p_1)\mathcal{K}_{\mu_4\mu_3;\alpha\beta}(p_4,p_3;p_1,p_2)\Sigma^0_{\beta\mu_2}(p_2)\Big]\\ &\quad-\frac{1}{2\Lambda^2}\Big[\mathcal{K}\mapsto C\;;\;\Sigma\mapsto\theta\Big] + \text{cycles}\,.\end{aligned} \tag{2.6.11}$$

In deriving these, we again exploit the interleave identities (2.4.29) up to and including the two-point level, as proven in appendix 2.A.

Although we will not need them here, let us note that these equations can be integrated to give explicit formulae for the (integrated) classical effective action vertices. Taking all the $S_0^{(3)}$ terms to the left hand side in (2.6.10), including the $S_0^{(3)}$ part of $\Sigma_0^{(3)}$, we find the same integrating factors $Z_{\mu_i\alpha}(p_i)$ as in (2.5.4). Thus the flow for $S_0^{(3)}$ can be similarly integrated. This results in an explicit formula for $S_0^{(3)}$ of the same form as (2.5.8) – except that in this case there is also an integration constant, namely $I_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$. The latter follows because the $\Lambda\to\infty$ limit of $S_0^{(3)}$ is $I^{(3)}$, as *e.g.* can be seen from (2.6.1) and (2.2.8). Similarly, taking all $S_0^{(4)}$ terms to the left hand side in (2.6.11) gives the same integrating factors and allows its flow to be integrated up to an explicit formula

for $S_0^{(4)}$. This time on the right hand side we have instances of the just computed $S_0^{(3)}$ vertex, and now the integration constant is $I^{(4)}$ as in (2.2.12).

2.6.2 Trace terms

To evaluate the trace terms in (2.6.7) above one needs to expand the log terms. The vertex expansion for the $B^\alpha{}_\beta$ terms, for example, can be done in the following way:

$$\begin{aligned}
&\mathrm{Tr}\ln B^\alpha{}_\beta \equiv \\
&\equiv \mathrm{Tr}\ln\Big[\underbrace{B_r}_{O(A_\mu^0)} + \underbrace{B_\mu(p;r,s)}_{O(A_\mu^1)} + \underbrace{B_{\mu\nu}(p,q;r,s)}_{O(A_\mu^2)} + \dots\Big] && (2.6.12)\\
&= \int_r \ln B_r + \mathrm{Tr}\ln\big[1 + B_r^{-1}B_\mu(p;r,s) + B_r^{-1}B_{\mu\nu}(p,q;r,s) + \dots\big] && (2.6.13)\\
&= 2N\int_q \overbrace{B_q^{-1}B_{\mu\nu}(p,-p;q,-q) - \frac{1}{2}B_q^{-1}B_\mu(p;q,-q-p)B_{p+q}^{-1}B_\nu(-p;p+q,-q)}^{O(A_\mu^2)} + \dots\,, && (2.6.14)
\end{aligned}$$

where N accounts for group combinatorics and the extra factor of two for the $p_\mu \leftrightarrow -p_\nu$ symmetry (or, equivalently, for the definition of the two-point function). Here note that in going from the third to the fourth line we have discarded the log term because it is just a vacuum contribution to the flow equation (or it is differentiated away), and we have also Taylor expanded the remaining log term using

$$\ln(1+x) = x - \frac{x^2}{2} + \cdots\,. \tag{2.6.15}$$

Similar expressions can be written for $\left(C^{-1}\right)^a{}_b$ and $b^\alpha{}_\beta$ terms, respectively.

To evaluate the remaining trace term $\delta\Psi_0^a/\delta\phi^a$ it is useful to use its covariant representation:

$$\begin{aligned}
\frac{\delta\Psi_0^a}{\delta\phi^a} &= \frac{\delta}{\delta\phi^a}\left(-\frac{1}{2}\mathcal{K}^{ab}\Sigma^0{}_{,b} + C^{ab}\theta^0{}_{,b}\right) && (2.6.16)\\
&\equiv -\frac{1}{2\Lambda^2}\frac{\delta}{\delta A_\alpha}\cdot\mathcal{K}_{\alpha\beta}\cdot\frac{\delta\Sigma^0}{\delta A_\beta} + \frac{1}{\Lambda^2}\frac{\delta}{\delta A_\alpha}\cdot C_{\alpha\beta}\cdot\frac{\delta\theta^0}{\delta A_\beta}\,. && (2.6.17)
\end{aligned}$$

This, in turn, allows us to compute it at the two-point level using (2.2.17), yielding:

$$\begin{aligned}\left.\frac{\delta\Psi_0^a}{\delta\phi^a}\right|_{\text{2-pt.function}} = \frac{1}{2\Lambda^2}\int_q \Big\{ & 2\Big(N-\frac{1}{N}\Big)C_{\alpha\beta}(q)\theta^0_{\beta\alpha\mu\nu}(q,-q,p,-p) \\ & -\frac{1}{N}C_{\alpha\beta}(q)\theta^0_{\beta\mu\alpha\nu}(q,p,-q,-p) \\ & +2NC_{\mu;\alpha\beta}(p;q,-p-q)\theta^0_{\beta\nu\alpha}(q+p,-p,-q) \\ & +2NC_{\mu\nu;\alpha\beta}(p,-p;q,-q)\theta^0_{\beta\alpha}(q) \\ & -2NC_{\mu\alpha;\alpha\beta}(p,-q;q,-p)\theta^0_{\beta\nu}(p)\Big\} \\ -\frac{1}{4\Lambda^2}\big[C\mapsto\mathcal{K}\,;\,\theta\mapsto\Sigma\big]\,. & \end{aligned} \tag{2.6.18}$$

This can be further refined. If we split the kernels into their transverse and longitudinal parts, respectively, then one can check explicitly that the $O(1/N)$ longitudinal pieces cancel each other after using Ward identities. One can also show that, under the q integral and by using Lorentz invariance ($q\to -q$, $p_\mu \leftrightarrow -p_\nu$ *etc.*) together with (2.4.6), (2.2.55), and (2.6.11), the following identities hold:

$$\theta^0_{\alpha\mu\alpha\nu}(q,p,-q,-p) = -2\,\theta^0_{\alpha\alpha\mu\nu}(q,-q,p,-p)\,, \tag{2.6.19}$$

$$\Sigma^0_{\alpha\mu\alpha\nu}(q,p,-q,-p) = -2\,\Sigma^0_{\alpha\alpha\mu\nu}(q,-q,p,-p)\,. \tag{2.6.20}$$

This means that the $O(1/N)$ transverse pieces vanish as well, and thus (2.6.18) becomes:

$$\begin{aligned}\left.\frac{\delta\Psi_0^a}{\delta\phi^a}\right|_{\text{2-pt.function}} = \frac{N}{\Lambda^2}\int_q \Big\{ & C_{\alpha\beta}(q)\theta^0_{\beta\alpha\mu\nu}(q,-q,p,-p) \\ & +C_{\mu;\alpha\beta}(p;q,-p-q)\theta^0_{\beta\nu\alpha}(q+p,-p,-q) \\ & +C_{\mu\nu;\alpha\beta}(p,-p;q,-q)\theta^0_{\beta\alpha}(q) \\ & -C_{\mu\alpha;\alpha\beta}(p,-q;q,-p)\theta^0_{\beta\nu}(p)\Big\} \\ -\frac{N}{2\Lambda^2}\big[C\mapsto\mathcal{K}\,;\,\theta\mapsto\Sigma\big]\,. & \end{aligned} \tag{2.6.21}$$

2.6.3 Two-point vertex at zeroth order in momentum

If the one-loop two-point vertex is transverse and quasi-local, as it should be, it can have no momentum independent part, *i.e.* at $O(p^0)$ it ought to vanish. However this property is strictly only true if the flow equation is properly regulated since, as we will see, it would then follow from the fact that this contribution can be cast as a momentum-space surface integral at large q, which vanishes if properly regularised. Actually with sufficient care the construction proposed in ref. [47] does ensure this, up to a divergent term that can be discarded since it is Λ independent but differentiated by Λ, because it in fact depends only on two-point action vertices or zero-point kernels, as we will see.

Thus we turn to computing the two-point $O(p^0)$ part of the one-loop flow equation (2.4.12), or equivalently (2.4.10). As we have just noted, it ought to vanish. The first line on the right-hand side is clearly of $O(p^2)$ and above and can therefore be discarded. This means that the only contribution comes from the functional trace terms, and hence, at $O(p^0)$, the right hand side of (2.4.12) becomes:

$$\begin{aligned}\mathrm{Tr}\Big\{&\Lambda\partial_\Lambda\big[\ln\big(C^{-1}\big)^a{}_b\big]-\Lambda\partial_\Lambda\big[\ln B^\alpha{}_\beta\big]-\frac{1}{2}\Lambda\partial_\Lambda\big[\ln b^\alpha{}_\beta\big]+\frac{\delta\Psi_0^a}{\delta\phi^b}\\&+\Psi_0^m\frac{\delta}{\delta\phi^m}\big[\ln\big(C^{-1}\big)^a{}_b\big]-\Psi_0^m\frac{\delta}{\delta\phi^m}\big[\ln B^\alpha{}_\beta\big]-\frac{1}{2}\Psi_0^m\frac{\delta}{\delta\phi^m}\big[\ln b^\alpha{}_\beta\big]\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^0)}}.\end{aligned}\tag{2.6.22}$$

Moreover, one can easily see that the second line above does not contribute at $O(p^0)$ because the blocking functional part (*i.e.* Ψ_0^m) has either a two-point function residue which is at least of $O(p^2)$, or a three-point function residue which is at least $O(p)$ (furthermore it is multiplied by a term that is antisymmetric in q, and hence vanishes under the q integral). The first three terms can be computed using trace expansions as described in section 2.6.2 above. For example, using (2.6.14) we will obtain the following:

$$\begin{aligned}&\mathrm{Tr}\Big\{\Lambda\partial_\Lambda\big[\ln B^\alpha{}_\beta\big]\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^0)}}\equiv\\&\quad\equiv 2N\Lambda\partial_\Lambda\int_q B_q^{-1}B_{\mu\nu}(0,0;q,-q)-\frac{1}{2}B_q^{-1}B_\mu(0;q,-q)B_q^{-1}B_\nu(0;q,-q)\,.\end{aligned}\tag{2.6.23}$$

Here note that quasi-locality ensures that the $p\to 0$ limit is straightforward. The above expression can be simplified by using differential Ward identities:

$$\begin{aligned}\mathrm{Tr}\Big\{\Lambda\partial_\Lambda\big[\ln B^\alpha{}_\beta\big]\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^0)}}&\equiv 2N\Lambda\partial_\Lambda\int_q B_q^{-1}\frac{1}{2}\partial_\mu^q\partial_\nu^q B_q-\frac{1}{2}B_q^{-1}\partial_\mu^q B_q\,B_q^{-1}\partial_\nu^q B_q\\&=N\Lambda\partial_\Lambda\int_q\partial_\mu^q\partial_\nu^q\big[\ln B_q\big]\,.\end{aligned}\tag{2.6.24}$$

Similar expressions can be computed for C and b terms, respectively. The remaining term can be computed starting from (2.6.21) and following a similar strategy, yields:

$$\begin{aligned}\frac{\delta\Psi_0^a}{\delta\phi^a}\bigg|_{\substack{\text{2-pt.function}\\ \text{at } O(p^0)}}&=\frac{N}{\Lambda^2}\int_q\Big\{C_{\alpha\beta}(q)\frac{1}{2}\partial_\mu^q\partial_\nu^q\theta^0_{\beta\alpha}(q)\\&\qquad+\partial_\mu^q C_{\alpha\beta}(q)\,\partial_\nu^q\theta^0_{\beta\alpha}(q)\\&\qquad+\frac{1}{2}\partial_\mu^q\partial_\nu^q C_{\alpha\beta}(q)\,\theta^0_{\beta\alpha}(q)\Big\}\\&\quad-\frac{N}{2\Lambda^2}\big[C\mapsto\mathcal{K}\,;\,\theta\mapsto\Sigma\big]\,,\end{aligned}\tag{2.6.25}$$

which can be recast as:

$$\left.\frac{\delta\Psi_0^a}{\delta\phi^a}\right|_{\substack{\text{2-pt.function}\\ \text{at } O(p^0)}} = \frac{N}{4\Lambda^2}\int_q \partial_\mu^q\partial_\nu^q\big[2C_T(q)\theta^0_{\alpha\alpha}(q) - \mathcal{K}_T(q)\Sigma^0_{\alpha\alpha}(q)\big] \tag{2.6.26}$$

$$= \frac{N}{2}(D-1)\int_q \partial_\mu^q\partial_\nu^q\Big[\tilde{q}^2\mathcal{K}_T(q)F_{\hat{S}}(q)\Big] \tag{2.6.27}$$

$$= \frac{N}{2}(D-1)\int_q \Lambda\partial_\Lambda\partial_\mu^q\partial_\nu^q\big[\ln\tilde{P}_q^\perp\big]\,, \tag{2.6.28}$$

where to get the second line we used (2.5.5). The result can be cast as a total Λ-derivative which we do in the final line, where

$$\tilde{P}_q^\perp := \Lambda^2 P^\perp(q)\,, \qquad P^\perp(q) := \frac{1}{2q^2F(q)}\,. \tag{2.6.29}$$

As we will see in sec. 2.6.4 this object plays the rôle of an effective action propagator (in the transverse space). Indeed from (2.4.39):

$$P^\perp(q)\,S^0_{\alpha\beta}(q) = \delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\,. \tag{2.6.30}$$

In (2.6.28) one can see that the $\Lambda\partial_\Lambda$ sits inside the integral, whereas in (2.6.24) it sits outside the momentum integral. In general, changing the integration and differentiation order is not trivial and to ensure consistency one must show that the undifferentiated expression is sufficiently well-behaved in both UV and IR. However as stressed in ref. [47], the flow equation is consistent only if we consider all trace terms together. Therefore, we should interpret this as having the $\Lambda\partial_\Lambda$ outside the integral in (2.6.28), as long as we add to it all the remaining terms similar to and including (2.6.24). This means that (2.6.22) becomes:

$$\Lambda\partial_\Lambda\int_q \partial_\mu^q\partial_\nu^q\Big\{\big[\ln C\big]_{\alpha\alpha}(q) + \ln B_q + \frac{1}{2}\ln b_q - \frac{D-1}{2}\ln\tilde{P}_q^\perp\Big\}\,, \tag{2.6.31}$$

where $\big[\ln C\big]_{\alpha\alpha}(q) = (D-1)\ln C_T(q) + \ln C_L(q)$. Given that the above integral is an integral of a total derivative, it amounts to a surface integral at large q, as advertised at the beginning of this section. This surface integral can be discarded, and thus the integral vanishes, if the term in braces above is UV finite. Using the definitions (2.2.50), (2.2.51) and (2.2.52), we can write it as follows:

$$\begin{aligned}(D-1)\ln\frac{c_q}{c_q+\tilde{q}^2} + \ln\frac{1}{1+\tilde{q}^2Y_q} + \ln\Big(1+\tilde{q}^2Y_q\Big) + \frac{1}{2}\ln\Big(t_q+\tilde{q}^2\Big) + \frac{D-1}{2}\ln\frac{2\tilde{q}^2\Big(c_q^2+\tilde{q}^2\Big)}{c_q^2}\\ = -(D-1)\ln\left(\frac{c_q}{\tilde{q}^2}+1\right) + \frac{D-1}{2}\ln\left(\frac{c_q^2}{\tilde{q}^2}+1\right) + \frac{1}{2}\ln\left(\frac{t_q}{\tilde{q}^2}+1\right) + \frac{1}{2}\ln\tilde{q}^2\,.\end{aligned} \tag{2.6.32}$$

All but the last term vanish rapidly for large momentum (from (2.6.15) and because

c_q and t_q do). In this calculation, the last term can be formally discarded because it vanishes under the combined action of q and Λ derivatives. In this sense we are justified in dropping all the $O(p^0)$ terms, confirming the arguments outlined in ref. [47] work to $O(p^0)$.

However it is unclear whether the part that leads to this unregulated $\ln \tilde{q}^2$ term can always be safely discarded. In particular it is unclear whether such a part could cause an unrecoverable failure of UV regularisation at higher loops. If we first compute the q derivatives, the resulting momentum integral is quadratically UV divergent. By dimensions one might expect to find an analogous logarithmic UV divergence in the $O(p^2)$ part. In the next section we confirm this expectation.

2.6.4 Two-point vertex at second order in momentum

The next step is to compute the one-loop beta function by analysing the flow equation (2.6.7). The coefficient of the beta function at one loop is a universal quantity and thus it should be independent of all artefacts of the regularisation scheme [71]. In the earlier successful construction [28, 62–83] this could be understood as follows. Since the beta function is dimensionless, the momentum integrals that compute it are also dimensionless. By trading the higher-point regularisation vertices in these integrals for Λ-derivatives of effective action vertices, using the classical flow equations, one finds that the results combine into terms that either vanish, because they are Λ-derivatives of regularised dimensionless integrals (which thus do not actually depend on Λ), or terms that survive but only because when cast in this way a finite result is obtained from a logarithmic IR divergence, where the effective action is universal – as determined by the renormalization condition (2.6.1) [71,88]. Here we will follow this route by manipulating the $\delta\Psi_0^a/\delta\phi^a$ term. However, before delving into the details of this, it is better for the ease of presenting to focus on the other trace terms first.

The remaining trace terms can be partitioned into two groups, namely $\Lambda\partial_\Lambda \ln K$ and $\Psi_0^m \frac{\delta}{\delta\phi^m} \ln K$, where $K = C^{-1}, B, b$. This means that in order to compute them it will be sufficient to do this for one value of K chosen for convenience, all the other contributions following from this one by substitution. It is straightforward to write down the first B trace terms using (2.6.14) as follows:

$$\begin{aligned}
&\mathrm{Tr}\Big\{\Lambda\partial_\Lambda\big[\ln B^\alpha{}_\beta\big]\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^2)}} \equiv \\
&\qquad \equiv 2N\Lambda\partial_\Lambda \int_q \Big\{ B_q^{-1} B_{\mu\nu}(p,-p;q,-q) \\
&\qquad\qquad - \frac{1}{2} B_q^{-1} B_\mu(p;q,-q-p) B_{p+q}^{-1} B_\nu(-p;p+q,-q) \Big\}_{O(p^2)} . \qquad (2.6.33)
\end{aligned}$$

The second B trace term can be written using $\mathrm{Tr}\ \Psi_0^m \frac{\delta}{\delta\phi^m} \ln B = \mathrm{Tr}\ \Psi_0^m B_{,m} B^{-1}$, yielding:

$$\begin{aligned}\mathrm{Tr}\Big\{\Psi_0^m \frac{\delta}{\delta\phi^m}\big[\ln B^{\alpha}{}_{\beta}\big]\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^2)}} \equiv \\ \equiv 2N\int_q \Big\{ & 2C_T(p)\theta^0_{\mu\alpha}(p)B_{\alpha\nu}(p,-p;q,-q)B_q^{-1} \\ & + C_T(p)\theta^0_{\mu\alpha}(p)B_{\alpha}(p;-p-q,q)B_{\nu}^{-1}(-p;p+q,-q) \\ & + C_{\alpha\beta}(0)\theta^0_{\mu\nu\alpha}(p,-p,0)B_{\beta}(0;q,-q)B_q^{-1} \\ & - \frac{1}{2}\big[C\mapsto\mathcal{K}\ ;\ \theta\mapsto\Sigma\big]\Big\}_{O(p^2)}\,. \end{aligned} \tag{2.6.34}$$

The third line is odd in p and thus has no $O(p^2)$ part. Furthermore the one-point kernel is antisymmetric in q, whereas B_q is symmetric, and so the entire row vanishes under the q integral. A similar argument holds for the $\mathcal{K}$ sector as well. From (2.4.6) we get that $\theta^0_{\mu\alpha}(p)\big|_{O(p^2)} = 2\,\Box_{\mu\alpha}(p)$, and, given that all the other functions appearing in (2.6.34) are quasi-local, we are free to set $p=0$ anywhere else in the C sector. In the $\mathcal{K}$ sector on the other hand, $\Sigma^0_{\mu\alpha}(p)$ is $O(p^4)$, which means that it does not contribute to (2.6.34) at all. Collecting everything together we have

$$\begin{aligned}\mathrm{Tr}\Big\{\Psi_0^m \frac{\delta}{\delta\phi^m}\big[\ln B^{\alpha}{}_{\beta}\big]\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^2)}} \equiv \\ \equiv 2N\int_q \Big\{ & 2C_T(0)2\Box_{\mu\alpha}(p)B_{\alpha\nu}(0,0;q,-q)B_q^{-1} \\ & + C_T(0)2\Box_{\mu\alpha}(p)B_{\alpha}(0;-q,q)B_{\nu}^{-1}(0;q,-q)\Big\}_{O(p^2)}\,, \end{aligned} \tag{2.6.35}$$

which we can recast if we use differential Ward identities for the kernel vertices:

$$\mathrm{Tr}\Big\{\Psi_0^m \frac{\delta}{\delta\phi^m}\big[\ln B^{\alpha}{}_{\beta}\big]\Big\}_{\substack{\text{2-pt.function}\\ \text{at } O(p^2)}} \equiv 4N\Box_{\mu\alpha}(p)\int_q \partial^q_\alpha\partial^q_\nu \ln B_q\,. \tag{2.6.36}$$

The flow equation (2.6.7) shows that if we add up all trace term contributions we ought to end up with a transverse expression, *i.e.* $\propto \Box_{\mu\nu}(p)$. Terms like (2.6.36) are transverse on the ν index if we use Lorentz invariance of the q integral. But such arguments make sense strictly speaking only if the integral is properly regularised. As already noted in sec. 2.6.3, trace terms should be considered together, not individually. However for more involved computations, this is not enough because the sum would still be ambiguous due to the so-called momentum routing problem (see app. 2.B). We need in general to make statements about *individual* terms, and to do so we need to apply some auxiliary regularisation for each momentum integral to give them a well defined meaning. In the end one can put all the parts back together again at which point, provided that the trace terms do actually fully regularise, the result is finite and the "pre-regularisation" can be

safely removed. From now on we will use dimensional regularisation for this purpose, *i.e.* compute in $d = 4 - \varepsilon$ dimensions. This means that terms similar to (2.6.36) are indeed transverse on both indices. If we contract p_μ into (2.6.33) and use Ward identities we will get (ignore the $O(p^2)$ requirement for now):

$$2N\Lambda\partial_\Lambda \int_q \Big\{ B_q^{-1}\Big(B_\mu(0;q,-q) - B_\mu(-p;p+q,-q)\Big) \\ - \frac{1}{2}B_q^{-1}\big(B_q - B_{p+q}\big)B_{p+q}^{-1}B_\nu(-p;p+q,-q)\Big\}\,. \tag{2.6.37}$$

The first term in the bracket on the first line is odd in q, and thus vanishes when integrated. In the second line we can relabel $q \to -q-p$ for the first term inside the bracket and use the antisymmetry of the one-point kernel vertex, $B_\nu(-p;p+q,-q) = -B_\nu(-p;-q,p+q)$, to arrive at

$$2N\Lambda\partial_\Lambda \int_q \Big\{ - B_q^{-1}B_\mu(-p;p+q,-q) + B_q^{-1}B_\nu(-p;p+q,-q)\Big\} \tag{2.6.38}$$

$$= 2N\Lambda\partial_\Lambda \int_q \big\{0\big\}\,. \tag{2.6.39}$$

Lorentz invariance implies that a similar result would have been obtained if p_ν had been used. Therefore, all terms with a similar structure to (2.6.33), when properly regularised, are individually transverse.

The last step is to compute $\delta\Psi_0^a/\delta\phi^a$. At this point we note that (2.6.21) is formally transverse prior to any substitution being done. One can easily check this by contracting it with p_μ (or equivalently p_ν), and then show that the expression vanishes at all orders, not only at $O(p^2)$. This means that formally all trace terms in (2.6.7) are individually transverse. This is gratifying but unsurprising: it is a consistency check on the formal preservation of gauge invariance by the flow equation. As we have just emphasised, for these manipulations to be meaningful we need to verify this with a gauge invariant pre-regularisation in place. If the expressions are in fact properly regularised through the PV trace terms, we should then be able to combine the results into a finite transverse, and in fact universal, answer.

Thus the real test is to see whether (2.6.21) remains transverse in dimensional regularisation after we trade four and three-point Σ^0 and θ^0 vertices for (Λ derivatives of) effective action vertices. If we substitute first (2.6.11) into (2.6.21) to trade four-point Σ^0 and θ^0 vertices, and then use (2.6.10) to trade away the Σ^0 and θ^0 three-point vertices,

we obtain eventually the following:

$$\begin{aligned}
&\frac{\delta\Psi_0^a}{\delta\phi^a}\Big|_{\substack{\text{2-pt.function}\\ \text{at } O(p^2)}} = \\
&= N\int_q \Bigg\{ -\Lambda\partial_\Lambda\Big[P^\perp(q)S^0_{\alpha\alpha\mu\nu}(q,-q,p,-p)\Big]_{O(p^2)} \\
&\qquad + \frac{1}{2}\Lambda\partial_\Lambda\Big[P^\perp(q)P^\perp(p-q)S^0_{\alpha\beta\mu}(p-q,q,-p)S^0_{\alpha\beta\nu}(p-q,q,-p)\Big]_{O(p^2)} \\
&\qquad + \frac{\Box_{\alpha\nu}(p)}{\Lambda^2 q^2 F_q}\Bigg[\frac{\big(1-\mathcal{K}_T(q)\big)\Sigma^0_{\alpha\mu}(q)-\big(1-2C_T(q)\big)\theta^0_{\alpha\mu}(q)}{q^2} \\
&\qquad\qquad - \frac{1}{2}\partial^q_\mu\partial^q_\alpha\big(S^0_{\beta\beta}(q)-\Sigma^0_{\beta\beta}(q)+\theta^0_{\beta\beta}(q)\big) \\
&\qquad\qquad + 2\frac{q_\alpha q_\mu}{q^2}\big(\mathcal{K}_T(q)F_\Sigma(q)-2C_T(q)F_\theta(q)-F_q\big) \\
&\qquad\qquad + \frac{1}{2\Lambda^2 q^2 F_q}\partial^q_\mu\big(\Box_{\beta\gamma}(q)F_q\big)\,\partial^q_\alpha\big(S^0_{\beta\gamma}(q)-\Sigma^0_{\beta\gamma}(q)+\theta^0_{\beta\gamma}(q)\big)\Bigg]\Bigg\},
\end{aligned} \tag{2.6.40}$$

where $P^\perp(q)$ is an effective propagator and was defined already in (2.6.29). To arrive at this expression we use the symmetries including differential Ward identities in a similar way to the trace terms above. As there, we do not need the explicit formulae except for the zero-point kernels and two-point action vertices.

All the terms appearing in the above expression are manifestly transverse except the Λ-derivative ones. We can evaluate the contribution of the latter in dimensional regularisation and check whether they are transverse or not. Again, we are free to move the Λ derivative outside the integral if the resulting momentum integrals are regularised. The first term (containing the four-point vertex) would then vanish if the UV regularisation (as also provided by the trace terms) is correctly in place, because the integral is dimensionless in four dimensions and thus actually independent of Λ. This is so because, although $P^\perp \sim 1/q^2$ for small q, this IR divergence is integrable in four dimensions. Similarly the second term (containing the three-point effective action vertices) has no surviving UV contribution if it is properly regulated there, but this time there is a logarithmic IR divergence which when differentiated with respect to Λ gives a finite universal answer dependent only on the renormalization condition (2.6.1).

