**University of Southampton**

Faculty of Social Sciences
School of Mathematical Sciences

# Super Yang-Mills $\alpha'$ Corrections in $D = 10$ from the Pure Spinor Formalism

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<u>Abstract</u>

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In this thesis we aim to utilise the powerful machinery, developed in the past decade, to calculate $D = 10$ Super Yang-Mills amplitudes in the pure spinor formalism and apply it to find $\alpha'$ corrections to Super Yang-Mills via an alternative cohomology-driven method. We begin by presenting a brief review of the Ramond-Neveu-Schwarz String and the Green-Schwarz String, discussing some of the issues when quantising such theories. We then move on to the current state-of-the-art methods, as well as a brief overview of the advantages of pure spinor. We then discuss how we determine higher-order corrections to Super Yang-Mills using BRST cohomology arguments and ansatzes. This involves using the simplicity of the BRST operator in pure spinor superspace to efficiently find generating series expressions which give rise to the $n-$point amplitudes corresponding to the corrections. In order to perform these calculations we derive a number of new identities relating higher mass non-linear fields to lower mass fields which facilitates the canonicalisation of ansatz variations. We also introduce a number of new higher mass fields and determine their variations and other identities that may be useful in future research. Furthermore, some simple calculations for the torodial compactification of the pure spinor method in $D = 4$ are presented - showing that the resulting spectrum replicates the expected spectrum. There are also some additional topics which cover experimental calculations aimed at simplifying some of the results presented within this thesis - however many of these experiments have failed to produce novel results and are presented here for posterity.

*I have thought it my duty to exhibit things as they are,*
*not as they ought to be.*

ALEXANDER HAMILTON (1782)

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# Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. None of this work has been published before submission

Signed:........................................................................ Date: 01/09/2023

# Acknowledgements

I would like to begin by thanking my supervisor, Dr. Carlos Mafra, for giving me to opportunity to undertake a PhD. While I have not completed the full course of this degree, I feel immensely privileged to have been selected by someone as competent as he is. I would also like to thank him for the countless hours spent working on code, derivations and the fiddly problems thrown up by some of the research attempted here.

I would further like to thank my partner, Hannah Dawson. She has been with me since before I started my PhD studies and was there during the interviews, the offers, and the work itself. Through the ups and downs of PhD life she has been a constant source of support and solace. Many of my achievements, both here and in broader life, are due their success to her and the tireless enthusiasm she gave me when I was down.

I would also like to thank my parents. Whilst we did not have much when I was young, they always ensured I had access to as many opportunities as possible. This tireless effort has led me to be the first person in my family to get a degree, the first to get a masters and the first to study (although not complete) a PhD. Without their sacrifice and belief in my goals, I would not have achieved so much from such an underprivileged position.

It would be remiss of me to not give a special mention and thanks to my grandmother (Nana Dotty). Without her I would not be where I am today. She gave me access to a huge number of opportunities and experiences that have given me the confidence and self-assurance to pursue my goals.

I should also thank Michael Kenna-Allison for the weekly chats, work on cosmology not covered here, and being a sounding board for some of my ideas.

I would also like to thank my office mates, as well as the wider community in STAG and the mathematical sciences department at Southampton. We have enjoyed many lunches together, some heated discussions, and lots of laughs. I will miss them all very much.

*To all those that have helped me get this far.*

# Preface

From as early as I can remember I had always wanted to become a scientist - although this seems like a clichèd statement it is true. I was lucky enough in the early part of 2021 to be offered to study for a PhD in theoretical physics, the last step before I could call myself a good and proper physicist. However, as I write this thesis, it is 2023 and I am leaving my PhD to go and work in politics. The reasons why are myriad and in many cases personal but I do not regret any of those choices and for the most part I am very happy with the choices I have made. They have allowed me to enter an career that is seldom easy for an outsider. Furthermore, these choices have given me the opportunity to study right at the edge of what is known about the universe and the theories we use to describe it. Although I might not be going on to study these topics any further, many of the skills I have learnt over the past two years will hopefully make me a more diligent and curious researcher in the private sector.

Throughout the two years I spent studying pure spinor superstring theory I have had opportunities which few get. I have attended conferences in Amsterdam, Prague and my hometown of Durham where I gave my first in person conference talk. I have had the opportunity to organise and run a summer school for fellow PhD students and postdocs. I am also happy to have been able to spend as much time teaching undergraduate students, a task I always enjoyed and will truly miss.

What follows in these pages is an account of some of the work I have carried out towards my PhD, now MPhil. This thesis presents a whole host of novel results, some more impressive than other, in the hopes that some of the work here can be used to answer the questions posed. There are many questions left unanswered here and I had genuinely hoped to finish the work to answer them, however I have chosen not to. I never thought I would not get a PhD, let alone not get a PhD due to my wilful walking away from it. But that is how things have landed.

# Chapter 1

# Introduction

The calculation of scattering amplitudes in String Theory has been pivotal in the discovery of many of the properties of the theory. These amplitudes have been used to investigate the low energy limit of string theories, giving rise to supergravity and supersymmetric field theories [1]. Furthermore, twistor string theory amplitudes have found applications in Yang-Mills theory and QCD calculations [2, 3]. The study of amplitudes in general theories is incredibly important as they have found use in a wide range of subjects from using amplitudes to reconstruct exact solutions [4, 5] to determining the ringdown of black hole mergers [6]. Given the prominence of amplitudes in the study of theoretical physics and particularly string theory, it is important to discover more efficient methods to calculate these objects. The main purpose of this thesis is to present a new method for determining the generating series of $\alpha'$ corrected amplitudes in the pure spinor formalism [7].

Since the inception of Super-Poincaré $D = 10$ Super Yang-Mills[1] [8, 9] and the Super-Poincaré covariant quantization of the superstring using the pure spinor formalism just over two decades ago [10, 11, 12] there have been many advances in the computation of both superstring scattering amplitudes as well as Super Yang-Mills scattering. Traditional methods for performing string theory calculations, such as the Ramond–Neveu–Schwarz (RNS) [13, 14, 15, 16] and Green–Schwarz (GS) formalisms [17, 18], lose some of the symmetry inherent in the theory when scattering amplitude

[1]Unless otherwise stated, Super Yang-Mills in this thesis always refers to $D = 10$ Super Yang-Mills.

calculations are performed. This loss of symmetry can often make performing calculations cumbersome or even impossible. Such issues are widely described in the textbook literature [19, 20, 21, 22, 23, 24]. There have been many ways in which Super Yang-Mills amplitudes have been calculated, including Feynman rules and the BCFW recursion procedure [25, 26] as well as amplituhedron methods [27, 28] in $\mathcal{N} = 4$ $D = 4$ Super Yang-Mills; however, the pure spinor formalism has greatly simplified these calculations in $D = 10$. The key to this success in recent years has been the introduction of non-linear Super Yang-Mills generating series [29, 30] which represent generic number multiparticle currents. These methods rely heavily upon the recursive nature of the string Operator Product Expansions[2] (OPEs) which were previously exploited to simplify scattering amplitudes in the pure spinor formalism [36, 37]. When calculating scattering amplitudes in the pure spinor superstring theory, one has to take OPEs between various worldsheet fields. It was realised that as higher-order OPEs were calculated, their form could be represented as recursive formulae for a generic $n$-point OPE - these fields are known as multiparticle superfields. One can sum these fields, weighting them by Lie-algebra generators, to give the non-linear generating series of Super Yang-Mills fields, these ten-dimensional non-linear fields are usually denoted by,

$$\mathbb{A}_\alpha(x,\theta), \quad \mathbb{A}^m(x,\theta), \quad \mathbb{F}^{mn}(x,\theta), \quad \mathbb{W}^\alpha(x,\theta), \tag{1.1}$$

where $\mathbb{A}_\alpha$ is the pre-spinor potential, $\mathbb{A}^m$ is the gauge field, $\mathbb{F}^{mn}$ is the Super Yang-Mills field strength and $\mathbb{W}^\alpha$ is the gaugino field. Note that we use Latin indices to represent spacetime directions and Greek indices to denote spinors. Such fields satisfy the full non-linear Super Yang-Mills equations of motion which will be given in the next chapter, furthermore, their linearised parts appear in the massless vertex operators of the pure spinor superstring [10]. Of course, $D = 10$ Super Yang-Mills is non-linear, however when calculating amplitudes via traditional methods the theory is linearised. Then interaction terms are treated as perturbations from the linear theory. Hence, when we refer to non-linear Super Yang-Mills we specifically mean we are calculating amplitudes using the full non-linear equations of motion. It is this fact which allows us to find

[2] For the original papers on OPEs see [31, 32] and their applications to conformal field theory [33, 34, 35].

generating series which can be used to find the $n-$point amplitude, essentially collapsing the usual Feynman method. Using these series dramatically reduces the complexity of calculating the amplitudes since any generating series expression that is in the BRST cohomology is part of an $n-$point amplitude. In essence this means that if one can write the kinematic factor of an amplitude using the non-linear fields in (1.1) then it is suspected that the component expansion of such a kinematic factor will be part of the full $n-$point kinematics factor. As a result, one can easily recurse the correct $n-$point amplitude from the kinematic factor. In later chapters we shall describe explicitly how one can leverage a combination of the non-linear fields, Berends-Giele currents [38] and the Harnad-Schnider gauge [39] to very efficiently compute such component expansion to arbitrary point. These tools, coupled with the computational power of `FORM` [40] now allow exploration of amplitudes in Super Yang-Mills via a new route. In fact, the main part of this work is to determine the generating series of $\alpha'$ corrections to Super Yang-Mills using the BRST cohomology of the pure spinor. Such $\alpha'$ corrections are so-called stringy corrections to the field theory amplitude result, expanded in powers of the inverse string tension ($\alpha'$).

One of the main achievements within the pure spinor methodology over the past decade has been the simplification and compact representation of the $n-$point tree-level scattering amplitude in superstring theory and Super Yang-Mills [36, 41, 42]. These amplitudes can be easily captured by the expression,

$$\frac{1}{3}\mathrm{Tr}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle, \tag{1.2}$$

where $\mathbb{V} = \lambda^\alpha \mathbb{A}_\alpha$ is the non-linear pure spinor unintegrated vertex operator and $\lambda$ are the pure spinors. It is these pure spinors that make finding the generating series and calculating amplitudes so simple. The introduction of pure spinors first allows one to covariantly quantise the superstring in $D = 10$ while keeping supersymmetry manifest [10], something that is not possible in the GS or RNS formalisms[3]. It also makes finding terms that contribute to the amplitude very simple due to the integration measure. The angled brackets, $\langle\rangle$, represent the integration over the pure spinor measure, which we

[3]Note that it is now known how the RNS [43, 44] and the GS [45] formalisms are related to the pure spinor formalism.

normalise here and throughout as,

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 2880. \tag{1.3}$$

This integration measure is one of the reasons working in pure spinor is significantly more efficient than other methods in $D = 10$. From a pragmatic viewpoint one simply needs to perform the $\theta-$expansion, as detailed in later chapters, and look for terms that contain $\lambda^3$ and $\theta^5$. This makes calculating the 4-point, tree-level superstring amplitude go from many pages of working to a single page - a dramatic simplification of the calculation [10, 11].

## 1.1 The Old String Theories

It is worthwhile introducing, from a qualitative point of view, the explicit problems that occur in older formalisms of string theory and why these issues pose problems when attempting to calculate higher-order processes. There are two perspectives from which to view a string in spacetime[4]: the first is to consider the string as sweeping out a worldsheet in spacetime and perform the analysis on the worldsheet; the second is to consider the motion of a string in a $D-$dimensional spacetime and analyse the string from the spacetime perspective.

The former of these interpretations is most closely associated with the Ramond-Neveu-Schwarz formalism [56, 57] and can be quantised using a number of methods that we shall cover in Chapters 2 and 3 of this work. The issue facing the RNS formalism is that manifest supersymmetry is lost upon quantisation making calculating amplitudes incredibly difficult. As a result, the formalism requires summing over spin structures [24, 58] in order to perform the GSO projection required to remove tachyonic modes from the theory. Summing over such spin structures increases in difficulty as the genus

[4]The same is true of particle physics. One can study Quantum Field Theory from the spacetime perspective which is found in most textbooks [46, 47, 48, 49], or one can use the much less popular world-line formalism [50, 51, 52, 53, 54, 55].

of the worldsheet increases and poses a practical limit on the loop amplitude calculations in this formalism[5].

The latter approach is more associated with the Green-Schwarz formalism [62, 63, 64] which includes the GSO projection to obtain a consistent theory of fermions. This theory cannot be quantised easily and often leads to issues if one tries to use covariant or BRST quantisation - it can only be quantised in light-cone gauge and hence the theory loses its manifest Lorentz invariance. Working in light-cone gauge yields calculations that are immensely complicated to perform and often takes many pages of detailed working in order to find [19, 20]. This essentially closes off working on high-point or -loop amplitudes as the technical finesse required to work in light-cone gauge at these orders is immense.

As a result of the issues present in both formalisms, it has taken decades to perform one- and two-loop calculations. Often such calculations have involved hundreds of pages of careful attention to the minutiae of the problem. At the turn of the millennium this changed and the pure spinor formalism was introduced.

## 1.2 The Pure Spinor and its Superspace

Here we shall present a brief, qualitative overview of the pure spinor and its corresponding superspace before we dive into the more technical chapters of this thesis. To solve the previous issues, Siegel introduced new variables to the GS string [65] which allowed it to be written in a quadratic form similar to free field theory. However, when one does this one obtains a series of constraints that are very similar to the pure spinor constraint [66, 67, 68, 69, 70] which when quantised breaks the $SO(9,1)$ symmetry of the theory. Further work was carried out by a variety of authors that led to various hybrid formalisms that allowed some covariant quantisation of the superstring in the desired manner. It was not until 2000 [10] that a covariantly quantise-able theory was found involving the pure spinor. The pure spinor is in effect a ghost particle, akin to those found in the usual BRST quantisation procedure, which essentially determines the cohomology of the BRST space.

[5] Even as recently as 2020 work is still being carried out at the two-loop level in this formalism [59, 60, 61]

One of the features of the pure spinor formalism is that its fields are free on the world-sheet, and so one can define OPEs of the free fields. Being able to calculate OPEs massively aids the calculation of string scattering amplitudes, as well as other observables, as Conformal Field Theory in two dimensions is well understood. As a result, one can go back to many of the techniques used in bosonic string theory [21] and use them in the calculation of superstring amplitudes. Furthermore, the introduction of the pure spinor makes calculating Super Yang-Mills amplitudes rather simple, as we shall see. In fact, this formalism, combined with the concept of perturbiners [71, 72, 73, 74], allows one to find expressions that encapsulate the entire $n-$point amplitudes of various processes. Finding such expressions is the main purpose of this thesis.

## 1.3 Higher Order Corrections

In this project we investigate whether the $\alpha'$ corrections to open superstring amplitudes can be written in a generating series form. Such corrections are interesting for a number of reasons - for example, such corrections may have links to loop-amplitudes and they help uncover the low energy behaviour of string theories. Until 2010 the state of the art single-particle representation was the up to $\alpha'^2$ correction, which was found in [75, 76]. This problem was first tackled up to order $\alpha'^4$ in the non-abelian theory in 2010 [77] in which the authors used a mix of cohomological arguments and superspace techniques to find the corrected action at $\alpha'^4$. However, the resulting answer was rather long and, once translated into the non-linear generating series form, did not naïvely close under the pure spinor BRST operator. It is not *a priori* expected that the corrections to the action found in [77], when promoted to generating series, will be BRST closed - hence the lack of closure of terms from that paper is not wholly unexpected. Furthermore, no $\alpha'^3$ corrections have been found owing to the fact that previous attempts only considered the abelian theory. In this work we intend to extend the these corrections to the pure spinor formalism by presenting the $\alpha'^2$ correction and finding both the $\alpha'^3$ and $\alpha'^4$ corrections as well as explore their connections to superstring scattering amplitudes and the associated kinematic factors. Currently, both the $\alpha'^2$ and $\alpha'^3$ corrections have been found, however, there are issues in determining the $\alpha'^4$ corrections which we shall detail later. At present, there is ongoing research into a new way of tackling the $\alpha'^4$ correction which has been shown to work at $\alpha'^2$.

In the pure spinor formalism, the BRST operator of the theory is given by [10],

$$Q = \lambda^\alpha D_\alpha, \tag{1.4}$$

where $D_\alpha$ is the spinor derivative. This operator follows from the OPE between the free fields on the worldsheet and the contour integral - something we shall detail in further chapters. The simplicity of this operator is the reason that finding higher order corrections can be boiled down to a very simple, if somewhat computationally intensive, method that can be applied at any order. The cohomology of the pure spinor formalism defines almost everything about the formalism (up to some rescalings that we shall discuss in later chapters) and the philosophy is that if something can exist in the cohomology of the operator $Q$ then it must necessarily take part in the kinematic factors in some way. As a result, finding the higher order corrections becomes the search for factors of the correct weight that exist in the cohomology of the BRST operator. Hence, it is of the utmost importance to understand how the BRST operator acts on the nonlinear fields, as it is these fields that will allow us to generalise our results to the $n-$point case. One can further simplify the use of this operator in the context of generating series by noting that generating series expressions involve the trace of the operators and the labelled vertex operator $\mathbb{V}_1$, which is trivially BRST closed. We shall detail more about $\mathbb{V}_1$ and its relation to the 1-loop superstring amplitudes in later chapters. However, $\mathbb{V}_1$ allows one to define the operator,

$$\mathcal{Q} = \lambda^\alpha \nabla_\alpha, \tag{1.5}$$

where $\nabla_\alpha$ is the spinor covariant derivative given by $\nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha$. Using this operator removes many of the terms that are produced by the BRST variation - for example, this allows one to remove terms that produce $\{\mathbb{V}, \mathbb{V}\}$ once varied. This will be shown in later chapters.

The process of finding the higher order terms begins by producing an ansatz at the correct weight to capture the correction. This task requires a number of new identities which allow us to express higher mass fields (that is, fields that contain a $D = 10$ covariant derivative $\nabla_m$, e.g. $\mathbb{W}^{mn}$) in terms of decomposition of a symmetric traceless part and terms involving quadratic nonlinear fields - in some cases using these symmetric traceless fields to produce the ansatz drastically reduces the number of terms in the ansatz. Furthermore, we also have to consider the types of terms that can appear at each order and capture all of them, as well as all vector index contractions compatible with the pure spinor formalism, in the ansatz. This is a rather delicate procedure and we shall explain in detail the methodology used in later chapters.

Such higher order corrections are intimately linked to higher-loop kinematic factors in the pure spinor superstring theory, and so determining their form in an efficient manner is an important task. These themes were initially explored in [30] which demonstrated that the complicated three-loop kinematic factors generating the operator $D^6R^4$ [78] can be written very efficiently in terms of the multiparticle superfields. Such simplifications will be the bread and butter of this project and will be the main focus. It is important to emphasise that the results in this project, when given in terms of the non-linear superfields, capture the kinematic factors of those terms to any number of external points. Hence representing such corrections in this form yields vast amounts of information about the low energy limit of string theory and the higher order corrections of Super Yang-Mills.

## 1.4 Outline

This thesis is laid out across another 14 chapters in four Parts which detail background information as well as novel research that has thus far been carried out.

Part I mainly focuses on introducing many of the concepts required to study the generating series of Super Yang-Mills amplitudes and their $\alpha'$ corrections. Chapters 2 and 3 give a brief review of the RNS and GS formalisms and some results from those theories - these chapters are predominantly based on [19, 20, 21, 22, 23]. Chapter 4 concerns non-linear Super Yang-Mills and the derivation of the field equations from the superspace constraint

$F_{\alpha\beta} = 0$, it also introduces the notion of higher mass fields as fundamentals. Chapter 5 reviews how to use the pure spinor formalism to find amplitudes and also demonstrates the use of the multiparticle superfield approach to finding such amplitudes. Chapter 6 details how the multiparticle superfields can be arranged into generating series and how those generating series solve the nonlinear equations of motion of Super Yang-Mills.

In Part II the main bulk of the novel research carried out as part of this thesis is presented, including the novel generating series of $\alpha'^2$ and $\alpha'^3$. Chapter 7 details the methods used to produce the ansatz for each correction, we pay particular attention to group theory arguments as well as graph theory to efficiently build the ansatz. Chapter 8 presents new identities for the non-linear generating series which allow one to decompose some of the higher mass fields into lower mass fields - this is exceptionally useful when canonicalizing BRST variations of the ansatzes. In Chapter 9 we briefly discuss the generating series of the $\alpha'$ and $\alpha'^2$ corrections - this generating series form is a novel result. Chapter 10 presents the novel corrections at $\alpha'^3$. This involves detailing two different sectors of the correction as well as new BRST exact terms which can be used to simplify the results of the analysis. In Chapter 10 we also detail the search for a generating series expression which shows that the two different sectors of $\alpha'^3$ corrections are indeed the same. This is not a necessity, as the component expansion of each of the expressions are the same, and since they are BRST closed they can only differ by a BRST-exact piece. In Chapter 11, we present the search for corrections which take the form $\mathcal{O}(\mathbb{VVV})$ where $\mathcal{O}$ is some differential operator; this section does not give the desired results but the equations derived therein may be useful in future research. Chapter 12 details the current state of the search for $\alpha'^4$ corrections, including some attempts at resolution of the issues currently plaguing the $\alpha'^4$ analysis. This section details some of the new methods that are currently being investigated and are likely to resolve the issues outlined.

Part III concerns a small exercise in looking at the compactification of the pure spinor machinery down to $D = 4$. Chapter 13 details a small research project whereby the machinery of $D = 10$ pure spinor Super Yang-Mills is compactified on the torus, deriving the required equations for computing amplitudes in lower dimensions.

Lastly, Part IV presents the conclusions of the thesis, including a quick review of the work presented here and some ideas for future research work. In Chapter 14 we review the work carried out in this thesis and begin to draw conclusions from the research undertaken. Finally, Chapter 15 details some of the potential routes forward from this thesis, as well as some issues that are present in the wider pure spinor formalism.

# Part I

# Introducing Concepts

# Chapter 2

# The Worldsheet Theory

In this chapter we present a review of the superstring using the worldsheet formalism, based on notes taken from [19, 20, 21, 22, 23]. We present this brief introduction here, as they add a more pedagogical element to the work presented here, plus we present work that fills in some of the gaps in the textbooks previously cited. As we shall see, this formalism has pros and cons and to see the whole picture we shall have to consider the theory from the spacetime point of view as well as the worldsheet view used in this chapter.

## 2.1 Ramond-Neveu-Schwarz Strings

In the Ramond-Neveu-Schwarz (RNS) formalism we pair each $X^\mu$ (these are the worldsheet scalars representing the spacetime position of the string) with a worldsheet spinor $\psi^\mu$, which transform as spinors on the worldsheet but act as $D-$dimensional Lorentz vectors in the $D-$dimensional spacetime. The $D = 10$ spin statistics are a little obscure, but they will become clearer as we go along. The action that results from the addition of the spinors in the conformal gauge[1], is given by,

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu), \tag{2.1}$$

[1]Here and in the following we set $\alpha' = \frac{1}{2}/T = \frac{1}{\pi}$

where $\alpha$ are the worldsheet indices and $\rho^\alpha$ are Dirac matrices that obey the usual algebra. The spinor has two components $\psi^\mu_A$, for $A = \pm$, and hence we can express it in the column vector form as $\psi^\mu = (\psi^\mu_-, \psi^\mu_+)$, where we define the Dirac conjugate of $\psi$ as,

$$\bar{\psi} = i\psi^\dagger \rho^0, \tag{2.2}$$

which for a Majorana spinor is $\psi^T\beta$, with $\beta = i\rho^0$. Here, and throughout, we shall take,

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

which is a Majorana representation and hence,

$$\psi^*_+ = \psi_+, \qquad \psi^*_- = \psi_-. \tag{2.4}$$

As a result of the above, the fermion action with suppressed Lorentz indices can be expressed as,

$$S_F = \frac{i}{\pi}\int \mathrm{d}^2\sigma(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+), \tag{2.5}$$

where we have introduced the lightcone coordinates[2]. The equations of motion resulting from the lightcone action are,

$$\partial_+\psi_- = 0, \qquad \partial_-\psi_+ = 0, \tag{2.6}$$

and hence we have left and right moving wave equations for massless particles. For spinors in $D = 2$ these are Weyl conditions so $\psi_\pm$ are Majorana-Weyl spinors[3].

The action in (2.1) is actually invariant under global supersymmetry transformations defined by the infinitesimal transforms,

$$\begin{aligned} \delta X^\mu &= \bar{\varepsilon}\psi^\mu \\ \delta\psi^\mu &= \rho^\alpha\partial_\alpha X^\mu \varepsilon, \end{aligned} \tag{2.7}$$

[2]Although not formally introduced in this work, the light cone formalism is relatively straightforward and amounts to setting $\sigma^\pm = \sigma^0 \pm \sigma^1$.

[3]They are actually two inequivalent one-dimensional spinor representations of $spin(1,1)$.

where $\varepsilon$ is a Majorana spinor. In lightcone coordinates this becomes,

$$\begin{aligned} \delta X^\mu &= i(\varepsilon_+ \psi_-^\mu - \varepsilon_- \psi_+^\mu) \\ \delta \psi_\pm^\mu &= \pm 2\partial_\pm X^\mu \varepsilon_\mp . \end{aligned} \tag{2.8}$$

We can now fairly easily show that (2.1) is invariant under (2.7) by going to lightcone coordinates and using (2.8). In lightcone coordinates, the action becomes,

$$S = \frac{1}{\pi}\int \mathrm{d}^2\sigma (2\partial_+ X \partial_- X + i\psi_-\partial_+\psi_- + i\psi_+\partial_-\psi_+). \tag{2.9}$$

The variation then gives,

$$\begin{aligned} \delta S &= \frac{1}{\pi}\int \mathrm{d}^2\sigma (2\partial_+\delta X \partial_- X + 2\partial_+ X\partial_-\delta X + i\delta\psi_-\partial_+\psi_- \\ &\qquad + i\psi_-\partial_+\delta\psi_- + i\delta\psi_+\partial_-\psi_+ + i\psi_+\partial_-\delta\psi_+) \\ &= \frac{1}{\pi}\int \mathrm{d}^2\sigma (2i\varepsilon_+\partial_+\psi_-\partial_- X - 2i\varepsilon_-\partial_+\psi_+\partial_- X + 2i\varepsilon_+\partial_+ X\partial_-\psi_- \\ &\qquad - 2i\varepsilon_-\partial_+ X\partial_-\psi_+ - 2i\varepsilon_+\partial_- X\partial_+\psi_- - 2i\psi_-\varepsilon_+\partial_+\partial_- X \\ &\qquad + 2i\partial_+ X\partial_-\psi_+\varepsilon_- + 2i\psi_+\varepsilon_-\partial_+\partial_- X) \\ &= \frac{1}{\pi}\int \mathrm{d}^2\sigma\; 2i\varepsilon_+(\partial_+\psi_-\partial_- X - \partial_-\partial_+\psi_-) - 2i\psi_-\varepsilon_+\partial_+\partial_- X \\ &\qquad + 2i\varepsilon_+\partial_+ X\partial_-\psi_- + (\varepsilon_-) \\ &= \frac{1}{\pi}\int \mathrm{d}^2\sigma\; 2i\varepsilon_+\partial_-(\psi_-\partial_+ X) + (\varepsilon_-) \\ &= 0 \end{aligned}$$

where in going from the prepenultimate line to the penultimate line we cancel the first term and pull the $\partial_-$ derivative our of the last two terms; the final line follows since the penultimate line is the integral of a total derivative. As a result, we have shown that the action is supersymmetric as expected.

### 2.1.1 The Local Action

The above discussion concerns global supersymmetry in conformal gauge, however there is a more fundamental formulation which will give rise to local supersymmetry. We shall take a detour here and turn to this formulation before we introduce superspace in two dimensions.

To make a locally supersymmetric theory we shall have to introduce a method that allows us to place fermions on a general worldsheet that is not flat. We can do this by introducing the vielbein $e_\mu^m$, which maps between the locally flat spacetime and the general curved spacetime. This is allowed since the equivalence principle holds, and as a result we may use our flat space convention in curved space as long as we use the vielbein to map between the flat inertial space and the general curved manifold. Essentially $e_\mu^m$ are orthonormal vectors at each spacetime point. We can relate the general metric $g^{\mu\nu}$ and the vielbein via the relations,

$$g_{\mu\nu} = \eta_{mn} e_\mu^m e_\nu^n, \qquad \eta^{mn} = g^{\mu\nu} e_\mu^m e_\nu^n, \tag{2.10}$$

where we also note that $e = \sqrt{g}$. Local Lorentz symmetry can be enforced in much the same way as a gauge group can be made local to give the Yang-Mills theory. Analogous to $A_\mu$ in Yang-Mills, one can introduce the so-called spin connection $\omega_\mu^{mn}$ as a gauge field of Lorentz transformations. Under local Lorentz transforms, $\Theta^{mn} = -\Theta^{nm}$, the spin connection becomes,

$$\begin{aligned} \delta\omega_\mu^{mn} &= \partial_\mu \Theta^{mn} + [\omega_\mu, \Theta]^{mn} \\ &= (D_\mu \Theta)^{mn}. \end{aligned} \tag{2.11}$$

We now introduce some general Dirac matrices $\Gamma^m$ that obey the usual Dirac anticommutation relations, and the local Lorentz transform of a spinor can be defined as,

$$\delta\psi = -\frac{1}{4}\Theta^{mn}\Gamma_{mn}. \tag{2.12}$$

The covariant derivative is given by,

$$D_\mu \psi = (\partial_\mu + \tfrac{1}{4}\omega_\mu^{mn}\Gamma_{mn})\psi. \tag{2.13}$$

Now in curved space we will have spacetime dependent Dirac matrices such that $\Gamma^\mu(x) = e^\mu_m \Gamma^m$. Then the action for a spinor in a curved background is given by,

$$S_\psi = \frac{i}{2}\int \mathrm{d}x\; e\left[\bar{\psi}\Gamma^\mu D_\mu \psi\right]. \tag{2.14}$$

If we wish to discuss standard General Relativity then the spin connection ought not to be some arbitrary new object, but it should be constructed from the metric. We can do this by noting that the metric is covariantly conserved and so the vielbein should also be,

$$D_\mu e^m_\nu = \partial_\mu e^m_\nu + \omega^m_{\mu n} e^n_\nu - \Gamma^\rho{}_{\mu\nu} e^m_\rho = 0. \tag{2.15}$$

This equation determines the spin connection uniquely, if one does a counting of unknowns. We can define the Riemann curvature tensor as a Yang-Mills-like field strength tensor,

$$R^{mn}_{\mu\nu} = \partial_\mu \omega^{mn}_\nu - \partial_\nu \omega^{mn}_\mu + [\omega_\mu, \omega_\nu]^{mn}, \tag{2.16}$$

which ought to be the Riemann tensor since the right hand side is at most given by two derivatives of the metric.

As an application of the above, let us quickly consider $\mathcal{N} = 1$ supergravity in $D = 4$. This contains a vierbein, spin-connection and Rarita-Schwinger field $\chi_{A\mu}$. The action for such a theory is given by,

$$S = \int \mathrm{d}^4x\; e\left[-\tfrac{1}{2\kappa^2}R - \tfrac{i}{2}\bar{\chi}_\mu \gamma^{\mu\nu\rho} D_\nu \chi_\rho\right], \tag{2.17}$$

which has manifest global coordinate invariance and local Lorentz invariance; the supersymmetric transformations are then given by,

$$\delta\chi_\mu = \tfrac{1}{\kappa} D_\mu \varepsilon, \qquad \delta^m_\mu = -\tfrac{i}{2}\kappa\bar{\varepsilon}\gamma^\mu \chi_\mu. \tag{2.18}$$

However, we are not currently concerned with $D = 4$ supergravity, we are concerned with our superstring. Now making the adjustments for the vielbein, the action in (2.1) becomes,

$$S_1 = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\; e[h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + \bar{\psi}^\mu \rho^\alpha \nabla_\alpha \psi_\mu]. \tag{2.19}$$

The Fermi statistics imply that $\nabla^\alpha$ cannot depend on the spin connection and hence $\nabla^\alpha \to \partial^\alpha$; this is a special feature of $D = 2$ Majorana spinors. This action does not currently have manifest local supersymmetry. If one varies $S_1$ with $\varepsilon$, a local function, then one finds the term,

$$\int (\nabla_\alpha \bar{\varepsilon})(\tfrac{1}{2}\rho^\beta \rho^\alpha \psi^\mu \partial_\beta X_\mu)\mathrm{d}^2\sigma. \tag{2.20}$$

We now add supersymmetry via the Noether method: let us introduce the gravitino $\chi_\alpha$ such that,

$$\delta\chi_\alpha = \nabla_\alpha \varepsilon, \tag{2.21}$$

where we have suppressed $\chi$'s worldsheet spinor index. The variation in (2.20) can be removed by varying $\chi_\alpha$ in the action,

$$S_2 = \frac{1}{\pi}\int \mathrm{d}^2\sigma\ e[\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \partial_\beta X_\mu], \tag{2.22}$$

however varying this with respect to $X^\mu$ gives,

$$\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi_\mu \bar{\psi}_\mu \nabla_\beta \varepsilon = -\frac{1}{2}\bar{\psi}_\mu \psi^\mu \bar{\chi}_\alpha \rho^\beta \rho^\alpha \nabla_\beta \varepsilon, \tag{2.23}$$

which can be cancelled with the addition of a third part of the action,

$$S_3 = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\ e\bar{\psi}_\mu \psi^\mu \bar{\chi}_\alpha \rho^\beta \rho^\alpha \chi_\beta. \tag{2.24}$$

The full locally supersymmetric action is then given by the sum of $S = S_1 + S_2 + S_3$, and it is invariant under,

$$\begin{aligned} \delta X^\mu &= \bar{\varepsilon}\psi^\mu, & \delta\psi^\mu &= \rho^\alpha \varepsilon(\partial_\alpha X^\mu - \bar{\psi}^\mu \chi_\alpha), \\ \delta e^a_\alpha &= 2\bar{\varepsilon}\rho^a \chi_\alpha, & \delta\chi_\alpha &= \nabla_\alpha \varepsilon. \end{aligned} \tag{2.25}$$

We have two further symmetries of of our $S$. The first is local Weyl symmetry given by,

$$\begin{aligned} \delta X^\mu &= 0, & \delta\psi^\mu &= -\frac{1}{2}\Lambda\psi^\mu, \\ \delta e^a_\alpha &= \Lambda e^a_\alpha, & \delta\chi_\alpha &= -\frac{1}{2}\Lambda\chi_\alpha, \end{aligned} \tag{2.26}$$

and the second is a local fermionic symmetry given by,

$$\delta\chi_\alpha = -\rho_\alpha \eta, \qquad \delta e^a_\alpha = \delta\psi^\mu = \delta X^\mu = 0, \tag{2.27}$$

where $\eta$ is a Majorana spinor. The proof of this symmetry requires the $D = 2$ identity $\rho^\alpha\rho_\beta\rho_\alpha = 0$. We can use the two $\varepsilon$ symmetries and $\eta$ symmetries to set all of $\chi_\alpha = 0$ which yields the globally supersymmetric model after applying the conformal gauge. The equations of motion of this global theory must then be supplemented by the equation of motion of $e^a_\alpha$ and $\chi_\alpha$ in the gauge $e^a_\alpha = \delta^a_\alpha$, $\chi_\alpha = 0$. These give the following equations,

$$J_\alpha \equiv -\frac{\pi}{2e}\frac{\delta S}{\delta\chi_\alpha} = -\frac{1}{2}\rho^\beta\rho_\alpha\psi^\mu\partial_\beta X_\mu = 0 \tag{2.28a}$$

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \tfrac{1}{2}\bar{\psi}^\mu\rho_{(\alpha}\partial_{\beta)}\psi_\mu - (\text{trace}) = 0. \tag{2.28b}$$

These will turn out to be the Supervirasoro constraints we need to quantise the theory - albeit with a loss of covariance.

### 2.1.2 Superspace and the Action

Now we shall write the action (2.1) in $\mathcal{N} = 1$ superspace which will have the bonus of making worldsheet supersymmetry manifest; in our case we will have a $\mathcal{N} = 2$ theory, but this can still be formulated in the $\mathcal{N} = 1$ superspace. For now, we introduce the superworldsheet coordinates $(\sigma^\alpha, \theta_A)$ where,

$$\theta_A = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix}, \tag{2.29}$$

which are anticommuting Grassmann numbers. The content of this section will follow the same kind of logic as in [19]. We note that the most general superfield in two dimensions is given by,

$$Y^\mu = X^\mu + \bar{\theta}\psi^\mu + \frac{1}{2}\bar{\theta}\theta B^\mu, \tag{2.30}$$

where $B^\mu$ is an auxiliary field that will allow us to make the supersymmetry manifest. We also note that since $\bar{\psi}\theta = \bar{\theta}\psi$, a term linear in $\theta$ is included in our $\bar{\theta}\psi$ above. The generators of worldsheet supersymmetry are given by,

$$Q_A = \bar{\partial}_A - (\rho^\alpha \theta)_A \partial_\alpha, \tag{2.31}$$

where acting on the superspace coordinates, the operator $\bar{\varepsilon} Q$ gives,

$$\begin{aligned} \delta\theta^A &= [\bar{\varepsilon}Q, \theta^A] = \varepsilon^A \\ \delta\sigma^\alpha &= [\bar{\varepsilon}Q, \sigma^\alpha] = -\bar{\varepsilon}\rho^\alpha\theta = \bar{\theta}\rho^\alpha\varepsilon, \end{aligned} \tag{2.32}$$

and hence supersymmetry transforms can be interpreted nicely as geometrical transforms of the superspace. The action of $Q_A$ on a superfield gives,

$$\delta Y^\mu = [\bar{\varepsilon}Q, Y^\mu] = \bar{\varepsilon}QY^\mu, \tag{2.33}$$

which implies the following,

$$\begin{aligned} \delta Y^\mu &= \bar{\varepsilon}^A Q_A Y^\mu \\ &= \bar{\varepsilon}^A Q_A X^\mu + \bar{\varepsilon}^A Q_A \bar{\theta}\psi^\mu + \frac{1}{2}\bar{\varepsilon}^A Q_A \bar{\theta}\theta B^\mu \\ &= -\bar{\varepsilon}^A (\rho^\alpha\theta)_A \partial_\alpha X^\mu + \bar{\varepsilon}^A \bar{\partial}_A (\bar{\theta}\psi^\mu) - \bar{\varepsilon}^A (\rho^\alpha\theta)_A \bar{\theta}\partial_\alpha \psi^\mu \\ &\quad + \frac{1}{2}\bar{\varepsilon}^A \bar{\partial}_A (\bar{\theta}\theta) B^\mu - \frac{1}{2}\bar{\varepsilon}^A (\rho^\alpha\theta)_A \bar{\theta}\theta\partial_\alpha B^\mu \\ &= -\bar{\varepsilon}^A (\rho^\alpha\theta)_a \partial_\alpha X^\mu + \bar{\varepsilon}\psi^\mu - \bar{\varepsilon}^A (\rho^\alpha\theta)_A \bar{\theta}\partial_\alpha \psi^\mu \\ &\quad + \frac{1}{2}\varepsilon\bar{\theta}B^\mu + \frac{1}{2}\varepsilon\theta B^\mu - \frac{1}{2}\varepsilon^A \rho^\alpha \theta_A \bar{\theta}\theta\partial_\alpha B^\mu \\ &= \bar{\theta}\rho^\alpha\varepsilon\partial_\alpha X^\mu + \bar{\theta}\varepsilon B^\mu + \bar{\varepsilon}\psi^\mu - \bar{\varepsilon}^A (\rho^\alpha \theta_A \bar{\theta}_B \partial_\alpha \psi^\mu_B) \\ &\qquad - \frac{1}{2}\varepsilon^A \rho^\alpha \cancelto{0}{\theta_A \bar{\theta}\theta}\partial_\alpha B^\mu \\ &= \bar{\varepsilon}\psi^\mu + \bar{\theta}(\rho^\alpha\partial_\alpha X^\mu + B^\mu)\varepsilon + \frac{1}{2}\bar{\theta}\theta(\bar{\varepsilon}\rho^\alpha\partial_\alpha\psi^\mu), \end{aligned}$$

where we have used $\theta_A\bar{\theta}_B = -\frac{1}{2}\delta_{AB}\bar{\theta}\theta$, $\bar{\psi}\theta = \bar{\theta}\psi$, $\bar{\theta}\theta = -\theta\bar{\theta}$ and $\bar{\varepsilon}\rho^\alpha\theta = -\bar{\theta}\rho^\alpha\varepsilon$. Hence we find that the supersymmetry transformations are,

$$\delta X^\mu = \bar{\varepsilon}\psi^\mu, \quad \delta\psi^\mu = \rho^\alpha\partial_\alpha X^\mu\varepsilon + B^\mu\varepsilon, \quad \delta B^\mu = \bar{\varepsilon}\rho^\alpha\partial_\alpha\psi^\mu, \tag{2.34}$$

which reduces to the usual supersymmetry transforms if we take $B_\mu = 0$, which will turn out to be the equation of motion for this field. Moving forward we can introduce the supercovariant derivative,

$$D_A = \bar{\partial}_A + (\rho^\alpha \theta)_A \partial_\alpha, \tag{2.35}$$

which obeys,

$$\{D_A, Q_B\} = 0. \tag{2.36}$$

Then the action that we need in $D = 2$ is given by,

$$S = \frac{i}{4\pi} \int \mathrm{d}^2\sigma \mathrm{d}^2\theta \; \bar{D}Y^\mu DY_\mu, \tag{2.37}$$

where we define the integration over the Grassmann variables in the following manner,

$$\int \mathrm{d}\theta(a + \theta b) = b, \qquad \int \mathrm{d}^2\theta \; \bar{\theta}\theta = 2i, \tag{2.38}$$

and so, where in $D = 4$ the superspace integral one picks out the $\theta\theta\bar{\theta}\bar{\theta}$ term [79], here the superspace integral shall pick out the $\theta\bar{\theta}$ terms of $S$. We also note that $S$ is manifestly supersymmetric since,

$$\delta S = \frac{i}{4\pi} \int \mathrm{d}^2\sigma \mathrm{d}^2\theta \; \bar{\varepsilon}Q(\bar{D}Y^\mu DY_\mu), \tag{2.39}$$

both terms from $Q$ give a total derivative, one in $\sigma^\alpha$ and one in $\theta_A$. The $\sigma^\alpha$ derivative can break supersymmetry, and that will be of interest later down the line. If one notes,

$$\begin{aligned} DY^\mu &= \psi^\mu + \theta B^\mu + \rho^\alpha \theta \partial_\alpha X^\mu - \frac{1}{2}\bar{\theta}\theta\rho^\alpha\partial_\alpha\psi^\mu \\ \bar{D}Y^\mu &= \bar{\psi}^\mu + B^\mu\bar{\theta} - \bar{\theta}\partial_\alpha X^\mu \rho^\alpha + \frac{1}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha, \end{aligned} \tag{2.40}$$

then upon performing the superspace integral of (2.37) one finds,

$$S = -\frac{1}{2\pi} \int \mathrm{d}^2\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - B_\mu B^\mu), \tag{2.41}$$

and so we can set $B_\mu = 0$; however in doing so we will lose the manifest supersymmetry.

To perform canonical quantisation we will need to find the constraints of the system such that we can eliminate any negative norm states. Since the RNS string has superconformal symmetry, we will have to find some supervirasoro constraints. As we have seen, the energy momentum tensor of the RNS string can be given by (2.28b),

$$T_{\alpha\beta} = \partial_\alpha X_\mu \partial_\beta X^\mu + \frac{1}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \frac{1}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - (\text{trace}). \tag{2.42}$$

The Noether current associated with the global supersymmetry is given by (2.28a),

$$J^\alpha_A = -\frac{1}{2}(\rho^\beta \rho^\alpha \psi_\mu)_A \partial_\beta X^\mu, \tag{2.43}$$

which satisfies $(\rho_\alpha)_{AB} J^\alpha_B = 0$ since $\rho^\alpha \rho_\beta \rho_\alpha = 0$. This equation is the analogue of the tracelessness of $T_{\alpha\beta}$ and is a consequence of local Super-Weyl invariance. Thus $J^\alpha_A$ has only two independent components $J_+$ and $J_-$. In terms of lightcone coordinates, the stress-energy tensor can be expressed as,

$$\begin{aligned} T_{++} &= \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2}\psi^\mu_+ \partial_+ \psi_{+\mu}, \\ T_{--} &= \partial_- X_\mu \partial_- X^\mu + \frac{i}{2}\psi^\mu_- \partial_- \psi_{-\mu}, \end{aligned} \tag{2.44}$$

and for $J_\pm$,

$$J_\pm = \psi^\mu_\pm \partial_\pm X^\mu. \tag{2.45}$$

We can show that the supercurrent $J^\alpha_A$ is conserved via the equations of motion,

$$\begin{aligned} \partial_\alpha J^\alpha &= -\frac{1}{2}\rho^\beta \rho^\alpha \partial_\alpha \psi_\mu \partial_\beta X^\mu - \frac{1}{2}\rho^\beta \rho^\alpha \psi_\mu \partial_\alpha \partial_\beta X^\mu \\ &\quad - \frac{1}{4}(\rho^\beta \rho^\alpha + \rho^\alpha \rho^\beta)\psi_\mu \partial_\alpha \partial_\beta X^\mu \\ &= -\frac{1}{2}\psi_\mu \partial^\alpha \partial_\alpha X^\mu = 0. \end{aligned} \tag{2.46}$$

This then leads to,

$$\partial_- J_+ = \partial_+ J_- = 0, \tag{2.47}$$

and the stress-energy tensor satisfies the analogous relations,

$$\partial_- T_{++} = \partial_+ T_{--} = 0. \tag{2.48}$$

The requirements of superconformal symmetry will actually lead to stronger constraints: the vanishing of $J$ and $T$. To quantise the theory, one can introduce the anticommutation relations for the worldsheet fermions[4],

$$\{\psi^{\mu}_{A}, \psi'^{\nu}_{B}\} = \pi\eta^{\mu\nu}\delta_{AB}\delta(\sigma - \sigma'), \tag{2.49}$$

and since $\eta^{00} = -1$ there are negative norm states. In bosonic theory, $T_{+-} = T_{-+} = 0$ follows from the Weyl invariance of the theory and $T_{++} = T_{--} = 0$ follows from the equations of motion; these are enough to eliminate ghosts in the bosonic case. In the RNS theory, the constraints we have are,

$$J_{+} = J_{-} = T_{++} = T_{--} = 0, \tag{2.50}$$

which follows from the discussion of local symmetry above. We now ought to consider the boundary conditions that we need to apply to the strings, to do so let us begin with,

$$S_F \sim \int \mathrm{d}^2\sigma(\psi_{-}\partial_{+}\psi_{-} + \psi_{+}\partial_{-}\psi_{+}), \tag{2.51}$$

whence the variation yields the boundary terms,

$$\delta S \sim \int \mathrm{d}\tau(\psi_{+}\delta\psi_{+} - \psi_{-}\delta\psi_{-})|_{\sigma=\pi} - (\psi_{+}\delta\psi_{+} - \psi_{-}\delta\psi_{-})|_{\sigma=0}, \tag{2.52}$$

which must also vanish. In the case of open strings, the variation must vanish at each end of the string independently and this is satisfied if,

$$\psi^{\mu}_{\pm} = \pm\psi^{\mu}_{\mp}, \tag{2.53}$$

at each end of the string. The overall sign is a matter of convention and we shall choose,

$$\psi^{\mu}_{+}|_{\sigma=0} = \psi^{\mu}_{-}|_{\sigma=0}, \tag{2.54}$$

[4]We follow the conventions set out in Becker, Becker and Schwarz [23] and so we introduce a spurious factor of $\pi$ in order to match their convention.

however, the sign at the other end of the string *does* matter. In the first case we have the so-called Ramond (R) boundary condition,

$$\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi}, \tag{2.55}$$

which, as we shall see later, gives rise to spacetime fermions. The mode expansion in this sector is given by,

$$\psi_\pm^\mu = \frac{1}{\sqrt{2}} \sum_n d_n^\mu e^{-in(\tau\pm\sigma)}, \tag{2.56}$$

where $n \in \mathbb{Z}$. The Majorana conditions forces these expansions to be real such that $d_{-n}^\mu = d_n^{\mu\dagger}$. We may also have the so-called Neveu-Schwarz (NS) boundary conditions, whereby,

$$\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi}, \tag{2.57}$$

which will give spacetime bosons. The NS mode expansion is then given by,

$$\psi_\pm = \frac{1}{\sqrt{2}} \sum_r b_r^\mu e^{-ir(\tau\pm\sigma)}, \tag{2.58}$$

where $r \in \mathbb{Z} + \frac{1}{2}$. In general we shall choose $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$. We can now turn to closed strings; there are two possible fermionic periodicity conditions,

$$\psi_\pm(\sigma) = \pm\psi_\pm(\sigma + \pi), \tag{2.59}$$

where the $+$ sign describes periodic R-strings and the $-$ sign describes antiperiodic NS-strings. We can impose R and NS periodicities on the left and right movers independently, and so for the right movers we have the following,

$$\psi_-^\mu = \frac{1}{\sqrt{2}} \sum_n d_n^\mu e^{-2in(\tau-\sigma)} \quad \text{or} \quad \psi_- = \frac{1}{\sqrt{2}} \sum_r b_r^\mu e^{-2ir(\tau-\sigma)} \tag{2.60}$$

while for the left movers we have,

$$\psi_+^\mu = \frac{1}{\sqrt{2}} \sum_n \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \quad \text{or} \quad \psi_+ = \frac{1}{\sqrt{2}} \sum_r \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}. \tag{2.61}$$

Thus, we have four distinct closed string sectors: the NS-NS and R-R sectors give spacetime bosons; the R-NS and NS-R sectors give spacetime fermions.

## 2.2 Canonical Quantisation

In what follows, we will quantise the expansion coefficients $\alpha$, $b$ and $d$; it is implicitly understood that for the closed string the following holds for the $\tilde{\alpha}$ etc. - that is, the right-moving fields. The commutation relations for the bosonic part of the theory are given by,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \tag{2.62}$$

and the fermionic modes obey the relations implied by (2.49),

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{m+n}, \qquad \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n}, \tag{2.63}$$

and hence we see the rise of ghost modes due to the 00 component of $\eta$. We can decouple these states by enforcing the superconformal symmetry. The ground state of the theory is defined by,

$$\begin{aligned} \alpha_m^\mu|0\rangle_R = d_m^\mu|0\rangle_R = 0, \qquad m > 0 \\ \alpha_m^\mu|0\rangle_{NS} = b_r^\mu|0\rangle_{NS} = 0, \quad r, m > 0. \end{aligned} \tag{2.64}$$

Now in the NS sector there is a unique ground state which has spin-0 in spacetime and since the oscillators transform as vectors, the states in the NS-sector represent spacetime bosons. However, in the R-sector the oscillator $d_0^\mu$ commutes with the number operator (defined below) $N$ and hence the R-sector ground state is degenerate. The $d_0^\mu$ modes satisfy,

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}, \tag{2.65}$$

which is essentially the Dirac algebra and hence the R-sector must furnish a representation of this algebra. Thus, we have a set of degenerate ground states $|a\rangle$, where $a$ is a spinor index such that,

$$d_0^\mu|a\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{ba}|b\rangle, \tag{2.66}$$

and so the R-sector ground state is a fermion. Once again, all the oscillators in the R-sector are spacetime vectors and so the R-sector gives us spacetime fermions. The Virasoro generators are given in a similar way to the bosonic string [21], that is,

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma \; e^{im\sigma} T_{++} = L_m^{(b)} + L_m^{(f)}, \tag{2.67}$$

for the open string. The bosonic contribution is given by,

$$L_m^{(b)} = \frac{1}{2}\sum_n : \alpha_{-n} \cdot \alpha_{m+n} :, \tag{2.68}$$

and for fermions in the NS sector we have,

$$L_m^{(f)} \frac{1}{2}\sum_r (r + \tfrac{m}{2}) : b_{-r} \cdot b_{m+r} :, \tag{2.69}$$

where :: represents normal ordering of the fields. The modes of the Supercurrent are then given by,

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma \; e^{ir\sigma} J_+ = \sum_n \alpha_{-n} \cdot b_{r+n}. \tag{2.70}$$

The $L_0$ operator is given by,

$$L_0 = \frac{\alpha_0^2}{2} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r} \cdot b_r = \frac{\alpha_0^2}{2} + N, \tag{2.71}$$

where $N$ is given by,

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r} \cdot b_r. \tag{2.72}$$

Now in the R-sector we have,

$$L_m^{(f)} = \frac{1}{2}\sum_n (n + \tfrac{m}{2}) : d_{-n} \cdot d_{m+n} :, \tag{2.73}$$

and for the supercurrent modes we have,

$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\sigma \; e^{im\sigma} J_+ = \sum_n \alpha_{-n} \cdot d_{m+n}, \tag{2.74}$$

where there is no $F_0$ ordering ambiguity. These modes then satisfy the Supervirasoro algebra. For the R-sector this is,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n}, \\
[L_m, F_n] &= (\tfrac{m}{2} - n)F_{m+n}, \\
\{F_m, F_n\} &= 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n},
\end{aligned} \tag{2.75}$$

and in the NS-sector the algebra is,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n}, \\
[L_m, G_r] &= (\tfrac{m}{2} - r)G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}(r^2 - \tfrac{1}{4})\delta_{m+n}.
\end{aligned} \tag{2.76}$$

The central extensions can be found in a similar way as in the bosonic case [19, 20, 21]. Now when we quantise the Ramond-Neveu-Schwarz string, we can only demand that positive Virasoro generators annihilate the physical states. In the NS-sector this gives,

$$G_r|\phi\rangle = 0 r > 0, \quad L_m|\phi\rangle = 0 m > 0, \quad (L_0 - a_{NS})|\phi\rangle = 0 \tag{2.77}$$

where the last statement implies $\alpha' M^2 = N - a_{NS}$ where $M$ is the mass of the state $|\phi\rangle$. In the R-sector these become,

$$\begin{aligned}
F_n|\phi\rangle &= 0 \qquad n \geq 0, \\
L_m|\phi\rangle &= 0 \qquad m > 0, \\
(L_0 - a_R)|\phi\rangle &= 0.
\end{aligned} \tag{2.78}$$

The $a_{NS}$ and $a_R$ are constants to account for normal ordering ambiguities, however $a_R = 0$ since there is no ambiguity in the R-sector. This can be deduced from the fact that $L_0 = F_0^2$ and the $F_0$ equation,

$$\begin{aligned}
(L_0 - a_R)^2|\phi\rangle &= 0 \\
(L_0^2 - 2a_R L_0 + a_R^2)|\phi\rangle = 0 \implies [a_R L_0 + a_R(L_0 - a_R)]|\phi\rangle &= 0 \\
\implies a_R^2 L_0|\phi\rangle = 0 \implies a_R = 0.&
\end{aligned} \tag{2.79}$$

Let us consider $F_0$ a little further for a moment, now since $\alpha_0^\mu = \frac{1}{2} l_s p^\mu$ and $d_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu$ we have [23],

$$\begin{aligned} F_0 &= \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot d_n = a_0 \cdot d_0 + \sum_{n=1}^{\infty} (\alpha_{-n} \cdot d_n + d_{-n} \cdot \alpha_n) \\ &= \frac{l_s p \cdot \Gamma}{2\sqrt{2}} + \sum_{n=1}^{\infty} (\alpha_{-n} \cdot d_n + d_{-n} \cdot \alpha_n) \end{aligned} \tag{2.80}$$

but $F_0|\phi\rangle = 0$ and hence,

$$\left( p \cdot \Gamma + \frac{2\sqrt{2}}{l_s} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot d_n + d_{-n} \cdot \alpha_n) \right) |\phi\rangle = 0. \tag{2.81}$$

This is the stringy generalisation of the Dirac equation and is called the *Dirac-Ramond equation.*

We can now perform the method in GSW [19] and BBS [23] for determining $a_{NS}$, $a_R$ and $D$. To begin with consider the NS-sector physical states that have the form $|\psi\rangle = G_{-1/2}|\chi\rangle$ where $|\chi\rangle$ satisfies,

$$G_{1/2}|\chi\rangle = G_{3/2}|\chi\rangle = (L_0 - a_{NS} + \tfrac{1}{2})|\chi\rangle = 0, \tag{2.82}$$

where the last term is just $(L_0 - a_{NS})|\psi\rangle = 0$. Now for $|\psi\rangle$ to be physical we ought to have $G_{1/2}|\psi\rangle = G_{3/2}|\psi\rangle = 0$. Thus,

$$G_{3/2}|\psi\rangle = G_{3/2}G_{-1/2}|\chi\rangle = (2L_0 - G_{-1/2}G_{1/2})|\chi\rangle = (2a_{NS} - 1)|\chi\rangle \stackrel{!}{=} 0 \tag{2.83}$$

and hence we have $a_{NS} = 1/2$. This gives a family of zero-norm spurious states such that $|\psi\rangle$ is just about physical and such states are orthogonal to all physical states including itself,

$$\langle\alpha|\psi\rangle = \langle\alpha|G_{-1/2}|\chi\rangle = \langle\chi|G_{1/2}|\alpha\rangle^* = 0. \tag{2.84}$$

Now let us consider a second class of these spurious states of the form,

$$|\psi\rangle = (G_{-3/2} + \lambda G_{-1/2}L_{-1})|\chi\rangle, \tag{2.85}$$

such that the following is satisfied,

$$G_{1/2}|\chi\rangle = G_{3/2}|\chi\rangle = (L_0+1)|\chi\rangle = 0 \tag{2.86}$$

where the last equation includes the fact that $a_{NS} = 1/2$. We can then use the Virasoro algebra to find,

$$\begin{aligned}
G_{1/2}|\psi\rangle &= G_{1/2}(G_{-3/2} + \lambda G_{-1/2}L_{-1})|\chi\rangle \\
&= (2L_{-1} + \lambda G_{1/2}G_{-1/2}L_{-1})|\chi\rangle \\
&= (2L_{-1} + 2\lambda L_0 L_{-1} - \lambda G_{-1/2}G_{1/2}L_{-1})|\chi\rangle \\
&= (2L_{-1} + 2\lambda L_{-1} + 2\lambda L_{-1}L_0 - \lambda G_{1/2}G_{1/2}L_{-1})|\chi\rangle \\
&= 2L_{-1}|\chi\rangle - \lambda G_{-1/2}G_{1/2}L_{-1}|\chi\rangle \\
&= 2L_{-1}|\chi\rangle - \lambda G_{-1/2}G_{-1/2}|\chi\rangle \\
&= (2-\lambda)L_{-1}|\chi\rangle
\end{aligned} \tag{2.87}$$

which gives us the condition that $\lambda = 2$. Now we can use the $G_{3/2}$ operator to find the value of $D$,

$$\begin{aligned}
G_{3/2}|\psi\rangle &= (2L_0 + D + \lambda G_{3/2}G_{-1/2}L_{-1})|\chi\rangle \\
&= (D - 2 + 2\lambda L_1 L_{-1} - \lambda G_{-1/2}G_{3/2}L_{-1})|\chi\rangle \\
&= (D - 2 - 4\lambda)|\chi\rangle,
\end{aligned} \tag{2.88}$$

and thus $D = 10$. We can also confirm that $D = 10$ in the R-sector; noting that $a_R = 0$ we consider the set of zero norm states such that, $|\psi\rangle = F_0 F_{-1}|\chi\rangle$ where the state satisfies,

$$F_1|\chi\rangle = (L_0+1)|\chi\rangle = 0. \tag{2.89}$$

The physical state $|\psi\rangle$ satisfies $F_0|\psi\rangle$ and it should be annihilated by $L_1$,

$$\begin{aligned}
L_1|\psi\rangle &= L_1 F_0 F_{-1}|\chi\rangle \\
&= (\tfrac{1}{2}F_1F_{-1} + F_0 L_1 F_{-1})|\chi\rangle \\
&= (L_0 + D/4 + F_0 L_1 F_{-1})|\chi\rangle \\
&= (L_0 + D/4 + 3/2 F_0 F_0)|\chi\rangle \\
&= \tfrac{1}{4}(10L_0 + D)|\chi\rangle = (D-10)|\chi\rangle,
\end{aligned} \tag{2.90}$$

and so $D = 10$ for this to be a spurious state and hence we have 10 dimensions in the R-sector also.

## 2.3 Light-Cone Quantisation and the Spectrum

As in the bosonic case [19, 21] it is possible to use the residual gauge symmetry after fixing the conformal gauge to impose the lightcone gauge,

$$X^+ = x^+ + p^+\tau, \tag{2.91}$$

for the RNS string. We also have some residual fermionic symmetry such that,

$$\psi^+ = 0. \tag{2.92}$$

In the lightcone gauge, the constraints we previously found imply that $X^-$ and $\psi^-$ are not independent and that the states are built out of transverse oscillators. In the following we shall identify the string spectrum using the lightcone gauge, one can rederive the dimension of the theory and the normal ordering constants, but that is not done here explicitly as we have already found them. This procedure is covered in Section 4.3 of Green, Schwarz and Witten Volume 1 [19].

We begin by looking at the open string spectrum, and to do so we note that in the NS-sector the mass formula is given by,

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n} \cdot \alpha^i_n + \sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r} \cdot b^i_r - \frac{1}{2}. \tag{2.93}$$

The ground state in this sector satisfies,

$$\alpha^i_n |0; k\rangle_{NS} = b^i_r |0; k\rangle_{NS} = 0 \; \forall \; n, r > 0 \tag{2.94}$$

as well as,

$$\alpha^\mu_0 |0; k\rangle_{NS} = \sqrt{2\alpha'} k^\mu |0; k\rangle_{NS}, \tag{2.95}$$

and so the NS-sector ground state is a scalar that has mass,

$$\alpha' M^2 = -\frac{1}{2}, \tag{2.96}$$

and thus it is a tachyon, but in the superstring this is an issue we can deal with as we shall see shortly. Now to get the first excited state, we act with the lowest frequency raising operator,

$$b^i_{-1/2}|0;k\rangle_{NS}. \tag{2.97}$$

Now, since this is in lightcone gauge the $i$ label only runs over $D-2=8$ transverse directions of oscillation and in the above case the operator is a vector operator since it is a spacetime vector. This operator acts on a bosonic scalar ground state and so produces a massless spacetime vector with eight degrees of freedom - the number required for a massless vector in $D=10$. We can thus rederive $a_{NS}$ by noting that the above vector must be a representation of $SO(8)$ and it must be massless and hence,

$$\alpha' M^2 = \frac{1}{2} - a_{NS} = 0. \tag{2.98}$$

We shall consider the R-sector in which the mass-shell condition gives,

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} \cdot d^i_n. \tag{2.99}$$

The ground state in the R-sector is a solution of the massless Dirac equation since there is not normal ordering constant and, since it is physical,

$$\alpha^i_n|0;k\rangle_R = d^i_n|0;k\rangle_R = 0 \;\forall\; n>0, \tag{2.100}$$

note that the solution will have a spinor index that is suppressed here. We know from our previous discussion that the R-sector ground state is not unique; it satisfies the $D=10$ Dirac algebra and thus the ground state will be a $spin(9,1)$ spinor. We can act with $d^\mu_0$ and change nothing, and since $d^\mu_0$ is a $D=10$ Dirac matrix, that has dimension $32\times 32$, the ground state must be a 32-component spinor. The Majorana condition is implicit here but the Weyl condition will give rise to two ground states corresponding to two different chiralities. The Majorana condition eliminates half of the 64 real degrees of freedom by forcing the spinor to be real, the Weyl condition then eliminates half of these again leaving 16 components. Finally, the Dirac equation eliminates half of these again leaving only eight fermionic degrees of freedom; thus the ground state is a *spin*(8) spinor. This is the minimum number of degrees of freedom we can consistently have - it also turns out to be the necessary number. Since $\alpha^i_{-n}$ and $d^i_{-n}$ are both spacetime vectors, and the ground state is a spacetime spinor, the R-sector generates fermions.

