An analytic approach to the RTA Boltzmann attractor

Inês Aniceto,[1,*] Jorge Noronha,[2,†] and Michał Spaliński[3,4,‡]

[1]School of Mathematical Sciences, University of Southampton, Southampton, UK

[2]Illinois Center for Advanced Studies of the Universe & Department of Physics, University of Illinois Urbana-Champaign, Urbana, IL 61801, USA

[3]National Centre for Nuclear Research, 02-093 Warsaw, Poland

[4]Physics Department, University of Białystok, 15-245 Białystok, Poland

We reformulate the Boltzmann equation in the relaxation time approximation undergoing Bjorken flow in terms of a novel partial differential equation for the generating function of the moments of the distribution function. This is used to obtain an approximate analytic description of this system's far-from-equilibrium attractor via a series expansion at early times. This expansion possesses a finite radius of convergence and can be analytically continued to late times. We find that this procedure reproduces the known values of shear viscosity and other transport coefficients to high accuracy. We also provide a simple approximate analytic expression that describes the attractor in the entire domain of interest for studies of quark-gluon plasma dynamics.

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Introduction– Studies of Bjorken flow [1] in models of quark-gluon plasma dynamics have led to important insights into the physics of relativistic non-equilibrium evolution. These include the significance of nonhydrodynamic modes and the role of far-from-equilibrium attractors [2] (see also the reviews [3, 4]). An important class of relativistic models describing the onset of hydrodynamic behavior is formulated in the language of kinetic theory. The simplest of these are based on the Boltzmann equation in the relaxation time approximation (RTA), with many studies devoted to the dynamics of Bjorken flow in this system. This is also our focus here.

A very concise way of expressing the dynamics of kinetic theory is in the form of an infinite hierarchy of ordinary differential equations for a set of moments of the distribution function [5]. There are various ways of defining appropriate moments for relativistic gases [6–9]. Since in this Letter we will focus exclusively on Bjorken flow, it will be convenient to adopt the definitions of Blaizot and Yan [10], which take advantage of the special features of boost-invariant and transversely homogeneous dynamics. An important element of our analysis is to reformulate the Blaizot-Yan hierarchy in terms of a new set of dimensionless moments and a dimensionless time variable $w \equiv \tau/\tau_R$, where τ is the proper time and τ_R is the relaxation time. It is then straightforward to make contact with earlier studies of attractors in systems undergoing Bjorken flow [3, 4, 11].

The physics of this system has been explored by truncating the hierarchy at some level L and studying the resulting finite set of coupled ODEs [10, 12, 13]. One may use these truncated systems to study the physics of this theory and find that some important features are well-accounted for [12]. At the same time, these truncations miss features of the dynamics, such as the large order behavior of the gradient expansion and the physics at early times. In particular, they reproduce the free streaming at early times only approximately and completely miss the properties of the large order behavior of the gradient expansion pointed out in Ref. [11].

Our main goal in this paper is to pursue an alternative route by introducing a generating function for the moments. This function satisfies a partial differential equation, which we derive in this Letter. This equation provides a new, compact description of the dynamics of this system; we study its solutions perturbatively in the early and late time regimes, which is equivalent to finding series expansions of the moments.

In the early time regime, we identify solutions that are regular at $w = 0$ in the form of power series expansions with a non-zero radius of convergence. In particular, we calculate the series expansion of the pressure anisotropy of Bjorken flow — this is the well-known far-from-equilibrium attractor of RTA kinetic theory. We show that the system is initially free-streaming, as expected on general grounds. We also find such solutions for the higher moments, which supports the claim that the entire distribution function follows a far-from-equilibrium attractor [14].