Now we contract the first two terms in (2.6.40) with p_μ and use Ward identities to further simplify it, following essentially the same steps that took us from (2.6.37) to (2.6.38). This time we get:

$$N\Lambda\partial_\Lambda\int_q P^\perp(q)\Big\{S^0_{\alpha\alpha\nu}(q,p-q,-p)-P^\perp(p-q)S^0_{\alpha\beta}(p-q)S^0_{\alpha\beta\nu}(q,p-q,-p)\Big\}. \tag{2.6.41}$$

Now using the propagator identity (2.6.30) and Ward identities recursively this simplifies to

$$\Lambda\partial_\Lambda \int_q \frac{(p+q)_\beta}{(p+q)^2}\left(\delta_{\beta\nu} - \frac{q_\beta q_\nu}{q^2}\right). \tag{2.6.42}$$

The $\delta_{\beta\nu}$ part is odd in q (one can easily see that this is the case after expanding the bracket and relabelling $q \to q-p$), and hence vanishes, whereas the second term can be recast using $q\cdot(p+q) = \frac{1}{2}[(p+q)^2 + q^2 - p^2]$, yielding:

$$-\frac{\Lambda}{2}\frac{\partial}{\partial\Lambda}\int_q \frac{q_\nu}{q^2} + \frac{q_\nu}{(p+q)^2} - \frac{p^2 q_\nu}{(p+q)^2 q^2}\,. \tag{2.6.43}$$

Again, the first term is odd in q. The second term is quadratically divergent but in dimensional regularisation it vanishes, whereas the third term is linearly divergent and gives a finite contribution in dimensionsal regularisation (due to a logarithmic subdivergence):

$$\Lambda\partial_\Lambda\left(-\frac{1}{2}\frac{p^2 p_\nu}{(4\pi)^2}\ln\Lambda\right) = -\frac{1}{2}\frac{p^2 p_\nu}{(4\pi)^2}\,, \tag{2.6.44}$$

where we recognise that the regularisation scale is set by Λ. This means that the original Λ derivative terms from (2.6.40) give a non-vanishing longitudinal contribution in dimensional regularisation, which amounts to

$$-\frac{1}{2}\frac{p_\mu p_\nu}{(4\pi)^2}\,, \tag{2.6.45}$$

and thus they cannot be transverse on their own. In fact, apart from an overall factor of $\frac{1}{2}$, the effective action terms in (2.6.40) have precisely the same structure as found in ref. [71] as part of the one-loop beta function computation using the manifestly gauge invariant flow equation developed there. Moreover, the computed longitudinal part (2.6.45) is precisely half the one found in ref. [71]. The same arguments in that paper can be used to extract the full IR contribution from the first two lines of (2.6.40):

$$\frac{19}{6}p^2\delta_{\mu\nu} - \frac{11}{3}p_\mu p_\nu\,, \tag{2.6.46}$$

and this has the same longitudinal part.

Thus we see that if we assume that (2.6.7) is sufficiently regularised in the UV we get a non-transverse answer. As we noted, the above result is arrived at independently of the detailed form of the regularisation structure, apart from the expressions for two-point action vertices and zero-point kernels. This means that the problem cannot be cured by a more careful choice of cutoff functions or an alternative covariantization, but lies at a deeper structural level.

Since formally the integrals start out transverse, it follows that there are unregulated UV divergences which provide a cancelling longitudinal contribution. In fact if the third

term in (2.6.43) has no UV regularisation we can (cavalierly) regard it as an infinite constant, independent of Λ, which is then annihilated by the Λ derivative.

The problem then is that we cannot extract universal information from the S_0 terms in (2.6.40). We can see this explicitly by substituting $S_0 = \Sigma_0 + \hat{S}$. Since $P^\perp(q) \approx c_q^2$, *cf.* (2.6.29) and (2.4.39), the resulting terms with a Σ_0 three-point or four-point vertex are UV regularised because $\Sigma_0^{(3)}$ and $\Sigma_0^{(4)}$ diverge only as c_q^{-1}, as we confirmed explicitly in sec. 2.5. As explained above this means that the $\Sigma_0^{(4)}$ contribution actually vanishes, because the integral is then well defined and dimensionless, and thus annihilated by $\Lambda\partial_\Lambda$. The contributions with a $\Sigma_0^{(3)}$ vertex vanish for the same reason. To see this we only need to show that there is no longer a logarithmic divergence at small q. Since $\Sigma_0^{(2)}$ is $O(p^4)$, *cf.* (2.4.40), we know by gauge invariance (or equivalently the Ward identities) that $\Sigma^0_{\alpha\beta\mu}(p-q,q,-p)$ starts cubic in momenta in a small momentum expansion. In other words, retaining powers of p up to a maximum of $O(p^2)$, as $q \to 0$ it vanishes with at least one power of q, and thus indeed there is no longer an IR logarithmic divergence from the second line of (2.6.40).

This means that, on substituting $S_0 = \Sigma_0 + \hat{S}$, the top two lines of (2.6.40) become precisely the same expression but with $S_0^{(3)}$ and $S_0^{(4)}$ replaced by $\hat{S}_0^{(3)}$ and $\hat{S}_0^{(4)}$ respectively. However $\hat{S}$ is freely designed by us as part of the regularisation scheme so cannot of itself contain the requisite universal information. Indeed we can go one step further and introduce a transverse space effective propagator for $\hat{S}_{\alpha\beta}(q)$:

$$\hat{P}(q) := \frac{1}{2q^2 F_{\hat{S}}(q)}\,, \tag{2.6.47}$$

which satisfies the obvious transverse projector relation, *cf.* (2.4.38) and (2.6.30). Substituting

$$P^\perp(q) = \hat{P}(q) + \Big[P^\perp(q) - \hat{P}(q)\Big] = \hat{P}(q) + \frac{c_q^3}{2q^2}\frac{1-c_q}{(c_q^2+\tilde{q}^2)(c_q+\tilde{q}^2)}\,, \tag{2.6.48}$$

for all propagators in the same top two lines, it is apparent that any part containing the last expression above is again both UV and IR finite (in the latter case because $c(0) = 1$) and therefore vanishes under the Λ derivative. Thus the top two lines only contain information determined by our choice of seed action.

We have shown that if we assume that (2.6.7) is sufficiently regularised in the UV we get a universal non-transverse answer from a certain IR contribution. Since formally the integrals start out transverse, it follows that there are actually unregulated UV divergences which can formally be viewed as providing a cancelling longitudinal contribution. Although one can indeed proceed in this way, it is then no longer possible to prove the result is universal. Indeed we show that this contribution can be expressed solely in terms of the seed action, a quantity that we are free to choose.

2.7 Summary and Conclusions

We conclude this chapter with a brief summary and discussion of our main findings. In sec. 2.6.3, we showed that all but one term in the $O(p^0)$ part of the two-point function at one-loop level is correctly regularised. Key to this, was the finding that the $O(p^0)$ actually only depends on zero-point kernels and two-point action vertices. If properly regularised, it has to vanish by gauge invariance. We show that it does vanish. For the properly regularised terms, this is so because they can be written as a vanishing UV surface term in momentum space. The one term that is not properly regularised can however be formally set to zero because it is the Λ derivative of a Λ independent quadratically divergent term.

The $O(p^2)$ part should give us the one-loop beta function, if properly regularised, and thus be universal (independent of regularisation artefacts). In sec. 2.6.4 we recast the result in a way that has previously allowed a universal answer to be extracted, if properly regularised [71, 88]. We saw that although the $O(p^2)$ part is formally transverse, if computed assuming complete UV regularisation there is actually a non-vanishing longitudinal part. The problem is caused by unregulated UV divergences, which appears to be associated to the one found at $O(p^0)$, and is independent of choice of cutoff functions or covariantization.

Similarly to the $O(p^0)$ case, if the incompletely regularised longitudinal part is regarded formally as an infinite constant, it can then be set to zero, since it is differentiated by Λ. The problem then is that the earlier techniques can no longer be used to extract a universal answer because it is no longer clear which IR divergent terms should provide a universal answer and which should be formally set to zero. Indeed we saw explicitly that in the current case these terms can then be made independent of the classical effective action. Instead they can be seen to depend only on the seed action, part of the regularisation structure which we are free to choose (and which is thus non-universal).

Note that these results are not at variance with ref. [47] where the one-loop beta function coefficient is successfully computed. It is there extracted from determinants of zero-point kernels and two-point vertices (the result obtained in 2.6.3), using a heat kernel approach. It is thus insensitive to the problems with UV regularisation that we are highlighting. Indeed the heat-kernel expression formally depends only on the value of these kernels at vanishing momentum.

Our results show that the one-loop off-shell contribution is not fully regularised. At one loop it appears to be possible to proceed formally and obtain valid answers, but it is unclear whether that would still be possible at higher loops. Also, the powerful techniques [71, 88] that previously allowed universal results to be extracted, unfortunately fail in this formulation. As we have already noted, a repair of this problem (if there is one which does not introduce explicit PV fields as done previously [28, 62–83]) would

seem to require structural changes. Hints lie in the insufficient regularisation at $O(p^0)$ seen in 2.6.3, and in the longitudinal sector as seen in sec. 2.6.4 and discussed in more general terms in sec. 2.3. Hopefully, following these hints, a way can be found to implement a more complete regularisation.

2.A Interleave identities for compound kernels

Interleave identites (2.4.29) hold for those kernels whose covariantisation is carried by gauge fields acting by commutation. This means, for example, that any well-behaved (or quasi-local) function of $\Delta^a{}_b$ or $\Delta_\parallel$ fulfills this condition. We exploit these identities for example in the one-loop beta function calculation where it allows the cancellation of certain terms. However, the two kernels $\mathcal{K}^a{}_b$ and $C^a{}_b$, are not solely constructed in this way, and it is not a priori evident that they still obey interleave identities. It may be that they do not at some higher point level. Fortunately for us they do obey the interleave identities up to the two-point level, and this is all that we need.

In this appendix we prove that the identities do hold up to this level using the other already established symmetries. At the same time we give more details on how to construct them. Given the definition of $\mathcal{K}^a{}_b$ in (2.2.64), it is sufficient to check the vertices of $C^a{}_b$. Then, since $\kappa^a{}_b$ is a function of $\Delta^a{}_b$, it is guaranteed that a similar behaviour is inherited by the $\mathcal{K}^a{}_b$ vertices. From (2.2.50) we can write $\left(C^{-1}\right)^a{}_b$ in the following way:

$$\left(C^{-1}\right)^a{}_b = \delta^a{}_b + R^a{}_b + z^a{}_b\,, \tag{2.A.1}$$

where $R^{ab} := \gamma^{am} A_{,mn} \gamma^{nb}$ and $z^{ab} := K^a_\alpha Y^{\alpha\beta} K^b_\beta$. It is easy to see that $z^a{}_b$ is made out of gauge fields which act by commutation, and hence its vertices obey interleave identities. However, checking whether R vertices satisfy these identities or not is non-trivial. From sec. 2.4.2.2 we see that for two (matrix valued) functions $J_\mu(x)$ and $J_\nu(y)$ the following equivalent statements can be written:

$$J_a R^{ab} J_b \equiv \int_x \int_y J^a_\mu(x) R^{ab}_{\mu\nu}(x,y) J^b_\nu(y) \tag{2.A.2}$$

$$= \frac{2}{\Lambda^2} \mathrm{tr} \int_x \int_y J_\mu(x) R_{\mu\nu}(x,y) J_\nu(y) \tag{2.A.3}$$

$$= \frac{2}{\Lambda^2} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \int_x \int_y \int_{x_1 \cdots x_k} R_{\mu_1 \ldots \mu_{k-m}, \mu_{k-m+1} \ldots \mu_k; \mu\nu}(x_1 \ldots x_{k-m}; x_{k-m+1} \ldots x_k; x, y) \times \mathrm{tr}\Big[J_\mu(x) A_{\mu_1}(x_1) \cdots A_{\mu_{k-m}}(x_{k-m}) J_\nu(y) A_{\mu_{k-m+1}}(x_{k-m+1}) \cdots A_{\mu_k}(x_k) \Big]\,, \tag{2.A.4}$$

where we note that the sum is half that in (2.4.27) and we used the definition of functional derivatives below (2.2.13) and normalisation of the generators. We also note that

$R^{ab}_{\mu\nu}(x,y)$ is given by[5]

$$R^{ab}_{\mu\nu}(x,y) = \frac{1}{\Lambda^4}\frac{\delta^2 A}{\delta A^b_\nu(y)\delta A^a_\mu(x)}\,, \tag{2.A.5}$$

We can look at the functional derivatives of A in more detail if we use the trace expansion (2.4.13). The first functional derivative takes the following form:

$$\frac{\delta A}{\delta A^a_\mu(x)} = \sum_{k=2}^{\infty}\int_{x_1\ldots x_{k-1}} A_{\mu\mu_1\ldots\mu_{k-1}}(x,x_1\ldots x_{k-1})\mathrm{tr}\Big[T^a A_{\mu_1}(x_1)\ldots A_{\mu_{k-1}}(x_{k-1})\Big]\,, \tag{2.A.6}$$

where we consider only the single-trace terms which are the only ones of interest to us. If we differentiate it a second time, we will obtain:

$$\begin{aligned}&\frac{\delta^2 A}{\delta A^b_\nu(y)\delta A^a_\mu(x)} = \\ &= \sum_{k=2}^{\infty}\sum_{m=0}^{k-2}\int_{x_1\ldots x_{k-2}} A_{\mu\mu_1\ldots\mu_{k-2-m}\nu\mu_{k-1-m}\ldots\mu_{k-2}}(x,x_1\ldots x_{k-2-m},y,x_{k-1-m}\ldots x_{k-2}) \\ &\quad \mathrm{tr}\Big[T^a A_{\mu_1}(x_1)\ldots A_{\mu_{k-2-m}}(x_{k-2-m})T^b A_{\mu_{k-1-m}}(x_{k-1-m})\ldots A_{\mu_{k-2}}(x_{k-2})\Big]\,.\end{aligned} \tag{2.A.7}$$

If we now relabel $k\mapsto k+2$ and substitute the result into (2.A.2) using (2.A.5), we will obtain the following identity:

$$\begin{aligned}&J_a R^{ab} J_b \equiv \\ &\equiv \frac{1}{\Lambda^4}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\int_x\int_y\int_{x_1\ldots x_k} A_{\mu\mu_1\ldots\mu_{k-m}\nu\mu_{k-m+1}\ldots\mu_k}(x,x_1\ldots x_{k-m},y,x_{k-m+1}\ldots x_k) \\ &\quad \mathrm{tr}\Big[J_\mu(x)A_{\mu_1}(x_1)\ldots A_{\mu_{k-2-m}}(x_{k-m})J_\nu(y)A_{\mu_{k-m+1}}(x_{k-m+1})\ldots A_{\mu_k}(x_k)\Big]\,.\end{aligned} \tag{2.A.8}$$

This means that (2.A.2) and (2.A.4) remain equivalent provided that we make the following identification:

$$\begin{aligned}&R_{\mu_1\ldots\mu_{k-m},\mu_{k-m+1}\ldots\mu_k;\mu\nu}(x_1\ldots x_{k-m};x_{k-m+1}\ldots x_k;x,y) = \\ &\qquad = \frac{1}{2\Lambda^2}A_{\mu\mu_1\ldots\mu_{k-m}\nu\mu_{k-m+1}\ldots\mu_k}(x,x_1\ldots x_{k-m},y,x_{k-m+1}\ldots x_k)\,,\end{aligned} \tag{2.A.9}$$

which take the exact same form in momentum space.

If we now look at the one-point R part of (2.A.1), we will obtain:

$$R_{\mu;\alpha\beta}(p;r,s) = \frac{1}{2\Lambda^2}A_{\mu\beta\alpha}(p,s,r)\,, \tag{2.A.10}$$

$$R_{,\mu;\alpha\beta}(;p;r,s) = \frac{1}{2\Lambda^2}A_{\mu\alpha\beta}(p,r,s)\,. \tag{2.A.11}$$

[5] Recall that we factor out the powers of g, *cf.* sec. 2.4.1.

Three-point action vertices are totally anti-symmetric in all their arguments, and thus the above expressions can be recast as

$$R_{\mu;\alpha\beta}(p;r,s) + R_{,\mu;\alpha\beta}(;p;r,s) = 0\,, \tag{2.A.12}$$

which is exactly (2.4.29) at the one-point level.

At the two-point level the R part of (2.A.1) becomes:

$$R_{\mu\nu;\alpha\beta}(p,q;r,s) = \frac{1}{2\Lambda^2} A_{\mu\nu\beta\alpha}(p,q,s,r)\,, \tag{2.A.13}$$

$$R_{\mu,\nu;\alpha\beta}(p;q;r,s) = \frac{1}{2\Lambda^2} A_{\mu\beta\nu\alpha}(p,s,q,r)\,, \tag{2.A.14}$$

$$R_{,\mu\nu;\alpha\beta}(;p,q;r,s) = \frac{1}{2\Lambda^2} A_{\mu\nu\alpha\beta}(p,q,r,s)\,. \tag{2.A.15}$$

From (2.A.13) and (2.A.15) we can write the following:

$$\begin{aligned} R_{,\mu\nu;\alpha\beta}(;p,q;r,s) - R_{\nu\mu;\alpha\beta}(q,p;r,s) &= \frac{1}{2\Lambda^2} A_{\mu\nu\alpha\beta}(p,q,r,s) - \frac{1}{2\Lambda^2} A_{\nu\mu\beta\alpha}(q,p,s,r) \\ &= \frac{1}{2\Lambda^2} A_{\mu\nu\alpha\beta}(p,q,r,s) - \frac{1}{2\Lambda^2} A_{\mu\nu\alpha\beta}(p,q,r,s) = 0\,, \end{aligned} \tag{2.A.16}$$

where in going from the first to the second line we have used charge conjugation invariance. Similarly, working out the explicit form of the four-point A vertices, one can show that the following expression also holds:

$$R_{\mu,\nu;\alpha\beta}(p;q;r,s) + R_{\mu\nu;\alpha\beta}(p,q;r,s) + R_{\nu\mu;\alpha\beta}(q,p;r,s) = 0\,. \tag{2.A.17}$$

These last two identities are exactly the interleave identities (2.4.29) at the two-point level for the R part of C^{-1}. This means that the full one-point and two-point vertices of C^{-1} also obey interleave identities.

We are now at the stage where we can inspect the vertices of ${C^a}_b$ using ${C^a}_m \big(C^{-1}\big)^m{}_b = {\delta^a}_b$. At $O(A_\mu)$, this can be expressed schematically in the following way:

$$\text{one-point } C = -\Big(\text{zero-point } C\Big) \times \Big(\text{one-point } C^{-1}\Big) \times \Big(\text{zero-point } C\Big)\,, \tag{2.A.18}$$

or explicitly as a set of two identities:

$$C_{\mu;\alpha\beta}(p;r,s) = -C_{\alpha\gamma}(r) C^{-1}_{\mu;\gamma\delta}(p;r,s) C_{\delta\beta}(s)\,, \tag{2.A.19}$$

$$C_{,\mu;\alpha\beta}(p;r,s) = -C_{\alpha\gamma}(r) C^{-1}_{,\mu;\gamma\delta}(p;r,s) C_{\delta\beta}(s)\,. \tag{2.A.20}$$

If we use (2.A.12) for the full one-point C^{-1} vertices, we can recast the expressions above as:

$$C_{\mu;\alpha\beta}(p;r,s) + C_{,\mu;\alpha\beta}(;p;r,s) = 0\,, \tag{2.A.21}$$

which is exactly the interleave identity (2.4.29) for one-point C vertices.

Similarly, at $O(A_\mu^2)$ one can write schematically:

$$\begin{aligned}\text{two-point } C = &-\Big(\text{zero-point } C\Big)\times\Big(\text{two-point } C^{-1}\Big)\times\Big(\text{zero-point } C\Big)\\ &+\Big(\text{zero-point } C\Big)\times\Big(\text{one-point } C^{-1}\Big)\times\Big(\text{zero-point } C\Big)\\ &\times\Big(\text{one-point } C^{-1}\Big)\times\Big(\text{zero-point } C\Big)\,. \qquad (2.A.22)\end{aligned}$$

From the above expression we can infer the explicit expression for the two-point C vertices:

$$\begin{aligned}C_{\mu,\nu;\alpha\beta}(p;q;r,s) = &-C_{\alpha\alpha_1}(r)C^{-1}_{\mu,\nu;\alpha_1\beta_1}(p;q;r,s)C_{\beta_1\beta}(s)\\ &+C_{\alpha\alpha_1}(r)C^{-1}_{\mu;\alpha_1\gamma}(p;r,s+q)C_{\gamma\delta}(s+q)C^{-1}_{,\nu;\delta\beta_1}(;q;p+r,s)C_{\beta_1\beta}(s)\\ &+C_{\alpha\alpha_1}(r)C^{-1}_{,\nu;\alpha_1\gamma}(;q;r,s+p)C_{\gamma\delta}(r+q)C^{-1}_{\mu;\delta\beta_1}(p;q+r,s)C_{\beta_1\beta}(s)\,, \qquad (2.A.23)\end{aligned}$$

$$\begin{aligned}C_{\mu\nu;\alpha\beta}(p,q;r,s) = &-C_{\alpha\alpha_1}(r)C^{-1}_{\mu\nu;\alpha_1\beta_1}(p,q;r,s)C_{\beta_1\beta}(s)\\ &+C_{\alpha\alpha_1}(r)C^{-1}_{\mu;\alpha_1\gamma}(p;r,s+q)C_{\gamma\delta}(p+r)C^{-1}_{\nu;\delta\beta_1}(q;p+r,s)C_{\beta_1\beta}(s)\,, \qquad (2.A.24)\end{aligned}$$

$$\begin{aligned}C_{,\mu\nu;\alpha\beta}(;p,q;r,s) = &-C_{\alpha\alpha_1}(r)C^{-1}_{,\mu\nu;\alpha_1\beta_1}(;p,q;r,s)C_{\beta_1\beta}(s)\\ &+C_{\alpha\alpha_1}(r)C^{-1}_{,\mu;\alpha_1\gamma}(;p;r,s+q)C_{\gamma\delta}(p+r)C^{-1}_{,\nu;\delta\beta_1}(;q;p+r,s)C_{\beta_1\beta}(s)\,. \qquad (2.A.25)\end{aligned}$$

The interleave identities $C_{,\mu\nu;\alpha\beta}(;p,q;r,s)-C_{\nu\mu;\alpha\beta}(q,p;r,s)=0$ and $C_{\mu,\nu;\alpha\beta}(p;q;r,s)+C_{\mu\nu;\alpha\beta}(p,q;r,s)+C_{\nu\mu;\alpha\beta}(q,p;r,s)=0$ follow trivially from (2.A.23), (2.A.24), and (2.A.25) if we use (2.A.12), (2.A.16) for the full C^{-1} vertices. This concludes the proof that the one and two-point functions of $C^a{}_b$ and $\mathcal{K}^a{}_b$, respectively, obey interleave identities.

2.B Why preregularisation is necessary

Consider the momentum integral [62]

$$\int_q\left\{\frac{1}{q^2+\Lambda^2}-\frac{1}{(q+p)^2+\Lambda^2}\right\}. \qquad (2.B.1)$$

Evidently this is zero, since in the second term we can change variables to $q\mapsto -q-p$ to make it equal and opposite to the first. These sort of cancelling terms are generic and unavoidable in a system of regularisation that uses gauge invariant PV regularisation. For example, just such a relabelling is used in (2.6.37) to cancel terms against each other.

Depending on how terms are unpacked and combined, they can then give contributions that appear in a similar way to above. Expanding the second term in (2.B.1) to $O(p^2)$, we obtain

$$\int_q \left\{ \frac{p^2}{(q^2+\Lambda^2)^2} - 4\frac{(p\cdot q)^2}{(q^2+\Lambda^2)^3} \right\} . \tag{2.B.2}$$

In four dimensions by Lorentz invariance (actually rotational invariance since we are in Euclidean signature) we can replace $(p\cdot q)^2$ by $q^2p^2/4$, so the above result simplifies to

$$\Lambda^2 p^2 \int_q \frac{1}{(q^2+\Lambda^2)^3} , \tag{2.B.3}$$

a manifestly positive convergent answer from a vanishing integral! The problem is that (2.B.1) is finite but ambiguous; the result (2.B.3) can be cast as a total derivative but with a finite surface term. Generally in situations where PV regularisation is used, we must first pre-regularise (in a way that is compatible with gauge invariance) to avoid this so-called momentum routing problem. This is done, not to subtract divergences with respect to the pre-regulator, but in order to ensure that the integrals are defined sufficiently carefully [86,95,98]. In the current chapter we use dimensional regularisation, working in $d = 4-\varepsilon$ dimensions when necessary. Then in (2.B.2), we effectively replace $(p\cdot q)^2$ by $q^2p^2/4 + \varepsilon\, q^2p^2/16$. Using dimensional regularisation it is straightforward to evaluate the new contribution and verify that it cancels (2.B.3), restoring agreement with (2.B.1).

Chapter 3

Universal scaling dimensions for highly irrelevant operators in the Local Potential Approximation

3.1 Introduction

In this chapter we investigate the properties of higher scaling dimension operators using the effective average action flow equation (1.3.19), which we have discussed at large in sec. 1.3. As mentioned before, one cannot solve exactly the flow equation, and hence, in practical applications, some form of approximation becomes necessary. Here we use the Local Potential Approximation [99–108]. This amounts to simplifying the flow equations by disregarding the momentum dependence of the effective action, except for a local potential term, V_Λ. Thus, for a single-component scalar field φ, the effective action Γ_Λ takes the form:

$$\Gamma_\Lambda[\varphi] = \int_x \left(\frac{1}{2}(\partial_\mu \varphi)^2 + V_\Lambda(\varphi) \right) . \tag{3.1.1}$$

While an exact analytical solution to this truncated FRG formulation is still not possible in general, the LPA enables numerical treatments that provide valuable insights into the system's behaviour. It allows for numerical estimates of various physical quantities, including critical exponents and the scaling equation of state [28, 29, 34, 103, 109–114]. Moreover, the LPA serves as the initial step in a systematic derivative expansion [34, 103, 109–111], which facilitates a more comprehensive exploration of the system's properties [28, 29, 34, 110, 112–114].

Nevertheless it is important to acknowledge the limitations of the LPA and, more generally, the derivative expansion. Since such truncations do not correspond to a controlled expansion in some small parameter, the errors incurred can be expected to be of the

same order in general as the quantities being computed. Furthermore, quantities that should be universal, and, thus, independent of the specific form of the cutoff, are not.

It has long been understood that an exception to this is the general form of a non-trivial fixed point potential $V(\varphi)$ in the large field regime [34,103,109,111], which follows from asymptotic analysis:

$$V(\varphi) = A|\varphi|^{d/d_\varphi} + \cdots \qquad \text{as} \qquad \varphi \to \pm\infty\,, \tag{3.1.2}$$

where the ellipses stand for subleading terms (see later). The leading term coincides with the scaling equation of state precisely at the fixed point. It is a simple consequence of dimensional analysis on using the scaling dimension $d_\varphi = \frac{1}{2}(d-2+\eta)$ for the field φ at the fixed point, η being its anomalous dimension. However asymptotic analysis does not fix the amplitude A or the anomalous dimension η, which have to be found by other means, for example by numerical solution of truncated fixed point equations.

In this chapter, we will show that within LPA, asymptotic analysis combined with Sturm-Liouville (SL) and Wentzel–Kramers–Brillouin (WKB) analysis,[1] also allows one to determine asymptotically the scaling dimension d_n of the highly irrelevant ($d_n \gg 1$) eigenoperators $\mathcal{O}_n = \mathcal{O}_n(\varphi)$ of potential-type (those containing no spacetime derivatives). Ordering them by increasing scaling dimension, we will show that $d_n = n(d-d_\varphi)$ to leading order in n. In the case of $O(N)$ invariant scalar field theory with fixed $N \geq 0$ the dimension d_n is doubled to $d_n = 2n(d-d_\varphi)$. The scaling dimension is thus independent of N. It agrees with the result for the single scalar field since these eigenoperators are functions of $\varphi^2 = \varphi^a\varphi^a$, and thus pick out only the even eigenoperators (those symmetric under $\varphi \leftrightarrow -\varphi$) in the $N = 1$ case. We also show that the scaling dimension is $d_n = 2n(d-d_\varphi)$ whenever $N = -2k$, where k is a non-negative integer.

Once again these results are independent of the choice of cutoff and thus universal. Indeed in this chapter, we will keep the cutoff function completely general throughout, subject only to some weak technical constraints that we derive later. Note that, like the fixed point equation of state (3.1.2), the d_n take the same form, independent of the choice of fixed point, provided only that $d_\varphi > 0$ and that the fixed point potential is non-vanishing. We also show that the next to leading correction to d_n behaves as a power of n. The power is universal although the coefficient of the subleading correction is not.