The spectrum we have just derived still has issues: the first of which is the presence of the tachyon, the second issue is the fact that spacetime supersymmetry is not satisfied. Supersymmetry ought to be unbroken since the system contains a massless gravitino - the quantum of the gauge field for local supersymmetry. We can fix this issue by applying the so-called *GSO Projection* and to do this we first have to define the *G-parity* operator. In the NS-sector it is given by,

$$G = (-1)^{F+1} = (-1)^{\sum_{r=\frac{1}{2}}^{\infty} b^i_{-r} \cdot b^i_r + 1}, \tag{2.101}$$

where $F$ essentially counts the worldsheet fermion number. Thus $G$ determines if a state has an even or odd number of worldsheet fermion excitations. In the R-sector we have,

$$G = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d^i_{-n} \cdot d^i_n}, \tag{2.102}$$

where,

$$(\Gamma_{11})^2 = 1, \qquad \{\Gamma_{11}, \Gamma^\mu\} = 0. \tag{2.103}$$

Now spinors that satisfy,

$$\Gamma_{11}\psi = \pm\psi, \tag{2.104}$$

have definite chirality and we can define the chirality projection operator via,

$$P_\pm = \frac{1}{2}(1 \pm \Gamma_{11}). \tag{2.105}$$

If a spinor has definite chirality, then as we have seen, it is a Weyl spinor. Applying both the Majorana and Weyl conditions is only possible in certain dimensions; fortuitously this occurs for $D = 2$ and $D = 10$. The GSO projection means that we only keep the states which have positive $G$-parity in the NS-sector and hence we must have,

$$(-1)^F = -1, \tag{2.106}$$

which corresponds to an odd number of $b$ oscillators. For the R-sector, the $G$ operator simply selects a ground state of definite chirality and the choice is a matter of convention. Acting $G$ on the NS ground state gives,

$$G|0\rangle_{NS} = -|0\rangle_{NS}, \tag{2.107}$$

thus the ground state, and hence the tachyon, is projected out! The first excited state in the NS-sector survives this projection and so this massless vector is now the ground state and is paired with the massless spinor in the R-sector. The GSO projection actually reintroduces supersymmetry and hence it is required for consistency[5]. One can also derive the GSO projection by demanding the modular invariance of one- and two-loop amplitudes.

In the closed string case, we have four possible combinations of left and right movers: R-R, R-NS, NS-R and NS-NS. By using the $G$-parity operator, we can get rid of the tachyon in the NS-sectors, and using the operator on the R-sector gets rid of the positive or negative chirality states depending on the ground state choice. Hence, we can obtain two theories based upon whether the $G$-parity of the left and right movers is the same or opposite. In the Type IIB theory, both R-sectors have the same chirality, which we take to be positive and hence the R-sector ground state is given by $|+\rangle_R$. As a result, the Type IIB massless spectrum is given by,

$$\begin{aligned} |+\rangle_R \otimes |+\rangle_R, \quad & \tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS}, \\ \tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R, \quad & \qquad |+\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}, \end{aligned} \tag{2.108}$$

and since $|+\rangle_R$ is an eight component spinor, each state has 64 degrees of freedom.

[5]One can check this by counting the degrees of freedom at each level in both sectors and comparing after the projection.

In the Type IIA theory the R-sectors have different chiralities and thus the massless states are,

$$\begin{aligned} |+\rangle_R \otimes |-\rangle_R, \quad \tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS}, \\ \tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |-\rangle_R, \qquad\quad |+\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}. \end{aligned} \tag{2.109}$$

Both of these theories contain two massless gravitinos and hence we have $\mathcal{N}=2$ supergravity multiplets. There are 64 massless states for each sector. For the NS-NS sector, the case is the same for Type IIA and IIB. There is a dilaton (1), an antisymmetric two-form gauge field (28) and a symmetric rank-two tensor, the graviton (35). In the NS-R and R-NS sectors we have a spin-3/2 gravitino (56) and spin-1/2 dilatino (8). In IIA, the gravitinos have opposite chirality. In the R-R sector we have bosons from tensoring two Majorana spinors. In IIA these have opposite chirality and one gets a vector gauge field (8) and a three-form gauge field (56). In IIB one obtains a scalar gauge field (1), a two-form gauge field (28), and a four-form gauge field with self-dual field strength (35).

## 2.4 SCFT and BRST

One can use the conformal gauge to treat the worldsheet theory as a conformal field theory, and this proves very useful so we shall repeat the same analysis here. The gauge fixed worldsheet action in $z$ is given by,

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z(2\partial X^\mu \bar{\partial} X_\mu + \tfrac{1}{2}\psi^\mu \bar{\partial}\psi_\mu + \tfrac{1}{2}\tilde{\psi}_\mu \partial \tilde{\psi}_\mu). \tag{2.110}$$

The holomorphic bosonic stress-energy tensor then takes the form,

$$T_B = -2\partial X_\mu \partial X^\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \tag{2.111}$$

which will have a central charge $c = 3D/2$. The $\psi^\mu$ is a conformal field with $h = \frac{1}{2}$ and the OPE,

$$\psi^\mu_z \psi^\nu_w \sim \frac{\eta^{\mu\nu}}{z-w}. \tag{2.112}$$

In the superconformal gauge we also have a $h = \frac{3}{2}$ conserved supercurrent whose holomorphic part is given by,

$$T_F = 2i\psi^\mu \partial X_\mu = \sum_{r=-\infty}^{\infty} \frac{G_r}{z^{r+\frac{3}{2}}}, \tag{2.113}$$

in the NS-sector. In the R-sector, we would replace $G_r$ with $F_n$. This current forms the supercurrent algebra via the OPE,

$$T_F(z)T_F(w) \sim \frac{D}{(z-w)^3} + \frac{2T_B(w)}{z-w}, \tag{2.114}$$

where $c = \frac{3}{2}\hat{c}$ such that $\hat{c} = D$ and hence $c = \frac{3}{2}D = 15$ where 10 comes from the bosonic degrees of freedom and 5 come from the fermions. If we introduce the Grassmann numbers and go into superspace, then we combine $T_F$ and $T_B$,

$$T(z,\theta) = T_F(z) + \theta T_B(z), \tag{2.115}$$

whose OPE is,

$$T_1 T_2 \sim \frac{\hat{c}}{4z_{12}^3} + \frac{3\theta_{12}T_2}{2z_{12}^2} + \frac{D_2 T_2}{2z_{12}} + \frac{\theta_{12}\partial_2 T_2}{z_{12}}, \tag{2.116}$$

where $z_{12} = z_1 - z_2 - \theta_1\theta_2$, $\theta_{12} = \theta_1 - \theta_2$ and,

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}. \tag{2.117}$$

This gives the entire superconformal algebra. The combinations $z_{12}$ and $\theta_{12}$ are invariant under the supersymmetry transforms $\delta\theta = \varepsilon$ and $\delta z = \theta\varepsilon$. Now given a superfield $\Phi$ with conformal dimension $h$ and $h + \frac{1}{2}$ the following holds,

$$T_1\Phi_2 \sim h\frac{\theta_{12}}{z_{12}^2}\Phi_2 + \frac{D_2\Phi_2}{2z_{12}} + \frac{\theta_{12}}{z_{12}}\partial_2\Phi_2. \tag{2.118}$$

We can now discuss the superconformal field theory that results from the Faddeev-Popov method and path integral quantisation, we will not discuss the actual path integral procedure here. In the bosonic case we introduced $b$ and $c$ ghosts to allow us to fix the diffeomorphism$\times$Weyl symmetry, but now we also need to gauge fix the supersymmetry. We can do this by introducing two bosonic ghosts $\beta$ and $\gamma$ which have $h = \frac{3}{2}$ and $-\frac{1}{2}$

respectively. They have the OPE,

$$\gamma(z)\beta(w) \sim \frac{1}{z-w}, \qquad \beta(z)\gamma(w) \sim -\frac{1}{z-w}. \tag{2.119}$$

The fully gauged fixed action then takes the form,

$$\begin{aligned} S = \frac{1}{2\pi}\int \mathrm{d}^2 z(2\partial X_\mu \bar{\partial} X^\mu + \tfrac{1}{2}\psi^\mu \bar{\partial}\psi_\mu + \tfrac{1}{2}\tilde{\psi}^\mu \partial \tilde{\psi}_\mu \\ + b\bar{\partial}c + \bar{b}\partial\bar{c} + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}). \end{aligned} \tag{2.120}$$

The $c$ and $\gamma$ ghosts have ghost number +1, the $b$ and $\beta$ ghosts have ghost number -1. They also contribute to the superconformal algebra,

$$\begin{aligned} T_B^g &= -2b\partial c + c\partial b - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\gamma\partial\beta, \\ T_F^g &= -2b\gamma + c\partial\beta + \frac{3}{2}\beta\partial c, \end{aligned} \tag{2.121}$$

which contribute $\hat{c} = -10$ which cancels the anomaly from the matter fields. The action (2.120) now has a global BRST symmetry, whose transformations are given by,

$$\begin{aligned} \delta X^\mu &= \eta(c\partial X^\mu - \tfrac{i}{2}\gamma\psi^\mu), \quad & \delta\psi^\mu &= \eta(c\partial\psi^\mu - \tfrac{1}{2}\psi^\mu\partial c + 2i\gamma\partial X^\mu) \\ \delta c &= \eta(c\partial c - \gamma^2), & \delta b &= \eta T_B \\ \delta\gamma &= \eta(c\partial\gamma - \tfrac{1}{2}\gamma\partial c), & \delta\beta &= \eta T_F. \end{aligned} \tag{2.122}$$

These transforms can be generated by the BRST charge,

$$Q_B = \frac{1}{2\pi i}\oint \mathrm{d}z(cT_B^{\mathrm{m}} + \gamma T_F^{\mathrm{m}} + bc\partial c - \tfrac{1}{2}c\gamma\partial\beta - \tfrac{3}{2}c\beta\partial\gamma - b\gamma^2), \tag{2.123}$$

where the m superscript denotes the matter part of the tensor. The $b$ and $\beta$ transformations correspond to simple equations,

$$\{Q_B, b_z\} = T_B(z), \qquad [Q_B, \beta_z] = T_F(z). \tag{2.124}$$

The BRST charge is in fact nilpotent such that $Q_B^2 = 0$ and we will show this is the case below. We can show an example of nilpotence in the above transforms where we note that $\eta$ is fermionic. Let us consider $\delta_2\delta_1 c$,

$$\begin{aligned}
\delta_2\delta_1 c &= \eta_1(\delta_2 c\partial c + c\partial\delta_2 c - \gamma\delta_2\gamma - \delta_2\gamma\gamma) \\
&= \eta_1(\eta_2(c\partial c - \gamma^2)\partial c + c\eta_2(\partial c\partial c + c\partial^2 c - 2\gamma\partial\gamma) \\
&\quad - \gamma\eta_2(c\partial\gamma - \tfrac{1}{2}\gamma\partial c) - \eta_2(c\partial\gamma - \tfrac{1}{2}\gamma\partial c)\gamma) \\
&= \eta_1\eta_2(c(\partial c)^2 - \gamma^2\partial c - c(\partial c)^2 - c^2\partial^2 c + 2c\gamma\partial\gamma \\
&\quad - \gamma c\partial\gamma + \tfrac{1}{2}\gamma^2\partial c - c\partial\gamma\gamma + \tfrac{1}{2}\gamma\partial c\gamma) \\
&= \eta_1\eta_2(-c^2\partial^2 c) = 0,
\end{aligned} \tag{2.125}$$

since $c^2 = 0$ due to its anticommutation relation, thus we have partially demonstrated the nilpotency of $Q_B$. It must be noted that $\gamma$ is a boson and so satisfies commutation relations. The BRST current is given by,

$$j_B = cT_B^{\mathrm{m}} + \gamma T_F^{\mathrm{m}} + bc\partial c + \frac{3}{4}\partial c\beta\gamma + \frac{1}{4}c\partial\beta\gamma - \frac{3}{4}c\beta\partial\gamma - b\gamma^2, \tag{2.126}$$

where as usual we can define (2.123) via,

$$Q_B = \frac{1}{2\pi i}\oint \mathrm{d}z\, j_B - \mathrm{d}\bar{z}\, \tilde{j}_B. \tag{2.127}$$

The current obeys the OPEs,

$$j_B(z)b(w) \sim \frac{T_B(w)}{z-w}, \qquad j_B(z)\beta(w) \sim \frac{T_F(w)}{z-w}, \tag{2.128}$$

so that the commutators of $Q_B$ in terms of the mode expansions are given by,

$$\{Q_B, b_n\} = L_n, \qquad [Q_B, \beta_r] = G_r, \tag{2.129}$$

where $G_r = F_n$ in the R-sector. The $Q_B$, expanded in modes, is given as,

$$\begin{aligned}
Q_B = \sum_m c_{-m}L_m^{\mathrm{m}} + \sum_r \gamma_{-r}G_r^{\mathrm{m}} - \sum_{m,n}(n-m) :b_{-n-m}c_m c_n: \\
+ \sum_{m,r}[\tfrac{1}{2}(2r-m) :\beta_{-m-r}c + m\gamma_r: - :b_{-m}\gamma_{m-r}\gamma_r:] + a^g c_0,
\end{aligned} \tag{2.130}$$

where in the R-sector $a^g = -5/8$ and in the NS-sector $a^g = -1/2$. We have defined :: as normal ordering of the creation and annihilation operators - we only use this notation here and it is not relevant to the rest of this thesis.

Since $Q_B$ is nilpotent we can again describe physical states in terms of the BRST cohomology classes. In the NS-sector the $\beta, \gamma$ system gives two-fold vacuum degeneracy due to the zero modes $b_0$ and $c_0$, just as in the bosonic string. The R-sector is more subtle; this is because there are extra $\beta_0$ and $\gamma_0$ zero modes which produce infinite degeneracy. Thus we can interpret the $\beta_0 - \gamma_0$ Fock space as containing infinitely many equivalent descriptions in different pictures. Each picture has an integer label and there are picture-changing operators that allow to map between these pictures. In formulating scattering amplitudes, there are restrictions of which pictures can be used - these operators are not discussed in this thesis as they are not required.

## 2.5 Vertex Operators and Bosonization

Let us consider the unit operator $|1\rangle$ and note that the fields remain holomorphic at the origin and they are single valued. The Laurent expansion of the fermionic fields is given by,

$$\psi^\mu = \sum_{r+\nu} \frac{\psi^\mu_r}{z^{r+\frac{1}{2}}}, \qquad \tilde{\psi}^\mu = \sum_{r+\nu} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+\frac{1}{2}}}, \tag{2.131}$$

where $\nu = 0$ for the NS-sector and $\frac{1}{2}$ for the R-sector. These expansions imply that the unit operator should be in the NS-sector; the conformal transform that takes the incoming string to $z = 0$ cancels the branch cut in the expansion for the NS-sector but introduces one for the R-sector. The holomorphicity of $\psi$ at the origin implies (using contour arguments) that the state corresponding to the unit operator ought to satisfy,

$$\psi^\mu_r|1\rangle = 0, \qquad r = \tfrac{1}{2}, \tfrac{3}{2}, ..., \tag{2.132}$$

and hence we have $|1\rangle = |0\rangle$. The $\psi\psi$ OPE is single-valued and hence all products of $\psi$ and its derivatives are in the NS-sector and the contour argument gives the map,

$$\psi^{\mu}_{-r} \longmapsto \frac{1}{(r-\frac{1}{2})!}\partial^{r-\frac{1}{2}}\psi^{\mu}(0) \tag{2.133}$$

and so we have a one-to-one map from products to the NS-states. We can find the analogue of the Noether relation between the superconformal variation of an NS operator and the OPE, which is,

$$\delta_{\eta}\mathcal{A}(z,\bar{z}) = -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}[\partial^{n}\eta G_{n-1/2} + (\partial^{n}\eta)\tilde{G}_{n-1/2}]\mathcal{A}(z,\bar{z}). \tag{2.134}$$

The R-sector is more complicated since it does contain a branch cut; in order to make sense of this we will find that in $D=2$ field theory the R-sector vertex operators can be related to bosonic operators which do not have this branch cut. Let $H(z)$ be the holomorphic part of some scalar field such that its OPE is given by,

$$H(z)H(0) \sim -\ln z. \tag{2.135}$$

Then consider the bosonic operators of the form $e^{\pm iH_z}$, these have the following OPEs,

$$e^{iH_z}e^{-iH_0} \sim \frac{1}{z}, \quad e^{iH_z}e^{iH_0} \sim \mathcal{O}(z), \quad e^{-iH_z}e^{-iH_0} \sim \mathcal{O}(z). \tag{2.136}$$

The poles and zeroes of the OPE along with the smoothness at infinity will determine the expectation values of such operators on the sphere up to some normalisation which we can set as,

$$\left\langle \prod_{j} e^{i\epsilon_j H_j}\right\rangle_{S^2} = \prod_{i<j} z_{ij}^{\epsilon_i\epsilon_j}, \qquad \sum_{i}\epsilon_i = 0, \tag{2.137}$$

where $\epsilon_i = \pm 1$ in the example, but the result does hold more generally. Now consider the CFT of $\psi_z^{1,2}$ and form the complex fields,

$$\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2), \qquad \bar{\psi} = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2), \tag{2.138}$$

which have the OPEs,

$$\begin{aligned} \psi(z)\bar{\psi}(0) &\sim \frac{1}{z} \\ \psi(z)\bar{\psi}(0) &\sim \mathcal{O}(z) \\ \bar{\psi}(z)\bar{\psi}(0) &\sim \mathcal{O}(z). \end{aligned} \tag{2.139}$$

These OPEs are identical to the scalar operators, and hence the expectation value of these fields on the sphere will be the same and so we can make the identification,

$$\psi(z) \cong e^{H_z}, \qquad \bar{\psi}(z) \cong e^{-iH_z}. \tag{2.140}$$

We can construct general operators out of these basic building blocks, so this identification will follow for all operators. To show that these are the same CFT we shall need to show that the stress-energy tensor of each theory is the same. Hence, consider,

$$\begin{aligned} e^{iH_z}e^{-iH_z} &= \frac{1}{2z} + i\partial H_0 + 2zT_B^H + \mathcal{O}(z^2) \\ \psi(z)\bar{\psi}(-z) &= \frac{1}{2z} + \psi_0\bar{\psi}_0 + 2zT_B^\psi + \mathcal{O}(z^2). \end{aligned} \tag{2.141}$$

As a result, for equivalence of the theories we shall require,

$$\psi\bar{\psi} \cong i\partial H, \qquad T_B^\psi \cong T_B^H. \tag{2.142}$$

So in the operator language, let us define the identification,

$$\psi(z) \cong \vdots e^{iH_z} \vdots, \tag{2.143}$$

and then from the Campbell-Baker-Hausdorf identity we have for equal times ($|z| = |z'|$),

$$\vdots e^{iH_z} \vdots\vdots e^{iH'_z} \vdots = \exp(-[H, H'])\vdots e^{iH'_z} \vdots\vdots e^{iH_z} \vdots = -\vdots e^{iH'_z} \vdots\vdots e^{iH_z} \vdots, \tag{2.144}$$

and so the bosonized operators anticommute as we would expect from the fermions. Let us now consider,

$$H(z)\vdots e^{iH_z} \vdots = \vdots e^{iH'_z} \vdots(H(z) + iH, H')\vdots e^{iH'_z} \vdots(H(z) - \pi\ \mathrm{sign}[\sigma - \sigma']), \tag{2.145}$$

and thus the fermion field produces a kink in the boson field. This method outlined above is known as *bosonization*.

We can extend this to our R-sector issue. Once we bring two fermions together to form a complex pair, we can have the periodicity condition,

$$\psi(w+2\pi) = e^{2\pi i\nu}\psi(w), \tag{2.146}$$

for $\nu \in \mathbb{R}$, but in $D = 10$ we will take $\nu = 0, \frac{1}{2}$. The expressions for the expansion are the same as before and the algebra is,

$$\{\psi_r, \bar{\psi}_s\} = \delta_{r+s}. \tag{2.147}$$

Now let us define a state $|0\rangle_\nu$ such that,

$$\psi_{n+\nu}|0\rangle_\nu = \bar{\psi}_{n+1-\nu}|0\rangle_\nu = 0, \tag{2.148}$$

for $n = 0, 1, ....$ The first nonzero term in the Laurent expansion will have $r = \nu - 1$ and $s = -\nu$ and so for some operator $\mathcal{A}_\nu$ the OPE will be,

$$\psi_z\mathcal{A}_{\nu 0} \sim \mathcal{O}(z^{-\nu+\frac{1}{2}}), \qquad \bar{\psi}_z\mathcal{A}_{\nu 0} \sim \mathcal{O}(z^{\nu-\frac{1}{2}}). \tag{2.149}$$

The condition in (2.148) uniquely selects $|0\rangle_\nu$ and so the OPEs above determine the bosonic equivalent, $\exp[i(\nu+\frac{1}{2})H] \cong \mathcal{A}_\nu$. The $\psi$ periodicity condition is the same for $\nu$ and $\nu+1$ but the reference states are not, it is only the ground state for $0 \leq \nu < 1$ and as we vary $\nu$ the $|0\rangle_\nu$ reference state changes continuously and we get back to the original one at $\nu+1$ and it becomes an excited state,

$$|0\rangle_{\nu+1} = \bar{\psi}_{-\nu}|0\rangle_\nu. \tag{2.150}$$

This is *spectral flow*. For R, $\nu = 0$ and hence the two degenerate ground states are,

$$|s\rangle \cong e^{isH}, \qquad s = \pm\frac{1}{2}. \tag{2.151}$$

For the $D = 10$ superstring we need five bosons $H^a$ such that,

$$\begin{aligned} \frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1) &\cong e^{\pm iH^0} \\ \frac{1}{\sqrt{2}}(\psi^{2a} \pm i\psi^{2a+1}) &\cong e^{\pm iH^a} \end{aligned} \tag{2.152}$$

for $a = 1, ..., 4$. Then the vertex operator $\Theta_s$ for an R state $|s\rangle$ is given by,

$$\Theta_s \cong \exp[i \textstyle\sum_a s_a H^a], \tag{2.153}$$

which is called the spin field.

We can get the $bc$ CFT by renaming $\psi \to b$ and $\bar{\psi} \to c$ and modifying the stress energy tensor,

$$T_B^{(\lambda)} = T_B^{(1/2)} - (\lambda - \tfrac{1}{2})\partial(bc), \tag{2.154}$$

and our previous equivalence gives the bosonic operator,

$$T_B^{(\lambda)} \cong T_B^H - i(\lambda - \tfrac{1}{2})\partial^2 H. \tag{2.155}$$

This is the linear dilaton CFT with $V = -i(\lambda - \frac{1}{2})$. Thus, the linear dilaton CFT and the $bc$ CFT are equivalent,

$$b \cong e^{iH}, \qquad c \cong e^{-iH}. \tag{2.156}$$

## 2.6 Nilpotency of $Q_B$

In this section we shall demonstrate the nilpotency of $Q_B$ - the BRST operator. First, we shall need the result of $[L_m, \Phi_n]$, where $\Phi$ is a conformal field with dimension $h$ with the expansion,

$$\Phi(z) = \sum_n \frac{\Phi_n}{z^{n+h}}. \tag{2.157}$$

This expansion then implies,

$$\frac{1}{2\pi i}\oint \mathrm{d}z\ \Phi(z) z^{n+h-1} = \Phi_n \implies \frac{1}{2\pi i}\oint \mathrm{d}z\ \partial\Phi(z) z^{n+h} = -(n+h)\Phi_n. \tag{2.158}$$

We also note that,

$$L_m = \frac{1}{2\pi i}\oint \mathrm{d}w\ T_w w^{m+1} \tag{2.159}$$

and thus,

$$[L_m, \Phi_n] = \frac{1}{2\pi i}\oint_C \mathrm{d}z\ z^{n+h-1}\frac{1}{2\pi i}\oint \mathrm{d}w\ T_w \Phi_z w^{m+1}, \tag{2.160}$$

where $C$ is contour around both $z$ and $w$. Using the $T\Phi$ OPE,

$$T_z\Phi_w = \frac{h}{(z-w)^2}\Phi_w + \frac{\partial\Phi_w}{z-w}, \tag{2.161}$$

we have,

$$\begin{aligned} [L_m, \Phi_n] &= \frac{1}{2\pi i}\oint \mathrm{d}z\ z^{n+h-1}[h(m+1)z^m\Phi_z + z^{m+1}\partial\Phi_z] \\ &= -(n+m(1-h))\Phi_{n+m}. \end{aligned} \tag{2.162}$$

Now we know that $h_b = 2$ and $h_\beta = \frac{1}{2}$ and so we have the relations,

$$[L_m, b_n] = (m-n)b_n, \qquad [L_m, \beta_r] = (\tfrac{m}{2} - r)\beta_r, \tag{2.163}$$

and we also know how these transform under the BRST charge,

$$\{Q_B, b_z\} = T_B(z), \qquad [Q_B, \beta_z] = T_F(z) \tag{2.164}$$

and thence,

$$\{Q_B, b_n\} = L_n - a_{NS}\delta_n, \qquad [Q_B, \beta_r] = G_r. \tag{2.165}$$

Thus if we also include,

$$[L_m, L_n] = (m-n)(L_{m+n} - a_{NS}\delta_{m+n}), \quad [L_m, G_r] = (\tfrac{m}{2} - r)G_{m+r}, \tag{2.166}$$

then we have a complete superextended superalgebra. Now from the graded Jacobi identity we have,

$$\begin{aligned} \{[Q_B, L_m], b_n\} &= \{[L_m, b_n], Q_B\} + [\{b_n, Q_B\}, L_m] \\ &= \{(m-n)b_{n+m}, Q_B\} + [L_n, L_m] \\ &= 0, \end{aligned} \tag{2.167}$$

due to the (anti)commutation relations. The same is true for,

$$\{[Q_B, L_m], \beta_r\} = \{[L_m, \beta_r], Q_B\} - [\{\beta_r, Q_B\}, L_m] = 0. \tag{2.168}$$

Consider also,

$$[\{Q_B, G_r\}, b_m] = \{[G_r, b_m], Q_B\} - \{[b_m, Q_B], G_r\} = 0, \tag{2.169}$$

and the same holds for,

$$[\{Q_B, G_r\}, \beta_s] = 0, \tag{2.170}$$

note that the above only holds if $\hat{c} = 0$, otherwise there would be central extensions. Now if $[Q_B, L_m] \neq 0$ then it cannot contain any $c$ or $\gamma$ ghosts otherwise the above relations would be violated, however we know that such a commutator has a positive ghost number. Thus, we must have $[Q_B, L_m] = 0$. This argument applies to all of the $Q_B$ relations above. Thus it follows that,

$$\begin{aligned} [Q_B^2, b_n] &= \tfrac{1}{2}[\{Q_B, Q_B\}, b_n] = \tfrac{1}{2}[\{Q_B, b_n\}, Q_B] + \tfrac{1}{2}[\{b_n, Q_B\}, Q_B] \\ &= [L_n, Q_B] = 0, \end{aligned} \tag{2.171}$$

and

$$\begin{aligned} [Q_B^2, \beta_r] &= \tfrac{1}{2}[Q_B, Q_B, \beta_r] = \tfrac{1}{2}\{[Q_B, \beta_r], Q_B\} + \tfrac{1}{2}[\{\beta_r, Q_B\}, Q_B] \\ &= [G_r, Q_B] = 0. \end{aligned} \tag{2.172}$$

If $Q_B^2$ is non-zero then it cannot contain any $\gamma$ or $c$ ghosts but it does so it must be zero. Hence $Q_B^2 = 0$ and the BRST charge is nilpotent.

In this chapter we briefly reviewed the worldsheet formulation of string theory, based on the RNS formalism. We have discussed the classical worldsheet theory and its supersymmetry as well as various quantisation procedures - all of which break some symmetry of the theory preventing covariant quantisation. We also discussed the idea of Superconformal field theories and BRST quantisation.

# Chapter 3

# Type I and II Strings in Spacetime

We now turn to discussing superstring theory from a spacetime point of view - this will make supersymmetry manifest at the cost of making quantisation far more difficult. In order to quantise this theory in the spacetime framework we shall require the pure spinor formalism, something we shall come onto in the coming chapters.

## 3.1 The Superparticle

The superparticle is generally considered as the worldline of a point particle moving through spacetime that has supersymmetry - in this section we briefly discuss the superparticle before moving on to the superstring. We can describe a particle of mass $m$ with the action

$$S = \frac{1}{2}\int \mathrm{d}\tau(e^{-1}\dot{x}^2 - em^2), \tag{3.1}$$

where $e$ is the einbein. Since we have no mass term in string theory, we shall consider the massless extension of (3.1) to the supersymmetric case. Not only is (3.1) reparameterization invariant under $\tau \to f(\tau)$ it is also Poincaré invariant. To add supersymmetry we previously introduced fermionic coordinates and if there are $\mathcal{N}$ supersymmetries we added $\mathcal{N}$ anticommuting $\theta^{Aa}$, where $A = 1, ..., \mathcal{N}$ and $a$ is a spacetime spinor index. We can then introduce supersymmetry by requiring our action to be invariant under the following transformations,

$$\begin{aligned} &\delta\theta^A = \varepsilon^A, \quad \delta x^\mu = \bar{\varepsilon}^A \cdot \Gamma^\mu \theta^A \\ &\delta\bar{\theta}^A = \bar{\varepsilon}^A, \qquad \delta e = 0, \end{aligned} \tag{3.2}$$

where $\varepsilon^A$ is a spinor. These and any following formulas are written in a way that is suitable for the Majorana-Weyl condition that we shall have to impose. We may build many actions out of the supersymmetric invariant combinations,

$$\dot{x}^\mu - \bar{\theta}^A \Gamma^\mu \dot{\theta}^A, \qquad \dot{\theta}^{Aa}, \tag{3.3}$$

and the most simple that allows both $\theta$ and $x$ to be dynamic is,

$$S = \frac{1}{2} \int \mathrm{d}\tau \; e^{-1} (\dot{x}^\mu - \bar{\theta}^A \Gamma^\mu \dot{\theta}^A)^2. \tag{3.4}$$

This is clearly Lorentz invariant, but let us quickly check its supersymmetric invariance; this amounts to confirming the claim that (3.3) is supersymmetric,

$$\begin{aligned} \delta(\dot{x}^\mu - \bar{\theta}^A \Gamma^\mu \dot{\theta}^A) &= \tfrac{\mathrm{d}}{\mathrm{d}\tau}(\delta x^\mu) - \delta\bar{\theta}^A \Gamma^\mu \dot{\theta}^A - \cancelto{0}{\bar{\theta}^A \Gamma^\mu \delta\dot{\theta}^A} \\ &= \bar{\varepsilon}^A \Gamma^\mu \dot{\theta}^A - \bar{\varepsilon}^A \Gamma^\mu \dot{\theta}^A = 0, \end{aligned} \tag{3.5}$$

and thus the action in (3.4) is supersymmetric. Its equations of motion are,

$$p^2 = 0, \qquad \dot{p}^\mu = 0, \qquad \Gamma \cdot p\theta = 0, \tag{3.6}$$

where,

$$p^\mu = \dot{x}^\mu - \bar{\theta}^A \Gamma^\mu \dot{\theta}^A. \tag{3.7}$$

Now we note that $(\Gamma \cdot p)^2 = -p^2 = 0$ and thus $\Gamma \cdot p$ has half the maximum rank; since $\theta$ is always multiplied into this combination half of its components will decouple from the theory. This is the result of a new fermionic symmetry called *Kappa-Symmetry*. Let

$\kappa^{Aa}$ be a spinor that is dependent on $\tau$ and consider the transformations given by,

$$\delta\theta^A = \Gamma \cdot p\kappa^A, \qquad \delta x^\mu = \bar{\theta}^A \Gamma^\mu \delta\theta^A, \qquad \delta e = 4e\dot{\bar{\theta}}^A \kappa^A. \tag{3.8}$$

The action in (3.4) is actually invariant under these transformations. We begin the proof of this by noting,

$$\begin{aligned} \delta p^\mu = 2\dot{\bar{\theta}}^A \Gamma^\mu \delta\theta^A \implies \delta p^2 &= 4\dot{\bar{\theta}}^A \Gamma \cdot p \delta\theta^A \\ &= 4p^2 \dot{\bar{\theta}}^A \kappa^A, \end{aligned} \tag{3.9}$$

and so $e^{-1}p^2$ will be invariant if,

$$\delta e^{-1} = -4e^{-1}\dot{\bar{\theta}}^A \kappa^A, \tag{3.10}$$

which is the same transformation as in (3.8). This kappa symmetry is not the usual supersymmetry on the worldline or spacetime; it contains no worldline spinors at all. To learn more about this symmetry, let us consider the commutator of two transforms to assess the algebra,

$$\begin{aligned} [\delta_1, \delta_2]\theta^A &= \Gamma^\mu \kappa_2^A \delta_1 p_\mu - \Gamma^\mu \kappa_1^A \delta_2 p_\mu \\ &= -2\Gamma_\mu \kappa_2^A \dot{\bar{\theta}}^B \Gamma^\mu \Gamma \cdot p \kappa_1^B + (1 \leftrightarrow 2) \\ &= (2\Gamma_\mu \kappa_2^A \dot{\bar{\theta}}^B \Gamma \cdot p \Gamma^\mu \kappa_1^B + 4\Gamma \cdot p \kappa_2^A \dot{\bar{\theta}}^B \kappa_1^B) - (1 \leftrightarrow 2). \end{aligned} \tag{3.11}$$

To close this algebra we must use the equations of motion, where $\Gamma \cdot p\dot{\theta} = 0$ sets the last term in the last line above to zero. Thus we are left with something of the form,

$$[\delta_1, \delta_2]\theta^A = \Gamma \cdot p\kappa^A + ..., \tag{3.12}$$

where the ... represents the equation of motion terms. For the above to be true, we must set,

$$\kappa^A = 4\kappa_2^A \dot{\bar{\theta}}^B \kappa_1^B - (1 \leftrightarrow 2). \tag{3.13}$$

Hence the commutator of two kappa transforms yields another kappa transform on-shell; there is no on-shell conserved kappa charge. The structure constants are not constant, but coordinate invariant as in supergravity theories. We have one more symmetry of the action in (3.4); this is a bosonic symmetry parameterised by $\lambda(\tau)$,

$$\delta\dot{\theta}^A = \lambda\theta^A, \quad \delta x^\mu = \bar{\theta}^A\Gamma^\mu\delta\theta^A, \qquad \delta e = 0, \tag{3.14}$$

which is very similar to the $\varepsilon$ reparametrisation of the $\theta$ coordinates and it has no additional implications.

The quantisation of the superparticle is nontrivial due to the constraint equation $\pi_\theta^A = \Gamma \cdot \pi_x\theta^A$, where $\pi_\theta$ and $\pi_x$ are the conjugate momenta. The same is true for the superstring where Dirac brackets become unwieldly; we will use the lightcone gauge to get around this.

## 3.2 The String Action and its Symmetries

Given the results of the bosonic string [21] and the above superparticle discussion, a reasonable guess for the superstring action would be given by,

$$S_1 = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma \ \sqrt{-h}h^{\alpha\beta}\Pi_\alpha \cdot \Pi_\beta, \tag{3.15}$$

where $\Pi_\alpha^\mu = \partial_\alpha X^\mu - \bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A$. This action has the usual symmetries we would *a priori* expect but it does not posses the kappa invariance we saw in the superparticle case and so the $\theta^A$ will have twice as many degrees of freedom we would like them to have. Also the action in (3.15) gives intractable equations of motion. To solve the first issue we must take $\mathcal{N} \leq 2$, and hence we formulate the kappa invariant action for $\mathcal{N} = 2$ and set $\theta^A = 0$ for the $\mathcal{N} = 0, 1$ cases. The extra term we require to make this action kappa invariant is,

$$S_2 = -\frac{1}{\pi}\int \mathrm{d}^2\sigma \ \epsilon^{\alpha\beta}\{\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial\theta^1 - \bar{\theta}^2\Gamma^\mu\partial_\beta\theta^2) + \bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\}, \tag{3.16}$$

which does not affect $T_{\alpha\beta}$. This term of course has local reparameterization and global Lorentz symmetry. We need to do a little work in order to check that is it $\mathcal{N} = 2$

supersymmetric. To check this we shall consider,

$$\delta\theta^A = \varepsilon^A, \qquad \delta X^\mu = \bar{\varepsilon}^A \Gamma^\mu \theta^A. \tag{3.17}$$

Upon variation, many terms in (3.16) cancel and terms such as,

$$\epsilon^{\alpha\beta}\partial_\alpha X^\mu \bar{\varepsilon}^1 \Gamma_\mu \partial_\beta \theta^1, \tag{3.18}$$

are total derivatives. We are only left with,

$$A = \epsilon^{\alpha\beta}\bar{\varepsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 \bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1, \tag{3.19}$$

and similarly for $\varepsilon^2$ and $\theta^2$. Dropping the superscripts and writing this in components we have,

$$A = \bar{\varepsilon}\Gamma^\mu \dot{\theta}\bar{\theta}\Gamma_\mu \theta' - \bar{\varepsilon}\Gamma^\mu \theta' \bar{\theta}\Gamma_\mu \dot{\theta} = A_1 + A_2, \tag{3.20}$$

where,

$$\begin{aligned} A_1 &= \tfrac{2}{3}[\bar{\varepsilon}\Gamma^\mu \dot{\theta}\bar{\theta}\Gamma_\mu \theta' + \bar{\varepsilon}\Gamma^\mu \theta' \dot{\bar{\theta}}\Gamma_\mu \theta + \bar{\varepsilon}\Gamma^\mu \theta \bar{\theta}' \Gamma_\mu \dot{\theta}] \\ A_2 &= \tfrac{1}{3}[\bar{\varepsilon}\Gamma^\mu \dot{\theta}\bar{\theta}\Gamma_\mu \theta' + \bar{\varepsilon}\Gamma^\mu \theta' \dot{\bar{\theta}}\Gamma_\mu \theta - 2\bar{\varepsilon}\Gamma^\mu \theta \bar{\theta}' \Gamma_\mu \dot{\theta}] \\ &= \tfrac{1}{3}\partial_\tau[\bar{\varepsilon}\Gamma^\mu \theta\bar{\theta}\Gamma_\mu \theta'] - \tfrac{1}{3}\partial_\sigma[\bar{\varepsilon}\Gamma^\mu \theta\bar{\theta}\Gamma_\mu \dot{\theta}], \end{aligned} \tag{3.21}$$

thus $A_2$ is a total derivative. We are then left with $A_1$ only, which can be written as,

$$A_1 = 2\bar{\varepsilon}\Gamma_\mu \psi_{[1}\bar{\psi}_2 \Gamma^\mu \psi_{3]}, \tag{3.22}$$

where $(\psi_1, \psi_2, \psi_3) = (\theta, \theta', \dot{\theta})$ and hence the spinors are antisymmetric. This is the exact structure in Yang-Mills[1] and it vanishes under certain circumstances. $S_2$ will only be supersymmetric if,

1. $D = 3$, $\theta$ is Majorana,
2. $D = 4$, $\theta$ is Majorana or Weyl,
3. $D = 6$, $\theta$ is Weyl,

[1]See Appendix 4A of GSW Volume 1 [19].

4. $D = 10$, $\theta$ is Majorana-Weyl,

and so the superstring can only classically exist with these conditions[2]. Quantum considerations, similar to those in previous sections, will then select $D = 10$ uniquely.

We now need to establish the kappa symmetry in the combination $S_1 + S_2$. The $\kappa$ parameters will now carry three indices $\kappa^{A\alpha a}$. In two dimensions, the vector representation ($\alpha$) is reducible and we can use projection operators to decompose such vectors into self-dual and anti-self-dual parts,

$$P_{\pm}^{\alpha\beta} = \tfrac{1}{2}\left(h^{\alpha\beta} \pm \epsilon^{\alpha\beta}/\sqrt{h}\right), \tag{3.23}$$

which satisfy,

$$\begin{aligned} P_{\pm}^{\alpha\beta} h_{\beta\gamma} P_{\pm}^{\gamma\delta} &= P_{\pm}^{\alpha\delta} \\ P^{\alpha\beta} h_{\beta\gamma} P_{\mp}^{\gamma\delta} &= 0. \end{aligned} \tag{3.24}$$

We then restrict $\kappa^A$ to be anti-self-dual for $A = 1$ and self-dual for $A = 2$ thence,

$$\kappa^{1\alpha} = P_{-}^{\alpha\beta}\kappa_{\beta}^{1}, \qquad \kappa^{2\alpha} = P_{+}^{\alpha\beta}\kappa_{\beta}^{2}, \tag{3.25}$$

and we shall see that $A = 1$ describes right moving modes and symmetries whereas $A = 2$ describes the left moving equivalents. Now suppose,

$$\delta\theta^A = 2\Gamma \cdot \Pi_\alpha \kappa^{A\alpha}, \qquad \delta X^\mu = \bar{\theta}^A \Gamma^\mu \delta\theta^A, \tag{3.26}$$

where we still need to find $\delta h_{\alpha\beta}$. The variation of the Lagrangian, $\mathcal{L}_1$, gives,

$$\delta\mathcal{L}_1 = -\sqrt{-h}h^{\alpha\beta}\Pi_\alpha \cdot \delta\Pi_\beta - \tfrac{1}{2}\delta(\sqrt{-h}h^{\alpha\beta})\Pi_\alpha \cdot \Pi_\beta, \tag{3.27}$$

where,

$$\delta\Pi_\beta^\mu = 2\partial_\beta \bar{\theta}^A \Gamma^\mu \delta\theta^A. \tag{3.28}$$

[2]Some of the equations need to be adjusted if we do not have $\theta$ Majorana-Weyl.

We can rewrite $\mathcal{L}_2$ as,

$$\mathcal{L}_2 = -\epsilon^{\alpha\beta}\Pi^\mu_\alpha(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + \mathcal{O}(\theta^4). \tag{3.29}$$

Now if we vary the $\theta$ in the $\Pi\theta^2$ terms of $\mathcal{L}_2$ and combine it with the first term of the $\mathcal{L}_1$ variation it gives a term that can be cancelled if,

$$\delta_\kappa(\sqrt{-h}h^{\alpha\beta}) = -16\sqrt{-h}(P_-^{\alpha\gamma}\bar{\kappa}^{1\beta}\partial_\gamma\theta^1 + P_+^{\alpha\gamma}\bar{\kappa}^{2\beta}\partial_\gamma\theta^2). \tag{3.30}$$

To complete the $\kappa-$symmetry proof we still need to show that the $\delta\Pi$ variation in (3.29) and the variation of the $\theta^4$ piece combine to give zero. If we perform the variations, add them together, and perform the obvious cancellations we are left,

$$2\bar{\theta}^1\Gamma_\mu\theta^{1'}\dot{\bar{\theta}}^1\Gamma^\mu\delta\theta^1 - 2\bar{\theta}^1\Gamma_\mu\dot{\theta}^1\bar{\theta}^{1'}\Gamma^\mu\delta\theta^1 - 2\bar{\theta}^1\Gamma_\mu\delta\theta^1\dot{\bar{\theta}}^1\Gamma^\mu\theta^{1'} + ..., \tag{3.31}$$

where ... corresponds to similar terms with $\theta^2$. This only cancels for the cases listed above, so local $\kappa$ symmetry can only be implemented if we can implement global $\varepsilon$ symmetry.

There is a futher bosonic symmetry that we saw in the superparticle which will also occur in the superstring, and the symmetry transforms are given by,

$$\begin{aligned} \delta\theta^1 &= \sqrt{-h}P_-^{\alpha\beta}\partial_\beta\theta^1\lambda_\alpha, & \delta\theta^2 &= \sqrt{-h}P_+^{\alpha\beta}\partial_\beta\theta^2\lambda_\alpha \\ \delta X^\mu &= \theta^A\Gamma^\mu\delta\theta^A, & \delta(\sqrt{-h}h^{\alpha\beta}) &= 0. \end{aligned} \tag{3.32}$$

The equations of motion for the superstring action are then given by,

$$\begin{aligned} \Pi_\alpha\cdot\Pi_\beta &= \tfrac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\Pi_\gamma\cdot\Pi_\delta, \\ \Gamma\cdot\Pi_\alpha P_-^{\alpha\beta}\partial_\beta\theta^1 &= 0, \\ \Gamma\cdot\Pi_\alpha P_+^{\alpha\beta}\partial_\beta\theta^2 &= 0, \\ \partial_\alpha\big[\sqrt{-h}(h^{\alpha\beta}\partial_\beta X^\mu - 2P_-^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\beta\theta^1 &- 2P_+^{\alpha\beta}\bar{\theta}^2\Gamma^\mu\partial_\beta\theta^2)\big] = 0. \end{aligned} \tag{3.33}$$

These equations are hard to quantise using a covariant gauge - in fact no one has found a way to do this - and the only gauge that one can use to quantise these equations is the light cone gauge.

### 3.2.1 A Note on Type I and Type II Strings

In $D = 10$ we know that for a consistent theory $\theta$ must be Majorana-Weyl and so $\theta^1$ and $\theta^2$ must be assigned a definite handedness. As before the overall meaning of left and right is a matter of convention but there are two possibilities: $\theta^1$ and $\theta^2$ have the same handedness or they have opposite handedness. For the closed string the only thing that matters is the periodicity and we can have either since this periodicity property does not relate $\theta^1$ and $\theta^2$. However, for the open string $\theta^1$ and $\theta^2$ must be equated at the ends of the string and so they must have the same handedness. Any superstring theory based on open strings is called a Type I superstring and we shall see that the open string boundary conditions reduce the system to $\mathcal{N} = 1$ spacetime supersymmetry; using Chan-Paton factors we can attach classical group charges to the end of the open strings. Any choice of gauge group is valid at classical level but due to quantum level consistency, only $SO(32)$ is allowed[3]. If we apply orthogonal or symplectic symmetries the theory will then become a theory of unoriented strings, and hence the only quantum mechanically consistent theory has $SO(32)$ symmetry. This theory will consist of open and closed strings; these closed strings are required since quantum mechanically an open string can close to form a closed string which is a Yang-Mills singlet of the gauge group. We can also consider theories based on closed strings alone. If $\theta^1$ and $\theta^2$ do not have equal handedness then the string is unoriented and the theory has two $D = 10$ conserved supersymmetries of opposite chirality and this is called the Type IIA theory. This theory ends up being nonchiral and does not have a Yang-Mills group. Finally, we can set $\theta^1$ and $\theta^2$ to have the same handedness. If we symmeterize the left and right moving sectors then we have unoriented strings and we are led to the closed theory of the Type I theory. However, if we do not make this restriction then we have an oriented theory with two spacetime symmetries of the same handedness - the so-called Type IIB theory. This is a chiral theory and cannot support a Yang-Mills group. We can just introduce one

[3] This follows from consistency of the one- and two- loop amplitudes [19, 20].

$\theta$ coordinate and obtain a Heterotic string theory - however this is not something we discuss in this thesis.

## 3.3 Quantisation

At the moment we have not developed the pure spinor formalism, as a result covariantly quantising the spacetime action in this section is impossible. Instead we have to pick a gauge which makes the complicated differential equations linear - this is our old friend the lightcone gauge. Whilst difficult to use for general purposes, this gauge does allow us to find the spectrum.

We can use the diffeomorphism invariance of the theory to set the metric to the conformally flat metric $h^{\alpha\beta} = e^{2\omega}\eta_{\alpha\beta}$ but even after this gauge fixing we still have superconformal invariance that we can use to select the lightcone gauge,

$$X^+ = x^+ + p^+\tau, \tag{3.34}$$

which leaves only transverse degrees of freedom. As a result we have eight bosonic degrees of freedom which appear as eight left movers and eight right movers. Now a generic spinor in eight dimensions has 32 complex components; imposing the Majorana-Weyl conditions reduces this to 16 real degrees of freedom as was the case of the worldsheet theory. Here we have two Majorana-Weyl spinors $\theta^A$ and thus we have 32 real components. We can use the $\kappa$ symmetry to reduce this to 16 and the final factor of two we need for supermultiplets comes from the equations of motion. Thus we have eight left fermionic movers and eight right movers. A good gauge choice we can make is,

$$\Gamma^+\theta^A = 0, \qquad \Gamma^{\pm} = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9), \tag{3.35}$$

where $\eta_{+-} = -1$ such that $\Gamma^+ = -\Gamma_-$ and $\Gamma^- = -\Gamma_+$. This gauge choice works well since in this gauge we find $\delta X^+ = \bar{\varepsilon}^A\Gamma^+\theta^A = 0$. Once this gauge has been imposed the equations of motion in (3.33) become linear. This is because the term,

$$\bar{\theta}^A \Gamma^\mu \partial_\alpha \theta^A = 0, \quad \mu = i, +, \tag{3.36}$$

and is only nonzero for $\mu = -$. In this gauge choice, the second equation in (3.33) becomes,

$$\begin{aligned} \Gamma \cdot \Pi_\alpha P_-^{\alpha\beta} \partial_\beta \theta^1 &= (\Gamma_+ \Pi_\alpha^+ + \Gamma_i \Pi_\alpha^i) P_-^{\alpha\beta} \partial +_\beta \theta^1 = 0 \\ \implies \Gamma^+ (\Gamma_+ \Pi_\alpha^+ + \Gamma_i \Pi_\alpha^i) P_-^{\alpha\beta} \partial_\beta \theta^1 &= 2 \Pi_\alpha^+ P_-^{\alpha\beta} \partial_\beta \theta^1 = 0, \end{aligned} \tag{3.37}$$

then using $\Pi_\alpha^+ = p^+ \delta_\alpha$ we find,

$$P_-^{\alpha\beta} \partial_\beta \theta^1 = 0, \tag{3.38}$$

and then applying the definition of $P_-^{\alpha\beta}$, the conformal gauge we find,

$$(\partial_\tau - \partial_\sigma) \theta^1 = 0. \tag{3.39}$$

One can repeat this analysis for the third equation in (3.33) and one finds,

$$(\partial_\tau + \partial_\sigma) \theta^2 = 0, \tag{3.40}$$

and hence $\theta^1$ and $\theta^2$ propagate in different directions and obey the wave equation.

The superstring theories we have explored so far are Lorentz invariant but in lightcone gauge only the $SO(8)$ transverse rotational symmetry is manifest. The remaining $\theta$ components form a spinor representation of $SO(8)$[4]. We have two representations of $spin(8)$ denoted by $\mathbf{8_s}$ and $\mathbf{8_c}$; these representations have opposite chirality. The full $D = 10$ chirality of $\theta^A$ determines whether the spinor ends up in $\mathbf{8_s}$ or $\mathbf{8_c}$ after gauge fixing. If we denote the components that survive the gauge fixing as $S$ then we find that the choices we have are,

$$\mathrm{IIA}: \sqrt{p^+}\theta^A \to \mathbf{8_s} \oplus \mathbf{8_c} = (S_1^a, S_2^{\dot{a}}), \qquad \mathrm{IIB}: \sqrt{p^+}\theta^A \to \mathbf{8_s} \oplus \mathbf{8_s} = (S_1^a, S_2^a), \tag{3.41}$$

[4]Actually they form a representation of its covering group $spin(8)$.

where $\{a, b, ...\} \in \mathbf{8_s}$ and $\{\dot{a}, \dot{b}, ...\} \in \mathbf{8_c}$. In the RNS formalism the fermions were spacetime vectors of $SO(8)$ which is not the case here. For the group $spin(8)$ there is a triality symmetry between the vector and two spinor representations. The equations of motion take the form,

$$\partial_+\partial_- X^i = 0, \qquad \partial_+ S_1^a = 0, \qquad \partial_- S_2^{a/\dot{a}} = 0, \tag{3.42}$$

which are the same equations of motion for $X^i$, $\psi_+^i$ and $\psi_-^i$ in the RNS formalism but now the fermions are in the spinor representation of $spin(8)$. The triality makes these equations of motion isomorphic but there are important differences which we do not discuss here. If we combine $S_1^a$ and $S_2^{a/\dot{a}}$ into a Majorana spinor $S^a$ then we can write the action that gives these equations of motion as,

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma(\partial_\alpha X_i \partial^\alpha X^i + \bar{S}^a \rho^\alpha \partial_\alpha S^a). \tag{3.43}$$

Originally $\theta^{Aa}$ transformed as worldsheet scalars but after gauge fixing they are now worldsheet spinors!

### 3.3.1 Canonical Quantisation

The bosonic part of the above action will still satisfy known expansions and relations [19] so we need only focus our attention on the fermionic part of the action. The fermionic coordinates satisfy the anticommutation relation,

$$\{S^{Aa}, S^{Bb'}\} = \pi\delta^{ab}\delta^{AB}\delta(\sigma - \sigma'). \tag{3.44}$$

To find the quantisation conditions for the modes we first need to discuss boundary conditions; there are a number of options for these conditions as before. First let us consider the Type I open superstring; the bosonic fields satisfy Neumann boundary conditions at $\sigma = 0, \pi$. This situation is a little more delicate if one allows for the existence of $D-$branes [23] - something we shall not do in this thesis. In our current case we need to keep the fermionic zero mode in order to keep supersymmetry unbroken in spacetime. Thus there is no choice for overall sign between the ends of the string as in the RNS case, we must have the same choice at both ends and so,

$$S^{1a}(\sigma=0,\tau)=S^{2a}(\sigma=0,\tau), \qquad S^{1a}(\sigma=\pi,\tau)=S^{2a}(\sigma=\pi,\tau), \tag{3.45}$$

and this is only consistent if $\varepsilon^1=\varepsilon^2$ and hence we only have $\mathcal{N}=1$ supersymmetry. The mode expansions for the fermionic fields are then,

$$S^{1a}=\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}S_n^a e^{-in(\tau-\sigma)}, \qquad S^{2a}=\tfrac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}S_n^a e^{-in(\tau+\sigma)}. \tag{3.46}$$

Then once we apply the quantisation condition to these expansions we find that the mode anticommutation relation,

$$\{S_m^a,S_n^b\}=\delta_{m+n}\delta^{ab}, \tag{3.47}$$

and reality implies $S_{-m}^a=(S_m^a)^\dagger$. The closed string only has the periodicity condition $S^{Aa}(\sigma,\tau)=S^{Aa}(\sigma+\pi,\tau)$ and hence the mode expansions are,

$$S^{1a}=\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}S_n^a e^{-2in(\tau-\sigma)}, \qquad S^{2a}=\tfrac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\tilde{S}_n^a e^{-2in(\tau+\sigma)}. \tag{3.48}$$

These modes then satisfy (3.47). A left-right symmeterization (or orientifold projection) of closed IIB gives the closed Type I theory with $\mathcal{N}=1$ supergravity.

### 3.3.2 The Equivalence of the Theories

Th equivalence of the RNS and GS formalisms can be understood using the triality of $SO(8)$. It is possible to relate the lightcone action (3.43) with the RNS lightcone action,

$$S'=-\frac{1}{2\pi}\int \mathrm{d}^2\sigma(\partial_\alpha X_i\partial^\alpha X_i-\bar{\psi}^i\rho^\alpha\partial_\alpha\psi^i), \tag{3.49}$$

since in one case $S^a\in\mathbf{8_s},\mathbf{8_c}$ of *spin*(8) and $\psi^i\in\mathbf{8_v}$ of *spin*(8); this is meaningful since $X_i\in\mathbf{8_v}$ in both cases. We think of the relationship by bosonizing the $\psi^i$ then refermionizing them and we can do this by introducing four real scalars,

$$\begin{aligned} \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_1 &= \bar{\psi}^1\rho^\alpha\psi^2 \\ \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_2 &= \bar{\psi}^3\rho^\alpha\psi^4 \\ \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_3 &= \bar{\psi}^5\rho^\alpha\psi^6 \\ \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_4 &= \bar{\psi}^7\rho^\alpha\psi^8. \end{aligned} \tag{3.50}$$

Then we can rewrite these scalars in the combinations,

$$\begin{aligned} \sigma_1 &= \tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4), \quad \sigma_2 = \tfrac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4) \\ \sigma_3 &= \tfrac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4), \quad \sigma_4 = \tfrac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4), \end{aligned} \tag{3.51}$$

which is an automorphism of $SO(8)$. We can then refermionize via,

$$\begin{aligned} \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_1 &= \bar{S}^1\rho^\alpha S^2 \\ \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_2 &= \bar{S}^3\rho^\alpha S^4 \\ \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_3 &= \bar{S}^3\rho^\alpha S^6 \\ \tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_4 &= \bar{S}^3\rho^\alpha S^8. \end{aligned} \tag{3.52}$$

This is a demonstration of the $SO(8)/spin(8)$ triality. A full treatment would require the methods not covered in this thesis [19, 20]; there are finite volume subtleties we do not address here and this is simply a heuristic exposition.

### 3.3.3 The SuperPoincaré Algebra

Here we wish to discuss how supersymmetry 'works' in the lightcone formalism. The $\delta\theta = \varepsilon$ transform does not preserve the gauge choice $\Gamma^+\theta = 0$. Then for such $\varepsilon$ we have $\bar{\varepsilon}\Gamma^i\theta = 0$ and so the eight supersymmetries can be written in $spin(8)$ notation as,

$$\delta S^a = \sqrt{2p^+}\eta^a, \qquad \delta X^i = 0, \tag{3.53}$$

which is an invariance of the lightcone action. The other eight components of the supersymmetry transform have $\Gamma^+\theta \neq 0$ and correspond to a $\mathbf{8_c}$ spinor $\varepsilon^{\dot\alpha}$. To preserve

$\Gamma^+\theta = 0$ we need to combine $\varepsilon$ with $\kappa$ to obtain $\delta\theta = \varepsilon + 2\Gamma \cdot \Pi_\alpha \kappa^\alpha$, where we choose $\kappa$ in terms of $\varepsilon$ such that the right hand side is annihilated by $\Gamma^+$. The new $\varepsilon$ transform is then given by,

$$\delta S^a = \rho \cdot \partial X^i \gamma^i_{a\dot{a}} \varepsilon^{\dot{a}} \sqrt{2p^+}, \qquad \delta X^i = 2\gamma^i_{a\dot{a}} \bar{\varepsilon}^{\dot{a}} S^a / \sqrt{2p^+}, \tag{3.54}$$

where the $\gamma^i_{a\dot{a}}$ are the Clebsch-Gordon coefficients of the triality. It is easy to show that the transformations leaves (3.43) invariant by using the equations of motion. We need to include both $\delta S^a$ and $\delta X^i$ transforms in order to account for both fermionic symmetries that are present in the theory.