We also study series solutions valid at late times, corresponding to the gradient expansion in hydrodynamics. The late-time solution of the PDE satisfied by the generating function leads to a very efficient method of generating the gradient expansion to very high order. The large-order behavior of this series extends the pattern of cuts in the Borel-Padé plane noted in Refs. [11, 15]. Since we generate many more terms of the gradient expansion, we also identify additional cuts. These features demonstrate the richness of RTA kinetic theory, showing that its early-time dynamics is very different from the Mueller-Israel-Stewart theory [6, 16] which describes its near-equilibrium regime.

The moment hierarchy– The RTA Boltzmann kinetic equation for the case of Bjorken flow takes the form

$$\partial_\tau f(\tau, p) = \frac{f_{\rm eq}(\tau, p) - f(\tau, p)}{\tau_R} \tag{1}$$

where f is the distribution function, $f_{\rm eq}$ is its equilibrium form (taken to be the Boltzmann distribution), p is the 4-momentum, and τ_R is the relaxation time. We will consider an implicit time dependence for the relaxation time of the form $\tau_R = \gamma T(\tau)^{-\Delta}$, where the effective temperature $T(\tau)$ is related to the energy density via $\mathcal{E} \propto T^4$. Note that $\Delta = 0$ corresponds to the case of constant relaxation time [17], while $\Delta = 1$ corresponds to the conformally invariant fluid [11, 15].

We follow Ref. [10], which defines a set of moments of the distribution function

$$\mathcal{L}_n \equiv \int \frac{d^3p}{(2\pi)^3} p_0 \; P_{2n}(\cos\psi) \; f(\tau, p) \; , \qquad \forall n \geq 0 \tag{2}$$

where $\cos\psi = p_z/p_0 = v_z$ is the particle velocity along the $z-$axis, and P_{2n} are Legendre polynomials of degree $2n$. Note that $\mathcal{E} \equiv \mathcal{L}_0$. The moments $\mathcal{L}_n$ obey a hierarchy of coupled, ordinary differential equations of the form

$$\tau \frac{d\mathcal{L}_n}{d\tau} = -a(n)\mathcal{L}_n - b(n)\mathcal{L}_{n-1} - c(n)\mathcal{L}_{n+1} - \frac{\tau}{\tau_R}\mathcal{L}_n \left(1 - \delta_{n,0}\right) \; , \quad n \geq 0 \, . \tag{3}$$

with

$$\begin{aligned} a(n) &= \frac{7}{4} + \frac{5}{16}\frac{1}{4n-1} - \frac{5}{16}\frac{1}{4n+3}; \\ b(n) &= \frac{1}{4} + \frac{n}{2} - \frac{5}{16}\frac{1}{4n-1} - \frac{9}{16}\frac{1}{4n+1}; \\ c(n) &= -\frac{n}{2} + \frac{9}{16}\frac{1}{4n+1} + \frac{5}{16}\frac{1}{4n+3}. \end{aligned} \tag{4}$$

We now introduce the dimensionless moments (similar to [8, 18–20])

$$\mathcal{M}_n \equiv \frac{\mathcal{L}_n}{\mathcal{L}_0} \, , \quad n \geq 0. \tag{5}$$

Note that $\mathcal{M}_0 = 1$, while $\mathcal{M}_1$ is related to the pressure anisotropy of Bjorken flow $\mathcal{A} = -3\mathcal{M}_1$ (see e.g. Refs. [3, 4]). This definition is motivated by earlier studies of hydrodynamization.

For $n \geq 1$, one then finds the following hierarchy of differential equations:

$$\hat{O}_w \mathcal{M}_n + a(n)\mathcal{M}_n + b(n)\mathcal{M}_{n-1} + c(n)\mathcal{M}_{n+1} \, , \quad n \geq 1, \tag{6}$$

where $\hat{O}_w$ is independent of n and given by

$$\hat{O}_w = \left(1 - \frac{\Delta}{3} - \frac{\Delta}{6}\mathcal{M}_1\right) w \frac{\partial}{\partial w} + \left(-\frac{4}{3} - \frac{2}{3}\mathcal{M}_1 + w\right) \, . \tag{7}$$

Note that to recover the energy density $\mathcal{E} \equiv \mathcal{L}_0$, one also needs the relation

$$\left(1 - \frac{\Delta}{3} - \frac{\Delta}{6}\mathcal{M}_1\right) \frac{\partial \ln \mathcal{L}_0}{\partial \ln w} + \frac{4}{3} + \frac{2}{3}\mathcal{M}_1 = 0. \tag{8}$$

These equations constitute a reformulation of the original Boltzmann equation in the RTA.