Actually this approach was first employed to determine the scaling dimension of highly irrelevant eigenoperators in an $f(R)$ approximation [117–120] to the asymptotic safety scenario [55–57] in quantum gravity. The $f(R)$ approximation serves as a close analogue to the LPA in this context [121–123]. However, while the resulting scaling dimensions d_n exhibit a simple nearly-universal form for large values of n, they nevertheless retained

[1] See *e.g.* ref. [115] for textbook discussion of SL methods and ref. [116] for WKB methods.

strong dependence on the choice of cutoff. This issue can be traced back [118] to the so-called single-metric (or background field) approximation [55], where the identification of the quantum metric with the background metric is made in order to close the equations. The work done in the present chapter will thus complete the circle by demonstrating that, indeed, without such an approximation, the results become truly universal. Additionally, it showcases the power of these methods in a simpler context.

The chapter is organised as follows. We first analyse the functional renormalisation group equations for a single scalar field in the LPA. From the eigenoperator equation we write the resulting SL equation in Schrödinger form and thus, by taking the large field limit, deduce the asymptotic form of the renormalisation group eigenvalues in the WKB limit. Sec. 3.3 extends the analysis to $O(N)$ scalar field theory using the same approach. Finally in sec. 3.4 we conclude and discuss the results, placing them in a wider context.

3.2 Flow equations in LPA

The LPA approximation amounts to setting the field φ in the Hessian to a spacetime constant, thus dropping from a derivative expansion all terms that do not take the form of a correction to the potential. The flow equation for $V_\Lambda(\varphi)$ then takes the form:

$$\left(\partial_t + d_\varphi \varphi \frac{\partial}{\partial \varphi} - d\right) V_\Lambda(\varphi) = -\frac{1}{2} \int_q \frac{\dot{\Delta}}{\Delta} \frac{1}{1 + \Delta V''_\Lambda(\varphi)}\,, \tag{3.2.1}$$

where $\partial_t = -\Lambda \partial_\Lambda$, t being the renormalisation group 'time' which, following [30], we have chosen to flow towards the IR. Here the momentum, potential and field are already scaled by the appropriate power of Λ to make them dimensionless. Then $\Delta = C(q^2)/q^2$ no longer depends on Λ. The same is true of $\partial_t \Delta_\Lambda$, which after scaling we write as $\dot{\Delta}$, where

$$\dot{\Delta} = 2\, C'(q^2)\,. \tag{3.2.2}$$

Since $C(q^2)$ is monotonically increasing, we have that $\dot{\Delta} > 0$.

The scaling dimension of the field is $d_\varphi = \frac{1}{2}(d - 2 + \eta)$, where η is the anomalous dimension. Since η arises from the renormalisation group running of the field, and is typically inferred from corrections to the kinetic term, one would naturally conclude that it vanishes in LPA [30, 99–106, 112]. Nevertheless, as noticed in refs. [107, 108], this assumption is not necessary. The flow equation (3.2.1) is still a mathematically consistent equation with $\eta \neq 0$. However, since we cannot determine η directly from (3.2.1), its value needs to be input from elsewhere (either from experiment or other theoretical studies). We will follow this strategy, in the expectation that it improves the accuracy of our final estimates for d_n.

At a fixed point, $V_\Lambda(\varphi) = V(\varphi)$ and η have no renormalisation group time dependence. The eigenoperator equation follows from linearising about a FP:

$$V_\Lambda(\varphi) = V(\varphi) + \varepsilon\, v(\varphi)\, \mathrm{e}^{\lambda t}\,, \tag{3.2.3}$$

ε being infinitesimal. Here λ is the RG eigenvalue. It is the scaling dimension of the corresponding coupling, and is positive (negative) for relevant (irrelevant) operators. The scaling dimension of the operator $v(\varphi)$ itself is then $d-\lambda$. We write the eigenoperator equation in the same form as refs. [117, 118, 123]:

$$-a_2(\varphi)v''(\varphi) + a_1(\varphi)v'(\varphi) + a_0(\varphi)v(\varphi) = (d-\lambda)v(\varphi)\,, \tag{3.2.4}$$

where the φ-dependent coefficients multiplying the eigenoperators are given by:

$$a_0(\varphi) = 0\,, \tag{3.2.5}$$

$$a_1(\varphi) = d_\varphi \varphi\,, \tag{3.2.6}$$

$$a_2(\varphi) = \frac{1}{2}\int_q \frac{\dot{\Delta}}{(1+\Delta V'')^2} > 0\,, \tag{3.2.7}$$

and we have noted that a_2 is positive. We can now repeat the analysis carried out in [117, 118, 123] to solve for λ in the case of high dimension eigenoperators.

3.2.1 Asymptotic solutions

For large φ, the RHS of (3.2.1) can be neglected. Thus, at a fixed point, the equation reduces to a first order ODE (ordinary differential equation) which is easily solved. It gives the first term (3.1.2) in an asymptotic series solution [109]:

$$V(\varphi) = A|\varphi|^m + O\left(|\varphi|^{2-m}\right) \quad \text{as} \quad \varphi \to \pm\infty\,, \tag{3.2.8}$$

where for convenience we introduce

$$m = d/d_\varphi\,, \tag{3.2.9}$$

and A is a real constant (that is determined by solving for the full FP solution). The subleading terms arise from iterating the leading order contribution to next order.

Of course there is always the trivial $V(\varphi) \equiv 0$ fixed point solution, corresponding to the Gaussian fixed point. We will not be interested in that (the scaling dimensions in that case are exactly known and reviewed in the discussion in sec. 3.4). Instead we focus on non-trivial FP solutions for which $A \neq 0$. In principle, A could be different in the two limits $\varphi \to \pm\infty$, although, in practice, the fixed point potentials (3.2.8) are symmetric. Anyway, we will see that A drops out of the analysis in a few further steps.

It is helpful for the following to note that $m > 3$. Neglecting η (typically $\eta \ll 1$, see *e.g.* [124]), we see that m is a decreasing function of d for all $d > 2$. In practice, non-trivial FP solutions only exist for $d < 4$ (see *e.g.* [103]). In the limit $d \to 4^-$, $\eta \to 0$ (by the ϵ expansion [124]) and thus $m \to 4$. In $d = 2$ dimensions, the asymptotic solution (3.2.8) corresponds to that of a unitary minimal model [125,126]. The one with the largest anomalous dimension is that of the Ising model universality class which has $\eta = 1/4$, thus in $d = 2$ dimensions we have $m \geq 8$ for all the unitary minimal models.

Note that the solution (3.2.8) has a single free parameter even though the FP equation is a (non-linear) second order ODE. The second parameter, if it exists, can be deduced by linearising around (3.2.8), writing $V(\varphi) \mapsto V(\varphi) + \delta V(\varphi)$, and solving the flow equation (3.2.1) at the FP this time for δV. Since δV satisfies a *linear* second order ODE and one solution is already known, namely $\delta V = \partial_A V(\varphi)$, it is easy to find the solution that corresponds at the linearised level to the missing parameter [103, 109]. However, one then discovers that these 'missing' linearised solutions are rapidly growing exponentials. Such a linearised perturbation is not valid asymptotically since for diverging φ it is much larger than the solution (3.2.8) we perturbed around. Hence, the FP asymptotic solutions only have the one free parameter, A.

Substituting (3.2.8) into (3.2.7), we see that asymptotically $a_2(\varphi)$ scales as follows:

$$a_2(\varphi) = F\,|\varphi|^{2(2-m)} + O\left(|\varphi|^{3(2-m)}\right) \quad \text{as} \quad \varphi \to \pm\infty\,, \tag{3.2.10}$$

where F is positive and cutoff dependent:

$$F = \frac{1}{2\,(m(m-1)A)^2} \int_q \frac{\dot{\Delta}}{\Delta^2} = -\frac{1}{(m(m-1)A)^2} \int_q q^4 \frac{\partial}{\partial q^2} C^{-1}(q^2)\,. \tag{3.2.11}$$

We will assume that the integral converges. This imposes some weak constraints on the cutoff profile. From (3.2.11), we see that we require $C(q^2)$ to vanish slower than q^{d+2} as $q \to 0$, and $C \to 1$ faster than $1/q^{d+2}$ as $q \to \infty$. This is true for example for the popular form of additive (*i.e.* mass-type) cutoff [45] (which was the one used in the analogous $f(R)$ analyses in refs. [117,118]):

$$r(q^2) = \frac{q^2}{\exp(aq^{2b}) - 1}\,, \quad a > 0,\, b \geq 1\,, \tag{3.2.12}$$

provided also we set $b < \frac{1}{2}(d+2)$, the relation to $C(q^2)$ being $q^2 C^{-1}(q^2) = q^2 + r(q^2)$.

Given that $a_2(\varphi)$ vanishes asymptotically, it is tempting to neglect the a_2 term in (3.2.4). We will shortly justify this. By neglecting the a_2 term, the ODE becomes linear first order giving a unique solution up to normalization. Thus we deduce that the eigenoperators asymptotically scale as a power of the field:

$$v(\varphi) \propto |\varphi|^{\frac{d-\lambda}{d_\varphi}} + \cdots\,, \tag{3.2.13}$$

where the ellipses stands for subleading corrections. The neglect of the a_2 is justified as follows. The missing solution is one that grows exponentially (again, so that $a_2(\varphi)v''(\varphi)$ cannot be neglected). Since the ODE is linear, these are allowed solutions to (3.2.4), but they are ruled out because they do not evolve multiplicatively in the RG [34, 110, 117, 118, 127, 128].

Now, the asymptotic solution (3.2.13) imposes two boundary conditions (one for each limit $\varphi \to \pm\infty$) on the second order ODE (3.2.4), but since the ODE is linear this overconstrains the equation[2] which thus leads to quantisation of the RG eigenvalue λ. We index the solutions as $v_n(\varphi)$, ordering them so that λ_n decreases as n increases. We can now perform an SL transformation and deduce the asymptotic dependence of the eigenvalues λ_n on n, as $n \to \infty$.

3.2.2 SL analysis

We can rewrite the eigenvalue equation (3.2.4) in a SL form by multiplying it with the SL weight function

$$w(\varphi) = \frac{1}{a_2(\varphi)} \exp\left\{ -\int_0^\varphi d\varphi' \frac{a_1(\varphi')}{a_2(\varphi')} d\varphi' \right\} , \tag{3.2.14}$$

which is always positive due to the positivity of a_2. Then the eigenvalue equation becomes

$$-\big(a_2(\varphi)w(\varphi)v'(\varphi)\big)' = (d-\lambda)w(\varphi)v(\varphi) . \tag{3.2.15}$$

The SL operator on the left, $L = -\frac{d}{d\varphi}\left(a_2 w \frac{d}{d\varphi}\cdot\right)$, is self adjoint when acting on the space spanned by the eigenoperators, *i.e.* it satisfies

$$\int_{-\infty}^{\infty} d\varphi\, u_1(\varphi)\, Lu_2(\varphi) = \int_{-\infty}^{\infty} d\varphi\, u_2(\varphi)\, Lu_1(\varphi) , \tag{3.2.16}$$

when the u_i are linear combinations of the eigenoperators. This is so because the boundary terms at infinity, generated by integration by parts, vanish in this case. This follows because, from (3.2.13), the u_i diverge at worst as a power of φ, whilst $w(\varphi) \to 0$ exponentially fast as $\varphi \to \pm\infty$.

Thus from SL analysis [115], we know that the eigenvalues λ_n are real, discrete, with a most positive (relevant) eigenvalue and an infinite tower of ever more negative (more irrelevant) eigenvalues, $\lambda_n \to -\infty$ as $n \to \infty$ [110]. Let us define a 'coordinate' x:

$$x = \int_0^\varphi \frac{1}{\sqrt{a_2(\varphi')}}\, d\varphi' \tag{3.2.17}$$

[2] We can see this for example by imposing a normalization condition on v.

(always taking the positive root in fractional powers). Defining the wave-function as

$$\psi(x) = a_2^{1/4}(\varphi) w^{1/2}(\varphi) v(\varphi)\,, \tag{3.2.18}$$

enables us to recast (3.2.15) in the following way:

$$-\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = (d-\lambda)\psi(x)\,. \tag{3.2.19}$$

This is a one-dimensional time-independent Schrödinger equation for a particle of mass $m = 1/2$, with energy $E = d - \lambda$ *i.e.* just the eigenoperator scaling dimension, and with potential [117, 118, 123]:

$$U(x) = \frac{a_1^2}{4a_2} - \frac{a_1'}{2} + a_2'\left(\frac{a_1}{2a_2} + \frac{3a_2'}{16a_2}\right) - \frac{a_2''}{4}\,, \tag{3.2.20}$$

where the terms on the right hand side are functions of φ.

From the limiting behaviour of $a_2(\varphi)$, (3.2.10), we see that asymptotically the coordinate x scales as

$$x = \int_0^\varphi \left(\frac{|\varphi'|^{m-2}}{\sqrt{F}} + O(1)\right) d\varphi' = \pm\frac{|\varphi|^{m-1}}{(m-1)\sqrt{F}} + O(|\varphi|) \quad \text{as} \quad \varphi \to \pm\infty\,, \tag{3.2.21}$$

so in particular when $\varphi \to \pm\infty$ we have $x \to \pm\infty$. On the right hand side of (3.2.20), the first term dominates at leading order (LO) and next-to-leading order (NLO). Since asymptotically,

$$\frac{a_1^2(\varphi)}{4a_2(\varphi)} = \frac{d_\varphi^2}{4F}|\varphi|^{2m-2} + O(|\varphi|^m)\,, \tag{3.2.22}$$

we thus find that

$$U(x) = \frac{1}{4}(d-d_\varphi)^2 x^2 + O(|x|^{1+\frac{1}{m-1}}) \quad \text{as} \quad x \to \pm\infty\,. \tag{3.2.23}$$

To LO, this is the potential of a simple harmonic oscillator of the form $\frac{1}{2}m\omega^2 x^2$, where

$$\omega = d - d_\varphi = \frac{1}{2}(d+2-\eta)\,. \tag{3.2.24}$$

3.2.3 WKB analysis

We can now use WKB analysis to compute the asymptotic form of the energy levels, a.k.a. operator scaling dimensions, E_n, at large n. This follows from solving the equality

$$\int_{-x_n}^{x_n} dx\,\sqrt{E_n - U(x)} = \left(n + \frac{1}{2}\right)\pi\,, \tag{3.2.25}$$

for the total phase of the wave oscillations described by $\psi(x)$, in the limit of large E_n [116]. Here x_n are the classical turning points, *i.e.* such that $E_n = U(\pm x_n)$. Now,

the above integral is dominated by the regions close to the turning points, where we can substitute the asymptotic form (3.2.23). Including the subleading correction proportional to some constant γ (that depends on the cutoff profile) the integral is

$$\frac{\omega}{2}\int_{-x_n}^{x_n} dx\,\sqrt{x_n^2+\gamma x_n^{1+\frac{1}{m-1}}-x^2-\gamma|x|^{1+\frac{1}{m-1}}} = \\ = \frac{\omega}{2}x_n^2\int_{-1}^{1} dy\,\sqrt{1-y^2+\gamma x_n^{\frac{1}{m-1}-1}(1-|y|^{1+\frac{1}{m-1}})}\,. \quad (3.2.26)$$

Since the x_n are also large we can now evaluate the right hand side and thus from (3.2.25) we get the asymptotic relation between x_n and n:

$$\frac{\omega\pi}{4}x_n^2+O\left(x_n^{1+\frac{1}{m-1}}\right)=n\pi\,. \quad (3.2.27)$$

Hence, using (3.2.23), (3.2.24) and (3.2.27), the scaling dimension of the eigenoperators takes the form

$$\begin{aligned} d_n &= E_n = d-\lambda_n = U(x_n) \\ &= n\omega+O\left(n^{\frac{m}{2(m-1)}}\right) = n(d-d_\varphi)+O\left(n^{\frac{m}{2(m-1)}}\right) \quad \text{as} \quad n\to\infty\,. \end{aligned} \quad (3.2.28)$$

The subleading correction to the critical exponents contain information about the cutoff via the constant γ introduced in (3.2.26). However, at leading order, the result is independent of the cutoff, and hence universal.

3.3 O(N) scalar field theory

Now let us apply the same treatment to N scalar fields φ^a $(a=1,\dots,N)$ with an $O(N)$ invariant potential $V_\Lambda(\varphi^2)=V_\Lambda(\rho)$ [129–132], in the LPA. We use the shorthand $\rho=\varphi^a\varphi^a=\varphi^2$. The flow equation (3.2.1) becomes [111,133]:

$$\left(\partial_t-d+2d_\varphi\rho\frac{\partial}{\partial\rho}\right)V_\Lambda(\rho)=-\frac{1}{2}\int_q\frac{\dot\Delta}{\Delta}\left(M^{-1}\right)^{aa}\,, \quad (3.3.1)$$

where the matrix M is given by:

$$M^{ab}=\delta^{ab}+\Delta\frac{\partial^2V_\Lambda(\rho)}{\partial\varphi^a\partial\varphi^b}=\delta^{ab}+2\Delta\left[\delta^{ab}V'_\Lambda(\rho)+2\varphi^a\varphi^bV''_\Lambda(\rho)\right]\,. \quad (3.3.2)$$

Inverting and tracing, yields:

$$\left(M^{-1}\right)^{aa}=\frac{N-1}{1+2\Delta V'_\Lambda(\rho)}+\frac{1}{1+2\Delta V'_\Lambda(\rho)+4\Delta\rho V''_\Lambda(\rho)}\,. \quad (3.3.3)$$

In the limit of large ρ, the right hand side of the flow equation (3.3.1) can be neglected at leading order. This implies that a FP solution $V_\Lambda(\rho) = V(\rho)$ takes the following asymptotic form:

$$V(\rho) = A\rho^{\frac{m}{2}} + O\left(\rho^{1-\frac{m}{2}}\right) \quad \text{as} \quad \rho \to \infty\,, \tag{3.3.4}$$

where as before the subleading term has been calculated by iterating the leading contribution to next order.

The RG eigenvalue equation follows by linearising (3.3.1) around the fixed point solution,

$$V_\Lambda(\rho) = V(\rho) + \varepsilon\, v(\rho)\, \mathrm{e}^{\lambda t}\,, \tag{3.3.5}$$

giving an equation for $v(\rho)$ with the same structure as (3.2.4), *i.e.*

$$-a_2(\rho)v'' + a_1(\rho)v' + a_0(\rho)v = (d-\lambda)v\,, \tag{3.3.6}$$

the same value for $a_0(\rho) = 0$, but different expressions for $a_1(\rho)$,

$$a_1(\rho) = 2d_\varphi\rho - \int_q \dot{\Delta}\left[\frac{1}{(1+2\Delta V' + 4\Delta\rho V'')^2} + \frac{N-1}{(1+2\Delta V')^2}\right], \tag{3.3.7}$$

and $a_2(\rho)$, which however is again always positive:

$$a_2(\rho) = \int_q \frac{2\dot{\Delta}\rho}{(1+2\Delta V' + 4\Delta\rho V'')^2}\,. \tag{3.3.8}$$

Using the asymptotic fixed point solution (3.3.4) (and assuming $A \neq 0$) we get that asymptotically a_2 scales as follows:

$$a_2(\rho) = 4F\rho^{3-m} + O\left(\rho^{4-\frac{3m}{2}}\right) \quad \text{as} \quad \rho \to \infty\,, \tag{3.3.9}$$

where F was already defined in (3.2.11). By similar arguments to before, we see that $m > 3$ in practice, so this implies $a_2(\rho) \to 0$. We also find that a_1 scales as follows:

$$a_1(\rho) = 2\, d_\varphi\rho + O\left(\rho^{2-m}\right) \quad \text{as} \quad \rho \to \infty\,. \tag{3.3.10}$$

If we substitute $\rho = \varphi^2$ into the above asymptotic expansions, they differ from the large φ behaviour (3.2.6) of $a_1(\varphi)$ and (3.2.10) of $a_2(\varphi)$. However they reproduce the previous results once we transform the ODE (3.3.6) by changing variables $\rho = \varphi^2$. Thus by the same arguments as before, *cf.* (3.2.13), we also know that for $\rho \to \infty$, we must have

$$v(\rho) \propto \rho^{\frac{d-\lambda}{2d_\varphi}} + \cdots\,. \tag{3.3.11}$$

However, this now imposes only one boundary condition on the linear ODE (3.3.6) since ρ is restricted to be non-negative. On the other hand we see from (3.3.8) that $a_2(0) = 0$, so the ODE has a so-called fixed singularity at $\rho = 0$. In order to ensure that $v(\rho)$

remains non-singular at this point, an additional boundary condition is then required:

$$a_1(0)v'(0) = (d-\lambda)v(0)\,. \tag{3.3.12}$$

Now we again have two boundary conditions, overconstraining the equation, and leading to quantisation of the RG eigenvalue λ.

3.3.1 SL analysis

The last step is to perform the SL analysis, which also differs because of the $\rho = 0$ boundary. For small ρ we have

$$a_2(\rho) = 2G\rho + O(\rho^2) \qquad \text{and} \qquad a_1(\rho) = -GN + O(\rho)\,, \tag{3.3.13}$$

where we have set

$$G = \int \frac{d^dq}{(2\pi)^d} \frac{\dot{\Delta}}{[1+2\Delta V'(0)]^2}\,. \tag{3.3.14}$$

Note that G is of course positive. (By Taylor expanding (3.3.1) one sees that its convergence is guaranteed for any such solution to the flow equation.) The SL weight function now takes the form

$$w(\rho) = \frac{1}{a_2(\rho)} \exp\left\{-\int_{\rho_0}^{\rho} d\rho' \, \frac{a_1(\rho')}{a_2(\rho')}\right\}\,, \tag{3.3.15}$$

where by (3.3.13) a non-zero lower limit, $\rho_0 > 0$, is required to avoid the integral diverging (when $N \neq 0$).

Using $w(\rho)$ we can now cast (3.3.6) in SL form (3.2.15). However, for the SL operator to be self-adjoint, we need the boundary contributions that appear on integration by parts, to vanish. This is still true for large field since as $\rho \to \infty$, the eigenoperators diverge at worst as a power, whilst from (3.3.9) we have $a_2(\rho) \to 0$, and thus $w(\rho) \to 0$ exponentially fast. At the $\rho = 0$ boundary we require:[3]

$$\lim_{\rho\to 0} a_2(\rho)w(\rho)\left(v_i(\rho)v_j'(\rho) - v_j(\rho)v_i'(\rho)\right) = 0\,, \tag{3.3.16}$$

for any two eigenfunctions $v_i(\rho)$ and $v_j(\rho)$. This is true for all $N > 0$ since by (3.3.13) and (3.3.15) we see that for small ρ,

$$a_2(\rho)w(\rho) \propto \rho^{N/2}\left[1 + O(\rho)\right]\,. \tag{3.3.17}$$

We have thus determined that the SL operator is self-adjoint for all $N > 0$.

Actually, $N = 0$ is also interesting since it corresponds to the universality class of fluctuating long polymers [124]. In this case, the above analysis shows that $a_2(0)w(0) >$

[3] Using (3.3.12) and (3.3.13), this can be reduced to $\lim_{\rho\to 0} a_2(\rho)w(\rho)(\lambda_i - \lambda_j)\, v_i(\rho)v_j(\rho) = 0$ (when $N \neq 0$).

0, which would appear to imply that (3.3.16) is no longer satisfied. However from (3.3.13) we see that $a_1(0) = 0$ now and thus, from (3.3.12), either $\lambda_i = d$ or $v_i(0) = 0$ [111]. The first possibility corresponds to the uninteresting solution $v(\rho) \equiv 1$, *i.e.* the unit operator, which we discard. All the other eigenoperators must thus satisfy $v_i(0) = 0$, and so (3.3.16) is satisfied in this reduced space. Therefore, with this one proviso, the SL operator is actually self-adjoint for all $N \geq 0$.

For general $N < 0$, the SL operator fails to be self-adjoint, and thus SL analysis is no longer applicable. However for $N = -2k$, k a non-negative integer, something special happens. The first $k+1$ eigenoperators with the lowest scaling dimension turn out to have exactly soluble scaling dimensions, in fact coinciding with the Gaussian ones [134–136]. (The case $N = 0$ above is the first example, the lowest dimension operator being the unit operator with scaling dimension zero.) Again, the SL operator is self-adjoint in the remainder of the space. For example for $N = -2$, one knows from ref. [111] that the remaining eigenoperators satisfy $v_i(0) = v_i'(0) = 0$, and thus $v_i(\rho) \propto \rho^2$ for small ρ, whilst for $N = -4$ boundary conditions force the remaining eigenoperators to satisfy $v_i(\rho) \propto \rho^3$ for small ρ. From that analysis it is clear that in general at $N = -2k$, we have that the remaining operators satisfy

$$v_i(\rho) \propto \rho^{k+1} \qquad \text{as} \qquad \rho \to 0\,. \tag{3.3.18}$$

Combining these observations with (3.3.16) and (3.3.17), we see that the SL operator is indeed self-adjoint in the reduced space defined by excluding the first $k+1$ operators.

The SL equation can now be recast in the same way as before, using (3.2.17) for x and (3.2.18) for $\psi(x)$ (except for the obvious replacement of φ by ρ). The resulting Schrödinger equation is then precisely as before, *viz.* (3.2.19), and the potential $U(x)$ also takes precisely the same form in terms of the a_i, *viz.* (3.2.20). However the $\rho = 0$ boundary turns into an $x = 0$ boundary since, by (3.3.13) and (3.2.17), we have

$$x = \sqrt{2\rho/G} + O\left(\rho^{\frac{3}{2}}\right) \quad \text{as} \quad \rho \to 0\,. \tag{3.3.19}$$

Thus, using a_2 from (3.3.13) and $a_2 w$ from (3.3.17), we see that

$$\psi(x) \propto x^{\frac{N-1}{2}} v(x) \tag{3.3.20}$$

for small x. Hence for all $N > 1$, $\psi(x)$ vanishes as $x \to 0$. On taking into account the behaviour (3.3.18) we see that in the reduced space, $\psi(x)$ also vanishes for the special cases $N = -2k$. In this limit the leading contributions to the potential come from the first, third and fourth terms in (3.2.20), and thus we find:

$$U(x) = \frac{(N-1)(N-3)}{4\,x^2} + O(1) \quad \text{as} \quad x \to 0\,. \tag{3.3.21}$$

The cases $N = 1, 3$ are exceptional since this leading behaviour then vanishes, whilst the range $1 < N < 3$ will need a separate treatment because the potential is then unbounded from below.

At the other end of x's range, we find that

$$x = \int_0^\rho d\rho' \left\{ \frac{(\rho')^{\frac{1}{2}(m-3)}}{2\sqrt{F}} + O\left(\rho'^{-\frac{1}{2}}\right) \right\} = \frac{\rho^{\frac{1}{2}(m-1)}}{(m-1)\sqrt{F}} + O\left(\rho^{\frac{1}{2}}\right) \quad \text{as} \quad \rho \to \infty\,. \tag{3.3.22}$$

Identifying $\rho = \varphi^2$, this is the same formula (3.2.21) as before. The potential $U(x)$ is again dominated by the first term in (3.2.20), both at LO and NLO. Substituting the asymptotic expressions (3.3.10) and (3.3.9) for a_1 and a_2, we find exactly the same formula (3.2.23) for the large x behaviour of $U(x)$. In particular the leading term is again that of a simple harmonic oscillator with angular frequency $\omega = d - d_\varphi$.

3.3.2 WKB analysis

For the cases $N > 3$, $0 < N < 1$ and $N = -2m$, we can now proceed with the WKB analysis in the usual way. In this case we have for the total phase of the wave function:

$$\int_{x_n^-}^{x_n^+} dx\, \sqrt{E_n - U(x)} = \left(n + \frac{1}{2}\right)\pi\,, \tag{3.3.23}$$

where x_n^- and x_n^+ are the classical turning points, *i.e.* $E_n = d - \lambda_n = U(x_n^-) = U(x_n^+)$. In contrast to the previous case, the potential is not symmetric and there is no simple relation between x_n^- and x_n^+.