As before, two supersymmetry transforms will yield a spacetime translation which in this case will be a translation in the worldsheet and a translation in the transverse direction. We can show this via the following calculations,

$$\begin{aligned}
[\delta_1, \delta_2] X^i &= \delta_1 \delta_2 X^i - (1 \leftrightarrow 2) \\
&= \delta_1 (2\gamma^i_{a\dot{a}} \bar{\varepsilon}^{\dot{a}}_2 S^a / \sqrt{2p^+}) - (1 \leftrightarrow 2) \\
&= 2\gamma^i_{a\dot{a}} \bar{\varepsilon}^{\dot{a}}_2 (\eta^a - i\rho \cdot \partial X^j \gamma^j_{a\dot{\beta}} \varepsilon^{\dot{\beta}}_1) - (1 \leftrightarrow 2) \\
&= 2\gamma^i_{a\dot{a}} \bar{\varepsilon}^{\dot{a}}_2 \eta^a - 2i\gamma^i_{a\dot{a}} \bar{\varepsilon}^{\dot{a}}_2 \rho \cdot \partial X^j \gamma_{\alpha\dot{\beta}} \varepsilon^{\dot{\beta}}_1 - (1 \leftrightarrow 2) \\
&= -2\eta^{(1)} \gamma^i \bar{\varepsilon}^{(2)} + 2\eta^{(2)} \gamma^i \bar{\varepsilon}^{(1)} - 4i\bar{\varepsilon}^{(1)} \rho^\alpha \varepsilon^{(2)} \partial_\alpha X^i \\
&= \xi^\alpha \partial_\alpha X^i + a^i,
\end{aligned} \tag{3.55}$$

and

$$\begin{aligned}
[\delta_1, \delta_2] S^a &= \delta_1 \delta_2 S^a - (1 \leftrightarrow 2) \\
&= \delta_1 (\sqrt{2p^+} \eta^a_2 - i\rho \cdot \partial X^i \gamma^i_{a\dot{a}} \varepsilon^{\dot{a}}_2 \sqrt{2p^+}) - (1 \leftrightarrow 2) \\
&= -i\rho \cdot \partial(\delta_1 X^i) \gamma^i_{a\dot{a}} \varepsilon^{\dot{a}}_2 \sqrt{2p^+} - (1 \leftrightarrow 2) \\
&= -2i\rho \cdot \partial(\gamma^i_{a\dot{\beta}} \bar{\varepsilon}^{\dot{\beta}}_1 S^a) \gamma^i_{a\dot{a}} \varepsilon^{\dot{a}}_2 - (1 \leftrightarrow 2) \\
&= -4i\bar{\varepsilon}^{(1)} \rho^\alpha \varepsilon^{(2)} \partial_\alpha S^a \\
&= \xi^\alpha \partial_\alpha S^a,
\end{aligned} \tag{3.56}$$

where,

$$\xi^\alpha = -4i\bar{\varepsilon}^{(1)} \rho^\alpha \varepsilon^{(2)}, \qquad a^i = 2\eta^{(2)} \gamma^i \bar{\varepsilon}^{(1)} - 2\eta^{(1)} \gamma^i \bar{\varepsilon}^{(2)} \tag{3.57}$$

We can find the $\eta^a$ and $\varepsilon^{\dot{a}}$ supersymmetry charges by noting that for the $\eta^a$ transforms the expressions in (3.53) are generated by,

$$Q^a = \sqrt{2p^+}S_0^a, \tag{3.58}$$

and for $\varepsilon^{\dot{a}}$ in (3.54),

$$Q^{\dot{a}} = \frac{1}{\sqrt{p^+}}\gamma^i_{\dot{a}a}\sum_{n=-\infty}^{\infty} S^a_{-n}\alpha^i_n. \tag{3.59}$$

These are the 16 components of a covariant Majorana-Weyl spinor which satisfy $\{Q,Q\} \sim (1 \pm \Gamma_{11})\Gamma \cdot p$. Now in our $spin(8)$ notation this splits and gives,

$$\begin{aligned}
\{Q^a, Q^b\} &= 2p^+\{S_0^a, S_0^b\} = 2p^+\delta^{ab} \\
\{Q^a, Q^{\dot{a}}\} &= \sqrt{2}\gamma^i_{\dot{a}b}\sum_n \{S_0^a, S_0^b\}\alpha^i_n \\
&= \sqrt{2}\gamma^i_{\dot{a}b}\sum_n \delta^{ab}\delta_{-n}\alpha^i_n \\
&= \sqrt{2}\gamma^i_{\dot{a}a}\alpha^i_0 = \sqrt{2}\gamma^i_{\dot{a}a}p^i \\
\{Q^{\dot{a}}, Q^{\dot{b}}\} &= \frac{1}{p^+}\gamma^i_{\dot{a}a}\gamma^j_{\dot{b}b}\sum_{n,m}\{S^a_{-n}\alpha^i_n, S^b_{-m}\alpha^j_m\} \\
&= \frac{1}{p^+}\gamma^i_{\dot{a}a}\gamma^j_{\dot{b}b}\sum_{n,m} S^a_{-n}S^b_{-m}[\alpha^i_n, \alpha^j_m] + \{S^a_{-n}, S^b_{-m}\}\alpha^j_m\alpha^i_n \\
&= \frac{1}{p^+}\gamma^i_{\dot{a}a}\gamma^j_{\dot{b}b}\sum_{n,m} nS^a_{-n}S^b_{-m}\delta_{m+n}\delta^{ij} + \delta_{n+m}\delta^{ab}\alpha^j_m\alpha^i_n \\
&= \frac{1}{p^+}\gamma^i_{\dot{a}a}\gamma^i_{\dot{b}b}\sum_m -mS^a_mS^b_{-m} + \tfrac{1}{p^+}\gamma^i_{\dot{a}a}\gamma^j_{\dot{b}a}\sum_m \alpha^j_m\alpha^i_{-m} \\
&= \frac{1}{p^+}\delta^{\dot{a}\dot{b}}\sum_{m=0}^{\infty} mS_{-m}\cdot S_m + \alpha_{-m}\cdot\alpha_m \\
&= \frac{1}{p^+}\delta^{\dot{a}\dot{b}}\left(p_ip^i + \sum_{m=1}^{\infty} mS_{-m}\cdot S_m + \alpha_{-m}\cdot\alpha_m\right) \\
&= 2H\delta^{\dot{a}\dot{b}},
\end{aligned}$$

where define the lightcone Hamiltonian as,

$$H = \frac{1}{2p^+}(p_ip^i + N), \tag{3.60}$$

where $N$ is the number operator,

$$N = \sum_{m=1}^{\infty} mS_{-m} \cdot S_m + \alpha_{-m} \cdot \alpha_m. \tag{3.61}$$

This gives the required information concerning the Super-Poincaré algebra of the Green-Schwarz string in light cone gauge. We now turn our attention to the resultin string spectrum.

## 3.4 The Spectrum

In this section we shall ignore the possible Chan-Paton charges that one can attach to each end of the open string - this is a simplification that will allow us to easily find the spectrum. We begin with the open string and examine the massless spectrum since there is no tachyon to form an unstable ground state. In our $spin(8)$ notation the ground state must furnish a representation of the algebra $\{S_0^a, S_0^b\} = \delta^{ab}$. Due to triality we can do this in the same way as the Clifford algebra - see [19],

$$S_0^a \sim \begin{pmatrix} 0 & \gamma^a_{i\dot{a}} \\ \gamma^a_{\dot{a}i} & 0 \end{pmatrix}. \tag{3.62}$$

In this case the representation is $\mathbf{8_v} + \mathbf{8_c}$ and is a complete supermultiplet. This state comprises eight bosons in the $\mathbf{8_v}$ representation $|i\rangle$ and eight fermions in the $\mathbf{8_c}$ representation $|\dot{a}\rangle$. These are normalised such that,

$$\langle i|j\rangle = \delta_{ij}, \qquad \langle \dot{a}|\dot{b}\rangle = \delta_{\dot{a}\dot{b}}, \tag{3.63}$$

and the identity in the space of the $S_0^a$ operators is $1 = |i\rangle\langle i| + |\dot{a}\rangle\langle\dot{a}|$. We now wish to study the properties of the $S_0$ operators, and to do so it is useful to note the Fierz identity for this operator,

$$S_0^a S_0^b = \frac{1}{2}\{S_0^a, S_0^b\} + \frac{1}{2}[S_0^a, S_b^0] = \frac{1}{2}\delta^{ab} + \frac{1}{16} S_0^c \gamma^{ij}_{cd} S_0^d \gamma^{ij}_{ab}, \tag{3.64}$$

and so the only tensors that can be made out so $S_0$ are $\delta^{ab}$ and,

$$R_0^{ij} = \frac{1}{4} S_0^a \gamma_{ab}^{ij} S_0^b. \tag{3.65}$$

This operator then satisfies the relation,

$$[R_0^{ij}, R_0^{kl}] = \delta^{il} R_0^{jk} - \delta^{ik} R_0^{jl} + \delta^{jk} R_0^{il} - \delta^{jl} R_0^{ik}, \tag{3.66}$$

it also is the operator that rotates the spin of a state but leaves its mass level unchanged. In particular, for a massless vector we find,

$$R_0^{ij} |k\rangle = \delta^{jk} |i\rangle - \delta^{ik} |j\rangle, \tag{3.67}$$

and for the massless spinor,

$$R_0^{ij} |\dot{a}\rangle = -\frac{1}{2} \gamma_{\dot{a}\dot{b}}^{ij} |\dot{b}\rangle. \tag{3.68}$$

Now since $S_0^a$ maps $|i\rangle$ and $|\dot{a}\rangle$ into one another we have,

$$S_0^a |\dot{a}\rangle = \frac{1}{\sqrt{2}} \gamma_{a\dot{a}}^i |i\rangle, \qquad S_0^a |i\rangle = \frac{1}{\sqrt{2}} \gamma_{a\dot{a}}^i |\dot{a}\rangle. \tag{3.69}$$

The normalisations follow from applying $S_0^b$, using the $R_0^{ij}$ commutator and that the action of the $R$ operators on $|i\rangle$ and $|\dot{a}\rangle$.

We now turn to the type II suprestring theories; the ground state is massless and is given by tensoring the left and right movers together. Due to this we essentially have a double copy of the open string states and thus two $\mathbf{8_v} + \mathbf{8_c}$ copies leading to $16 \times 16 = 256$ states. The supermultiplets will be different for the IIA and IIB theories. For IIA, the spinors should have different chirality and so the particle spectrum is given by decomposing the tensor product,

$$(\mathbf{8_v} + \mathbf{8_c}) \otimes (\mathbf{8_v} + \mathbf{8_s}). \tag{3.70}$$

For the bosonic part of the spectrum, this decomposition gives,

$$(\mathbf{8_v} \otimes \mathbf{8_v}) = \mathbf{1} + \mathbf{28} + \mathbf{35}, \qquad (\mathbf{8_s} \otimes \mathbf{8_c}) = \mathbf{8_v} + \mathbf{56}_t, \tag{3.71}$$

where the first decomposition follows from the decomposition of a rank two tensor into a trace, an antisymmetric tensor and a symmetric tensor. The $\mathbf{8_v}$ in $\mathbf{8_s} \otimes \mathbf{8_c}$ comes from the combination $\bar{\xi}\gamma_i\chi$ and the remaining $\mathbf{56}_t$ is a third-rank antisymmetric tensor that comes from the combination $\bar{\xi}\gamma_{ijk}\chi$. The $\mathbf{8_v} \otimes \mathbf{8_s}$ and $\mathbf{8_v} \otimes \mathbf{8_c}$ parts of the tensor product give the corresponding fermions required for supersymmetry. The above bosonic content is the same as $D = 11$ supergravity once dimensional reduction is applied, the fermionic content also matches and this has deep implications linked to M-Theory.

For the type IIB theory, the fermionic parts must have the same chirality and hence the ground state is a decomposition of,

$$(\mathbf{8_v} + \mathbf{8_c}) \otimes (\mathbf{8_v} + \mathbf{8_c}). \tag{3.72}$$

Then for the bosonic decomposition we find,

$$(\mathbf{8_v} \otimes \mathbf{8_v}) = \mathbf{1} + \mathbf{28} + \mathbf{35}, \qquad (\mathbf{8_c} \otimes \mathbf{8_c}) = \mathbf{1} + \mathbf{28} + \mathbf{35}_+, \tag{3.73}$$

where $\mathbf{35}_+$ is a fourth-rank self-dual antisymmetric tensor. This *does not* follow from the dimensional reduction of some higher dimensional theory. We note that in the NS-NS sector[5] both type II theories have the same content,

$$(\mathbf{8_v} \otimes \mathbf{8_v}) = \phi \oplus B_{\mu\nu} \oplus G_{\mu\nu}, \tag{3.74}$$

which are the dilaton, antisymmetric tensor and graviton. In the R-R sector, IIA has odd rank potentials: one-form and three-form. However, the IIB theory has even rank potentials: a zero-form R-R scalar, a two-form potential and a four-form potential with a self-dual field strength. This fact determines that dimension $Dp-$brane each theory can couple to, but in fact, these theories and branes are related by T-duality[6].

In this chapter we have briefly covered many of the topics in the Green-Schwarz formalism of string theory. We have discussed the superparticle and its associated kappa

[5]Although the Green-Schwarz does not have sectors, we can use the Ramond-Neveu-Schwarz classification here to compare between the two formalisms.

[6]See BBS Chapter 6 [23] and Chapter 13 of Polchinski Volume 2 [22].

symmetry, using the things learned here to inform the discussion on the string action. Using the lightcone gauge we were able to find the Super-Poincaré algebra and the string spectrum. There are many things we have glanced over in this chapter and we refer the interested reader to the wealth of fantastic textbooks on this area [19, 20, 21, 22].

# Chapter 4

# Overview of Non-Linear Super Yang-Mills

The study of $D = 10$ Super Yang-Mills is fruitful owing to the fact that is the low energy limit of the open superstring [80] and and secondly, it is the simplest of all the Super Yang-Mills theories as it contains only two particles - the gluon and gluino. The gluon, and its superpartner the gluino, are linked via sixteen supercharges such that they forms a Majorana-Weyl spinor of $SO(1,9)$ [81]. In this chapter we cover the formulation of Super Yang-Mills as well as the non-linear wave equations which can be derived in Lorenz gauge. These equations underpin everything that follows, and the wave equations turn out to be fundamental to the derivation of the Berends-Giele current equations of motion - in fact these currents end up being the solution to these equations of motion [30].

## 4.1 Non-Linear Super Yang-Mills

Super Yang-Mills in $D = 10$ has its origins in two papers [9, 82] in which the author(s) describe Super Yang-Mills in a super-Poincaré invariant way using $\mathcal{N} = 1$ superspace variables $\{x^m, \theta^\alpha\}$ where $m \in \{0, ..., 9\}$ and $\alpha \in \{0, ..., 16\}$ represent the vector and spinor indices of the Lorentz group respectively. The basis of this formalism are the vector and spinor covariant derivatives given by,

$$\nabla_m \equiv \partial_m - \mathbb{A}_m, \qquad \nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha, \tag{4.1}$$

where $\mathbb{A}_m$ and $\mathbb{A}_\alpha$ play the role of Lie algebra valued connections in the theory. The fermionic deriative operator $D_\alpha$ is given in terms of partial derivatives as,

$$D_\alpha \equiv \partial_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m, \qquad \{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m, \tag{4.2}$$

where $\gamma^m_{\alpha\beta}$ denote the $16\times 16$ Pauli matrices that obey the Clifford algebra with convention,

$$\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta^\gamma_\alpha, \tag{4.3}$$

note that in general (anti)symmetrization does not include a factor of $1/n!$ unless explicitly stated. In order to obtain the non-linear equations of motion we begin with the spinor covariant derivative constraint equation which is given by [9], $\{\nabla_\alpha, \nabla_\beta\} = \gamma^m_{\alpha\beta}\nabla_m$. Taking this together with Bianchi identities and the definitions $\mathbb{F}^{mn} \equiv -[\nabla_m, \nabla_n]$ and $\mathbb{W}^\alpha_m \equiv [\nabla_m, \mathbb{W}^\alpha]$ the equations of motion are,

$$\begin{aligned} \{\nabla_\alpha, \nabla_\beta\} &= \gamma^m_{\alpha\beta}\nabla_m, \qquad \{\nabla_\alpha, \mathbb{W}^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn}, \\ [\nabla_\alpha, \nabla_m] &= -(\gamma_m\mathbb{W})_\alpha, \quad [\nabla_\alpha, \mathbb{F}^{mn}] = (\mathbb{W}^{[m}\gamma^{n]})_\alpha, \end{aligned} \tag{4.4}$$

which can be rewritten as,

$$\begin{aligned} \{D_{(\alpha}, \mathbb{A}_{\beta)}\} &= \gamma^m_{\alpha\beta}\mathbb{A}_m + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\}, \qquad \{D_\alpha, \mathbb{W}^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn} + \{\mathbb{A}_\alpha, \mathbb{W}^\beta\}, \\ [D_\alpha, \mathbb{F}^{mn}] &= (\mathbb{W}^{[m}\gamma^{n]})_\alpha + [\mathbb{A}_\alpha, \mathbb{F}^{mn}], \quad [D_\alpha, \mathbb{A}_m] = [\partial_m, \mathbb{A}_\alpha] + (\gamma_m\mathbb{W})_\alpha + [\mathbb{A}_\alpha, \mathbb{A}_m], \end{aligned} \tag{4.5}$$

using (4.1) in (4.4). It is worth demonstrating how one can arrive at these equations using the constraint and Jacobi/Bianchi identities. Consider the Jacobi identity between $\nabla_\alpha, \nabla_\beta$ and $\nabla_m$ given by,

$$[\nabla_m, \{\nabla_\alpha, \nabla_\beta\}] - \{\nabla_\alpha, [\nabla_m, \nabla_\beta]\} - \{\nabla_\beta, [\nabla_m, \nabla_\alpha]\} = 0, \tag{4.6}$$

using[1] $[\nabla_m, \nabla_\alpha] = (\gamma_m \mathbb{W})_\alpha$ one can show,

$$\gamma^n_{\alpha\beta}[\nabla_m, \nabla_n] - \gamma_{m\beta\rho}\{\nabla_\alpha, \mathbb{W}^\rho\} - \gamma_{m\alpha\rho}\{\nabla_\beta, \mathbb{W}^\rho\} = 0. \tag{4.7}$$

Now one can contract $\gamma^{m\beta\sigma}$ and obtain,

$$(\gamma^{mn})_\alpha{}^\sigma \mathbb{F}_{mn} - 10\{\nabla_\alpha, \mathbb{W}^\sigma\} - \gamma^{m\beta\sigma}\gamma_{m\alpha\rho}\{\nabla_\beta, \mathbb{W}^\rho\}. = 0 \tag{4.8}$$

Now one can consider contracting this equation with $\gamma^p_{\sigma\kappa}\gamma^{\alpha\rho}_p$, noting the following identities,

$$\gamma^{m\beta\sigma}\gamma_{m\alpha\rho}\gamma^p_{\sigma\kappa}\gamma^{\alpha\delta}_p = -4\gamma^{\beta\delta}_q\gamma^q_{\rho\kappa} + 12\delta^\beta_\kappa\delta^\delta_\rho + 8\delta^\beta_\rho\delta^\delta_\kappa \tag{4.9}$$

and $(\gamma^p\gamma^{mn}\gamma_p)^\rho{}_\kappa = 6(\gamma^{mn})^\rho{}_\kappa$. Using these, one can show that the following holds true,

$$-6\gamma^p_{\sigma\kappa}\gamma^{\alpha\delta}_p\{\nabla_\alpha, \mathbb{W}^\sigma\} - 12\{\nabla_\kappa, \mathbb{W}^\delta\} - 6(\gamma^{mn})_\kappa{}^\delta \mathbb{F}_{mn} = 0, \tag{4.10}$$

then using (4.8) again, one can show the following,

$$\{\nabla_\alpha, \mathbb{W}^\beta\} = \frac{1}{4}\mathbb{F}^{mn}(\gamma_{mn})_\alpha{}^\beta. \tag{4.11}$$

Similar manipulations lead to the other equations of motion above, for more detail see [9, 83]. Such equations of motion are invariant under a Lie algebra-valued gauge parameter generalisation [29] - these variations are given by,

$$\delta\mathbb{A}_\alpha = [\nabla_\alpha, \Delta], \qquad \delta\mathbb{A}_m = [\nabla_m, \Delta], \tag{4.12}$$

and,

$$\delta\mathbb{W}^\alpha = [\Delta, \mathbb{W}^\alpha], \qquad \delta\mathbb{F}^{mn} = [\Delta, \mathbb{F}^{mn}], \tag{4.13}$$

where $\Delta$ is the Lie-algebra valued variation. Note that the transformations of the higher mass fields follow trivially from the above relations. The linearised version of the equations of motion in (4.5) yield the field equations of linearised Super Yang-Mills and they are invariant under the usual gauge symmetry of that theory. These linear equations of

[1] Note that this equation can be derived by considering the Jacobi identity involving $\nabla_\alpha$, $\nabla_\beta$ and $\nabla_\gamma$, see [9].

motion guarantee the BRST invariance of the linear superfields [84] and this gives one way to prove that the massless vertex operators in the pure spinor superstring theory are also gauge invariant. We draw the distinction in ten dimensions between linearised Super Yang-Mills and non-linear Super Yang-Mills as the former is typically used in amplitude calculations - interactions are considered perturbations from the free field theory. Note that in line with the literature, when we refer to a general superfield we will denote it by $\mathbb{K}$, that is $\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}_m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$. It is also important to point out that (4.4) implies the Dirac and Yang-Mills equations,

$$\gamma^m_{\alpha\beta}[\nabla_m, \mathbb{W}^\beta] = 0, \qquad [\nabla_m, \mathbb{F}^{mn}] = \gamma^n_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\}, \tag{4.14}$$

for the full details of this proof we point the reader to Section 2.1 of [7]. The proof uses the Jacobi identity and a number of $\gamma$ matrix identities, such manipulations are synonymous with this research area and will be a staple of the work carried out here.

It is possible to generalise such fields to higher mass dimension, and in fact at higher loop orders of Super Yang-Mills it becomes necessary to introduce such higher mass fields[2]. In fact, one can introduce these higher mass fields as fundamental[3] fields in their own right. In order to preserve gauge invariance we define these higher mass fields as [7, 30],

$$\begin{aligned} \mathbb{W}^{m_1\ldots m_k\alpha} &\equiv [\nabla^{m_1}, \mathbb{W}^{m_2\ldots m_k\alpha}] \\ \mathbb{F}^{m_1\ldots m_k|pq} &\equiv [\nabla^{m_1}, \mathbb{F}^{m_2\ldots m_k|pq}], \end{aligned} \tag{4.15}$$

which can be used to recursively define the higher mass fields in terms of their lowest mass partners. One can then show that these higher mass equations obey a set of equations of motion similar to their lowest mass counterparts [7],

$$\begin{aligned} \{\nabla_\alpha, \mathbb{W}^{N\beta}\} &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}^N_{mn} - \sum_{\substack{\delta(N)=R\otimes S \\ R\neq\emptyset}} \{(\mathbb{W}\gamma)^R_\alpha, \mathbb{W}^{S\beta}\}, \\ [\nabla_\alpha, \mathbb{F}^{N|pq}] &= (\mathbb{W}^{N[p}\gamma^{q]})_\alpha - \sum_{\substack{\delta(N)=R\otimes S \\ R\neq\emptyset}} \big[(\mathbb{W}\gamma)^R_\alpha, \mathbb{F}^{S|pq}\big]. \end{aligned} \tag{4.16}$$

[2]It is not strictly true that they need to be added - one can work with commutators on these fields but it makes computations much simpler if these fields are introduced.

[3]Fundamental in the sense that they have their own equations of motion that are analogous to the original superfields and as such there is no need to refer to lower mass fields once they have been defined.

Note that $\delta(N)$ is the deshuffle map defined by,

$$\delta(P) = \sum_{X,Y} \langle P, X \sqcup\!\sqcup Y \rangle X \otimes Y, \tag{4.17}$$

where $\sqcup\!\sqcup$ denotes the shuffle product and $\langle \cdot, \cdot \rangle$ is the scalar product on words such that $\langle A, B \rangle = \delta_{A,B}$. As a concrete example take,

$$\delta(mn) = mn \otimes \emptyset + m \otimes n + n \otimes m + \emptyset \otimes mn. \tag{4.18}$$

**The Shuffle Product**

The shuffle product is the inverse of the deshuffle product and the latter is defined recursively in the following manner [7],

$$\emptyset \sqcup\!\sqcup P = P \sqcup\!\sqcup \emptyset := P, \qquad iP \sqcup\!\sqcup jQ := i(P \sqcup\!\sqcup jQ) + j(Q \sqcup\!\sqcup iP), \tag{4.19}$$

where $i$, $j$ are letters and $P$, $Q$ are words. As an example, consider the following $12 \sqcup\!\sqcup 345$, which comes out as,

$$\begin{aligned} 12 \sqcup\!\sqcup 345 = &12345 + 13245 + 13425 + 13452 \\ &+ 31245 + 31425 + 34125 \\ &+ 31452 + 34152 + 34512. \end{aligned} \tag{4.20}$$

This combinatorical method exhibits a symmetry on the currents - this is known as the *shuffle symmetry* and has been proved for gluons [38] and supersymmetric extensions [29].

Given that $Q = \lambda^\alpha D_\alpha$, we can use the equations of motion in (4.5) and (4.16) to find the BRST variation of the fields. However, as we shall see in later chapters we can use the simpler $\mathcal{Q} = \lambda^\alpha \nabla_\alpha$ operator to reduce the number of fields that are present in variations. This follows for two reasons, firstly all generating series are under a trace and we have the following relations,

$$\mathrm{Tr}\{\mathbb{V}, \mathrm{fermion}\} = 0, \qquad \mathrm{Tr}[\mathbb{V}, \mathrm{boson}] = 0, \tag{4.21}$$

which allows us to set many of the terms to zero that we obtain after variation using $Q = \lambda^\alpha D_\alpha$. Secondly, we shall find that invoking a labelled vertex operator $\mathbb{V}_1$ will simplify the higher order correction expressions we obtain and this operator is BRST closed under $\mathcal{Q}$. For example, consider the variation of $\mathbb{F}^{mn}$, which can be expressed as,

$$[Q, \mathbb{F}^{mn}] = -(\lambda\gamma^{[m}\mathbb{W}^{n]}) + [\mathbb{V}, \mathbb{F}^{mn}], \tag{4.22}$$

which introduces new $\mathbb{V}$ fields into our expressions. Alternatively, one can use $\mathcal{Q}$ and find,

$$\mathrm{Tr}[\mathcal{Q}, \mathbb{F}^{mn}] = -\mathrm{Tr}\big[(\lambda\gamma^{[m}\mathbb{W}^{n]})\big], \tag{4.23}$$

this works as all of the generating series expressions we work with are under the trace of the fields - thus using the statistics of the fields one can always eliminate the extra $\mathbb{V}$ terms.

The Dirac and Yang-Mills equations for the lowest mass fields can be generalised to the higher mass case using [7]

$$\begin{aligned} [\nabla_m, (\gamma^m\mathbb{W}^N)^\alpha] &= \sum_{\delta(N)=R\otimes S} \big[\mathbb{F}^{Rm}, (\gamma_m\mathbb{W}^S)_\alpha\big], \\ [\nabla_m, \mathbb{F}^{N|pm}] &= \delta_{mn} \sum_{\substack{\delta(N)=R\otimes S \\ R\neq\emptyset}} \big[\mathbb{F}^{Rm}, \mathbb{F}^{S|pn}\big] - \sum_{\substack{\delta(N)=R\otimes S \\ R\neq\emptyset}} \big\{\mathbb{W}^{R\alpha}, (\gamma^p\mathbb{W}^S)_\alpha\big\}, \end{aligned} \tag{4.24}$$

where $R = p_1p_2...p_n|q$. Since the higher mass fields can be written as nested commutators they lend themselves very naturally to obeying certain Jacobi-like relations. One of the first consequences of these relations is the antisymmetrization of vector indices. We will require a handful of these identities in later chapters and so we give them here,

$$\begin{aligned} \mathbb{W}^{[mn]} &= [\mathbb{W}, \mathbb{F}^{mn}], \quad \mathbb{F}^{[mn]|pq} = [\mathbb{F}^{pq}, \mathbb{F}^{mn}], \\ \mathbb{W}^{[mnp]} &= [\mathbb{W}^m, \mathbb{F}^{np}] + [\mathbb{W}^n, \mathbb{F}^{pm}] + [\mathbb{W}^p, \mathbb{F}^{mn}], \end{aligned} \tag{4.25}$$

where the last identity gives a concrete example of the generalised Jacobi identities that indices of superfields obey, that is,

$$\mathbb{F}^{[mn]|pq} + \mathbb{F}^{[pq]|mn} = 0. \tag{4.26}$$

Such symmetries can be summarised by using the left-to-right Dynkin bracket defined as [7],

$$l(123...n) := l(123...n-1)n - nl(123...n-1), \quad l(i) := i, \quad l(\emptyset) := 0. \tag{4.27}$$

These symmetries are then given by,

$$K_{Al(B)C} + K_{Bl(A)C} = 0, \qquad A, B \neq \emptyset, \quad \forall C, \tag{4.28}$$

and such generalised Jacobi identities arise due to the nature of the structure constants inherent in the $D = 10$ Super Yang-Mills structure. Note that, when we are only concerned with vector index manipulations we will suppress the spinor indices on fields. In places where confusion may arise we will put all indices back onto the fields. In Chapter 8 we will demonstrate that $\mathbb{F}^{mn|pq}$, $\mathbb{W}^{mn\alpha}$ and $\mathbb{W}^{mnp\alpha}$ can be decomposed in a different way into the lowest mass partners and an irreducible traceless symmetric part. It is this exercise which will require the use of the above anti-symmetrized indices and the Jacobi like nature of higher mass fields.

## 4.2 Wave Equations

It is now possible to take the above equations of motion and, by applying the Lie-algebra d'Alembertian operator, find a series of wave equations. These are important as the solutions to these wave equations end up being the Berends-Giele currents of the multiparticle superfields, a fact we will demonstrate in Chapter 5. In order to do this we have to choose a gauge to work in and here, as in [7], we use the Lorenz gauge,

$$[\partial_m, \mathbb{A}^m] = -[\partial_m, \nabla^m] = 0. \tag{4.29}$$

Using the d'Alembertian operator,

$$\Box\mathbb{K} \equiv [\partial^m, [\partial_m, \mathbb{K}]], \tag{4.30}$$

one can show,

$$\Box\mathbb{K} = [\mathbb{A}^m, [\partial_m, \mathbb{K}]] + [\mathbb{A}^m, [\nabla_m, \mathbb{K}]] + [\nabla^m, [\nabla_m, \mathbb{K}]]. \tag{4.31}$$

The proof of this is as follows,

$$\begin{aligned}
\Box\mathbb{K} &= [\nabla^m, [\partial_m, \mathbb{K}]] + [\mathbb{A}^m, [\partial_m, \mathbb{K}]] \\
&= [\partial_m, [\nabla^m, \mathbb{K}]] - [\mathbb{K}, [\mathbb{A}^m, \partial_m]] + [\partial_m, [\mathbb{A}^m, \mathbb{K}]] \\
&= [\partial_m, [\nabla^m, \mathbb{K}]] + [\mathbb{A}^m, [\partial_m, \mathbb{K}]] \\
&= [\mathbb{A}^m, [\partial_m, \mathbb{K}]] + [\mathbb{A}^m, [\nabla_m, \mathbb{K}]] + [\nabla^m, [\nabla_m, \mathbb{K}]],
\end{aligned} \tag{4.32}$$

where in the third line we use the Jacobi identity for $[\mathbb{K}, [\mathbb{A}^m, \partial_m]]$. Now let us consider setting $\mathbb{K} = \nabla^n$ which gives the following,

$$\begin{aligned}
\Box\mathbb{A}^n &= -[\mathbb{A}_m, [\partial^m, \nabla^n]] - [\mathbb{A}_m, [\nabla^m, \nabla^n]] - [\nabla^m, [\nabla_m, \nabla^n]] \\
&= [\mathbb{A}^m, [\partial_m, \mathbb{A}^n]] + [\mathbb{A}^m, \mathbb{F}^{mn}] + \gamma^n_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\},
\end{aligned} \tag{4.33}$$

where for the last term we use the Yang-Mills equation of motion. Similar wave equations are also given for the other fields, although higher mass Dirac and Yang-Mills equations are required [7],

$$\begin{aligned}
\Box\mathbb{A}_\alpha &= [\mathbb{A}^m, [\partial^m, \mathbb{A}_\alpha]] + [(\gamma^m\mathbb{W})_\alpha, \mathbb{A}_m], \\
\Box\mathbb{W}^\alpha &= [\mathbb{A}_m, [\partial^m, \mathbb{W}^\alpha]] + [\mathbb{A}_m, \mathbb{W}^{m\alpha}] + \frac{1}{2}[\mathbb{F}_{mn}, (\gamma^{mn}\mathbb{W})^\alpha], \\
\Box\mathbb{F}^{mn} &= [\mathbb{A}_p, [\partial^p, \mathbb{F}^{mn}]] + [\mathbb{A}_p, \mathbb{F}^{p|mn}] + 2[\mathbb{F}^{mp}, \mathbb{F}^{pn}] \\
&\quad + 2\{(\mathbb{W}^{[m}\gamma^{n]})_\alpha, \mathbb{W}^\alpha\}.
\end{aligned} \tag{4.34}$$

One can extend these wave equations to the higher mass fields by considering $[\nabla^m, \Box\mathbb{K}]$,

$$\Box\mathbb{K}^m = [\nabla^m, \Box\mathbb{K}] + [\mathbb{K}, \Box\mathbb{A}^m] + 2[[\partial^n, \mathbb{K}], [\partial^n, \mathbb{A}^m]], \tag{4.35}$$

and so on. These wave equations can then be used to determine the higher mass Berends-Giele current expansions - something which is needed in order to perform component expansions efficiently. As an example, let us consider a field we shall utilise later $\mathbb{V}^m = [\nabla^m, \mathbb{V}]$, where $\mathbb{V} = \lambda^\alpha \mathbb{A}_\alpha$ as usual. The field equation for this higher mass field is given by,

$$\begin{aligned}\Box\mathbb{V}^m =& - [\mathbb{A}^n, [\partial^n, \mathbb{V}^m]] - [\mathbb{A}^n, (\lambda\gamma^n\mathbb{W}^m)] - [\mathbb{A}^{m|n}, (\lambda\gamma^n\mathbb{W})] - [\mathbb{V}, [\partial^n, \mathbb{A}^{n|m}]] \\ &+ [\mathbb{V}^n, [\partial^n, \mathbb{A}^m]] + [[\partial^n, \mathbb{V}], [\mathbb{A}^n, \mathbb{A}^m]] + 2[\mathbb{A}^n, [\mathbb{V}, [\partial^n, \mathbb{A}^m]]],\end{aligned} \tag{4.36}$$

where $\mathbb{A}^{m|n} = [\nabla^m, \mathbb{A}^n]$ is the higher-mass generalisation of the gauge field - another field which we will formally define and utilise in later chapters. These equations can then used in order to determine the deconcatenation of higher-point Berends-Giele currents - this is something we shall cover in Chapter 6.

This chapter has introduced the necessary machinery to begin to work with the generating series and perturbiners of later chapters. We reviewed the derivation of the $D = 10$ Super Yang-Mills equations and the associated higher mass equations of motion, as well as the vector index symmetries of such fields. Furthermore, we have demonstrated that one can use the above equations to determine non-linear wave equations for each of the fields. We briefly extended this method to higher mass fields, giving an equation for general field of mass dimension $k+1$ where $k$ is the mass dimension of a general field $\mathbb{K}$. We now move on to discussing Super Yang-Mills in the pure spinor formalism as well as some aspects of the pure spinor superstring.

# Chapter 5

# Super Yang-Mills in Pure Spinor Superspace

In this chapter we introduce a good proportion of the machinery required to begin performing amplitude calculations in the pure spinor formalism. We briefly review some of the issues with previous superstring formalisms and outline an attempted solution. Then we discuss pure spinor superspace and the superstring action in this formalism, demonstrating the simplicity of many aspects of the formalism. We also succinctly discuss how to calculate string scattering amplitudes using the free field OPEs. Finally, we discuss one of the recent new technologies in the pure spinor formalism - so-called multiparticle superfields which greatly aid the calculation and expression of amplitudes in the pure spinor formalism. Finally, we give a short introduction to Berends-Giele currents which are the back bone of much of the computational techniques in Super Yang-Mills.

In the conformal gauge, the Green-Schwarz action is given by the following,

$$S_{\mathrm{GS}} = \int \mathrm{d}^2 z [\Pi^m \bar{\Pi}_m + \tfrac{1}{2}\Pi_m(\theta\gamma^m\bar{\partial}\theta) - \tfrac{1}{4}\bar{\Pi}_m(\theta\gamma^m\partial\theta)], \tag{5.1}$$

where the conjugate momenta are given by,

$$\Pi^m = \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta), \qquad \bar{\Pi}^m = \bar{\partial} x^m + \frac{1}{2}(\theta\gamma^m\bar{\partial}\theta). \tag{5.2}$$

Alongside this, we also need to include the Virasoro constraint,

$$T = -\frac{1}{2}\Pi^m \Pi_m = 0. \tag{5.3}$$

Note that the canonical momentum of $\theta^\alpha$ does not appear in the action so we need to define it as,

$$p_\alpha = \frac{1}{2}(\Pi_m - \tfrac{1}{4}(\theta\gamma_m \partial\theta))(\gamma^m\theta)_\alpha. \tag{5.4}$$

Since it does not appear in the action we need to introduce a Dirac constraint of the form,

$$d_\alpha = p_\alpha - \frac{1}{2}(\Pi_m - \tfrac{1}{4}(\theta\gamma_m \partial\theta))(\gamma^m\theta)_\alpha, \tag{5.5}$$

whose anti-commutator is given by,

$$\{d_\alpha, d_\beta\} = i\gamma^m_{\alpha\beta}\Pi_m. \tag{5.6}$$

However, the Virasoro constraint then causes problems - it mixes first and second class constraints and thus covaraint quantisation is not possible[19, 20]. One can use light-cone gauge but this is cumbersome and calculations are inefficient [85], as we saw in previous chapters[1].

When considering the quantisation of the superparticle or superstring, one encounters the above issues with the nature of the constraints on the system. It turns out that, in attempting to quantise the Brink-Schwarz superparticle [81] there are a set of second order constraints that cannot be quantised covariantly. The same sort of issue occurs in string theory. In 1986 Siegel proposed an alternative to the GS formalism [65] whose action is given by,

$$S_{\text{Siegel}} = \int \mathrm{d}^2 z[\tfrac{1}{2}\partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial}\theta^\alpha], \tag{5.7}$$

where $x^m$ are the worldsheet scalars (spacetime coordinates), $p_\alpha$ is a free field and $\theta^\alpha$ is a free Grassmann field. In this formulation, the second class constraints that plague

[1]For further information see the Chapter 7 bibliographic discussion in [19].

the Brink-Schwarz superparticle can be replaced by a field that is assumed to be free,

$$d_\alpha = p_\alpha - \tfrac{1}{2}(\partial x^m + \tfrac{1}{4}(\theta\gamma^m\partial\theta))(\gamma^m\theta)_\alpha, \tag{5.8}$$

hence the issues of first/second class constraints disappear in Siegel's approach. This method is successful in quantising the superparticle [86], however, applying this method directly to the superstring does not solve any issues, it simply moves the issue from one area to another. Despite these issue, there is one large advantage of Siegel's approach - all of the fields are free and hence one can determine the OPEs of the fields,

$$\begin{aligned}
x^m(y)x^n(z) &\sim -2\eta^{mn}\ln|y-z|, \\
p_\alpha(y)\theta^\beta(z) &\sim \frac{\delta^\beta_\alpha}{y-z}, \\
d_\alpha(y)d_\beta(z) &\sim \frac{\gamma^m_{\alpha\beta}}{y-z}\Pi_m(z), \\
d_\alpha(y)\Pi^m(z) &\sim \frac{\gamma^m_{\alpha\beta}}{y-z}\theta^\beta(z),
\end{aligned} \tag{5.9}$$

where $\Pi_m$ is the conjugate momenta to $x^m$. These OPEs give us the hint that such a formulation will allow the simple calculation of worldsheet terms, in order to perform these calculations one first needs to introduce the pure spinor.

## 5.1 Pure Spinor Superspace

There is an $SO(10)$ irreducible representation which contributes a central charge of 22, and cancels the matter central charge [10], whose Lorentz generator double pole gives -3 [10]. This irreducible representation consists of the bosonic pure spinor $\lambda^\alpha$, which is subject to the constraint [87],

$$\lambda\gamma^m\lambda = 0, \tag{5.10}$$

this is the pure spinor constraint and is the most important equation in this work. The $SO(10)$ covariant worldsheet action is then given by,

$$S = \int \mathrm{d}^2 z\left(\frac{1}{2}\partial X^m\bar{\partial}X_m + p_\alpha\bar{\partial}\theta^\alpha - w_\alpha\bar{\partial}\lambda^\alpha\right), \tag{5.11}$$

where $w_\alpha$ is now the pure spinor's conjugate momentum. We need to also consider the extra definitions required to fully define the theory,

$$\begin{aligned}
\Pi^m &= \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta), \\
d_\alpha &= p_\alpha - \frac{1}{2}\left(\partial X^m + \frac{1}{4}(\theta\gamma^m\partial\theta)\right)(\gamma_m\theta)_\alpha, \\
D_\alpha &= \partial_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m,
\end{aligned} \tag{5.12}$$

where $D_\alpha$ is the supersymmetric derivative. One of the most important definitions to this work is that of the BRST charge which is given by,

$$Q = \oint \mathrm{d}z\lambda^\alpha d_\alpha. \tag{5.13}$$

This operator is nilpotent, $Q^2 = 0$, the proof of which can be found in (3.24) of [7] - the main argument follows from the pure spinor constraint. Such nilpotency allows one to identify which states of the theory are within the same cohomolgy classes and hence are physically the same - that is states that differ by $Q\Phi$. Although we will typically use this in the form,

$$Q = \lambda^\alpha D_\alpha, \tag{5.14}$$

for a formal derivation of the BRST operator see [88]. One can use the latter form due to the OPE of the $d_\alpha$ operator contracted with a superfield $K$,

$$d_\alpha(z)K(w) \sim \frac{D_\alpha K(w)}{z-w}, \tag{5.15}$$

hence applying this, one finds that when one performs the contour integral one is effectively left with $Q = \lambda^\alpha D_\alpha$.

Before moving on, it is instructive to look at decomposing the $SO(10)$ symmetry into its subgroup $U(5)$ as this highlights how many degrees of freedom are actually eliminated by the pure spinor constraint - naïvely one would think this is 10 but this is not the case. We can parameterise $\lambda^\alpha$ by,

$$\lambda^\alpha \to (\lambda^+, \lambda^a, \lambda_{ab}), \tag{5.16}$$

where $\lambda^+ \neq 0$ and,

$$\lambda_{ab} = -\lambda_{ba}, \qquad \lambda^a = -\frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de}, \tag{5.17}$$

the pure spinor constraint only eliminates $\lambda^a \in \mathbf{5}$ and hence leaves 11 degrees of freedom.

### 5.1.1 Vertex Operators

Physical states in the pure spinor formalism all have ghost number 1 and are in the cohomology of the BRST operator - this cohomology is unique and so if two expressions at the same weight are BRST closed then they are assumed to be physically the same. Two such physical states which are important to calculations in string theory are the integrated and unintegrated vertex operators. Here we discuss these operators and their form.

Owing to the pure spinor constraint, the conjugate momenta of the pure spinor, $w_\alpha$, is invariant under the transformation [84],

$$\delta w_\alpha = (\gamma^m \lambda)_\alpha \Lambda_m, \tag{5.18}$$

and as a result, $w_\alpha$ can only appear in Lorentz covariant combinations,

$$N_{mn} = \frac{1}{2}(w\gamma_{mn}\lambda), \qquad J = (w\lambda), \tag{5.19}$$

where $N_{mn}$ are the Lorentz currents and $J$ is the ghost number operator. We can construct vertex operators from combinations of

$$[X^m, \theta^\alpha, d_\alpha, \lambda^\alpha, N_{mn}, J],$$

any combination that is physical must have ghost number 1, a central charge of 0 and the mass must obey $m^2 = n/2$ at zero momentum, where $n$ is the conformal weight of the fields [84]. The proposal for the unintegrated operator is [10],

$$V = \lambda^\alpha A_\alpha(x, \theta), \tag{5.20}$$

this has the correct ghost number and central charge. Note that $QV = 0$ implies $\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0$, that is it give the pre-spinor potential's equation of motions and $\delta V = Q\Delta$ implies $\delta A_\alpha = D_\alpha \Delta$. In order to calculate scattering amplitudes in the string theory we also need an integrated operator, $U$, given by,

$$U = \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha w^\alpha + \frac{1}{2} N_{mn} F^{mn}, \tag{5.21}$$

where $A_m$ is the gauge field and $F_{mn}$ is the gauge field strength. The actual form of this integrated operator is not required for the work carried in the next parts of this thesis as we do not use the integrated operator - we present it here for completeness. Note that any state that is physical must not be exact. That is, if $Q\phi = 0$ and $\phi = Q\psi$ then $\phi$ is BRST exact and is not a physical state - much like the spurious states found in the canonical quantisation of previous chapters.

### 5.1.2 A Note on String Amplitudes

In order to calculate string scattering amplitudes, one needs to integrate over the zero modes of the worldsheet fields, however the correlation functions will still contain $\lambda$ and $\theta$ and so one needs a rule to integrate over these. Such a rule is given by,

$$(\lambda^3\theta^5) = \langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma^{mnp}\theta)\rangle = 2880, \tag{5.22}$$

thus when we perform component expansions of the fields we only care about terms that involve $\lambda^3\theta^5$. This is the zero mode prescription for the tree level amplitudes and showing that this pure spinor integrate measure is in the BRST cohomology is relatively straightforward. One simply needs to apply the operator $Q = \lambda^\alpha D_\alpha$, noting that $D_\alpha = \partial_\alpha - \frac{1}{2}(\theta\gamma^m)_\alpha \partial_m$, and apply the integration measure - doing so shown the expression is zero. In order to find the tree-level scattering of massless states we can take inspiration from the methods used to find them in the bosonic string theory. That is we fix three unintegrated vertex operators using the usual $SL(2\mathbb{C})$ invariance and integrate

$n-3$ such that the $n-$point amplitude is given by,

$$A_n = \Big(\prod_{i=4}^{n} \int \mathrm{d}z_i\Big)\Big\langle V_1 V_2 V_3 \prod_{r=4}^{n} U_r\Big\rangle, \tag{5.23}$$

and then we use the form of the vertex operators from above to calculate the OPEs. The relevant OPEs can be found in [7], but we repeat them here for convenience. The worldsheet matter variables obey the following,

$$\begin{aligned}
X^m(y,\bar{y})X^n(z,\bar{z}) &\sim -\delta^{mn}\ln|y-z|^2,\\
d_\alpha(y)d_\beta(z) &\sim -\frac{\gamma^m_{\alpha\beta}}{z-w}\Pi_m(z),\\
\Pi^m(y)\Pi^n(z) &\sim -\frac{\delta^{mn}}{(y-z)^2},\\
d_\alpha(y)\theta^\beta(z) &\sim \frac{\delta^\beta_\alpha}{y-z},\\
d_\alpha(y)\Pi^m(z) &\sim \frac{(\gamma^m\partial\theta(z))_\alpha}{y-z}.
\end{aligned} \tag{5.24}$$

In order to determine the OPEs between the Lorentz currents, we first have to introduce the fermionic Lorentz current which is given by,

$$M^{mn} = -\frac{1}{2}(p\gamma^{mn}\theta) + \frac{1}{2}(w\gamma^{mn}\lambda), \tag{5.25}$$

and hence the OPEs are given by [7],

$$\begin{aligned}
M^{mn}(y)M^{pq}(z) &\sim \frac{\delta^{p[m}M^{n]q}(z) - \delta^{q[m}M^{n]p}(z)}{y-z} + \frac{\delta^{m[q}\delta^{p]n}}{(y-z)^2},\\
N^{mn}(y)N^{pq}(z) &\sim \frac{\delta^{p[m}N^{n]q}(z) - \delta^{q[m}N^{n]p}(z)}{y-z} - 3\frac{\delta^{m[q}\delta^{p]n}}{(y-z)^2},\\
N^{mn}(y)\lambda^\alpha(z) &\sim \frac{1}{2}\frac{(\gamma^{mn}\lambda(z))^\alpha}{y-z},
\end{aligned} \tag{5.26}$$

where all other OPEs are regular. Finally, for a general superfield $K(x,\theta)$, that has no dependence on derivatives we have,

$$\begin{aligned}
d_\alpha(y)K(z) &\sim \frac{D_\alpha K(z)}{y-z},\\
\Pi^m(y)K(z) &\sim -\frac{\partial^m K(z)}{y-z},
\end{aligned} \tag{5.27}$$

where $K(z) = K(X(z,\bar{z}), \theta(z))$ and all other OPEs are regular. The superfields can be decomposed into plane waves such that $K(X,\theta) = K(\theta)e^{ik\cdot X}$, and hence we will eventually need the $\theta-$expansions of the superfields in order to perform the pure spinor space integration that was outlined above. Using the OPEs we can eliminate worldsheet fields with non-zero dimension, and after integrating over the $X^m$ zero modes and dropping the Koba-Nielsen type factors we obtain,

$$A_n = \left(\prod_{i=4}^{n}\int \mathrm{d}z_i\right)\langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}\rangle, \tag{5.28}$$

where $f_{\alpha\beta\gamma}$ is some function of superfields, $\theta$, and $z$. This prescription, without modification, only works for tree-level amplitudes; in order to compute loop-level amplitudes one needs to either include picture changing operators or modify the formalism [88, 89, 90, 91]. This modification is not important for the things we wish to discuss here and so we refer the reader to the previous references.

As an example of an amplitude, let us consider calculating the three-point superstring amplitude - usually this would require a quite involved set of calculations but in the pure spinor formalism it is very simple. The three point amplitude for the open string in the pure spinor formalism amounts to calculating,

$$A_3 = \mathrm{Tr}\langle\lambda A_1\lambda A_2\lambda A_3\rangle, \tag{5.29}$$

and in order to do this we need the $\theta$ expansion of the pre-spinor potential which is given by [29],

$$A_\alpha = \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\chi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}f_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + ... \tag{5.30}$$

We do not require the higher order parts of the expansion because $A_\alpha$ starts at order $\theta^1$, there are three $V$ in the amplitude and we only include terms with $\theta^5$ in them. If we label the $\theta^n$ piece of $A_\alpha$ as $A_\alpha^{(n)}$ then the only terms that give a $\theta^5$ in the bosonic sector are,

$$A_\alpha^{1(3)}A_\beta^{2(1)}A_\gamma^{2(1)}, \qquad A_\alpha^{1(1)}A_\beta^{2(3)}A_\gamma^{2(1)}, \qquad A_\alpha^{1(1)}A_\beta^{2(1)}A_\gamma^{2(3)}, \tag{5.31}$$

and evaluating this we find,

$$A_{BBB} = -[(e_1 \cdot e_2)(e_3 \cdot k_2) + (e_1 \cdot e_3)(e_2 \cdot k_1) + (e_2 \cdot e_3)(e_1 \cdot k_3)], \tag{5.32}$$

this vanishes for photons due to the anti-symmetry in $2 \leftrightarrow 3$, however for gluons it is non-zero owing to the Chan-Paton factors. Note that in the expansion of $A_\alpha$, $a_m = e_m e^{ik\cdot x}$, $\chi^\alpha = g^\alpha e^{ik\cdot x}$ and $f_{mn} = (k_m e_n - k_n e_m)e^{ik\cdot x}$. In order to get to this result we perform the following calculation,

$$\begin{aligned} \lambda^\alpha A^{1(1)}_\alpha \lambda^\beta A^{2(1)}_\beta \lambda^\gamma A^{2(3)}_\gamma &\propto e^1_m e^2_n f^3_{rs}(\lambda\gamma^m\theta)(\lambda\gamma_p\theta)(\theta\gamma^{rsp}\theta) \\ &\propto 2k^3_r e^1_m e^2_n e^3_s(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{rsp}\theta) \\ &= -\frac{1}{64}k^3_r e^1_m e^2_n e^3_s(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{rsp}\theta), \end{aligned} \tag{5.33}$$

where we have used $\gamma^{rsp} = -\gamma^{srp}$. Note that the following holds for the pure spinor integration measure,

$$\langle(\lambda\gamma_r\theta)(\lambda\gamma_s\theta)(\lambda\gamma_p\theta)(\theta\gamma^{pmn}\theta)\rangle = \frac{1}{45}\delta^{mn}_{rs}, \tag{5.34}$$

where [7],

$$\delta^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_n} = \frac{1}{n!}\delta^{[a_1}_{b_1}\delta^{a_2}_{b_2}\ldots\delta^{a_n]}_{b_n}. \tag{5.35}$$

The boson, fermion, fermion amplitude is similarly given by,

$$A_{BFF} = -e^1_m(g^2\gamma^m g^3) + (2 \leftrightarrow 3). \tag{5.36}$$

Calculating higher point amplitudes is much that same, however one now needs to calculate OPEs of between fields since the integrated vertex operator requires such calculations. We do not show them here but there are a wealth of papers demonstrating these calculations [84, 92, 93, 83]. The main thrust of this chapter is to demonstrate just how easy it is to calculate the scattering amplitudes of strings in the pure spinor formalism.

## 5.2 Multiparticle Superfields

The superfields in the above calculations were all single particle superfields - that is they represented just one particle current. However, if one calculates OPEs between multiple vertex operators then a new structure appears that become incredibly powerful for finding amplitudes - such structures were first observed in [94, 95] and later expanded upon in [29, 30, 96]. Let us consider a recursive definition of the OPEs of the form [36, 37],

$$\begin{aligned} \lim_{z_1 \to z_2} V^1(z_1) U^2(z_2) &\sim z_{21}^{-k_1 \cdot k_2} \frac{L_{21}}{z_{12}}, \\ \lim_{z_1 \to z_2} L_{12\ldots(p-1)}(z_1) U^p(z_p) &\sim z_{p1}^{-(k_1+\ldots k_{p-1})\cdot k_p} \frac{L_{2131\ldots(p-1)1}}{z_{p1}}, \end{aligned} \tag{5.37}$$

and we define fermionic, ghost number 1 BRST building blocks by removing the BRST exact terms from these expressions. The BRST building blocks, $T_P$, are BRST-covariant, that is their BRST variation contains themselves, for example, $QT_{123} = (k^1 \cdot k^2)(T_1 T_{23} + T_{13} T_2) + (k^{12} \cdot k^3) T_{12} T_3$. More generally, their BRST variation is given by,

$$QT_{1\ldots p} = \sum_{j=2}^{p} \sum_{\substack{P = XjY \\ \delta(Y) = R \otimes S}} (k_X \cdot k_j) T_{XR} T_{jS}, \tag{5.38}$$

note that this terminates with $T_i = V_i$. After the $Q$ exact terms are removed from $L_P$ to give $T_P$, the $T_P$ exhibit Lie symmetries in their indices. That is they obey the following [37],

$$\begin{aligned} \mathcal{L}_{k=2n+1} : T_{12\ldots n+1[n+2,[\ldots,[2n-1,[2n,2n+1]]\ldots]]} - T_{2n+1\ldots n+2[n+1,[\ldots,[3,[2,1]]\ldots]]} &= 0, \\ \mathcal{L}_{k=2n} : T_{12\ldots n[n+1,[\ldots,[2n-2,[2n-1,2n]]\ldots]]} + T_{2n\ldots n+1[n,[\ldots,[3,[2,1]]\ldots]]} &= 0. \end{aligned} \tag{5.39}$$

The first two of these are just a statement of anti-symmetry and the Jacobi identity, that is,

$$\mathcal{L}_2 \circ T_{12} = T_{12} + T_{21} = 0, \quad \mathcal{L}_2 \circ T_{123} = T_{123} + T_{231} + T_{312} = 0, \tag{5.40}$$

however the higher order terms lead on to the generalised Jacobi identities, in the case of $n = 2$ we find,

$$\mathcal{L}_4 \circ T_{1234} = T_{1234} - T_{1243} + T_{3412} - T_{3421} = 0, \tag{5.41}$$

and as a result of these symmetries, only $(p-1)!$ permutations of the indices are independent instead of the usual $p!$. Furthermore, these identities imply the nested commutation relations and colour factor structure that occurs in Super Yang-Mills, that is $T_{12\ldots p} \leftrightarrow f^{12a_1}f^{a_1 3a_2}\ldots f^{a_{p-1}pa_p}$. It is imperative that the BRST operator respect these symmetries, so let us consider $Q(T_{12}+T_{21})$ [37, 7],

$$Q(T_{12}+T_{21}) = (k_1 \cdot k_2)\{T_1, T_2\} = 0, \tag{5.42}$$

which follows from the fact these are single particle fermionic objects and hence anticommute.

It would be useful now to define a relation that allows us to extend this recursive technology and build *multiparticle superfields*, denoted by $A^P_\alpha, A^m_P, W^\alpha_P, F^{mn}_P$, where $P$ is a word[2]. They must obey the Super Yang-Mills equations of motion (or at least a modified version that takes account of their multiparticle nature). They must also carry the same Lie symmetries as above - those fields that do satisfy these relations are accordingly called BRST building blocks. The rank two fields can be ascertained by considering the pure spinor superstring OPE between two integrated vertex operators. The results of this calculation are the following two-particle superfields [30, 37],

$$\begin{aligned}
A^{12}_\alpha &= \frac{1}{2}[A^2_\alpha(k_2 \cdot A_1) + A^m_2(\gamma_m W_1)_\alpha - (1 \leftrightarrow 2)], \\
A^m_{12} &= \frac{1}{2}[A^m_2(k_2 \cdot A_1) + A^1_p F^{pm}_2 + (W_1 \gamma^m W_2) - (1 \leftrightarrow 2)], \\
W^\alpha_{12} &= \frac{1}{4}(\gamma_{mn} W_2)^\alpha F^{mn}_1 + W^\alpha_2(k_2 \cdot A_1) - (1 \leftrightarrow 2), \\
F^{mn}_{12} &= k^m_{12} A^n_{12} - k^n_{12} A^m_{12} - (k_1 \cdot k_2)(A^m_1 A^n_2 - A^n_1 A^m_2),
\end{aligned} \tag{5.43}$$

where $k_{12} = k_1 + k_2$ and the $F^{mn}_{12}$ clearly generalises the single particle expression to a multiparticle expression in the obvious way, apart from the contact terms stemming from the single particle fields. In fact, these multiparticle superfields satisfy some equations of motion very similar to the single particle Super Yang-Mills equations. That is, they

[2]A word is an object that is composed of letters representing the particle in a field or object. For example, $P = 12345$ is a word and so is $P = mnpqrst$ - such objects obey a number of interesting mathematical identities that we do not cover here [7, 97].

satisfy,

$$\begin{aligned}
D_{(\alpha}A^{12}_{\beta)} &= \gamma^m_{\alpha\beta}A^{12}_m + (k_1\cdot k_2)(A^1_\alpha A^2_\beta + A^1_\beta A^2_\alpha),\\
D_\alpha A^{12}_m &= (\gamma_m W^{12})_\alpha + k^{12}_m A^{12}_\alpha + (k_1\cdot k_2)(A^1_\alpha A^2_m - A^2_\alpha A^1_m),\\
D_\alpha W^\beta_{12} &= \frac{1}{4}(\gamma_{mn})_\alpha{}^\beta F^{mn}_{12} + (k_1\cdot k_2)(A^1_\alpha W^\beta_2 - A^2_\alpha W^\beta_1),\\
D_\alpha F^{12}_{mn} &= k^{[m}_{12}(\gamma^{n]}W_{12})_\alpha + (k_1\cdot k_2)[A^1_\alpha F^{mn}_2 + A^{[n}_1(\gamma^{m]}W_2)_\alpha - (1\leftrightarrow 2)],
\end{aligned} \tag{5.44}$$

which are clearly generalisations of the original equations of motion. The single particle relations propagate up to the multiparticle level and imply the following,

$$k^m_{12}A^{12}_m = 0, \tag{5.45}$$

which is the Lorentz condition. The Dirac and Yang-Mills equations also propagate to this level, with some non-linear corrections, given by,

$$\begin{aligned}
k^{12}_m(\gamma^m W^{12})_\alpha &= (k_1\cdot k_2)[A^1_m(\gamma^m W^2)_\alpha - (1\leftrightarrow 2)]\\
k^m_{12}F^{12}_{mn} &= (k_1\cdot k_2)[A^{12}_n + A^1_n(k^1\cdot A^2) - (1\leftrightarrow 2)].
\end{aligned} \tag{5.46}$$

Since we want to calculate amplitudes, and those amplitudes are built out of (un)integrated vertex operators, it is useful to consider the multiparticle extensions of these operators also. The rank $P$ unintegrated vertex operator is defined in the obvious way as,

$$V_P = \lambda^\alpha A^P_\alpha, \tag{5.47}$$

note that the integrated vertex operator is defined in a similar manner - that is to promote the single particle fields to multiparticle fields, although we do not need its explicit form in the rest of this work. Note that when one builds a BRST invariant out of these $T_P$ building blocks, one can globally redefine $T_P$ to $V_P$ - the resulting expression will still be BRST invariant [37]. Taking the rank 2 fields as example, their BRST variations are,

$$QV^{12} = (k_1\cdot k_2)V_1V_2, \qquad QU^{12} = \partial V^{12} + (k^1\cdot k^2)(V^1U^2 - V^2U^1). \tag{5.48}$$

To obtain $QV^{12}$ we perform the following quick calculation using (5.44),

$$QV^{12} = \frac{1}{2}\lambda^\alpha\lambda^\beta D_{(\beta}A^{12}_{\alpha)} = \frac{1}{2}(k_1 \cdot k_2)(V_1V_2 + V_1V_2) = (k_1 \cdot k_2)(V_1V_2). \tag{5.49}$$

Generalising to rank three and beyond involves many of the same ideas save for one modification required in order to ensure the fields satisfy the Lie symmetries. Naïvely following the structure of the OPEs does produce three field and beyond superfields, however one has to add on extra supplementary pieces, collectively denoted as $H$, in order to ensure the Lie symmetries are obey and the fields can be BRST building blocks. Ultimately, the details of this redefinition are not required for the majority of the work here and only crop up when computing scattering amplitudes in `FORM` - something that forms only a minor part of the methods used here. As such we do not discuss the technicalities here, however we will mention them in passing below, for more information see [29, 96].

## 5.3 Berends-Giele Currents

The above superfields are local and as a result do not contain the kinematic pole structure required for an amplitude. One way around this is to introduce the notion of multiparticle Berends-Giele currents. These currents were originated as a way to compute, recursively, gluon scattering amplitudes [38]. These methods were then extended to give a $D = 10$, supersymmetric version that can be used in Super Yang-Mills. Hence for each of the superfields $K_P$, we define a ghost number zero Berends-Giele current denoted by $\mathcal{K}_P$ such that the following are true:

- They decorate cubic diagram involving $K_B$ with the correct propagator, and;
- they combine such as to give a colour-ordered Yang-Mills tree amplitude.

Such objects are the Berends-Giele currents defined in [7]. Let us consider the $p \leq 4$, in this case the Berends-Giele currents are defined as,

$$\begin{aligned}\mathcal{K}_{12} &= \frac{K_{12}}{s_{12}}, \quad \mathcal{K}_{123} = \frac{K_{123}}{s_{12}s_{123}} + \frac{K_{321}}{s_{23}s_{123}} \\ \mathcal{K}_{1234} &= \frac{1}{s_{1234}}\left(\frac{K_{1234}}{s_{12}s_{123}} + \frac{K_{3214}}{s_{23}s_{123}} + \frac{K_{3421}}{s_{34}s_{234}} + \frac{K_{3241}}{s_{23}s_{234}} + \frac{K_{[12,34]}}{s_{12}s_{34}}\right).\end{aligned} \tag{5.50}$$

Now we can then generalise the unintegrated vertex operator to the Berends-Giele equivalent,

$$\mathcal{V}_A := \lambda^\alpha \mathcal{A}^A_\alpha \equiv M_A, \tag{5.51}$$

where the BRST variation of this object is just given by a deconcatenation of words,

$$QM_P = \sum_{XY=P} M_X M_Y. \tag{5.52}$$

For example, we find that,

$$QM_{123} = M_1 M_{23} + M_{12} M_3 = \frac{V^1 V^{23}}{s_{23}} + \frac{V^{12}V^3}{s_{12}} = \frac{V_{[1,2],3]}}{s_{12}s_{123}} + \frac{V_{[1,[2,3]]}}{s_{23}s_{123}}, \tag{5.53}$$

this follows from the use of (5.38) to collect terms. The extension of this to the other matter fields is as follows [30],

$$\begin{aligned} Q\mathcal{A}^m_P &= (\lambda\gamma^m \mathcal{W}_P) + k^m_P \mathcal{V}_P + \sum_{j=1}^{|P|-1} (\mathcal{V}_{12\ldots j}\mathcal{A}^m_{j+1\ldots p} - \mathcal{V}_{j+1\ldots p}\mathcal{A}^m_{12\ldots j}), \\ Q\mathcal{W}^\alpha_P &= \frac{1}{4}(\lambda\gamma_{mn})^\alpha \mathcal{F}^{mn}_P + \sum_{j=1}^{|P|-1} (\mathcal{V}_{12\ldots j}\mathcal{W}^\alpha_{j+1\ldots p} - \mathcal{V}_{j+1\ldots p}\mathcal{W}^\alpha_{12\ldots j}), \\ Q\mathcal{F}^{mn}_P &= 2k^{[m}_P(\lambda\gamma^{n]}\mathcal{W}_P) + \sum_{j=1}^{|P|-1} (\mathcal{V}_{12\ldots j}\mathcal{F}^{mn}_{j+1\ldots p} - \mathcal{V}_{j+1\ldots p}\mathcal{F}^{mn}_{12\ldots j}) \\ &\quad + 2\sum_{j=1}^{|P|-1} (\mathcal{A}^{[n}_{12\ldots j}(\lambda\gamma^{m]}\mathcal{W}_{j+1\ldots p} - \mathcal{A}^{[n}_{j+1\ldots p}(\lambda\gamma^{m]}\mathcal{W}_{1\ldots j}), \end{aligned} \tag{5.54}$$

note that all of these Berends-Giele currents obey the shuffle symmetry in their indices, that is [30],

$$\mathcal{K}_{A\sqcup B} = 0 \quad \forall A, B \neq \emptyset, \tag{5.55}$$

as well as the Berends-Giele symmetry,

$$\mathcal{K}_{B1A} = (-1)^{|B|}\mathcal{K}_{1(A\sqcup B^t)}, \tag{5.56}$$

where $B^t$ represents the reverse ordering of the word $B$. These symmetries reduce the number of independent Feynman diagrams that contribute at any given point. At tree-level, the Yang-Mills amplitudes can be written as in terms of such blocks as [36],

$$A^{\text{YM}} = \langle E_{12\ldots n-1}V_n\rangle, \tag{5.57}$$

where $E_P$ is a BRST-exact superfield defined as,

$$E_P = \sum_{XY=P} M_X M_Y = QM_P, \tag{5.58}$$

and as we shall see in the next chapter, this structure is crucial to expressing the $n-$point tree-level amplitude in a compact manner.