Generating functions for the moments– We now introduce a generating function for the moments,

$$G_{\mathcal{M}}(x, w) = \sum_{n=0}^{+\infty} x^n \, \mathcal{M}_n(w) \, , \tag{9}$$

where x is a formal variable. This generating function satisfies a partial differential equation, which will be the basis of our subsequent investigations. To determine it we follow the general approach of Ref. [21] (see also [18, 19]): we multiply Eq. (6) by x^n and sum over all $n \geq 0$, obtaining

$$\hat{O}_w G_{\mathcal{M}}(x,w) + \frac{7+x}{4} G_{\mathcal{M}}(x,w) - \frac{5}{16}\mathcal{M}_0 + \frac{9}{16}\mathcal{M}_1 + \frac{5}{16}(1+x)\sum_{n\geq 1}\frac{x^n}{4n+3}(\mathcal{M}_{n+1}-\mathcal{M}_n) + \\ + \frac{9}{16}\sum_{n\geq 1}\frac{x^n}{4n+1}(\mathcal{M}_{n+1}-\mathcal{M}_{n-1}) - \sum_{n\geq 1} x^n \frac{n}{2}(\mathcal{M}_{n+1}-\mathcal{M}_{n-1}) = 0\,.$$

We now change variables by setting $x = \xi^4$. A simple calculation reveals

$$\frac{5}{16}(1+\xi^4)\sum_{n\geq 1}\frac{\xi^{4n}}{4n+3}(\mathcal{M}_{n+1}-\mathcal{M}_n) = \frac{5}{16}\frac{(1+\xi^4)}{\xi^3}\left[\int^{\xi} d\eta\,(\eta^{-2}-\eta^2)\,G_{\mathcal{M}}(\xi,w) - \mathcal{M}_0\int^{\xi} d\eta\,\eta^{-2}\right];$$

$$\frac{9}{16}\sum_{n\geq 1}\frac{\xi^{4n}}{4n+1}(\mathcal{M}_{n+1}-\mathcal{M}_{n-1}) = \frac{9}{16}\xi^{-1}\left[\int^{\xi} d\eta\,(\eta^{-4}-\eta^4)\,G_{\mathcal{M}}(\xi,w) - \int^{\xi} d\eta\,(\eta^{-4}\mathcal{M}_0+\mathcal{M}_1)\right];$$

$$-\sum_{n\geq 1}\xi^{4n}\frac{n}{2}(\mathcal{M}_{n+1}-\mathcal{M}_{n-1}) = -\frac{\xi}{8}\frac{d}{d\xi}\left[(\xi^{-4}-\xi^4)\,G_{\mathcal{M}}(\xi,w) - \xi^{-4}\mathcal{M}_0 - \mathcal{M}_1\right].$$

To turn these differential/integral equations into a single partial differential equation, we need to multiply the full equation by ξ^3 and then differentiate it 5 times with respect to ξ, so that the indefinite integrals no longer appear. Proceeding this way we find that Eq. (6) can be rewritten as the following PDE

$$\left(\hat{O}_w\hat{O}_x^{(5)} + \hat{O}_x^{(6)}\right) G_{\mathcal{M}}(x,w) = 0\,, \tag{10}$$

where

$$\hat{O}_x^{(5)} = \sum_{\ell=1}^{5} k_\ell \left(4\frac{\partial}{\partial \ln x}\right)^{\ell},$$
$$\hat{O}_x^{(6)} = \sum_{\ell=0}^{6}\left(a_{-1}^{(\ell)}\frac{1}{x} + a_0^{(\ell)} + a_1^{(\ell)} x\right)\left(4\frac{\partial}{\partial \ln x}\right)^{\ell}. \tag{11}$$