In the large n limit, the contribution from the right hand boundary gives half of what we obtained before. To see this in detail, let x_0^+ be some fixed finite value but sufficiently large to trust the asymptotic form (3.2.23) of the potential, then the contribution from the right hand boundary is

$$\int_{x_0^+}^{x_n^+} dx\, \sqrt{E_n - U(x)} = \frac{\omega}{2}(x_n^+)^2 \int_{x_0^+/x_n^+}^1 dy\, \sqrt{1 - y^2 + \gamma (x_n^+)^{\frac{1}{m-1}-1}(1 - |y|^{1+\frac{1}{m-1}})}\,. \tag{3.3.24}$$

Taking into account the multiplying factor of $(x_n^+)^2$ we see that the lower limit x_0^+/x_n^+ of the integral can be set to zero, since the correction is of order $O(x_n^+)$ which is smaller than that given by the γ correction. Thus we get half the integral in (3.2.26) (with x_n replaced by x_n^+) giving half the left hand side of (3.2.27):

$$\int_{x_0^+}^{x_n^+} dx\, \sqrt{E_n - U(x)} = \frac{\omega\pi}{8}(x_n^+)^2 + O\left((x_n^+)^{1+\frac{1}{m-1}}\right)\,. \tag{3.3.25}$$

Using the asymptotic form of the potential, we see that the leading term can be written as $\pi E_n/(2\omega)$. In the large n limit, the left hand boundary makes a contribution that can

be neglected in comparison. To see this let x_0^- be some fixed finite value but sufficiently small to use (3.3.21). Then the contribution from the left hand boundary is

$$\int_{x_n^-}^{x_0^-} dx\, \sqrt{E_n - U(x)} = \frac{1}{2}\sqrt{(N-1)(N-3)} \int_1^{x_0^-/x_n^-} dy \left(\frac{\sqrt{y^2-1}}{y} + O(x_n^-) \right) . \quad (3.3.26)$$

Since x_n^- is vanishing for large E_n, we see that this integral is $O(1/x_n^-)$ or, using again the relation (3.3.21), $O(E_n^{1/2})$. That only leaves the portion of the integral that goes from x_0^- to x_0^+, but since these boundaries are fixed and finite, we see that this part also grows as $\sqrt{E_n}$ and thus it too can be neglected in comparison to (3.3.25).

Therefore asymptotically the integral in (3.3.23) is given by (3.3.25). Inverting the relation to find $(x_n^+)^2$ asymptotically in terms of n, we thus find

$$\begin{aligned} d_n = E_n &= d - \lambda_n = U(x_n^+) \\ &= 2n\omega + O\left(n^{\frac{m}{2(m-1)}}\right) = 2n(d-d_\varphi) + O\left(n^{\frac{m}{2(m-1)}}\right) \quad \text{as} \quad n \to \infty\,, \end{aligned} \quad (3.3.27)$$

i.e. precisely double the value we found for a single component field in (3.2.28) and independent of N.

We see that technically this arises because the WKB integral is precisely half as large in the $O(N)$ case, the leading contribution coming from the x_n^+ boundary only. Recall that at $N = 1, 3$, the leading behaviour (3.3.21) of $U(x)$ is no longer applicable. Since the potential is now finite as $x \to 0$, it is clear from the above analysis that the left hand boundary continues to contribute at most $O(E_n^{1/2}) \sim \sqrt{n}$ and so can be neglected. Thus we see that (3.3.27) applies also to these exceptional cases. Thus also for $N = 1$ we find twice the previous scaling dimension as a function of large index n. This is in agreement with that single field result however, because these eigenoperators are a function of φ^2 only. Hence for a single component field, the current n indexes only the even eigenoperators (those symmetric under $\varphi \leftrightarrow -\varphi$).

Finally, let us show that our result (3.3.27) is also applicable to the range $1 < N < 3$. Although in this case, from (3.3.21), the potential $U(x) \to -\infty$ as $x \to 0$, we know from (3.3.20) that the solutions we need, have $\psi(x)$ vanishing there. These solutions are consistent with the Schrödinger equation (3.2.19) because for small x we have, by (3.3.20), a diverging second derivative:

$$-\frac{d^2\psi(x)}{dx^2} \propto -\frac{(N-1)(N-3)}{4\,x^2}\psi(x)\,, \quad (3.3.28)$$

which is precisely the right behaviour to cancel the divergence in the Schrödinger equation coming from the $U(x)\psi(x)$ term. Meanwhile the $v(x)$ term in (3.3.20) is well behaved in terms of oscillations at small x, behaving similarly to the above cases. Therefore we are only neglecting a subleading contribution to the total phase, if we work instead with a modified WKB integral where we replace the lower limit in (3.3.23) with some

finite value x_0^-. By the above analysis we then recover (3.3.27) again. In this way we have shown that the result (3.3.27) is actually applicable for all $N \geq 0$ and to the special cases $N = -2k$ (where k is a non-negative integer).

3.4 Summary and discussion

We have used SL theory and WKB methods to derive the scaling dimension d_n of highly irrelevant operators $\mathcal{O}_n$ around a non-trivial fixed point for scalar field theory in LPA. The scaling dimensions d_n are ordered so that they increase with increasing index n. The d_n are derived following the methods developed in [117]. They are given to leading order in n, together with the power-law dependence on n of the next-to-leading order. The results apply to all the non-trivial (multi)critical fixed points in $2 < d < 4$, for single component scalar field theory and for $O(N)$ invariant scalar field theory, and also to the unitary minimal models in $d = 2$ dimensions. The d_n are universal, independent of the choice of fixed point (except through the anomalous dimension η) and independent of the cutoff choice which we have left general throughout, apart from the weak technical constraints discussed below eqn. (3.2.11). In particular these constraints allow for the popular smooth cutoff choice (3.2.12). The crucial property leading to universality is that the results depend only on asymptotic solutions at large field, which can be derived analytically, and are also universal in the same sense. Although non-universal cutoff-dependent terms, in particular (3.2.11) and (3.3.14), enter into the calculation at intermediate stages, they drop out in the final stages. For a single component real scalar field, d_n is given in (3.2.28). For $O(N)$ scalar field theory, the d_n are just twice this, *cf.* (3.3.27), independent of N. This is in agreement with the single field result because here n indexes the eigenoperators that are a function of φ^2 only.

The first steps in deriving these results is to recast the eigenoperator equation in SL form, and then establish that the SL operator is self-adjoint in the space spanned by the eigenoperators. For a single component scalar field this follows after demonstrating that the SL weight decays exponentially for large field, since the eigenoperators grow at most as a power of the field. For the $O(N)$ case the analysis is more subtle because the relevant space is now the positive real line (parametrised by $\rho = \varphi^2 \geq 0$) and thus the SL operator is self-adjoint only if the boundary terms at $\rho = 0$ also vanish. By analytically determining the small ρ dependence of the relevant quantities we see that the SL operator is self-adjoint when $N > 0$. For $N \leq 0$, the SL operator is not self-adjoint and the analysis does not apply. Presumably in these cases one would find that the scaling dimensions d_n are no longer real. However for a sequence of special cases $N = -2k$, k a non-negative integer, the SL operator is self-adjoint on a reduced space spanned by all eigenoperators apart from the first $k + 1$. The analysis can then proceed on this reduced space. As we already noted, while most of these special cases are

presumably only of theoretical interest, the $N = 0$ case describes the statistical physics of long polymers.

The next step is to cast the SL equation in the form of a one-dimensional time-independent Schrödinger equation with energy levels $E_n = d_n$ and potential $U(x)$. For the single component field this potential is symmetric, and in order to determine the energy levels E_n asymptotically at large n, using the WKB approximation, we need only the behaviour of $U(x)$ at large x. The latter follows from our asymptotic analysis. For $O(N)$ scalar field theory, the space is the positive real line $x \geq 0$, and thus for WKB analysis we need also the behaviour of the potential $U(x)$ at small x. Here we find that the range $1 \leq N \leq 3$ requires a separate treatment because the leading term in $U(x)$ turns negative leading to a potential unbounded from below. Nevertheless we are able to treat this case and the end result for d_n, (3.3.27), is the same, thus applying universally to all $N \geq 0$ and the $N = -2k$ special cases.

Although these results are universal, they are still derived within LPA, which is an uncontrolled model approximation. One might reasonably hope however that the fact that these results are universal in the sense of being independent of the detailed choice of cutoff, is an indication that they are nevertheless close to the truth. On the other hand the LPA [102] of the Polchinski flow equation [36] is in fact completely cutoff independent, although this property arises rather trivially. It is actually equivalent under a Legendre transformation [137] to the flow equation (3.2.1) for the Legendre effective action in LPA, as we study here, but only for a special (but actually popular) choice of additive cutoff known as the optimised cutoff [44]. However the optimised cutoff does not satisfy our technical constraints given below (3.2.11) so our analysis is invalid for this case. Nor in fact does a sharp cutoff [35, 100, 103, 138] or power-law cutoff [109] satisfy the technical constraints. What this means is that these particular cutoffs fail to regularise completely the region of large fields, in the sense that a_2, defined by (3.2.7) or (3.3.8), no longer has an asymptotic expansion given simply by integrating over the asymptotic expansion of its integrand. For these three particular cutoffs, regions of momenta far from Λ alter the asymptotic expansion of a_2 so that it is no longer of the form (3.2.10) or (3.3.9), and for this reason these cutoffs are less satisfactory.

Nevertheless, following our methods, it would be straightforward to derive the asymptotic scaling dimensions d_n in LPA for any or all of these three special choices of cutoff, by using the particular form of the LPA flow equation in these cases (which are known in closed form, since the momentum integrals can be calculated analytically in these cases). The results will differ from the d_n derived here and amongst themselves, but their investigation would improve insight into the accuracy of the LPA in this regime. Furthermore it would seem possible to generalise any of these special choices of cutoff to their own class of cutoffs with similar properties, and thus understand the extent to which the results could still be cutoff independent, up to some appropriate constraints, in these cases, and gain a more detailed understanding of why the d_n differ.

Unfortunately our d_n do not seem to match in a useful way to existing results in the literature. The LPA restricts us to eigenoperators that contain no spacetime derivatives, and thus our index n counts only over these. In reality all eigenoperators (apart from the unit operator) contain spacetime derivatives, so in particular it is not clear how our index n would map into the exact sequence.

However in some special limits the LPA is effectively exact. This is true for the Gaussian fixed point for example, where $d_n = nd_\varphi$ (with $\eta = 0$). Our scaling dimensions d_n differ from this, but the Gaussian fixed point is specifically excluded from our analysis since our results apply only to non-trivial fixed points, such that the asymptotic expansion of the fixed point potential takes the form (3.1.2) or (3.3.4) with $A \neq 0$.

The LPA also becomes effectively exact in the large N limit [133], and there the scaling dimensions are $d_n = 2n$ (with $\eta = 0$) which again differs from our result (as well as differing from the Gaussian fixed point result). Furthermore they continue to disagree even if we now take a second limit such that both n and N are sent to infinity. However in this case we have an example where the order of the limits matters. The $N \to \infty$ result is derived for d_n whilst first holding n fixed, while our result applies first for fixed N while $n \to \infty$.

The difference can be seen at the technical level. The first term on the right hand side of the flow equation (3.3.1) is proportional to N. In our analysis however it is the denominators that dominate. On the other hand in the large N analysis, only the first term survives, resulting in a first order ODE with no SL properties (or Schrödinger equation representation). The universal results fall out on the one hand in our analysis from the asymptotic behaviour at large field, but on the other hand in large N they fall out from a Taylor expansion around the minimum of the fixed point potential [133]. There seems unfortunately to be no way to bridge the gap between these two limiting regimes.

An even clearer example where the exchange of limits do not commute, is provided by the special cases $N = -2k$. As we recalled in sec. 3.3, in these cases the first $k+1$ eigenoperators become degenerate, gaining Gaussian scaling dimensions. But our d_n apply to the highly irrelevant eigenoperators that are found in the reduced space, which excludes these first $k+1$ operators, and hence have non-trivial scaling dimensions. However if instead we fix on the n^{th} eigenoperator and let $N \to -\infty$ by sending $k \to \infty$, we see that this n^{th} eigenoperator will fall into the excluded space and, thus, end up with Gaussian scaling dimensions. The disagreement between the two results will then remain even if we choose next to send $n \to \infty$.

Chapter 4

Off-shell divergences in quantum gravity

4.1 Introduction

In this chapter we investigate the four dimensional perturbative quantum gravity, constructed by quantising the Einstein-Hilbert action. As previously emphasised in section 1.1.3, its perturbative non-renormalisability implies that, at each new loop order ℓ, counterterms have to be added to the bare action to cancel UV divergences, and associated with these counterterms are new operators and renormalised couplings that did not exist in the bare action at lower loop order. Nevertheless, perturbative quantum gravity can be consistently treated as an effective theory in this way [139], see also [54,140], in much the same way as the (similarly non-renormalisable) chiral perturbation theory of low energy pions [141–146].

In light of this, it might seem odd at first that we turn our attention towards investigating gravity perturbatively. However, there are good reasons for such a choice [147–149]. Our original motivation was to explore the possibility that the renormalisation group in this context might provide a route to learning something useful about the *non-perturbative* behaviour of quantum gravity. In particular, even in a perturbatively non-renormalisable theory, the RG relates the leading UV divergence at each new loop order ℓ to one-loop ($\ell = 1$) divergences [146]. More physically, it allows us to compute in this way the leading log power $(\ln \mu)^\ell$, of the standard arbitrary RG energy scale μ, at each loop order ℓ. (These are called chiral logs in pionic perturbation theory [141–146].) If it were possible to use the RG relations to compute these leading terms to arbitrarily high loop order, and resum them, we would get a powerful insight into the UV behaviour of quantum gravity at the non-perturbative level.

In perturbative quantum gravity the leading divergences actually vanish on-shell. They are therefore field reparametrisations, and have no effect on the S-matrix. However, if we keep in mind that the UV behaviour of the full two-point correlator is characterised by its *off-shell* dependence, we see that these leading divergences and associated powers of $\ln \mu$ could nevertheless be important. For example, after resumming them, one might find that the non-perturbative UV behaviour of the two-point correlator, and potentially, thus, that of quantum gravity more generally, is very different from what one would naively conclude order by order in perturbation theory.

In a non-renormalisable gauge theory, divergences that vanish on the equations of motion (of the quantum fields), are related to modifications of the BRST algebra [139] (see also [150–154]).[1] At each loop order the corresponding counterterms modify the BRST algebra in a way that remains consistent with the Zinn-Justin identities [124,155]. They do this by generating *canonical* reparametrisations of the antifields (sources for BRST transformations) [156–159] and quantum fields.

On the other hand, as we already mentioned, in a generic non-renormalisable theory the RG tells us that the leading divergences can be expressed recursively in terms of divergences in one-loop diagrams, namely one-loop counterterm diagrams, being those that contain at least one counterterm vertex [146]. As we demonstrate in sec. 4.2.5, these recurrence relations are actually crucial for consistency of the above canonical transformations. Unfortunately, for a non-renormalisable theory, the one-loop counterterm diagrams are themselves new and non-trivial at each new loop order, and thus provide a practical obstruction to deriving the leading divergence at arbitrary order.

Viewed in this light, the proposal of ref. [160] would appear to potentially provide a breakthrough. The key idea is to exploit the pole equations that follow from assuming finite generalised β-functions for the field reparametrisations. As we will see in sec. 4.4, they imply that the leading divergences at higher loops ($\ell > 1$) should actually be computable by recursive differentiation, in particular without computing any more Feynman diagrams. Unfortunately, the proposal is not correct as will become clear throughout this chapter. We spell this out in detail in sec. 4.4.

One problem with exploring these ideas is that there are effectively no explicit higher-loop off-shell leading divergences in the literature that one can test against. Some purely background field off-shell two-loop $1/\varepsilon^2$ divergences appear in the famous paper ref. [18], but unfortunately they contain an error, as pointed out in ref. [160].

All of the above considerations motivated us to compute explicitly (in Feynman – De Donder gauge and dimensional regularisation) the leading off-shell divergences for the two-point vertex up to two loops, and in particular to draw out their intimate relation to the one-loop counterterm diagrams [146] and to canonical transformations in the BRST

[1] Actually this was established only for vanishing background field. We treat the non-vanishing case in sec. 4.2.9.

algebra [139]. Since this necessitates computing, as an intermediate step, of the off-shell divergences in one-loop diagrams with three external legs, two of which are quantum, we widened our investigation so as to provide explicit results for all off-shell one-loop divergences with up to three fields.

In fact even for just the graviton one-loop two-point divergence, the complete results do not appear in the literature. Famously, the pure background part appears in ref. [161]. The pure quantum part appears in ref. [162], *cf.* also app. 4.A, and ref. [163]. But to our knowledge the divergence in the mixed quantum background vertex has not appeared before in the literature. These three divergences can be expressed in terms of appropriately defined linearised curvatures. (For the quantum field, this is an accident of Feynman – De Donder gauge, *cf.* sec. 4.3.1.2.) However, the three expressions are all different (thus not as assumed in ref. [164]). Although they are all different, they are not independent. Their relation is precisely such that all three are removed by a canonical transformation of the quantum fields (and antifields).

This may come as a surprise since *a priori* one might expect that a separate reparametrisation of the background metric should also be performed (in fact this is what is assumed and employed in ref. [160]). However in sec. 4.2.9 we show in general that this does not happen. New divergences at each loop order which involve background and quantum fluctuations, and do not vanish on the equations of motion, are purely a function of the total metric (that combines background and fluctuation), whilst all other divergences are renormalised by a canonical transformation of the quantum fields and antifields.

We show explicitly that this scenario continues to hold at the three-point level, where now thousands of vertices are divergent. We verify that the divergence in the Gauss-Bonnet topological term [18, 165] is indeed a function only of the total metric, whilst all other divergences are removed by a canonical transformation on the antifields and quantum fields.

Then in sec. 4.3.3 we use the one-loop counterterm diagrams to derive the leading divergence at two loops in the pure background, pure quantum, and mixed, two-point vertices. At this stage the dependence on the quantum field can no longer be written in terms of linearised curvatures, reflecting the fact that BRST transformations are now modified to the extent that they do not reduce to diffeomorphisms. Nevertheless, taking proper account of non-linearities in the Zinn-Justin equations, we verify again that all these divergences can be removed by a canonical transformation on the antifields and quantum fields.

The chapter is structured as follows. In sec. 4.2 we define the BRST transformations for the quantum fluctuation field and ghosts in the presence of a background metric. We develop the formalism that is needed to cope with the fact that BRST invariance is significantly altered in the process of renormalisation. Consistency is maintained

by preserving the Zinn-Justin equation [124, 155] a.k.a. CME (Classical Master Equation) [156–158]. We work with so-called *off-shell BRST* and display results in so-called *minimal basis*, since it provides the most elegant and powerful realisation, but in sec. 4.2.3 we explain why the calculations themselves are essentially the usual ones. Both the bare action and the Legendre effective action satisfy the Zinn-Justin equation [124, 155] as we review in secs. 4.2.1 and 4.2.4 respectively, but beyond one loop this leads to a tension and this tension is resolved by the RG relations for counterterm diagrams, as we explain in secs. 4.2.5 and 4.2.6.

New divergences are invariant under the total classical BRST charge s_0 which incorporates not only the BRST transformations but also the action of the Koszul-Tate operator. Taking into account the presence of the background metric, their properties are developed in sec. 4.2.7. Since s_0 is nilpotent, solutions are classified according to its cohomology. As we recall in sec. 4.2.8, those solutions that are s_0-exact are first order canonical transformations of the CME. At two loops we need also the canonical transformations to second order and their relation to the perturbatively expanded CME. This is derived in sec. 4.2.8. Then in sec. 4.2.9 we derive the general solution for s_0-closed divergences. We show that cohomologically non-trivial solutions can be taken to be functions of only the total metric, with the rest being s_0-exact, in particular there are no separate purely background metric divergences.

As already mentioned, in sec. 4.3 we compute for the first time many off-shell counterterms that appear up to two loops, and use them to verify all these properties. In this way also we provide a concrete example of how the BRST transformations get appreciably modified by loop corrections. In sec. 4.4 we investigate the proposal for generalised beta-functions for field reparametrisations. We start by assuming as in the original proposal that it is the background metric that should be reparametrised and then, given the results of this chapter, put forward a more natural scenario where the beta functions are built on the canonical transformations. Unfortunately neither of these ideas lead to finite beta functions, and we explain why they cannot. Finally in sec. 4.5 we draw our conclusions.

4.2 BRST in perturbative quantum gravity and its renormalisation

In this section we first set up the BRST framework that we will use, and then develop its properties. Along the way we make a number of new observations. In particular we will see in sec. 4.2.5 that RG invariance is actually essential to ensure that the BRST symmetry can be renormalised successfully, whilst in sec. 4.2.9 we prove the absence of a separate background field divergence in new divergences at each loop order.

4.2.1 The CME for the bare action

In a perturbative setting we work with a quantum, a.k.a. fluctuation, field $h_{\mu\nu}$. This field is defined by our choice of expansion of the (total) metric $g_{\mu\nu}$ around a background metric $\bar{g}_{\mu\nu}$ [166]. Here we simply set

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa\, h_{\mu\nu}\,, \tag{4.2.1}$$

where $\kappa = \sqrt{32\pi G}$ is the natural expansion parameter, G being Newton's gravitational constant. We are interested in off-shell divergences, and their value depends on the choice of expansion. Using the above allows us to compare with previous results in the literature [18, 161–163].

We will work with so-called *off-shell* BRST [167–170]. In this way we can fully exploit BRST invariance at every step, and keep track of how it changes under quantum corrections. Although we only actually need the Zinn-Justin equation [124, 155] for this, it is convenient to phrase the calculation in terms of the Batalin-Vilkovisky formalism [156–158], employing known identities for the antibracket [156–159]:

$$(X, Y) = \frac{\partial_r X}{\partial \phi^A}\frac{\partial_l Y}{\partial \phi^*_A} - \frac{\partial_r X}{\partial \phi^*_A}\frac{\partial_l Y}{\partial \phi^A}\,, \tag{4.2.2}$$

where X and Y are two functionals, ϕ^A are the quantum fields (including ghosts c^μ) and ϕ^*_A are the antifields (opposite statistics sources for the BRST transformations $Q\phi^A$ of the corresponding fields). As we will see, the resulting framework allows calculations that are no more onerous than standard ones employing only on-shell BRST invariance [163, 171, 172]. Furthermore, we can then display the results more compactly by using the so-called minimal basis [156, 163, 171, 172].

We choose the bare action $S[\phi, \phi^*]$ to include these sources. It will be made up of the classical action S_0 plus a series of local counterterms S_ℓ chosen to cancel the divergences that appear at each loop order ℓ, whilst introducing the new renormalised couplings (*cf.* sec. 4.2.6 [146]), which, because they run with μ, must also be introduced at that order:

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots\,. \tag{4.2.3}$$

By including the sources ϕ^* we will additionally incorporate the counterterms necessary to render finite the BRST transformations [124, 155].

At the classical level the bare action is thus given by

$$S_0 = -\int_x \left\{ \frac{2}{\kappa^2}\sqrt{g}R + (Qh_{\mu\nu})\, h^{*\mu\nu} + (Qc^\mu)\, c^*_\mu \right\}\,. \tag{4.2.4}$$

The first term is the Einstein-Hilbert action in Euclidean signature. In this chapter we consider the pure gravity case only, and thus we take the cosmological constant to

vanish. At the perturbative level, divergences do not force its introduction, so working in this simplified setting is consistent. The integral is over

$$d = 4 - 2\varepsilon \tag{4.2.5}$$

dimensional spacetime (we will be using dimensional regularisation). Our conventions for curvatures are $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, and $[\nabla_\mu, \nabla_\nu]v^\lambda = R_{\mu\nu}{}^{\lambda}{}_{\sigma}v^\sigma$.

For convenience, we choose to define the antifields to have indices in the position shown and to transform as tensor densities of weight -1 so that no metric is required above for these terms. Also for convenience, a minus sign is included so that none appears in the identity:

$$Q\phi^A = (S_0, \phi^A)\,. \tag{4.2.6}$$

Note that this defines our charges to act from the left. Classically, the BRST charge Q can be defined in terms of the Lie derivative along κc^μ:

$$Qh_{\mu\nu} = \frac{1}{\kappa}Qg_{\mu\nu} = \mathcal{L}_c g_{\mu\nu} = 2\partial_{(\mu}c^\alpha g_{\nu)\alpha} + c^\alpha \partial_\alpha g_{\mu\nu}\,, \tag{4.2.7}$$

$$Qc^\mu = \frac{\kappa}{2}\mathcal{L}_c c^\mu = \kappa c^\nu \partial_\nu c^\mu\,. \tag{4.2.8}$$

Its nilpotence ($Q^2 = 0$), and diffeomorphism invariance of the Einstein-Hilbert action, implies

$$0 = QS_0 = Q\phi^A \frac{\partial_l S_0}{\partial \phi^A} = -\frac{\partial_r S_0}{\partial \phi^*_A}\frac{\partial_l S_0}{\partial \phi^A} = \frac{1}{2}(S_0, S_0)\,, \tag{4.2.9}$$

and thus that the classical bare action $S = S_0$ satisfies the so-called CME (Classical Master Equation) [156–158], a.k.a. Zinn-Justin equation [124, 155]. Once we consider quantum corrections, it is not the BRST transformations (4.2.7) and (4.2.8) that we can preserve but only the CME, *i.e.* we will ensure that to any loop order ℓ the bare action satisfies:

$$(S, S) = 0\,. \tag{4.2.10}$$

4.2.2 Canonical transformation to gauge fixed basis

To get the gauge fixed version, we need to work in the so-called extended basis, which introduces a new field and antifield over and above what we already have (the so-called minimal basis) [156–158]:

$$S^{(ext)} = S + \int_x \left\{ \frac{1}{2\alpha}\sqrt{\bar{g}}\bar{g}_{\mu\nu}b^\mu b^\nu + ib^\mu \bar{c}^*_\mu \right\}\,, \tag{4.2.11}$$

where α is the gauge parameter, b^μ is a bosonic auxiliary field, and $\bar{c}^*_\mu$ sources the BRST transformation for the antighost. From (4.2.6) we have $Q\bar{c}^\mu = -ib^\mu$ and $Qb^\mu = 0$. Trivially, the CME and $Q^2 = 0$ continue to hold. The next step is to introduce a

suitable gauge fixing fermion $\Psi[\phi]$. In the Batalin-Vilkovisky treatment this is used to eliminate the antifields [156–159]. We keep them however, because of their crucial rôle in renormalisation, and in particular in the Zinn-Justin identities, and instead get the same effect by performing an exact canonical transformation [159]

$$\begin{aligned} \check{\phi}^A &= \frac{\partial_l}{\partial \check{\phi}^*_A} \mathcal{K}[\phi, \check{\phi}^*] \,, \\ \phi^*_A &= \frac{\partial_r}{\partial \phi^A} \mathcal{K}[\phi, \check{\phi}^*] \,, \end{aligned} \tag{4.2.12}$$

from the above gauge invariant (g.i.) basis $\{\phi, \phi^*\}$, to a gauge fixed (g.f.) basis $\{\check{\phi}, \check{\phi}^*\}$, setting [163, 171, 172]

$$\mathcal{K} = \check{\phi}^*_A \phi^A - \Psi[\phi] \,. \tag{4.2.13}$$

The advantage of employing a canonical transformation is that by definition it leaves the antibracket invariant and thus in the new basis the CME continues to hold. We choose

$$\Psi = \int_x \sqrt{\bar{g}} F_\mu \bar{c}^\mu \,, \tag{4.2.14}$$

and choose DeDonder gauge by setting F_μ to

$$F_\mu = \bar{\nabla}_\nu h^\nu{}_\mu - \bar{\nabla}_\mu \varphi \,, \tag{4.2.15}$$

$$\varphi = \frac{1}{2} h^\mu{}_\mu = \frac{1}{2} \bar{g}^{\mu\nu} h_{\mu\nu} \,. \tag{4.2.16}$$

This breaks the diffeomorphism invariance as realised through the total metric $g_{\mu\nu}$ (as required) but leaves it realised as "background diffeomorphism" invariance, using the background metric $\bar{g}_{\mu\nu}$. From here on we raise and lower indices using the background metric, unless explicitly mentioned otherwise, and employ the background covariant derivative $\bar{\nabla}_\mu$ (using the background metric Levi-Civita connection). As is well known, we can put a connection in for free in Lie derivatives, so to make background diffeomorphism invariance manifest in (4.2.7) and (4.2.8) we can write the classical BRST transformations (in minimal basis) instead as

$$\begin{aligned} Q h_{\mu\nu} &= 2\bar{\nabla}_{(\mu} c^\alpha g_{\nu)\alpha} + c^\alpha \bar{\nabla}_\alpha g_{\mu\nu} \\ &= 2\bar{\nabla}_{(\mu} c_{\nu)} + 2\kappa \bar{\nabla}_{(\mu} c^\alpha h_{\nu)\alpha} + \kappa c^\alpha \bar{\nabla}_\alpha h_{\mu\nu} \,, \\ Q c^\mu &= \kappa c^\nu \bar{\nabla}_\nu c^\mu \,. \end{aligned} \tag{4.2.17}$$

Applying the canonical transformation we see that only the following antifields change:

$$h^{*\mu\nu}\big|_{g.f.} = h^{*\mu\nu}\big|_{g.i.} - \sqrt{\bar{g}} \left(\bar{\nabla}^{(\mu} \bar{c}^{\nu)} - \frac{1}{2} \bar{\nabla}_\alpha \bar{c}^\alpha \bar{g}^{\mu\nu} \right) \,, \tag{4.2.18}$$

$$\bar{c}^{*\mu}\big|_{g.f.} = \bar{c}^{*\mu}\big|_{g.i.} + \sqrt{\bar{g}} F^\mu \,, \tag{4.2.19}$$

thus mapping the extended action (4.2.11) at the classical level to

$$S_0^{(ext)}\big|_{g.f.} = S_0 + \int_x \left\{ \frac{1}{2\alpha}\sqrt{\bar{g}}\bar{g}_{\mu\nu}b^\mu b^\nu - i\sqrt{\bar{g}}F_\mu b^\mu + ib^\mu \bar{c}^*_\mu \right\} + \int_x \sqrt{\bar{g}}\left(\bar{\nabla}^{(\mu}\bar{c}^{\nu)} - \frac{1}{2}\bar{\nabla}_\alpha \bar{c}^\alpha \bar{g}^{\mu\nu}\right) Q h_{\mu\nu}\,. \quad (4.2.20)$$

The first term is (4.2.4), the classical action in minimal basis, and the last term is the usual ghost action (in DeDonder gauge). The middle term is purely quadratic in b^μ. We could thus integrate it out. Dropping the $\bar{c}^*_\mu$, the integrand is:

$$\frac{\sqrt{\bar{g}}}{2\alpha}(b_\mu - iF_\mu)^2 + \frac{\alpha}{2}\sqrt{\bar{g}}F^\mu F_\mu\,. \quad (4.2.21)$$

The b_μ integral over the first term vanishes in dimensional regularisation, whilst the second term is the standard gauge fixing term. In fact this is now the textbook *on-shell BRST* treatment. The action S_0 is still BRST invariant if we now set $Q\bar{c}^\mu = \alpha F^\mu$. But this is not quite as powerful because $Q^2\bar{c}^\mu = \alpha Q F^\mu$, only vanishes on shell ($QF_\mu = 0$ is the $\bar{c}$ equation of motion). For this reason we keep b^μ and stick with this off-shell BRST treatment.