Before closing this chapter it is worth emphasising what the Berends-Giele currents mean in an intuitive sense. These currents represent multiparticle fields, with the correct pole structure, and so one can imagine that they represent sub-diagrams of some larger process. This is immensely powerful because, as we shall see, it allows one to define any $n-$point tree-level amplitude in terms of the fundamental three-point vertex of Super Yang-Mills. Hence, any tree-level diagram can be drawn as a three-point interaction where each of the legs is a multiparticle superfield, representing many other tree-level interactions. This is what will lead to the notion of generating series, recursive amplitudes and the perturbiner in the next chapter.

# Chapter 6

# Generating Series

In this chapter we discuss further the connection between the Super Yang-Mills equations of motion in Chapter 4 and the Berends-Giele currents of the previous chapter. This connection is based upon the perturbiner method [71, 72, 73, 74] for solving the wave equations in Chapter 4 - one can expand the non-linear fields in terms of Lie algebra generators and at each order of the expansion, the equations of motion are solved.

## 6.1 Generating Series as Perturbiners

The superfields $\mathbb{K}$ can be cast a generating series of in which the superfield is expanded in terms of Lie-algebra generators $t^i$ with coefficients that are identified as multiparticle Berends-Giele currents [30]. Such a series can be represented as [29, 30],

$$\mathbb{K} \equiv \sum_{p=1}^{\infty} \sum_{i_1,\ldots,i_p} \mathcal{K}_{i_1\ldots i_p} t^{i_1} \ldots t^{i_p}, \tag{6.1}$$

and such an expansion can be shown to solve the non-linear Super Yang-Mills equations at each expansion level [30]. Hence these generating series are a solution to the non-linear equations of motion. The non-linear gauge transformations that accompany these fields can also be decomposed into a generating series of the form [29],

$$\Delta = \sum_{p=1}^{\infty} \sum_{i_1,i_2,\ldots,i_p} \Delta_{i_1 i_2 \ldots i_p} t^{i_1} t^{i_2} \ldots t^{i_p}, \qquad \Delta_{A \sqcup\!\sqcup B} = 0, \quad \forall A, B \neq \emptyset, \tag{6.2}$$

this allows us to change the gauge of the generating series - later in this chapter we shall discuss the Harnd-Shnider gauge which simplifies the process for finding the $\theta-$expansions of the superfields. We could also choose other gauges, such as the BCJ gauge which makes gauge-gravity duality manifest. In fact, one can also solve the equations using the local vertex prescription outlined in [36], however the Berends-Giele current approach has a slight advantage in the fact that it automatically contains the required pole structure due to the recursion,

$$\mathcal{K}_P \equiv \frac{1}{s_P} \sum_{XY=P} \mathcal{K}_{[X,Y]}, \tag{6.3}$$

where $P = i_1 i_2 ... i_p$ is a multiparticle label referring to a multiparticle external state. The equations $XY = P$ states that the word $P$ should be deconcatenated into smaller words, preserving the order. For example, let $P = 123$ we would then have, $X = \emptyset, Y = 123$, $X = 1, Y = 23$, $X = 12, Y = 3$ and $X = 123, Y = \emptyset$. Multiparticle momenta is defined as $k_P^m \equiv \sum_j k_{i_j}^m$ and the associated Mandelstam invariant $s_P \equiv \frac{1}{2}k_P^2$. Note that, even though the Super Yang-Mills theory contains a quartic interaction, it is captured by the cubic interactions in the Berends-Giele currents [98]. In fact, on a conceptual level, the recursion outlined above starts a tree-level interaction with a cubic interaction with multiparticle legs and then expands them down to single particle legs. In this sense, using the multiparticle fields, all tree-level diagrams can be written as a cubic interaction and in Super Yang-Mills that cubic interaction is given by (1.2)[1]. The form of $\mathcal{K}_{[X,Y]}$, in Lorentz gauge, for each of the fields are given by [29],

$$\begin{aligned}
\mathcal{A}_\alpha^{[X,Y]} &= \frac{1}{2}[\mathcal{A}_\alpha^X (k_X \cdot \mathcal{A}_Y) + \mathcal{A}_X^m (\gamma_m \mathcal{W}_Y)_\alpha - (X \leftrightarrow Y)], \\
\mathcal{A}^m_{[X,Y]} &= \frac{1}{2}[\mathcal{A}_X^m (k_X \cdot \mathcal{A}_Y) + \mathcal{A}_n^X \mathcal{F}_Y^{nm} + (\mathcal{W}_X \gamma_m \mathcal{W}_Y) - (X \leftrightarrow Y)], \\
\mathcal{W}^\alpha_{[X,Y]} &= \frac{1}{4}\mathcal{F}_X^{rs}(\gamma_{rs}\mathcal{W}_Y)^\alpha + \frac{1}{2}\mathcal{W}_Y^\alpha (k_Y \cdot \mathcal{A}_X) + \frac{1}{2}\mathcal{W}_Y^{m\alpha}\mathcal{A}_X^m - (X \leftrightarrow Y), \\
\mathcal{F}_P^{mn} &= k_P^m \mathcal{A}_P^n - k_P^n \mathcal{A}_P^m - \sum_{XY=P} (\mathcal{A}_X^m \mathcal{A}_Y^n - \mathcal{A}_Y^m \mathcal{A}_X^n),
\end{aligned} \tag{6.4}$$

[1]This fact allows one to use the powerful mathematics of planar binary trees when considering these amplitudes [99].

where higher mass fields can be defined using their commutator definitions, for example,

$$\mathcal{W}_P^{m\alpha} = k_P^m \mathcal{W}_P^\alpha + \sum_{XY=P} (\mathcal{W}_X^\alpha \mathcal{A}_Y^n - \mathcal{W}_Y^\alpha \mathcal{A}_X^m). \tag{6.5}$$

In general, we replace $[\partial_m, \mathcal{K}]_P = k_m^P \mathcal{K}_P$ and $[\mathcal{K}, \mathcal{K}']_P = \sum_{XY=P}(\mathcal{K}_X \mathcal{K}'_Y - \mathcal{K}_Y \mathcal{K}'_X)$. In fact these deconcatentations follow directly from non-linear wave equations in Section 4.2, noting that $\Box e^{k_P \cdot X} = 2s_P e^{k_P \cdot X}$. Noting this we can then determine the expression that gives $\mathcal{V}^m_{[X,Y]}$ - which will follow from (4.36),

$$\begin{aligned}
\mathcal{V}_P^m = -\frac{1}{2} \sum_{XY=P} \big[&\mathcal{V}_X^m (k_X \cdot \mathcal{A}_Y) + \mathcal{A}_X^n (\lambda \gamma^n \mathcal{W}_Y^m) + \mathcal{A}_X^{m|n} (\lambda \gamma^n \mathcal{W}_Y) \\
&+ \mathcal{V}_X (k \cdot \mathcal{A}_Y^{|m}) - \mathcal{A}_X^m (k_X \cdot \mathcal{V}_Y) - (X \leftrightarrow Y)\big] \\
+ \frac{1}{2} \sum_{XYZ=P} \big[&\mathcal{V}_X (k_X \cdot \mathcal{A}_Y) \mathcal{A}_Z^m + (k_Z \cdot \mathcal{A}_X) \mathcal{V}_Y \mathcal{A}_Z^m - (X \leftrightarrow (Y \leftrightarrow Z))\big],
\end{aligned} \tag{6.6}$$

where $(X \leftrightarrow (Y \leftrightarrow Z))$ denotes swapping $X \leftrightarrow YZ$ as well as $Y \leftrightarrow Z$. For example, consider $P = 1234$ split in the first instance as $X = 12$, $Y = 3$, $Z = 4$. Then $(X \leftrightarrow (Y \leftrightarrow Z))$ implies the following set of combinations: $(12, 4, 3)$, $(34, 1, 2)$, $(34, 2, 1)$. Of course, one could simplify this noting (6.5) and find,

$$\mathcal{V}_P^m = k_P^m \mathcal{V}_P + \sum_{XY=P} (\mathcal{V}_X \mathcal{A}_Y^m - \mathcal{V}_Y \mathcal{A}_X^m). \tag{6.7}$$

Now to demonstrate that the generating series is in fact a perturbiner we simply substitute the series (6.1) into the non-linear equations of motion (4.5), upon inspecting each order of the expansion we find the Berends-Giele equations of motion. This is essentially what a perturbiner is: a sum of solutions to the equations of motion at each order. The Berends-Giele equations of motion are then given by [29],

$$\begin{aligned}
D_{(\alpha} \mathcal{A}_{\beta)}^P &= \gamma_{\alpha\beta}^m \mathcal{A}_\alpha^P + \sum_{XY=P} (\mathcal{A}_\alpha^X \mathcal{A}_\beta^Y - \mathcal{A}_\alpha^Y \mathcal{A}_\beta^X), \\
D_\alpha \mathcal{A}_m^P &= k_m^P \mathcal{A}_\alpha^P + (\gamma_m \mathcal{W}_P)_\alpha + \sum_{XY=P} (\mathcal{A}_\alpha^X \mathcal{A}_m^Y - \mathcal{A}_\alpha^Y \mathcal{A}_m^X), \\
D_\alpha \mathcal{W}_P^\beta &= \frac{1}{4} (\gamma_{mn})_\alpha{}^\beta \mathcal{F}_P^{mn} + \sum_{XY=P} (\mathcal{A}_\alpha^X \mathcal{W}_Y^\beta - \mathcal{A}_\alpha^Y \mathcal{W}_X^\beta), \\
D_\alpha \mathcal{F}_P^{mn} &= -(\gamma^{[m} \mathcal{W}_P^{n]})_\alpha + \sum_{XY=P} (\mathcal{A}_\alpha^X \mathcal{F}_Y^{mn} - \mathcal{A}_\alpha^Y \mathcal{F}_X^{mn}),
\end{aligned}$$

hence the Berends-Giele currents, summed into a generating series, solve the non-linear equations of motion of Super Yang-Mills in $D = 10$. Note also that the Lorenz gauge, as well as the Dirac and Yang-Mills equations, are also satisfied [29],

$$\begin{aligned} k_m^P \mathcal{A}_P^m &= 0, \\ k_m^p (\gamma^m \mathcal{W}_P)_\alpha &= \sum_{XY=P} [\mathcal{A}_m^X (\gamma^m \mathcal{W}_Y)_\alpha - \mathcal{A}_m^Y (\gamma^m \mathcal{W}_X)_\alpha], \\ k_m^P \mathcal{F}_P^{mn} &= \sum_{XY=P} [2(\mathcal{W}_X \gamma^n \mathcal{W}_Y) + \mathcal{A}_m^X \mathcal{F}_Y^{mn} - \mathcal{A}_m^Y \mathcal{F}_X^{mn}], \end{aligned} \tag{6.8}$$

and so the generating series solves all of the equations we reqiure. One can define finite gauge transformations for the Berends-Giele currents in order to move them into BCJ gauge, or any other gauge of choice. However, this is not particularly important here, so we refer to the reader to Section 4.2.5 in [7] for further information. We will, however, discuss gauge transformations into the Harnad-Shnider gauge as this greatly simplifies the calculation of amplitudes.

## 6.2 Harnad-Shnider Gauge

During the calculation of the string theory tree-level amplitude, we showed a $\theta$ expansion for $A_\alpha$ in terms of single particle polarisation tensors. One can extend this to multiparticle fields, however doing so in Lorenz gauge is quite difficult and in order to avoid this one can perform a finite gauge transformation into the so-called Harnad-Shnider gauge which obeys [39],

$$\theta^\alpha \mathbb{A}_\alpha = 0. \tag{6.9}$$

This gauge is not preserved in Lorenz gauge - this requires us to perform the non-linear transform,

$$\mathbb{A}_\alpha^{\mathrm{HS}} = \mathbb{A}_\alpha^{\mathrm{L}} - [D_\alpha, \mathbb{L}] + [\mathbb{A}_\alpha^{\mathrm{L}}, \mathbb{L}], \tag{6.10}$$

where $\mathbb{L}$ is the generating series of the gauge transformation. If we now define the Euler operator as $\mathcal{D} = \theta^\alpha \partial_\alpha$, then the above equation can be re-written as,

$$[\mathcal{D}, \mathbb{L}] = \theta^\alpha \mathbb{A}_\alpha^{\mathrm{L}} + [\theta^\alpha \mathbb{A}_\alpha^{\mathrm{L}}, \mathbb{L}], \tag{6.11}$$

and so the Berends-Giele current transformations $\mathcal{L}_P$ must satisfy,

$$\mathcal{D}\mathcal{L}_P = \theta^\alpha \mathcal{A}^P_\alpha + \sum_{XY=P} (\theta^\alpha \mathcal{A}^X_\alpha \mathcal{L}_Y - \theta^\alpha \mathcal{A}^Y_\alpha \mathcal{L}_X), \tag{6.12}$$

where $\mathcal{L}_i = 0$. Thus for $P = 12$, the gauge transformation must satisfy,

$$\mathcal{D}\mathcal{L}_{12} = \theta^\alpha \mathcal{A}^{12}_\alpha \implies \mathcal{L}_{12} = \frac{1}{s_{12}} \theta^\alpha \mathcal{A}^{12}_\alpha, \tag{6.13}$$

and so in this way we can transform the fields from one gauge to another. Now, let us assume that this step has been taken such that all of the fields in the following are in Harnad-Shnider gauge. Upon contracting $\theta^\alpha$ into (4.5) we find the following,

$$\begin{aligned} (\mathcal{D}+1)\mathbb{A}_\alpha &= (\theta\gamma^m)_\alpha \mathbb{A}_m, \qquad & \mathcal{D}\mathbb{A}_m &= (\theta\gamma_m \mathbb{W}), \\ \mathcal{D}\mathbb{W}^\beta &= \frac{1}{4}(\theta\gamma^{mn})^\beta \mathbb{F}_{mn}, & \mathcal{D}\mathbb{F}^{mn} &= (\theta\gamma^{[m}\mathbb{W}^{n]}) \end{aligned} \tag{6.14}$$

which can be inverted to give the recursive definition of the entire set of $\theta$ expansions for the non-linear fields. Inverting these gives the following set of definitions [100],

$$\begin{aligned} [\mathbb{A}_\alpha]_k &= \frac{1}{k+1}(\theta\gamma^m)_\alpha [\mathbb{A}_m]_{k-1}, \quad & [\mathbb{A}_m]_k &= \frac{1}{k}(\theta\gamma_m [\mathbb{W}]_{k-1}), \\ [\mathbb{W}^\alpha]_k &= \frac{1}{4k}(\theta\gamma^{mn})^\alpha [\mathbb{F}_{mn}]_{k-1}, & [\mathbb{F}^{mn}]_k &= \frac{1}{k}(\theta\gamma^{[m}[\mathbb{W}^{n]}]_{k-1}), \end{aligned} \tag{6.15}$$

where the higher mass terms follow from (4.16). The recursions defined above begin with the $0^{th}$ order part of the non-linear fields, that is $[\mathbb{K}]_0$ and their corresponding multiparticle Berends-Giele equivalent $[\mathcal{K}_P]_0$ is initiated by the following components,

$$[\mathcal{A}^m_P]_0 \equiv \mathfrak{e}^m_p e^{k_P\cdot x}, \quad [\mathcal{W}^\alpha_P]_0 \equiv \mathfrak{g}^\alpha_P e^{k_P\cdot x}, \quad \mathfrak{f}^{mn}_P \equiv k^m_P \mathfrak{e}^n_P - k^n_P \mathfrak{e}^m_P - \sum_{XY=P} (\mathfrak{e}^m_X \mathfrak{e}^n_Y - \mathfrak{e}^m_Y \mathfrak{e}^n_X), \tag{6.16}$$

where $\mathfrak{e}^m_P$ and $\chi^\alpha_P$ represent multiparticle polarisation tensors - these can be deconcatenated to single particle polarisation tensors by using analogues of (6.3) and (6.4). However, as we shall see in the next section, they allow one to express the $n-$point tree-level amplitude in a succinct and familiar form. Note that these mutliparticle polarisations inherit the Lorenz gauge condition, that is they obey $k_P \cdot \mathfrak{e}_P = 0$. Taking all of this in hand, and using (6.15), one can determine the Berends-Giele current $\theta$ expansions very quickly, they are given by,

$$\begin{aligned}
\mathcal{A}_\alpha^P &= \frac{1}{2}(\theta\gamma_m)_\alpha \mathfrak{e}_P^m + \frac{1}{3}(\theta\gamma_m)_\alpha(\theta\gamma^m \mathfrak{g}_P) - \frac{1}{32}(\theta\gamma^m)_\alpha(\theta\gamma_{mnp}\theta)\mathfrak{f}_P^{np} + ..., \\
\mathcal{A}_m^P &= \mathfrak{e}_P^m + (\theta\gamma^m \mathfrak{g}_P) - \frac{1}{8}(\theta\gamma^{mnp}\theta)\mathfrak{f}_{np}^P + \frac{1}{12}(\theta\gamma^{mnp}\theta)(\theta\gamma_n \mathfrak{g}_p^P) + ..., \\
\mathcal{W}_P^\alpha &= \mathfrak{g}_P^\alpha + \frac{1}{4}(\theta\gamma^{mn})^\alpha \mathfrak{f}_{mn}^P - \frac{1}{4}(\theta\gamma_{mn})^\alpha(\theta\gamma^m \mathfrak{g}_P^n) - \frac{1}{48}(\theta\gamma^{mq})^\alpha(\theta\gamma_{qnp}\theta)\mathfrak{f}_P^{m|np} + ..., \\
\mathcal{F}_P^{mn} &= \mathfrak{f}_P^{mn} - (\theta\gamma^{[m}\mathfrak{g}_P^{n]}) + \frac{1}{8}(\theta\gamma^{pq[m}\theta)\mathfrak{f}_P^{n]|pq} - \frac{1}{12}(\theta\gamma^{pq[m}\theta)(\mathfrak{g}_P^{n]p}\gamma^q\theta) + ...,
\end{aligned} \tag{6.17}$$

where we give the expansion to $\theta^3$, although in actually calculations we often need to go to $\theta^5$. Going to higher orders is only required when we start to investigate corrections, because as we shall see, tree-level SYM amplitudes only involves $\langle\mathbb{VVV}\rangle$ - and so we will only require the $\mathcal{A}_\alpha^P$ expansion in this chapter owing to the pure spinor integration measure.

## 6.3 Tree Level Amplitudes

The Harnad-Shnider gauge drastically reduces the amount of work required in order to obtain $n-$point amplitudes. This follows because we can work at the level of multiparticle superfields to find any terms that contain $\lambda^3\theta^5$ - which is easily given the above $\theta-$expansions. Then from these expressions we can use the multiparticle polarisation deconcatenations to find the amplitudes. Tree-level amplitudes are summarised by the BRST block [98],

$$\langle M_A M_B M_C \rangle, \tag{6.18}$$

for $M_A \equiv \lambda^\alpha \mathcal{A}_\alpha^A$ - where $\mathcal{A}_\alpha^A$ is the multiparticle Berends-Giele current for the prespinor potential. The BRST invariant combinations of such blocks will descend from the generating series given by,

$$\mathrm{Tr}\langle\mathbb{VVV}\rangle \propto \sum_{n=3}^{\infty}(n-2)\sum_{i_1<i_2<...<i_n}\mathcal{M}_{SYM}(i_1, i_2, ..., i_n), \tag{6.19}$$

where $\mathcal{M}_{SYM}$ are the colour-dressed super Yang-Mills amplitudes. It is rather simple to show that the generating series $\mathrm{Tr}\langle\mathbb{VVV}\rangle$ is BRST closed. The BRST variation of $\mathbb{V}$ is

| $\theta$ Component | | | |
|---|---|---|---|
| $\mathcal{A}^A_\alpha$ | $\mathcal{A}^B_\beta$ | $\mathcal{A}^C_\gamma$ | Components |
| 1 | 1 | 3 | $-\frac{1}{128}(\theta\gamma_m)_\alpha(\theta\gamma_n)_\beta(\theta\gamma_p)_\gamma(\theta\gamma^{pqr}\theta)\mathfrak{e}^m_A\mathfrak{e}^n_B\mathfrak{f}^C_{qr}$ |
| 1 | 3 | 1 | $-\frac{1}{128}(\theta\gamma_m)_\alpha(\theta\gamma_n)_\beta(\theta\gamma^{npq}\theta)(\theta\gamma_p)_\gamma\mathfrak{e}^m_A\mathfrak{f}^B_{pq}\mathfrak{e}^r_C$ |
| 3 | 1 | 1 | $-\frac{1}{128}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\theta\gamma_q)_\beta(\theta\gamma_r)_\gamma\mathfrak{f}^{np}_A\mathfrak{e}^B_q\mathfrak{e}^r_C$ |
| 2 | 2 | 1 | $\frac{1}{18}(\theta\gamma_m)_\alpha(\theta\gamma^m\mathfrak{g}_A)(\theta\gamma_n)_\beta(\theta\gamma^n\mathfrak{g}_B)(\theta\gamma_p)_\gamma\mathfrak{e}^p_C$ |
| 2 | 1 | 2 | $\frac{1}{18}(\theta\gamma_m)_\alpha(\theta\gamma^m\mathfrak{g}_A)(\theta\gamma_n)_\beta\mathfrak{e}^n_B(\theta\gamma_p)_\gamma(\theta\gamma^p\mathfrak{g}_C)$ |
| 1 | 2 | 2 | $\frac{1}{18}(\theta\gamma_m)_\alpha\mathfrak{e}^m_A(\theta\gamma_n)_\beta(\theta\gamma^n\mathfrak{g}_B)(\theta\gamma_p)_\gamma(\theta\gamma^p\mathfrak{g}_C)$ |

Table 6.1: Various multiparticle component expressions for the tree-level amplitude.

given by $\{Q,\mathbb{V}\} = \frac{1}{2}\{\mathbb{V},\mathbb{V}\}$ and hence the variation of the tree level generating series is $\mathrm{Tr}\langle\mathbb{V}\mathbb{V}\mathbb{V}\mathbb{V}\rangle$, but owing to the cyclicity of the trace as well as the fermionic statistics of $\mathbb{V}$ the above result is zero. Now note that (6.19) is invariant under gauge transformations and so we can apply Harnad-Shnider gauge and expand the BRST blocks to give,

$$\langle M_A M_B M_C\rangle = \frac{1}{2}\mathfrak{e}^m_A\mathfrak{e}^n_B\mathfrak{f}^C_{mn} + (\mathfrak{g}_A\gamma_m\mathfrak{g}_B)\mathfrak{e}^m_C, \tag{6.20}$$

and so the $n-$point amplitude can be written succinctly in terms of the three-point amplitudes, where the single particle polarisations are replaced with multiparticle polarisations. In the following, we detail the calculations required to get to this.

### Finding the BRST Block in Components

Since $\langle M_A M_B M_C\rangle = \langle\lambda^\alpha\lambda^\beta\lambda^\gamma\mathcal{A}^A_\alpha\mathcal{A}^B_\beta\mathcal{A}^C_\gamma\rangle$ we have already satisfied the $\lambda^3$ part of the pure spinor integration measure. Hence we only need to consider finding the $\theta^5$ part of the $\mathcal{A}^P_\alpha$ product. We show these in Table 6.1. Let us take the first and fourth entries as examples to demonstrate how to go from these raw expansions to the required expressions. Contracting the $\lambda$ into the expressions in the table, we find that the first entry gives,

$$\begin{aligned} A_{113} &= \frac{1}{128}\langle(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{qrp}\theta)\rangle\mathfrak{e}^m_A\mathfrak{e}^n_B\mathfrak{f}^C_{qr} \\ &= 32(\delta^q_m\delta^r_n - \delta^r_m\delta^q_n)\mathfrak{e}^m_A\mathfrak{e}^n_B\mathfrak{f}^C_{qr} = \frac{1}{2}\mathfrak{e}^m_A\mathfrak{e}^n_B\mathfrak{f}^C_{qr}. \end{aligned} \tag{6.21}$$

In the case of the fourth entry in the table we have the following,

$$\begin{aligned} A_{221} &= -\frac{1}{18}\langle(\lambda\gamma_p\theta)(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\theta\gamma^m)_\alpha(\theta\gamma^n)_\beta\rangle\mathfrak{g}_A^\alpha\mathfrak{g}_B^\beta\mathfrak{e}_C^p \\ &= (\mathfrak{g}_A\gamma_p\mathfrak{g}_B)\mathfrak{e}_C^p, \end{aligned} \tag{6.22}$$

where $\langle(\lambda\gamma_p\theta)(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\theta\gamma^m)_\alpha(\theta\gamma^n)_\beta\rangle = -18\gamma^p_{\alpha\beta}$. Repeating these steps for the other parts of the table gives the BRST building block in multiparticle polarisations.

For the three point amplitude we simply replace $\mathfrak{e}_A^m \to e_m^i$, $\mathfrak{f}_{mn}^C = k_m^i e_n^i - k_n e_m^i$, and $\mathfrak{g}_A \to \chi_i$ and one obtains,

$$\langle M_1M_2M_3\rangle = \frac{1}{2}(e_1\cdot k_3)(e_2\cdot e_3) - (e_2\cdot k_3)(e_1\cdot e_3) + \chi_1\gamma^m\chi_2 e_m^3 + \mathrm{cyc}(1,2,3), \tag{6.23}$$

however, note that $k_3 = -k_2 - k_1$, due to momentum conservation and hence we obtain the amplitude found in the string theory case,

$$\begin{aligned} \langle M_1M_2M_3\rangle &= \frac{1}{2}(e_1\cdot k_3)(e_2\cdot e_3) + (e_2\cdot k_1)(e_1\cdot e_3) + \chi_1\gamma^m\chi_2 e_m^3 + \mathrm{cyc}(1,2,3), \\ &= \frac{1}{2}e_1^m f_2^{mn} e_3^n + (\chi_1\gamma^m\chi_2)e_3^m + \mathrm{cyc}(1,2,3). \end{aligned} \tag{6.24}$$

## 6.4 Introducing $\mathbb{V}_i$

Many of the expressions used in the next chapter are not actually constructed with the field $\mathbb{V}$ - which can contain any particle labels. Rather, the $\mathbb{V}$ in many of the expressions below is a labelled field denoted by $\mathbb{V}_i$. This means than the expansion of $\mathbb{V}_i$ into Berends-Giele currents and components must contain the particle $i$ in its expansion. The main reason for the introduction of this non-linear field is that it is BRST closed and we are able to find generating series expressions which contain it, hence $\{\mathcal{Q}, \mathbb{V}_i\} = 0$. Without this field the BRST variation of the ansatz would produce terms with $\mathbb{V}\mathbb{V}$ which are hard to eliminate. A more important motivation can be found when looking at 1-loop amplitudes in the pure spinor superstring theory [41, 42] and Super Yang-Mills [101]. It was found that upon calculating the disk amplitudes the BRST closed expressions contained $\mathbb{V}_1$ [41]. Given the potential link between $\alpha'$ corrections and loop kinematic factors, it does not seem unreasonable to expect the presence of such a field in the corrections. However, as we shall elucidate in later chapters, there is a potential way to define the corrections without use of $\mathbb{V}_1$ - this technique involving interpreting the Chern-Simons-like action of the theory in a novel way. This method is in its infancy but initial investigations are promising.

The key takeaway here is that if one finds a generating series expression, such as $\langle \mathbb{V}\mathbb{V}\mathbb{V} \rangle$, then going from this expression to the component expansions at arbitrary order is quite simple. In fact, in `FORM` applying these methods can lead to very quick computations. As such, in order to determine $\alpha'$ corrections, that stem from the field theory limit of string amplitudes, one simply has to find BRST closed generating series expressions. This is what we turn to in the next Part.

# Part II

# Finding Corrections

# Introduction to Part II

This part details the majority of original research in this project and covers a number of topics which have been used to attempt to find the $\alpha'$ corrections. The first chapter details how one can build the ansatz using a mixture of group theory arguments and graph theory tricks in order to capture all the terms that could possible contribute. The next chapter details some of the novel identities that have been derived in order to canonicalise the ansatz as well as its variation. Chapter 9 details the calculations used in order to find the $\alpha'$ and $\alpha'^2$ generating series. The calculations are done by hand and give a very good demonstration of the techniques used throughout this work. The following chapter give the novel $\alpha'^3$ corrections in a number of forms and details some BRST exact identities that may be useful in future research. Chapter 11 generalises the idea of the higher mass fields to $\mathbb{V}$ and $\mathbb{A}^m$ - this was done in the hope of uncovering some new way of expressing the corrections and as a remedy to some of the issues highlighted in other chapters. The final chapter of this part presents the current state-of-the-art knowledge concerning the $\alpha'^4$ corrections as well as associated issues with this correction's derivation and a few possible ways to overcome these issues using novel techniques.

# Chapter 7

# Building the Ansatz

To find the higher order corrections we first generate an ansatz that produces all of the terms that can appear at a given order by group theory arguments. We initially assume that any correction must be composed of $\mathbb{W}$ and $\mathbb{F}$ as these are the gauge invariant combinations of the gauge field, as well as any terms that contain a single $\mathbb{V}_i$. Then we apply the BRST operator to the ansatz and set the result to zero, thus implying BRST closure. This produces a number of linear equations of the unknown coefficients in front of each term in the ansatz. That is, the BRST variation of the ansatz produces a series of terms of the form,

$$f_i(x_j)\mathcal{S}_i(\mathbb{K}) = 0, \tag{7.1}$$

where we assume $\mathcal{S}_i(\mathbb{K}) \neq 0$. Ensuring that we capture all the cases where $\mathcal{S}_i(\mathbb{K}) = 0$ is the motivation behind the canonicaliztion we shall undertake in the coming chapters as we want to guarantee that the only solution to the above equation follows from $f_i(x_j) = 0$. These linear equations can be solved to give the terms which are BRST closed and hence form the higher order correction. There is some redundancy in these linear equations, a point we shall come to in the next chapter. Typically one has to then compare the result from the BRST variation with a known result from string theory in order to fix the last constant - this is due to various Mandelstam invariants being invariant under the action of the BRST operator. Effectively, this means we can multiply a generating series expression by certain function of the generalised Mandelstam variable

$s_P$ and the result remains BRST closed. However, this new generating series may not give the correct component expansion. Hence we use a result from the field theory limit of string theory to fix this freedom, such that the component expansion reproduces the known result. It is clear that the structure of the ansatz is of fundamental importance to finding the correct higher order generating series, thus simplifying this ansatz as much as possible is a rather important part of finding any correction - without these simplifications one cannot guarantee that the linear equations are correct. Furthermore, without such simplifications the ansatz could contain many thousands of terms that are redundant or automatically zero due to the pure spinor constraint.

This chapter will focus primarily on the methods used to build the ansatz and in order to build the ansatz we need to answer three questions: what terms are allowed by group theoretic constraints?; are there any relations between fields that allow us to put everything in a simple basis?; and what are the possible vector index contractions between the allowed fields? In this chapter we aim to answer the first and last questions - giving the most general ansatz we can, considering only the most basic restrictions. The following chapter details a number of new identities which can be used to simplify the ansatz and canonicalise any BRST variation of the superfields. These identities are not only useful here but can be used in other areas of research including loop calculations in string theory.

## 7.1 Dynkin Labels of $SO(10)$

In order to begin restricting what generating series expressions can appear at each correction order of the ansatz we first begin by determining which terms are allowed by the $SO(10)$ group. This part of the procedure allows us to determine which terms are non-zero under the $SO(10)$ group structure and so this generally takes account of the Bianchi identities between $\gamma$ and $\mathbb{F}$, as well as other group based identities. In order to do this we turn to the Dynkin label decomposition of each ansatz term - see [7, 102, 103] for more detail. Note that all of the decompositions here are performed using `LiE` [104] - a programme that performs Dynkin label decomposition automatically. The fundamental

Dynkin labels for $\mathfrak{so}(10)$ algebra are given by the following [7, 105],

$$\begin{aligned} \text{Scalar } A &= (0,0,0,0,0), \\ \text{Vector } A^{m} &= (1,0,0,0,0), \\ \text{2-Form } A^{[mn]} &= (0,1,0,0,0), \\ \text{3-Form } A^{[mnp]} &= (0,0,1,0,0), \\ \text{Anti-Weyl } \xi_{\alpha} &= (0,0,0,1,0), \\ \text{Weyl } \xi^{\alpha} &= (0,0,0,0,1), \end{aligned} \tag{7.2}$$

where the last five correspond to the five eigenvalues associated with the rank five Lie algebra $D_5$ - these are the basis labels that we use to determine the labels of higher mass fields. The pure spinor, $\lambda^{\alpha}$, is a Weyl spinor and a product of $n$ such pure spinors has the Dynkin label,

$$\text{Pure Spinor } (\lambda^{\alpha})^{n} = (0,0,0,0,n). \tag{7.3}$$

One can then use the anti-symmetric tensor product of these labels to find higher mass objects, for example the antisymmetric four tensor has the form, $A^{[mnpq]} = (0,0,0,1,1)$, which is the anti-symmetric tensor product of four vector labels - exactly what one expects. We can also consider the 5-form which is given by $A^{[mnpqr]} = (0,0,0,2,0) \oplus (0,0,0,0,2)$, which is the sum of two other representations. To understand the individual pieces of this decomposition we can work with a slightly simpler situation which combines the above with the knowledge of Fierz decompositions. We have the following decompositions between Weyl and anti-Weyl spinors,

$$\begin{aligned} \psi^{\alpha}\chi^{\beta} &= \frac{1}{16}\gamma_{m_1}^{\alpha\beta}(\psi\gamma_{m_1}\chi) + \frac{1}{96}(\gamma_{m_1\ldots m_3})^{\alpha\beta}(\psi\gamma^{m_1\ldots m_3}\chi) \\ &\quad + \frac{1}{3840}(\gamma_{m_1\ldots m_5})^{\alpha\beta}(\psi\gamma^{m_1\ldots m_5}\chi), \\ \psi_{\alpha}\chi^{\beta} &= \frac{1}{16}\delta_{\alpha}^{\beta}(\psi\chi) + \frac{1}{32}(\gamma_{m_1m_2})_{\alpha}{}^{\beta}(\psi\gamma^{m_1m_2}\chi) \\ &\quad + \frac{1}{3840}(\gamma_{m_1\ldots m_4})_{\alpha}{}^{\beta}(\psi\gamma^{m_1\ldots m_4}\chi), \\ \psi_{\alpha}\chi_{\beta} &= \frac{1}{16}\gamma_{\alpha\beta}^{m_1}(\psi\gamma_{m_1}\chi) + \frac{1}{96}(\gamma_{m_1\ldots m_3})_{\alpha\beta}(\psi\gamma^{m_1\ldots m_3}\chi) \\ &\quad + \frac{1}{3840}(\gamma_{m_1\ldots m_5})_{\alpha\beta}(\psi\gamma^{m_1\ldots m_5}\chi). \end{aligned} \tag{7.4}$$

The Dynkin labels of these products can be found using `LiE` [104] and are given by the tensor sums,

$$\begin{aligned}\psi^\alpha\chi^\beta &= (1,0,0,0,0)\oplus(0,0,1,0,0)\oplus(0,0,0,0,2),\\ \psi_\alpha\chi^\beta &= (0,0,0,0,0)\oplus(0,1,0,0,0)\oplus(0,0,0,1,1),\\ \psi_\alpha\chi_\beta &= (1,0,0,0,0)\oplus(0,0,1,0,0)\oplus(0,0,0,2,0).\end{aligned}\tag{7.5}$$

One can then compare the results of the 5-form decomposition with the results from the Fierz decompositions and determine their form. These results then imply the following useful relations,

$$\begin{aligned}(\psi\gamma^m\chi) &\leftrightarrow (1,0,0,0,0),\\ (\psi\gamma^{mn}\chi) &\leftrightarrow (0,1,0,0,0),\\ (\psi\gamma^{mnp}\chi) &\leftrightarrow (0,0,1,0,0),\\ (\psi\gamma^{mnpq}\chi) &\leftrightarrow (0,0,0,1,1),\\ \psi^\alpha\gamma^{mnpqr}_{\alpha\beta}\chi^\beta &\leftrightarrow (0,0,0,0,2),\\ \psi_\alpha\gamma_{mnpqr}^{\alpha\beta}\chi_\beta &\leftrightarrow (0,0,0,2,0),\end{aligned}\tag{7.6}$$

where we note that there are two different types of five form: the self dual form $(0,0,0,0,2)$ and the anti-self dual form $(0,0,0,2,0)$ which correspond to whether the $\gamma$ matrix spinor indices are anti-Weyl or Weyl respectively. Note that, owing to the self-duality of $\gamma^{mnpqr}$ in $D=10$, that is,

$$\gamma^{m_1m_2m_3m_4m_5} = \frac{1}{5!}\epsilon^{m_1\ldots m_5n_1\ldots n_5}\gamma_{n_1n_2n_3n_4n_5},\tag{7.7}$$

we have $\gamma^{mnpqr}\gamma^{mnpqr}=0$. There are other duality property identities similar to the above which link $\gamma^3$ with $\gamma^7$ and $\gamma^1$ with $\gamma^9$ - hence we can usually eliminate $\gamma^7$ and $\gamma^9$ in favour of $\gamma^3$ and $\gamma^1$ respectively [7].

In order to determine the correct ansatz we are ultimately interested in the number, and type, of scalar structures that can be formed from various different fields - scalar terms are terms that are fully contracted and do not have any free indices. Again we can use `LiE` to determine the number of such contractions possible from a given set of fields. For example, consider the product of two 2-form tensors, the decomposition of such a product is given by,

$$\begin{aligned} A^{[mn]} \otimes A^{[pq]} \leftrightarrow & (0,0,0,0,0) \oplus (0,0,0,1,1) \oplus (0,1,0,0,0) \\ & \oplus (0,2,0,0,0) \oplus (1,0,1,0,0) \oplus (2,0,0,0,0), \end{aligned} \tag{7.8}$$

hence there is only one independent scalar term that can be constructed from two 2-forms, $F_{mn}F_{mn}$; the same is true when the product is between three 2-forms, the only scalar that can be constructed is $F_{mn}F_{np}F_{pm}$. The other parts of the decompositions, for example $(2,0,0,0,0)$, come from the other group objects that can be constructed out of the fields but these have free indices and hence for our ansatz they do not contribute. Things become more interesting when we take the product of four 2-forms as there are many more ways to generate scalars; the decomposition of this is,

$$A^{[mn]} \otimes A^{[pq]} \otimes A^{[rs]} \otimes A^{[ab]} \leftrightarrow 6(0,0,0,0,0) \oplus 27(0,0,0,1,1) \oplus ..., \tag{7.9}$$

and hence there are six possible scalar combinations. These six scalar combinations are given by,

$$\begin{aligned} & F_{ma}F_{ab}F_{bc}F_{cm}, \quad F_{mn}F_{mn}F_{pq}F_{pq}, \qquad F_{ma}F_{nb}F_{am}F_{bn}, \\ & F_{mn}F_{bc}F_{bc}F_{mn}, \quad F_{ma}F_{nb}F_{an}F_{bm}, \qquad F_{ma}F_{an}F_{bm}F_{nb}. \end{aligned} \tag{7.10}$$

In order to find the possible scalar combinations for higher order corrections we simply repeat this procedure for all field combinations that are of the correct dimension and determine which terms can produce a scalar combination - each of these combinations can then be used to generate terms in the ansatz.

Before we move on to discussing the higher mass superfields and their scalar combinations we ought to say a word or two about how to determine which terms may contribute to a given correction order. To do this we first need the dimensions of each of the fields, these are given in Table 7.1. The tree-level terms are given by $\mathbb{VVV}$ which has weight 1.5, this is the only combination at weight 1.5 that has ghost number three and is non-zero; the order $\alpha'^2$ terms have weight 5.5, $\alpha'^3$ has weight 7.5 and $\alpha'^4$ has weight 9.5. These restrictions help to pin down the handful of structures that may contribute to the ansatz, before applying the Dynkin decomposition we first determine which terms can contribute at a given order using their dimensions. For example, a term of the

| Field | Dimension |
|---|---|
| $\mathbb{A}_\alpha$ | 0.5 |
| $\mathbb{A}_m$ | 1 |
| $\nabla_m$ | 1 |
| $\mathbb{W}^\alpha$ | 1.5 |
| $\mathbb{F}_{mn}$ | 2 |

TABLE 7.1: The dimensions of each of the fields. These are used to determine whether a term can contribute to the correction.

form $\mathbb{V}(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})\mathbb{F}\mathbb{F}$ could contribute to the $\alpha'^3$ term - it has ghost number three and weight 7.5, there are also combinations of the indices which are non-zero. Note that we only consider terms built from $\mathbb{W}$ and $\mathbb{F}$ as these fields are gauge invariant - as mentioned previously.

What is more, recognise that we can generate terms which contain $\mathbb{V}_i$ and those that do not contain $\mathbb{V}_i$ and at $\alpha'^3$ these terms decouple and one can use either representation. However, at $\alpha'^4$ we suspect that these terms no longer decouple and both are required in order to give the correct kinematic factor component expansion. The $\alpha'^2$ correction has no representation which does not contain a $\mathbb{V}_i$, this correction can only be expressed as an operator involving $\mathbb{V}_i$. This splitting is curious and it is an interesting subtly to note that, at least up to $\alpha'^4$, only the $\alpha'^3$ correction has this decoupling effect. In fact, at $\alpha'^4$ it is likely that there is no notion of decoupling and that $\mathbb{V}_i$ and no-$\mathbb{V}_i$ terms must contribute to the correction.

### 7.1.1 Higher Mass Superfield Labels

Finding the Dynkin labels of higher mass superfields is relatively straightforward. We take the tensor products of the required labels and then subtract any representations associated with the constraints. As an example, take $\mathbb{W}^{m\alpha}$, this can be built from a vector representation and a Weyl spinor representation and hence we conclude $\mathbb{W}^{m\alpha} \leftrightarrow (0,0,0,1,0) \oplus (1,0,0,0,1)$. Now we are left with determining what each of these labels could represent. The first label represents an irreducible representation that can be constructed from a gamma matrix and $\mathbb{W}^{m\alpha}$ as the representation is that of an anti-Weyl spinor. Hence we find that, $\gamma^m_{\alpha\beta}\mathbb{W}^{m\beta} \leftrightarrow (0,0,0,1,0)$, however, this term must vanish due to the Dirac equation - `LiE` has no notion of equations of motion and so we have to remove these kinds of terms post-decomposition although this is automated in our procedure. Hence we are left with,

$$\mathbb{W}^{m\alpha} \leftrightarrow (1,0,0,0,1), \tag{7.11}$$

as the label for the higher mass $\mathbb{W}^{m\alpha}$ field, this feels 'natural' as it somewhat looks like the vector sum of a vector $SO(10)$ label and a Weyl label. We can also look at the case of $\mathbb{F}^{m|pq}$, which is the product of a vector and 2-form, upon performing the decomposition one finds the following,

$$\mathbb{F}^{m|pq} \leftrightarrow (1,0,0,0,0) \oplus (0,0,1,0,0) \oplus (1,1,0,0,0). \tag{7.12}$$

The first term corresponds to a vector label and so it can only be composed[1] of $\mathbb{F}^{m|mq}$, but this is *zero*[2] due to the Yang-Mills equation of motion. The second part of the decomposition is a 3-form, which can only correspond to $\mathbb{F}^{[m|pq]} = 0$ due to the Bianchi/Jacobi identity. As a result, we must conclude that,

$$\mathbb{F}^{m|pq} \leftrightarrow (1,1,0,0,0), \tag{7.13}$$

which is again intuitive as it resembles the vector addition of the vector and 2-form Dynkin labels. One can continue in this manner to determine the higher mass Dynkin labels [105],

$$\hat{\mathbb{F}}^{(m_1\ldots m_n)|pq} \leftrightarrow (n,1,0,0,0), \qquad \hat{\mathbb{W}}^{(m_1\ldots m_n)\alpha} \leftrightarrow (n,0,0,0,1), \tag{7.14}$$

where $\hat{\mathbb{W}}^{(m_1\ldots m_n)\alpha}$ is the traceless symmetric part of the field - we will show in the next chapter that all of the other parts of the field can be decomposed in terms of lower mass dimension fields.

[1] Note that $\mathbb{F}^{m|pp}$ is already set to zero since this is zero under the $SO(10)$ group structure and is not the result of an equation of motion.

[2] Actually, it technically is not zero since $\mathbb{F}^{m|mn} = \gamma^n_{\alpha\beta}\{\mathbb{W}^\alpha, \mathbb{W}^\beta\}$, however these terms will be generated elsewhere in the ansatz and hence we set $\mathbb{F}^{m|mn} = 0$ so we can reduce the number of terms that we need to deal with in the ansatz code.

## 7.2 A Graph Theory Trick for Ansatz Building

Here we lay out a method to quickly build an ansatz using a graph theory based approach and the `dd_` command in `FORM`. From the `FORM` manual [106] the `dd_` command,

> *...is a combinatorics function. The tensor* `dd_` *with an even number of indices is equal to the totally symmetric tensor built up from products of kronecker delta's. Each term in this symmetric combination is normalised to one. In principle there are* $n!/(2n/2(n/2)!$ *terms in this combination. The profit comes when some or all the indices are contracted with vectors and some of these vectors are identical. In that case FORM will use combinatorics to generate only different terms, each with the proper prefactor. This can result in great time and space savings.*

In practice, this generates all possible contractions between vector indices on our superfields. Hence the premise is based on writing index contractions as a graph, for example the unique seed $(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^{s}\mathbb{W})\mathbb{F}_{mn}\mathbb{F}_{pq}\mathbb{F}_{rs}$ would be displayed as in Figure 7.1 and $(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^{r}\mathbb{W})\mathbb{F}_{am}\mathbb{F}_{np}\mathbb{F}_{qa}$ would be as in Figure 7.2. Each index contraction between two vector indices is denoted by an edge of the graph, the nodes are characterised by the *position* and the number of vector indices it has, this is crucial as generating series do not commute and position of $\gamma$-matrices very much matters. For example a gamma matrix in the third position with five vector indices is denoted by $\gamma_3^{(5)}$. The basic building block of the graphs are $\gamma$, $F$, $W$ and $D$ with the correct position and index numbers. In the case of derivatives we would display a term like $\mathbb{W}\mathbb{W}\mathbb{W}\mathbb{W}D\mathbb{F}$ as,

$$(\lambda\gamma^{(5)}\lambda)(\mathbb{W}\gamma^{(n)}\mathbb{W})(\mathbb{W}\gamma^{(m)}\mathbb{W})D^{a}\mathbb{F}_{bc} \longrightarrow \gamma_1^{(5)}\gamma_2^{(m)}\gamma_3^{(n)}D_4^{(1)}F_4^{(2)} \tag{7.15}$$

on the implicit understanding that all other $\gamma$ matrices other than $\gamma_1$ contract two $\mathbb{W}$s together. Generating these graphs in `FORM` is very straightforward owing to the `dd_` command already built in, one can then apply some of the relevant restrictions on the terms (e.g. from the PS constraint) - we will detail some of these later in this chapter. Loops have been forbidden as these are typically zero due to some equation of motion

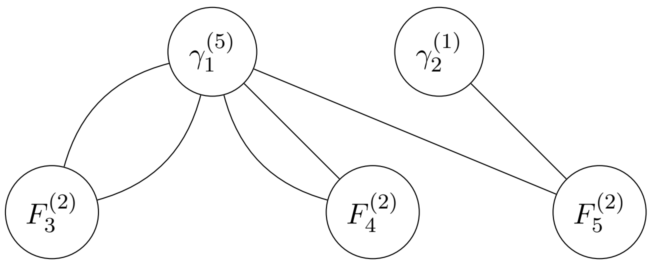


FIGURE 7.1: Graph of $(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^s\mathbb{W})\mathbb{F}_{mn}\mathbb{F}_{pq}\mathbb{F}_{rs}$ and its permutations.

and can be written as some other set of fields or are zero due to gamma self-contractions.

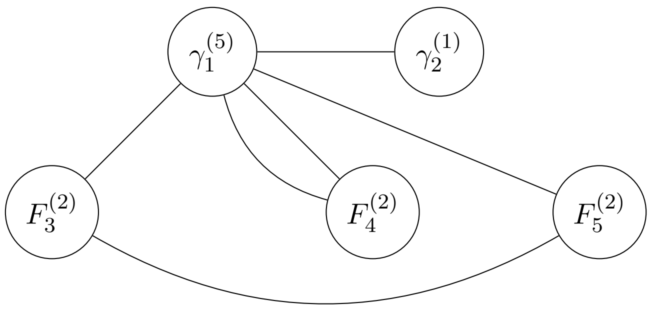


FIGURE 7.2: Graph of $(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^r\mathbb{W})\mathbb{F}_{am}\mathbb{F}_{np}\mathbb{F}_{qa}$ and its permutations.

Let us consider working through an example at $\alpha'^4$ by considering terms of the form $\mathbb{W}\mathbb{W}\mathbb{F}\mathbb{F}\mathbb{F}$. Note that in the building of our ansatz we ignore the $\mathbb{V}_i$ at the start of the $\mathbb{V}_i$ expressions as it is already a scalar and so we do not need to consider it when building the vector contractions. Furthermore, because expressions are under the trace we can always move $\mathbb{V}_i$ to the front of any expression. Throughout this chapter we shall assume that we are working in the basis whereby expressions involving two $\mathbb{W}$ are expressed as $(\lambda\gamma\lambda)(\mathbb{W}\gamma\mathbb{W})$ rather than $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})$. We do this here as it reduces the number of terms one has to consider because we only have to cycle through $\gamma^1$, $\gamma^3$, $\gamma^5$ for one $\gamma$ matrix. This follows because $(\lambda\gamma\lambda) = (\lambda\gamma^{mnpqr}\lambda)$. To implement the code that generates the graphs for the $\mathbb{W}\mathbb{W}\mathbb{F}\mathbb{F}\mathbb{F}$ term with $\gamma^1$ we use the following `FORM` code[3],

```
AutoDeclare Vectors g;
AutoDeclare Vectors D;
AutoDeclare Vectors F;
```

[3]This is a very simplified version of the full code, which transforms the vector indices to generating series expressions and applies equations of motion and other restrictions.

```
Local F = dd_(g1,g1,g1,g1,g1,g2,F3,F3,F4,F4,F5,F5);

id g?.g?=0;
id (F?.F?)=0;

Print F +s;
.end
```

let us take a moment to explain some of these lines of code. The first `id` statement is in place to prevent any gamma matrix self-contraction as these are zero on the spot. The second `id` statement prevents $\mathbb{F}$ self contractions as these are also zero on the spot due to anti-symmetry. Putting all of this together produces,

```
F =
+ 480*g1.g2*g1.F3*g1.F4*g1.F5^2*F3.F4
+ 480*g1.g2*g1.F3*g1.F4^2*g1.F5*F3.F5
+ 480*g1.g2*g1.F3^2*g1.F4*g1.F5*F4.F5
+ 240*g1.F3*g1.F4^2*g1.F5^2*g2.F3
+ 240*g1.F3^2*g1.F4*g1.F5^2*g2.F4
+ 240*g1.F3^2*g1.F4^2*g1.F5*g2.F5
;
```

The second three terms of the above reproduce the terms from Figure 7.1 and the first three terms reproduce the terms in Figure 7.2. At this basic level the code works well and produces many of the terms we will need, the same is true for the case with $\mathbb{W}\gamma^{rsp}\mathbb{W}$, as this produces,

```
F =
    + 2880*g1.g2*g1.F3*g1.F4*g1.F5^2*g2.F3*g2.F4
    + 2880*g1.g2*g1.F3*g1.F4^2*g1.F5*g2.F3*g2.F5
```

```
    + 2880*g1.g2*g1.F3^2*g1.F4*g1.F5*g2.F4*g2.F5
    + 720*g1.g2*g1.F3^2*g1.F4^2*g2.F5^2
    + 720*g1.g2*g1.F3^2*g1.F5^2*g2.F4^2
    + 720*g1.g2*g1.F4^2*g1.F5^2*g2.F3^2
    + 2880*g1.g2^2*g1.F3*g1.F4*g1.F5*g2.F3*F4.F5
    + 2880*g1.g2^2*g1.F3*g1.F4*g1.F5*g2.F4*F3.F5
    + 2880*g1.g2^2*g1.F3*g1.F4*g1.F5*g2.F5*F3.F4
    + 1440*g1.g2^2*g1.F3*g1.F4^2*g2.F5*F3.F5
    + 1440*g1.g2^2*g1.F3*g1.F5^2*g2.F4*F3.F4
    + 1440*g1.g2^2*g1.F3^2*g1.F4*g2.F5*F4.F5
    + 1440*g1.g2^2*g1.F3^2*g1.F5*g2.F4*F4.F5
    + 1440*g1.g2^2*g1.F4*g1.F5^2*g2.F3*F3.F4
    + 1440*g1.g2^2*g1.F4^2*g1.F5*g2.F3*F3.F5
    + 960*g1.g2^3*g1.F3*g1.F4*F3.F5*F4.F5
    + 960*g1.g2^3*g1.F3*g1.F5*F3.F4*F4.F5
    + 240*g1.g2^3*g1.F3^2*F4.F5^2
    + 960*g1.g2^3*g1.F4*g1.F5*F3.F4*F3.F5
    + 240*g1.g2^3*g1.F4^2*F3.F5^2
    + 240*g1.g2^3*g1.F5^2*F3.F4^2
   ;
```

which gives all 21 permutations of index contractions that we expect from group theory. It is simply a matter of scanning the result and finding the unique contractions and then writing them in a suggestive form which can them be permuted by the ansatz file. The situation breaks down a little when we have two $\gamma^{mnpqrs}$ matrices appearing in the contraction. In this case we obtain,

```
F =
    + 57600*g1.g2^2*g1.F3*g1.F4*g1.F5*g2.F3*g2.F4*g2.F5
    + 14400*g1.g2^2*g1.F3*g1.F4^2*g2.F3*g2.F5^2
    + 14400*g1.g2^2*g1.F3*g1.F5^2*g2.F3*g2.F4^2
    + 14400*g1.g2^2*g1.F3^2*g1.F4*g2.F4*g2.F5^2
```

```
+ 14400*g1.g2^2*g1.F3^2*g1.F5*g2.F4^2*g2.F5

+ 14400*g1.g2^2*g1.F4*g1.F5^2*g2.F3^2*g2.F4

+ 14400*g1.g2^2*g1.F4^2*g1.F5*g2.F3^2*g2.F5

+ 19200*g1.g2^3*g1.F3*g1.F4*g2.F3*g2.F5*F4.F5

+ 19200*g1.g2^3*g1.F3*g1.F4*g2.F4*g2.F5*F3.F5

+ 9600*g1.g2^3*g1.F3*g1.F4*g2.F5^2*F3.F4

+ 19200*g1.g2^3*g1.F3*g1.F5*g2.F3*g2.F4*F4.F5

+ 19200*g1.g2^3*g1.F3*g1.F5*g2.F4*g2.F5*F3.F4

+ 9600*g1.g2^3*g1.F3*g1.F5*g2.F4^2*F3.F5

+ 9600*g1.g2^3*g1.F3^2*g2.F4*g2.F5*F4.F5

+ 19200*g1.g2^3*g1.F4*g1.F5*g2.F3*g2.F4*F3.F5

+ 19200*g1.g2^3*g1.F4*g1.F5*g2.F3*g2.F5*F3.F4

+ 9600*g1.g2^3*g1.F4*g1.F5*g2.F3^2*F4.F5

+ 9600*g1.g2^3*g1.F4^2*g2.F3*g2.F5*F3.F5

+ 9600*g1.g2^3*g1.F5^2*g2.F3*g2.F4*F3.F4

+ 4800*g1.g2^4*g1.F3*g2.F4*F3.F5*F4.F5

+ 4800*g1.g2^4*g1.F3*g2.F5*F3.F4*F4.F5

+ 4800*g1.g2^4*g1.F4*g2.F3*F3.F5*F4.F5

+ 4800*g1.g2^4*g1.F4*g2.F5*F3.F4*F3.F5

+ 4800*g1.g2^4*g1.F5*g2.F3*F3.F4*F4.F5

+ 4800*g1.g2^4*g1.F5*g2.F4*F3.F4*F3.F5

;
```

which gives 25 permutations despite there only being 13 independent ones. This is quite easy to explain: the code is summing the permutations coming from the diagrams in Figure 7.3. The more general code generally eliminates these kinds of graphs and so we do not generate these kinds of terms in the ansatz - this is important as a simplified ansatz greatly reduces the computational power/time required to perform the BRST variation. So we extend this to the $(\lambda\gamma^{(5)}\lambda)(\mathbb{W}\gamma^{(n)}\mathbb{W})(\mathbb{W}\gamma^{(m)}\mathbb{W})D^a\mathbb{F}_{bc}$ case and look at the case where $m = n = 1$, the code for this is,

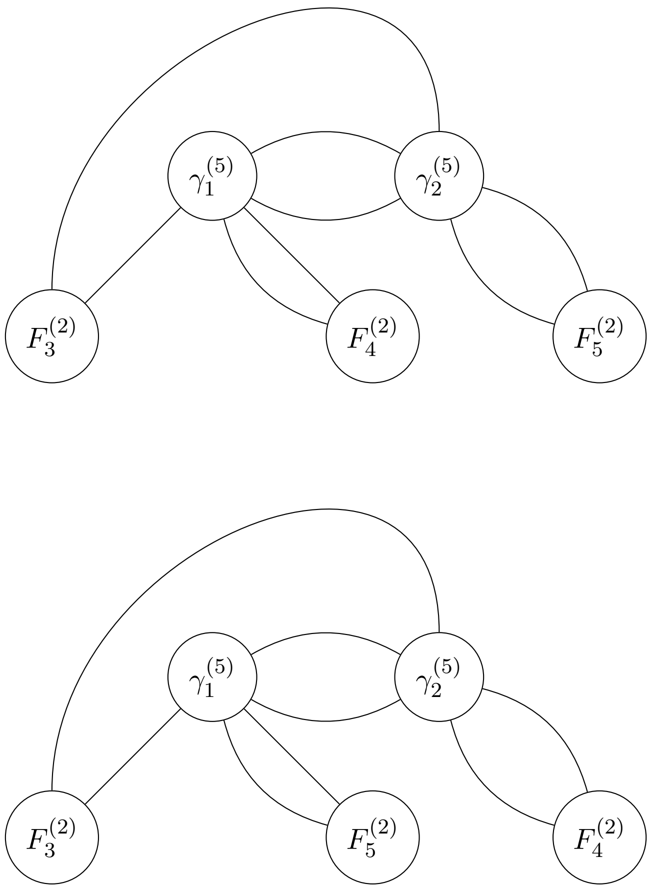


FIGURE 7.3: The two graphs that contribute to permutations of $(\lambda\gamma^{(5)}\lambda)(\mathbb{W}\gamma^{(5)}\mathbb{W})\mathbb{F}\mathbb{F}\mathbb{F}$. Note that $F_4$ and $F_5$ has changed places in the second diagram.

```
AutoDeclare Vectors g;
AutoDeclare Vectors D;
AutoDeclare Vectors F;

Local F = dd_(g1,g1,g1,g1,g1,g2,g3,D4,F4,F4);

id g?.g?=0;
id (F?.F?)=0;
id (D?.F?)=0;

Print F +s;
.end
```

where the $D4.F4 = 0$ statement takes care of the equations of motion - recall the Yang-Mills equation generates fields at order $p+1$ and so we set these terms to zero when consider $p-$fields. This can be performed because we generate our ansatz by counting the number of fields in each term. So for example, we begin by generating all terms that can involve three fields. Hence if we can use an equation of motion to eliminate a three field term in favour of a four field term we simply discard it. This is because such a term will appear when we consider ansatz terms with four fields. In the case outlined above there is only one seed given by,

```
F =
   + 120*g1.g2*g1.g3*g1.D4*g1.F4^2
  ;
```

which corresponds to,

$$(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^{m}\mathbb{W})(\mathbb{W}\gamma^{n}\mathbb{W})(\mathbb{F}_{p|qr}). \tag{7.16}$$

We can then extend to allowing $m = 3, n = 1$, and we obtain,

```
F =
+ 720*g1.g2^2*g1.g3*g1.D4*g1.F4*g2.F4
+ 360*g1.g2^2*g1.g3*g1.F4^2*g2.D4
+ 360*g1.g2^2*g1.D4*g1.F4^2*g2.g3
+ 240*g1.g2^3*g1.D4*g1.F4*g3.F4
+ 120*g1.g2^3*g1.F4^2*g3.D4
;
```

giving us four seeds of the form,

$$\begin{aligned}
&(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^{mns}\mathbb{W})(\mathbb{W}\gamma^{p}\mathbb{W})(\mathbb{F}_{q|rs})\\
&(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^{mns}\mathbb{W})(\mathbb{W}\gamma^{p}\mathbb{W})(\mathbb{F}_{s|qr})\\
&(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^{mnp}\mathbb{W})(\mathbb{W}\gamma^{s}\mathbb{W})(\mathbb{F}_{q|rs})\\
&(\lambda\gamma^{mnpqr}\lambda)(\mathbb{W}\gamma^{mnp}\mathbb{W})(\mathbb{W}\gamma^{s}\mathbb{W})(\mathbb{F}_{s|qr}),
\end{aligned} \tag{7.17}$$

these same seeds will appear for $m = 1, n = 3$ but in the correct order for the fields. We can also consider $m = 5, n = 1$ to obtain,

```
F =
+ 2400*g1.g2^3*g1.g3*g1.D4*g2.F4^2
+ 4800*g1.g2^3*g1.g3*g1.F4*g2.D4*g2.F4
+ 4800*g1.g2^3*g1.D4*g1.F4*g2.g3*g2.F4
+ 2400*g1.g2^3*g1.F4^2*g2.g3*g2.D4
+ 1200*g1.g2^4*g1.D4*g2.F4*g3.F4
+ 1200*g1.g2^4*g1.F4*g2.D4*g3.F4
+ 1200*g1.g2^4*g1.F4*g2.F4*g3.D4
;
```

giving three more seeds, again these are symmetric with the $m = 1, n = 5$ case. For $m = 3, n = 3$ we obtain,

```
F =
+ 2160*g1.g2*g1.g3*g1.D4*g1.F4^2*g2.g3^2
+ 4320*g1.g2*g1.g3^2*g1.D4*g1.F4*g2.g3*g2.F4
+ 2160*g1.g2*g1.g3^2*g1.F4^2*g2.g3*g2.D4
+ 720*g1.g2*g1.g3^3*g1.D4*g2.F4^2
+ 1440*g1.g2*g1.g3^3*g1.F4*g2.D4*g2.F4
+ 4320*g1.g2^2*g1.g3*g1.D4*g1.F4*g2.g3*g3.F4
+ 2160*g1.g2^2*g1.g3*g1.F4^2*g2.g3*g3.D4
+ 2160*g1.g2^2*g1.g3^2*g1.D4*g2.F4*g3.F4
+ 2160*g1.g2^2*g1.g3^2*g1.F4*g2.D4*g3.F4
+ 2160*g1.g2^2*g1.g3^2*g1.F4*g2.F4*g3.D4
+ 720*g1.g2^3*g1.g3*g1.D4*g3.F4^2
+ 1440*g1.g2^3*g1.g3*g1.F4*g3.D4*g3.F4
;
```

for four more seeds. The $m = 5, n = 3$ case gives,

```
F =
+ 43200*g1.g2^2*g1.g3*g1.D4*g1.F4*g2.g3^2*g2.F4
+ 21600*g1.g2^2*g1.g3*g1.F4^2*g2.g3^2*g2.D4
+ 21600*g1.g2^2*g1.g3^2*g1.D4*g2.g3*g2.F4^2
+ 43200*g1.g2^2*g1.g3^2*g1.F4*g2.g3*g2.D4*g2.F4
+ 7200*g1.g2^2*g1.g3^3*g2.D4*g2.F4^2
+ 7200*g1.g2^2*g1.D4*g1.F4^2*g2.g3^3
+ 28800*g1.g2^3*g1.g3*g1.D4*g2.g3*g2.F4*g3.F4
+ 28800*g1.g2^3*g1.g3*g1.F4*g2.g3*g2.D4*g3.F4
+ 28800*g1.g2^3*g1.g3*g1.F4*g2.g3*g2.F4*g3.D4
+ 14400*g1.g2^3*g1.g3^2*g2.D4*g2.F4*g3.F4
+ 7200*g1.g2^3*g1.g3^2*g2.F4^2*g3.D4
+ 14400*g1.g2^3*g1.D4*g1.F4*g2.g3^2*g3.F4
+ 7200*g1.g2^3*g1.F4^2*g2.g3^2*g3.D4
+ 3600*g1.g2^4*g1.g3*g2.D4*g3.F4^2
+ 7200*g1.g2^4*g1.g3*g2.F4*g3.D4*g3.F4
+ 3600*g1.g2^4*g1.D4*g2.g3*g3.F4^2
+ 7200*g1.g2^4*g1.F4*g2.g3*g3.D4*g3.F4
;
```

giving another six seeds. Finally, the $m = n = 5$ case gives,

```
F =
+ 72000*g1.g2*g1.g3*g1.D4*g1.F4^2*g2.g3^4
+ 288000*g1.g2*g1.g3^2*g1.D4*g1.F4*g2.g3^3*g2.F4
+ 144000*g1.g2*g1.g3^2*g1.F4^2*g2.g3^3*g2.D4
+ 144000*g1.g2*g1.g3^3*g1.D4*g2.g3^2*g2.F4^2
+ 288000*g1.g2*g1.g3^3*g1.F4*g2.g3^2*g2.D4*g2.F4
+ 72000*g1.g2*g1.g3^4*g2.g3*g2.D4*g2.F4^2
+ 288000*g1.g2^2*g1.g3*g1.D4*g1.F4*g2.g3^3*g3.F4
+ 144000*g1.g2^2*g1.g3*g1.F4^2*g2.g3^3*g3.D4
+ 432000*g1.g2^2*g1.g3^2*g1.D4*g2.g3^2*g2.F4*g3.F4
```

```
+ 432000*g1.g2^2*g1.g3^2*g1.F4*g2.g3^2*g2.D4*g3.F4
+ 432000*g1.g2^2*g1.g3^2*g1.F4*g2.g3^2*g2.F4*g3.D4
+ 288000*g1.g2^2*g1.g3^3*g2.g3*g2.D4*g2.F4*g3.F4
+ 144000*g1.g2^2*g1.g3^3*g2.g3*g2.F4^2*g3.D4
+ 144000*g1.g2^3*g1.g3*g1.D4*g2.g3^2*g3.F4^2
+ 288000*g1.g2^3*g1.g3*g1.F4*g2.g3^2*g3.D4*g3.F4
+ 144000*g1.g2^3*g1.g3^2*g2.g3*g2.D4*g3.F4^2
+ 288000*g1.g2^3*g1.g3^2*g2.g3*g2.F4*g3.D4*g3.F4
+ 72000*g1.g2^4*g1.g3*g2.g3*g3.D4*g3.F4^2
;
```

and so we have another four seeds. Thus in total for this term there are 21 possible seeds we should consider in the ansatz. This automation makes it very quick to identify the number of seeds as well as their structural form although care is to be taken when analysing these terms so that they come out correctly. Once this has been set up it is then possible to code up various identities to transform there graph outputs to superfield outputs. Then one can employ much of the other machinery developed here and in other works in order to canonicalise any ansatz into a form that contains as few higher mass fields as possible.