Here k_ℓ and $a_i^{(\ell)}$ are numerical coefficients given by:

$$\{k_\ell\} = \{0,-6,-5,5,5,1,0\}\,; \tag{12}$$
$$\{a_{-1}^{(\ell)}\} = \left\{0,60,-97,\frac{119}{2},-\frac{69}{4},\frac{19}{8},-\frac{1}{8}\right\}; \tag{13}$$
$$\{a_0^{(\ell)}\} = \left\{0,-8,-5,10,\frac{35}{4},\frac{7}{4},0\right\}; \tag{14}$$
$$\{a_1^{(\ell)}\} = \left\{1344,1928,1088,\frac{625}{2},\frac{97}{2},\frac{38}{8},\frac{1}{8}\right\}, \tag{15}$$

with $\ell \in \{0,1,\cdots,6\}$. Note that the operator $\hat{O}_w$ appearing in Eq. (10) depends on $\mathcal{M}_1 = \partial_x\, G_{\mathcal{M}}(x,w)|_{x=0}$.

Early time solutions– Our basic observation is that Eq. (10) can be used to calculate the expansion of all the moments in powers of w. We will see that there are exactly two such solutions regular at $w = 0$, one of which is an attractor, while the other is repulsive — this is similar to what is known from studies of MIS theory [2, 22]. The system also possesses solutions which are divergent at $w = 0$. Their series representation starts with $\mathcal{M}_n = c_n/w^4 + \dots$, where the c_n's are integration constants. These are the generic solutions seen in numerical studies [14], which quickly approach the attractor already in the far-from-equilibrium regime, again in a way reminiscent of what is seen in MIS theory.

We look for regular solutions by expanding the moments in powers of w:

$$\mathcal{M}_n = \sum_{k=0}^{+\infty} w^k\, m_k^{(n)} \quad , \qquad n \geq 1. \tag{16}$$

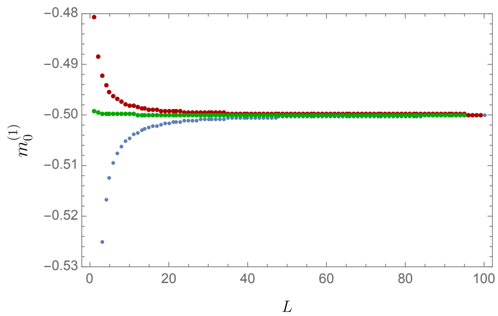

FIG. 1. Convergence-of the truncated calculation of coefficient $m_0^{(1)}$ as a function of truncation order L. Accelerated convergence via Richardson transforms is also shown (red - order 1; green - order 5).

where $m_0^{(0)} = 1$ and $m_k^{(0)} = 0$ for $k \geq 1$. The generating function can then be written as

$$G_{\mathcal{M}}(x,w) = \sum_{k\geq 0} w^k\, h_k(x) \ , \quad \text{where} \quad h_k(x) \equiv \sum_{n=0}^{+\infty} x^n\, m_k^{(n)} \, . \tag{17}$$