Since we will be working with a perturbative expansion over quantum fields and anti-fields, we may as well treat the background metric perturbatively also. Following (4.2.1), we write:

$$\bar{g}_{\mu\nu} = \delta_{\mu\nu} + \kappa\bar{h}_{\mu\nu} \qquad \Longrightarrow \qquad g_{\mu\nu} = \delta_{\mu\nu} + \kappa\, h_{\mu\nu} + \kappa\bar{h}_{\mu\nu}\,. \quad (4.2.22)$$

At this stage we can invert the terms bilinear in the quantum fields to get the propagators. For general α gauge see *e.g.* ref. [171]. We will use Feynman gauge, $\alpha = 2$, which gives the simplest propagators. Once again, the coefficients of off-shell divergences depend on these choices. By using Feynman DeDonder gauge we make the same choices as in older works [18, 161–163], and can thus compare our results. Writing

$$\phi^A(x) = \int \frac{d^dp}{(2\pi)^d}\, \mathrm{e}^{-ip\cdot x}\, \phi^A(p)\,, \quad (4.2.23)$$

we have:

$$\langle h_{\mu\nu}(p)\, h_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} - \frac{1}{d-2}\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2}\,, \quad (4.2.24)$$

$$\langle b_\mu(p)\, h_{\alpha\beta}(-p)\rangle = -\langle h_{\alpha\beta}(p)\, b_\mu(-p)\rangle = 2\,\delta_{\mu(\alpha}p_{\beta)}/p^2\,, \quad (4.2.25)$$

$$\langle b_\mu(p)\, b_\nu(-p)\rangle = 0\,, \quad (4.2.26)$$

$$\langle c_\mu(p)\, \bar{c}_\nu(-p)\rangle = -\langle \bar{c}_\mu(p)\, c_\nu(-p)\rangle = \delta_{\mu\nu}/p^2\,. \quad (4.2.27)$$

4.2.3 Minimal basis and comparisons to on-shell BRST

We will be computing quantum corrections to the one-particle irreducible, a.k.a. Legendre, effective action Γ. Since we have an auxiliary field b^μ and the extra propagator $\langle b_\mu h_{\alpha\beta}\rangle$, at first sight this formalism complicates the computation and cannot be directly compared to earlier results using on-shell BRST [18, 161, 162]. However, this is not the case.

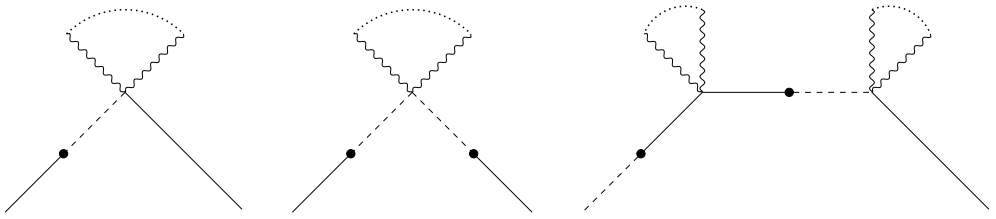

FIGURE 4.2.1: Examples that illustrate that one-particle irreducible Feynman diagrams involving b interactions with an unspecified number of external background metric $\bar{h}$ legs (fan of wavy lines), arise by starting with an internal h (solid line) propagating into b (dashed line) and eventually back to h. These implement in diagrammatic language the effect (4.2.21) of integrating out the b field.

First note that the h propagator (4.2.24) is the same as in the usual treatment. (This is actually guaranteed in any gauge, but we omit the proof.) Setting $\bar{h}_{\mu\nu} = 0$ for the moment, we note that the interaction terms (*i.e.* with three or more fields) in (4.2.20) do not contain b^μ or $\bar{c}^*_\mu$. Feynman diagram contributions to Γ therefore have the same property and coincide with those computed in the usual (on-shell BRST) treatment. Switching back on the background metric, we do now have interactions involving the background metric and either b^2, or b and h. However it is not possible then to draw one-particle irreducible diagrams with external b-field legs. The interactions only contribute in diagrams by having h propagate to b and back again, see fig. 4.2.1, and the net effect of including all these corrections is to incorporate in diagrammatic language the result of integrating out b. Thus these Feynman diagrams simply reproduce the corrections we get from the second term in (4.2.21), *i.e.* the standard gauge fixing term. So we see that we can continue to ignore b^μ and $\bar{c}^*_\mu$ provided we include the interactions from the standard gauge fixing term. Furthermore, we get in this way the same results as the standard treatment.

Next note that the corrections only depend on $\bar{c}^\mu$ through the combination on the right-hand side of (4.2.18). This means that we can shift back to g.i. basis after computing loop contributions to Γ, the only dependence on b and $\bar{c}^*$ then being as in the extended action (4.2.11). Furthermore, we can then display results in minimal basis by removing the b and $\bar{c}^*$ terms.

This all means that we can construct Γ order by order in the minimal basis, never needing b or $\bar{c}^*$. To do so we shift $h^{*\mu\nu}\big|_{g.i.}$ to $h^{*\mu\nu}\big|_{g.f.}$ in interactions and use the $\langle h_{\alpha\beta} h_{\mu\nu}\rangle$ and

$\langle c_\mu \bar{c}_\nu \rangle$ propagators and include the interaction vertices from the standard gauge fixing term (4.2.21) as appropriate, and afterwards shift back to g.i. basis [163,172]. Of course, this does not mean that off-shell quantum corrections are independent of our choice of gauge. However, the results are sometimes much simpler when cast back in (minimal) g.i. basis in this way, which is why we use it.

4.2.4 The CME for the Legendre effective action

Since the BRST transformations (4.2.17), or (4.2.7) and (4.2.8), involve products of fields at the same spacetime point, they are not preserved under renormalisation. Order by order in the loop expansion not only must the action be modified, but also the BRST transformations themselves, and since the theory is non-renormalisable, the changes involve in fact an infinite series in powers of the fields and antifields. The Zinn-Justin equation [124, 155–157] can keep track of all this. We start with the fact that the partition function

$$\mathcal{Z} \equiv \mathcal{Z}[J, \phi^*] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi,\phi^*]+\phi^A J_A} \,, \tag{4.2.28}$$

satisfies the identity

$$\frac{\partial_r \mathcal{Z}}{\partial \phi^*_A} J_A = 0 \,. \tag{4.2.29}$$

To prove this at the classical level it is sufficient to use the fact that $QS_0 = 0$, assuming invariance of the measure:

$$0 = \int \mathcal{D}\phi \, Q\Big(\mathrm{e}^{-S_0+\phi^A J_A}\Big) = \int \mathcal{D}\phi \, \mathrm{e}^{-S_0+\phi^A J_A} \, \big(Q\phi^A\big) \, J_A = -\frac{\partial_r \mathcal{Z}}{\partial \phi^*_A} J_A \,. \tag{4.2.30}$$

But at the quantum level we need to derive it via preservation of the CME (4.2.10):

$$\begin{aligned} 0 &= \int \mathcal{D}\phi \, \frac{\partial_l}{\partial \phi^A} \frac{\partial_r}{\partial \phi^*_A} \, \mathrm{e}^{-S[\phi,\phi^*]+\phi^A J_A} \\ &= -\int \mathcal{D}\phi \, \left\{ J_A \frac{\partial_r S}{\partial \phi^*_A} + \frac{1}{2}(S,S) + \frac{\partial_l}{\partial \phi^A} \frac{\partial_r}{\partial \phi^*_A} S \right\} \mathrm{e}^{-S[\phi,\phi^*]+\phi^A J_A} \,. \end{aligned} \tag{4.2.31}$$

Here the first equality follows because it is an integral of a total derivative. After rearranging the result using the statistics of the (anti)fields, we get the three terms inside the braces. The first term gives the required identity, the second term vanishes by the CME, whilst the third term is the Batalin-Vilkovisky measure term [156–158]. In general we need to take this into account (giving the Quantum Master Equation) [156–158, 163, 171, 172], however, since S is local, this term always contains $\delta(x)|_{x=0}$ or its spacetime derivatives. These vanish in dimensional regularisation. Therefore, we can discard the measure term.

Introducing the generator $W[J,\phi^*]$ of connected diagrams, through $\mathcal{Z} = \mathrm{e}^W$, we define the Legendre effective action in the usual way:

$$\Gamma[\Phi,\Phi^*] = -W + \Phi^A J_A\,, \qquad \Phi^A = \frac{\partial_r W}{\partial J_A}\,, \quad J_A = \frac{\partial_l \Gamma}{\partial \Phi^A}\,, \tag{4.2.32}$$

where Φ^A is the so-called classical field, and we have renamed $\phi^*_A \equiv \Phi^*_A$ just because it looks better. Then, by standard manipulations, (4.2.29) turns into the Zinn-Justin equation:

$$(\Gamma,\Gamma) = 0\,, \tag{4.2.33}$$

i.e. again the CME (4.2.10), now applied to $\Gamma[\Phi,\Phi^*]$, the antibracket taking the same form as (4.2.2) but with $\{\phi,\phi^*\}$ replaced with $\{\Phi,\Phi^*\}$.

The Legendre effective action

$$\Gamma = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots\,, \tag{4.2.34}$$

is built up recursively, where Γ_ℓ is the ℓ-loop contribution, starting with $\Gamma_0 = S_0$, the classical bare action. The logic now is to introduce at each new loop order ℓ, a local counterterm action S_ℓ to the bare action in order to cancel the divergences $\Gamma_\ell|_\infty$ that arise in Γ_ℓ, leaving behind an arbitrary finite part which is parametrised by the new renormalised couplings that appear at this order. Provided we introduce S_ℓ in such a way as to preserve $(S,S) = 0$ we also have that $(\Gamma,\Gamma) = 0$ is satisfied. However, although both the bare action S and the Legendre effective action Γ satisfy the CME, the CME plays a different rôle in each case so that it is in fact not trivial that the two are consistent beyond one loop. As we will see, what makes them nevertheless consistent is the RG.

4.2.5 How the RG is needed for consistent solutions to both versions of the CME

Expanding the CME (4.2.33) for Γ, we see that the one-loop contribution satisfies $(\Gamma_0,\Gamma_1) = 0$. It is useful to define the total classical BRST charge s_0 acting on any functional X as

$$s_0 X = (S_0, X)\,, \tag{4.2.35}$$

which, thus, acts also on antifields (see sec. 4.2.7). Then, the one-loop BRST identity is simply $s_0\Gamma_1 = 0$. Since dimensional regularisation is a gauge invariant regulator, the infinite part, which at one loop is proportional to a single pole (*i.e.* $\propto 1/\varepsilon$), also satisfies this identity, *i.e.*

$$s_0\,\Gamma_{1/1}[\Phi,\Phi^*] = 0\,. \tag{4.2.36}$$

(We label terms proportional to divergences $1/\varepsilon^k$, by appending $/k$ to the subscript.)

It is simplest for our purposes to now consider the identity satisfied by the two-loop contribution, Γ_2, before any renormalisation. From the CME (4.2.33) we see that it satisfies

$$s_0\Gamma_2 = -\frac{1}{2}(\Gamma_1,\Gamma_1)\,. \tag{4.2.37}$$

In particular this implies for the double-pole divergence:

$$s_0\,\Gamma_{2/2} = -\frac{1}{2}\,(\Gamma_{1/1},\Gamma_{1/1})\ . \tag{4.2.38}$$

Given that the right hand side does not vanish, this is a non-trivial relation between the $1/\varepsilon^2$ divergences at two loops and the $1/\varepsilon$ divergences at one loop.

Now we consider the process of renormalisation. At one loop, if we add a counterterm action S_1, then in order to preserve the CME (4.2.10) for S, we find in the same way that S_1 must be chosen so that it is also annihilated by the total classical BRST charge:

$$s_0 S_1 = 0\,. \tag{4.2.39}$$

Since the one-loop divergence is local we can then render the one-loop result finite by setting

$$S_1 = -\Gamma_{1/1}[\phi,\phi^*] + S_{c_1}[\phi,\phi^*]\,, \tag{4.2.40}$$

where the finite remainder S_{c_1} contains the new renormalised couplings $c_1^j(\mu)$ that appear at one loop, *cf.* sec. 4.2.6, in particular they are needed for the curvature-squared terms but also for antifield vertices, see secs. 4.3.1 and 4.3.2. Clearly we must also have $s_0 S_{c_1} = 0$.

Expanding the CME (4.2.10) to $O(\hbar^2)$, we find of course an algebraically identical formula to (4.2.37) and (4.2.38), *i.e.*

$$s_0 S_2 = -\frac{1}{2}(S_1,S_1)\,. \tag{4.2.41}$$

This must be satisfied by the counterterm action S_2. It relates the $1/\varepsilon^2$ divergence in this two-loop counterterm to the $1/\varepsilon$ divergence in the one-loop counterterms. Then by (4.2.40), we see that the $1/\varepsilon^2$ divergence on the right hand side is precisely the same as in the Γ identity (4.2.38). But this is in apparent contradiction with the fact that S_2 *must cancel* the divergence in Γ_2. In particular, the latter implies that $s_0(S_2+\Gamma_2)$ must be finite.

The resolution is that, once we add the one-loop counterterm from S_1 to the bare action, at $O(\hbar^2)$ we also have one-loop counterterm diagrams from one-loop diagrams $\Gamma_1[S_1]$ with one S_1 vertex inserted (as illustrated in fig. 4.3.3 of sec. 4.3.3). The two-loop divergence in (4.2.38) comes from diagrams containing only tree level vertices. It must be that the $1/\varepsilon^2$ contribution from the one-loop counterterm diagrams, is in fact precisely right to

flip the sign so that, in full, the double-pole part satisfies

$$s_0\left(\Gamma_{2/2}+\Gamma_{1/2}[S_1]\right)=+\frac{1}{2}\left(\Gamma_{1/1},\Gamma_{1/1}\right)\,. \tag{4.2.42}$$

As we will see in the next subsection, RG invariance tells us that we have the relation

$$\Gamma_{1/2}[S_1]=-2\Gamma_{2/2}\,, \tag{4.2.43}$$

and thus for the full double-pole contribution, $\Gamma_{2/2}+\Gamma_{1/2}[S_1]=-\Gamma_{2/2}$, we indeed have the required change of sign (even before the application of s_0). We see therefore that the RG relations are responsible for restoring consistency between the two versions of the CME.

Although the relations above constrain the form of the double-pole divergences, we still have to compute some Feynman integrals to determine them. Nevertheless we can simplify the process by exchanging the genuinely two-loop diagrams for one-loop counterterm diagrams. The corresponding double-pole counterterm action will automatically satisfy the constraint (4.2.41). This latter constraint does not uniquely determine S_2 since it is invariant under adding a piece, S_2', provided it is annihilated by the total classical BRST charge: $s_0S_2'=0$. Since this constraint is linear homogeneous, S_2' has finite remainders parametrised by new two-loop couplings $c_2^j(\mu)$.

We finish this section with some comments about the two-loop *single-pole* divergences. Firstly note that, before adding the one-loop counterterm diagrams, the two-loop single-pole divergences are actually non-local. Indeed, this must be the case since the right hand side of (4.2.37) has such non-local divergences in the antibracket contribution $(finite,\Gamma_{1/1})$, where we have written $\Gamma_1=\Gamma_{1/1}+finite$, and recognised that the finite part is non-local. On adding the counterterm diagrams, the same RG invariance identity that resolves the above putative puzzle is also responsible for eliminating the non-local divergences (see the argument of Chase [164], which we review in the next subsection). In a similar vein, the two-loop counterterm action S_2 has single-pole divergences that depend on the one-loop couplings c_1^j, as it must in order to renormalise the $\Gamma_{1/1}[S_1]$ contribution. The fact that S_2 must have dependence on c_1^j can also be seen through (4.2.40) and the two-loop CME relation, (4.2.41). These two constraints must again be related through similar RG identities.

Finally note that there are two-loop single-pole divergences that are not fixed by the RG or by the CME. These will include the famous Goroff and Sagnotti term (4.4.11), but also further terms that vanish on the equations of motion. Renormalising them requires new counterterms whose finite remainder introduces further two-loop renormalised couplings $c_2^j(\mu)$. As before, from (4.2.41) we see that this new part S_2' must be chosen so that it is annihilated by the total classical BRST charge: $s_0S_2'=0$. Thus, despite the fact that BRST invariance is significantly altered by the quantum corrections, a central role

is played, order by order in the loop expansion, by the total classical BRST charge s_0. We will develop the properties of s_0 in sec. 4.2.7.

4.2.6 Relating counterterms via the RG

Adapting ref. [146] to quantum gravity, we prove the RG relation (4.2.43), which was used in the previous subsection to demonstrate consistency at two loops of the two rôles for the CME. This key equation relates the double-pole $\Gamma_{1/2}[S_1]$ from the one-loop counterterm diagrams, to the double-pole $\Gamma_{2/2}$ generated by two-loop diagrams using only tree-level vertices. In this subsection, we also review the alternative proof in ref. [164] for this relation. Rearranging (4.2.43) we see that it implies that the $1/\varepsilon^2$ part of the two-loop counterterm is $-1/2$ times the $1/\varepsilon^2$ pole in the one-loop counterterm diagrams:

$$S_{2/2} = -\big(\Gamma_{2/2} + \Gamma_{1/2}[S_1]\big) = -\frac{1}{2}\Gamma_{1/2}[S_1]\,. \tag{4.2.44}$$

It is this form that falls out most naturally from the RG analysis, and it is also this form that we use in sec. 4.3.3 to compute the $1/\varepsilon^2$ divergence in the two-loop graviton self-energy.

To adapt [146], it proves convenient to absorb Newton's constant into the operators so that the $O(\hbar^0)$ (*i.e.* classical) bare action has pure fluctuation field vertices ($n \geq 2$):

$$\mathcal{O}_{0\,i} \sim \kappa^{n-2} h^n p^2\,. \tag{4.2.45}$$

The numerical subscript on $\mathcal{O}$ refers to $\hbar$ order [146], and here we are just counting the number of instances of the fluctuation field $h_{\mu\nu}$, κ and momentum p, where the latter stands for any momentum (or spacetime derivative) in the vertex, in order to track their dimensions and motivate the formulae below. Working with pure $h_{\mu\nu}$ vertices will be sufficient to derive (4.2.44) in this case. Then we will justify why it is clear that (4.2.44) continues to hold when the background, ghosts and antifields are included.

In $d = 4 - 2\varepsilon$ dimensions, the mass dimensions are $[h] = -[\kappa] = 1 - \varepsilon$. *A priori* both κ and the fluctuation field should be taken to be bare, in the expectation that they will have a divergent expansion in renormalised quantities, but the divergences that are generated involve ever greater powers of momentum, so the vertices in (4.2.45) are never reproduced and, thus, neither κ nor h require renormalisation. The classical bare action is therefore being written as

$$S_0 = \Gamma_0 = \int_x c_0^i\, \mathcal{O}_{0\,i}\,. \tag{4.2.46}$$

The c_0^i are the classical couplings with κ factored out. They are fixed up to choice of expansion of the metric, choice of gauge fixing, and the value of the cosmological constant if there is one. As mentioned below (4.2.4), we set the cosmological constant to zero throughout this chapter.

The divergent one-loop quantum corrections then take the form (H is the vacuum expectation value of h):

$$\Gamma_{1/1} \sim \frac{1}{\varepsilon}\kappa^n H^n p^{4-2\varepsilon}\,, \tag{4.2.47}$$

i.e. in terms of counting overall powers there is an extra factor of $\kappa^2 p^{2-2\varepsilon}$. To renormalise we thus have to add to the bare action the local action (4.2.40):

$$S_1 = \mu^{-2\varepsilon}\int_x \left\{c_1^i\,\mathcal{O}_{1\,i} + \frac{1}{\varepsilon}a^i_{1/1}\,\mathcal{O}_{1\,i}\right\}\,, \tag{4.2.48}$$

where the second set are the counterterms $-\Gamma_{1/1}$, and the first set is the expansion of S_{c_1} and contains the new $O(\hbar^1)$ renormalised couplings. The new operators take the form

$$\mathcal{O}_{1\,i} \sim \kappa^n h^n p^4\,, \tag{4.2.49}$$

i.e. with an extra $\kappa^2 p^2$ compared to $O(\hbar^0)$ vertices. At this stage the arbitrary RG scale μ is needed so that $\mu^{-2\varepsilon}$ in (4.2.48) can restore dimensions. Since the bare action (4.2.3) is independent of μ, the renormalised couplings c_1^i run with μ. By differentiating (4.2.48) we see that they satisfy:

$$\beta_1^i = \dot{c}_1^i - 2\varepsilon\, c_1^i = 2\,a^i_{1/1}\,, \tag{4.2.50}$$

where $\dot{c} := \mu\partial_\mu c$. The one-loop counterterm diagrams formed by using one $a^i_{1/1}$ vertex (corresponding to one copy of S_1 being inserted) give in particular double pole divergences

$$\Gamma_{1/2}[S_1] \sim \frac{1}{\varepsilon^2}\,a_{1/1}\,\mu^{-2\varepsilon}\kappa^{n+2}H^n p^{6-2\varepsilon}\,, \tag{4.2.51}$$

that must satisfy relation (4.2.43): $\Gamma_{1/2}[S_1] = -2\Gamma_{2/2}$. As noted by Chase [164], the easiest way to see why this is so, is to recognise that the latter take the form

$$\Gamma_{2/2} \sim \kappa^{n+2}H^n p^{6-4\varepsilon}\left[\frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right)\right]\,, \tag{4.2.52}$$

but divergences must be local and thus the $(\ln p)/\varepsilon$ terms must cancel between (4.2.51) and (4.2.52).

We get the same conclusion another way by following Buchler and Colangelo [146] whilst also deriving some more useful identities. At $O(\hbar^2)$ the divergences generate the operators

$$\mathcal{O}_{2\,i} \sim \kappa^{n+2}h^n p^6\,, \tag{4.2.53}$$

so we have to add to the bare action

$$S_2 = \mu^{-4\varepsilon}\int_x \left\{c_2^i\,\mathcal{O}_{2\,i} + \frac{1}{\varepsilon^2}a^i_{2/2}\,\mathcal{O}_{2\,i} + \frac{1}{\varepsilon}\left(a^i_{2/1} + a^i_{1/1\,j}c_1^j\right)\,\mathcal{O}_{2\,i}\right\}\,, \tag{4.2.54}$$

where we now have counterterms with both single and double ε-poles, and c_2^i are the

new $O(\hbar^2)$ renormalised couplings. The $a^i_{2/2}$ counterterms cancel the full set of $1/\varepsilon^2$ divergences at $O(\hbar^2)$, *i.e.* from the sum of two-loop diagrams and the one-loop counterterm diagrams. The single poles $a^i_{2/1}/\varepsilon$ arise from two-loop diagrams using only vertices from (4.2.45), whilst the $a^i_{1/1\,j}c^j_1/\varepsilon$ are generated by one-loop diagrams containing one c_1 vertex. Now μ-independence of the bare action implies

$$\begin{aligned}\beta^i_2 &= \dot{c}^i_2 - 4\varepsilon c^i_2 = \frac{4}{\varepsilon}a^i_{2/2} + 4\left(a^i_{2/1} + a^i_{1/1\,j}c^j_1\right) - \frac{1}{\varepsilon}a^i_{1/1\,j}\,\dot{c}^j_1\,,\\ &= \frac{4}{\varepsilon}a^i_{2/2} - \frac{2}{\varepsilon}a^i_{1/1\,j}a^j_{1/1} + 4a^i_{2/1} + 2a^i_{1/1\,j}c^j_1\,,\end{aligned} \tag{4.2.55}$$

where in the second line we substituted the one-loop β function (4.2.50). Since this equation is expressed in terms of renormalised quantities, it must be finite, and therefore the single poles must cancel. Thus we see that

$$a^i_{2/2} = \frac{1}{2}a^i_{1/1\,j}a^j_{1/1}\,. \tag{4.2.56}$$

This is the same conclusion as before, but we are now proving it in the form given in (4.2.44). The left hand side is the coefficient of the $\mathcal{O}_{2\,i}$ in $S_{2/2}$ while on the right hand side we have replaced the c^j_1 coupling in (4.2.54) by the counterterm coefficient $a^j_{1/1}$. The right hand side is thus the coefficient of $\mathcal{O}_{2\,i}$ in $-\frac{1}{2}\Gamma_{1/2}[S_1]$.

Finally, let us show that (4.2.44) will continue to hold when the background, ghosts and antifields are included. Firstly, vertices can now include ghost antighost pairs, but at this schematic level it is not necessary to track these separately from h: what really matters in this analysis are the powers of p^ε and μ^ε, and they are unchanged if c and $\bar{c}$ are included. Secondly, it is clear that any instance of h (or H) can trivially be exchanged for the background $\bar{h}$ in the above schematic formulae, though of course operators $\mathcal{O}_{1\,j}$ with less than two quantum fields in gauge fixed basis, cannot contribute to the relation (4.2.44) (their coefficients $a^i_{1/1\,j}$ vanish). Finally from the minimal classical action (4.2.4), we see that whenever an antifield is involved in an action vertex there is one less power of p (compensated dimensionally by the fact that they have $[\phi^*] = 2-\varepsilon$, *cf.* table 4.1). This observation is useful for finding the general form of the corrections, but again for this analysis what actually matters is the tracking of non-integer powers.