This chapter has discussed some of the basic things we can do when determining the ansatz at a given order. By first considering dimensions and the $SO(10)$ group structure we can greatly reduce the number of terms we need to consider in the ansatz. Then using the above graph theory trick we can easily generate all of the relevant contractions and begin to apply some of the equations of motion not captured by the Dynkin label decomposition. From here we can apply further identities, as we discuss in the next chapter, to not only simplify the ansatz but also canonicalise the BRST variation and hence the system of linear equations.

# Chapter 8

# New Higher Mass Identities

In this chapter we derive and highlight some of the higher mass identities used to canoncalise the BRST variations of the ansatz. These identities can also be used to greatly simplify the ansatz as these identities demonstrate that some field combinations are not independent. As a result, one can reduce the number of independent ansatz terms generated by the methods in the previous chapter. We also discuss some identities that can be derived from the pure spinor constraint, $\lambda\gamma^m\lambda = 0$, as this can also greatly reduce the number of terms considered in the ansatz. The next few sections are devoted to the way in which it is possible to canonicalise the BRST variations of the ansatzes - this is a crucial step in uncovering the corrections as it is this canonicalisation that gives the correct linear equations. It is fruitful to review here the methodology which we are employing in order to determine the $\alpha'$ corrections. Initially we determine the most general ansatz using group theory constraints and the pure spinor identity. This process produces many hundreds or thousands of terms. The nature of the BRST cohomology is such that an operator at the correct mass dimension for a correction, that is BRST closed, is necessarily a generating series representation of the correction. Thus, in order to determine the corrections we need to take the BRST variation of our ansatz and set it equal to zero. In doing so, we will obtain a number of linear equations in the coefficients of the terms in the ansatz. This system of linear equations must then be solved in order to find the correction. It is for this reason that canonicalisation of the generating series variation is so important. In fact, it is the core issue that prevents us from quickly determining the correct generating series expression for the correction.

## 8.1 Jacobi Identities

One of the ways in which we can perform this kind of canonicalisation is using Jacobi identities, and this amounts to picking a specific ordering basis for the indices and applying this in `FORM`. This is a must when using the generating series formalism as it is quite possible that any ansatz that is generated contains terms that are redundant due to Jacobi identities. For example, let us consider $\mathbb{W}^{nm}$ where we wish to order all terms such that we have $\mathbb{W}^{mn}$ along with other quadratic fields. In this case we can simply use the Jacobi identity given by,

$$\begin{aligned}\mathbb{W}^{nm} &= [\nabla^n, [\nabla^m, \mathbb{W}]] \\ &= -[\nabla^m, [\mathbb{W}, \nabla^n]] - [\mathbb{W}, [\nabla^n, \nabla^m]] \\ &= \mathbb{W}^{mn} - [\mathbb{W}, \mathbb{F}^{mn}].\end{aligned} \tag{8.1}$$

Ultimately, our goal here and throughout subsequent sections on canoncalisation is to, where possible, reduce higher mass fields to lower mass quadratic or cubic fields. In the example above we can replace, for example, $\mathbb{W}^{mn} + \mathbb{W}^{nm}$ with $2\mathbb{W}^{mn} - [\mathbb{W}, \mathbb{F}^{mn}]$. This induces mixing between the ansatz terms which help to give a fuller set of linear equations.

This can be generalised to the three-vector index case, whereby there are three possible positions for our canonical index $m$, $\mathbb{W}^{mnp}$, $\mathbb{W}^{nmp}$ and $\mathbb{W}^{npm}$. In the first instance we do nothing, however in the second instance we can use the results from the above equations and replace $\mathbb{W}$ with $\mathbb{W}^p$ in the right-hand side,

$$\mathbb{W}^{nmp} = \mathbb{W}^{mnp} - [\mathbb{W}^p, \mathbb{F}^{mn}]. \tag{8.2}$$

For the $\mathbb{W}^{npm}$ case, we have to do a little more manipulation but it is just applying the Jacobi identity a couple of times,

$$\begin{aligned}
\mathbb{W}^{npm} &= [\nabla^n, [\nabla^p, [\nabla^m, \mathbb{W}]]] \\
&= [\nabla^n, [\nabla^m, \mathbb{W}^p]] - [\nabla^n, [\mathbb{W}, \mathbb{F}^{mp}]] \\
&= -[\nabla^m, [\mathbb{W}^p, \nabla^n]] - [\mathbb{W}^p, [\nabla^n, \nabla^m]] \\
&\qquad + [\mathbb{W}, [\mathbb{F}^{mp}, \nabla^n]] + [\mathbb{F}^{mp}, [\nabla^n, \mathbb{W}]] \\
&= \mathbb{W}^{mpn} - [\mathbb{W}^p, \mathbb{F}^{mn}] - [\mathbb{W}, \mathbb{F}^{n|mp}] - [\mathbb{W}^n, \mathbb{F}^{mp}].
\end{aligned} \tag{8.3}$$

These are the only identities we need for $\mathbb{W}$ for the $\alpha'^3$ & $\alpha'^4$ corrections, however, we can generalise this such that we can determine $\mathbb{W}^{Pm}$ in terms of $\mathbb{W}^{mP}$ as well as other fields. This is given by,

$$\mathbb{W}^{Pm} = \mathbb{W}^{mP} - \sum_{\substack{\delta(P)=R\otimes S \\ S\neq\emptyset}} [\mathbb{W}^R, (\mathbb{F})^{Sm}], \tag{8.4}$$

where $(\mathbb{F})^{Sm} = \mathbb{F}^{s_1 s_2 \ldots, s_{k-1}|m s_k}$ for $|S| = k$ - for example consider $S = npqr$, then $(\mathbb{F})^{Sm} = \mathbb{F}^{npq|mr}$. Let us consider the example whereby we set $P = npq$, the result is given by,

$$\begin{aligned}
\mathbb{W}^{npqm} = \mathbb{W}^{mnpq} &- [\mathbb{W}^{np}, \mathbb{F}^{mq}] - [\mathbb{W}^{nq}, \mathbb{F}^{mp}] - [\mathbb{W}^{pq}, \mathbb{F}^{mn}] - [\mathbb{W}^q, \mathbb{F}^{n|mp}] \\
&- [\mathbb{W}^p, \mathbb{F}^{n|mq}] - [\mathbb{W}^n, \mathbb{F}^{p|mq}] - [\mathbb{W}, \mathbb{F}^{np|mq}],
\end{aligned} \tag{8.5}$$

which can be readily shown by considering $\mathbb{W}^{npqm} = [\nabla^n, [\nabla^p, [\nabla^q, [\nabla^m, \mathbb{W}]]]]$ and the relevant Jacobi identities. We shall also need some identities for the $\mathbb{F}$ field. For this non-linear field we choose to send the canonicalised index to the end - this is mainly to do with calculational ease - in order to obtain a basis. For $\mathbb{F}^{nm} = -\mathbb{F}^{mn}$ this is easy, however for higher mass fields we need to use Jacobi. Consider the case where $a$ is the canonicalised index and we have $\mathbb{F}^{a|bc}$

$$0 = \mathbb{F}^{a|bc} + \mathbb{F}^{b|ca} + \mathbb{F}^{c|ab} \implies \mathbb{F}^{a|bc} = \mathbb{F}^{c|ba} - \mathbb{F}^{b|ca}. \tag{8.6}$$

This can be trivially extended to the case $\mathbb{F}^{ba|cd}$,

$$\mathbb{F}^{ba|cd} = \mathbb{F}^{bd|ca} - \mathbb{F}^{bc|da}, \tag{8.7}$$

however for the $\mathbb{F}^{ab|cd}$ case we get an extra quadratic factor in the fields,

$$\mathbb{F}^{ab|cd} = \mathbb{F}^{bd|ca} - \mathbb{F}^{bc|da} + [\mathbb{F}^{ba}, \mathbb{F}^{cd}]. \tag{8.8}$$

These are the only identities required to canonicalise the BRST variation using the Jacobi identities; they successfully capture all of the Jacobi triplets in the variation.

We further need the identities to handle terms such as $\mathbb{F}^{nm|pm}$ which can be decomposed into two quadratic field terms. The equation of motion we begin with is given by the higher mass Yang-Mills equation of motion,

$$\mathbb{F}^{mn|pm} = [\nabla^m, \mathbb{F}^{n|pm}] = [\mathbb{F}^{nm}, \mathbb{F}^{pm}] - \{\mathbb{W}^n, (\gamma^p \mathbb{W})\} - \{\mathbb{W}, (\gamma^n \mathbb{W}^p)\}, \tag{8.9}$$

in order to determine the equation of motion for $\mathbb{F}^{nm|pm}$ we need to use the following Jacobi identity,

$$[\nabla^n, [\nabla^m, \mathbb{F}^{pm}]] = \mathbb{F}^{nm|pm} - [\mathbb{F}^{pm}, \mathbb{F}^{mn}], \tag{8.10}$$

and so we have,

$$\mathbb{F}^{nm|pm} = 2[\nabla^m, \mathbb{F}^{n|pm}] - \{\mathbb{W}^n, (\gamma^p \mathbb{W})\} - \{\mathbb{W}, (\gamma^n) \mathbb{W}^p)\}. \tag{8.11}$$

The premise behind such canonicalisation is to remove any such terms in the non-linear superfields and replace them with lower dimensional fields. This ensures that we have proper canonicalisation across the entire variation.

## 8.2 Tensor Decomposition

When performing the Dynkin label decomposition we are dealing with the irreducible parts of the various higher mass fields. It turns out, as we will show below, that it is actually the symmetric, traceless part of the fields (with more than two covariant derivatives acting on them) that is the irreducible. All of the other parts of the field can be decomposed into lower mass dimension fields. Hence, in the most proper sense the ansatz is formed from the symmetric, traceless parts of the higher mass fields and as such we need to be able to project from these symmetric parts to the full non-linear fields and then back again. Projecting to the full non-linear fields is trivial and can be done by substituting the defining equations of the symmetric parts in terms of the full-fields. However, to project from these full fields back to the symmetric part is much harder and requires knowledge of the full decomposition in terms of the symmetric parts of the field and quadratic terms that make up the rest of the field. In this chapter we derive the tensor decompositions of these higher mass fields and demonstrate that everything but the symmetric traceless part can be written in terms of lower mass dimension fields. In order to perform this decomposition we take inspiration from [107, 108].

An easy example is $\mathbb{W}^{mn}$ which can be decomposed into a symmetric traceless part $\hat{\mathbb{W}}^{(mn)}$, a symmetric trace $\tilde{\mathbb{W}}^{(mn)}$ and an anti-symmetric part $\mathbb{W}^{[mn]}$,

$$\mathbb{W}^{mn} = \hat{\mathbb{W}}^{(mn)} + \frac{1}{10}\tilde{\mathbb{W}}^{(mn)} + \frac{1}{2}\mathbb{W}^{[mn]}. \tag{8.12}$$

The last two parts of the decomposition can be expressed in terms of lower mass fields,

$$\begin{aligned} \tilde{\mathbb{W}}^{(mn)} &= \frac{\delta^{mn}}{2}[\mathbb{F}_{ab}, (\gamma^{ab})\mathbb{W}] \\ \mathbb{W}^{[mn]} &= [\mathbb{W}, \mathbb{F}^{mn}], \end{aligned} \tag{8.13}$$

where $(\gamma^{ab}\mathbb{W}) = (\gamma^{ab}\mathbb{W})^{\alpha}$ - we use this convention frequently below. Using the above we have the following decomposition,

$$\mathbb{W}^{mn} = \hat{\mathbb{W}}^{(mn)} + \frac{\delta^{mn}}{20}[\mathbb{F}_{ab}, (\gamma^{ab})\mathbb{W}] + [\mathbb{W}, \mathbb{F}^{mn}]. \tag{8.14}$$

We can also play this game for $\mathbb{F}^{ab|mn}$ since we shall deal with the part of this tensor that is symmetric and traceless in $a, b$. The symmetric, trace part becomes,

$$\tilde{\mathbb{F}}^{(ab)|mn} = 2\delta^{ab}([\mathbb{F}^{mp}, \mathbb{F}^{pn}] + \{(\mathbb{W}^{[m}\gamma^{n]}), \mathbb{W}\}), \tag{8.15}$$

and, in keeping with our conventions, the anti-symmetric part can be written as,

$$\mathbb{F}^{[ab]|mn} = \frac{1}{2}[\mathbb{F}^{mn}, \mathbb{F}^{ab}], \tag{8.16}$$

which follows from the Jacobi identity. As a result, we note that,

$$\mathbb{F}^{ab|mn} = \hat{\mathbb{F}}^{(ab)|mn} + \frac{\delta^{ab}}{5}([\mathbb{F}^{mp}, \mathbb{F}^{pn}] + \{(\mathbb{W}^{[m}\gamma^{n]}), \mathbb{W}\}) + \frac{1}{2}[\mathbb{F}^{mn}, \mathbb{F}^{ab}] \tag{8.17}$$

This becomes more complicated for $\mathbb{W}^{mnp}$ as this tensor can be decomposed into a symmetric traceless part, a symmetric trace, an anti-symmetric part, and three 2-forms which are anti-symmetric in two of the indices. Thus the decomposition schematically looks like,

$$\mathbb{W}^{mnp} = \hat{\mathbb{W}}^{(mnp)} + \frac{1}{X}\tilde{\mathbb{W}}^{(mnp)} + \mathbb{W}^{[mnp]} + \mathbb{W}^{[mn]p} + \mathbb{W}^{[m|n|p]} + \mathbb{W}^{m[np]}, \tag{8.18}$$

where again $\hat{\mathbb{W}}$ is the traceless symmetric part and $\tilde{\mathbb{W}}$ is the symmetric trace part - note we have not fixed the constants yet. We can, and do, show that only two of the 2-forms are independent and so the decomposition actually becomes,

$$\mathbb{W}^{mnp} = \hat{\mathbb{W}}^{(mnp)} + \frac{1}{X}\tilde{\mathbb{W}}^{(mnp)} + \mathbb{W}^{[mnp]} + \mathbb{W}^{[mn]p} + \mathbb{W}^{m[np]}, \tag{8.19}$$

where the constants can be determined such that the decomposition is an irreducible representation. This is given below. The anti-symmetric part of this is well-known in the literature and is given by [7],

$$\mathbb{W}^{[mnp]} = [\mathbb{W}^{p}, \mathbb{F}^{mn}] + [\mathbb{W}^{n}, \mathbb{F}^{pm}] + [\mathbb{W}^{m}, \mathbb{F}^{np}]. \tag{8.20}$$

The 2-forms require a little more work and we choose as the independent basis structures $\mathbb{W}^{[mn]p}$ and $\mathbb{W}^{m[np]}$ as we can express $\mathbb{W}^{[m|n|p]}$ in terms of this in a simple manner. Before we show this, however, we shall find the form of $\mathbb{W}^{[mn]p}$ and $\mathbb{W}^{m[np]}$ in terms of lower mass fields,

$$\begin{aligned} \mathbb{W}^{[mn]p} &= \mathbb{W}^{mnp} - \mathbb{W}^{nmp} \\ \implies \mathbb{W}^{nmp} &= -[\nabla^m, [\mathbb{W}^p, \nabla^n]] - [\mathbb{W}^p, [\nabla^n, \nabla^m]] \\ &= \mathbb{W}^{mnp} + [\mathbb{W}^p, \mathbb{F}^{nm}] \\ \implies \mathbb{W}^{[mn]p} &= [\mathbb{W}^p, \mathbb{F}^{mn}]. \end{aligned} \tag{8.21}$$

For $\mathbb{W}^{m[np]}$ we obtain,

$$\begin{aligned} \mathbb{W}^{m[np]} &= \mathbb{W}^{mnp} - \mathbb{W}^{mpn} \\ \implies \mathbb{W}^{mpn} &= -[\nabla^m, [\nabla^n, [\mathbb{W}, \nabla^p]]] - [\nabla^m, [\mathbb{W}, [\nabla^p, \nabla^n]]] \\ &= \mathbb{W}^{mnp} + [\nabla^m, [\mathbb{W}, \mathbb{F}^{pn}]] \\ \implies \mathbb{W}^{m[np]} &= [\mathbb{W}, \mathbb{F}^{m|np}] + [\mathbb{W}^m, \mathbb{F}^{np}] \\ &= [\mathbb{W}, \mathbb{F}^{m|np}] + \mathbb{W}^{[np]m}. \end{aligned} \tag{8.22}$$

Now we can show that the case of $\mathbb{W}^{[m|n|p]}$ can be written as a linear combination of $\mathbb{W}^{[mn]p}$ and $\mathbb{W}^{m[np]}$. To do this, consider $\mathbb{W}^{pnm}$ and apply the Jacobi identity to $[\nabla^n, [\nabla^m, \mathbb{W}]]$,

$$\mathbb{W}^{pnm} = [\nabla^p, [\nabla^m, \mathbb{W}^n]] - [\nabla^p, [\mathbb{W}, \mathbb{F}^{mn}]], \tag{8.23}$$

the second term of this equation is in a form we are happy with, however the first term corresponds to $\mathbb{W}^{pmn}$ which we cannot do much with until we apply the Jacobi identity one more time to give,

$$\mathbb{W}^{pnm} = \mathbb{W}^{mnp} - [\nabla^m, [\mathbb{W}, [\nabla^p, \nabla^n]]] - [\mathbb{W}^n, \mathbb{F}^{mp}] - [\nabla^p, [\mathbb{W}, \mathbb{F}^{mn}]] \tag{8.24}$$

thence we find,

$$\begin{aligned} \mathbb{W}^{[m|n|p]} &= [\mathbb{W}, \mathbb{F}^{m|np}] + [\mathbb{W}^m, \mathbb{F}^{np}] + [\mathbb{W}^n, \mathbb{F}^{mp}] + [\mathbb{W}, \mathbb{F}^{p|mn}] + [\mathbb{W}^p, \mathbb{F}^{mn}] \\ &= \mathbb{W}^{m[np]} + \mathbb{W}^{[mp]n} + \mathbb{W}^{p[mn]} \end{aligned} \tag{8.25}$$

and so there are only two 2-form tensors that are independent and here we choose $\mathbb{W}^{[mn]p}$ and $\mathbb{W}^{m[np]}$ to be the basis. Now the trace symmetric part we shall deal with in due course, but first let us look at the mixed symmetry terms. After removing the symmetric and anti-symmetric terms from $\mathbb{W}^{mnp}$ we are left with some terms proportional to,

$$2\mathbb{W}^{mnp} - \mathbb{W}^{npm} - \mathbb{W}^{pmn} = 2(\mathbb{W}^{[mn]p} + \mathbb{W}^{m[np]} - \mathbb{W}^{n[pm]} - \mathbb{W}^{[pm]n}). \tag{8.26}$$

We can now move on to the symmetric trace part of the decomposition. We have that,

$$\tilde{\mathbb{W}}^{(mnp)} = (\beta^m \delta^{np} + \beta^n \delta^{mp} + \beta^p \delta^{mn}), \tag{8.27}$$

where,

$$\beta^m = \frac{1}{3}(\mathbb{W}^{qqm} + \mathbb{W}^{qmq} + \mathbb{W}^{mqq}). \tag{8.28}$$

We are now in a position to fix the normalisation $X$ in (8.19), to do so we note that we can write the symmetric, traceless part of $\mathbb{W}^{mnp}$ as,

$$\hat{\mathbb{W}}^{(mnp)} = \mathbb{W}^{(mnp)} - \frac{1}{X}(\beta^m \delta^{np} + \beta^n \delta^{mp} + \beta^p \delta^{mn}). \tag{8.29}$$

Now taking the trace of both sides and expanding $\mathbb{W}^{(mnp)}$ in terms of $\mathbb{W}^{mnp}$ we find,

$$\begin{aligned} 0 &= \frac{1}{3}(\mathbb{W}^{mpp} + \mathbb{W}^{pmp} + \mathbb{W}^{ppm}) - \frac{1}{X}(10\beta^m + \beta^m + \beta^m), \\ 0 &= X\beta^m - 12\beta^m, \\ \implies X &= 12 \end{aligned} \tag{8.30}$$

We need to determine the form of the trace parts of the non-linear fields. The first of these is rather trivial as it is simply an extension of (8.13),

$$\begin{aligned} \mathbb{W}^{mqq} = [\nabla^m, \mathbb{W}^{qq}] &= \frac{1}{2}[\nabla^m, [\mathbb{F}_{ab}, (\gamma^{ab})\mathbb{W}]] \\ &= -\frac{1}{2}[(\gamma^{ab})\mathbb{W}^m, \mathbb{F}_{ab}] - \frac{1}{2}[(\gamma_{ab})\mathbb{W}, \mathbb{F}^{m|ab}]. \end{aligned} \tag{8.31}$$

To obtain the other trace vectors we can use the Jacobi identity to massage the indices into the form $\mathbb{W}^{mqq}$, for example,

$$\begin{aligned} \mathbb{W}^{qnq} = [\nabla^q, [\nabla^n, \mathbb{W}^q]] &= [\nabla^n, \mathbb{W}^{qq}] - [\mathbb{W}^q, [\nabla^q, \nabla^n]] \\ &= -\frac{1}{2}[(\gamma^{ab})\mathbb{W}^n, \mathbb{F}_{ab}] - \frac{1}{2}[(\gamma_{ab})\mathbb{W}, \mathbb{F}^{n|ab}] - [\mathbb{W}^q, \mathbb{F}^{nq}], \end{aligned} \tag{8.32}$$

and finally,

$$\begin{aligned}\mathbb{W}^{qqp} &= [\nabla^q,[\nabla^q,[\nabla^p,\mathbb{W}]]] = [\nabla^q,[\nabla^p,\mathbb{W}^q]] - [\nabla^q,[\mathbb{W},\mathbb{F}^{pq}]] \\ &= [\nabla^p,\mathbb{W}^{qq}] - [\mathbb{W}^q,\mathbb{F}^{pq}] - [\mathbb{W}^q,\mathbb{F}^{pq}] - [\mathbb{W},\mathbb{F}^{q|pq}],\end{aligned} \tag{8.33}$$

where we have,

$$\mathbb{F}^{q|pq} = -\gamma^p_{\alpha\beta}\{\mathbb{W}^\alpha,\mathbb{W}^\beta\}, \tag{8.34}$$

thence,

$$\mathbb{W}^{qqp} = -\frac{1}{2}[(\gamma^{ab})\mathbb{W}^p,\mathbb{F}_{ab}] - \frac{1}{2}[(\gamma_{ab})\mathbb{W},\mathbb{F}^{p|ab}] + [\mathbb{W},\gamma^p_{\alpha\beta}\{\mathbb{W}^\alpha,\mathbb{W}^\beta\}]. \tag{8.35}$$

Since $\gamma^p$ is symmetric in its spinor indices we have,

$$\begin{aligned}\gamma^p_{\alpha\beta}\{\mathbb{W}^\alpha,\mathbb{W}^\beta\} &= \gamma^p_{\alpha\beta}\mathbb{W}^\alpha\mathbb{W}^\beta + \gamma^p_{\alpha\beta}\mathbb{W}^\beta\mathbb{W}^\alpha \\ &= 2(\mathbb{W}\gamma^p\mathbb{W}),\end{aligned} \tag{8.36}$$

So finally,

$$\mathbb{W}^{qqp} = -\frac{1}{2}[(\gamma^{ab})\mathbb{W}^p,\mathbb{F}_{ab}] - \frac{1}{2}[(\gamma_{ab})\mathbb{W},\mathbb{F}^{p|ab}] + 2[\mathbb{W},(\mathbb{W}\gamma^p\mathbb{W})], \tag{8.37}$$

or alternatively,

$$\mathbb{W}^{qqp} = -\frac{1}{2}[(\gamma^{ab})\mathbb{W}^p,\mathbb{F}_{ab}] - \frac{1}{2}[(\gamma_{ab})\mathbb{W},\mathbb{F}^{p|ab}] - 2[\mathbb{W}^q,\mathbb{F}^{pq}]. \tag{8.38}$$

To make (8.19) an irreducible representation of the $SO(10)$ symmetry group we have to choose the constants in a particular way (there is a freedom to change the constants in the partially symmetric term due to the fact only two of the 2-forms are independent), hence the irreducible representation is given by,

$$\begin{aligned}\mathbb{W}^{mnp} &= \hat{\mathbb{W}}^{(mnp)} + \frac{1}{12}\tilde{\mathbb{W}}^{(mnp)} + \mathbb{W}^{[mnp]} \\ &\quad + \frac{2}{3}(2\mathbb{W}^{m[np]} + \mathbb{W}^{p[mn]} - \mathbb{W}^{[mn]p} - 2\mathbb{W}^{[pm]n}),\end{aligned} \tag{8.39}$$

alternatively one can take,

$$\begin{aligned}\mathbb{W}^{mnp} &= \hat{\mathbb{W}}^{(mnp)} + \frac{1}{12}\tilde{\mathbb{W}}^{(mnp)} + \mathbb{W}^{[mnp]} \\ &\quad + \frac{2}{3}(\mathbb{W}^{[mn]p} + \mathbb{W}^{m[np]} - \mathbb{W}^{n[pm]} - \mathbb{W}^{[pm]n}).\end{aligned} \tag{8.40}$$

These decompositions allow us to write much simpler ansatzes as we can assume symmetry properties about the indices which eliminates many terms from the ansatz. It also reduces the number of unknowns that need to be solved for, thus making the whole process of finding corrections slightly easier.

## 8.3 Pure Spinor Identities

In this section we shall derive some identities that follow from the application of the pure spinor constraint - these identities are absolutely necessary in order to properly canonicalise the BRST variations. These identities allow us to canonicalise the $\gamma$-matrices in various generating series expressions, it is the $\gamma-$matrices that often causes issues when finding the correct set of linear equations. The first identity that we need to consider, which will underpin many of the other identities presented here, stems from the $\gamma$-matrix identity,

$$\gamma^m_{\alpha(\beta}\gamma^m_{\rho\sigma)} = 0. \tag{8.41}$$

Consider the following,

$$\begin{aligned}\lambda^\alpha\lambda^\rho\gamma^m_{\alpha(\beta}\gamma^m_{\rho\sigma)} &= (\lambda\gamma^m)_\beta(\lambda\gamma^m)_\sigma + (\lambda\gamma^m)_\sigma(\lambda\gamma^m)_\beta + \gamma^m_{\beta\sigma}(\lambda\gamma^m\lambda) \\ &= 2(\lambda\gamma^m)_\beta(\lambda\gamma^m)_\sigma = 0,\end{aligned} \tag{8.42}$$

hence we have the following important identity [7, 87],

$$(\lambda\gamma^m)_\alpha(\lambda\gamma^m)_\beta = 0. \tag{8.43}$$

As a result, any terms of the form $(\lambda\gamma^m\mathbb{W})(\lambda\gamma^m\mathbb{W}) = 0$, but it also implies that there are identities that can simplify terms such as $(\lambda\gamma^m)_\alpha(\lambda\gamma^{mnp})_\beta$. To see this, consider the

$\gamma-$matrix decomposition into $\gamma^m$ and $\gamma^{np}$, that is,

$$\gamma^{mnp} = \gamma^m \gamma^{np} + \delta^m_p \gamma^n - \delta^m_n \gamma^p. \tag{8.44}$$

Using this, one can easily show the following,

$$(\lambda\gamma^m)_\alpha(\lambda\gamma^{mnp})_\beta = (\lambda\gamma^n)_\alpha(\lambda\gamma^p)_\beta - (\lambda\gamma^p)_\alpha(\lambda\gamma^n)_\beta, \tag{8.45}$$

and hence we can break the contraction $(\lambda\gamma^m)_\alpha(\lambda\gamma^{mnp})_\beta$ into terms involving only $\gamma^m$ - thus reducing the number of terms that need to be considered in the ansatz. One can extend this to higher order $\gamma$ matrices, for example consider the case of $(\lambda\gamma^{mnp})_\alpha(\lambda\gamma^{mqr})_\beta$ which simplifies to,

$$\begin{aligned}(\lambda\gamma^{mnp})_\alpha(\lambda\gamma^{mqr})_\beta =& (\lambda\gamma^{npr})_\alpha(\lambda\gamma^q)_\beta - (\lambda\gamma^{npq})_\alpha(\lambda\gamma^r)_\beta + (\lambda\gamma^n)_\alpha(\lambda\gamma^{pqr})_\beta \\ & - (\lambda\gamma^p)_\alpha(\lambda\gamma^{nqr})_\beta + \eta^{nr}(\lambda\gamma^p)_\alpha(\lambda\gamma^q)_\beta - \eta^{rp}(\lambda\gamma^n)_\alpha(\lambda\gamma^q)_\beta \\ & - \eta^{nq}(\lambda\gamma^p)_\alpha(\lambda\gamma^r)_\beta + \delta^{pq}(\lambda\gamma^n)_\alpha(\lambda\gamma^r)_\beta,\end{aligned} \tag{8.46}$$

which again allows us to write terms generated in the ansatz, that contain $(\lambda\gamma^{mnp})_\alpha(\lambda\gamma^{mqr})_\beta$, can be absorbed into other terms in the ansatz. It also allows us to canonicalise the BRST variation of the ansatz as we will have to apply these identities to the variation in order to obtain the correct set of linear equations. For higher order $\gamma-$matrices we can play the same game by noting that, in the case of $\gamma^{mnpqr}$, we have the decomposition,

$$\gamma^{mnpqr} = \gamma^m\gamma^{npqr} + \eta^{mr}\gamma^{npq} - \eta^{mq}\gamma^{npr} + \eta^{mp}\gamma^{nqr} - \eta^{mn}\gamma^{pqr} \tag{8.47}$$

and so we have identities such as,

$$\begin{aligned}(\lambda\gamma^m)_\alpha(\lambda\gamma^{mnpqr})_\beta =& (\lambda\gamma^r)_\alpha(\lambda\gamma^{npq})_\beta - (\lambda\gamma^q)_\alpha(\lambda\gamma^{npr})_\beta \\ & + (\lambda\gamma^p)_\alpha(\lambda\gamma^{nqr})_\beta - (\lambda\gamma^n)_\alpha(\lambda\gamma^{pqr})_\beta\end{aligned} \tag{8.48}$$

and,

$$\begin{aligned}(\lambda\gamma^{mnp})_\alpha(\lambda\gamma^{mqrst})_\beta =& (\lambda\gamma^n)_\alpha(\lambda\gamma^{pqrst})_\beta - (\lambda\gamma^p)_\alpha(\lambda\gamma^{nqrst})_\beta + (\lambda\gamma^{npt})_\alpha(\lambda\gamma^{qrs})_\beta \quad (8.49) \\ & - (\lambda\gamma^{nps})_\alpha(\lambda\gamma^{qrt})_\beta + (\lambda\gamma^{npr})_\alpha(\lambda\gamma^{qst})_\beta - (\lambda\gamma^{npq})_\alpha(\lambda\gamma^{rst})_\beta\end{aligned}$$

$$
\begin{aligned}
&- \eta^{nq}(\lambda\gamma^{p})_{\alpha}(\lambda\gamma^{rst})_{\beta} + \eta^{nr}(\lambda\gamma^{p})_{\alpha}(\lambda\gamma^{qst})_{\beta} - \eta^{ns}(\lambda\gamma^{p})_{\alpha}(\lambda\gamma^{qrt})_{\beta} \\
&+ \eta^{nt}(\lambda\gamma^{p})_{\alpha}(\lambda\gamma^{qrs})_{\beta} + \eta^{pq}(\lambda\gamma^{n})_{\alpha}(\lambda\gamma^{rst})_{\beta} - \eta^{pr}(\lambda\gamma^{n})_{\alpha}(\lambda\gamma^{qst})_{\beta} \\
&+ \eta^{ps}(\lambda\gamma^{n})_{\alpha}(\lambda\gamma^{qrt})_{\beta} - \eta^{pt}(\lambda\gamma^{n})_{\alpha}(\lambda\gamma^{qrs})_{\beta}.
\end{aligned}
$$

One can continue to give the $\gamma^5\gamma^5$ decomposition, but this contains 24 different terms. The point of this section is to show that the pure spinor condition can have very intricate effects and link ansatz terms together in a way that is not *a priori* obvious. It is only by utilising all of the identities and methods outlined above that one can determine these $\alpha'$ corrections in a somewhat efficient manner.

In this chapter we have introduced a number of new identities which can be used to canonicalise the indices of various higher mass fields. These new identities included some new Jacobi based identities which allow us to pick a specific index to canonicalise on, allowing us to find Jacobi identities that result from the BRST variation of the ansatz. Furthermore, we determined the decomposition of some of the higher mass fields into their irreducible symmetric, traceless part and other fields. Finally, we reviewed some of the identities that follow from the pure spinor identity. All of the identities are pivotal to properly ordering the generating series expressions that result from BRST variation of complex ansatzes.

# Chapter 9

# $\alpha'$ and $\alpha'^2$ Results

Determining the $\alpha'$ and $\alpha'^2$ results is a much simpler task than working at higher orders, mainly due to the fact that the ansatz grows exponentially at each mass dimension[1]. Hence, at lower $\alpha'$ orders it is easier to canonicalise the variation of the ansatz because there are fewer terms to deal with. In fact, in the case of the $\alpha'$ corrections it is almost trivial to demonstrate that there are no corrections at this order. The constraints from group theory and dimensional analysis demonstrate that there is no ansatz at this order that meets the requirements. The $\alpha'^2$ corrections are a little more involved, but they are much more tractable than higher order corrections and so we spend a little time analysing these terms in depth. In fact we can find this correction by hand as a demonstration of the techniques outlined here.

## 9.1 A Brief Note on the $\alpha'$ Correction

It is worth briefly outlining the the results of applying dimensional analysis and group theory arguments at the $\alpha'$ order to show there is no ansatz. We can also come to this conclusion using dimensional arguments only, and a little bit of calculation. For completeness, we shall explore both avenues.

[1]Note that the dimensions of each $\alpha'$ corrections grows as $\dim(\alpha'^n) = (2n + \frac{3}{2})$.

The corrections at $\alpha'$ would have mass dimension equal to 3.5 (each $\alpha'$ order adds mass dimension 2 - the $\alpha'^0$ correction has mass dimension 1.5). At this mass dimension there are only a handful of terms we can write down[2],

$$\begin{aligned}&\mathbb{V}(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W}),\\&\mathbb{V}(\lambda\gamma\lambda)\nabla\mathbb{F},\\&(\lambda\gamma\lambda)(\lambda\gamma\mathbb{W})\mathbb{F}.\end{aligned} \tag{9.1}$$

With a little thought, as well as some trial and error, it becomes obvious that these are the only terms that one can consider. If one then use `LiE` to analyse these terms one finds that there are no scalar combinations in $SO(10)$ and hence there can be no ansatz at this order and as a result there can be no $\alpha'$ correction [77, 109]. This is a well known result and comes as no surprise, but it is always worthwhile checking the obvious results to ensure the method is sound.

There is another way to come to this conclusion using some knowledge about $\gamma$ matrices in $D = 10$. Firstly, note that the pure spinor constraint implies that the $\gamma$ matrix in $(\lambda\gamma\lambda)$ must have five vector indices. It is obvious that this matrix cannot have one index as this is the definition of the pure spinor. The $\gamma$ matrix cannot have three vector indices because $\gamma^{mnp}_{\alpha\beta} = -\gamma^{mnp}_{\beta\alpha}$ and the pure spinor is a bosonic spinor. Thus $(\lambda\gamma^{mnp}\lambda) = 0$ due to the antisymmetry of the matrix and the bosonic statistics of the pure spinor. Hence $(\lambda\gamma\lambda) \equiv (\lambda\gamma^{mnpqr}\lambda)$ in all of our ansatzes. This immediately implies that the second term in (9.1) must be zero since there is no way to fully contract all of the vector indices without a $\gamma^5$ self contraction which sets the term to zero anyway. Hence the second term is out. As for the third term, this a little more involved, but one can use pure spinor identities to remove this term as well. Note that the $\gamma$ matrix in $(\lambda\gamma\mathbb{W})$ must have $\gamma^3$ in order to saturate all of the vector indices. Hence, consider the identity given in (8.49) - when one sums over two more indices in this identity the terms involving $\gamma^1\gamma^5$ disappear due to the self-contractions within the $\gamma^5$ matrix. Hence the only terms left are ones which will involve $(\lambda\gamma^m\lambda) = 0$ and $(\lambda\gamma^{mnp}\lambda) = 0$ and hence there are no corrections at this order from this kind of term. The first term in (9.1) requires a

[2]Note that we may occasionally use this kind of abstract notation where we do not explicitly include the trace and $\mathbb{V}_i$ but this is implicit throughout.

little more explanation but the theme basically runs the same - it's all the pure spinor identity. In the case when we have $\gamma^1\gamma^1$ the pure spinor identity trivially kicks in. For other combinations one needs to simply contract all of the indices on $\gamma^3\gamma^3$ in (8.46) and one obtains zero via self-contractions and the pure spinor identity. Finally, we know that due to self-duality,

$$\gamma^{mnpqr}_{\alpha\beta}\gamma^{mnpqr}_{\rho\sigma} = 0, \tag{9.2}$$

hence there can be no corrections at this order due solely to the pure spinor identity. This ought to make clear the power of the pure spinor identity when considering these ansatzes - many of the terms one would presume could contribute are actually zero. In many cases, this is not immediately obvious but a little tracking of $\gamma$ matrices and many of the terms simply disappear - although such tracking can become quite difficult at higher orders.

## 9.2 Determining the $\alpha'^2$ Correction

When working to $\alpha'^2$ we have to consider operators which have mass dimension 5.5, this increases the possible number of terms we can consider, on dimensional grounds, to nine. In the following we shall analyse these terms using group theory analysis and then explicitly show the BRST calculations that one performs in order to obtain BRST closure. The possible $\mathbb{V}$ and no-$\mathbb{V}$ terms are given by the following set,

$$\begin{aligned}
&\mathbb{V}(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})\mathbb{F}, && \mathbb{V}(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\nabla\mathbb{W}),\\
&\mathbb{V}(\lambda\gamma\mathbb{W})(\lambda\gamma\nabla^2\mathbb{W}), && \mathbb{V}(\lambda\gamma\lambda)\mathbb{F}\nabla\mathbb{F},\\
&(\lambda\gamma\lambda)(\lambda\gamma\nabla\mathbb{W})\nabla\mathbb{F}, && (\lambda\gamma\lambda)(\lambda\gamma\nabla^2\mathbb{W})\mathbb{F},\\
&(\lambda\gamma\lambda)(\lambda\gamma\mathbb{W})\nabla^2\mathbb{F}, && (\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})(\lambda\gamma\nabla\mathbb{W}),
\end{aligned}$$

however, using the $SO(10)$ group structure we find that all but two of these structures are zero[3]. The only two non-zero terms are, $\mathbb{V}(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})\mathbb{F}$ and $\mathbb{V}(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\nabla\mathbb{W})$ - there is only one independent scalar for each of these terms. One thus concludes that

[3]Note that the term $\mathbb{V}(\lambda\gamma\lambda)\mathbb{F}\nabla\mathbb{F}$ is zero, but using the equations of motion it is linked to the first term in the set.

there is not a no-$\mathbb{V}$ sector correction at this order and any correction must involve $\mathbb{V}$. In the case of the first term, the only independent scalar one can write down is[4],

$$\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}, \tag{9.3}$$

since any other scalar form would either be zero or can be broken down into this form using the pure spinor identity. As for the second term, one has to keep in mind all of the pure spinor machinery as well as the Dirac equation. These facts imply that the only scalar we can construct that is non-zero (at least at the level of group theory and pure spinor arguments) is,

$$\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^m). \tag{9.4}$$

As a result our ansatz is quite easy to write down and gives a flavour of the kind of behind-the-scenes calculations required to obtain these ansatzes. The ansatz will be given by,

$$\begin{aligned}\mathbb{Y} =&\mathrm{Tr}\big(\mathbb{V}_1[x_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn} + x_2(\lambda\gamma^m\mathbb{W})\mathbb{F}^{mn}(\lambda\gamma^n\mathbb{W})\\ &+ x_3\mathbb{F}^{mn}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})] + x_4\mathbb{V}(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^m)\big),\end{aligned} \tag{9.5}$$

where we have dropped terms that are the same up to anti-symmetry of $\mathbb{F}$. Let us consider the first term, with $x_1$ and look at the BRST variation of this term. The calculation goes along as follows,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q},(\mathbb{V}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn})\} =& -\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\{\mathcal{Q},\mathbb{W}\})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}]\\ &+\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\{\mathcal{Q},\mathbb{W}\})\mathbb{F}^{mn}]\\ &-\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\{\mathcal{Q},\mathbb{F}^{mn}\}]\\ =& -\frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^{pq}\gamma^m\lambda)\mathbb{F}_{pq}(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}]\\ &+\frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^{pq}\gamma^n\lambda)\mathbb{F}_{pq}\mathbb{F}^{mn}]\\ &-\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})(\lambda\gamma^{[m}\mathbb{W}^{n]})],\end{aligned} \tag{9.6}$$

[4]Note the explicit label on the $\mathbb{V}$ in these expressions. The trace is implicit in these expressions.

now note that $\gamma^{pq}\gamma^m = \gamma^{mpq} + \delta^{mq}\gamma^p - \delta^{mp}\gamma^q$ and hence the pure spinor constraint and anti-symmetry gets rid of the first two terms[5]. The final term is zero by the pure spinor constraint also, this is clear from the $(\lambda\gamma^m)_\alpha(\lambda\gamma^n)_\beta$ identity. Hence, we have actually stumbled[6] onto one of the forms of the $\alpha'^2$ corrections, that is,

$$L_{\alpha'^2} = \mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}], \tag{9.7}$$

note of course that all other combinations of the fields will have a BRST variation that is zero as our arguments are based off of symmetry and the pure spinor constraint and hence they are not affected by field ordering. In fact, we can show that these field orderings do not matter by looking at BRST ancestors of the expressions. For example, consider,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}_1(\lambda\gamma^{mnp}\mathbb{W})(\mathbb{W}\gamma^{mnp}\mathbb{W})\} = {}& \frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^{mnp})_\beta(\lambda\gamma^{qr})^\beta\mathbb{F}_{qr}(\mathbb{W}\gamma^{mnp}\mathbb{W})] \\ & - \frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^{mnp}\mathbb{W})\mathbb{F}_{qr}(\lambda\gamma^{qr})^\beta(\gamma^{mnp}\mathbb{W})_\beta] \\ & + \frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^{mnp}\mathbb{W})(\mathbb{W}\gamma^{mnp})_\beta(\gamma^{qr}\lambda)^\beta],\end{aligned} \tag{9.8}$$

let us take the first line of this variation and manipulate it into a form that will be useful here. First begin by noting the identity $\gamma^{mnp}_{\alpha\beta}\gamma^{mnp}_{\rho\sigma} = 12(\gamma^m_{\alpha\sigma}\gamma^m_{\beta\rho} - \gamma^m_{\alpha\rho}\gamma^m_{\beta\rho})$, as a result of this identity we can write the first term in the following manner,

$$\begin{aligned}\frac{1}{4}(\lambda\gamma^{mnp})_\beta(\lambda\gamma^{qr})^\beta\mathbb{F}_{qr}(\mathbb{W}\gamma^{mnp}\mathbb{W}) = {}& 4\mathbb{F}_{qr}(\lambda\gamma^{qr}\gamma^m\mathbb{W})(\lambda\gamma^m\mathbb{W}) \\ & - 4\mathbb{F}_{qr}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^{qr}\gamma^m\mathbb{W}),\end{aligned} \tag{9.9}$$

and now one can apply $\gamma^{qr}\gamma^m = \gamma^{mqr} - \delta^{mq}\gamma^r + \delta^{mr}\gamma^q$ as well as (8.45) to obtain,

$$\frac{1}{4}(\lambda\gamma^{mnp})_\beta(\lambda\gamma^{qr})^\beta\mathbb{F}_{qr}(\mathbb{W}\gamma^{mnp}\mathbb{W}) = 24\mathbb{F}_{mn}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W}). \tag{9.10}$$

The same sort of calculation follows for the other terms, except two small caveats. The first being that one obtains terms of the form $(\lambda\gamma^{mqr}\lambda) = 0$ which follows as before from

[5] One could see this immediately and there really was no need to invoke the $\gamma$ matrix identity. We know that $\lambda\gamma\lambda$ must be $\lambda\gamma^5\lambda$ to be non-zero and it is not possible form this combination out of three $\gamma$ matrices hence these terms had to be zero on the spot.

[6] Of course there was no stumbling here - this term was chosen on purpose!

the antisymmetry of $\gamma^3$. The second caveat is taking care of the antisymmetry of $\gamma^2$ under swapping the spinor indices. Bearing these points in mind one finds,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}_1(\lambda\gamma^{mnp}\mathbb{W})(\mathbb{W}\gamma^{mnp}\mathbb{W})\} = 24\mathrm{Tr}[\mathbb{V}_1\mathbb{F}_{mn}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})]\\ - 12\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})\mathbb{F}_{mn}(\lambda\gamma^n\mathbb{W})]\\ - 12\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}_{mn}].\end{aligned} \tag{9.11}$$

Of course the expression we started with is BRST exact and hence we find,

$$2\langle\mathbb{V}_1\mathbb{F}_{mn}(\lambda\gamma^m\mathbb{W})(\mathbb{W}\gamma^n\mathbb{W})\rangle = \langle\mathbb{V}_1(\lambda\gamma^m\mathbb{W})\mathbb{F}_{mn}(\lambda\gamma^n\mathbb{W})\rangle + \langle\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}_{mn}\rangle, \tag{9.12}$$

where the pure spinor integration measure $\langle\rangle$ projects out the BRST exact piece. One can then perform the same manipulations in order to demonstrate,

$$\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^m\mathbb{W})\} = \mathrm{Tr}\big[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})[\mathbb{F}_{mn}, (\lambda\gamma^n\mathbb{W})]\big] \tag{9.13}$$

which then implies,

$$\langle\mathbb{V}_1(\lambda\gamma^m\mathbb{W})\mathbb{F}_{mn}(\lambda\gamma^n\mathbb{W})\rangle = \langle\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}_{mn}\rangle, \tag{9.14}$$

and hence field ordering does not matter as all field ordering only differ by a BRST exact term(s). Given that all of these field ordering are the same one could take a linear combination of each ordering weighted by 1/3. However, this allows us to use the symmetric trace [110, 111] to express the correction in line with the literature,

$$L_{\alpha'^2} = \mathrm{sTr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}], \tag{9.15}$$

where the symmetry acts on $\mathbb{W}$ and $\mathbb{F}$ as we can use cyclicity to always bring $\mathbb{V}_1$ to the front of the trace[7]. Note that the symmetric trace implicitly contains the correct factors when expressed in terms of normal traces - that is including the relevant $1/n!$ factors.

Now, let us quickly look at the second kind of term that we can add to the ansatz, namely $\mathbb{V}(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^m)$ and determine its BRST variation. The variation goes as

[7]Essentially, the symmetric trace here represents $\mathrm{sTr}(ABC) = \frac{1}{3}\mathrm{Tr}(ABC + CAB + BCA)$.

follows,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q},\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^m)\} =& -\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\{\mathcal{Q},\mathbb{W}^n\})(\lambda\gamma^n\mathbb{W}^m)] \\ &+\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\{\mathcal{Q},\mathbb{W}^m\})]\end{aligned} \tag{9.16}$$

which gives the following,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q},\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^m)\} =& -\frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^{pq}\gamma^m\lambda)\mathbb{F}^{n|pq}(\lambda\gamma^n\mathbb{W}^m)] \\ &+\mathrm{Tr}[\mathbb{V}_1\{(\lambda\gamma^n\mathbb{W}),(\lambda\gamma^m\mathbb{W})\}(\lambda\gamma^n\mathbb{W}^m)] \\ &+\frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^{pq}\gamma^n\lambda)(\lambda\gamma^m\mathbb{W}^n)\mathbb{F}^{m|pq}] \\ &-\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^n\mathbb{W}^m)\{(\lambda\gamma^m\mathbb{W}),(\lambda\gamma^n\mathbb{W})\}],\end{aligned} \tag{9.17}$$

which is zero due to the pure spinor identity. Let us explain this a little further. We know that $(\lambda\gamma^{pq}\gamma^n\lambda)$ terms are zero due to the antisymmetry of $\gamma^3$ as well as the plain pure spinor identity. The anti-commutator terms are zero due to $(\lambda\gamma^m)_\alpha(\lambda\gamma^m)_\beta = 0$ hence both of these terms disappear. The same occurs with the variation $\{\mathcal{Q},\mathbb{W}^m\}$ and hence the total BRST variation is zero. This is another BRST-closed expression and hence this, by cohomology arguments alone, is another expression for the $\alpha'^2$ corrections,

$$L_{\alpha'^2} = \mathrm{Tr}(\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^m)). \tag{9.18}$$

However, if one considers the term,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q},\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)\mathbb{F}^{mn}\} =& \frac{1}{4}\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m)_\beta(\lambda\gamma^{pq})_\beta\mathbb{F}^{n|pq}] - \mathrm{Tr}[\mathbb{V}_1\{(\lambda\gamma^n\mathbb{W}),(\lambda\gamma^m\mathbb{W})\}\mathbb{F}^{mn}] \\ &+\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^{[m}\mathbb{W}^{n]})],\end{aligned} \tag{9.19}$$

one can demonstrate,

$$\mathrm{Tr}\{\mathcal{Q},\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)\mathbb{F}^{mn}\} = -\mathrm{Tr}[\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^m)], \tag{9.20}$$

and hence this term is BRST exact, as a result this term is not actually a physical generating series of the correction at this order and so the only valid expression is the previous one.

Using BRST closure and component expansions it appears that the first expression is the form of the $\alpha'^2$ correction. We can connect the correction to an effective generating series action, at tree-level the amplitudes are generated by the action [109, 112, 113],

$$S = \int \mathrm{d}[X]\mathrm{Tr}\langle \tfrac{1}{2}\mathbb{V}Q\mathbb{V} + \tfrac{1}{3}\mathbb{V}\mathbb{V}\mathbb{V}\rangle, \tag{9.21}$$

where $Q\mathbb{V}$ is a kind of kinematic term and $\mathbb{V}\mathbb{V}\mathbb{V}$ is the generating series of tree-level amplitudes - the generating series $\mathbb{V} = \lambda^\alpha \mathbb{A}_\alpha$ satisfies the equation of motion $Q\mathbb{V} = \mathbb{V}\mathbb{V}$. Hence, we have an action for $\mathbb{V}$, which has some equations of motion so we can treat the $\alpha'$ corrections are perturbations to this action [77]. Thence we find that, up to $\alpha'^2$ we have the action,

$$S = \int \mathrm{d}[X]\mathrm{Tr}\langle \tfrac{1}{2}\mathbb{V}Q\mathbb{V} + \tfrac{1}{3}\mathbb{V}\mathbb{V}\mathbb{V} + \alpha'^2\mathbb{V}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}\rangle. \tag{9.22}$$

Hence, at this order, the generating series have to satisfy the following equation of motion,

$$Q\mathbb{V} = \mathbb{V}\mathbb{V} + \alpha'^2(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn} \tag{9.23}$$

which can be used to determine effective equations of motion. Each of the interaction terms that are added to this effective action represent the generating series of the amplitudes at that order. One can perform further analysis of such actions along Batalin-Vilkovisky [114, 115, 116, 117, 118, 119] which allows one to consider off-shell actions. This premise is the basis of one potential solution to the issues raised by the $\alpha'^4$ correction, we shall detail this in the coming chapters.

In this chapter we have found the $\alpha'^2$ corrections to the Super Yang-Mills generating series using BRST cohomology arguments. The calculations in this chapter have been rather tractable, a luxury which is lost once we move to higher order corrections as the ansatz exponentially grows. This chapter has shown the essence of the method used to determine the results presented, or attempted, in later chapters and serves as a good example of how one can use BRST cohomology arguments to determine the corrections. We also detailed how the generating series can be used to form an action that yields equations of motion that the generating series satisfies. The generating series expressions found in this chapter can then be added to this action and one obtains an *effective generating series* whose interaction terms are the various amplitude generating series.

# Chapter 10

# $\alpha'^3$ Results

Working at the level of $\alpha'$ and $\alpha'^2$ corrections is rather straightforward. In the former case, there is simply no correction and in the latter case the weight of the correction, as well as group theory constraints, mean that the ansatz is incredibly simple. As a result, we only have to begin to employ more complex machinery once we arrive at $\alpha'^3$. This correction demonstrates a decoupling of the generating series into terms which involve $\mathbb{V}_1$ and those that do not. This decoupling is interesting as it appears to be the only $\alpha'$ order that such a decoupling occurs at. In the $\alpha'^2$ case there was no way to express the correction in terms of no-$\mathbb{V}_1$ terms and at $\alpha'^4$, as we shall see, one needs to mix both $\mathbb{V}_1$ and no-$\mathbb{V}_1$ terms in order to find the correction. First, we present the solution involving $\mathbb{V}_1$ terms as well as a number of BRST-exact expressions at this order that are found as a consequence of enforcing BRST closure on the ansatz. These BRST-exact terms are quite useful as they allow us to simplify the result from the ansatz.

Note that BRST closure is not enough to completely fix all of the constants, as mentioned previously - one can add BRST exact expressions to the solution and change the form of the solution. In fact, it is this that allows us to simplifies the initial solution. The BRST-closure property also does not fix completely the component expansion at a given order, this is because some Mandelstam invariants are BRST invariant and hence one needs to properly compare any solution to the BRST-closure with a know amplitude. Luckily these amplitudes have been previously derived as the field theory limit of string scattering amplitudes [7].

## 10.1 $\mathbb{V}$ Corrections

In order to build the ansatz, one first needs to determine the terms that can contribute in the $\mathbb{V}$ sector according to their weight and group theory properties. Recalling that $\mathbb{V}$ has weight 0.5, and $\alpha'^3$ corrections have weight 7.5, we need to look for terms that are dimension 7 scalars under $SO(10)$. This is simply a matter of pen and paper and the results of the possible terms are displayed[1] in the first half of Table 10.2. After finding all of the possible vector index contractions that are non-zero, and taking all relevant permutations of the fields, the ansatz contains 168 terms. Note that we need not use the tensor decompositions of Chapter 8, but we can use the pure spinor identities as well as the Jacobi identities to perform some canonicalisation to simplify the ansatz and the subsequent BRST variation.