4.2.7 Properties of the total classical BRST charge

We now develop the properties of the total classical BRST charge s_0. Using the identity [156–159]:

$$(X,(Y,Z)) = ((X,Y),Z) + (-1)^{(X+1)(Y+1)}(Y,(X,Z))\,, \tag{4.2.57}$$

where $(-1)^X = \pm 1$ if X bosonic (fermionic), we have

$$s_0^2 X[\phi,\phi^*] = (S_0,(S_0,X)) = \tfrac{1}{2}((S_0,S_0),X) = 0\,, \tag{4.2.58}$$

where the last equality follows by the CME. Therefore s_0 is nilpotent just like the BRST charge Q. From (4.2.6), we see that on ϕ^A it reduces to the BRST charge Q. However from (4.2.35), s_0 also acts on antifields:

$$s_0\phi^*_A = \big(S_0, \phi^*_A\big) = \frac{\partial_r S_0}{\partial \phi^A}\,. \tag{4.2.59}$$

This is called the Koszul-Tate differential [163, 171, 173–176]. In minimal basis we get explicitly that:

$$s_0 h^{*\mu\nu} = -2\sqrt{g}G^{\mu\nu}/\kappa + 2\kappa h^{*\alpha(\mu}\bar{\nabla}_\alpha c^{\nu)} + \kappa\bar{\nabla}_\alpha\big(c^\alpha h^{*\mu\nu}\big)\,, \tag{4.2.60}$$

$$s_0 c^*_\mu = \kappa\bar{\nabla}_\mu c^\nu c^*_\nu + \kappa\bar{\nabla}_\nu\big(c^\nu c^*_\mu\big) - 2\bar{\nabla}_\nu h^{*\nu}{}_\mu - 2\kappa\bar{\nabla}_\alpha\big(h_{\mu\nu}h^{*\alpha\nu}\big) + \kappa\bar{\nabla}_\mu h_{\alpha\beta}h^{*\alpha\beta}\,. \tag{4.2.61}$$

Here $G_{\mu\nu} = -R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor. (Note that it inherits an overall minus sign from the Euclidean action compared to the usual definition.) Its indices are raised in (4.2.60) using $G^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}G_{\alpha\beta}$. As we noted earlier we are raising and lowering indices with the background metric unless explicitly stated otherwise. This case is the one exception.

	ϵ	gh #	ag #	pure gh #	dimension
$h_{\mu\nu}$	0	0	0	0	$(d-2)/2$
c_μ	1	1	0	1	$(d-2)/2$
$\bar{c}_\mu$	1	-1	1	0	$(d-2)/2$
b_μ	0	0	1	1	$d/2$
$h^*_{\mu\nu}$	1	-1	1	0	$d/2$
c^*_μ	0	-2	2	0	$d/2$
$\bar{c}^*_\mu$	0	0	0	0	$d/2$
Q	1	1	0	1	1
Q^-	1	1	-1	0	1

TABLE 4.1: The various Abelian charges (a.k.a. gradings) carried by the fields and operators. ϵ is the Grassmann grading, being 1(0) if the object is fermionic (bosonic). gh # is the ghost number, ag # the antighost/antifield number, pure gh # = gh # + ag #, and dimension is the engineering dimension. The first two rows are the minimal set of fields, the next two make it up to the non-minimal set, then the ensuing two rows are the minimal set of antifields, and $\bar{c}^*_\mu$ is needed for the non-minimal set. Finally, the charges are determined in order to ensure that Q and Q^- can also be assigned definite charges.

It is useful to assign antighost/antifield number to each field and operator [171, 176, 177], see table 4.1. The reason this is useful is precisely because it is *not* preserved by interactions, which then split into pieces according to their antighost level. For example one sees from (4.2.4), that the three parts of the minimal classical action split into levels 0, 1, and 2, respectively. The Koszul-Tate differential also splits, in this case into two pieces, one that preserves antighost number and one that lowers it by one. We call these

pieces respectively, Q and Q^-, and thus write:

$$s_0\phi^*_A = \big(Q + Q^-\big)\phi^*_A\,. \tag{4.2.62}$$

From (4.2.60) and (4.2.61) we see that

$$Qh^{*\mu\nu} = 2\kappa h^{*\alpha(\mu}\bar\nabla_\alpha c^{\nu)} + \kappa\bar\nabla_\alpha\big(c^\alpha h^{*\mu\nu}\big)\,, \tag{4.2.63}$$

$$Q^- h^{*\mu\nu} = -2\sqrt{g}G^{\mu\nu}/\kappa\,, \tag{4.2.64}$$

$$Qc^*_\mu = \kappa\bar\nabla_\mu c^\nu c^*_\nu + \kappa\bar\nabla_\nu\big(c^\nu c^*_\mu\big)\,, \tag{4.2.65}$$

$$Q^- c^*_\mu = -2\bar\nabla_\nu h^{*\nu}{}_\mu - 2\kappa\bar\nabla_\alpha\big(h_{\mu\nu}h^{*\alpha\nu}\big) + \kappa\bar\nabla_\mu h_{\alpha\beta}h^{*\alpha\beta}\,. \tag{4.2.66}$$

Since Q here acts on antifields there is no reason to confuse it with the previously defined BRST charge in (4.2.6) and (4.2.17). Its extension to antifields is natural since $Qh^{*\mu\nu}$ and Qc^*_μ are in fact the correct Lie derivative expressions for these tensor densities. The advantage of the antighost grading becomes clear when we consider the nilpotency of s_0:

$$0 = s_0^2 = Q^2 + \{Q, Q^-\} + (Q^-)^2\,. \tag{4.2.67}$$

These terms must vanish separately since they lower the antighost number by 0, 1 and 2 respectively. Therefore we know that our definitions of Q and Q^- are such that they are nilpotent and they anticommute.

4.2.8 Canonical transformations up to second order

We saw in sec. 4.2.5 that a central role is played by counterterms that are s_0-closed, for example at one loop we have exactly this relation (4.2.39): $s_0S_1 = 0$. We saw in the previous subsection that s_0 is nilpotent, so one solution to this is that S_1 is exact: $S_1 = s_0K_1$, where K_1 is a local functional of ghost number -1. In the next subsection we derive the general solution for such s_0-closed counterterms, but for that we will need the relation between s_0-exact solutions and canonical transformations. Taking the general canonical transformation (4.2.12), and setting

$$\mathcal{K} = \check\phi^*_A\phi^A + K_1[\phi, \check\phi^*]\,, \tag{4.2.68}$$

and then treating K_1 to first order, one gets the following field and source reparametrisations

$$\delta\phi^A = \frac{\partial_l K_1}{\partial\phi^*_A}\,, \qquad \delta\phi^*_A = -\frac{\partial_l K_1}{\partial\phi^A}\,. \tag{4.2.69}$$

That these correspond to s_0-exact solutions, can then be seen by writing out the change in the classical action:

$$\delta S_0 = \frac{\partial_r S_0}{\partial \phi^A}\delta\phi^A + \frac{\partial_r S_0}{\partial \phi^*_A}\delta\phi^*_A = \frac{\partial_r S_0}{\partial \phi^A}\frac{\partial_l K_1}{\partial \phi^*_A} - \frac{\partial_r S_0}{\partial \phi^*_A}\frac{\partial_l K_1}{\partial \phi^A} = s_0 K_1 \,. \tag{4.2.70}$$

This interpretation extends to higher orders [139], see also [150–154]. For sec. 4.4 we will want their explicit form to second order. Given that $S_1 = s_0 K_1$, one solution to the CME to second order (4.2.41), *i.e.* $s_0 S_2 = -\frac{1}{2}(S_1, S_1)$, is:

$$S_2 = \frac{1}{2}(S_1, K_1) + s_0 K_2 \tag{4.2.71}$$

where K_2 is a second-order local functional of ghost number -1. This follows from the antibracket identity (4.2.57) because

$$s_0(S_1, K_1) = (s_0 S_1, K_1) - (S_1, s_0 K_1) = -(S_1, S_1) \,. \tag{4.2.72}$$

In fact the relation (4.2.71) is just the result of taking the K_1 canonical transformation to second order and adding the new part K_2 which appears linearly at this order. To see this we set

$$\mathcal{K} = \check{\phi}^*_A \phi^A + K_1[\phi, \check{\phi}^*] + K_2[\phi, \check{\phi}^*] \,, \tag{4.2.73}$$

and solve the exact canonical transformation (4.2.12) perturbatively for $\delta\phi^{(*)} = \check{\phi}^{(*)} - \phi^{(*)}$, starting with the first order expression (4.2.69). We get the following:

$$\begin{aligned}
\delta\phi^A &= \frac{\partial_l K_1}{\partial \phi^*_A} + \frac{1}{2}\frac{\partial_l}{\partial \phi^*_A}\frac{\partial_r K_1}{\partial \phi^B}\frac{\partial_l K_1}{\partial \phi^*_B} - \frac{1}{2}\frac{\partial_l}{\partial \phi^*_A}\frac{\partial_r K_1}{\partial \phi^*_B}\frac{\partial_l K_1}{\partial \phi^B} + \frac{\partial_l K_2}{\partial \phi^*_A} \,, \\
\delta\phi^*_A &= -\frac{\partial_l K_1}{\partial \phi^A} + \frac{1}{2}\frac{\partial_l}{\partial \phi^A}\frac{\partial_r K_1}{\partial \phi^*_B}\frac{\partial_l K_1}{\partial \phi^B} - \frac{1}{2}\frac{\partial_l}{\partial \phi^A}\frac{\partial_r K_1}{\partial \phi^B}\frac{\partial_l K_1}{\partial \phi^*_B} - \frac{\partial_l K_2}{\partial \phi^A} \,.
\end{aligned} \tag{4.2.74}$$

Taylor expanding the classical action to second order gives

$$\begin{aligned}
\delta S_0 \;=\; & \frac{\partial_r S_0}{\partial \phi^A}\delta\phi^A + \frac{1}{2}\frac{\partial_r}{\partial \phi^B}\left(\frac{\partial_r S_0}{\partial \phi^A}\delta\phi^A\right)\delta\phi^B + \frac{1}{2}\frac{\partial_r}{\partial \phi^*_B}\left(\frac{\partial_r S_0}{\partial \phi^A}\delta\phi^A\right)\delta\phi^*_B \\
& + \frac{\partial_r S_0}{\partial \phi^*_A}\delta\phi^*_A + \frac{1}{2}\frac{\partial_r}{\partial \phi^B}\left(\frac{\partial_r S_0}{\partial \phi^*_A}\delta\phi^*_A\right)\delta\phi^B + \frac{1}{2}\frac{\partial_r}{\partial \phi^*_B}\left(\frac{\partial_r S_0}{\partial \phi^*_A}\delta\phi^*_A\right)\delta\phi^*_B \\
& - \frac{1}{2}\frac{\partial_r S_0}{\partial \phi^A}\left(\frac{\partial_r}{\partial \phi^B}\delta\phi^A\right)\delta\phi^B - \frac{1}{2}\frac{\partial_r S_0}{\partial \phi^A}\left(\frac{\partial_r}{\partial \phi^*_B}\delta\phi^A\right)\delta\phi^*_B \\
& - \frac{1}{2}\frac{\partial_r S_0}{\partial \phi^*_A}\left(\frac{\partial_r}{\partial \phi^B}\delta\phi^*_A\right)\delta\phi^B - \frac{1}{2}\frac{\partial_r S_0}{\partial \phi^*_A}\left(\frac{\partial_r}{\partial \phi^*_B}\delta\phi^*_A\right)\delta\phi^*_B \,.
\end{aligned} \tag{4.2.75}$$

Substituting (4.2.74), its non-linear terms cancel the final two lines, whilst the first two lines organise into antibrackets, and thus we find that

$$\delta S_0 = (S_0, K_1 + K_2) + \frac{1}{2}((S_0, K_1), K_1) = s_0 K_1 + \frac{1}{2}(S_1, K_1) + s_0 K_2 \,, \tag{4.2.76}$$

showing that the non-linear term in (4.2.71), is indeed the result (4.2.74) of carrying the canonical transformation to second order.

4.2.9 General form of s_0-closed divergences

On the other hand, at each new loop order the s_0-closed counterterms are associated to the 'new' part Γ_∞ of the divergences. Their form can be classified by the cohomology of s_0 in the space of local functionals. As we have seen, one possibility is that it is a local s_0-exact solution: $\Gamma_\infty = s_0 K_\infty[\Phi, \Phi^*]$, where K_∞ is a functional with ghost number -1. However another possibility is that the divergence is a local functional $\Gamma_\infty[g_{\mu\nu}]$ of only the total metric,[2] $g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa H_{\mu\nu}$, and is diffeomorphism invariant. Ref. [178], see also [139], proves from the cohomological properties of s_0 that if the background metric is flat, *viz.* $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$, then in fact the general local s_0-closed solution is a linear combination of these two possibilities, *i.e.*

$$s_0\Gamma_\infty[\Phi, \Phi^*] = 0 \qquad \Longrightarrow \qquad \Gamma_\infty[\Phi, \Phi^*] = \Gamma_\infty[g_{\mu\nu}] + s_0 K_\infty[\Phi, \Phi^*]\,. \tag{4.2.77}$$

However in a non-flat background, as a statement on s_0-cohomology, this result is no longer true, since clearly one can now add to this a local functional $\Gamma_\infty[\bar{g}_{\mu\nu}]$ of only the background field (such a functional being trivially annihilated by s_0). Nevertheless it is true as a statement about s_0-closed divergences, as we show below.

Before doing so, we note that it is useful here to grade the solution (4.2.77) by antighost number. The first part, $\Gamma[g]$, has of course zero antighost number, but since K has ghost number -1, we see from table 4.1 that it splits up as $K = K^1 + K^2 + \cdots$, where the superscript denotes antighost number. Thanks to the perturbative non-renormalisability of quantum gravity, already at one loop one finds that all these infinitely many K^n functionals are non-vanishing. In minimal basis, K^1 is characterised by having one copy of H^*, K^2 by containing one copy of C^* or two copies of H^* whilst also being linear in the ghost C^μ, and so on, with the higher level K^n containing ever greater numbers of antifields and compensating powers of ghosts.

Now we show that (4.2.77) is indeed the general form of an s_0-closed divergence, even in a non-trivial background. Although this is effectively a small extension of the proof in flat background, it has not, to our knowledge, been noticed before. Following [48], first we observe that, up to a choice of gauge, the Legendre effective action can equivalently be computed by shifting

$$h_{\mu\nu} \mapsto h_{\mu\nu} - \bar{h}_{\mu\nu} \tag{4.2.78}$$

which, by (4.2.22), amounts to expanding around flat space. Indeed this shift makes no difference to the minimal classical action (4.2.4), since it depends only on the total

[2] We write the vacuum expectation value of the quantum fields in capitals, thus in minimal basis $\Phi^A = H_{\mu\nu}, C^\xi$.

metric $g_{\mu\nu}$. Differences arise only because separate $h_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ dependence enters via the canonical transformation induced by the gauge fixing fermion (4.2.14), which from (4.2.12) and (4.2.13) takes the form

$$Q\phi^A \frac{\partial \Psi}{\partial \phi^A} = Q\Psi[\phi]\,, \tag{4.2.79}$$

and enters via the quadratic b^μ term from the extension (4.2.11), which can however also be written in Q-exact form:

$$\frac{1}{2\alpha}\sqrt{\bar{g}}\bar{g}_{\mu\nu}b^\mu b^\nu = \frac{i}{2\alpha}Q\left(\sqrt{\bar{g}}\bar{g}_{\mu\nu}\bar{c}^\mu b^\nu\right) = Q\Psi_b[\phi]\,. \tag{4.2.80}$$

Thus, the entire $\bar{g}$ (equivalently $\bar{h}$) dependence can be seen as being just part of the parametrisation of our choice of gauge, *i.e.* of $\Psi_{\rm tot}[\phi] = \Psi[\phi] + \Psi_b[\phi]$.

Now in the shifted basis (4.2.78) we are expanding around flat space. If we also use an $\bar{h}$-independent gauge, then we can be sure that (4.2.77) holds. We cannot use this result directly to rule out a separate $\Gamma_\infty[\bar{g}_{\mu\nu}]$ piece, because we have changed the gauge. However we can proceed by comparing physical quantities since they are independent of the choice of gauge. We do this by setting $\Phi^*_A = 0$ and setting $H_{\mu\nu}$ on shell. Note that since we are dealing with new divergences appearing at some given loop order, it is the classical equations of motion for $g_{\mu\nu}$ that one needs. Then $\Gamma_\infty[g_{\mu\nu}]$ is independent of the background, whilst $s_0 K_\infty$ vanishes. The latter follows because

$$s_0 K_\infty = \frac{\partial_r S_0}{\partial \Phi^A}\frac{\partial_l K_\infty}{\partial \Phi^*_A} - \frac{\partial_r S_0}{\partial \Phi^*_A}\frac{\partial_l K_\infty}{\partial \Phi^A}\,. \tag{4.2.81}$$

Given that $\Phi^*_A = 0$, on the right hand side the first term vanishes (in minimal basis) by the equations of motion of $H_{\mu\nu}$, and the second term because K_∞ has non-vanishing antighost number. Now comparing the results in flat background and non-flat background, we see that they must have the same total metric part $\Gamma_\infty[g_{\mu\nu}]$, whilst for a non-flat background the purely background part must vanish: $\Gamma_\infty[\bar{g}_{\mu\nu}] = 0$.

We finish with some important remarks. Firstly, to avoid over-counting, the counterterm $S_\ell[g]$ for the pure metric part of the s_0-closed solution (4.2.77) should be restricted to terms that do not vanish on the classical equations of motion (or more generally to a specific choice, as in (4.3.15), the Gauss-Bonnet term). To see this we note that if $S_\ell[g]$ does vanish on the classical equations of motion, it can be written as

$$S_\ell[g_{\mu\nu}] = -\frac{2}{\kappa}\int_x \sqrt{g}\, G^{\mu\nu}T_{\mu\nu}[g_{\mu\nu}] = Q^- \int_x h^{*\mu\nu}T_{\mu\nu} = s_0 \int_x h^{*\mu\nu}T_{\mu\nu} \tag{4.2.82}$$

for some tensor $T_{\mu\nu}[g_{\mu\nu}]$. In the last step we used the fact that both $h^{*\mu\nu}$ and $T_{\mu\nu}$ transform properly as tensor densities under Q. Thus any part of $S_\ell[g]$ that vanishes on the classical equations of motion can be written instead as part of the s_0-exact piece, $s_0 K_\ell$, *i.e.* to a canonical transformation taken to first order.

Secondly, notice that it is important for the above arguments that we are setting $H_{\mu\nu}$ on shell, but not the background metric $\bar{g}_{\mu\nu}$. This is what allows us to deduce that there cannot be any purely background part. On the other hand in the background field method one sets all the classical fields to zero and keeps only the background metric. Although this technique is not the primary focus of the chapter (apart from in sec. 4.4), the proof here tells us something important about it. Since on shell the background field effective action gives the same results [48], we know that divergences that do not vanish on the background equations of motion descend from functionals of the total metric $g_{\mu\nu}$, whilst those divergences that vanish on the background equations of motion belong to canonical transformations and are, thus, removed by reparametrising $h_{\mu\nu}$, not the background field.

4.3 Explicit expressions for counterterms

We now verify these results in explicit loop computations, up to two loops. In particular, we draw out the intimate relationship between the leading off-shell divergences for the two-point vertex up to two loops and the one-loop counterterm diagrams [146] and, in turn, to canonical transformations in the BRST algebra [139]. Since this necessitates computing, as an intermediate step, of the off-shell divergences in one-loop diagrams with three external legs, two of which are fluctuation fields, we widened our investigation so as to compute explicitly all off-shell one-loop divergences with up to three (anti)fields. Below we express these divergences in terms of the minimal-basis counterterms in S_ℓ ($\ell = 1, 2$) that one needs to add to the bare action. In minimal subtraction, which we follow, the counterterms are just minus the divergences. However, since the bare action is a μ-independent local functional, the RG and CME relationships are most naturally expressed in terms of the counterterms, as we have seen in secs. 4.2.5 and 4.2.6.

In fact it was in the process of computing these that we noticed that purely background metric pieces were not generated, which motivated the general proof in sec. 4.2.9. It was also whilst analysing these that we noticed that the RG relations for counterterms are actually crucial for consistency of the BRST algebra as realised on the Legendre effective action versus as realised on the counterterms. This is explained in sec. 4.2.5. Finally these results allowed explicit verification that the generalised β function proposal of ref. [160] cannot be correct, which led to us formulating the detailed analysis provided in sec. 4.4. We similarly hope that these examples will prove useful in future studies of perturbative quantum gravity.

Just like for K in sec. 4.2.9, it is useful to split the Legendre effective action and bare action according to antighost number. All antighost levels S^n depend on the graviton fields $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$, but their dependence on (anti)ghosts is restricted by the quantum numbers, *cf.* table 4.1. Thus S^0 depends only on the graviton fields, whilst $\Gamma^0[H_{\mu\nu}, \bar{H}_{\mu\nu}]$

is the physical part that ultimately provides the S-matrix, S^1 is linear in $h^{*\mu\nu}$ and c^μ (in gauge fixed basis (4.2.18), S^1 renormalises the ghost action), S^2 is made of vertices containing two c^μ and either one c^*_μ or two $h^{*\alpha\beta}$, and so on.

4.3.1 One-loop two-point counterterms

4.3.1.1 Level zero, *i.e.* graviton, counterterms

Recall from sec. 4.2.1 that we are using Feynman DeDonder gauge. As explained in the next subsection, in this case it turns out that the result for the one-loop two-point graviton counterterm can be expressed entirely in terms of curvatures linearised around the flat metric. In particular let us introduce for the quantum fluctuation the linearised 'quantum curvature'

$$R_{\mu\alpha\nu\beta} = \kappa \mathbb{R}_{\mu\alpha\nu\beta} + O(\kappa^2)\,, \tag{4.3.1}$$

where we are expanding $g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$, and thus

$$\begin{aligned} R^{(1)}_{\mu\alpha\nu\beta} &= -2\partial_{[\mu|}\,\partial_{[\nu}h_{\beta]\,|\alpha]}\,,\\ R^{(1)}_{\mu\nu} &= -\partial^2_{\mu\nu}\varphi + \partial_{(\mu}\partial^\alpha h_{\nu)\alpha} - \tfrac{1}{2}\,\Box\, h_{\mu\nu}\,,\\ R^{(1)} &= \partial^2_{\alpha\beta}h_{\alpha\beta} - 2\,\Box\,\varphi \end{aligned} \tag{4.3.2}$$

(defining $\frac{1}{2}(t_{\mu\nu} \pm t_{\nu\mu})$ for symmetrisation $t_{(\mu\nu)}$, respectively antisymmetrisation $t_{[\mu\nu]}$). Here we are using $\varphi = \frac{1}{2}\delta^{\mu\nu}h_{\mu\nu}$[3] and indices are raised and lowered with the flat metric $\delta_{\mu\nu}$. Following the definition below (4.2.61), the linearised Einstein tensor is then $G^{(1)}_{\mu\nu} = -\mathbb{R}_{\mu\nu} + \frac{1}{2}\delta_{\mu\nu}\mathbb{R}$. Similarly we introduce the corresponding linearised background curvatures $\bar{R}^{(1)}_{\mu\alpha\nu\beta}$ *etc.* and linearised background Einstein tensor $\bar{G}^{(1)}_{\mu\nu}$, by replacing $h_{\mu\nu}$ with $\bar{h}_{\mu\nu}$.

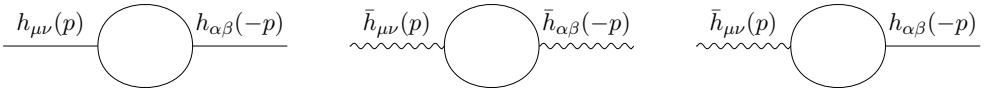

FIGURE 4.3.1: Two-point graviton diagrams at one loop. The wavy line represents the background field and the external plain line represents the quantum graviton field. The internal lines represent both a graviton loop and a ghost loop.

Computing the diagrams in fig. 4.3.1 we find

$$S^0_{1/1} = \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \left\{\frac{61}{60}\left(R^{(1)}_{\mu\nu}\right)^2 - \frac{19}{120}\left(R^{(1)}\right)^2 + \frac{7}{20}\left(\bar{R}^{(1)}_{\mu\nu}\right)^2 + \frac{1}{120}\left(\bar{R}^{(1)}\right)^2 + \frac{41}{30}R^{(1)}_{\mu\nu}\bar{R}^{(1)\mu\nu} - \frac{3}{20}R^{(1)}\bar{R}^{(1)}\right\}. \tag{4.3.3}$$

[3] This definition is the previous one (4.2.16) after linearisation.

The first diagram gives the first two terms, *i.e.* the pure quantum terms. The result agrees with ref. [163]. It was calculated in a general two parameter gauge in ref. [162]. After correcting some typos and specialising to Feynman DeDonder gauge, it also agrees. The next two terms, the purely background terms, agree with the famous result in [161] and (up to a factor of 1/2) with [19]. For more details on these comparisons, see app. 4.A. To our knowledge the last two terms, *i.e.* the mixed terms, have not appeared in the literature before.

By (4.2.39), the terms (4.3.3) must be part of an s_0-closed counterterm action $S_{1/1}$. Furthermore according to the proof given in sec. 4.2.9, since the quantum curvature pieces vanish on the equations of motion and since there cannot be a separate purely background part, we must be able to express the entire result as s_0-exact, and thus in fact the terms must collect into

$$S^0_{1/1} = Q^- K^1_{1/1} \,. \tag{4.3.4}$$

Given that (4.3.3) is made solely of linearised curvatures, at the two-point level the only possible terms in $K^1_{1/1}$ that can contribute, are:

$$K^1_{1/1} \ \ni \ \frac{\kappa^2 \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \left\{ \beta h^{*\mu\nu} R^{(1)}_{\mu\nu} + \gamma \varphi^* R^{(1)} + \bar{\beta} h^{*\mu\nu} \bar{R}^{(1)}_{\mu\nu} + \bar{\gamma} \varphi^* \bar{R}^{(1)} \right\} , \tag{4.3.5}$$

where β, γ, $\bar{\beta}$ and $\bar{\gamma}$ are parameters to be determined, and we have introduced

$$\varphi^* = \tfrac{1}{2} \bar{g}_{\mu\nu} h^{*\mu\nu} \tag{4.3.6}$$

by analogy with (4.2.16) (although here $\bar{g}_{\mu\nu}$ can be replaced by $\delta_{\mu\nu}$). It is apparent that we have six numbers in (4.3.3) to reproduce with only four parameters, and therefore this relation is a non-trivial check on the formalism. From (4.2.64), the action of Q^- reduces in this case to

$$Q^- h^{*\mu\nu} = -2 \left(G^{(1)\mu\nu} + \bar{G}^{(1)\mu\nu} \right) , \tag{4.3.7}$$

and thus from (4.3.4) and (4.3.5),

$$\begin{aligned} S^0_{1/1} = \frac{\kappa^2 \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \Big\{ & 2\beta \left(R^{(1)}_{\mu\nu} \right)^2 - [\beta + \gamma] \left(R^{(1)} \right)^2 + 2\bar{\beta} \left(\bar{R}^{(1)}_{\mu\nu} \right)^2 - [\bar{\beta} + \bar{\gamma}] \left(\bar{R}^{(1)} \right)^2 \\ & + 2[\beta + \bar{\beta}] R^{(1)}_{\mu\nu} \bar{R}^{(1)\mu\nu} - [\beta + \bar{\beta} + \gamma + \bar{\gamma}] R^{(1)} \bar{R}^{(1)} \Big\} . \end{aligned} \tag{4.3.8}$$

We see that the mixed Ricci-squared terms must have a coefficient which is simply the sum of the coefficients of the pure quantum and pure background Ricci-squared terms, and likewise for the scalar-curvature-squared terms. One can easily verify from (4.3.3) that these two constraints are indeed satisfied. Thereforem there are four independent

constraints and we can find a consistent (and unique) solution. It is:

$$\beta = \frac{61}{120}\,, \quad \gamma = -\frac{7}{20}\,, \quad \bar{\beta} = \frac{7}{40}\,, \quad \bar{\gamma} = -\frac{11}{60}\,. \tag{4.3.9}$$

4.3.1.2 Level one (a.k.a. ghost) counterterms

The level one two-point counterterm is computed by using the classical three-point vertices involving $h^{*\mu\nu}$, and transferring to gauge fixed basis using (4.2.18). We display the result in minimal basis where it takes its simplest form, since it then contains only the divergent corrections to $Qh_{\mu\nu}$ (at the linearised level, compare (4.3.10) to (4.2.4) and (4.2.17)), but in gauge fixed basis the generated $\bar{c}^\alpha$ terms are the counterterms necessary to renormalise the ghost action (4.2.20). We find that:

$$S^1_{1/1} = \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \left\{\frac{1}{2}h^{*\mu\nu}\partial^3_{\mu\nu\alpha}c^\alpha - \frac{3}{4}h^{*\mu\nu}\Box\partial_\mu c_\nu\right\}\,, \tag{4.3.10}$$

in agreement with ref. [163], *cf.* app. 4.A. Again these must belong to $s_0K_{1/1}$ for a suitable choice of $K_{1/1}$, which means that we must add to what we have in (4.3.5). A solution is to add

$$K^1_{1/1} \ni -\frac{1}{2}\frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x h^{*\mu\nu}\partial^2_{\mu\nu}\varphi\,, \qquad K^2_{1/1} \ni -\frac{3}{8}\frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x c^*_\mu\Box c^\mu\,. \tag{4.3.11}$$

At the two-point level, it is straightforward to see that the level-two part gives the second term in (4.3.10) via $Q^-K^2_{1/1}$, whilst the level-one part gives the first term via $QK^1_{1/1}$, (4.3.5) making no contribution because it is annihilated by Q. On the other hand, (4.3.5) is still correct for reproducing $S^0_{1/1}$ because the level-one part above is annihilated by Q^-, as follows by the Bianchi identity for the Einstein tensor or by recognising that the above level-one part is proportional to $Q^-(\partial^\alpha c^*_\alpha\varphi)$. Indeed at this stage one has to face the issue that the solution for K is unique only in the cohomology. One can always add an s_0-exact piece to K, in particular one can add $s_0(\partial^\alpha c^*_\alpha\varphi)$. The above solution is one choice, in fact the same as that made in ref. [163].

Now let us comment on the results of the previous subsection. The fact that they can be written covariantly, in terms of curvatures of the background metric, is of course no accident: this is guaranteed by background diffeomorphism invariance. The fact that one can also do so in terms of $g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$, is however an accident of Feynman DeDonder gauge. At the level of the action it is a consequence of the fact that $Q^-S^1_{1/1} = 0$ in this gauge, and thus the graviton counterterm action must be annihilated by Q:

$$0 = s_0S_{1/1} = QS^0_{1/1} + Q^-S^1_{1/1} = QS^0_{1/1}\,. \tag{4.3.12}$$

Up to cohomology and normalisation, there is a unique term $\varphi^*\Box\varphi \in K_{1/1}$ that could arise in the one-loop calculation which would break this 'quantum diffeomorphism' invariance. Equivalently in S_1^0 we would find a term proportional to

$$Q^- \int_x \varphi^*\Box\varphi = -\int_x (\mathbb{R} + \bar{R}^{(1)})\Box\varphi\,. \tag{4.3.13}$$

Indeed from [162], *cf.* app. 4.A, we know this term is present in a more general gauge. Furthermore we will see in sec. 4.3.3 that at two loops an analogous term is generated even in Feynman DeDonder gauge, while at one loop but beyond the two-point level many terms ensure that $QS^0_{1/1} \neq 0$.

This completes the calculation at the two-point level because it is not possible to generate two-point higher level counterterms $S^{n>1}_\ell$ (since n is also the pure ghost number).

4.3.2 One-loop three-point counterterms

This involves computing one-loop diagrams with the topologies given in fig. 4.3.2. Already at this stage there are thousands of divergent vertices, and computer algebra becomes essential. We proceed by comparing the results with the general structure (4.2.77), *i.e.* we should find that the counterterm action takes the form:

$$S_{1/1}[\Phi,\Phi^*] = S^0_{1/1}[g_{\mu\nu}] + s_0 K_{1/1}[\Phi,\Phi^*]\,. \tag{4.3.14}$$

As explained in ref. [18], dimensional regularisation allows for the computation of the Gauss-Bonnet topological term:

$$\begin{aligned} S^0_{1/1}[g_{\mu\nu}] &= \frac{\tau\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \sqrt{g}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + R^2 - 4R^{\mu\nu}R_{\mu\nu}\right) \\ &= \frac{\tau\mu^{-2\varepsilon}}{(8\pi)^2\varepsilon}\int_x \sqrt{g}\,\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\rho\sigma}{}_{\gamma\delta}\,, \end{aligned} \tag{4.3.15}$$

which is the unique possibility for $S^0_{1/1}[g_{\mu\nu}]$ up to choice of coefficient τ and terms that vanish on shell (*cf.* the discussion at the end of sec. 4.2.9).

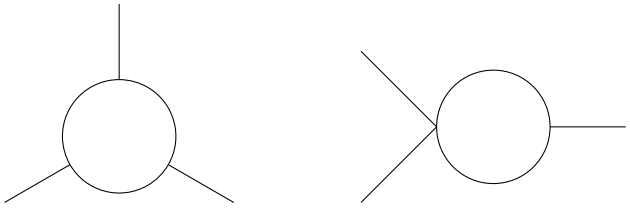

FIGURE 4.3.2: Topologies of three-point Feynman diagrams at one loop.

Up to the three-point level, $K_{1/1}$ has no more than antighost number two. The two antighost levels have the following general parametrisation:

$$\begin{aligned}
K^1_{1/1} = {}& \frac{\kappa\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \left(\bar\beta h^{*\mu\nu}\bar R_{\mu\nu} + \bar\gamma\varphi^*\bar R\right) \\
&+ \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \Big\{\beta h^{*\mu\nu}\left(\bar\nabla_\mu\bar\nabla^\alpha h_{\alpha\nu} - \tfrac{1}{2}\bar\Box h_{\mu\nu}\right) \\
&\qquad + (c_1-\beta)\, h^{*\mu\nu}\bar\nabla_\mu\bar\nabla_\nu\varphi + \gamma\varphi^*\left(\bar\nabla_\alpha\bar\nabla_\beta h^{\alpha\beta} - 2\bar\Box\varphi\right) \\
&\qquad + \alpha_3 h^{*\mu\nu}\bar R_\mu{}^\alpha h_{\alpha\nu} + \alpha_4 h^{*\mu\nu}\bar R_{\alpha\mu\nu\beta}h^{\alpha\beta} + \alpha_5 h^{*\mu\nu}\bar R_{\mu\nu}\varphi \\
&\qquad + \alpha_6\varphi^*\bar R^{\alpha\beta}h_{\alpha\beta} + \alpha_7\bar R\varphi^*\varphi\Big\} \\
&+ \frac{\kappa^3\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \sum_{i=1}^{27} b_i\left(h^*h^2\partial^2\right)_i\,,
\end{aligned} \tag{4.3.16}$$

$$\begin{aligned}
K^2_{1/1} = {}& \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \left(c_2 c^*_\mu\bar\Box c^\mu + \alpha_1 c^*_\mu c^\mu\bar R + \alpha_2 c^*_\mu c^\nu\bar R^\mu{}_\nu\right) \\
&+ \frac{\kappa^3\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \frac{1}{\sqrt{\bar g}}\Big(\alpha_8\varphi^*\bar\nabla_\mu\varphi^* c^\mu + \alpha_9 h^{*\alpha\beta}\bar\nabla_\mu h^*_{\alpha\beta}c^\mu + \alpha_{10}\varphi^* h^*_{\mu\nu}\bar\nabla^\mu c^\nu \\
&\qquad + \alpha_{11} h^*_{\alpha\mu}h^{*\alpha}{}_\nu\bar\nabla^\mu c^\nu\Big) \\
&+ \frac{\kappa^3\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \sum_{i=1}^{21} d_i\left(c^*ch\partial^2\right)_i\,.
\end{aligned} \tag{4.3.17}$$

Here we have used the symmetries and statistics of the (anti)fields. In particular, the result must be background diffeomorphism invariant (which implies the factor of $1/\sqrt{\bar g}$ in the terms with two antifields, because we defined them to transform as tensor densities of weight -1). Furthermore, we know that the terms with one antifield have two spacetime derivatives whilst those with two antifields have one spacetime derivative. The power of κ and μ then follow from $[K] = -1$.

The parametrisation must be consistent with the results at the two-point level, hence the appearance of parameters $\bar\beta$, $\bar\gamma$, β and γ from (4.3.5). We similarly introduce parameters c_1 and c_2 where, from (4.3.11), we know that

$$c_1 = -\frac{1}{2}\,, \qquad \text{and} \qquad c_2 = -\frac{3}{8}\,. \tag{4.3.18}$$

Background diffeomorphism invariance tells us that the linearised curvatures accompanying $\bar\beta$ and $\bar\gamma$ simply become full curvatures (by (4.3.1) they absorb one power of κ) but, as discussed in sec. 4.3.1.2, the appearance of the linearised quantum curvatures in (4.3.5) is accidental, so it is more appropriate for the β and γ pieces to appear with their separate parts covariantised, following (4.3.2). Even though all these parameters are known, and that includes τ [18, 165], we leave them general when we match to the three-point one-loop results, as extra checks on the formalism.

The remaining eleven α_i, twenty-seven b_i, and twenty-one d_i, are genuinely free parameters to be determined. The schematic representation for the d_i terms means that one sums over the vertices with coefficients d_i, these vertices being the twenty-one linearly independent combinations of two spacetime derivatives and one c^*_α, c^β, and $h_{\gamma\delta}$. We ensure independence under integration by parts by taking as representatives those vertices where c^*_α is undifferentiated. Since the d_i terms are already three-point vertices, as are the b_i terms, background covariantisation is ignored there. For the same reason, we actually do not need diffeomorphism invariant expressions for the $\alpha_8, \cdots, \alpha_{11}$ terms, whilst in the other α_i terms we actually only need the linearised background curvature.

The sum over b_i vertices is defined in the same way as for the d_i vertices, except that all terms involving $\partial_\alpha h^{*\alpha\beta}$ are discarded, and likewise any two vertices should be considered equal if they only differ by such terms on using integration by parts. (This can be implemented straightforwardly by deriving the vertices in momentum space.) The reason for this restriction is because at the three-point level, vertices containing $\partial_\alpha h^{*\alpha\beta}$ are already accounted for in the d_i sum. As in the discussion in sec. 4.3.1.2, this is a consequence of the fact that we can add an s_0-exact part to $K_{1/1}$ without altering $S_{1/1}$, *cf.* (4.3.14). At the three-point level we can add $(Q + Q^-)(c^* h^2 \partial)$, but Q^- generates the $\partial_\alpha h^{*\alpha\beta}$ terms while Q maps onto combinations in the d_i sum that contain $\partial_{(\alpha} c_{\beta)}$. Finally, for the same reason we do not want a free parameter for the combination

$$-\frac{\kappa}{\sqrt{\bar{g}}} s_0 \left(\varphi^* h^{*\mu\nu} h_{\mu\nu}\right) = \bar{R} h^{*\mu\nu} h_{\mu\nu} + 2\varphi^* \bar{R}^{\alpha\beta} h_{\alpha\beta} - 2\bar{R}\varphi^*\varphi - 2\frac{\kappa}{\sqrt{\bar{g}}} \varphi^* h^{*\mu\nu} \bar{\nabla}_\mu c_\nu \,. \quad (4.3.19)$$

The last three terms on the right hand side appear in our parametrisation, but this is why the first term is missing from it.

Although the resulting parametrisation is long, it is a dramatic reduction compared to the thousands of vertices from the Feynman diagram calculation, and therefore in fact the parameters are vastly overdetermined. That we nevertheless find a consistent solution for all vertices is thus a highly non-trivial verification of the formalism.

Matching to just the (antighost level zero) pure background $\bar{h}^3$ vertices, we reproduce well-known results: we confirm that the pure background curvature-squared terms at the two-point level, *cf.* (4.3.3), are covariantised to full background curvatures, as is in fact clear here from our $K^1_{1/1}$ (4.3.16), and confirm that the remaining part is the Gauss-Bonnet term given in (4.3.15). In this way we reaffirm the $\bar{\beta}$ and $\bar{\gamma}$ values from (4.3.9) and also find

$$\tau = \frac{53}{90} \,, \quad (4.3.20)$$

in agreement with previous calculations [18, 165].

One can determine all the coefficients in $K^1_{1/1}$ by matching to antighost level zero vertices, up to several vertices parametrised by c_1. In fact just using the $h^2\bar{h}$ and h^3 vertices is sufficient to determine all that can be found at this level, but we matched also to $\bar{h}^2 h$

vertices to verify the result and further confirm consistency. The $K^2_{1/1}$ parameters cannot of course be determined by matching to antighost level zero vertices, because the lowest antighost level it generates is level one, via $Q^- K^2_{1/1}$, while c_1 and some vertices in the b_i sum also remain undetermined because in $K^1_{1/1}$ at the three-point level they can be collected into $\frac{1}{2}c_1 Q^-(c^{*\nu}\bar{\nabla}_\nu\varphi)$.

Now all the parameters in $K^2_{1/1}$, and c_1, can be (over)determined by matching to the full set of level-one three-point Feynman diagrams with topology of fig. 4.3.2, *i.e.* such that one external leg is a ghost c^μ, one external leg is $h^{*\alpha\beta}$ and the remaining leg is h or $\bar{h}$. In this way we recover the previously stated values for c_1, c_2, $\bar{\beta}$, $\bar{\gamma}$, β and γ, and determine that

$$\begin{aligned}
&\alpha_1 = -\frac{1}{8}\,, \quad \alpha_2 = -\frac{1}{24}\,, \quad \alpha_3 = \frac{161}{120}\,, \quad \alpha_4 = \frac{1}{120}\,, \quad \alpha_5 = -\frac{3}{4}\,, \quad \alpha_6 = -\frac{7}{15}\,,\\
&\alpha_7 = \frac{19}{60}\,, \quad \alpha_8 = -\frac{1}{6}\,, \quad \alpha_9 = -\frac{1}{12}\,, \quad \alpha_{10} = -\frac{4}{15}\,, \quad \alpha_{11} = -\frac{1}{6}\,,
\end{aligned} \tag{4.3.21}$$

and also the b_i and d_i parameters as given below:

$$\begin{aligned}
\sum_{i=1}^{27} b_i \left(h^* h^2 \partial^2\right)_i &= \frac{5}{12}h^{*\mu\nu}\varphi\partial^2_{\mu\nu}\varphi - \frac{13}{160}h^{*\mu\nu}\partial^2_{\mu\nu}h^\beta{}_\alpha h^\alpha{}_\beta + \frac{1}{4}h^{*\mu\alpha}\left(\partial^\nu h_{\alpha\nu}\partial_\mu\varphi - \partial_\mu\partial^\nu h_{\alpha\nu}\varphi\right)\\
&+ \frac{61}{240}h^{*\mu\alpha}\left(\partial_\mu h_{\alpha\nu}\partial^\nu\varphi - h_{\alpha\nu}\partial^\nu\partial_\mu\varphi\right) + \frac{7}{80}h^{*\mu\alpha}\left(\partial_\mu h_{\beta\nu}\partial^\nu h^\beta{}_\alpha - h_{\beta\nu}\partial^\nu\partial_\mu h^\beta{}_\alpha\right)\\
&- \frac{61}{240}h^{*\mu\alpha}\left(\partial_\nu h_\beta{}^\nu\partial_\mu h^\beta{}_\alpha - \partial^2_{\mu\nu}h_\beta{}^\nu h^\beta{}_\alpha\right) + \frac{13}{60}\varphi^*\partial^\alpha h_{\beta\nu}\partial^\nu h^\beta{}_\alpha + \frac{43}{60}\varphi^* h_{\beta\nu}\partial^\nu\partial^\alpha h^\beta{}_\alpha\\
&+ \frac{77}{120}\varphi^*\partial^\nu h_{\beta\nu}\partial^\alpha h^\beta{}_\alpha - \frac{53}{60}\varphi^* h_{\alpha\nu}\partial^\nu\partial^\alpha\varphi - \frac{17}{10}\varphi^*\partial^\nu h_{\alpha\nu}\partial^\alpha\varphi - \frac{3}{10}\varphi^*\varphi\Box\varphi\\
&- \frac{11}{60}\varphi^*\varphi\partial^2_{\alpha\nu}h^{\alpha\nu} + \frac{9}{40}\varphi^* h^\alpha{}_\beta\Box h^\beta{}_\alpha + \frac{14}{15}\varphi^*\partial_\nu\varphi\partial^\nu\varphi - \frac{11}{80}\varphi^*\partial_\nu h^\alpha{}_\beta\partial^\nu h^\beta{}_\alpha\\
&- \frac{131}{240}h^{*\mu\nu}\partial^\alpha h_{\mu\nu}\partial_\alpha\varphi - \frac{1}{4}h^{*\mu\nu}h^\alpha{}_\mu\partial^2_{\alpha\beta}h^\beta{}_\nu - \frac{1}{12}h^{*\mu\nu}\partial_\alpha h^\alpha{}_\mu\partial_\beta h^\beta{}_\nu - \frac{27}{80}h^{*\mu\nu}\partial_\beta h^\alpha{}_\mu\partial_\alpha h^\beta{}_\nu\\
&+ \frac{17}{80}h^{*\mu\nu}\partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta{}_\nu + \frac{7}{80}h^{*\mu\nu}\partial^2_{\alpha\beta}h_{\mu\nu}h^{\alpha\beta} - \frac{1}{2}h^{*\mu\nu}\Box h_{\mu\beta}h^\beta{}_\nu + \frac{37}{80}h^{*\mu\nu}\partial^\alpha h_{\mu\nu}\partial^\beta h_{\alpha\beta}\\
&- \frac{1}{3}h^{*\mu\nu}h_{\mu\nu}\Box\varphi + \frac{1}{3}h^{*\mu\nu}h_{\mu\nu}\partial^2_{\alpha\beta}h^{\alpha\beta} + \frac{11}{24}h^{*\mu\nu}\Box h_{\mu\nu}\varphi\,.
\end{aligned} \tag{4.3.22}$$

$$\begin{aligned}
\sum_{i=1}^{21} d_i \left(c^* c h \partial^2\right)_i &= \frac{1}{12}c^{*\mu}\partial^2_{\mu\nu}c^\nu\varphi - \frac{121}{480}c^*_\mu\partial^\mu c^\nu\partial_\nu\varphi + \frac{61}{480}c^*_\mu\partial^\mu c^\nu\partial_\alpha h^\alpha{}_\nu - \frac{11}{24}c^{*\mu}\partial^2_{\mu\alpha}c^\nu h^\alpha{}_\nu\\
&- \frac{1}{3}c^{*\mu}c^\nu\partial^2_{\mu\nu}\varphi + \frac{1}{6}c^{*\mu}c^\nu\partial^2_{\alpha\mu}h^\alpha{}_\nu - \frac{1}{24}c^*_\mu\partial_\nu c^\nu\partial^\mu\varphi - \frac{101}{480}c^*_\mu\partial_\alpha c^\nu\partial^\mu h^\alpha{}_\nu - \frac{1}{8}c^*_\alpha\partial^2_{\mu\nu}c^\nu h^{\alpha\mu}\\
&- \frac{119}{480}c^*_\alpha\partial^\mu c^\nu\partial_\nu h^\alpha{}_\mu + \frac{1}{12}c^*_\alpha c^\nu\partial^2_{\mu\nu}h^{\alpha\mu} + \frac{1}{8}c^*_\alpha\partial_\nu c^\nu\partial^\mu h^\alpha{}_\mu - \frac{301}{480}c^*_\alpha\partial^\nu c^\alpha\partial_\nu\varphi + \frac{1}{4}c^*_\alpha c^\alpha\Box\varphi\\
&+ \frac{1}{3}c^*_\alpha\Box c^\alpha\varphi - \frac{1}{12}c^*_\alpha c^\alpha\partial^2_{\mu\nu}h^{\mu\nu} + \frac{27}{160}c^*_\alpha\Box c^\mu h^\alpha{}_\mu - \frac{239}{480}c^*_\alpha\partial_\nu c^\mu\partial^\nu h^\alpha{}_\mu - \frac{1}{4}c^*_\alpha c^\mu\Box h^\alpha{}_\mu\\
&+ \frac{7}{160}c^*_\alpha\partial^2_{\mu\nu}c^\alpha h^{\mu\nu} + \frac{241}{480}c^*_\alpha\partial^\nu c^\alpha\partial^\mu h_{\mu\nu}\,.
\end{aligned} \tag{4.3.23}$$

Since the above provides us with the full expression for $K_{1/1}$ up to the three-point

level, we get as a bonus the full expression up to three-point level for the antighost level-two counterterm, without having to compute it from Feynman diagrams, since it is given by $S^2_{1/1} = QK^2_{1/1}$. This completes the explicit calculation of all off-shell one-loop divergences with up to three (anti)fields.

4.3.3 Two-loop double-pole two-point graviton counterterms

Now as advertised we use the one-loop counterterm diagrams, illustrated in fig. 4.3.3, to compute the two-loop $1/\varepsilon^2$ counterterm via the RG relation (4.2.44). We limit ourselves to the two-point diagrams at antighost level zero, *i.e.* with either a quantum or background graviton external leg. This is already enough for a non-trivial explicit test of the second order canonical expansion relation (4.2.71).

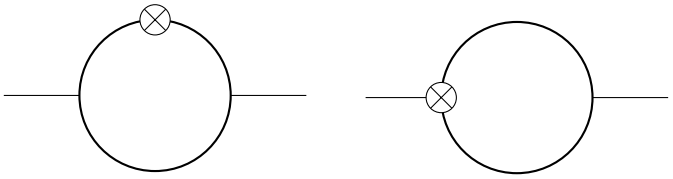

FIGURE 4.3.3: RG relates the $1/\varepsilon^2$ pole in the two-loop two-point counterterm vertices to one-loop counterterm vertices, represented by the crossed circles, via one-loop counterterm diagrams with the above topologies.

For the first diagram in fig. 4.3.3, we need the one-loop two-point counterterm vertices with purely quantum legs. They are given by (4.3.10) and the first two terms in (4.3.3) for ghosts and graviton respectively. (In the former case we need to shift to gauge fixed basis using relation (4.2.18) at the linearised level.) For the second diagram we need the one-loop three-point counterterm vertices with two quantum legs and either an external $h_{\alpha\beta}$ or $\bar{h}_{\alpha\beta}$. These can be ported directly from intermediate results created as a side-product of the computation reported in the previous subsection. Alternatively, they can be generated by evaluating $s_0 K_{1/1}$ using the explicit expressions given there. (As expected the topological counterterm (4.3.15) can be disregarded since it makes no contribution to the Feynman integrals.)

The result we find is that for two-point vertices:

$$\begin{aligned} S^0_{2/2} = -\frac{1}{2}\frac{\kappa^4\mu^{4\varepsilon}}{(4\pi)^4\varepsilon^2}\int_x \Big\{ &\frac{11}{36}\bar{R}^{(1)}\Box\bar{R}^{(1)} + \frac{5}{72}\bar{R}^{(1)\mu\nu}\Box\bar{R}^{(1)}_{\mu\nu} - \frac{469}{3600}R^{(1)}\Box R^{(1)} \\ &+ \frac{79}{200}R^{(1)\mu\nu}\Box R^{(1)}_{\mu\nu} + \frac{781}{3600}\bar{R}^{(1)}\Box R^{(1)} + \frac{53}{150}\bar{R}^{(1)\mu\nu}\Box R^{(1)}_{\mu\nu} \\ &- \frac{31}{720}\left(\bar{R}^{(1)} + R^{(1)}\right)\Box^2\varphi \Big\}, \end{aligned} \tag{4.3.24}$$

where the overall factor of $-\frac{1}{2}$ is the conversion (4.2.44) from the double-pole in fig. 4.3.3 to the two-loop counterterm $S_{2/2}$. As we will see this result passes a highly non-trivial consistency check in that it satisfies the second order canonical transformation relation (4.2.71). As far as we know the above result has not appeared in the literature before, except for the one term: $\bar{R}^{\mu\nu}\bar{\Box}\bar{R}_{\mu\nu}$ [19]. However this was quoted there as part of some partial results that unfortunately contain an error [160]. Nevertheless comparing the coefficients for this one term, we find that they agree up to a factor of half, see app. 4.A.

Recall that the one-loop level-zero two-point result (4.3.3) can be written entirely in terms of linearised curvatures (4.3.2) and is thus invariant under (linearised) diffeomorphisms, in particular also for the fluctuation field $h_{\mu\nu}$. This latter invariance is a consequence of invariance under the linearised BRST charge $Qh_{\mu\nu} = \partial_{(\mu}c_{\nu)}$. Recall also from sec. 4.3.1.2 that this property is actually an accident of Feynman DeDonder gauge. The presence of the $\Box^2\varphi$ term above shows that at two loops, one's luck runs out and this property is violated. As is evident from the form of this last term, it just corresponds to inserting another $\Box$ into the unique one-loop Q-invariance-breaking possibility (4.3.13).

In the remainder of this subsection we will show that the double-pole (4.3.24) corresponds to a canonical transformation taken to second order, *i.e.* can be expressed as in (4.2.71):

$$S_2 = \frac{1}{2}(S_1, K_1) + s_0 K_2 \,. \tag{4.3.25}$$

Actually recall that this expression follows from the non-linear CME relation $s_0 S_2 = -\frac{1}{2}(S_1, S_1)$, *viz.* (4.2.41), on assuming that S_1 is given only by the exact piece $s_0 K_1$, whereas the one-loop solution (4.3.14) also contains the Gauss-Bonnet term (4.3.15). However since the latter is topological it makes no contribution to the antibracket and thus (4.3.25) is indeed the correct solution.

From sec. 4.3.2, it is clear that (S_1, S_1) cannot vanish at the three-point level, and thus the non-linear CME relation itself is highly non-trivial. However for the two-point vertices (S_1, S_1) in fact does vanish. This is straightforward to see by inspection since for the two-point vertices we only have the pure curvature antighost level zero part, $S^0_{1/1}$, as given in (4.3.3), and the antighost level one part, $S^1_{1/1}$, as given in (4.3.10). But substituting these into $(S_{1/1}, S_{1/1})$ the net effect is to replace $h_{\mu\nu}$ in a 'quantum curvature' by either $\partial^3_{\mu\nu\alpha}c^\alpha$ or $\Box\partial_\mu c_\nu$ (up to some coefficient of proportionality), causing the result to vanish since both of these are pure gauge.

Thus the non-linear CME relation (4.2.41) only implies that the two-point vertex in S_2 is s_0-closed. The problem is that the two-point level is to a certain extent degenerate. A related point is that if we take the action only to have an antighost level zero piece, and take this to be any product of linearised curvatures, that is any one of the terms in $S^0_{1/1}$ of (4.3.3), then this action is s_0-closed at the two-point level since the linearised quantum curvatures are invariant under linearised diffeomorphisms. Nevertheless as we

will see, this test is still non-trivial because although at the level of two-point vertices $\frac{1}{2}(S_1, K_1)$ in the general solution (4.3.25) is s_0-closed, it is not s_0-exact.