After taking the BRST variation of the 168 terms, we must canonicalise the ansatz as far as possible in order to obtain the correct set of linear equations. After the variation there are often a number of terms where the generating series expression is zero and hence the linear equation attached to such an expression need not be zero. We must catch all of these instances otherwise we risk setting certain coefficients to zero. Once we have done this and compared the component expansion of the result with the known 4-point amplitude we obtain the following (non-unique) form for the $\alpha'^3$ term,

$$\begin{aligned} L_{\alpha'^3} = 2880\mathrm{Tr}\Big[\mathbb{V}_1\big(&\mathbb{F}^{mn}(\lambda\gamma^n\mathbb{W}^p)(\lambda\gamma^p\mathbb{W}^m) + (\lambda\gamma^m\mathbb{W}^n)\mathbb{F}^{np}(\lambda\gamma^p\mathbb{W}^m) \\ &+ \frac{1}{2}\big[\mathbb{F}^{mn}\mathbb{F}^{np}, \{(\lambda\gamma^m\mathbb{W}), (\lambda\gamma^p\mathbb{W})\}\big] \\ &+ \frac{1}{4}\mathbb{F}^{mn}\mathbb{F}^{pq}\{(\lambda\gamma^m\mathbb{W}), (\lambda\gamma^{npq}\mathbb{W})\} \\ &+ \frac{1}{4}\mathbb{F}^{mn}(\lambda\gamma^{npq}\mathbb{W})[(\lambda\gamma^m\mathbb{W}), \mathbb{F}^{pq}] \\ &- \frac{1}{4}\mathbb{F}^{pq}(\lambda\gamma^m\mathbb{W})[(\lambda\gamma^{npq}\mathbb{W}), \mathbb{F}^{mn}] \\ &+ \frac{1}{4}(\lambda\gamma^{npq}\mathbb{W})(\lambda\gamma^m\mathbb{W})\mathbb{F}^{[mn]|pq}\big)\Big] \end{aligned} \tag{10.1}$$

However, during the procedure to find this result we obtain a number of BRST-exact expressions which can be used to further simplify this answer. For example, we can use

[1] Note that in the $\alpha'^3$ ansatz we use terms from Table 10.2 that do not contain a raw $\mathbb{A}$ field and are above the double line.

the following BRST-exact expression to remove the first five-field terms in the above result[2],

$$\begin{aligned}\langle \mathbb{V}_1[\mathbb{F}^{mn}\mathbb{F}^{np},\{(\lambda\gamma^n\mathbb{W}^p),(\lambda\gamma^p\mathbb{W}^m)\}]\rangle &= \langle \mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)\mathbb{F}^{mp}(\lambda\gamma^n\mathbb{W}^p)\rangle \\ &- \langle \mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^p\mathbb{W}^m)\mathbb{F}^{np}\rangle - \langle \mathbb{V}_1\mathbb{F}^{mn}(\lambda\gamma^n\mathbb{W}^p)(\lambda\gamma^p\mathbb{W}^m)\rangle \\ &- \langle \mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)\mathbb{F}^{np}(\lambda\gamma^p\mathbb{W}^m)\rangle,\end{aligned} \tag{10.2}$$

hence we can replace the five-field term with four-field terms, reducing the number of terms in the $\alpha'^3$ answer by two. There are many more BRST closed identities that can be determined from analysis like this, however it appears that these identities cannot be used to simplify the $\alpha'^3$ answer any further. For now it is best to content oneself with the simpler solution of the form,

$$\begin{aligned}L_{\alpha'^3} = 1440\mathrm{Tr}\Big[\mathbb{V}_1\big(&\{[\mathbb{F}^{mn},(\lambda\gamma^n\mathbb{W}^p)],(\lambda\gamma^p\mathbb{W}^m)\} \\ &+ \frac{1}{2}\mathbb{F}^{mn}\mathbb{F}^{pq}\{(\lambda\gamma^m\mathbb{W}),(\lambda\gamma^{npq}\mathbb{W})\} \\ &+ \frac{1}{2}\mathbb{F}^{mn}(\lambda\gamma^{npq}\mathbb{W})[(\lambda\gamma^m\mathbb{W}),\mathbb{F}^{pq}] \\ &- \frac{1}{2}\mathbb{F}^{pq}(\lambda\gamma^m\mathbb{W})[(\lambda\gamma^{npq}\mathbb{W}),\mathbb{F}^{mn}] \\ &+ \frac{1}{2}(\lambda\gamma^{npq}\mathbb{W})(\lambda\gamma^m\mathbb{W})\mathbb{F}^{[mn]|pq}\big)\Big].\end{aligned} \tag{10.3}$$

We do not believe that any other terms can be used to simplify this expression (after a systematic attempt to introduce more BRST-exact expressions we only introduce more terms). One could potentially use the equations of motion to simplify this result further, but is currently not clear whether this is worthwhile. However, it is worthwhile giving some of the other expressions that link terms with 4- and 5-fields as this may be useful in future research. One such term that connects these orders of fields is,

$$\begin{aligned}\langle \mathbb{V}_1\mathbb{F}^{mn}\{(\lambda\gamma^n\mathbb{W}),(\lambda\gamma^p\mathbb{W})\}\mathbb{F}^{mp}\rangle =&\langle \mathbb{V}_1\mathbb{F}^{mn}(\lambda\gamma^n\mathbb{W}^p)(\lambda\gamma^{[m}\mathbb{W}^{p]})\rangle \\ &- \langle \mathbb{V}_1(\lambda\gamma^m\mathbb{W}^n)(\lambda\gamma^n\mathbb{W}^p)\mathbb{F}^{mp}\rangle.\end{aligned} \tag{10.4}$$

[2]The equality between the two expressions here is not exact in superfield terms but the pure spinor integration kills any BRST exact terms

We can also find BRST-exact terms which link $\mathbb{W}^{mn}$ with $\mathbb{F}^{m|np}$, that is terms which look like integration by parts,

$$\langle \mathbb{V}_1(\lambda\gamma^m\mathbb{W})\mathbb{F}^{np}(\lambda\gamma^p\mathbb{W}^{mn})\rangle = \langle \mathbb{V}_1(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W}^p)\mathbb{F}^{m|np}\rangle, \tag{10.5}$$

but do not seem to follow the obvious integration by parts-like pattern discussed in Section 10.3.

Before we move on it is worth briefly discussing the connection between the correction presented above and the amplitudes it is able to produce. The first term in (10.3) corresponds to the generating series generalisation of the 1-loop Super Yang-Mills amplitude, that is the box diagram [29]. Hence this generating series ought to be able to generate the $n-$point kinematic factor for the 1-loop box diagram in Super Yang-Mills - this is a very profound result. Without reference to amplitudes, loops or string theory we are able, using the cohomology of pure spinor space alone, able to determine expressions that generate $n-$point kinematic factors of loops. As we shall see in the next section, the $\alpha'^3$ corrections in the no-$\mathbb{V}$ sector are able to generate a number of other diagrams at loop level. Other terms in (10.3) represent stringy corrections that arise in order to correct Super Yang-Mills amplitudes with string theory behaviours.

## 10.2 No-$\mathbb{V}$ Ansatz

One can also generate an ansatz that does not contain $\mathbb{V}_1$ terms and then apply the same methodologies in order to determine the correction. One now has a much more restricted set of terms that can be used to generate the ansatz as one has to find dimension 7.5 terms which contain 3 ghosts, without the use of $\mathbb{V}_1$. The terms that one can consider are given in Table 10.1 - these terms and their permutations constitute the ansatz. At $\alpha'^3$ there are actually many possible solutions corresponding to a number of different way of expressing the amplitudes and each of these makes some loop diagram manifest. One possible expression is,

$$L_{\alpha'^3} = \mathbb{F}^{m|pq}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^p\mathbb{W})(\lambda\gamma^q\mathbb{W}), \tag{10.6}$$

| Term | # Scalars |
|---|---|
| $(\lambda\gamma\lambda)(\lambda\gamma\nabla\mathbb{W})\nabla\mathbb{F}\mathbb{F}$ | 1 |
| $(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\nabla\mathbb{W})$ | 2 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\nabla^2\mathbb{W})$ | 1 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})(\lambda\gamma\nabla\mathbb{W})\mathbb{F}$ | 2 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})\nabla\mathbb{F}$ | 2 |
| $(\lambda\gamma\lambda)(\lambda\gamma\mathbb{W})\mathbb{F}\mathbb{F}\mathbb{F}$ | 2 |

TABLE 10.1: The only terms that are non-zero under $SO(10)$ that are allows by dimensional consideration in the no-$\mathbb{V}$ ansatz.

which is similar in construction to the Mercedes star multiparticle expression given in [29] - hence we can generate the 3-loop amplitude kinematic factor for arbitrary points using this form. In fact this is not quite directly true. One has to apply the *integration by parts* relations outlined in the next section to get from the above generating series expression to the 3-loop expression, which in terms of Berends-Giele currents is given by [29],

$$(\lambda\gamma_m\mathcal{W}_A^n)(\lambda\gamma_n\mathcal{W}_B^p)(\lambda\gamma_p\mathcal{W}_C^m). \tag{10.7}$$

When one starts to apply these integration by parts-like relations this term will be generated, along with a whole host of other terms that include other kinematic factors. The key point here is that these corrections include loop kinematic factors but they mix a number of loop kinematic factors together. This is obvious from the fact that in the $\mathbb{V}_1$ sector we can make manifest the box-diagram but in the no-$\mathbb{V}_1$ sector we can make the 3-loop diagram manifest. These two expressions are the same and so we must conclude that they have a mixture of loop kinematic factors in their composition. Another possible no$-\mathbb{V}_1$ solution is given by the expression,

$$L_{\alpha'^3} = (\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^a\mathbb{W})\mathbb{F}^{mn}\mathbb{F}^{pq}\mathbb{F}^{ra}, \tag{10.8}$$

which actually now makes the 2-loop double box-diagram manifest. It is important to note that these are generating series expressions and the relation to multiparticle Berends-Giele currents is not a direct one as there are other terms at higher point that will introduce the other diagram we have discussed.

The relation between the $\mathbb{V}$ and no-$\mathbb{V}$ sector is an ongoing piece of research - the results of which are covered in the next section. However, note that at a given mass dimension if $\mathcal{Q}\mathbb{X} = 0$ and $\mathcal{Q}\mathbb{Y} = 0$ then $\mathbb{X} = \mathbb{Y} + \mathcal{Q}\Omega$, thus they are equal from the cohomology viewpoint. Hence, whilst we may not have directly proven the equivalence at the level of superspace, we know that the expressions presented in this section and the last must be the same owing to the uniqueness of BRST cohomology.

## 10.3 Looking for $\Omega$

At $\alpha'^3$ there is a generating series expression involving an overall $\mathbb{V}$ and a number of expressions that have no $\mathbb{V}$. Owing to the uniqueness of the BRST cohomology, these different ways of expressing the $\alpha'^3$ corrections must be the same. However, it is interesting to also show that these expressions are the same in superspace. This involves finding a BRST ancestor that generates both terms such that the following equation is satisfied,

$$\mathcal{Q}\Omega = \mathbb{L}_{\mathbb{V}} - \mathbb{L}_{\hat{\mathbb{V}}}, \tag{10.9}$$

where $\mathbb{L}_{\mathbb{V}}$ is the generating series expression for the correction involving $\mathbb{V}_1$ and $\mathbb{L}_{\hat{\mathbb{V}}}$ is one of the expressions not containing $\mathbb{V}_1$. Theoretically, one can automate the solving of this equation by generating an ansatz for $\Omega$ which contains all of the dimension 7, ghost number 2 generating series expressions. Ultimately, $\Omega$ ought to contain terms whose BRST variations mix $\mathbb{V}$ and no-$\mathbb{V}$ terms. The simplest way to do this is to allow bare gauge fields, $\mathbb{A}^m$, in the ansatz; the variation of $\mathbb{A}^m$ is as follows [7],

$$[\mathcal{Q}, \mathbb{A}^m] = [\partial^m, \mathbb{V}_1] + (\lambda\gamma^m\mathbb{W}). \tag{10.10}$$

Note that we can replace the $\partial^m$ in the above expression with $\nabla^m = \partial^m - \mathbb{A}^m$ such that we find,

$$[\mathcal{Q}, \mathbb{A}^m] = \mathbb{V}_1^m + [\mathbb{A}^m, \mathbb{V}_1] + (\lambda\gamma^m\mathbb{W}), \tag{10.11}$$

where we can now use an integration by parts-like relation,

$$\mathrm{Tr}\big[[\mathbb{A}, \mathbb{B}]\mathbb{C}\mathbb{D}\big] = -\mathrm{Tr}\big[\mathbb{B}[\mathbb{A}, \mathbb{C}]\mathbb{D}\big] - \mathrm{Tr}\big[\mathbb{B}\mathbb{C}[\mathbb{A}, \mathbb{D}]\big], \tag{10.12}$$

to remove the derivative on $\mathbb{V}_1$ in the ansatz. Note that this integration by parts-like relation only holds once momentum conservation has been taken into account. Take the example of $\mathbb{V}_1^m(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}$, using the above relation we find[3],

$$\mathrm{Tr}\big[\mathbb{V}_1^m(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}\big] = -\mathrm{Tr}\big[\mathbb{V}_1(\lambda\gamma^n\mathbb{W}^m)\mathbb{F}^{mn}\big] - \mathrm{Tr}\big[\mathbb{V}_1(\lambda\gamma^n\mathbb{W})\mathbb{F}^{m|mn}\big]. \tag{10.13}$$

Initially, we consider the simplest ansatz possible which involves adding terms with a single $\mathbb{A}$ field to the no-$\mathbb{V}_1$ sector of the $\Omega$ ansatz, an example term would be $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})(\mathbb{W}\gamma\mathbb{W})\mathbb{A}$. No $\mathbb{A}$ fields are considered in the $\mathbb{V}_1$ sector of the ansatz since this will produce $\mathbb{V}^2$ terms which are difficult to cancel without introducing a larger ansatz. Hence the set of terms initially considered is rather simple and given in Table 10.2. The ansatz does not assume any symmetry of the higher mass field indices thus we have to apply the projections that are implied by the anti-symmetry of the $\gamma$ matrices. For example, we apply,

$$\gamma^{mnp}\mathbb{W}^{mnp} = \frac{1}{2}[\gamma^{mnp}\mathbb{W}^p, \mathbb{F}^{mn}]. \tag{10.14}$$

Note that we also use the Dirac equation to simplify terms of the form $\lambda\gamma^{mnp}\mathbb{W}^p$, which becomes,

$$(\lambda\gamma^{mnp}\mathbb{W}^p) = (\lambda\gamma^{[n}\mathbb{W}^{p]}). \tag{10.15}$$

Finally, we can also implement the Fierz identity,

$$\begin{aligned}\lambda^\alpha\mathbb{W}^\beta =& \frac{1}{16}(\gamma^m)^{\alpha\beta}(\lambda\gamma^m\mathbb{W}) + \frac{1}{96}(\gamma^{mnp})^{\alpha\beta}(\lambda\gamma^{mnp}\mathbb{W}) \\ &+ \frac{1}{3840}(\gamma^{mnpqr})^{\alpha\beta}(\lambda\gamma^{mnpqr}\mathbb{W}),\end{aligned} \tag{10.16}$$

which is useful when dealing with terms whose BRST variation produces terms of the form $\mathbb{W}^\alpha(\lambda\gamma\mathbb{W})\mathbb{W}^\beta\gamma_{\alpha\beta}$, as it allows us to write these terms as fully contracted spinor-scalars. For example, one can rewrite $\mathbb{W}^\alpha(\lambda\gamma^m\mathbb{W})\mathbb{W}^\beta\gamma^n_{\alpha\beta}$ in the following manner,

$$\begin{aligned}\mathbb{W}^\alpha(\lambda\gamma^m\mathbb{W})\mathbb{W}^\beta\gamma^n_{\alpha\beta} =& \frac{1}{16}(\lambda\gamma^p\mathbb{W})(\mathbb{W}\gamma^m\gamma^p\gamma^n\mathbb{W}) + \frac{1}{96}(\lambda\gamma^{pqr}\mathbb{W})(\mathbb{W}\gamma^m\gamma^{pqr}\gamma^n\mathbb{W}) \\ &+ \frac{1}{3840}(\lambda\gamma^{pqrst}\mathbb{W})(\mathbb{W}\gamma^m\gamma^{pqrst}\gamma^n\mathbb{W}),\end{aligned} \tag{10.17}$$

[3]Here we explicitly put the trace back into the equation to make the point that this only holds for expressions involving the trace.

| Term | # Scalars |
|---|---|
| $(\lambda\gamma\nabla^2\mathbb{W})(\lambda\gamma\nabla^2\mathbb{W})$ | 1 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\nabla^2\mathbb{W})\mathbb{F}$ | 2 |
| $(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\nabla\mathbb{W})\mathbb{F}$ | 5 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\nabla\mathbb{W})\nabla\mathbb{F}$ | 3 |
| $(\lambda\gamma\lambda)\mathbb{F}\mathbb{F}\nabla\mathbb{F}$ | 2 |
| $(\lambda\gamma\nabla^2\mathbb{W})(\lambda\gamma\nabla\mathbb{W})\mathbb{A}$ | 2 |
| $(\lambda\gamma\lambda)\nabla\mathbb{F}\nabla\mathbb{F}\mathbb{A}$ | 1 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})\mathbb{F}\mathbb{F}$ | 6 |
| $(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\mathbb{W})(\mathbb{W}\gamma\mathbb{W})$ | 6 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})(\mathbb{W}\gamma\nabla\mathbb{W})$ | 6 |
| $(\lambda\gamma\nabla\mathbb{W})(\lambda\gamma\mathbb{W})\mathbb{F}\mathbb{A}$ | 7 |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})\nabla\mathbb{F}\mathbb{A}$ | 4 |
| $(\lambda\gamma\lambda)\mathbb{F}\mathbb{F}\mathbb{F}\mathbb{A}$ | - |
| $(\lambda\gamma\mathbb{W})(\lambda\gamma\mathbb{W})(\mathbb{W}\gamma\mathbb{W})\mathbb{A}$ | 9 |
| $\mathbb{V}(\lambda\gamma\nabla\mathbb{W})\nabla\mathbb{F}$ | 1 |
| $\mathbb{V}(\lambda\gamma\nabla\mathbb{W})\mathbb{F}\mathbb{F}$ | 4 |
| $\mathbb{V}(\lambda\gamma\mathbb{W})\mathbb{F}\nabla\mathbb{F}$ | 2 |
| $\mathbb{V}(\lambda\gamma\mathbb{W})(\nabla\mathbb{W}\gamma\nabla\mathbb{W})$ | 4 |
| $\mathbb{V}(\lambda\gamma\nabla\mathbb{W})(\mathbb{W}\gamma\nabla\mathbb{W})$ | 4 |

TABLE 10.2: The terms that appear in the most basic ansatz for $\Omega$ - currently there is no solution with these terms only.

where one can then simplify the $\gamma$ matrices. We use all of these techniques on top of the others developed, including Jacobi identities and higher mass equations of motion. We also use cyclicity of the trace at the level of $\Omega$ to move $\mathbb{A}$ to the front of the no-$\mathbb{V}_1$ sector. We also use this cyclicity after the variation to move $\mathbb{V}_1$ to the front of any terms. At the current level, there is no solution for $\Omega$ indicating we may need to expand the ansatz to include higher $\mathbb{A}$ terms. It may be necessary to make the most general ansatz possible involving any number of $\mathbb{A}^m$ fields that are allowed - or include higher mass generalisations of the field $\mathbb{A}^m$.

This chapter has presented the $\alpha'^3$ generating series which encapsulates all $n-$point kinematic factors resulting from taking the field theory limit of string scattering amplitudes. We have seen that there are ways to express these corrections including the unintegrated vertex operator, and also a number of ways to express it without that operator. All of these expressions make various loop diagram manifest at the generating series level - and due to this we conclude that these $\alpha'$ corrections often contain a mixture of loop diagram kinematic factors in their expansions. We have also explained the attempt to demonstrate that these expressions are the same at the level of superspace, however there are a number of obstacles that need to be overcome before this can be shown concretely. Some of these have been suggested in the above. That being said, we know that all of the corrections expressed in this chapter have to be the same owing to the uniqueness of the BRST cohomology.

# Chapter 11

# New Higher Mass Operators

In this chapter we focus on determining the behaviour of the higher mass counterparts of the generating series $\mathbb{V}$ - that is the behaviour of $\mathbb{V}^M$. Whilst the results of this chapter initially appear without purpose we present the result of the analysis for two reasons: the first being posterity and the second being that we use some of the variations outlined here in the search for the unknown function $\Omega$ in the previous chapter. With that being said, in general the motivation for studying these higher mass unintegrated vertex operators is the following: the $n-$point tree-level amplitude in pure spinor Super Yang-Mills can be determined from $\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$, thus it is possible that higher mass $\alpha'$ corrections can be generated by $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$, where $\mathcal{O}$ is some differential operator built from $\nabla^m$. This operator descends from the Drinfeld Associator [120, 121] which it is assumed can be used to generate the corrections we seek. The basis for this expansion will be terms of the form $[\nabla^m, \mathbb{V}]$ and so for convenience we define $\mathbb{V}^m \equiv [\nabla^m, \mathbb{V}]$ in analogy with the higher mass $\mathbb{W}$ and $\mathbb{F}$ fields above. In order to work with these fields we first have to derive their behaviour in terms of BRST variations - this is a relatively straightforward exercise and the results are presented below. The results of this chapter detail the necessary machinery to test this method at $\alpha'$ and $\alpha'^2$ - two fairly simple generating series.

As an initial test we shall work to find a potential set of terms that can be used to generate the $\alpha'^2$ amplitudes and as such the highest mass field we will deal with is $\mathbb{V}^{mnpq}$ and so this is the highest mass dimension field we derive here. Although it is possible using the integration by parts-like identity of the previous chapter to remove some of the differentials from this field and distribute them to other terms, we shall use such terms here.

We begin our analysis with the graded Lie algebra identity,

$$(-1)^{|X||Z|}[X,[Y,Z]] + (-1)^{|X||Y|}[Y,[Z,X]] + (-1)^{|Y||Z|}[Z,[X,Y]] = 0, \tag{11.1}$$

where $|X| = 0$ if $X$ is a bosonic field and $|X| = 1$ if $X$ is a fermionic field. This identity has been used implicitly many times throughout this thesis - it underpins many of the Jacobi like manipulations we have made as well as the Bianchi identities. However, in this chapter the only graded identity that we actually need to find the BRST variation of $\mathbb{V}^M$ is,

$$[X,\{Y,Z\}] - \{Y,[X,Z]\} - \{Z,[X,Y]\} = 0, \tag{11.2}$$

where $X$ is bosonic and $Y, Z$ are fermionic. We also require the spinor equations of motion which give us the BRST variations of the required fields [7],

$$[Q,\nabla_m] = -(\lambda\gamma_m\mathbb{W}) - \mathbb{V}_m, \qquad \{Q,\mathbb{V}\} = \{\mathbb{V},\mathbb{V}\}, \tag{11.3}$$

and from these three equations we can derive the BRST variations of higher mass $\mathbb{V}$ fields. In the following we work with both $Q = \lambda^\alpha D_\alpha$ and $\mathcal{Q} = \lambda^\alpha\nabla_\alpha$ where the former needs to be used when expressions are not surrounded by a trace or do not include $\mathbb{V}_1$. Fortunately, all of the expressions used here are surround by traces and hence we can used the simpler variations with respect to $\mathcal{Q}$ as we have done in previous chapters. Note that this only holds, despite not using $\mathbb{V}_1$, because we can use the cyclicity of the trace and the fermionic statistics to remove the $\{\mathbb{V},\mathbb{V}\}$ parts of the variation.

## 11.1 Working with $Q$

We initially work with fields terms that involve $Q = \lambda^\alpha D_\alpha$ which is the usual BRST operator in pure spinor superstring theory - this is done mainly for completeness. However, since the corrections are given as traces over the fields we will later define $\mathcal{Q} = \lambda^\alpha\nabla_\alpha$

as a modified operator which will somewhat simplify the equations given below. That being said, the same structures will appear as this operator gives minor modifications to the equations. Namely, we obtain modifications that are proportional to $\{\mathbb{V}^M, \mathbb{V}^N\}$, but once we use $\mathcal{Q}$ these terms drop out of the variation due to the action of the trace.

Let us begin by considering $\{Q, \mathbb{V}^m\}$, using (11.2) we can show that the following graded Jacobi identity holds,

$$[\nabla^m, \{Q, \mathbb{V}\}] - \{Q, [\nabla^m, \mathbb{V}]\} + \{\mathbb{V}, [Q, \nabla^m]\} = 0, \tag{11.4}$$

from which it follows using (11.3) that,

$$\{Q, \mathbb{V}^m\} = -\{\mathbb{V}, (\lambda\gamma^m\mathbb{W})\} - \{\mathbb{V}, \mathbb{V}^m\} + [\nabla^m, \{\mathbb{V}, \mathbb{V}\}], \tag{11.5}$$

finally using the graded identity on the last term in the above equation we can show $[\nabla^m, \{\mathbb{V}, \mathbb{V}\}] - \{\mathbb{V}, [\nabla^m, \mathbb{V}]\} - \{\mathbb{V}, [\nabla^m, \mathbb{V}]\} = 0$, hence $[\nabla^m, \{\mathbb{V}, \mathbb{V}\}] = 2\{\mathbb{V}, \mathbb{V}^m\}$ and thus,

$$\{Q, \mathbb{V}^m\} = -\{\mathbb{V}, (\lambda\gamma^m\mathbb{W})\} + \{\mathbb{V}, \mathbb{V}^m\}. \tag{11.6}$$

From here the calculation follows rather trivially, albeit in a time-consuming manner and all higher mass identities follow from (11.4). For example, let us consider $\mathbb{V}^{mn}$. Using the first equation in (11.3), (11.6) and (11.4) we find,

$$\{Q, \mathbb{V}^{mn}\} = -\{\mathbb{V}^n, (\lambda\gamma^m\mathbb{W})\} - \{\mathbb{V}^n, \mathbb{V}^m\} + [\nabla^m, \{Q, \mathbb{V}^n\}], \tag{11.7}$$

and once again we need to use (11.4) on the first line. After doing this, and noting that $\{X, Y\} = \{Y, X\}$ we find,

$$\{Q, \mathbb{V}^{mn}\} = -\,\{\mathbb{V}^{(m}, (\lambda\gamma^{n)}\mathbb{W})\} + \{\mathbb{V}, \mathbb{V}^{mn}\} - \{\mathbb{V}, (\lambda\gamma^n\mathbb{W}^m)\}. \tag{11.8}$$

Note, however, that $\mathbb{V}^M$ exhibits the same anti-symmetry[1] properties as $\mathbb{W}^M$ (see (2.29) in [7]) and hence we note,

$$\mathbb{V}^{[mn]} = [\mathbb{V}, \mathbb{F}^{mn}]. \tag{11.9}$$

Furthermore, note that the trace of (11.8) gives,

$$\{Q, \mathbb{V}^{mm}\} = -2\{\mathbb{V}^m, (\lambda\gamma^m\mathbb{W})\} - 2\mathbb{V}^m\mathbb{V}^m. \tag{11.10}$$

Carrying this on to higher mass $\mathbb{V}$ we generate more terms but many of them have roughly the same structure. So, for example $\mathbb{V}^{mnp}$ is given by,

$$\begin{aligned}\{Q, \mathbb{V}^{mnp}\} = &- \{\mathbb{V}, (\lambda\gamma^p\mathbb{W}^{mn})\} - \{\mathbb{V}^m, (\lambda\gamma^p\mathbb{W}^n)\} \\ &- \{\mathbb{V}^{(n}, (\lambda\gamma^{p)}\mathbb{W}^m)\} - \{\mathbb{V}^{m(n}(\lambda\gamma^{p)}\mathbb{W})\} \\ &- \{\mathbb{V}^{np}, (\lambda\gamma^m\mathbb{W})\} + \{\mathbb{V}, \mathbb{V}^{mnp}\}.\end{aligned} \tag{11.11}$$

However, at this dimension when we take the variation we can generate terms that do not package nicely due to the appearance of $\mathbb{W}^{mm}$ in the anti-commutator with $\mathbb{V}$. These are terms that have a non-simple structure, by this we mean terms such as $\mathbb{W}^\alpha\mathbb{X}^m\mathbb{W}^\beta\gamma^m_{\alpha\beta}$. That is terms where the $\gamma$ matrix cannot be contracted to give a term of the form $(\lambda\gamma\mathbb{W})$ or $(\mathbb{W}\gamma\mathbb{W})$ due to commutation issues. This can pose issues if we try to use BRST closure to find the generating series and do not properly deal with these terms in a careful manner, for example using the Fierz identity or some canonicalisation procedure. We also note that the anti-symmetrization of the indices at this order gives,

$$\mathbb{V}^{[mnp]} = [\mathbb{V}^m, \mathbb{F}^{np}] + [\mathbb{V}^n, \mathbb{F}^{pm}] + [\mathbb{V}^p, \mathbb{F}^{mn}], \tag{11.12}$$

again respecting the symmetry of the indices from the $\mathbb{W}$ field. Finally, for $\mathbb{V}^{mnpq}$ we find,

$$\begin{aligned}\{Q, \mathbb{V}^{mnpq}\} = &- \{\mathbb{V}, (\lambda\gamma^q\mathbb{W}^{mnp})\} - \{\mathbb{V}^m, (\lambda\gamma^q\mathbb{W}^{np})\} \\ &- \{\mathbb{V}^n, (\lambda\gamma^q\mathbb{W}^{mp})\} - \{\mathbb{V}^{(p}, (\lambda\gamma^{q)}\mathbb{W}^{mn})\}\end{aligned} \tag{11.13}$$

[1] This is not exactly true, they are both higher mass spinor, albeit different kinds of spinors. Ultimately, the symmetry properties of the vector indices are not at all dependent upon the spinor type.

$$
\begin{aligned}
&- \{\mathbb{V}^{mn}, (\lambda\gamma^q\mathbb{W}^p\} - \{\mathbb{V}^{m(p}, (\lambda\gamma^{q)}\mathbb{W}^n)\} \\
&- \{\mathbb{V}^{n(p}, (\lambda\gamma^{q)}\mathbb{W}^m)\} - \{\mathbb{V}^{pq}, (\lambda\gamma^n\mathbb{W}^m)\} \\
&- \{\mathbb{V}^{mn(p}, (\lambda\gamma^{q)}\mathbb{W})\} - \{\mathbb{V}^{mpq}, (\lambda\gamma^n\mathbb{W})\} \\
&- \{\mathbb{V}^{npq}, (\lambda\gamma^m\mathbb{W})\} + \{\mathbb{V}, \mathbb{V}^{mnpq}\},
\end{aligned}
$$

which has a corresponding anti-symmeterized identity which can be found using (2.29) in [7]. It ought to be noted that $\mathbb{V}$ also satisfies the same decomposition as $\mathbb{W}$ - this simply follow from the fact that the Dynkin label decomposition of vector indices into the respective forms cares very little about the nature of the spinor the derivatives are acting on.

## 11.2 Working with $\mathcal{Q}$

When working with traces of operators in the form $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$, as we will do here, we can exchange $Q$ for $\mathcal{Q}$, due to the trace as this will perform the same action in the trace[2] - ultimately the extra $\{\mathbb{V}^M, \mathbb{V}^N\}$ terms that appear with $Q$ are zero under the trace. The BRST variations with this modified operator do very little to change the equations, in fact the only change is that the $\mathbb{V}$ only commutators in the previous section disappear. For example,

$$\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}^m\} = -\mathrm{Tr}\{\mathbb{V}, (\lambda\gamma^m\mathbb{W})\}, \tag{11.14}$$

and,

$$\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}^{mn}\} = -\mathrm{Tr}\{\mathbb{V}^{(m}, (\lambda\gamma^{n)}\mathbb{W})\} - \mathrm{Tr}\{\mathbb{V}, (\lambda\gamma^n\mathbb{W}^m)\}. \tag{11.15}$$

This continues to higher mass orders and we obtain,

$$
\begin{aligned}
\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}^{mnp}\} = &- \mathrm{Tr}\{\mathbb{V}, (\lambda\gamma^p\mathbb{W}^{mn})\} - \mathrm{Tr}\{\mathbb{V}^m, (\lambda\gamma^p\mathbb{W}^n)\} \\
&- \mathrm{Tr}\{\mathbb{V}^{(n}, (\lambda\gamma^{p)}\mathbb{W}^m)\} - \mathrm{Tr}\{\mathbb{V}^{m(n}(\lambda\gamma^{p)}\mathbb{W})\} \\
&- \mathrm{Tr}\{\mathbb{V}^{np}, (\lambda\gamma^m\mathbb{W})\}
\end{aligned} \tag{11.16}
$$

[2]One can easily get $\{\mathcal{Q}, \mathbb{V}^M\}$ from the previous section by noting that $\mathcal{Q} = Q - \mathbb{V}$.

and,

$$\begin{aligned}
\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}^{mnpq}\} = &- \mathrm{Tr}\{\mathbb{V}, (\lambda\gamma^q \mathbb{W}^{mnp})\} - \mathrm{Tr}\{\mathbb{V}^m, (\lambda\gamma^q \mathbb{W}^{np})\} \\
&- \mathrm{Tr}\{\mathbb{V}^n, (\lambda\gamma^q \mathbb{W}^{mp})\} - \mathrm{Tr}\{\mathbb{V}^{(p}, (\lambda\gamma^{q)} \mathbb{W}^{mn})\} \\
&- \mathrm{Tr}\{\mathbb{V}^{mn}, (\lambda\gamma^q \mathbb{W}^{p})\} - \mathrm{Tr}\{\mathbb{V}^{m(p}, (\lambda\gamma^{q)} \mathbb{W}^{n})\} \\
&- \mathrm{Tr}\{\mathbb{V}^{n(p}, (\lambda\gamma^{q)} \mathbb{W}^{m})\} - \mathrm{Tr}\{\mathbb{V}^{pq}, (\lambda\gamma^n \mathbb{W}^{m})\} \\
&- \mathrm{Tr}\{\mathbb{V}^{mn(p}, (\lambda\gamma^{q)} \mathbb{W})\} - \mathrm{Tr}\{\mathbb{V}^{mpq}, (\lambda\gamma^n \mathbb{W})\} \\
&- \mathrm{Tr}\{\mathbb{V}^{npq}, (\lambda\gamma^m \mathbb{W})\}.
\end{aligned} \tag{11.17}$$

These are the variations we shall use in the $\alpha'^2$ case as it reduces the number of terms required in the variation and thus reduces the possibility of something going awry. We can extend these variations to arbitrary order by recalling the similarities of symmetry between $\mathbb{V}^M$ and $\mathbb{W}^M$. If we take the higher mass equation of motion for $\mathbb{W}^M$ - see (4.16) - and naïvely adjust it for $\mathbb{V}^M$ we find,

$$\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}^N\} = - \sum_{\substack{\delta(N)=R\otimes S \\ R\neq\emptyset}} \mathrm{Tr}\big\{\mathbb{V}^S, (\lambda\gamma\mathbb{W})^R\big\}, \tag{11.18}$$

which correctly reproduces the above equations. We can further leverage the information already known about higher mass $\mathbb{W}$ fields in order to perform the decompositions of these higher mass $\mathbb{V}$ fields into other fields - this is what we turn to now.

## 11.3 Decomposition of Higher Mass $\mathbb{V}$

It will be useful to decompose the higher mass $\mathbb{V}$ fields as this allows us to reduce the number of terms we generate in the ansatz. For this we need to determine the symmetric trace part of the tensors as this is a required part of the decomposition that cannot be easily inferred from the $\mathbb{W}$ case. We can do this by splitting $\nabla^m$ as $\nabla^m = \partial^m - \mathbb{A}^m$ and as such we get,

$$\mathbb{V}^{mm} = \Box\mathbb{V} - [\partial^m, [\mathbb{A}^m, \mathbb{V}]] - [\mathbb{A}^m, [\partial^m, \mathbb{V}]] + [\mathbb{A}^m, [\mathbb{A}^m, \mathbb{V}]], \tag{11.19}$$

now we can use (2.27) and (2.28) in [29] which tells us[3],

$$\Box\mathbb{V} = [\mathbb{A}^m, [\partial^m, \mathbb{V}]] + [(\lambda\gamma^m\mathbb{W}), \mathbb{A}^m]. \tag{11.20}$$

Plugging this into the expression for $\mathbb{V}^{mm}$ and using the Jacobi identity we obtain,

$$\mathbb{V}^{mm} = [(\lambda\gamma^m\mathbb{W}), \mathbb{A}^m] - [\mathbb{A}^m, \mathbb{V}^m] - [\mathbb{A}^{m|m}, \mathbb{V}], \tag{11.21}$$

where one can use the Lorenz gauge $\mathbb{A}^{m|m} = [\partial^m, \mathbb{A}^m] = 0$[4] to show that,

$$\mathbb{V}^{mm} = [\mathbb{V}^m + (\lambda\gamma^m\mathbb{W}), \mathbb{A}^m], \tag{11.22}$$

and hence using this we can find the decomposition of $\mathbb{V}^{mn}$ into symmetric traceless, symmetric trace and anti-symmetric parts in much the same way we did in (8.14). This gives,

$$\mathbb{V}^{mn} = \hat{\mathbb{V}}^{(mn)} + [\mathbb{V}, \mathbb{F}^{mn}] + \frac{\delta^{mn}}{10}\big([\mathbb{V}^p, \mathbb{A}^p] + [(\lambda\gamma^p\mathbb{W}), \mathbb{A}^p]\big). \tag{11.23}$$

We can continue as we did for $\mathbb{W}^{mnp}$ and obtain the decomposition of $\mathbb{V}^{mnp}$, note that the anti-symmetric part follows from $\mathbb{W}^{[mnp]}$; in fact much of the structure is exactly the same as $\mathbb{W}^{mnp}$ and hence we have only to substitute in the correct expressions in order to obtain the correct decomposition. Thus we need to determine the trace symmetric piece of $\mathbb{V}^{mnp}$ and the following anti-symmetric pieces: $\mathbb{V}^{[mn]p}$ and $\mathbb{V}^{m[np]}$ which can be easily found after playing around with the Jacobi identity. To find the trace symmetric piece we need to find the $\beta^m$ outlined in (8.28). Using Jacobi identities one can show the following,

$$\begin{aligned} \mathbb{V}^{pmp} &= \mathbb{V}^{mpp} - [\mathbb{V}^p, \mathbb{F}^{mp}] \\ \mathbb{V}^{ppm} &= \mathbb{V}^{mpp} - 2[\mathbb{V}, (\mathbb{W}\gamma^m\mathbb{W})] - 2[\mathbb{V}^p, \mathbb{F}^{mp}]. \end{aligned} \tag{11.24}$$

Thus we have that,

$$\beta^m = \mathbb{V}^{mpp} - [\mathbb{V}^p, \mathbb{F}^{mp}] - \frac{2}{3}[\mathbb{V}, (\mathbb{W}\gamma^m\mathbb{W})], \tag{11.25}$$

[3] This relies on applying Lorenz gauge. This is the gauge we have performed all of the calculations thus far in and so it does not matter in a pragmatic sense.

[4] This follows since $\mathbb{A}^{m|m} = [\nabla^m, \mathbb{A}^m] = [\partial^m, \mathbb{A}^m] - [\mathbb{A}^m, \mathbb{A}^m] = [\partial^m, \mathbb{A}^m]$.

and now we only need to calculate $\mathbb{V}^{mpp}$ which is straightforward given that we now know $\mathbb{V}^{pp}$. After using the Jacobi identity again, we find the following form for $\mathbb{V}^{mpp}$,

$$\mathbb{V}^{mpp} = [\mathbb{V}^{mp}, \mathbb{A}^{p}] + [(\lambda\gamma^{p}\mathbb{W}^{m}), \mathbb{A}^{p}] + [\mathbb{V}^{p}, \mathbb{A}^{m|p}] + [(\lambda\gamma^{p}\mathbb{W}), \mathbb{A}^{m|p}], \tag{11.26}$$

and hence we have the correct form for $\alpha^m$ and thus for $\tilde{\mathbb{V}}^{(mnp)}$ once everything is plugged in. Now we need to find $\mathbb{V}^{[mn]p}$ and $\mathbb{V}^{m[np]}$ which is easily done by, yet again, using the Jacobi identity,

$$\begin{aligned} \mathbb{V}^{[mn]p} &= [\mathbb{V}^{p}, \mathbb{F}^{mn}] \\ \mathbb{V}^{m[np]} &= [\mathbb{V}, \mathbb{F}^{m|np}] + [\mathbb{V}^{m}, \mathbb{F}^{np}] = [\mathbb{V}, \mathbb{F}^{m|np}] + \mathbb{V}^{[np]m}, \end{aligned} \tag{11.27}$$

which follows exactly what one would expect from the corresponding $\mathbb{W}$ calculations. Finally, we have the fully anti-symmetric part of the tensor,

$$\mathbb{V}^{[mnp]} = [\mathbb{V}^{p}, \mathbb{F}^{mn}] + [\mathbb{V}^{n}, \mathbb{F}^{pm}] + [\mathbb{V}^{m}, \mathbb{F}^{np}], \tag{11.28}$$

again following directly from the previous $\mathbb{W}$ calculations. Hence the irreducible representation is given by,

$$\begin{aligned} \mathbb{V}^{mnp} &= \hat{\mathbb{V}}^{(mnp)} + \frac{1}{12}\tilde{\mathbb{V}}^{(mnp)} + \mathbb{V}^{[mnp]} \\ &\quad + \frac{2}{3}(2\mathbb{V}^{m[np]} + \mathbb{V}^{p[mn]} - \mathbb{V}^{[mn]p} - 2\mathbb{V}^{[pm]n}), \end{aligned} \tag{11.29}$$

where $\tilde{\mathbb{V}}^{(mnp)} = (\beta^m\delta^{np} + \beta^n\delta^{mp} + \beta^p\delta^{mn})$. This kind of decomposition allows us to work with the symmetric traceless part in the ansatz and utilise the index symmetries to simplify things. Then one can use the projections outlined here to go to the full field expression.

## 11.4 Working at Order $\alpha'$

Working at order $\alpha'$ means that the weight of the expression must be 3.5, hence the possible terms are restricted at this order. Once one considers various integration-by-parts-like identities and Jacobi identities, the basis ansatz that we can construct for this order is the following,

$$\mathcal{L} = A\mathrm{Tr}[\mathbb{V}^m \mathbb{V}^m \mathbb{V}], \tag{11.30}$$

which is a fairly tractable problem - that is, we can just vary this by hand using the above equations,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}^m\mathbb{V}^m\mathbb{V}\} &= -\mathrm{Tr}[\{\mathbb{V}, (\lambda\gamma^m)\mathbb{W}\}\mathbb{V}^m\mathbb{V}] + \mathrm{Tr}[\mathbb{V}^m\{\mathbb{V}, (\lambda\gamma^m\mathbb{W})\}\mathbb{V}] \\ &= -\mathrm{Tr}[\mathbb{V}(\lambda\gamma^m\mathbb{W})\mathbb{V}^m\mathbb{V}] - \mathrm{Tr}[(\lambda\gamma^m\mathbb{W})\mathbb{V}\mathbb{V}^m\mathbb{V}] \\ &\quad + \mathrm{Tr}[\mathbb{V}^m\mathbb{V}(\lambda\gamma^m\mathbb{W})\mathbb{V}] + \mathrm{Tr}[\mathbb{V}^m(\lambda\gamma^m\mathbb{W})\mathbb{V}\mathbb{V}],\end{aligned} \tag{11.31}$$

which implies $A = 0$: there is no BRST closed solution using the above ansatz. This is expected since, as we saw in previous chapters, there is no $\alpha'$ correction to Super Yang-Mills; the first correction enters at $\alpha'^2$.

## 11.5 The Harnad-Shnider Gauge

The Harnad-Shnider gauge is a very useful gauge to work in at is dramatically simplifies the $\theta$ expansion of the non-linear fields into components by enforcing $\theta^\alpha \mathbb{A}_\alpha = 0$. As such, it is useful to see the form of $\mathbb{V}^M$ at various dimensions in order to determine the usefulness of employing these higher mass fields. To do this is fairly straightforward and simply requires the application of previously used formulae. We begin by considering the equation of motion in Harnad-Shnider for the plain $\mathbb{V}$ field,

$$\{\mathcal{D} + 1, \mathbb{V}\} = (\mathcal{D} + 1)\mathbb{V} = (\lambda\gamma^m\theta)\mathbb{A}_m \tag{11.32}$$

and applying the graded Jacobi identity we obtain the following,

$$[\nabla^m, \{\mathcal{D}, \mathbb{V}\}] + \mathbb{V}^m = \{\mathcal{D}, \mathbb{V}^m\} + \{\mathbb{V}, (\theta\gamma^m\mathbb{W})\} + \mathbb{V}^m. \tag{11.33}$$

Now looking at the right-hand side of (11.32) we find,

$$[\nabla^m, (\lambda\gamma^n\theta)\mathbb{A}_n] = (\lambda\gamma^n\theta)[\nabla^m, \mathbb{A}_n], \tag{11.34}$$

and hence for $\mathbb{V}^m$ the equation of motion in Harnad-Shnider gauge becomes,

$$(\mathcal{D}+1)\mathbb{V}^m = (\lambda\gamma^n\theta)[\nabla^m, \mathbb{A}_n] - \{\mathbb{V}, (\theta\gamma^m\mathbb{W})\}. \tag{11.35}$$

We can now invert this in order to obtain the $k^{th}$ order expansion [29],

$$\mathbb{V}^m|_k = \frac{1}{k+1}\Big((\lambda\gamma^n\theta)\mathbb{A}^{m|n}|_{k-1} - \sum_{l=0}^{k-1}\{\mathbb{V}|_l, (\theta\gamma^m\mathbb{W}|_{k-l-1})\}\Big]\Big). \tag{11.36}$$

As a result of the appearance of $\mathbb{A}^{m|n} \equiv [\nabla^m, \mathbb{A}^n]$ we must define a higher mass $\mathbb{A}^m$ field in order to determine the component expansions of $\mathcal{V}^m_P$. At the level of multiparticle polarisation tensors this higher mass $\mathbb{A}^{M|n}$ field would correspond to,

$$\mathfrak{e}_P^{m_1\ldots m_i|n} = k_P^{m_1\ldots m_i}\mathfrak{e}_P^n - \sum_{XY=P}(\mathfrak{e}_X^n\mathfrak{e}_Y^{m_1\ldots m_i} - \mathfrak{e}_X^{m_1\ldots m_i}\mathfrak{e}_Y^n), \tag{11.37}$$

where intuitively $\partial^m\mathbb{A}^n$ corresponds to $k_P^{m_1\ldots m_i}\mathfrak{e}_P^n$. Working to higher mass dimension in $\mathbb{V}$ requires a similar sort of game with Jacobi identities as the derivation of the BRST variations. So let us consider the Harnad-Shnider gauge equation of motion for $\mathbb{V}^{mn}$, upon doing so one find the following,

$$(\mathcal{D}+1)\mathbb{V}^{mn} = (\lambda\gamma^p\theta)\mathbb{A}^{mn|p} - \{\mathbb{V}^n, (\theta\gamma^m\mathbb{W})\}, \tag{11.38}$$

and so it appears like defining a higher mass version of $\mathbb{A}^p$ becomes a reasonably sensible option because higher mass $\mathbb{V}$ will contain higher order commutators acting on $\mathbb{A}$. If one does this, then guessing the Harnad-Shnider gauge equation of motion for $\mathbb{V}^M$ becomes fairly trivial as it follows a similar kind of logic as (C.8) in [29]. Hence,

$$(\mathcal{D}+1)\mathbb{V}^N = (\lambda\gamma^m\theta)\mathbb{A}^{N|m} - \{\mathbb{V}^M, (\theta\gamma^i\mathbb{W})\}, \tag{11.39}$$

where $M = N \setminus i$ for $N = iM$, this is $i$ is the first index in $N$. For example, $\mathbb{V}^{mnp}$ will be given by,

$$(\mathcal{D}+1)\mathbb{V}^{mnp} = (\lambda\gamma^q\theta)\mathbb{A}^{mnp|q} - \{\mathbb{V}^{np}, (\theta\gamma^m\mathbb{W})\}, \tag{11.40}$$

which is readily verified using the identity,

$$[\nabla^M\{\mathcal{D},\mathbb{V}^N\}] = \{\mathcal{D},\mathbb{V}^{MN}\} + \{\mathbb{V}^{(M\backslash i)N},(\theta\gamma^i\mathbb{W})\}, \tag{11.41}$$

which follows from the graded Jacobi identity. We can then invert this general equation to give the $k^{th}$ order expansion of $\mathbb{V}^M$,

$$\mathbb{V}^M|_k = \frac{1}{k+1}\left((\lambda\gamma^m\theta)\mathbb{A}^{N|m}|_{k-1} - \sum_{l=0}^{k-1}\{\mathbb{V}^M|_l,(\theta\gamma^i\mathbb{W})|_{k-l-1}\}\right). \tag{11.42}$$

Hence we can find the Harnad-Shnider theta expansion of the fields $\mathbb{V}$ once we know the equivalent formulas for $\mathbb{A}^{M|n}$. This is quite straight forward since, $\mathcal{D}\mathbb{A}_m = (\theta\gamma_m\mathbb{W})$. Now using Jacobi identity on $[\nabla^m,[\mathcal{D},\mathbb{A}^n]]$ we find the following,

$$\mathcal{D}\mathbb{A}^{m|n} = (\theta\gamma^m\mathbb{W}^n) + [\mathbb{A}^n,(\theta\gamma^m\mathbb{W})], \tag{11.43}$$

which is somewhat expected given (4.13) in [29]. We need to continue this pattern onto higher mass $\mathbb{A}^p$ since we will need to work up to $\mathbb{V}^{mnpq}$ for $\alpha'^2$, as such we will require the expansion of $\mathbb{A}^{M|n}$ up to $M = m_1m_2m_3m_4$. For $\mathbb{A}^{mn|p}$ we obtain the following,

$$\mathcal{D}\mathbb{A}^{mn|p} = (\theta\gamma^n\mathbb{W}^{mp}) + [\mathbb{A}^p,(\theta\gamma^n\mathbb{W}^m)] - [\mathbb{A}^{(m||p},(\theta\gamma^{n)}\mathbb{W})], \tag{11.44}$$

where $(m||$ simultaneously denote symmetrization and the higher mass nature of $m$. Note that this follows from the Jacobi identity,

$$[\nabla^m[\mathcal{D},\mathbb{A}^{n|p}]] - [\mathcal{D},\mathbb{A}^{mn|p}] - [\mathbb{A}^{n|p},[\mathcal{D},\nabla^m]] = 0, \tag{11.45}$$

and in general, in order to find the higher mass Harnad-Shnider gauge equations of motion we use the following identity that follows directly from the Jacobi identity,

$$[\mathcal{D},\mathbb{A}^{mN|p}] = [\nabla^m,[\mathcal{D},\mathbb{A}^{N|p}]] + [\mathbb{A}^{N|p},(\theta\gamma^m\mathbb{W})], \tag{11.46}$$

where we have used $[\mathcal{D}, \nabla^m] = -(\theta\gamma^m\mathbb{W})$. We can continue to higher order mass terms in order to obtain $\mathbb{A}^{mnp|q}$,

$$\begin{aligned}\mathcal{D}\mathbb{A}^{mnp|q} =&(\theta\gamma^p\mathbb{W}^{mnq}) + [\mathbb{A}^{np|q}, (\theta\gamma^m\mathbb{W})] - [\mathbb{A}^{m(n||q}, (\theta\gamma^{|p)}\mathbb{W})] \\ &+ [\mathbb{A}^{m|q}, (\theta\gamma^p\mathbb{W}^n)] - [\mathbb{A}^{(n||q}, (\theta\gamma^{|p)}\mathbb{W}^m)] + [\mathbb{A}^q, (\theta\gamma^p\mathbb{W}^{mn})].\end{aligned} \tag{11.47}$$

Finally, we can find $\mathbb{A}^{mnpq|r}$, which is given by,

$$\begin{aligned}\mathcal{D}\mathbb{A}^{mnpq|r} =&(\theta\gamma^q\mathbb{W}^{mnpr}) + [\mathbb{A}^{(m|pq|r}, (\theta\gamma^{|n)}\mathbb{W})] \\ &- [\mathbb{A}^{mn(p||r}, (\theta\gamma^{|q)}\mathbb{W})] + [\mathbb{A}^{pq|r}, (\theta\gamma^n\mathbb{W}^m)] \\ &+ [\mathbb{A}^{mn|r}, (\theta\gamma^q\mathbb{W}^p)] - [\mathbb{A}^{n(p||r}, (\theta\gamma^{|q)}\mathbb{W}^m)] \\ &- [\mathbb{A}^{m(p||r}, (\theta\gamma^{|q)}\mathbb{W}^n)] + [\mathbb{A}^{(m||r}, (\theta\gamma^q\mathbb{W}^{|n)p})] \\ &- [\mathbb{A}^{(p||r}, (\theta\gamma^{|q)}\mathbb{W}^{mn})] + [\mathbb{A}^r, (\theta\gamma^q\mathbb{W}^{mnp})],\end{aligned} \tag{11.48}$$

note that we also require knowledge of $\mathbb{W}^M$, however this is available in Appendix C of [29]. It is possible to generalise this series of results to $\mathbb{A}^{M|q}$ and invert the equations in order to find the general recursion,

$$\mathbb{A}^{M|q}|_k = \frac{1}{k}\left((\theta\gamma\mathbb{W})^{Mq}|_{k-1} + \sum_{\substack{\delta(M)=X\otimes Y \\ Y\neq\emptyset}} (-1)^{|Y|+\zeta}\left[\mathbb{A}^{X|q}|_{k-1}, (\lambda\gamma\mathbb{W})^Y|_{k-l-1}\right]\right), \tag{11.49}$$

where $\zeta = 0$ if $m_1 \notin Y$ and $\zeta = 1$ if $m_1 \in Y$, also note that $(\lambda\gamma\mathbb{W})^{Mq} = (\lambda\gamma^{m_n}\mathbb{W}^{m_1m_2...m_{n-1}q})$ and $(\lambda\gamma\mathbb{W})^M = (\lambda\gamma^{m_n}\mathbb{W}^{m_1m_2...m_{n-1}})$, where $M = m_1m_2...m_n$ and $n = |M|$. As an example let us take $M = m$ one finds,

$$\mathbb{A}^{m|n}|_k = \frac{1}{k}\left((\theta\gamma^m\mathbb{W}^n)_{k-1} + \sum_{l=0}^{k-1}[\mathbb{A}^n|_l, (\theta\gamma^m\mathbb{W}|_{k-l-1})]\right) \tag{11.50}$$

which gives the following $\theta-$expansion up to $\theta^2$,

$$\mathcal{A}_P^{m|n} = \mathfrak{e}_P^{m|n} + (\theta\gamma^m\chi_P^n) + \frac{1}{8}(\theta\gamma^{mpq}\theta)\mathfrak{f}_P^{n|pq} + \sum_{XY=P}\left[\mathcal{A}_{X,Y}^{m|n}\right]_1 + \left[\mathcal{A}_{X,Y}^{m|n}\right]_2 + ..., \tag{11.51}$$

where,

$$\left[\mathcal{A}^{m|n}_{X,Y}\right]_1 = \left[\mathfrak{e}^n_X(\theta\gamma^m\chi_Y) - (X \leftrightarrow Y)\right],$$
$$\left[\mathcal{A}^{m|n}_{X,Y}\right]_2 = \frac{1}{8}(\theta\gamma^{mpq}\theta)\left[\mathfrak{e}^n_X\mathfrak{f}^{pq}_Y - (X \leftrightarrow Y)\right] - \left[(\theta\gamma^{[m}\chi_X)(\theta\gamma^{n]}\chi_Y)\right]. \tag{11.52}$$

This looks very similar to the expansion of $\mathcal{A}^m_P$ in [29] (see (4.20) of the reference), where each of the multiparticle polarisations is of a higher mass. The only difference to this naïve extension is are the deconcatenation terms that appear above - these do not appear in the expansion of $\mathcal{A}^m_P$ - hence these are new non-linearities associated with the higher mass field. Using this equation we can then return to (11.36) in order to determine the $\theta$ expansion of $\mathbb{V}^m$ which is given by,

$$\begin{aligned}\mathcal{V}^m_P =& \frac{1}{2}(\lambda\gamma^n\theta)\mathfrak{e}^{m|n}_P + \frac{1}{3}(\lambda\gamma^n\theta)(\theta\gamma^m\chi^n_P) + \frac{1}{8}(\lambda\gamma^n\theta)(\theta\gamma^{mpq}\theta)\mathfrak{f}^{n|pq}_P \\ &+ \sum_{XY=P}\left(\left[\mathcal{V}^m_{X,Y}\right]_2 + \left[\mathcal{V}^m_{X,Y}\right]_3\right) + \ldots,\end{aligned} \tag{11.53}$$

where we have worked to $\theta^3$. Working to this order is fine for terms of the form $\mathcal{O}\langle\mathbb{VVV}\rangle$ as each expansion begins with $\theta^1$. However if one involves other fields then one would need to go to $\theta^5$ in order to saturate the pure spinor integration measure. The deconcatenation terms for $\theta^2$ and $\theta^3$ are given by,

$$\left[\mathcal{V}^m_{X,Y}\right]_2 = \frac{1}{6}(\lambda\gamma^n\theta)\left[\mathfrak{e}^n_X(\theta\gamma^m\chi_Y) - (X \leftrightarrow Y)\right]$$
$$\left[\mathcal{V}^m_{X,Y}\right]_3 = -\frac{1}{16}(\lambda\gamma^n\theta)(\theta\gamma^{mpq}\theta)\mathfrak{f}^{pq}_X\mathfrak{e}^n_Y - \frac{1}{6}(\lambda\gamma^n\theta)(\theta\gamma^{[m}\chi_X)(\theta\gamma^{n]}\chi_Y). \tag{11.54}$$

Again this looks very similar to applying a derivative to the expansions given in [29] - however there are new non-linear deconcatenation terms that appear as with $\mathcal{A}^{m|n}$ above.

It is clear that working in the Harnad-Shnider gauge is very fruitful as it allows the derivation of the $\theta-$expansions of various higher mass fields in a fairly speedy manner. This gauge has allowed us to determine the $k^{th}$ order $\theta-$expansion for $\mathbb{V}^M$ and $\mathbb{A}^{M|q}$ for $M$ of any size - it is unclear whether such definitions will be fruitful in the future. They will likely come in very handy when looking more deeply into corrections of the form $\mathcal{O}\langle\mathbb{VVV}\rangle$ as they allow one to perform the component expansions and check that any ansatz contains the necessary term. However, these definitions present a novel extension to the works summarised in [7].

## 11.6 Working at $\alpha'^2$

Working at $\alpha'^2$ requires us to generate a larger ansatz than the one generated in the $\alpha'$ case owing to the fact that we have two extra derivatives to add to the $\mathbb{V}$. The obvious outcome of this is that terms involving $\mathbb{V}^{mnpq}$ and $\mathbb{V}^{mnp}$ are possible, however, owing to Jacobi identities, we also need to consider terms that contain $[\mathbb{V}^p, \mathbb{F}^{mn}]$ and $[\mathbb{V}^m, \mathbb{F}^{n|pq}]$. We shall show this in due to course, but first let us consider the kind of terms that can be generated using $\mathbb{V}^M$ only.

Excluding the permutations required for a general ansatz, we can have terms of the form,

$$\begin{array}{lll} \mathbb{V}\mathbb{V}\mathbb{V}^{mmnn}, & \mathbb{V}\mathbb{V}\mathbb{V}^{mnmn}, & \mathbb{V}\mathbb{V}\mathbb{V}^{mnnm} \\ \mathbb{V}\mathbb{V}^{mn}\mathbb{V}^{mn}, & \mathbb{V}\mathbb{V}^{mn}\mathbb{V}^{nm}, & \mathbb{V}\mathbb{V}^{mm}\mathbb{V}^{nn} \\ \mathbb{V}^{m}\mathbb{V}^{n}\mathbb{V}^{mn}, & \mathbb{V}^{m}\mathbb{V}^{n}\mathbb{V}^{nm}, & \mathbb{V}^{m}\mathbb{V}^{m}\mathbb{V}^{nn} \\ \mathbb{V}\mathbb{V}^{m}\mathbb{V}^{mnn}, & \mathbb{V}\mathbb{V}^{m}\mathbb{V}^{nmn}, & \mathbb{V}\mathbb{V}^{m}\mathbb{V}^{nnm}, \end{array} \tag{11.55}$$

however many of these terms are linked by Jacobi identities. The obvious case is the anti-symmetry of $\mathbb{V}^{mn}$ such that $\mathbb{V}^{mn} - \mathbb{V}^{nm} = [\mathbb{V}, \mathbb{F}^{mn}]$ and due to this linear relation between these terms we need not include terms of the form $\mathbb{V}^m\mathbb{V}^n[\mathbb{V}, \mathbb{F}^{mn}]$ or $\mathbb{V}\mathbb{V}^{mn}[\mathbb{V}, \mathbb{F}^{mn}]$ as they are already implicitly included in the ansatz. This allows us to keep using the $\mathcal{Q}$ operator as we still have an ansatz of the form $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$. Furthermore, many of the terms can have the integration by parts-like relation applied in order to reduce the number of terms that we can consider. It should now be clear that a lot of the Jacobi identities found for $\mathbb{W}$ in Chapter 8 also hold for $\mathbb{V}$. For example, we can show that,

$$\mathbb{V}^{[m|n|p]} = \mathbb{V}^{m[np]} + \mathbb{V}^{[mp]n} + \mathbb{V}^{p[mn]}, \tag{11.56}$$

and this together with (11.27) tell us that we need to only consider terms of the form $\mathbb{V}\mathbb{V}^m\mathbb{V}^{mnn}$ and $\mathbb{V}\mathbb{V}^m[\mathbb{V}^n, \mathbb{F}^{mn}]$ since $\mathbb{V}\mathbb{V}^n[\mathbb{V}, \mathbb{F}^{m|mn}]$ will be implicitly included due to the Jacobi identity. Then at $\mathbb{V}^{mnpq}$ we need to include all of the terms highlighted above plus, $\mathbb{V}\mathbb{V}[\mathbb{V}^m, \mathbb{F}^{n|mn}]$ and $\mathbb{V}\mathbb{V}[\mathbb{V}^{mn}, \mathbb{F}^{mn}]$ since terms including $\mathbb{V}\mathbb{V}[\mathbb{V}, \mathbb{F}^{mn|mn}]$ are implicitly included. All of this greatly simplifies the number of terms that we need to consider in the ansatz and allows us to write the most efficient ansatz possible. We can generate the ansatz using a similar method as for $\alpha'^4$ - except now we can do a direct replacement of terms since the ansatz is tractable.

Applying all that we know so far does not give an answer - currently there is no BRST closed generating series at dimension 5.5 using $\mathbb{V}\mathbb{V}\mathbb{V}$. It is possible that there is no way to use only three $\mathbb{V}$ fields to perform this analysis (at least as far as we have worked there appears to be no solution at this order). One could check this using component expansions and determine if the ansatz produces the terms we expect from string theory calculations. More work ought to be done in this vein in order to determine whether an expression in this form is possible.

In this chapter we have attempted to determine whether one can express the $\alpha'^2$ correction in the form of $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$, where $\mathcal{O}$ is some differential operator. Having derived all of the machinery to vary higher mass $\mathbb{V}$, and checking the absence of $\alpha'$ corrections, we have found no way to express the $\alpha'^2$ corrections as $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$. It may be interesting to revisit this work in the future and investigate the component expansions of the ansatz to see if we can produce component expressions that appear in the known amplitudes. For now, however, we leave this research as an open question with a partial answer.

# Chapter 12

# Current $\alpha'^4$ Results

We now turn to the $\alpha'^4$ corrections. These are significantly more difficult than the previous orders. The single particle expression for these amplitudes is known, and has been known for over a decade [77]. However, simply promoting this answer to a generating series is not enough and more work has to be done in order to find a BRST closed generating series expression. This has a number of challenges associated with it. Firstly, the ansatz is now thousands of terms long, in lieu of a handful or a few hundred. Secondly, the canonicalisation of so many terms is technically and computationally difficult and without this canonicalisation it is not possible to obtain the correct system of linear equations. Finally, the fields we have been using to build our ansatzes so far do not seem to be enough, and we have to potentially resort to more exotic constructions in order to obtain the correct generating series. This final point seems to be a consequence of the factorisation channel of $\alpha'^4$ into $\alpha'^2 \times \alpha'^2$ generating series. In the following section we shall explain each of these issues individually before turning to some potential solutions to these issues in the remainder of this chapter.

## 12.1 The Ansatz, the Variation and the Problem

At $\alpha'^4$ we have to work with generating series expressions that have mass dimension 9.5 and this greatly increases the number of possible terms that can be added to the ansatz. We do not list them all here as it is cumbersome and would take up a lot of unnecessary space. However, it is worth saying that at this order the ansatz generates many more terms than the 168 required for the $\alpha'^3$ ansatz. This ansatz is much larger as the mass dimension has increased and this exponentially increases the number of terms one has to consider. The ansatz also does not appear to have a separation between the $\mathbb{V}_1$ and no$-\mathbb{V}_1$ sectors and they must occur together - as we shall explain later, it is this mixing that can be used as a conceptual explanation for the need to include more exotic field content in the ansatz. Owing to this mixing, the number of terms that are in the ansatz is 'double'[1] that would occur if we had the splitting.