Specialising (4.3.25) to antighost level zero and divergences we have

$$S^0_{2/2} = \frac{1}{2}(S^0_{1/1}, K^1_{1/1}) + Q^- K^1_{2/2}\,. \tag{4.3.26}$$

Substituting $\int_x h^{*\mu\nu}\partial^2_{\mu\nu}\varphi$ for $K^1_{1/1}$ into the antibracket, we see that it vanishes for the same reasons as above. Therefore the (4.3.11) part of $K_{1/1}$ makes no contribution. Since the remaining part of $K_{1/1}$, *viz.* (4.3.5), is made of linearised curvatures, we see that the antibracket contributes terms with linearised curvatures only. Explicitly, we find

$$\begin{aligned}\frac{1}{2}(S^0_{1/1}, K^1_{1/1}) = \frac{\kappa^4\mu^{4\varepsilon}}{(4\pi)^4\varepsilon^2}\int_x \Big\{ &-\frac{1}{2}\bar\beta(\beta+\bar\beta)\bar R^{(1)}_{\mu\nu}\Box\bar R^{(1)\mu\nu} - \beta^2\mathbb{R}_{\mu\nu}\Box R^{(1)\mu\nu}\\ &+\frac{1}{2}(3\gamma^2+2\beta\gamma+\beta^2)\mathbb{R}\Box\mathbb{R} + \frac{1}{4}(3\bar\gamma^2+3\bar\gamma\gamma+2\bar\beta\bar\gamma+\bar\beta\gamma+\bar\beta^2+\beta\bar\gamma+\beta\bar\beta)\bar R^{(1)}\Box\bar R^{(1)}\\ &-\frac{1}{2}\beta(\beta+3\bar\beta)\mathbb{R}_{\mu\nu}\Box\bar R^{(1)\mu\nu} + \frac{1}{4}(9\gamma\bar\gamma+3\gamma^2+3\bar\beta\gamma+3\beta\bar\gamma+2\beta\gamma+3\beta\bar\beta+\beta^2)\mathbb{R}\Box\bar R^{(1)}\Big\}\,,\end{aligned} \tag{4.3.27}$$

where recall that the parameters were determined as in (4.3.9). Now this cannot come from an s_0-exact expression because if it did, we could write it as Q^-K^1 for some K^1. We can check if this is so by using the same rule discussed in sec. 4.3.1.1, *i.e.* from (4.3.7) we know that this would imply that the coefficient of the mixed terms above must be equal to the sum of the coefficients of the equivalent pure quantum and pure background pieces. It is easy to see that this does not work. Similarly one can verify that the curvature terms in (4.3.24) do not sum to something that is Q^--exact.

But according to (4.3.26), on subtracting (4.3.27) from (4.3.24) we should be left with a Q^--exact piece. We have already seen that this is true of the non-covariant term, the last term, in (4.3.24). The remaining parts are pure curvature terms and must thus have the parametrisation (4.3.5) except with an extra $\Box$ inserted (and different coefficients), up to some Q^--exact remainder, $Q^-R \in K^1_{2/2}$ (which does not contribute to (4.3.26) because Q^- is nilpotent). Matching to the above results, we find that this is indeed the case and thus we derive $K^1_{2/2}$ at the two-point level in the form

$$\begin{aligned}K^1_{2/2} \;=\; \frac{\kappa^4\mu^{4\varepsilon}}{(4\pi)^2\varepsilon^2}\int_x \Big\{ &\frac{877}{28800}h^{*\mu\nu}\Box R^{(1)}_{\mu\nu} + \frac{71}{1800}\varphi^*\Box R^{(1)}\\ &+\frac{361}{28800}h^{*\mu\nu}\Box\bar R^{(1)}_{\mu\nu} + \frac{2719}{14400}\varphi^*\Box\bar R^{(1)} - \frac{31}{1440}\varphi^*\Box^2\varphi\Big\} + Q^-R\,.\end{aligned} \tag{4.3.28}$$

Like in (4.3.5), the remainder term Q^-R has $\partial_\alpha h^{*\alpha\beta}$ as a factor. It could also be derived by matching to the two-loop double-pole level-one counterterm diagrams, and they can be computed using the results we have already obtained. However the above form for $K_{2/2}$ is sufficient for our purposes.

4.4 Generalised beta functions and why they are not finite

In this final section we comment on some ideas for generalised β-functions, where the field is taken to play the rôle of a collection of couplings. The key idea is to exploit relations that follow from assuming that these β-functions are finite. Unfortunately this assumption is incorrect. We explain why natural generalisations that respect the BRST symmetry also fail to work.

Inspired by ref. [179] and its many follow-ups *e.g.* [180,181], which themselves are inspired by refs. [182–184], the main proposal of ref. [160] consists of two key steps. The first key step is to allow for a non-linear renormalisation of the metric, replacing $g_{\mu\nu}$ in the Einstein-Hilbert term of the classical action (4.2.4) with a bare metric $g^0_{\mu\nu}$ which is then expanded as

$$g^0_{\mu\nu}(x) = g_{\mu\nu}(x) + \sum_{k=1} \frac{1}{\varepsilon^k}\, \mathfrak{g}^k_{\mu\nu}(x)\,. \tag{4.4.1}$$

The $\mathfrak{g}^k_{\mu\nu}$ are assumed to be local diffeomorphism covariant combinations constructed from covariant derivatives and curvatures using the renormalised metric $g_{\mu\nu}$. With this assumption, the proposal only applies to non-linear renormalisation of the background metric.

In ref. [160] the μ dependence in (4.4.1) is simplified to an overall multiplicative $\mu^{-2\varepsilon}$ on the right hand side, by taking the mass dimensions to be $[g^0_{\mu\nu}] = -2\varepsilon$, while $[g_{\mu\nu}] = 0$ and $[\kappa] = -1$ (also in d dimensions). However the same physics can be arrived at by including μ in the more conventional way, as we do. Thus, our metrics are taken to be dimensionless, while $[\kappa] = -1 + \varepsilon$. Then by dimensions, the $\mathfrak{g}^k_{\mu\nu}$ are forced to have explicit dependence on μ, *cf.* sec. 4.2.6 and sec. 4.3. In fact the ℓ-loop contribution is constructed from 2ℓ covariant derivatives, rendered dimensionless by the factor $(\kappa\mu^{-\varepsilon})^{2\ell}$.

A renormalisation of form (4.4.1) can provide all the covariant counterterms in the bare action that vanish on the equations of motion. For example, the purely background metric counterterms (in Feynman – De Donder gauge) are [161], *cf.* (4.3.3) and below (4.3.18),

$$S_1 = \frac{\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon} \int_x \sqrt{\bar{g}} \left(\frac{1}{120}\bar{R}^2 + \frac{7}{20}\bar{R}^2_{\mu\nu} \right) . \tag{4.4.2}$$

These counterterms can be generated by defining

$$\bar{g}^0_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\, \bar{\mathfrak{g}}^1_{\mu\nu}\,, \qquad \text{where} \qquad \bar{\mathfrak{g}}^1_{\mu\nu} = \frac{7}{40}\bar{R}_{\mu\nu} + \frac{11}{120}\bar{g}_{\mu\nu}\bar{R} \tag{4.4.3}$$

(where, from here on, we make explicit the $\kappa\mu^{-\varepsilon}/(4\pi)$ dependence in $\bar{\mathfrak{g}}^k_{\mu\nu}$).

Now by insisting that the bare metric is independent of μ, and differentiating both sides with respect to μ, one obtains a kind of generalised "beta function", $\beta_{\alpha\beta} = \mu\partial_\mu\, g_{\alpha\beta}$ for the renormalised metric (non-linear wavefunction renormalisation might be a better

term). For the above example, from (4.4.3), we have for the background metric to one loop,

$$\bar{\beta}_{\mu\nu} = 2\,\frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2}\,\bar{\mathfrak{g}}^1_{\mu\nu}\,. \tag{4.4.4}$$

The second key step is actually implicit in ref. [160]. It is the assumption that such generalised beta functions are finite in the limit $\varepsilon \to 0$. We have just seen that this is trivially true at one loop, but at higher loops this is a powerful assumption. Just as with the usual beta functions in a renormalisable theory, the one-loop result would then be enough to determine the leading pole $1/\varepsilon^\ell$ at each loop order ℓ without computing any more Feynman diagrams. To see this in our example, assume we already know the leading two-loop purely background counterterm and have chosen $\bar{\mathfrak{g}}^2_{\mu\nu}$ to generate it via

$$\bar{g}^0_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\,\bar{\mathfrak{g}}^1_{\mu\nu} + \frac{\kappa^4\mu^{-4\varepsilon}}{(4\pi)^4\varepsilon^2}\,\bar{\mathfrak{g}}^2_{\mu\nu}\,, \tag{4.4.5}$$

where the prefactor follows because $\bar{\mathfrak{g}}^2_{\mu\nu}$ will be formed from four background covariant derivatives. Then cancellation of the $1/\varepsilon$ single-pole in $\bar{\beta}_{\mu\nu}$ tells us that

$$\bar{\mathfrak{g}}^2_{\alpha\beta} = \frac{4\pi^2\mu^{2\varepsilon}}{\kappa^2}\,\mu\partial_\mu\bar{\mathfrak{g}}^1_{\alpha\beta}[\bar{g}]\,, \tag{4.4.6}$$

Applying the Leibniz rule and using (4.4.4), we see that $\bar{\mathfrak{g}}^2_{\alpha\beta}$ should in fact be computable simply by applying a first order shift of the background metric on the one-loop result:

$$\bar{\mathfrak{g}}^2_{\alpha\beta} = \delta\bar{\mathfrak{g}}^1_{\alpha\beta}[\bar{g}]\,, \qquad \text{where} \qquad \delta\bar{g}_{\mu\nu} = \frac{1}{2}\,\bar{\mathfrak{g}}^1_{\mu\nu}\,. \tag{4.4.7}$$

Unfortunately this does not work as can be verified explicitly at the two-point level by using the pure background terms from (4.3.24) (for higher order see the discussion below that equation). The reason is that the second key step, the assumption that these generalised beta functions are finite, is incorrect. In the original incarnation as applied to the target metric of the two-dimensional sigma model [182–184], it was correct, because the target metric actually represents an infinite set of couplings. But applied to the fields themselves, as in the proposal of ref. [160], it is not correct.

The obstruction to finiteness of $\bar{\beta}_{\mu\nu}$ shows up most clearly in the gauge fixing. The result (4.4.2) is derived using De Donder gauge (4.2.15). Clearly the transformation (4.4.5) alters the gauge (4.2.15) (by a divergent amount). That is a problem because the Legendre effective action is not the same in different gauges except on shell. But $\bar{\mathfrak{g}}^2_{\mu\nu}$ in (4.4.5) has been chosen to cancel a part that only exists off shell.

In fact let us now recall that counterterms are required that depend on all combinations of the fields, in particular the quantum fields, as we have seen. In the background field method it is possible to work exclusively with diagrams that have only external background field legs (as in *e.g.* [18]). However even if we do not explicitly track the

value of counterterms that cancel divergences in vertices involving quantum fields, they must be there in practice because they cancel sub-divergences in higher loops, and higher loop divergences are local, as required, only if all these sub-divergences have been cancelled [164, 185, 186], as we recalled in sec. 4.2.6.

Then as we saw in sec. 4.2.9, the 'new' divergences at each loop order are s_0-closed. Those that vanish on the equations of motion, are s_0-exact and correspond to infinitesimal canonical transformations (4.2.69) between the antifields and quantum fields. As we proved there, and also verified in sec. 4.3, there is no separate purely background renormalisation. What happens instead is that purely background counterterms also get absorbed by these canonical transformations. This extends to the non-linear terms that appear beyond one loop order. For example we saw that the leading (*i.e.* double-pole) counterterm at two loops, (4.3.26), also involves carrying the one-loop canonical transformation to second order, as we saw in sec. 4.2.8.

Now it is clear that if the proposal of [160] is going to work, it should apply not to the background metric, but to the antifields and quantum fields. Indeed the second-order canonical transformation $\delta\phi^{(*)}$ given in eqn. (4.2.74), is the correct non-linear transformation between bare (anti)fields

$$\phi_0^{(*)} = \phi^{(*)} + \delta\phi^{(*)} \tag{4.4.8}$$

and renormalised (anti)fields $\phi^{(*)}$, such that it will generate through Taylor expansion (4.2.76) of the classical action, all the required counterterms that vanish on shell, up to two loops.[4]

Independence of $\phi_0^{(*)}$ on μ, then implies the generalised beta functions

$$\beta^A[\phi,\phi^*] = \mu\partial_\mu\phi^A \qquad \text{and} \qquad \beta^*_A[\phi,\phi^*] = \mu\partial_\mu\phi^*_A\,. \tag{4.4.9}$$

Following the previous argument, if we assume that these beta functions are finite, we can derive K_2 from K_1 without computing Feynman diagrams. Once again we can check this idea explicitly using the results for K_1 from sec. 4.3.1. It turns out that it implies that at the two-point level K_2 must vanish. But from (4.3.28) this is incorrect. In fact, irrespective of the details, this proposal cannot work because the K_1 terms just furnish linearised curvatures for K_2, whereas K_2 has the explicitly non-covariant piece – the last term under the integral in (4.3.28). Again, the mistake in this reasoning is the assumption that the generalised beta functions are finite.

To see why they cannot be finite, note that the partition function (4.2.28) now takes the form

$$\mathcal{Z}[J,\phi^*] = \int\mathcal{D}\phi\,\mathrm{e}^{-S[\phi_0,\phi_0^*]+\phi^A J_A}\,, \tag{4.4.10}$$

[4] The Jacobian for this local transformation vanishes in dimensional regularisation, recall below (4.2.31).

Here the bare antifields are responsible for generating all the counterterms that vanish on shell, via canonical transformations (4.2.74), whilst S itself contains the counterterms for cohomologically non-trivial pieces which depend only on the total metric, such as the topological term (4.3.15) at one loop, and the Goroff-Sagnotti term [19]

$$S_2 \ni \frac{209}{5760}\frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^4\varepsilon}\int_x \sqrt{g}\, R_{\alpha\beta}{}^{\gamma\delta} R_{\gamma\delta}{}^{\epsilon\zeta} R_{\epsilon\zeta}{}^{\alpha\beta} \tag{4.4.11}$$

at two loops. All Green's functions are then finite (in particular this is so for the Legendre effective action, which is a functional of the classical fields Φ^A and the renormalised antifields $\Phi^*_A = \phi^*_A$). However for the operators that vanish on shell, we are now attributing μ dependence to the renormalised (anti)fields $\phi^{(*)}$ rather than renormalised couplings c^i_ℓ as before. Unfortunately μ-independence of the bare action $S[\phi_0, \phi^*_0]$ then implies that β^A cannot be finite since:

$$\mu\partial_\mu \mathcal{Z}[J,\phi^*] = \int \mathcal{D}\phi\ \beta^A[\phi,\phi^*]J_A\ \mathrm{e}^{-S[\phi_0,\phi^*_0]+\phi^A J_A}\,. \tag{4.4.12}$$

Indeed the left hand side is finite by construction, but the right hand side involves the insertion of β^A which is local and non-linear in renormalised quantum fields. The insertion of such terms generates new divergences, and the only way they can be cancelled is if in fact β^A already contains precisely the right divergences to cancel them.

4.5 Discussion and Conclusions

Off-shell counterterms in quantum gravity, defined perturbatively as an effective theory about a background metric $\bar{g}_{\mu\nu}$, are invariant under background diffeomorphisms, BRST, and the RG. In this chapter we have drawn out some of the consequences of the way these symmetries are interwoven with each other.

In particular we have shown in sec. 4.2.9 that at each new loop order the new divergences, those that are annihilated by the total classical BRST charge s_0, can be characterised as being either diffeomorphism invariant functionals of the total metric $g_{\mu\nu}$ which do not vanish on the classical equations of motion (*i.e.* do not vanish when $G_{\mu\nu} = 0$) or as s_0-exact functionals which are thus first order canonical transformations of the antifields and quantum fields (*cf.* sec. 4.2.8). In particular we show that there are no separate purely background field divergences. Then it follows that those background field terms that do not vanish on the equations of motion $\bar{G}_{\mu\nu} = 0$, are part of the diffeomorphism invariant functionals of the total metric, whilst those that do vanish on the equations of motion are renormalised by reparametrising the quantum fluctuation $h_{\mu\nu}$ as part of the canonical transformations. The background metric itself is never renormalised.

By adding the antifield sources for BRST transformations, we keep track of the deformations of the BRST algebra induced by renormalisation. These appear as part of the

s_0-exact counterterms. Whilst the Zinn-Justin/CME equation is preserved at each loop order ℓ for both the bare action and the Legendre effective action, the BRST transformations are altered in a non-linear way beyond one loop. As we demonstrated in sec. 4.2.5 this brings the Legendre effective action and bare action realisations of the CME equation into tension with each other. This tension is resolved by the RG identities for a perturbatively non-renormalisable theory, which relate lower loop $\ell' < \ell$ counterterm diagrams to higher order poles at ℓ-loop order [146].

In this chapter we only demonstrate how this works at two loops. A fully general understanding of how the RG ensures consistency for BRST seems possible, following the general understanding of the RG identities [146] and *e.g.* the proof of renormalisability put forward in ref. [139] for effective theories with gauge invariance (the latter does not address the above tension but proceeds assuming both realisations of the CME remain consistent with each other).

Let us emphasise that the way the RG and BRST relations work together is quite remarkable. On the one hand the RG relates the two-loop double-pole vertices to the one-loop single pole vertices through a linear map which however involves computing further one-loop Feynman diagrams (the counterterm diagrams). On the other hand BRST, through the second-order CME relation (4.2.41), directly relates the two-loop double-pole vertices to the square of the one-loop single-pole vertices, *i.e.* without involving further loop calculations. In a sense then the BRST relations achieve what generalised beta function proposals, *cf.* sec. 4.4, fail to do.

However the BRST relations do not determine the higher pole vertices completely but only up to an s_0-closed piece, for example this is evident in the two-loop relation (4.2.41): $s_0 S_{2/2} = -\frac{1}{2}(S_{1/1}, S_{1/1})$. They are thus less powerful than the RG identities. In fact in sec. 4.3.3, we saw in Feynman – De Donder gauge that the non-linear term on the right hand side starts only at the three point level. As we explained, at the two-point level the equations degenerate, although they still allow a unique determination of the second order canonical transformations, and, thus, also the new s_0-exact piece.

Let us note that the way the RG works to ensure consistency of BRST invariance, is not unique to non-renormalisable gauge theories. However, in renormalisable theories, the divergent vertices are those in the original action. The RG identities for counterterm diagrams then play a less dramatic rôle in that they just ensure that these divergences appear with the correct sign so that they can be renormalised multiplicatively.

For quantum gravity, we verified the assertions above in sec. 4.3 by computing counterterms at one-loop up to the three-point level and up to two-loops for the graviton two-point vertex. Exploiting the BRST properties we gave a general parametrisation of the one-loop three-point counterterms and determined the parameters by matching to the graviton and ghost one-loop integrals. The antighost level two counterterms (which

renormalise the BRST transformation of the ghosts) then follow without further Feynman diagram computations.

These results could be readily extended, for example the ghost two-loop double-pole two-point counterterms can be computed using the vertices presented here and this would allow the form of the two-point $K_{2/2}$ to be fully determined, *cf.* eqn. (4.3.28). An interesting but more challenging project would be to work out the form of the one-loop counterterms to the next order in $\bar{h}_{\mu\nu}$ since this would allow one to determine the two-loop double-pole three-point background field vertices which would then allow a complete comparison with the off-shell results reported in ref. [18]. The parametrisation we give for $K_{1/1}$ in (4.3.16) and (4.3.17) looks sufficient to compute the corresponding one-loop counterterm diagrams, if the d_i and b_i terms are covariantised, however this introduces a number of new terms with undetermined coefficients, in particular we would need to determine the $h^*h^2\bar{R}^{(1)}$ terms. The simplest way to do that would appear to be by matching to one-loop $h^*ch\bar{h}$ divergences.

In our discussion of generalised beta functions in sec. 4.4, we explained why they cannot be finite and verified this using our explicit results from sec. 4.3. In particular for generalised beta functions based on the canonical transformations we obtained the formula (4.4.12) which shows why they cannot be finite. Nevertheless, this formula implies some interesting relations between the divergent higher order coefficients and the divergences generated by expectation values of the lower coefficients. It would be interesting to verify these and explore further their consequences.

Finally let us return to our original motivation and note that the counterterms we have derived give directly the leading log behaviour at large euclidean momentum. Indeed, the one-loop divergence (4.2.47) and counterterm (4.2.48) taken together determine the $\ln(p^2/\mu^2)$ part. One can check explicitly that the two-loop double pole (4.2.52) from diagrams using only tree level vertices, together with divergences (4.2.51) in one-loop counterterm diagrams and the double-pole counterterm from (4.2.54), conspire to cancel all but a remaining $[\ln(p^2/\mu^2)]^2$ term. Thus from the explicit results (4.3.3) and (4.3.24) we see that the leading log contribution of for example the two-point $h_{\mu\nu}$ vertex is given to two loops, in Feynman – De Donder gauge, as:

$$\begin{aligned} h_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}(p)h_{\alpha\beta} = p^2\left(\varphi^2 - \frac{1}{2}h_{\mu\nu}^2\right) \\ + \frac{\kappa^2}{(4\pi)^2}\ln\left(\frac{p^2}{\mu^2}\right)\left(\frac{61}{60}(R^{(1)}_{\mu\nu})^2 - \frac{19}{120}(R^{(1)})^2\right) \\ - \frac{\kappa^4 p^2}{(4\pi)^4}\left[\ln\left(\frac{p^2}{\mu^2}\right)\right]^2\left(\frac{469}{7200}(R^{(1)})^2 - \frac{79}{400}(R^{(1)}_{\mu\nu})^2 + \frac{31}{1440}p^2R^{(1)}\varphi\right), \end{aligned} \tag{4.5.1}$$

where $h_{\mu\nu}$ and $\varphi = \frac{1}{2}h^{\mu}{}_{\mu}$ here just provide the polarisations, and the linearised curvatures (4.3.2) should be similarly understood and cast in momentum space, thus $R^{(1)}_{\mu\alpha\nu\beta} = 2p_{[\mu|}\,p_{[\nu}h_{\beta]\,|\alpha]}$ *etc.*

Of course as physical amplitudes these corrections vanish on shell, while for the moment it remains just a dream that a way can be found to resum these leading contributions to all orders, where one might get powerful insights into the non-perturbative UV behaviour of quantum gravity. Nevertheless we hope that the detailed understanding we have gained of some of the consequences of combining background diffeomorphism invariance, RG invariance, and BRST invariance, bring that dream a step closer to reality.

4.A Comparisons with the literature

Here we outline the differences in convention and notation that need to be taken into account in order to compare with other results in the literature.

The two-point purely quantum one-loop counterterm given in (4.3.3), corresponding to the first diagram in fig. 4.3.1, was computed in a general two parameter gauge $(\tilde{\alpha}\partial^{\mu}h_{\mu\nu} + \tilde{\beta}\partial_{\nu}h^{\rho}{}_{\rho})^2$ in ref. [162]. (We put a tilde over his parameters so as not to confuse with the ones used in this chapter.) After taking into account the Minkowski signature and that factors of $1/(2\pi)^4$ are accounted for differently, it should coincide with the first two terms in (4.3.3) on specialising $\tilde{\alpha} = 1$ and $\tilde{\beta} = -\frac{1}{2}$ to get Feynman DeDonder gauge.

Initially the results did not coincide. Recomputing the two-point vertex in this general gauge we found the following typos in ref. [162]: in the square brackets of his T_3 there should be an extra term: $+\frac{45}{8}\tilde{\beta}^4/\tilde{\alpha}^2$, and in T_4 the term $-135(\tilde{\beta}^2/\tilde{\alpha})$ should read $-135(\tilde{\beta}^2/\tilde{\alpha}^2)$. Finally his parameter a should be defined as $a = \frac{1}{2}T_2 - T_3$, rather than $\frac{1}{2}E_4$ as stated. Once these are fixed, we find complete agreement.

The result for the purely quantum pieces in (4.3.3) also agrees with the result quoted in ref. [163] on recognising that there the divergence can be recovered by setting $\ln(1/\mu_R) = 1/2\varepsilon$. This mapping is also the one to use to compare the level one divergence with (4.3.10).

The purely background terms in (4.3.3) agree with ref. [161] on recognising that their $\varepsilon = 8\pi^2(d-4)$, their definition of Ricci curvature is minus ours, *cf.* below (4.2.5), and that their action is defined to be the opposite sign from the usually defined Euclidean action, *cf.* (4.2.4). Their normalisation of the scalar curvature term is also non-standard but this is repaired by mapping $g_{\mu\nu} \mapsto \sqrt{2}\kappa g_{\mu\nu}$ and has no effect on the one-loop result, since it is a curvature-squared action.

In the famous paper [18], this result is reproduced but the value quoted is half that of (4.3.3). To see this one should note that it is Minkowski signature and their $\varepsilon = 4-d$ *i.e.* is twice ours. (There is also an accidental extra factor of $1/\varepsilon$ in their quoted equation.) They also quote a value for some two-loop double-pole divergences. The one point of comparison is the result (4.3.24) for the $\bar{R}^{\mu\nu}\Box\bar{R}_{\mu\nu}$ counterterm. Using these translations we see that their result is again half of what we find.

Chapter 5

Summary and concluding remarks

The research presented in this thesis considers a broad range of applications of the renormalisation group to the study of gauge theories and gravity. Let us summarise our main findings.

In chapter 2 we explored the properties of the background independent ERG flow equation constructed in ref. [47], which has some remarkable advantages: preserves gauge invariance manifestly, avoids introducing unphysical fields and does not require gauge-fixing. This would make it suitable for studying gauge theories or gravity non-perturbatively. However, we showed that in the simple case of $SU(N)$ Yang-Mills it fails to completely regularise the one-loop off-shell longitudinal part of the two-point vertex, thus invalidating powerful techniques previously used to uncover universal results [71, 96]. In particular, we proved that the problem is two-fold. First, we argued in section 2.6.3 that the convergence of the $O(p^0)$ part of the two-point vertex depends on whether we perform the momentum integral before or after differentiating with respect to Λ, and although we can formally set the divergent Λ independent piece to zero, it is unclear if this could be consistently carried to higher orders. Second, we demonstrated in section 2.6.4 that the $O(p^2)$ part of the two-point vertex cannot be simultaneously transverse, as required by gauge invariance, fully regularised and universal, regardless of the covariantisation or the cutoff profiles used.

In chapter 3 we investigated the spectrum of eigenoperators around a non-trivial fixed point for $O(N)$ and single component scalar field theory. We demonstrated that analytic results can be obtained within LPA and in the large field limit. The key idea was to recast the flow equation as a Schrödinger-type equation using SL theory, and then analyse it using WKB methods. This in turn allowed us to compute in (3.3.27) and (3.2.28) the leading order contribution and power-law growth of the subleading contribution to the scaling dimension of highly irrelevant operators for $O(N)$ and single component scalar field, respectively. We also proved that the leading contribution for the $O(N)$ case is

twice that for a single scalar, and that both results are independent of N or the cutoff profiles used, thus universal.

Finally, in chapter 4 we delved into the perturbative off-shell structure of pure gravity using the background field formalism. We argued in section 4.2.3 that the use off-shell BRST over the textbook on-shell BRST is more appropriate. In this way, we were able to uncover a series of surprising and important results. First, we showed in section 4.2.5 that RG is already at the perturbative level the crucial element that ensures the consistency of the CME. And thus we were able argue and show that the BRST transformations get drastically modified. Second, we showed that divergences that do not vanish on-shell are constructed from only the total metric, whilst those that vanish on-shell are renormalised by canonical transformations involving the quantum fields. We proved this by computing in section 4.3 explicit expressions for the leading off-shell divergences up to two-loop level. This, in turn, allowed us to invalidate in section 4.4 the generalised beta functions proposal of ref. [160].

There is no doubt that more work needs to be done regarding the topics covered in this thesis. As outlined above, understanding the interplay between different scales is surely of importance and central to this is the renormalisation group, which featured in its various incarnations throughout this thesis. This comes with its own challenges and difficulties, a few of which we tried to address. As emphasised in the introduction, what sparked our interest for it, to begin with, was the possibility to delve deeper into the quantum structure of spacetime. It would be a triumph for the RG program, as originally envisaged by Wilson, to achieve such a goal. Thus, we hope that the perspective constructed in this thesis is broad and compelling enough to argue in a convincing manner in its favour. This is not to say that it is the only way forward or that other frameworks should be ignored. As mentioned before, many of these are promising and have interesting features worth exploring. However, the versatility of the renormalisation group that we have highlighted throughout this thesis makes it a powerful tool capable of investigating quantum gravity, gauge theories or, more generally, for exploring new physics.

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