The second issue we encounter is that of canonicalisation - the spectre that is always haunting any of the BRST calculations performed here. With so many extra terms the system of linear equations one can generate is much larger than previous systems and hence proper canonicalisation, in order to obtain the correct system, is paramount. However, the terms that are generated via the BRST variation are more complex than previous terms owing to the dimensions of the higher mass fields that can appear. Take, for example, a term like $\mathbb{V}(\lambda\gamma^m\mathbb{W}^{pq})(\lambda\gamma^n\mathbb{W}^{pq})\mathbb{F}^{mn}$, this produces many terms owing to the variation of the $(\lambda\gamma^m\mathbb{W}^{pq})$ terms,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q},(\lambda\gamma^m\mathbb{W}^{pq})\} = -\mathrm{Tr}\{(\lambda\gamma^q\mathbb{W}^p),(\lambda\gamma^m\mathbb{W})\} - \mathrm{Tr}\{(\lambda\gamma^p\mathbb{W}),(\lambda\gamma^m\mathbb{W}^q)\}\\ - \mathrm{Tr}\{(\lambda\gamma^q\mathbb{W}),(\lambda\gamma^m\mathbb{W}^p)\},\end{aligned} \tag{12.1}$$

and hence the variation above generates the following,

$$\begin{aligned}\mathrm{Tr}\{\mathcal{Q},\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^{pq})(\lambda\gamma^n\mathbb{W}^{pq})\mathbb{F}^{mn}\} = \mathrm{Tr}\Big[&\mathbb{V}_1\big[\{(\lambda\gamma^q\mathbb{W}^p),(\lambda\gamma^m\mathbb{W})\} + \{(\lambda\gamma^p\mathbb{W}),(\lambda\gamma^m\mathbb{W}^q)\}\\ &+\{(\lambda\gamma^q\mathbb{W}),(\lambda\gamma^m\mathbb{W}^p)\}\big](\lambda\gamma^n\mathbb{W}^{pq})\mathbb{F}^{mn}\Big]\\ &-\mathrm{Tr}\Big[\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^{pq})\big[\{(\lambda\gamma^q\mathbb{W}^p),(\lambda\gamma^n\mathbb{W})\}\\ &+\{(\lambda\gamma^p\mathbb{W}),(\lambda\gamma^n\mathbb{W}^q)\}+\{(\lambda\gamma^q\mathbb{W}),(\lambda\gamma^n\mathbb{W}^p)\}\big]\mathbb{F}^{mn}\Big]\\ &+\mathrm{Tr}\Big[\mathbb{V}_1(\lambda\gamma^m\mathbb{W}^{pq})(\lambda\gamma^n\mathbb{W}^{pq})(\lambda\gamma^{[m}\mathbb{W}^{n]})\Big],\end{aligned} \tag{12.2}$$

yielding 14 terms. This makes it clear how the thousands of terms in the ansatz can snowball into a huge variation which needs to be canonicalised. Whilst we have a

[1] It is not exactly double, but as an order of magnitude estimate it is.

number of tools developed than can be used to canonicalise the ansatz, the $\gamma$ matrices often remain an issue and it is not possible to check that all of these matrices have been properly canonicalised. Thus, dealing with the generating series in their $SO(10)$ form may not be possible. One option here is to perform the $U(5)$ decomposition from previous chapters - this eliminates the uncertainty in the $\gamma$ matrix canonicalisation. This occurs because the $\gamma$ matrices decompose into Levi-Civita symbols and other objects that are far simpler to deal with than the full Clifford algebra. [7]. However, this is an ongoing project and it is not possible to determine whether this solves the issue.

Once one generates an ansatz containing all of the terms, it is a useful exercise to determine whether this ansatz can produce the required amplitudes. That is, given that we know the field theory limit of a number of amplitudes and we can calculate amplitudes from generating series, does performing the latter on our ansatz give us every term we need to generate the former? If the answer to this question is no then our ansatz cannot be complete and we have to expand it further. This is the situation when one considers ansatzes formed from just $\mathbb{V}$, $\mathbb{F}$ and $\mathbb{W}$. We are able to generate the correct 4- and 5-point amplitudes but we fail to generate all the terms required at 6-points. To some extent this should come as no surprise - we are mixing $\mathbb{V}_1$ and no-$\mathbb{V}_1$ terms[2] and the only field that can induce this mixing is the raw gauge field $\mathbb{A}^m$. We believe that the source of these terms is down to the factorisation that can take place due to $\alpha'^4 \sim \alpha'^2 \times \alpha'^2$. Determining the form of this channel is not simply a case of squaring the $\alpha'^2$ result and we only have partial evidence as to the form such a channel might take. As we shall see in the next section, it is very likely that any such factorisation channel will need to include $\mathbb{A}^m$. Hence, it is possible that at this order one has to generalise the ansatz to include terms that contain $\mathbb{A}^m$ - this also will help induce the mixing mentioned previously.

## 12.2 Looking for $\mathbb{L}$

In the 6-point amplitude for the $\alpha'^4$ corrections there are some component expansions missing - despite the generality of the ansatz. Some experimentation shows that these

[2]We have to do this in order to produce the 4- and 5- point component expansions we need.

| Term | # Scalars |
|---|---|
| $(\lambda\gamma\mathbb{W})\partial\mathbb{F}$ | 2 |
| $(\lambda\gamma\partial\mathbb{W})\mathbb{F}$ | 2 |
| $(\lambda\gamma\mathbb{W})\mathbb{A}\mathbb{F}$ | 2 |
| $(\lambda\gamma\mathbb{W})(\mathbb{W}\gamma\mathbb{W})$ | 2 |
| $(\lambda\gamma\mathbb{W})\mathbb{A}\mathbb{A}\mathbb{A}$ | 4 |
| $(\lambda\gamma\mathbb{W})\mathbb{A}\partial\mathbb{A}$ | 4 |
| $(\lambda\gamma\partial\mathbb{W})\mathbb{A}\mathbb{A}\ )$ | 4 |
| $(\lambda\gamma\partial\mathbb{W})\partial\mathbb{A}$ | 4 |
| $(\lambda\gamma\partial\partial\mathbb{W})\mathbb{A}$ | 4 |
| $(\lambda\gamma\partial\partial\partial\mathbb{W})$ | 4 |
| $\mathbb{V}(\mathbb{W}\gamma\mathbb{W})\mathbb{A}$ | 5 |
| $\mathbb{V}(\mathbb{W}\gamma\partial\mathbb{W})$ | 5 |
| $\partial\mathbb{V}(\mathbb{W}\gamma\mathbb{W})$ | 5 |
| $\mathbb{V}\mathbb{A}\partial\mathbb{F}$ | 5 |
| $\mathbb{V}\mathbb{A}\mathbb{A}\mathbb{F}$ | 5 |
| $\mathbb{V}\partial\mathbb{A}\mathbb{F}$ | 5 |
| $(\partial\mathbb{V})\mathbb{A}\mathbb{F}$ | 5 |
| $\mathbb{V}\mathbb{A}\mathbb{A}\mathbb{A}\mathbb{A}$ | 10 |
| $\mathbb{V}\mathbb{A}\mathbb{A}\partial\mathbb{A}$ | 10 |
| $(\partial\mathbb{V})\mathbb{A}\mathbb{A}\mathbb{A}$ | 10 |
| $\mathbb{V}\mathbb{A}\partial\partial\mathbb{A}$ | 10 |
| $\mathbb{V}\partial\mathbb{A}\partial\mathbb{A}$ | 10 |
| $(\partial\mathbb{V})\mathbb{A}\partial\mathbb{A}$ | 10 |
| $(\partial\partial\mathbb{V})\mathbb{A}\mathbb{A}$ | 10 |
| $\mathbb{V}\partial\partial\partial\mathbb{A}$ | 10 |
| $\partial\mathbb{V}\partial\partial\mathbb{A}$ | 10 |
| $\partial\partial\mathbb{V}\partial\mathbb{A}$ | 10 |
| $(\partial\partial\partial\mathbb{V})\mathbb{A}$ | 10 |
| $(\partial\partial\partial\partial\mathbb{V})$ | 10 |

TABLE 12.1: The number of scalars for the ansatz of $\mathbb{L}$ as predicted by group theory arguments.

terms may arise from the factorisation channel of $\alpha'^4$ amplitudes into $\alpha'^2 \times \alpha'^2$. There is an indication that the generating series for these terms is given by,

$$\mathrm{Tr}[\mathbb{L}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}], \tag{12.3}$$

where the following relation holds,

$$\mathrm{Tr}\{\mathcal{Q},\mathbb{L}\} = \mathrm{Tr}[(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}]. \tag{12.4}$$

In order to find the generating series for this factorisation channel we need to determine $\mathbb{L}$. From the component expansions, we find that $QL_{123} = M_{123}$ and thus indicates that at this level $\mathbb{L}$ may be in some way related to $\mathbb{V}$. As an initial starting point one potential way to start is to look at the single particle expansion of $\mathbb{V}$ at three points, that is expand $V_{123}$ in terms of single particle fields. Whilst $M_{123}$ has the pole structure, the generating series encapsulates this and so once we have $V_{123}$ in single particle form, we promote the unique terms that appear to generating series. This yields terms of the schematic form,

$$(\lambda\gamma^m\mathbb{W})\mathbb{A}^p\mathbb{F}^{mp}, \qquad \mathbb{V}(\partial^m\mathbb{A}^n)\mathbb{F}^{mn}, \tag{12.5}$$

which is promising because the first term here contributes $(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}$ under BRST variation from $\{\mathcal{Q}, \mathbb{A}^m\}$. The question then is: why not use the gauge invariant $\nabla^m$ rather than working with non-gauge invariant $\mathbb{A}^m$ and $\partial^m$? This is simply a matter of choice as ultimately one of $\nabla^m$, $\partial^m$ and $\mathbb{A}^m$ is dependent on the others following the defintion of $\nabla^m$. So one could pick any two of these. In the end, the final answer must be gauge invariant as we will impose BRST invariance on the entire generating series - it may well be that each individual term is not gauge invariant. This does not matter provided the entire expression is. Also note that since $\alpha'^4$ corrections have mass dimension 9.5, the mass dimension of $\mathbb{L}$ must be 4.5 and it also must have ghost number one so that the entire expression has ghost number three.

In order to determine which terms we ought to consider out of the multitude that one can construct we use the group theory program `LiE` in a similar as we did for the full ansatz previously. Since `LiE` does not care whether a term contains $\partial$ or $\mathbb{A}$ we can just use the same `LiE` expression for both - that is $(\lambda\gamma^m\mathbb{W})\partial^n\mathbb{F}^{mn}$ will have the same number of scalar contractions as $(\lambda\gamma^m\mathbb{W})\mathbb{A}^n\mathbb{F}^{mn}$. These terms each have two scalar contractions that are non-zero and they correspond to $(\lambda\gamma^m\mathbb{W})\partial^n\mathbb{F}^{mn}$ and $(\lambda\gamma^{mnp}\mathbb{W})\partial^m\mathbb{F}^{np}$. The results of LiE are given in Table 12.1. Of course, there may be some over counting of terms here as it is unclear how one can generate 10 terms from $\partial\partial\partial\partial\mathbb{V}$ - we can at least generate one type naïvely of the schematic form $\Box\Box\mathbb{V}$ which would correspond to $k_P^4 V_P$ at the level of components. In terms of gauge invariant terms, this would arise from $\mathbb{V}^{mmnn}$ and may contribute to the process. Perhaps it is due to us ignoring terms of the form $\mathbb{V}^M$ that the ansatz does not have the correct form?

Now that we have the terms that can belong in the ansatz, it is a simple matter of generating the full ansatz and performing two things simultaneously. The first is to use the knowledge that $L_1 = L_{12} = 0$, and $L_{123} = M_{123}$ to fix some of the terms. For example, the only terms that contribute to $L_1$ are $\partial^4\mathbb{V}$ and $\lambda\gamma\partial^3\mathbb{W}$ and hence their sum must be zero. It is likely that at least three field will be required for $\mathbb{L}$ which removes a lot of the ansatz terms. Then one can try to determine higher order values, such as $L_{1234}$ and $L_{12345}$ to fix the entire form of $\mathbb{L}$ or one can use BRST arguments to fix the unknown constants. Once this is done we will hopefully have the form of $\mathbb{L}$ - it ought to be much simpler than the full $\alpha'^4$ ansatz since we are only working to dimension 4.5 in the operators. In fact, it would be interesting to see if one could use the higher mass field $\mathbb{V}^{mmnn}$ and its permutations to obtain the correct form for $\mathbb{L}$.

This is one method to determine the $\alpha'^2 \times \alpha'^2$ factorisation channel. However, one can take inspiration from [77], as well as general effective field theory methods, and suppose that the equations of motion for the fields that we have at this order are corrected by the $\alpha'^2$ corrections previously found. Consider (9.23) from Chapter 9, this equation of motion for $\mathbb{V}$ is suspiciously similar to the expected BRST variation of the unknown $\mathbb{L}$ we require to determine the $\alpha'^4$ correction. In fact, if we use $\mathbb{V}_1$ in lieu of $\mathbb{V}$ in (9.23) we find the following BRST variation,

$$\mathrm{Tr}\{\mathcal{Q}, \mathbb{V}_1\} = \mathrm{Tr}[(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}], \tag{12.6}$$

which suggests that $\mathbb{V}_1$ is the $\mathbb{L}$ we are looking for. Taken *prima facie* this result seems contradictory to the evidence we thus far have - we have the most general ansatz involving $\mathbb{V}_1$ and the fields in the above variation. The subtly here is that we have to think about this all in terms of effective field theories. We have *solved* for the $\alpha'^2$ correction and this corrects the equations of motion, and hence the BRST variation of the fields. As a result, we need to factor this change into our analysis and essentially follow the working of Howe et al. in [77]. That is, we need to take the $\alpha'^2$ correction we found previously and use the new BRST variations when varying it (that is the $\alpha'^2$ correction) then the $\alpha'^4$ corrections are the terms that cancel the new terms coming from the corrected equations of motion. Such corrected equations of motion would also alter the Harnad-Shnider gauge $\theta-$expansions. These alterations may give us the terms in the component expansion which we have been previously missing in the ansatz - thus solving all of the issues that we found previously. At the current time of writing the other equations of motion are unknown to the author and it is currently only conjecture that following this method will yield the correction. If this is the case then the answer will likely look very similar to that found in [77] - however the correction found using the techniques laid out in these pages will give us the generating series of the correction. Some work has already been carried out in this vein - this method has been correctly used to determine the $\alpha'^2$ correction [122]. This method actually allows one to drop the concept of $\mathbb{V}_1$ all together and simply proceed using the following variations,

$$\begin{aligned}\{\mathcal{Q}, \mathbb{V}\} &= -\mathbb{V}\mathbb{V} + \alpha'^2(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn} + ...\\ \{\mathcal{Q}, (\lambda\gamma^m\mathbb{W})\} &= \alpha'^2(\lambda\psi_2^m\lambda) + ...,\\ [\mathcal{Q}, \mathbb{F}^{mn}] &= -(\lambda\gamma^{[m}\mathbb{W}^{n]}) + \alpha'^2(\lambda\mathbb{H}_2^{mn}) + ...,\end{aligned} \tag{12.7}$$

where $\psi_2^m$ and $\mathbb{H}_2^{mn}$ are as yet unknown superfield functions. This then allows one to work with the Chern-Simons-like action in (9.23) from Chapter 9 as a true action for the generating series representing a novel approach to the problem. Following this methodology is very promising and will hopefully lead to an expression for the $\alpha'^4$ generating series. Before then, it will be instructive to look at the $\alpha'^3$ correction and determine if this follows a similar pattern. If so, then it is highly suggestive that one can interpret the Chern-Simons-like action as a genuine action for the generating series of Super Yang-Mills amplitudes[3].

This chapter has briefly reviewed the state-of-the-art knowledge regarding $\alpha'^4$ generating series. There are significant challenges to overcome in order to find the correct generating series and current methods we have are not sufficient to move past these obstacles. Hopefully some of the techniques outlined here and others that are yet to be tested, will allow us to find the full generating series expression at this order. It may be the case that in lieu of working with $\partial^m$ and $\mathbb{A}^m$ it may be better to use $\nabla^m$ and $\mathbb{A}^m$ as we have full knowledge of the variation of higher mass fields defined using $\nabla^m$. There is strong

[3]Note that at the moment there are some subtleties in this approach that we do not elucidate here owing to the state of current research. Hopefully these subtleties will be addressed in [122].

evidence to suggest that generalising the ansatz to include $\mathbb{A}^m$ fields in some of terms is the key to unlocking the $\alpha'^4$ correction. There is also evidence that these issues could be solved by considering the corrections that occur due to the corrections to the equations of motion which arise due to the $\alpha'^2$ correction found previously. This would present a novel interpretation of the generating series action and allow one to use it as a 'true' action to determine amplitude structures more efficiently.

# Part III

# Further Topics

# Chapter 13

# Compactifying on the Torus

## 13.1 An Experiment in Nine Dimensions

In this chapter we explicitly demonstrate the method of finding the lower dimensional amplitudes by compactifying the equations of motion to nine dimensions. This gives the $\theta$ expansions of the fields from which the BRST building blocks can be defined, of course one can use these expansions for many of the calculations performed in the pure spinor methodology beyond those presented here. Here and it what follows, upper case Latin letters $(M, N, P, ...)$ typically denote the full $D = 10$ theory vectors, lower case Latin letters $(m, n, p, ...)$ are used for lower dimensional vectors and Greek letters $\alpha, \beta, \gamma, \delta$ are used for spinors. We may use some greek letters for vector indices but we will always clarify this use.

### 13.1.1 Compactifying the Field Equations

We begin by taking the ten-dimensional equations of motion given in (4.5) and compactifying them on a torus, this will not affect the spinors but will cause a splitting of the gauge field into a nine-dimensional gauge field and a scalar field which we denote as $\varphi$. We restrict to the massless fields and ignore any Kaluza-Klein modes that are generated by the compactification. Thus the 'splitting' that occurs in going to nine dimensions only affects the vectors in the theory and leaves the spinors untouched, as a result $\mathbb{A}_M \to (\mathbb{A}_m, \varphi)$ and,

$$\nabla_\alpha \to \nabla_\alpha = D_\alpha - \mathbb{A}_\alpha, \qquad \nabla_M \to \nabla_m = \partial_m - \mathbb{A}_m, \quad \nabla_9 = -\varphi, \tag{13.1}$$

where now $D_\alpha = \partial_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$. The $\gamma^M$ splits as $\gamma^m_{\alpha\beta}, \eta_{\alpha\beta}$, where the $\eta_{\alpha\beta}$ acts like a spinorial metric which allows us to raise and lower spinor indices. As a result, the equations in (4.5) become,

$$\{D_{(\alpha}, \mathbb{A}_{\beta)}\} = \gamma^m_{\alpha\beta}\mathbb{A}_m + \eta_{\alpha\beta}\varphi + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\} \tag{13.2a}$$

$$\{D_\alpha, \mathbb{W}^\beta\} = \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn} + \{\mathbb{A}_\alpha, \mathbb{W}^\beta\} \tag{13.2b}$$

$$[D_\alpha, \mathbb{A}_m] = [\partial_m, \mathbb{A}_\alpha] + (\gamma_m\mathbb{W})_\alpha + [\mathbb{A}_\alpha, \mathbb{A}_m] \tag{13.2c}$$

$$[D_\alpha, \varphi] = \mathbb{W}_\alpha + [\mathbb{A}_\alpha, \varphi] \tag{13.2d}$$

$$[D_\alpha, \mathbb{F}_{mn}] = [\partial_{[m}, (\gamma_{n]}\mathbb{W})_\alpha] + [\mathbb{A}_\alpha, \mathbb{F}_{mn}] \tag{13.2e}$$

$$[D_\alpha, \varphi^m] = \mathbb{W}^m_\alpha + [\mathbb{A}_\alpha, \varphi^m], \tag{13.2f}$$

where $\varphi^m = [\nabla^m, \varphi]$ is the higher mass analogue for $\varphi$. Now we can apply the Harnad-Shnider gauge to the above field equations in order to obtain,

$$(\mathcal{D} + 1)\mathbb{A}_\alpha = (\theta\gamma^m)_\alpha\mathbb{A}_m + \theta_\alpha\varphi \tag{13.3a}$$

$$\mathcal{D}\mathbb{W}_\alpha = \tfrac{1}{4}(\theta\gamma^{mn})_\alpha\mathbb{F}_{mn} \tag{13.3b}$$

$$\mathcal{D}\mathbb{A}_m = (\theta\gamma_m\mathbb{W}) \tag{13.3c}$$

$$\mathcal{D}\varphi = [\theta\mathbb{W}] \tag{13.3d}$$

$$\mathcal{D}\mathbb{F}_{mn} = -(\mathbb{W}^{[m}\gamma^{n]}\theta) \tag{13.3e}$$

$$\mathcal{D}\varphi^m = [\theta\mathbb{W}^m], \tag{13.3f}$$

where the fourth and sixth equations are essentially the same equation but with the higher mass definitions substituted in. Note that $[\theta\mathbb{W}] = \theta_\alpha\mathbb{W}^\alpha$. It is then quite simple to invert these equations to give the recursive definitions à la (6.15), which are very similar and are given by,

$$\mathbb{A}_\alpha|_k = \frac{1}{k+1}\big((\theta\gamma^m)_\alpha\mathbb{A}_m|_{k-1} + \theta_\alpha\varphi|_{k-1}\big), \tag{13.4a}$$

$$\mathbb{W}_\alpha|_k = \frac{1}{4k}(\theta\gamma^{mn})_\alpha \mathbb{F}_{mn}|_{k-1}, \tag{13.4b}$$

$$\mathbb{A}_m|_k = (\theta\gamma_m \mathbb{W}|_{k-1}), \tag{13.4c}$$

$$\varphi|_k = [\theta\mathbb{W}|_{k-1}], \tag{13.4d}$$

$$\mathbb{F}_{mn}|_k = -(\mathbb{W}|_{k-1}^{[m}\gamma^{n]}\theta). \tag{13.4e}$$

It is then not hard to see that the $\theta$ expansion of the $\mathcal{A}_\alpha$ multiparticle superfield in $D = 9$ is given by the naïve torodial compactification of the expansion of the $D = 10$ case, that is sending $\mathbb{A}_M \to (\mathbb{A}_m, \varphi)$. One may wonder why we went through the motions of compactifying the field equations at all, and then found the $\theta$ expansion using these, following this method means that we have the recursive definitions to all orders and can show that the field equations behave as one expects under such compactification. In $D = 9$ the higher mass equations of motion behave as expected, for example,

$$\{\nabla_\alpha, \mathbb{W}^{m\beta}\} = \tfrac{1}{4}(\gamma_{pq})_\alpha{}^\beta \mathbb{F}^{m|pq} - \{(\mathbb{W}\gamma^m)_\alpha, \mathbb{W}^\beta\}. \tag{13.5}$$

Note that $\gamma_{99} = \gamma_{m9} = 0$ and so we do not have any other contributions from the $\mathbb{F}$ term in the above equation; also note that $\mathbb{W}^{9\beta} = [\varphi, \mathbb{W}^\beta] = 0$ and so if we set $M = 9$ we obtain the relation,

$$\{\mathbb{W}_\alpha, \mathbb{W}^\beta\} = 0, \tag{13.6}$$

and so the $\mathbb{W}$ fields anti-commute and this will occur at lower dimensions but with internal $\gamma$ matrices contracting the $\mathbb{W}$ non-linear fields. Now we apply Harnad-Shnider gauge to the higher mass $\mathbb{W}$ equation and we find the recursion,

$$\begin{aligned} \mathcal{D}\mathbb{W}^{m\alpha} &= \tfrac{1}{4}(\theta\gamma_{pq})^\alpha \mathbb{F}^{m|pq} - \{(\mathbb{W}\gamma^m\theta), \mathbb{W}^\alpha\} \\ \implies [\mathbb{W}^{m\alpha}]_k &= \frac{1}{k}\big((\theta\gamma_{pq})^\alpha[\mathbb{F}^{m|pq}]_{k-1} - (\theta\gamma^m)_\beta \sum_{l=0}^{k-1}\{[\mathbb{W}^\beta]_l, [\mathbb{W}^\alpha]_{k-l-1}\}\big), \end{aligned} \tag{13.7}$$

which is identical to those found in [29]. Note that we do not need the higher mass $\mathbb{F}$ equation as this will not appear in the $\theta$ expansion such that the zero mode integration is non-zero. We also must deal with the compactification of the multiparticle polarisation vectors. The polarisation tensors are then,

$$[\mathcal{A}_P^m]_0 = \mathfrak{e}_P^m, \quad [\varphi_P]_0 = \mathfrak{v}_P, \quad [\mathcal{W}_P^\alpha]_0 = \mathfrak{g}_P^\alpha, \quad [\mathcal{F}_P^{mn}]_0 = \mathfrak{f}_P^{mn}, \quad [\varphi^m]_0 = \mathfrak{v}_P^m, \tag{13.8}$$

where,

$$\begin{aligned} \mathfrak{f}_P^{mn} &= k_P^m \mathfrak{e}_P^n - k_P^n \mathfrak{e}_P^m - \sum_{XY=P} (\mathfrak{e}_X^m \mathfrak{e}_Y^n - \mathfrak{e}_X^n \mathfrak{e}_Y^m) \\ \mathfrak{v}_P^m &= k_P^m \mathfrak{v}_P, \end{aligned} \tag{13.9}$$

where $\mathfrak{v}_P$ is the vacuum expectation value of the scalar gauge field. All of the above does not depend on the pure spinor formalism and is quite general. In order to calculate the amplitudes of this theory, however, we shall re-introduce the pure spinor zero mode integration measure and find its lower dimensional analogues and this will allow us to find the BRST building blocks.

### 13.1.2 Amplitudes

In order to determine the amplitudes using the methods outlined in Section 6 we require four things: the zero mode integration measure in nine-dimensions, the pure spinor constraint, the $\theta$ expansion of the multiparticle superfields and the polarisation tensors, and the form of the BRST building blocks. We have already found the lower dimensional $\theta$ expansions as well as the polarisation tensors, hence one needs to determine the form of the BRST building blocks, the integration measures and the pure spinor constraints.

The first of these three things is straight forward to define since the building blocks descend from $\langle \mathbb{V}\mathbb{V}\mathbb{V} \rangle$ which is an inherently spinorial quantity and from $D = 10$ to 9 the spinors do not change. Hence the building blocks still descend from,

$$\mathrm{Tr}\langle M_A M_B M_C \rangle, \tag{13.10}$$

which is unmodified[1]. The pure spinor constraint splits into two and we find,

$$\lambda\lambda = 0, \quad \lambda\gamma^m\lambda = 0. \tag{13.11}$$

[1] In lower dimensions as we shall see, the form is still the same but we must expand the BRST building blocks out into their lower dimensional blocks.

Hence the only object one needs to determine in order to find the BRST building blocks in nine dimensions is the form of the integration measure(s). In ten-dimensions the integration measure is given by (1.3), which in nine dimensions naïvely splits into,

$$\begin{aligned}\langle(\lambda\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{np}\theta)\rangle &= 288,\\ \langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle &= 2592,\end{aligned} \tag{13.12}$$

which follow from the ten-dimensional identity [29],

$$\langle(\lambda\gamma^M\theta)(\lambda\gamma^N\theta)(\lambda\gamma^R\theta)(\theta\gamma^{PQR}\theta)\rangle = 32(\delta^{MP}\delta^{NQ} - \delta^{MQ}\delta^{NP}). \tag{13.13}$$

One now needs to determine two things: first one needs to determine if the measures in (13.12) are BRST closed and not BRST exact. We can show that these are BRST closed quite easily. In order to do so we note the $D = 10$ identity,

$$(\lambda\gamma^M)_\alpha(\lambda\gamma^M)_\beta = 0 \to \lambda_\alpha\lambda_\beta + (\lambda\gamma^m)_\alpha(\lambda\gamma^m)_\beta = 0 \tag{13.14}$$

where the arrow denotes compactification. Since $\theta_\alpha$ is an anticommuting variable and $\lambda^\alpha$ is a bosonic Weyl-spinor we note that $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta) = -(\lambda\gamma^n\theta)(\lambda\gamma^m\theta)$. Using this as well as (13.14) one can manipulate the structures in (13.12) into the form $(\lambda\theta)^4$, after BRST variation, which is zero due to the fact that $(\lambda\theta)$ is an anti-commuting scalar. It is slightly more difficult to show that the two terms in (13.12) are not BRST exact index structures. To do this we must test if it is possible to obtain (13.12) by applying a BRST variation to a term of the form $\lambda^2\theta^6$. It is easiest to test this for the second term in (13.12) since the only possible BRST ancestors of this structure are,

$$\begin{aligned}\mathcal{Z}_1 &=(\lambda\theta)(\lambda\gamma^n\theta)(\theta\gamma^m\theta)(\theta\gamma^{mn}\theta)\\ \mathcal{Z}_2 &=(\lambda\theta)(\theta\gamma^n\theta)(\lambda\gamma^m\theta)(\theta\gamma^{mn}\theta)\\ \mathcal{Z}_3 &=(\lambda\theta)(\theta\gamma^n\theta)(\theta\gamma^m\theta)(\lambda\gamma^{mn}\theta),\end{aligned} \tag{13.15}$$

and linear combinations thereof. We include the second term for completeness though of course we do not need to, we also note that the first $\lambda^\alpha$ must always come with a $\theta$ since the other option would be to have $\theta^2 = 0$. However, we note that $\gamma^m_{\alpha\beta}$ is symmetric in its spinorial indices and so,

$$\theta\gamma^m\theta = \theta^\alpha\gamma^m_{\alpha\beta}\theta^\beta = \frac{1}{2}(\theta^\alpha\gamma^m_{\alpha\beta}\theta^\beta - \theta^\beta\gamma^m_{\alpha\beta}\theta^\alpha) = \frac{1}{2}(\theta\gamma^m\theta - \theta\gamma^m\theta) = 0, \tag{13.16}$$

and thus the possible ancestors of this term are zero. Thus this index structure cannot be BRST exact. We now check the ancestors of the first measure in (13.12); the independent ancestors of this term are,

$$\begin{aligned}
\mathcal{Y}_1 &= (\lambda\gamma^p\theta)(\lambda\gamma^n\theta)(\theta\gamma^m\theta)(\theta\gamma^{mnp}\theta)\\
\mathcal{Y}_2 &= (\lambda\gamma^p\theta)(\theta\gamma^n\theta)(\lambda\gamma^m\theta)(\theta\gamma^{mnp}\theta)\\
\mathcal{Y}_3 &= (\lambda\gamma^p\theta)(\theta\gamma^n\theta)(\theta\gamma^m\theta)(\lambda\gamma^{mnp}\theta)\\
\mathcal{Y}_4 &= (\theta\gamma^p\theta)(\lambda\gamma^n\theta)(\lambda\gamma^m\theta)(\theta\gamma^{mnp}\theta)\\
\mathcal{Y}_5 &= (\theta\gamma^p\theta)(\lambda\gamma^n\theta)(\theta\gamma^m\theta)(\lambda\gamma^{mnp}\theta)\\
\mathcal{Y}_6 &= (\theta\gamma^p\theta)(\theta\gamma^n\theta)(\lambda\gamma^m\theta)(\lambda\gamma^{mnp}\theta),
\end{aligned}$$

where there are an extra three terms with $(\theta\gamma^{mnp}\lambda)$ due to the spinor anti-symmetry of $\gamma^{mnp}$. The same argument runs through the same in this case, since $\theta\gamma^m\theta = 0$ all of the above terms are automatically zero. This shows that there are no ancestors of the terms in (13.12).

With all of this in hand we can find the BRST building blocks of the colour-stripped amplitudes, the $\theta$ expansion of the pre-spinor potential is given by,

$$\begin{aligned}
\mathcal{A}^P_\alpha =& \frac{1}{2}(\gamma^m\theta)_\alpha\mathfrak{e}^P_m + \frac{1}{2}\theta_\alpha\mathfrak{v}^P + \frac{1}{3}(\gamma^m\theta)_\alpha(\theta\gamma_m\mathfrak{g}^P) + \frac{1}{3}\theta_\alpha(\theta\mathfrak{g}^P)\\
&- \frac{1}{32}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)\mathfrak{f}^P_{np} - \frac{1}{32}\theta_\alpha(\theta\gamma^{np}\theta)\mathfrak{f}^P_{np}\\
&+ \frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\mathfrak{g}^P_n\gamma_p\theta) + \frac{1}{60}\theta_\alpha(\theta\gamma^{np}\theta)(\mathfrak{g}^P_n\gamma_p\theta),
\end{aligned} \tag{13.17}$$

where the decontanations of the polarisation vectors which follow from the compactified (4.19) in [29] and the deconcatenation of $\mathfrak{v}^P$ follows from $\mathfrak{e}^m_{[P,Q]}$. If one brings all of the above together one finds that the BRST building blocks are the expected compactification of the $D = 10$ results.

## 13.2 $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills

In this section we compactify the ten-dimensional non-linear theory to four dimensions on a 6-torus, $T^6$, and determine the BRST building blocks of the theory. This requires the compactification of all of the machinery presented in Section 13.1; combined with the usual notational complexity of four-dimensional supersymmetric theories, we present the explicit notation used in this section here for ease of reading. The ten-dimensional spinor indices now take the form $\underline{\alpha}, \underline{\beta}$ and the four-dimensional spinors are denoted by the usual Greek indices $\alpha, \beta, \dot{\alpha}, \dot{\beta}$, where the dotted and undotted indices denote different helicities; $\mathsf{a}, \mathsf{b}$ are the six-dimensional, internal, spinor indices; they also label the supersymmetry related $R-$Symmetry, which is $SU(4)$ in this case [123, 124, 125]. The ten-dimensional vector indices are still denote by $M, N$ and we split the vectors as $\mu, \nu$ for the four-dimensional part of theory and $m, n$ for the six-dimensional $T^6$ manifold. Hence the four-dimensional Pauli matrices are denoted by $(\sigma^\mu)_{\alpha\dot{\beta}}$ and the $T^6$ gamma matrices are denoted by $(\sigma^m)_{\mathsf{ab}}$.

### 13.2.1 Four-Dimensional Spinors

We now turn to the discussion of how the ten dimensional theory gives rise to the usual four dimensional splitting of chirality and so forth. Typically this occurs vis à vis the usual formulas,

$$
\begin{aligned}
\chi^{\underline{\alpha}}\chi_{\underline{\alpha}} &= \chi_\alpha\bar{\chi}^\alpha - \chi_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} - \chi^{I\alpha}\bar{\chi}_{I\alpha} + \chi_{I\dot{\alpha}}\bar{\chi}^{I\dot{\alpha}}, \\
\chi^{\underline{\alpha}}\gamma^\mu_{\underline{\alpha}\underline{\beta}}\xi^{\underline{\beta}} &= -\chi^\alpha\sigma^\mu_{\alpha\dot{\beta}}\xi^{\dot{\beta}} - \sigma^\mu_{\alpha\dot{\beta}}\chi^{\dot{\beta}}\xi^\alpha + \chi^{I\alpha}\sigma^\mu_{\alpha\dot{\beta}}\xi^{\dot{\beta}}_I + \sigma^\mu_{\alpha\dot{\beta}}\chi^{\dot{\beta}}_I\xi^{I\alpha}, \\
\chi^{\underline{\alpha}}\gamma^I_{\underline{\alpha}\underline{\beta}}\xi^{\underline{\beta}} &= \chi^\alpha\xi^I_\alpha - \chi^{I\alpha}\xi_\alpha + \varepsilon^{IJK}\chi_{J\dot{\alpha}}\xi^{\dot{\alpha}}_K,
\end{aligned}
\tag{13.18}
$$

where $I, J, K$ run from 1 to 3. We can however package these expression in a more compact way which will make working in four-dimensions much easier. We define,

$$
\chi^{\mathsf{a}\alpha} = (\chi^\alpha, \chi^{I\alpha}),
\tag{13.19}
$$

where $\mathsf{a}, \mathsf{b}$ now run from 1 to 4. Hence we package the four dimensional and six dimensional spinors into one. This will simplify many of the expressions we will require in this section. For example, in this notation we find,

$$\begin{aligned}
\chi^{\underline{\alpha}}\xi_{\underline{\alpha}} &= \chi^{\mathsf{a}\alpha}\xi_{\mathsf{a}\alpha} - \overline{\chi}_{\mathsf{a}\dot{\alpha}}\overline{\xi}^{\mathsf{a}\dot{\alpha}}, \\
\chi^{\underline{\alpha}}\gamma^{\mu}_{\underline{\alpha}\underline{\beta}}\xi^{\underline{\beta}} &= -\sigma^{\mu}_{\alpha\dot{\beta}}(\chi^{\mathsf{a}\alpha}\overline{\xi}^{\dot{\beta}}_{\mathsf{a}} + \overline{\chi}^{\dot{\beta}}_{\mathsf{a}}\xi^{\mathsf{a}\alpha}), \\
\chi^{\underline{\alpha}}\gamma^{m}_{\underline{\alpha}\underline{\beta}}\xi^{\underline{\beta}} &= \sigma^{m}_{\mathsf{ab}}(\chi^{\mathsf{a}\alpha}\xi^{\mathsf{b}}_{\alpha}) + \sigma^{m}_{\mathsf{ab}}(\overline{\chi}_{\mathsf{a}\dot{\alpha}}\overline{\xi}^{\dot{\alpha}}_{\mathsf{b}}),
\end{aligned} \tag{13.20}$$

which are much simpler than their original counter parts, note that we use the overline notation for negative chirality component spinors, we will not use this notation on the non-linear fields. This will become even more useful when we start to consider the zero mode integration measure [126] and the BRST building blocks as they will take on a much simpler form. Note that we adopt angle and square bracket notation in this section such that $\chi^{\mathsf{a}\alpha}\xi_{\mathsf{b}\alpha} = \langle\chi^{\mathsf{a}}\xi_{\mathsf{b}}\rangle$ and $\overline{\chi}_{\mathsf{a}\dot{\alpha}}\overline{\xi}^{\mathsf{b}\dot{\alpha}} = [\overline{\chi}_{\mathsf{a}}\overline{\xi}^{\mathsf{b}}]$. In this section we shall denote pure spinor zero mode integration by $\langle||\rangle$ to distinguish it from the spinor-helicity angle brackets.

Before discussing the field equations, it is worth addressing the two 'elephant in the room': what does the pure spinor constraint look like in this theory and what is the BRST operator? It is fairly easy to read off what the pure spinor constraint becomes by applying (13.20) which splits the constraint in two[2],

$$\lambda^{\mathsf{a}\alpha}\overline{\lambda}^{\dot{\alpha}}_{\mathsf{a}} = 0, \qquad \langle\lambda^{\mathsf{a}}\lambda^{\mathsf{b}}\rangle = \frac{1}{2}\varepsilon^{\mathsf{abcd}}[\overline{\lambda}_{\mathsf{c}}\overline{\lambda}_{\mathsf{d}}], \tag{13.21}$$

as usual in pure spinor, these formulas will be pivotal to future simplifications of the results. It is rather interesting to note that the first formula replicates the momentum conservation equation in spinor-helicity variables for the pure spinor, although rather than summing over particle labels, the sum is over the internal $SU(4)$ $R-$symmetry. Since there is no notion of pure spinor in $D = 4$ these restrictions on the ghost variables are rather interesting and potentially warrant some further investigations. It is also useful to note the form of the BRST operator in the lower dimensional case, it is given

[2]This was first found in [126].

by the simple form,

$$Q = \langle \lambda^{\mathsf{a}} D_{\mathsf{a}} \rangle - [\overline{\lambda}_{\mathsf{a}} \overline{D}^{\mathsf{a}}], \tag{13.22}$$

where $\overline{D}$ is defined in the obvious way. Having derived all of the above machinery we are in a position to begin discussing how to find the BRST building blocks for amplitudes in the $D = 4$ theory.

### 13.2.2 The Field Equations and Gauge Variation

In order to determine how the field equations are modified in lower dimensions we pre-contract and post-contract the general spinors $\zeta_\alpha$, $\xi_\alpha$ in $D = 10$ and then use the results of the previous subsection to determine the four-dimensional field equations. By using these general spinors in $D = 10$ we can isolate the various helicities of the equations of motion under compactification using known identities. To demonstrate this we shall give an example of one of the field equations and then quote the rest of the equations. Let us consider the $\mathbb{A}_M$ equation of motion for the $\mu$ indices, performing the general spinor contractions and expanding into four dimensional variables we find that each of the terms in this equation become,

$$\begin{aligned}
\zeta^{\underline{\alpha}}[D_{\underline{\alpha}}, \mathbb{A}_\mu] &= \zeta^{\alpha\mathsf{a}}[D_{\alpha\mathsf{a}}, \mathbb{A}_\mu] - \zeta_{\dot{\alpha}\mathsf{a}}[D^{\dot{\alpha}\mathsf{a}}, \mathbb{A}_\mu], \\
\zeta^{\underline{\alpha}}[\partial_\mu, \mathbb{A}_{\underline{\alpha}}] &= \zeta^{\alpha\mathsf{a}}[\partial_\mu, \mathbb{A}_{\alpha\mathsf{a}}] - \zeta_{\dot{\alpha}\mathsf{a}}[\partial_\mu, \mathbb{A}^{\dot{\alpha}\mathsf{a}}], \\
\zeta^{\underline{\alpha}}(\gamma_\mu)_{\underline{\alpha}\underline{\beta}}\mathbb{W}^{\underline{\beta}} &= -\sigma^\mu_{\alpha\dot{\beta}}(\zeta^{\alpha\mathsf{a}}\mathbb{W}^{\dot{\beta}}_{\mathsf{a}} + \zeta^{\dot{\beta}}_{\mathsf{a}}\mathbb{W}^{\alpha\mathsf{a}}), \\
\zeta^{\underline{\alpha}}[\mathbb{A}_{\underline{\alpha}}, \mathbb{A}_\mu] &= \zeta^{\alpha}[\mathbb{A}_\alpha, \mathbb{A}_\mu] - \zeta_{\dot{\alpha}\mathsf{a}}[\mathbb{A}^{\dot{\alpha}\mathsf{a}}, \mathbb{A}_\mu],
\end{aligned} \tag{13.23}$$

where we now extract the $\zeta_{\dot{\alpha}A}$ and $\zeta_{\dot{\alpha}A}$ coefficients and find the field equations,

$$\begin{aligned}
[D_{\mathsf{a}\alpha}, \mathbb{A}_\mu] &= [\partial_\mu, \mathbb{A}_{\mathsf{a}\alpha}] - (\sigma^\mu\overline{\mathbb{W}})_{\mathsf{a}\alpha} + [\mathbb{A}_{\mathsf{a}\alpha}, \mathbb{A}_\mu], \\
[D^{\mathsf{a}\dot{\alpha}}, \mathbb{A}_\mu] &= [\partial_\mu, \mathbb{A}^{\mathsf{a}\dot{\alpha}}] + (\sigma^\mu\mathbb{W})^{\mathsf{a}\dot{\alpha}} + [\mathbb{A}^{\mathsf{a}\dot{\alpha}}, \mathbb{A}_\mu],
\end{aligned} \tag{13.24}$$

where $(\sigma^\mu\overline{\mathbb{W}})_{\mathsf{a}\alpha} = \sigma^\mu_{\alpha\dot{\beta}}\overline{\mathbb{W}}^{\dot{\beta}}_{\mathsf{a}}$ and $(\sigma^\mu\mathbb{W})^{\mathsf{a}\dot{\alpha}} = (\sigma^\mu)^{\dot{\beta}\alpha}\mathbb{W}^{\mathsf{a}}_\alpha$ and hence we take $\sigma^\mu_{\alpha\dot{\beta}} = \sigma^\mu_{\dot{\beta}\alpha}$. In the four dimensional notation the Harnad-Shnider gauge and Euler operator become,

$$\theta^{\mathsf{a}\alpha}\mathbb{A}_{\mathsf{a}\alpha} - \theta_{\mathsf{a}\dot{\alpha}}\mathbb{A}^{\mathsf{a}\dot{\alpha}} = 0, \qquad \mathcal{D} = \theta^{\mathsf{a}\alpha}D_{\mathsf{a}\alpha} - \theta_{\mathsf{a}\dot{\alpha}}D^{\mathsf{a}\dot{\alpha}}, \tag{13.25}$$

and thus in Harnad-Shnider gauge we can collect the two equations in (13.24) to give,

$$\mathcal{D}\mathbb{A}_\mu = -(\theta\sigma_\mu\overline{\mathbb{W}}) - (\bar{\theta}\sigma_\mu\mathbb{W}), \tag{13.26}$$

this implies the recursive definition,

$$\mathbb{A}_\mu|_k = -\frac{1}{k}\big[(\theta\sigma_\mu\overline{\mathbb{W}})_{k-1} + (\bar{\theta}\sigma_\mu\mathbb{W})_{k-1}\big]. \tag{13.27}$$

We can then employ this definition for the rest of the equations we require, however these calculations are not shown explicitly as they follow the above. The $\Phi_m$ equations of motion become,

$$\begin{aligned}
&[D_{\mathsf{a}\alpha}, \Phi_m] = (\sigma_m\mathbb{W})_{\mathsf{a}\alpha} + [\mathbb{A}_{\mathsf{a}\alpha}, \Phi_m], \\
\implies\ &\mathcal{D}\Phi_m = (\sigma_{\mathsf{ab}})_m\langle\theta^{\mathsf{a}}\mathbb{W}^{\mathsf{b}}\rangle + (\sigma^{\mathsf{ab}})_m[\bar{\theta}_{\mathsf{a}}\overline{\mathbb{W}}_{\mathsf{b}}], \\
\implies\ &\Phi_m|_k = \frac{1}{k}\big((\sigma_{\mathsf{ab}})_m\langle\theta^{\mathsf{a}}\mathbb{W}^{\mathsf{b}}\rangle_{k-1} + (\sigma^{\mathsf{ab}})_m[\bar{\theta}_{\mathsf{a}}\overline{\mathbb{W}}_{\mathsf{b}}]_{k-1}\big),
\end{aligned} \tag{13.28}$$

where $(\sigma_m\mathbb{W})_{\mathsf{a}\alpha} = (\sigma_{\mathsf{ab}})_m\mathbb{W}^{\mathsf{b}}_\alpha$, and similarly for $(\sigma_m\overline{\mathbb{W}})^{\mathsf{a}\dot{\alpha}}$. However, in order to align more clearly with the usual $\mathcal{N} = 4, D = 4$ Super Yang-Mills supermultiplet we shall contract $\Phi_m$ with $(\sigma^m)_{\mathsf{ab}}$ such that we deal with the scalar $\Phi_{\mathsf{ab}} = -\Phi_{\mathsf{ba}}$[3]. In doing so we find that the Harnad-Shnider gauge equation of motion is given by,

$$\begin{aligned}
&\mathcal{D}\Phi_{\mathsf{ab}} = 2(\varepsilon_{\mathsf{abcd}}\langle\theta^{\mathsf{c}}\mathbb{W}^{\mathsf{d}}\rangle - [\bar{\theta}_{[\mathsf{a}|}\overline{\mathbb{W}}_{|\mathsf{b}]}]), \\
\implies\ &\Phi_{\mathsf{ab}}|_k = \frac{2}{k}\big(\varepsilon_{\mathsf{abcd}}\langle\theta^{\mathsf{c}}\mathbb{W}^{\mathsf{d}}\rangle_{k-1} - [\bar{\theta}_{[\mathsf{a}|}\overline{\mathbb{W}}_{|\mathsf{b}]}]_{k-1}\big).
\end{aligned} \tag{13.29}$$

The above equation is a natural modification of the nine-dimensional recursion from the previous section. Most of the novel parts of the recursion in four dimensions can be understood as the splitting of chirality which does not appear in the nine dimensional theory; it is this splitting that makes the $\theta$ expansion quite cumbersome as we shall see.

Now we can move on to the rest of the equations of motion, we will display these in the same manner as we did for the $\Phi_{\mathsf{ab}}$ equations for brevity. Let us consider the equation

[3]We use the identities $(\sigma^m)_{\mathsf{ab}}(\sigma^m)_{\mathsf{cd}} = 2\varepsilon_{\mathsf{abcd}}$ and $(\sigma^m)_{\mathsf{ab}}(\sigma^m)^{\mathsf{cd}} = -2(\delta^{\mathsf{c}}_{\mathsf{a}}\delta^{\mathsf{d}}_{\mathsf{b}} - \delta^{\mathsf{c}}_{\mathsf{b}}\delta^{\mathsf{d}}_{\mathsf{a}})$, which are used throughout this section in order to eliminate the six-dimensional matrices.

of motion for $\mathbb{A}_{\underline{\alpha}}$, in the four plus six splitting we find that,

$$
\begin{aligned}
\{D_{(\mathsf{a}\alpha}, \mathbb{A}_{\mathsf{b}\beta)}\} &= \varepsilon_{\alpha\beta}\Phi_{\mathsf{ab}} + \{\mathbb{A}_{\mathsf{a}\alpha}, \mathbb{A}_{\mathsf{b}\beta}\}\\
\{D_{(\mathsf{a}\alpha}, \overline{\mathbb{A}}^{\mathsf{b}\dot\beta)}\} &= -\varepsilon^{\dot\gamma\dot\beta}(\sigma_{\alpha\dot\gamma})^{\mu}\delta^{\mathsf{b}}_{\mathsf{a}}\mathbb{A}_{\mu} + \{\mathbb{A}_{\mathsf{a}\alpha}, \overline{\mathbb{A}}^{\mathsf{b}\dot\beta}\}\\
\{D^{(\mathsf{a}\dot\alpha}, \mathbb{A}_{\mathsf{b}\beta)}\} &= -\varepsilon^{\dot\gamma\dot\alpha}(\sigma_{\beta\dot\gamma})^{\mu}\delta^{\mathsf{a}}_{\mathsf{b}}\mathbb{A}_{\mu} + \{\overline{\mathbb{A}}^{\mathsf{a}\dot\alpha}, \mathbb{A}_{\mathsf{b}\beta}\}\\
\{D^{(\mathsf{a}\dot\alpha}, \overline{\mathbb{A}}^{\mathsf{b}\dot\beta)}\} &= \varepsilon^{\dot\alpha\dot\beta}\Phi_{\mathsf{ab}} + \{\overline{\mathbb{A}}^{\mathsf{a}\dot\alpha}, \overline{\mathbb{A}}^{\mathsf{b}\dot\beta}\},
\end{aligned}
\tag{13.30}
$$

these can then be packaged up thanks to the Harnd-Shnider gauge to give,

$$
(\mathcal{D}+1)\mathbb{A}_{\mathsf{a}\alpha} = \theta^{\mathsf{b}}_{\alpha}\Phi_{\mathsf{ba}} - (\sigma^{\mu}\overline{\theta})_{\mathsf{a}\alpha}\mathbb{A}_{\mu}, \qquad (\mathcal{D}+1)\overline{\mathbb{A}}^{\mathsf{a}\dot\alpha} = \theta^{\dot\alpha}_{\mathsf{b}}\Phi^{\mathsf{ba}} - (\theta\sigma^{\mu})^{\mathsf{a}\dot\alpha}\mathbb{A}_{\mu}, \tag{13.31}
$$

where $(\sigma^{\mu}\overline{\theta})_{\mathsf{a}\alpha} = (\sigma_{\alpha\dot\beta})^{\mu}\overline{\theta}^{\dot\beta}_{\mathsf{a}}$ and similarly for $(\theta\sigma^{\mu})^{\mathsf{a}\dot\alpha}$. From here we can deduce the recursion relations for $\mathbb{A}_{\mathsf{a}\alpha}$ and $\mathbb{A}^{\mathsf{a}\dot\alpha}$,

$$
\mathbb{A}_{\mathsf{a}\alpha}|_k = \frac{1}{k+1}\big(\theta^{\mathsf{b}}_{\alpha}\Phi_{\mathsf{ba}}|_{k-1} - (\sigma^{\mu}\overline{\theta})_{\mathsf{a}\alpha}\mathbb{A}_{\mu}|_{k-1}\big), \quad \overline{\mathbb{A}}^{\mathsf{a}\dot\alpha}|_k = \frac{1}{k+1}\big(\theta^{\dot\alpha}_{\mathsf{b}}\Phi^{\mathsf{ba}}|_{k-1} - (\theta\sigma^{\mu})^{\mathsf{a}\dot\alpha}\mathbb{A}_{\mu}|_{k-1}\big), \tag{13.32}
$$

a quick check against the $k=1$ component of the $D=10$ $\theta-$expansion [7] ensures we have the correct signs. The equations of motion for $\mathbb{W}$ are given by,

$$
\begin{aligned}
\{D_{\mathsf{a}\alpha}, \mathbb{W}^{\mathsf{b}\beta}\} &= \frac{1}{4}(\sigma^{\mu\nu})_{\alpha}{}^{\beta}\delta^{\mathsf{b}}_{\mathsf{a}}\mathbb{F}_{\mu\nu} + \frac{1}{2}\delta^{\beta}_{\alpha}[\Phi_{\mathsf{ac}}, \Phi^{\mathsf{cb}}] + \{\mathbb{A}_{\mathsf{a}\alpha}, \mathbb{W}^{\mathsf{b}\beta}\},\\
\{D^{\mathsf{a}\dot\alpha}, \overline{\mathbb{W}}_{\mathsf{b}\dot\beta}\} &= -\frac{1}{4}(\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\delta^{\mathsf{a}}_{\mathsf{b}}\mathbb{F}_{\mu\nu} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}[\Phi^{\mathsf{ac}}, \Phi_{\mathsf{cb}}] + \{\overline{\mathbb{A}}^{\mathsf{a}\dot\alpha}, \overline{\mathbb{W}}_{\mathsf{b}\dot\beta}\},\\
\{D_{\mathsf{a}\alpha}, \overline{\mathbb{W}}_{\mathsf{b}\dot\beta}\} &= \frac{1}{2}(\sigma_{\mu})_{\alpha\dot\beta}\Phi^{\mu}_{\mathsf{ab}} + \{\mathbb{A}_{\mathsf{a}\alpha}, \overline{\mathbb{W}}_{\mathsf{b}\dot\beta}\},\\
\{D^{\mathsf{a}\dot\alpha}, \mathbb{W}^{\mathsf{b}\beta}\} &= -\frac{1}{2}(\sigma_{\mu})^{\dot\alpha\beta}\Phi^{\mu}_{\mathsf{ab}} + \{\overline{\mathbb{A}}^{\mathsf{a}\dot\alpha}, \mathbb{W}^{\mathsf{b}\beta}\},
\end{aligned}
\tag{13.33}
$$

where we have defined $\mathbb{F}^{\mu m} = [\nabla^{\mu}, \Phi^{m}] = \Phi^{\mu|m}$ and applied the $(\sigma^{m})_{\mathsf{ab}}\Phi_{m} = \Phi_{\mathsf{ab}}$ identification to the scalar field. Now these can then be combined in Harnad-Shnider gauge to give,

$$
\begin{aligned}
\mathcal{D}\mathbb{W}^{\mathsf{a}\alpha} &= \frac{1}{4}(\theta\sigma^{\mu\nu})^{\mathsf{a}\alpha}\mathbb{F}_{\mu\nu} + \frac{1}{2}\theta^{\alpha\mathsf{b}}[\Phi_{\mathsf{bc}}, \Phi^{\mathsf{ca}}] + \frac{1}{2}(\sigma_{\mu}\overline{\theta})_{\mathsf{b}\alpha}\Phi^{\mu|\mathsf{ba}},\\
\mathcal{D}\overline{\mathbb{W}}_{\mathsf{a}\dot\alpha} &= \frac{1}{4}(\overline{\theta}\sigma^{\mu\nu})_{\mathsf{a}\dot\alpha}\mathbb{F}_{\mu\nu} + \frac{1}{2}\overline{\theta}_{\mathsf{b}\dot\alpha}[\Phi^{\mathsf{bc}}, \Phi_{\mathsf{ca}}] + \frac{1}{2}(\theta\sigma^{\mu})^{\mathsf{b}\dot\alpha}\Phi_{\mu|\mathsf{ba}}
\end{aligned}
\tag{13.34}
$$

and as a result,

$$
\begin{aligned}
\mathbb{W}^{\mathsf{a}\alpha}|_k &= \frac{1}{4k}(\theta\sigma^{\mu\nu})^{\mathsf{a}\alpha}\mathbb{F}_{\mu\nu}|_{k-1} + \frac{1}{2k}\theta^\beta_{\mathsf{a}}\sum_{l=0}^{k-1}[\Phi_{\mathsf{bc}}|_l, \Phi^{\mathsf{ca}}|_{k-l-1}] + \frac{1}{2k}(\sigma_\mu\bar{\theta})_{\mathsf{b}\alpha}\Phi^{\mu|\mathsf{ba}}|_{k-1},\\
\overline{\mathbb{W}}_{\mathsf{a}\dot{\alpha}}|_k &= \frac{1}{4k}(\bar{\theta}\sigma^{\mu\nu})_{\mathsf{a}\dot{\alpha}}\mathbb{F}_{\mu\nu}|_{k-1} + \frac{1}{2k}\bar{\theta}_{\mathsf{b}\dot{\alpha}}\sum_{l=0}^{k-1}[\Phi^{\mathsf{bc}}|_l, \Phi_{\mathsf{ca}}|_{k-l-1}] + \frac{1}{2k}(\theta\sigma^\mu)^{\mathsf{b}\dot{\alpha}}\Phi_{\mu|\mathsf{ba}}|_{k-1}.
\end{aligned}
\tag{13.35}
$$

Next we have the equations for $\mathbb{F}$ and $\Phi^{\mu|\mathsf{ab}}$,

$$
\begin{aligned}
[D_{\mathsf{a}\alpha}, \mathbb{F}^{\mu\nu}] &= -(\sigma^{[\mu}\overline{\mathbb{W}}^{\nu]})_{\mathsf{a}\alpha} + [\mathbb{A}_{\mathsf{a}\alpha}, \mathbb{F}^{\mu\nu}],\\
[D^{\mathsf{a}\dot{\alpha}}, \mathbb{F}^{\mu\nu}] &= -(\sigma^{[\mu}\mathbb{W}^{\nu]})^{\mathsf{a}\dot{\alpha}} + [\overline{\mathbb{A}}^{\mathsf{a}\dot{\alpha}}, \mathbb{F}^{\mu\nu}],\\
[D_{\mathsf{a}\alpha}, \Phi^\mu_{\mathsf{cd}}] &= 2\varepsilon_{\mathsf{abcd}}\mathbb{W}^{\mathsf{b}\mu}_\alpha + [(\sigma^\mu\overline{\mathbb{W}})_{\mathsf{a}\alpha}, \Phi_{\mathsf{cd}}] + [\mathbb{A}_{\mathsf{a}\alpha}, \Phi^\mu_{\mathsf{cd}}],\\
[D^{\mathsf{a}\dot{\alpha}}, \Phi^\mu_{\mathsf{cd}}] &= 2\delta^{\mathsf{a}}_{[\mathsf{d}|}\overline{\mathbb{W}}^{\dot{\alpha}\mu}_{|\mathsf{c}]} + [(\sigma^\mu\mathbb{W})^{\mathsf{a}\dot{\alpha}}, \Phi_{\mathsf{cd}}] + [\overline{\mathbb{A}}^{\mathsf{a}\dot{\alpha}}, \Phi^\mu_{\mathsf{cd}}].
\end{aligned}
\tag{13.36}
$$

One could have defined similar relations for $[\Phi_{ab}, \Phi_{cd}]$ using the $\mathbb{F}^{mn}$ equations of motion, however this would have been redundant as we can already defined $\Phi_{ab}$, furthermore the equation becomes the Jacobi identity once one eliminates $\Phi$ in favour of $\mathbb{W}$. Performing the usual trick we find that the Harnad-Shnider equations of motion are,

$$
\begin{aligned}
\mathcal{D}\mathbb{F}^{\mu\nu} &= (\theta\sigma^{[\mu}\overline{\mathbb{W}}^{\nu]}) - (\bar{\theta}\sigma^{[\mu}\mathbb{W}^{\nu]}),\\
\mathcal{D}\Phi^\mu_{\mathsf{cd}} &= 2\varepsilon_{\mathsf{abcd}}\langle\theta^{\mathsf{a}}\mathbb{W}^{\mathsf{b}\mu}\rangle + [(\theta\sigma^\mu\overline{\mathbb{W}}), \Phi_{\mathsf{cd}}] - 2[\bar{\theta}_{[\mathsf{d}|}\overline{\mathbb{W}}^\mu_{|\mathsf{c}]}] + [(\bar{\theta}\sigma^\mu\mathbb{W}), \Phi_{\mathsf{cd}}]
\end{aligned}
\tag{13.37}
$$

thence it follows as expected that,

$$
\begin{aligned}
\mathbb{F}^{\mu\nu}|_k &= \frac{1}{k}(\theta\sigma^{[\mu}\mathbb{W}^{\nu]})|_{k-1} - (\bar{\theta}\sigma^{[\mu}\mathbb{W}^{\nu]})|_{k-1},\\
\Phi^\mu_{\mathsf{ab}}|_k &= \frac{2}{k}\varepsilon_{\mathsf{abcd}}\langle\theta^{\mathsf{c}}\mathbb{W}^{\mathsf{d}\mu}\rangle|_{k-1} + \frac{2}{k}[\bar{\theta}_{[\mathsf{a}|}\overline{\mathbb{W}}^\mu_{|\mathsf{b}]}]|_{k-1}\\
&\quad + \frac{1}{k}\sum_{l=0}^{k-1}[(\theta\sigma^\mu\overline{\mathbb{W}})|_l, \Phi_{\mathsf{ab}}|_{k-l-1}] + \frac{1}{k}\sum_{l=0}^{k-1}[(\bar{\theta}\sigma^\mu\mathbb{W})|_l, \Phi_{\mathsf{ab}}|_{k-l-1}].
\end{aligned}
\tag{13.38}
$$

These may look intimidating a hard to handle, but for pragmatic purposes we only need the case where $k = 1$ for our work here as we only need up to $\theta^3$ in the pre-spinor potential expansion. For higher orders this would of course become very cumbersome to do by hand. Finally, for the work we consider in this paper we do not necessarily need the $\mathbb{W}^\mu$ equation of motion. However, we find them for completeness when compared to

[29], in four dimensions this yields,

$$
\begin{aligned}
\{D_{\mathsf{a}\alpha}, \overline{\mathbb{W}}^{\mu}_{\mathsf{b}\dot{\beta}}\} &= \frac{1}{2}(\sigma_\nu)_{\alpha\dot{\beta}}\Phi^{\mu\nu}_{\mathsf{ab}} + \{(\sigma^\mu\overline{\mathbb{W}})_{\mathsf{a}\alpha}, \overline{\mathbb{W}}_{\mathsf{b}\dot{\beta}}\} + \{\mathbb{A}_{\mathsf{a}\alpha}, \overline{\mathbb{W}}^{\mu}_{\mathsf{b}\dot{\beta}}\},\\
\{D^{\mathsf{a}\dot{\alpha}}, \mathbb{W}^{\mu\mathsf{b}\beta}\} &= -\frac{1}{2}(\sigma_\nu)^{\dot{\alpha}\beta}\Phi^{\mu\nu|\mathsf{ab}} - \{(\sigma^\mu\mathbb{W})^{\mathsf{a}\dot{\alpha}}, \mathbb{W}^{\mathsf{b}\beta}\} + \{\overline{\mathbb{A}}^{\mathsf{a}\dot{\alpha}}, \mathbb{W}^{\mu\mathsf{b}\beta}\},\\
\{D_{\mathsf{a}\alpha}, \mathbb{W}^{\mu\mathsf{b}\beta}\} &= \frac{1}{4}(\sigma_{\rho\sigma})_\alpha{}^\beta\delta^{\mathsf{b}}_{\mathsf{a}}\mathbb{F}^{\mu|\rho\sigma} + \frac{1}{2}\delta^\beta_\alpha\big([\Phi^\mu_{\mathsf{ac}}, \Phi^{\mathsf{cb}}] + [\Phi_{\mathsf{ac}}, \Phi^{\mu|\mathsf{cb}}]\big)\\
&\quad + \{(\sigma^\mu\overline{\mathbb{W}})_{\mathsf{a}\alpha}, \mathbb{W}^{\mathsf{b}\beta}\} + \{\mathbb{A}_{\mathsf{a}\alpha}, \mathbb{W}^{\mu\mathsf{b}\beta}\},\\
\{D^{\mathsf{a}\dot{\alpha}}, \mathbb{W}^{\mu}_{\mathsf{b}\dot{\beta}}\} &= -\frac{1}{4}(\sigma_{\rho\sigma})^{\dot{\alpha}}{}_{\dot{\beta}}\delta^{\mathsf{a}}_{\mathsf{b}}\mathbb{F}^{\mu|\rho\sigma} - \frac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\big([\Phi^{\mu|\mathsf{ac}}, \Phi_{\mathsf{cb}}] + [\Phi^{\mathsf{ac}}, \Phi^\mu_{\mathsf{cb}}]\big)\\
&\quad - \{(\sigma^\mu\mathbb{W})^{\mathsf{a}\dot{\alpha}}, \overline{\mathbb{W}}_{\mathsf{b}\dot{\beta}}\} + \{\overline{\mathbb{A}}^{\mathsf{a}\dot{\alpha}}, \overline{\mathbb{W}}^{\mu}_{\mathsf{b}\dot{\beta}}\},
\end{aligned}
\tag{13.39}
$$

which in Harnad-Shnider gauge can be collected to give,

$$
\begin{aligned}
\mathcal{D}\mathbb{W}^{\mu\mathsf{a}\alpha} = {}& \frac{1}{4}(\theta\sigma_{\rho\sigma})^{\mathsf{a}\alpha}\mathbb{F}^{\mu|\rho\sigma} + \frac{1}{2}\theta^{\mathsf{b}\alpha}\big([\Phi^\mu_{\mathsf{bc}}, \Phi^{\mathsf{ca}}] + [\Phi_{\mathsf{bc}}, \Phi^{\mu|\mathsf{ca}}]\big) + \frac{1}{2}(\sigma_\nu\overline{\theta})^{\mathsf{b}\alpha}\Phi^{\mu\nu}_{\mathsf{ba}}\\
&+ \{(\theta\sigma^\mu\overline{\mathbb{W}}), \mathbb{W}^{\mathsf{a}\alpha}\} - \{(\overline{\theta}\sigma^\mu\mathbb{W}), \mathbb{W}^{\mathsf{a}\alpha}\},
\end{aligned}
\tag{13.40}
$$

the analogous expression for $\mathbb{W}^{\mu}_{\mathsf{a}\dot{\alpha}}$ is identical witht he relevant index changes. From these equations we can then define the $k^{th}$ order $\theta$ expansion for the higher mass $\mathbb{W}$ field, which we do not display here simply for brevity. Note that, since $\mathbb{W}^{m\alpha} = [\mathbb{W}^\alpha, \Phi^m]$, we do not need to consider the $\mathbb{W}^{m\alpha}$ equation.

Before we move on to giving the $\theta$ expansion we will require to find the $D=4$, $\mathcal{N}=4$ BRST building blocks we say a brief few words about gauge invariance of the equations here. It follows from the non-linear gauge transformations that the gauge transformations of most of the fields follow trivially, the only one that requires a little calculation is the $\Phi_{\mathsf{ab}}$ field, whose transform is given simply by,

$$
\delta_\Omega\Phi_{\mathsf{ab}} = [\Omega, \Phi_{\mathsf{ab}}],
\tag{13.41}
$$

using this and the gauge transforms that follow from the $D=10$ case it is straightforward, if somewhat tedious, to show that these equations are gauge invariant - not a surprising result but a comforting one.

**Pre-Spinor Potential Expansion**

We can now turn to the expansions we require to find the BRST building blocks, these expansions are only needed up to fourth order in $\theta$ since $\mathcal{A}_{\alpha/\dot{\alpha}}$ starts at order $\theta^1$. The expansion for the spinor pre-potential MPSF up to $\theta^3$ is given by,

$$\begin{aligned}\mathcal{A}^P_{\mathsf{a}\alpha} =& -(\sigma^\mu\overline{\theta})_{\mathsf{a}\alpha}\mathfrak{e}^P_\mu - \mathfrak{v}^P_{\mathsf{ab}}\theta^{\mathsf{b}}_\alpha + \frac{2}{3}\varepsilon_{\mathsf{bacd}}\theta^{\mathsf{b}}_\alpha\langle\theta^{\mathsf{c}}\mathfrak{g}^{\mathsf{d}P}\rangle - \frac{1}{3}\theta^{\mathsf{b}}_\alpha[\overline{\theta}_{(\mathsf{a}|}\overline{\mathfrak{g}}^P_{|\mathsf{b})}] + \frac{2}{3}\mathfrak{g}^{\mathsf{b}P}_\alpha[\overline{\theta}_{\mathsf{a}}\overline{\theta}_{\mathsf{b}}]\\ &+ \frac{1}{16}\big[(\overline{\theta}_{[\mathsf{a}|}\sigma^{\mu\nu}\overline{\theta}_{|\mathsf{b}]}) - \varepsilon_{\mathsf{abcd}}(\theta^{\mathsf{d}}\sigma^{\mu\nu}\theta^{\mathsf{c}})\big]\theta^{\mathsf{b}}_\alpha\mathfrak{f}^P_{\mu\nu} - \frac{1}{32}(\sigma_\rho\overline{\theta})_{\mathsf{a}\alpha}(\theta^{\mathsf{b}}\sigma^{\rho\mu\nu}\overline{\theta}_{\mathsf{b}})\mathfrak{f}^P_{\mu\nu} + ...,\end{aligned} \tag{13.42}$$

where $\mathcal{A}^{P\dot{\alpha}}_{\mathsf{a}}$ is given by the hermitian conjugate of this; $[\mathsf{a}|...|\mathsf{b}] = \mathsf{ab} - \mathsf{ba}$ and $(\mathsf{a}|...|\mathsf{b}) = \mathsf{ab} + \mathsf{ba}$ for the indices with no weight. One can quite easily check that the expansion given here is by the dimensional reduction of the $\theta-$expansion - of course this is not unexpected but it is again comforting to see that our work is consistent.

**Berends-Giele Current Equations**

Having compactified the non-linear equations of motion it is worth looking at the Berends-Giele recursion relations and their equations of motion à la [29]; note that here and in the next subsection we work in Lorenz gauge. These relations are important since, first of all the recursion relations will define the supersymmetric generalisation of the Berends-Giele currents in the lower dimensional theory [98]; and secondly the equations of motion are important to show the BRST invariance of objects in field and string theory [36, 41, 127]. The Berends-Giele recursions are given in $D = 4$ by,

$$\begin{aligned}\mathcal{A}^{[P,Q]}_{\mathsf{a}\alpha} &\equiv -\frac{1}{2}[\mathcal{A}^P_{\mathsf{a}\alpha}(k^P\cdot\mathcal{A}^Q) - \mathcal{A}^P_\mu(\sigma^\mu\overline{\mathcal{W}})_{\mathsf{a}\alpha} + \varphi^P_{\mathsf{ab}}\mathcal{W}^{Q\mathsf{b}}_\alpha - (P\leftrightarrow Q)],\\ \mathcal{A}^{[P,Q]}_\mu &\equiv -\frac{1}{2}[\mathcal{A}^P_\mu(k^P\cdot\mathcal{A}^Q) + \mathcal{A}^{P\nu}\mathcal{F}^Q_{\mu\nu} + \varphi^P_{\mathsf{ab}}\varphi^{Q\mathsf{ab}}_\mu + (\mathcal{W}^P\sigma^\mu\overline{\mathcal{W}}^Q)\\ &\qquad + (\overline{\mathcal{W}}^Q\sigma^\mu\mathcal{W}^P) - (P\leftrightarrow Q)],\\ \varphi^{[P,Q]}_{\mathsf{ab}} &\equiv -\frac{1}{2}\big[\varphi^P_{\mathsf{ab}}(k^p\cdot\mathcal{A}^Q) + \mathcal{A}^{P\mu}\varphi^Q_{\mu|\mathsf{ab}} + \varphi^P_{\mathsf{cd}}[\varphi^Q_{\mathsf{ab}},\varphi^{Q\mathsf{cd}}]\\ &\qquad - 2\varepsilon_{\mathsf{abcd}}\langle\mathcal{W}^{P\mathsf{c}}\mathcal{W}^{Q\mathsf{d}}\rangle + 2[\overline{\mathcal{W}}^P_{[\mathsf{a}|}\overline{\mathcal{W}}^Q_{|\mathsf{b}]}] - (P\leftrightarrow Q)\big],\\ \mathcal{W}^{\mathsf{a}\alpha}_{[P,Q]} &\equiv \frac{1}{2}(k^\mu_P + k^\mu_Q)(\sigma_\mu)^{\alpha\dot{\alpha}}[\mathcal{A}^\nu_P(\sigma_\nu\mathcal{W})^{\mathsf{a}}_{\dot{\alpha}} + \varphi^{\mathsf{ab}}_P\overline{\mathcal{W}}^Q_{\mathsf{b}\dot{\alpha}} - (P\leftrightarrow Q)],\\ \mathcal{F}^{\mu\nu}_P &\equiv k^\mu_P\mathcal{A}^\nu_P - k^\nu_P\mathcal{A}^\mu_P - \sum_{XY=P}(\mathcal{A}^\mu_X\mathcal{A}^\nu_Y - \mathcal{A}^\nu_X\mathcal{A}^\mu_Y),\end{aligned} \tag{13.43}$$

where $\varphi_{\mathsf{ab}}$ is the four-dimensional Berends-Giele current for the scalar fields. Following [29, 37] there are other representations for the last two currents given by,

$$\begin{aligned}
\mathcal{W}^{\mathsf{a}\alpha}_{[P,Q]} &= -\frac{1}{2}\big[\mathcal{W}^{\mathsf{a}\alpha}_P(k_p\cdot\mathcal{A}_Q) + \mathcal{W}^{\mu|\mathsf{a}\alpha}_P\mathcal{A}_{Q\mu} + [\mathcal{W}^{\mathsf{a}\alpha}_P,\varphi^P_{\mathsf{cd}}]\varphi^{\mathsf{cd}}_Q + \tfrac{1}{2}(\sigma_{\mu\nu}\mathcal{W})^{\mathsf{a}\alpha}_P\mathcal{F}^{\mu\nu}_Q \\
&\quad + \tfrac{1}{2}(\sigma_\mu\overline{\mathcal{W}})^{\alpha}_{P\mathsf{b}}\varphi^{Q\mu|\mathsf{ab}} + \tfrac{1}{2}\mathcal{W}^{\mathsf{b}\alpha}_P[\varphi^{\mathsf{ac}}_Q,\varphi_{Q\mathsf{cb}}] - (P\leftrightarrow Q)\big], \\
\mathcal{F}^{\mu\nu}_{[P,Q]} &= -\frac{1}{2}\big[\mathcal{F}^{\mu\nu}_P(k_P\cdot\mathcal{A}_Q) + \mathcal{F}^{\rho|\mu\nu}_P\mathcal{A}^Q_\rho + 2\mathcal{F}^{\mu\rho}_P\mathcal{F}^{\nu}_{Q\rho} + 2[\mathcal{F}^{\mu\nu}_P,\varphi^{\mathsf{ab}}_P]\varphi^Q_{\mathsf{ab}} \\
&\quad + 2\varphi^{\mu|\mathsf{ab}}_P\varphi^{\nu}_{Q\mathsf{ab}} + 4(\mathcal{W}^{[\mu}_P\sigma^{\nu]}\overline{\mathcal{W}}_Q) + 4(\overline{\mathcal{W}}^{[\mu}_P\sigma^{\nu]}\mathcal{W}_Q) - (P\leftrightarrow Q)\big],
\end{aligned} \tag{13.44}$$

which appear more complicated but as we shall give rise to a much nicer generating series wave equation. The higher mass Berends-Giele currents are analogously defined such that,

$$\begin{aligned}
\varphi^{\mu|\mathsf{ab}}_P &\equiv k^\mu_P\varphi^{\mathsf{ab}}_P - \sum_{XY=P}(\mathcal{A}^\mu_X\varphi^{\mathsf{ab}}_Y - \varphi^{\mathsf{ab}}_X\mathcal{A}^\mu_Y), \\
\mathcal{W}^{\mu|\mathsf{a}\alpha}_P &\equiv k^\mu_P\mathcal{W}^{\mathsf{a}\alpha}_P + \sum_{XY=P}(\mathcal{W}^{\mathsf{a}\alpha}_X\mathcal{A}^\mu_Y - \mathcal{W}^{\mathsf{a}\alpha}_Y\mathcal{A}^\mu_X), \\
\mathcal{F}^{\mu|\rho\sigma}_P &\equiv k^\mu_P\mathcal{F}^{\rho\sigma}_P + \sum_{XY=P}(\mathcal{F}^{\rho\sigma}_X\mathcal{A}^\nu_Y - \mathcal{F}^{\rho\sigma}_Y\mathcal{A}^\mu_Y),
\end{aligned} \tag{13.45}$$

where in both of the above equations the opposite chirality currents for $\mathcal{W}$ and $\mathcal{A}$ can be found by taking the Hermitian conjugate of the above results. These currents also define the deconcatenation of higher point polarisation tensors which will be used in order to construct amplitudes from the BRST building blocks. Such polarisation tensors are given by the analogous formula to the $D = 10$ case,

$$\mathfrak{q}^\mu_P = \frac{1}{s_P}\sum_{XY=P}\mathfrak{q}^\mu_{[X,Y]}, \tag{13.46}$$

where we use $\mathfrak{q}$ to denote a general polarisation tensor and $\mu$ to denote a general index. These can easily be read off from (13.43), for example the $\mathfrak{g}^{\mathsf{a}\alpha}_{[X,Y]}$ is given by,

$$\mathfrak{g}^{\mathsf{a}\alpha}_{[X,Y]} = -\frac{1}{2}[\mathfrak{g}^{\mathsf{a}\alpha}_X(k^X\cdot\mathfrak{e}^Y) - \mathfrak{e}^\mu_X(\sigma_\mu\overline{\mathfrak{g}})^{\mathsf{a}\alpha}_Y + \mathfrak{v}^{\mathsf{ab}}_X\mathfrak{g}^\alpha_{Y\mathsf{b}} - (X\leftrightarrow Y)], \tag{13.47}$$

it is this deconcatenation of polarisation vectors that will give rise to the correct pole structure in the amplitudes. This pole structure would also result from the non-local Berends-Giele currents in (13.43) if one were to use the traditional Berends-Giele procedure, however the pure spinor methodology allows us to define the BRST block in terms of polarisation tensors and then build amplitudes in a much simpler way.

Furthermore, in $D = 10$ these currents satisfy a number of relations, including equations of motion which are inductively satisfied as well as a Lorentz gauge condition [29]. We do not show these here as they can be inferred from the non-linear equations of motion found in this subsection (not in Harnad-Shnider gauge) and since these currents satisfy the equations in $D = 10$ it is expected that the analogous $D = 4$ currents found from torodial compactification will also satisfy the equations of motion. We do however note the Lorentz gauge conditions on the currents which follow from (2.20-2.22) in [29],

$$\begin{aligned}
k^P_\mu \mathcal{A}^\mu_P &= 0, \\
k^P_\mu (\sigma^\mu \mathcal{W}^P)^{\mathsf{a}}_{\dot\alpha} &= \sum_{XY=P} \big[\mathcal{A}^X_\mu (\sigma^\mu \mathcal{W}^Y)^{\mathsf{a}}_{\dot\alpha} - \mathcal{A}^Y_\mu (\sigma^\mu \mathcal{W}^X)^{\mathsf{a}}_{\dot\alpha} + \varphi^{X\mathsf{ab}} \overline{\mathcal{W}}^Y_{\mathsf{b}\dot\alpha} - \varphi^{Y\mathsf{ab}} \overline{\mathcal{W}}^X_{\mathsf{b}\dot\alpha}\big], \\
k^P_\mu \mathcal{F}^{\mu\nu}_P &= \sum_{XY=P} \big[2(\mathcal{W}_Y \sigma^\nu \overline{\mathcal{W}}_X) - 2(\overline{\mathcal{W}}_Y \sigma^\nu \mathcal{W}_X) + \mathcal{A}^X_\mu \mathcal{F}^{\mu\nu}_Y - \mathcal{A}^Y_\mu \mathcal{F}^{\mu\nu}_X \\
&\qquad + \varphi^Y_{\mathsf{ab}} \varphi^{\nu|\mathsf{ab}}_X - \varphi^X_{\mathsf{ab}} \varphi^{\nu|\mathsf{ab}}_Y\big], \\
k^P_\mu \varphi^{\mu|\mathsf{ab}}_P &= \sum_{XY=P} \big[4\varepsilon^{\mathsf{abcd}} [\overline{\mathcal{W}}^X_{\mathsf{c}} \overline{\mathcal{W}}^Y_{\mathsf{d}}] - 4\langle \mathcal{W}^{[\mathsf{a}|}_X \mathcal{W}^{|\mathsf{b}]}_Y \rangle + \mathcal{A}^X_\mu \varphi^{\mu|\mathsf{ab}}_Y - \mathcal{A}^Y_\mu \varphi^{\mu|\mathsf{ab}}_X \\
&\qquad + \varphi^X_{\mathsf{cd}} [\varphi^{\mathsf{cd}}_Y, \varphi^{\mathsf{ab}}_Y] - \varphi^Y_{\mathsf{cd}} [\varphi^{\mathsf{cd}}_X, \varphi^{\mathsf{ab}}_X]\big],
\end{aligned} \tag{13.48}$$

which are particularly cumbersome. Luckily we will not explicitly use them in this work as we shall rely on the results from the higher dimensional theory being preserved where needs be. That being said we will need the transversailty of the gluon polarisation and the Dirac equation in order to find the correct amplitudes, these can be easily read off in much the same way as the deconcatenation of the polarisation tensors above. That is one obtains the Lorentz condition $k^P_\mu \mathfrak{e}^\mu_P = 0$ as well as,

$$k^P_\mu (\sigma^\mu \mathfrak{g}^P)^{\mathsf{a}}_{\dot\alpha} = \sum_{XY=P} \big[2\mathfrak{e}^X_\mu (\sigma^\mu \mathfrak{g}^Y)^{\mathsf{a}}_{\dot\alpha} - 2\mathfrak{e}^Y_\mu (\sigma^\mu \mathfrak{g}^X)^{\mathsf{a}}_{\dot\alpha} + \mathfrak{v}^{X\mathsf{ab}} \bar{\mathfrak{g}}^Y_{\mathsf{b}\dot\alpha} - \mathfrak{v}^{Y\mathsf{ab}} \bar{\mathfrak{g}}^X_{\mathsf{b}\dot\alpha}\big]. \tag{13.49}$$

The opposite chirality equations are easy to infer from the above and so we do not explicitly give these equations for any of the fields. It is useful to see that the equations laid out in this section are the supersymmeterised Berends-Giele currents for $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory and as a consequence these results are somewhat novel (although of course they were simply hiding in the higher dimensional theory all along).

This is all we wish to say in reference to the formalities of the Berends-Giele currents; such currents underpin much of the work presented in this paper however owing to the work carried out for the higher dimensional theory we need not spend too much time analysing such currents. Nevertheless, it is important to press the fact that these currents are really at the heart of the methodlogy laid out here, and elsewhere, and without such currents the generating series approach to Super Yang-Mills amplitudes would not be possible.

**The Non-Linear Wave Equations**

One can actually find the generating series of (13.43) in terms of non-linear wave equations for the full non-linear fields, noting that $\Box\mathbb{K} = [\partial_m, [\partial^m, \mathbb{K}]]$ one can infer from the first four equations in (13.43) that the non-linear wave equations are,

$$\begin{aligned}
\Box\mathbb{A}_{\mathsf{a}\alpha} =& \big[\mathbb{A}_\mu, [\partial^\mu, \mathbb{A}_{\mathsf{a}\alpha}]\big] - \big[(\sigma^\mu\overline{\mathbb{W}})_{\mathsf{a}\alpha}, \mathbb{A}_\mu\big] + \big[\mathbb{W}^{\mathsf{b}}_\alpha, \Phi_{\mathsf{ab}}\big],\\
\Box\mathbb{A}_\mu =& \big[\mathbb{A}_\mu, [\partial^\mu, \mathbb{A}^\nu]\big] + \big[\mathbb{F}^{\mu\nu}, \mathbb{A}_\nu\big] + \big[\Phi_{\mu|\mathsf{ab}}, \Phi^{\mathsf{ab}}\big] + 2\sigma^\mu_{\alpha\dot\alpha}\{\mathbb{W}^\alpha, \overline{\mathbb{W}}^{\dot\beta}\},\\
\Box\Phi_{\mathsf{ab}} =& \big[\mathbb{A}_\mu, [\partial^\mu, \Phi_{\mathsf{ab}}]\big] + \big[\Phi_{\mu|\mathsf{ab}}, \mathbb{A}^\mu\big] - \big[\Phi^{\mathsf{cd}}, [\Phi_{\mathsf{cd}}, \Phi_{\mathsf{ab}}]\big]\\
&+ 2\varepsilon_{\mathsf{abcd}}\{\mathbb{W}^{\mathsf{c}}_\alpha, \mathbb{W}^{\mathsf{d}\alpha}\} - 2\{\overline{\mathbb{W}}^{\dot\alpha}_{[\mathsf{a}|}, \overline{\mathbb{W}}_{|\mathsf{b}]\dot\alpha}\}.
\end{aligned} \tag{13.50}$$

Using the alternative definitions in (13.44) we find more explicit wave equations for $\mathbb{W}$ and $\mathbb{F}$,

$$\begin{aligned}
\Box\mathbb{W}^{\mathsf{a}\alpha} =& \big[\mathbb{A}_\mu, [\partial^\mu, \mathbb{W}^{\mathsf{a}\alpha}]\big] + \big[\mathbb{A}_\mu, \mathbb{W}^{\mu|\mathsf{a}\alpha}\big] - \big[\Phi^{\mathsf{bc}}, [\Phi_{\mathsf{bc}}, \mathbb{W}^{\mathsf{a}\alpha}]\big] + \big[\mathbb{F}^{\mu\nu}, (\sigma_{\mu\nu}\mathbb{W})^{\mathsf{a}\alpha}\big]\\
&+ \big[\Phi^{\mu|\mathsf{ab}}, (\sigma_\mu\overline{\mathbb{W}})^\alpha_{\mathsf{b}}\big] + \big[[\Phi^{\mathsf{ac}}, \Phi_{\mathsf{cb}}], \mathbb{W}^{\mathsf{b}\alpha}\big],\\
\Box\mathbb{F}^{\mu\nu} =& \big[\mathbb{A}_\rho, [\partial^\rho, \mathbb{F}^{\mu\nu}]\big] + \big[\mathbb{A}_\rho, \mathbb{F}^{\rho|\mu\nu}\big] - \big[\Phi_{\mathsf{ab}}, [\Phi^{\mathsf{ab}}, \mathbb{F}^{\mu\nu}]\big] + 2\big[\mathbb{F}^{\mu\rho}, \mathbb{F}_\rho{}^\nu\big]\\
&- 2\big[\Phi^{\mu|\mathsf{ab}}, \Phi^\nu_{\mathsf{ab}}\big] + 4\{(\mathbb{W}^{[\mu}\sigma^{\nu]}), \overline{\mathbb{W}}\} + 4\{(\overline{\mathbb{W}}^{[\mu}\sigma^{\nu]}), \mathbb{W}\}.
\end{aligned}$$

Deriving these equations can be done in one of three ways: first one can simply read of the equations from the Berends-Giele currents as we have done here; alternatively one can compactify the well-known $D = 10$ results using the methods laid out earlier in this section; or one can use the relations in Section 2.3.1 of [29] to find the wave equations from the general action of $\Box\mathbb{K}$. The equations laid out in this section have been checked in all three ways and all reproduce the same wave equations. The natural way to find this equations would be to begin with the non-linear equations of motion, derive the non-linear field equations, use the perturbiner solution and then find the Berends-Giele current relations of the previous section. Of course, pragmatically this makes no difference to the resulting calculations.

### 13.2.3 Amplitudes

In order to begin constructing the BRST building blocks of the SYM amplitudes we first need to specify the zero mode integration measures, for the $\mathcal{N} = 4$ theory these zero mode integration measures have already been found in [126]. There are 12 of these measures however, six of them are related by Hermitian conjugation of the spinors and so there are really only six independent measures. They are given by [126],

$$\begin{aligned}
&\varepsilon_{\mathsf{abcd}}[\overline{\lambda}_{\mathsf{e}}\overline{\lambda}_{\mathsf{f}}]\langle\lambda^{\mathsf{a}}\theta^{\mathsf{b}}\rangle\langle\theta^{\mathsf{e}}\theta^{\mathsf{c}}\rangle\langle\theta^{\mathsf{f}}\theta^{\mathsf{d}}\rangle, \quad [\overline{\lambda}_{\mathsf{a}}\overline{\lambda}_{\mathsf{b}}][\overline{\lambda}_{\mathsf{c}}\overline{\theta}_{\mathsf{d}}]\langle\theta^{\mathsf{d}}\theta^{\mathsf{a}}\rangle\langle\theta^{\mathsf{b}}\theta^{\mathsf{c}}\rangle,\\
&\varepsilon_{\mathsf{abcd}}[\overline{\lambda}_{\mathsf{e}}\overline{\theta}_{\mathsf{f}}]\langle\lambda^{\mathsf{f}}\theta^{\mathsf{d}}\rangle\langle\lambda^{\mathsf{a}}\theta^{\mathsf{b}}\rangle\langle\theta^{\mathsf{e}}\theta^{\mathsf{c}}\rangle, \quad [\overline{\lambda}_{\mathsf{a}}\overline{\lambda}_{\mathsf{b}}]\langle\lambda^{\mathsf{c}}\theta^{\mathsf{a}}\rangle\langle\theta^{\mathsf{d}}\theta^{\mathsf{b}}\rangle[\overline{\theta}_{\mathsf{d}}\overline{\theta}_{\mathsf{c}}],\\
&\varepsilon_{\mathsf{abcd}}\langle\lambda^{\mathsf{e}}\theta^{\mathsf{d}}\rangle\langle\lambda^{\mathsf{f}}\theta^{\mathsf{c}}\rangle\langle\lambda^{\mathsf{a}}\theta^{\mathsf{b}}\rangle[\overline{\theta}_{\mathsf{e}}\overline{\theta}_{\mathsf{f}}], \quad [\overline{\lambda}_{\mathsf{a}}\overline{\theta}_{\mathsf{b}}][\overline{\lambda}_{\mathsf{c}}\overline{\theta}_{\mathsf{d}}]\langle\lambda^{\mathsf{b}}\theta^{\mathsf{d}}\rangle\langle\theta^{\mathsf{a}}\theta^{\mathsf{c}}\rangle,
\end{aligned} \tag{13.51}$$

along with their hermitian conjugates. The issue we need to tackle now is to define a consistent normalisation of these terms with respect to the $D = 10$ measure, we can do this by considering (13.13) and looking at the expansion of this in terms of lower dimensional indices. The generating series of the BRST blocks splits in a simple way to give,

$$\langle|\mathbb{VVV}|\rangle \to \langle|\mathbb{VVV}|\rangle - 3\langle|\mathbb{VV}\overline{\mathbb{V}}|\rangle + 3\langle|\mathbb{V}\overline{\mathbb{VV}}|\rangle - \langle|\overline{\mathbb{VVV}}|\rangle, \tag{13.52}$$

where $\langle||\rangle$ represents zero mode integration over the $\lambda$ and $\theta$. Now following [29] we can define the BRST building blocks by, $M^A = \langle\lambda^{\mathsf{a}}\mathcal{A}^A_{\mathsf{a}}\rangle$, and $\overline{M}_A = [\lambda_{\mathsf{a}}\mathcal{A}^{\mathsf{a}}_A]$ and then when we apply the four-dimensional rules to the generic building block we find a relation similar to (13.52),

$$\begin{aligned}
\langle A, B, C\rangle =& \big\langle\big|\langle\lambda^{\mathsf{a}}\mathcal{A}^A_{\mathsf{a}}\rangle\langle\lambda^{\mathsf{b}}\mathcal{A}^B_{\mathsf{b}}\rangle\langle\lambda^{\mathsf{c}}\mathcal{A}^C_{\mathsf{c}}\rangle - 3\langle\lambda^{\mathsf{a}}\mathcal{A}^A_{\mathsf{a}}\rangle\langle\lambda^{\mathsf{b}}\mathcal{A}^B_{\mathsf{b}}\rangle[\overline{\lambda}_{\mathsf{c}}\overline{\mathcal{A}}^{\mathsf{c}}_C]\\
&+ 3\langle\lambda^{\mathsf{a}}\mathcal{A}^A_{\mathsf{a}}\rangle[\overline{\lambda}_{\mathsf{b}}\overline{\mathcal{A}}^{\mathsf{b}}_B][\overline{\lambda}_{\mathsf{c}}\overline{\mathcal{A}}^{\mathsf{c}}_C] - [\overline{\lambda}_{\mathsf{a}}\overline{\mathcal{A}}^{\mathsf{a}}_A][\overline{\lambda}_{\mathsf{b}}\overline{\mathcal{A}}^{\mathsf{b}}_B][\overline{\lambda}_{\mathsf{c}}\overline{\mathcal{A}}^{\mathsf{c}}_C]\big|\big\rangle
\end{aligned} \tag{13.53}$$

Whilst this chapter has not presented anything novel, in the sense of a new interpretation, the consistency of the pure spinor machinery with known results in $D = 4$ is comforting. This represents a rather technical, but simple, demonstration of how one can compactify the pure spinor machinery in a simple way. It would be interesting to investigate how one can compactify the $D = 10$ pure spinor machinery on more interesting manifolds to see if one can develop some tools in those situations.

# Part IV

# Conclusions

# Chapter 14

# A Look Back

The aim of this thesis was to present a methodology for determining $\alpha'$ corrections to Super Yang-Mills in the pure spinor formalism of string theory. The underlying concept allowing one to determine the $\alpha'$-corrections, without resorting to lengthy string theory calculations, is the uniqueness of the BRST cohomology of the pure spinor BRST operator. This fact implies that any expression that is BRST closed under the action of the operator must be in the cohomology of the pure spinor and hence must represent some physical state. This fact has been used in numerous works in order to determine expressions in the pure spinor formalism [7, 41, 42, 105] - as well as other formalisms of string theory. In fact, such BRST methods were originally used to determine the physical states of string theory, see [19, 20]. However, finding the $\alpha'$ corrections to specific, $n-$point amplitudes was not the aim here. Rather, the aim was to determine the generating series expressions which generate the $\alpha'$ correction to any amplitude order $n$. That is, finding the $\alpha'^m$ generating series allows one to calculate the $m^{th}$ order correction to any $n-$point amplitude. It is the use of generating series that make the problem tackled here more intricate than simply looking for the corrections to a specific point amplitude. In order to do this, a number of novel identities and formulae were found - these were required in order to properly canonicalise many of the expressions found in this work. Furthermore, novel equations were found for higher mass fields that had previously not been handled - it is unclear whether the higher mass $\mathbb{V}$ and $\mathbb{A}^m$ fields will play a role in determining higher order corrections but it is suspected they may. In this section we shall review the key outcomes of the work in this thesis - highlighting novel results and current issues. The final section of this work will be dedicated to discussing potential ways forward and new avenues of research that may be pursued.

## 14.1 Part I

Part I of this work was a review of some aspects of the Ramond-Neveu Schwarz and Green-Schwarz formalisms of the superstring as well as *non-linear Super Yang-Mills* in the pure spinor formalism. Of course, all Super Yang-Mills theories are non-linear in the sense that they explicitly involve interactions between the fields, however this distinction is made here owing to the difference between the usual method for finding amplitudes in field theories [128, 129, 130, 131] and the methods used employed in the pure spinor formalism. In 'typical' amplitude calculation one linearises the equations of motion and then treats interactions as perturbations from the free state. It is this concept that leads to the notion of Feynman diagrams and the coupling constant expansion which is quintessential in quantum field theory calculations.

However, in the pure spinor formalism one can employ the perturbiner method [71, 72, 73, 74] which allows one to determine the generating series expression for amplitudes[1]. As such, working with these generating series means one never linearises the equations of motion and the full non-linear equations are worked with throughout the calculation. Whilst this saves time calculating amplitudes, once one has the generating series expression, it does mean one deals with the extra complexities that come with the non-linear fields. For example, the single particle fields can commute rather nicely with one another as there are no quadratic field terms in the linearised method. Conversely, in the generating series method, one has to retain all of the quadratic terms that arise due to the commutation relations - this makes canonicalizing expressions rather cumbersome and non-trivial. Fortunately, these manipulations can be handled rather elegantly and quickly in `FORM` [40].

[1] It should be noted that this is not special to the pure spinor formalism - the perturbiner methodology can be used for any field theory. It is the pure spinor superspace integration measure which makes this method so appealing. One simply has to play a game of finding generating series/perturbiner terms in the BRST cohomology with ghost number 3.

The action of the BRST operator on these non-linear fields in captured in the equations of motion, for example,

$$\{\nabla_\alpha, \mathbb{W}^\beta\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}^{mn}, \tag{14.1}$$

implies the following BRST variation for the gluino field,

$$\{Q, \mathbb{W}^\beta\} = \frac{1}{4}(\lambda\gamma^{mn})^\beta \mathbb{F}^{mn} + \{\mathbb{V}, \mathbb{W}^\beta\}, \tag{14.2}$$

where we recall $\mathbb{V} = \lambda^\alpha \mathbb{A}_\alpha$ and $Q = \lambda^\alpha D_\alpha$. Through the use of this equation, and its counterparts for other fields, it is then possible to determine the BRST variation of any expression. One can then apply the various other Dirac and Yang-Mills equations for the higher mass fields to simplify the outcome of the variation. From there one can then check BRST closure - that is check that the expression obeys $Q\phi = 0$ such that $\phi \neq Q\psi$. The first such expression found was the tree-level generating series, given by [29],

$$\mathrm{Tr}\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle, \tag{14.3}$$

from which the $n-$point amplitude can be determined.

In order to go from the generating series expression given above, one needs to note that such generating series are the sum of Lie algebra generators and *Berends-Giele* currents [29], summarised by (6.1) and it is these Berends-Giele currents which take us from the $n-$point generating series expression to the component expansion of the kinematic factor of the amplitude. These currents represent multiparticle states - they find their origins in the computation of operator product expansions in string theory. These currents are non-local in the sense that they have poles in the $s-$plane on-shell. It is the inclusion of this pole structure which makes such currents appealing - when one calculates a kinematic factor the pole structure is already manifest. At tree-level (see the above generating series expression) these currents allow one to collapse an entire $n-$point diagram down to a single 3-point vertex where each leg of the vertex is a multiparticle current. These multiparticle currents can then be broken down to the single particle expressions by using Harnad-Shnider gauge and the expressions in (6.17).

## 14.2 Part II

Part II of this thesis concerned the main body of original research in the form of ansatz building techniques; new higher mass identities used in canonicalisation of the BRST variation; the current results for $\alpha'$, $\alpha'^2$ and $\alpha'^3$; attempts to find higher mass corrections in the form $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$, where $\mathcal{O}$ is the Drinfeld associator [120, 121], and the current state of the $\alpha'^4$ correction research.

The method to find corrections, laid out in Part II of this work, was a relatively simple procedure. First one has to determine the dimension of the $\alpha'$ correction one wishes to find, this is given by the simple equation,

$$[\alpha'^n] = \frac{3}{2} + 2n, \tag{14.4}$$

and then using this dimension construct all possible combinations of fields that give the correct dimension and have ghost number three. Using this one can then build an ansatz which contains all possible terms that can appear at a given dimension - let the number of terms in a given ansatz be denoted by $p$. Then this ansatz is canonicalised using various higher mass identities to simplify it as far as possible, and put it into a consistent basis that allows the application of Jacobi identities. The BRST variation of this ansatz is then taken and set to zero. This yields $p$ linear equations which can be solved for the unknown coefficients in the ansatz. Once this is performed, the result is compared to known amplitudes in order to fix the last redundancies in the answer. This then yields a BRST closed expression which generates the $\alpha'^n$ corrections.

Generating an ansatz that captures all of the possible terms is a rather important step in determining the correction - missing any type of term can lead to an incorrect result of zero. Once one has constructed all possible combinations of fields that give the correct dimension and have ghost number three, one need not pay much attention to the matching of indices as this is taken care of with the Dynkin label decomposition. In order to ensure that the terms being considered exist in the $SO(10)$ group structure, a Dynkin label decomposition is performed to determine whether any scalars can be constructed out of the field combinations allowed by symmetry and ghost number. If there are scalars of a given set of fields then we need to determine all possible combinations of the order of the fields, as well as all possible contractions of the vector indices of the fields. Performing the former of these tasks is a combinatorial problem and was already taken care of in `FORM`, however the latter required some new ingenuity. Representing the vector indices as node of a graph one can use the `dd_` [40] function built into `FORM` to determine all possible graphs - that is all possible vector contractions. Applying simple rules, such as $\mathbb{F}^{mm} = 0$ and other equations, allows one to throw away many of the terms that are zero. It should be mentioned that we only look for terms order-by-order in the number of fields. For example, one could consider a term like $\mathbb{V}(\lambda\gamma^m\mathbb{W}^{mn})(\lambda\gamma^n\mathbb{W})$ which contributes to the ansatz at $\alpha'^2$. This three-field term can be written as a four-field term by use of the higher-mass Dirac equation (4.24). However, this term would be set to zero as the four-field term it generates is automatically generated by our four-field analysis. This prevents the repetition of terms causing longer run times for canonicalisation and BRST variation. Once such considerations are taken, the ansatz is formed and can be canonicalised into a given basis.

This raises a question: "What do we mean by basis?" Once the ansatz is generated, and indeed after variation, there are terms which naïvely have nothing to do with one another (specifically higher mass fields). However, by canonicalizing their vector indices one sees that they are actually related by Jacobi identities. This allows one to remove certain terms using the Jacobi identity hence reducing the time taken to perform the variation of the ansatz. This is what we mean by basis - it is a way of arranging the vector indices that allows us to make Jacobi identities manifest. The identities used to do this are outlined in Section 8.1 - such identities are novel.

There are, however, further pieces of the puzzle that we have neglected to concern ourselves with. Such a piece is the nature of the terms analysed when one perform a Dynkin label decomposition on higher mass fields. These decompositions only trouble themselves with analysing those parts of the field that are independent. As was seen in Section 8.2, only the symmetric, traceless part of the higher mass fields are independent. All the other 'bits' of the field can be expressed in terms of lower dimensional fields. Hence, one can build an ansatz out of the symmetric, traceless parts of the fields (as these are the only independent parts) and exploit such vector index symmetries to simplify the ansatz. Once one has done this, these higher mass, symmetric and traceless fields have to projected out. This is for computational ease - the BRST variations of the 'normal' fields are known and so can be easy computed. The tensor decompositions for $\mathbb{W}^{mn}$, $\mathbb{W}^{mnp}$ and $\mathbb{F}^{mn|pq}$ were first derived in this work and are given in Section 8.2. One can then canonicalise these terms in the manner outlined above. Furthermore, the pure spinor identities outlined in the main body of the work can be applied to drastically simplify many of the terms. This produces the simplest ansatz - as far as we can determine.

Once the ansatz for a given correction order is determined, projected and canonicalised one can then get down to the business of performing the variation. After this variation then one has to perform the canonicalisation procedure again in order to be sure the resulting linear equations are *correct*. By *correct*, we mean that we obtain a series of equations of the form,

$$f_i(x_j)\mathcal{S}_i(\mathbb{K}) = 0, \tag{14.5}$$

where $\mathcal{S}_i(\mathbb{K})$ is a function of the generating series *that is non-zero* and $f_i(x_j)$ is a linear function of the unknown ansatz constant $x_j$. It is the non-zero nature of $\mathcal{S}_i(\mathbb{K})$ that makes the set of linear equations correct. If $\mathcal{S}_i(\mathbb{K}) = 0$, due to a pure spinor identity for example, then one would incorrectly assume $f_i(x_j) = 0$ which may not be true. This would then cause issues when solving the linear system. Hence the post-variation canonicalisation ensures the linear equations are of the correct form.

The first correction we attempted in this manner was the $\alpha'$ correction. This is a particularly good check of the method as it is well known in the literature that there is no $\alpha'$ correction to Super Yang-Mills - thus if the method returns zero then the method has passed a slightly non-trivial test. One can in fact perform this analysis by hand, and this was done briefly in Section 9.1. The 'by hand' method simply employs the pure spinor identity and saturation of the vector indices to show that there is no way to form an ansatz at this level. Alternatively, one can use `LiE` to demonstrate that the potential terms do not generate $SO(10)$ scalars. Hence there are no $\alpha'$ corrections.

Working at the next order presents some novelty as the $\alpha'^2$ corrections have not been found in this manner previously. At this order, there are two sectors one can write terms down for: a sector containing $\mathbb{V}_1$ and a sector containing no $\mathbb{V}_1$ - yielding nine different expression seeds. Fortunately, when one investigates the Dynkin decomposition, it turns out that only two of the seeds can produce $SO(10)$ scalars. These seeds are in the $\mathbb{V}_1$ sector - hence there is no no-$\mathbb{V}_1$ sector. In fact, one finds that when the ansatz terms are found, there are only four independent scalars and this is then a problem that is tractable and can be done by hand. Once the variation is performed it turns out that there are two ways of expressing the $\alpha'^2$ correction. However, it turns out that one of these expressions was BRST exact and hence could not be a correct form of the correction. The expression of the generating series of $\alpha'^2$ corrections in Super Yang-Mills represents a novel result. In fact, this term can be used to form an effective action extension to the generating series action of Super Yang-Mills as seen in (9.21) and (9.22). Such an action can then be used to determine the effective equations of motion for the non-linear fields [116, 117, 118, 119].

The following order, $\alpha'^3$, represents a genuine application of all the machinery outlined in this work. At this order there was a distinct splitting between terms that contained a $\mathbb{V}_1$ and those that did not. In fact, one is able to generate an expression for this correction which contains no $\mathbb{V}_1$ fields and is much simpler than the correction involving $\mathbb{V}_1$. Performing the method above using the current `FORM` code produces the expression given in (10.1). This correction is completely novel and correctly reproduces the kinematic factors of the $n-$point $\alpha'^3$ correction. This fact been checked by manually comparing the component expansion of (10.1) to known results. Of course, owing to the uniqueness of the BRST cohomology at each dimension, this *must* be the $\alpha'^3$ correction. The method outlined in this work not only gives the correction, it also gives a series of BRST exact expressions which can be used to simplify the correction (or potentially other expressions in future research). This occurs because the test of BRST closure does not eliminate those terms/expressions that are BRST exact. Hence, when the method above is performed a series of BRST exact expressions are generated along with the BRST closed expression[2]. One can use of these BRST exact expressions, namely (10.2), to simplify

[2]When the component expansion the BRST exact pieces is performed the answer is zero at any point - this is a strong indication that these terms are BRST exact.

(10.1) and in doing so obtain (10.3) - the final form of the $\alpha'^3$ correction. One can also play this game with the no-$\mathbb{V}_1$ sector. The result of this analysis is much simpler and produces a generating series with just one term[3]. The two no-$\mathbb{V}$ sector expressions are connected to the kinematic factors of loop diagrams in Super Yang-Mills. For example, (10.6) is similar to the Mercedes star diagram and can generate the kinematic factors of 3-loop diagrams. The second expression, (10.8) replicates the form of the 2-loop double-box diagram. Note that by virtue of the BRST cohomology the $\mathbb{V}_1$ and no-$\mathbb{V}_1$ sectors are related by a BRST-exact piece. Explicitly, if we let the $\mathbb{V}_1$ correction be denoted by $\mathbb{X}$ and the no-$\mathbb{V}_1$ sector be denoted by $\mathbb{Y}$ then these are related by,

$$\mathbb{X} = \mathbb{Y} + \mathcal{Q}\Omega, \tag{14.6}$$

where $\Omega$ is some unknown generating series expression. Determining $\Omega$ follows the same procedure as previously, except now we wish to solve the equation $\mathcal{Q}\Omega = \mathbb{X} - \mathbb{Y}$. This is an ongoing piece of research and required the introduction of terms containing bare $\mathbb{A}^m$ fields. These $\mathbb{A}^m$ fields are required as they are the only terms that can induce mixing between the $\mathbb{V}_1$ and no-$\mathbb{V}_1$ sectors. Ultimately, we do not need to find $\Omega$ in order to prove the 'equality' of the $\mathbb{V}_1$ and no-$\mathbb{V}_1$ sectors as this is guaranteed by the cohomology - however it is useful for completeness.

A further line of research proposed in this work required the introduction of higher mass generalisations of the vertex operator $\mathbb{V}$ and the gluon field $\mathbb{A}^m$. The former fields $\mathbb{V}^M$ were required in order to begin work on a potential new way of expressing the $\alpha'$ corrections using the so-called Drinfeld associator [120, 121]. The latter fields, $\mathbb{A}^{M|n}$, were required to perform the component expansion of $\mathbb{V}^M$ in Harnad-Shnider gauge. The Drinfeld associator is a differential operator $\mathcal{O}$ such that, once applied to $\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$ it produces the $\alpha'$ corrections - allowing all corrections to be expressed as an operator acting on the fundamental interaction term. To begin work on this possibility, we attempted to find the $\alpha'^2$ correction in this form by generating an ansatz containing $\mathbb{V}^M$ where $M \in \{\emptyset, m, mn, mnp, mnpq\}$. In order to work with the ansatz method we of course needed to determine the BRST variation of such fields. These variations were

[3]There are two generating series in this sector, one can use either in order to compute component expansions.

first presented in Sections 11.1 and 11.2 - where the former are the variations for general $\mathbb{V}^M$ and the latter are variations for $\mathbb{V}^M$. When generating the ansatz, using Dynkin label decomposition, only the traceless symmetric part of these higher mass fields are identified and hence we again had to derive the projection of higher mass $\mathbb{V}^M$. Luckily, we were able to borrow some of the information from the decomposition of $\mathbb{W}^M$ and apply it there giving us the decomposition rather quickly. After finding this machinery it was rather simple to demonstrate that there was no expression of the form $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$ at $\alpha'$ - as expected. In order to test higher order corrections we first derived the necessary results in order to perform component expansions. Working in the Harnad-Shnider gauge we were able to derive the recursive expression for the $\theta-$expansion of $\mathbb{V}^M$, for any $M$, see (11.42). Owing to the presence of $\mathbb{A}^{M|n}$ in the $\theta-$expansion of (11.42) we also required the general $\mathbb{A}^{M|n}$ $\theta-$expansion, this is given in (11.49). Both of these expressions are novel and are presented here for the first time, along with the first few terms of the expansion. The expressions in (11.42) is an extension of (C.8) in [29], however (11.49) is completely novel and has been tested up to $M = mnpq$. So far we have been unable to find $\mathcal{O}\langle\mathbb{V}\mathbb{V}\mathbb{V}\rangle$ for $\alpha'^2$ corrections, however there are further avenues to be explored here and they are detailed in the next section.

The final section of Part II detailed the state of the derivation of the $\alpha'^4$ correction. At the moment, there are numerous impasses in determining the correction at this order. First, the ansatz for the $\alpha'^4$ correction contains thousands of terms making canonicalisation slow and difficult - for example, ensuring we have simplified $\gamma$ matrices as far as possible is an obstacle. Unless we are confident that the ansatz, and subsequent variation, are simplified as far as possible, we cannot be sure whether the linear equations resulting from the variation are correct. Furthermore, at this order there is no splitting between the $\mathbb{V}$ and no-$\mathbb{V}$ sectors, in fact there is a mixing between these sectors. This prompts us to speculate that the bare gluon field, $\mathbb{A}^m$, may have to be included in order to correctly mix $\mathbb{V}$ and no-$\mathbb{V}$ terms. Finally, when the ansatz we have is expanded into components, there are terms missing that appear in known kinematic factors [132]. This further prompts us to believe that there are extra fields we ought to include. This is issue seems to be connected to the potential factorisation of $\alpha'^2 \sim \alpha'^2 \times \alpha'^2$. This section closed with a brief introduction of a new method - partially based on the method in [77] - which is promising. Currently, this effective action method has been able to reproduce the $\alpha'^2$ corrections. More work is required to determine if it can be used to find corrections at higher orders.

## 14.3 Part III

Part III of this work presents the results of a study carried out in order to determine the outcome of the compactifying the machinery of previous sections on the torus. Whilst this is a conceptually simple task to perform, it allows one to apply some of the higher dimensional machinery to lower dimensions.

This part began with the compactification of the fields equations on $T^1$ in order to obtain the nine-dimensional theory - these were then used to determine the equations required for the $\theta-$expansions. These expansions could also be found by simply compactifying the $D = 10$ expansions. In order to calculate amplitudes in the aforementioned manner, one needs to know what the $D = 10$ pure spinor superspace integration measure compactifies to. Using $D = 10$ identities, the $D = 9$ integration measures were found, and by probing possible BRST ancestors of these terms we were able to show that they are in fact the correct terms. Following this, we worked to find the $D = 4$, $\mathcal{N} = 4$ theory from the $D = 10$ theory. After compactifying the pure spinor constraint (which now has two equations) as well as the BRST operator, we found the well known $D = 4$ equations of motion for Super Yang-Mills. From this, using Harnad-Shnider gauge, we were able to determine the $\theta-$expansions. Again, these could be found by simply compactifying the $D = 10$ expansions. Further results were found including the lower dimensional wave equations and Berends-Giele Currents. Finally, we briefly discussed how one could obtain the $D = 4$, $\mathcal{N} = 4$ amplitudes using the derived machinery. This exercise demonstrates that one can recover the expected theory[4] as well as the pieces required to find amplitudes in the $D = 4$ theory using $D = 10$ pure spinor superspace compactified on $T^6$.

[4]This fact is not particularly surprising but investigating the compactification of the theory would have been incomplete without it.

# Chapter 15

# A View Forward

This work has made progress in finding the $\alpha'$ corrections up to order $\alpha'^4$ - in fact only $\alpha'^4$ currently eludes us. However, the $\alpha'^4$ correction has not been found, and there are a number of reasons, detailed above, as to why this is the case. In this section we lay out some possible solutions to the issues facing the $\alpha'^4$ calculation. Furthermore, we offer some further comments on possible paths forward to find the operator $\mathcal{O}$ that allows one to express the $\alpha'^2$ correction as $\mathcal{O}\langle\mathbb{VVV}\rangle$ - presenting a possible new avenue of research into the Drinfeld Associator. Finally, we make some comments on other potential avenues of research that may be carried out in the future by other intrepid researchers.

## 15.1 The $\alpha'^4$ Correction

The $\alpha'^4$ correction presents a number of issues that have to be overcome before the correction can be found. These issues pose quite substantial problems - this is not unexpected owing to the size of the ansatz. Overcoming these issues will be no easy achievement, but once they are surmounted that machinery one can deploy to solve other similar problems will be greatly enhanced. Hence, it is very worthwhile to complete this problem.

The first and, practically speaking, the most important is to generalise the ansatz even further. The current form of the ansatz does not contain terms that are present in the known field theory limit calculations from string theory. Upon expanding the current ansatz at 6-points we find many of the terms that do appear in the field theory limit but there are a number of terms which do not appear. Evidence points to this being an artefact of a possible factorisation channel existing in the amplitudes. That is, we expect the terms that are missing in the ansatz to come from considering the possible expression that generates a factorisation channel coming from $\alpha'^4 \sim \alpha'^2 \times \alpha'^2$. These terms are unlikely to come from the simple ansatz building fields we have previously considered, that is $\mathbb{F}^{M|pq}$, $\mathbb{W}^N$ and $\mathbb{V}$ are not enough to generate the terms that produce this factorisation. There are some clues as to how one can generalise the ansatz. Experimentation tells us that these terms can be produces by a generating series expression of the form,

$$\mathrm{Tr}(\mathbb{L}(\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn}), \tag{15.1}$$

where $\mathbb{L}$ is an unknown expression. It is known, from experimentation with the component expansions, that $QL_{123} = M_{123}$ which suggests that $\mathbb{L}$ is somehow related to $\mathbb{V}$. Using this information, as well as $\mathrm{Tr}\{\mathcal{Q},\mathbb{L}\} = \mathrm{Tr}((\lambda\gamma^m\mathbb{W})(\lambda\gamma^n\mathbb{W})\mathbb{F}^{mn})$, one can potentially find $\mathbb{L}$ using an ansatz method. This has yet to be attempted using the most general ansatz - if this is done one may then find the missing terms and have an idea of how to generalise the ansatz in the correct manner. In fact, we now know that at $\alpha'^4$ there is no decoupling of terms involving $\mathbb{V}$ and those with no-$\mathbb{V}$. Owing to this lack of decoupling, it is perfectly possible that in order to generalise we shall have to introduce terms involving $\mathbb{A}^m$ - as this is the field that can induce mixing between the two sectors after variation. As a result, the ansatz will have to be expanded to include such terms.

Further issues with the $\alpha'^4$ corrections reside with the canonicalisation of the ansatz and its variation. Much of the early work of this project was dedicated to understanding how one can simplify the ansatz and its variation in order to properly obtain the correct set of linear equations. Whilst the procedures outlined here have been incredibly useful for finding simplified versions of the $\alpha'^2$ and $\alpha'^3$ they do not seem to be enough at $\alpha'^4$. One of the issues here is recognising the Jacobi identities that frequently appear, owing to the general nature of the ansatz. At lower order the procedures used seem to work well and recognise the Jacobi identities and eliminate the terms required. However, at the fourth order this does not always seem to be the case - the procedure fails to catch and eliminate some of the terms that can be eliminated. This is very much a technical issue and overcoming this is less about new identities but rather about the time required to alter the procedures. Aside from the identification of Jacobi identities, there are also issues with $\gamma$ matrices and ensuring these are as simplified as possible. Unfortunately, this is a fundamental issue when working with Super Yang-Mills and it is rather difficult to perform this manipulation using the $SO(10)$ representation. However, one can use the $U(5)$ pure spinor representation to simplify this task. In this representation gathering terms, to obtain the correct linear equations, becomes much simpler. Unfortunately, the splitting of higher mass fields is currently unknown and these are required to perform this analysis. Obtaining these splittings would be a great help when tackling problems pertaining to Super Yang-Mills in the pure spinor formalism.

Once these issues are tackled, the $\alpha'^4$ correction should come much closer to being realised. Whilst more issues may still be unknown, solving the above issues would give us confidence that the ansatz contained all the relevant terms and that its variation could be correctly simplified, giving the correct linear equations.

Of course there is another new method - that is based on an old method - that may allow one to find these corrections in a relatively straightforward manner. This method was introduced very briefly at the end of Section 12 and essentially relied upon treating the generating series action from (9.23) in Section 9 as an effective action for the equations of motion. One can then use this philosophy to re-derive the $\alpha'^2$ corrections [122] and potentially the $\alpha'^3$ corrections by matching terms at each $\alpha'$ order. This method is still a work in progress and represents a novel way of interpreting such Chern-Simons-like actions. However, if this method does work then it provides a systematic way of finding higher order corrections, albeit with some subtleties not discussed in this work.

## 15.2 The Drinfeld Associator

As of the completion of this report the $\alpha'^2$ correction in the form $\mathcal{O}\langle\mathbb{VVV}\rangle$ is still unknown. Generating the ansatz in the manner outlined at the start of Part II and Section 11.6, performing the variation and then enforcing BRST closure only gives zero. It may be the case that it is not possible to express the corrections in the form $\mathcal{O}\langle\mathbb{VVV}\rangle$, however before giving up on this line of research there are a few things one can try to check if it is possible.

It would be incredibly instructive to find the component expansion of the ansatz outlined in Section 11.6. Once this is found, one can compare it to the known expansions from the form of the correction that is known. If there are missing terms then this suggests that there may be a way to generalise the ansatz further. This may not be the case, but knowing whether the current ansatz produces all the required terms would be a good step towards understanding whether this is possible.

Linked to the component expansion issue is the issues of generalising the ansatz. One should revisit the ansatz terms and systematically determine if all the terms that one can think of have been added. One could also consider whether using covariant $\nabla^m$ is the right approach. For example, could some of the terms require $\partial^m$ instead? This would require thorough investigation of the ansatz and the component expansions thereof. Expressing the corrections in the $\mathcal{O}\langle\mathbb{VVV}\rangle$ form would represent a great simplification of the corrections. Furthermore, it would give some initial information about the form of the Drinfeld associator operator.

## 15.3 ...And Beyond

There are still many open avenues (and some that have yet to be discovered) of research within the pure spinor formalism. A vast number of them we have not been able to touch upon in this relatively short and focused work. In this subsection we briefly outline some of the work that could be realised as an extension of the programme set out here, as well as a few of the other issues currently unanswered within the formalism. A great resource detailing some of these issues in greater depth can be found in the concluding Section of [7].

One potential avenue for very fruitful research is, once the issues surrounding the $\alpha'^2$ correction in the form $\mathcal{O}\langle\mathbb{VVV}\rangle$ have been solve, to extend this $\mathcal{O}$ to higher order corrections. Given the structure of such corrections, they promise to be potentially much simpler than the ones found in this work. This follows since the operator $\mathcal{O}$ must have mass dimension,

$$[\mathcal{O}]_n = 2n, \tag{15.2}$$

where $n$ is the order of the correction. Hence at each order one has to search for ansatzes with mass dimension 3/2 less than the current method. As we have seen, increasing the mass dimension of the ansatz can greatly increase the number of terms one has to consider in order to find the correction. Thus, working with this operator may drastically reduce the number of terms that need to be considered, and hence speed up the process of finding corrections. Furthermore, it would allow one to express the corrected generating series action in the form,

$$\int [\mathrm{d}Z]\langle \mathbb{V}Q\mathbb{V} + \mathcal{O}\mathbb{VVV}\rangle, \tag{15.3}$$

where we have the following operator expansion,

$$\mathcal{O} = \sum_{i=0}^{N} \mathcal{O}_i, \qquad \mathcal{O}_0 = 1, \quad \mathcal{O}_1 = 0, \tag{15.4}$$

and $\mathcal{O}_n$ for $n \geq 2$ is as of yet unknown. This form would then allow one to find the $\alpha'$ corrected equations of motion up to order $N$ in a very simple way. This may then give a hint to the associator that generates all $\alpha'$ corrections in the theory. Of course this is highly speculative but it is a potential source of a wealth of knowledge about the theory and formalism.

Taking the results of Section 13 a few steps further may also be quite interesting. One may wish to pose the question: "What does the theory look like if it is compactified on a more interesting manifold?" The answers to such questions may allow one to use some of the machinery of the $D = 10$ theory in low dimensions with less supersymmetry than the maximally supersymmetric cases that result from torodial compactification. This may prove fruitful in computing $\alpha'$ corrections in other types of Super Yang-Mills theory. Although it is unclear whether using the pure spinor machinery in lower dimensions actually makes the process of calculating easier. It would also be interesting the use the $\alpha'$ corrections to see if one can use the double copy to find some form of generating series of supergravity corrections. There are known issues with this, first and foremost being a general form for the gauge transform that allows one to put the generating series expressions into BCJ gauge [29].

There are also issues pertaining to the calculation of the higher order string theory amplitudes in the pure spinor formalism. Here we choose to highlight one particular problem that stands in the way of calculating higher order loop diagrams[1]. When calculating string loop scattering amplitudes, one has to integrate over the entire moduli space of possible worldsheets. However, there is some reparametrization invariance due to the symmetries of the worldsheet topologies. To fix this, one typically introduces a new ghost system, called the $(b, c)$ ghosts, which allows one to gauge fix this symmetry via the usual BRST method. In the pure spinor formalism one is able to effectively get rid of the need for the $c$ ghost [133], however the $b$ ghost remains. The current form of the $b$ ghost is rather complicated (hence we do not display it here), but the main feature of concern here is that there is a pole involving the pure spinor, $\lambda$. At 1- and 2-loops this poses no issues as the other parts of the amplitude expressions have enough $\lambda$ terms to saturate this pole and effectively get rid of it[2]. However, at 3-points the pole cannot be saturated, and since one has to integrate over $\mathrm{d}\lambda \subset [\mathrm{d}Z]$ one ends up obtaining a non-regular answer. This is one of the obstacles preventing a pure derivation of the 3-loop amplitude in the pure spinor formalism, currently one has to use a hybrid formalism [78, 134]. Finding a representation of the $b$ ghost without this pole structure would allow the efficient calculation of higher loop string amplitudes - this would unlock much more of the underlying structure implicit in string theory.

[1] We do not give the mathematical detail here, simply describe the issue and why it poses a problem for the formalism. For more information we direct the reader to [133], and references therein.

[2] Although, in order to do this one needs to adopt the non-minimal pure spinor formalism not covered in this work

This work has not been as wide in scope as was initially intended and leaves many unanswered questions as a result. These questions are worth answering as they will provide insight into areas covering string theory, Super Yang-Mills, amplitudes and combinatorics to list but a few. The pure spinor formalism represents one of the most efficient and effective ways to answer many of the questions that lie at the heart of the work presented here. It is hoped, that in the future, someone will pick up the mantel of the ideas presented here and take them to completion. Something the author has been unable to do.

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