The PDE given in Eq. (6) can be used to determine the expansion coefficients $m_k^{(n)}$ appearing above. Using it, one obtains the following relations:

$$\frac{4}{3}\left(1+\frac{m_0^{(1)}}{2}\right)\hat{O}_x^{(5)}h_0(x) - \hat{O}_x^{(6)}h_0(x) \ = \ 0\,; \tag{18}$$

$$\left(\left(\frac{4}{3}+\Delta\frac{k}{3}\right)\left(1+\frac{m_0^{(1)}}{2}\right)-k\right)\hat{O}_x^{(5)}h_k(x) - \hat{O}_x^{(6)}h_k(x) \ = \ \hat{O}_x^{(5)}h_{k-1}(x) - \sum_{\ell=0}^{k-1}\left(\frac{2}{3}+\Delta\frac{\ell}{6}\right)m_{k-\ell}^{(1)}\,\hat{O}_x^{(5)}h_\ell(x). \tag{19}$$

One first needs to find the initial condition $G_{\mathcal{M}}(x,0) = h_0(x)$, which amounts to finding the initial values for the moments $m_0^{(n)}$ for $n \geq 0$. This is determined by Eq. (18), which leads directly to the recursion relation

$$-K^{(n)}\left(\frac{4}{3}+\frac{3}{2}m_0^{(1)}\right)m_0^{(n)} + A_{-1}^{(n+1)}\,m_0^{(n+1)} + A_0^{(n)}\,m_0^{(n)} + A_1^{(n-1)}\,m_0^{(n-1)} = 0\,, \tag{20}$$

where

$$K^{(n)} \ \equiv \ \sum_{\ell=1}^{5} k_\ell\,(4n)^\ell \ ; \tag{21}$$

$$A_i^{(n)} \ \equiv \ \sum_{\ell=0}^{6} a_i^{(\ell)}\,(4n)^\ell \ , \quad i=-1,0,1\,. \tag{22}$$

The recursion relations Eq. (20) possess two solutions; they can be obtained numerically by truncating the sequence at some level L by setting $m_0^{(n)} = 0$ for $n > L$. Here, we focus on the one that corresponds to the attractor. This way one finds that $m_0^{(1)} = -1/2$ (see Fig. 1). Given this value, all the remaining $m_0^{(n)}$ can be generated from the recursion relation Eq. (20). The resulting sequence turns out to be

$$m_0^{(n)} = \frac{(-1)^n}{2^{2n-1}}\frac{(2n-1)!}{n!\,(n-1)!}\,, \quad n \geq 1, \tag{23}$$

which translates into an exact form for the $w = 0$ contribution to the generating function:

$$h_0(x) = \frac{1}{\sqrt{1+x}}. \tag{24}$$

One can verify directly that this solves Eq. (18) exactly. This is a very important result: it shows that there is a regular solution (attractor) for *all* the moments $\mathcal{M}_n$. This demonstrates, for the first time, that the entire distribution function

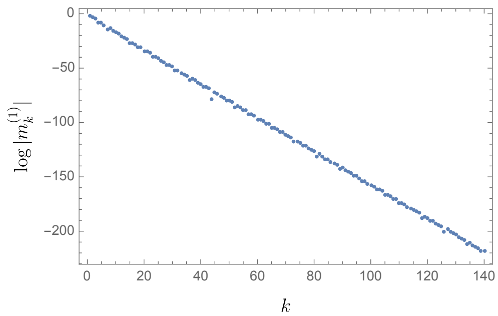

FIG. 2. Convergence of the early time expansion of the pressure anisotropy: this plot shows the exponential growth of the coefficients $m_k^{(1)}$ as a function of k.

follows a far-from-equilibrium attractor already at early times, corroborating the observation made by Strickland in Ref. [14] on the basis of numerical simulations.

The higher contributions $h_k(x), k > 0$ can be obtained by solving the remaining recursion relations which follow from Eq. (19). Using the fact that $m_0^{(1)} = -1/2$, we obtain

$$K^{(n)}\left(k\left(1-\frac{\Delta}{4}\right)-1\right)m_k^{(n)} + A_{-1}^{(n+1)}\,m_k^{(n+1)} + A_0^{(n)}\,m_k^{(n)} + A_1^{(n-1)}\,m_k^{(n-1)} = = -K^{(n)}\,m_{k-1}^{(n)} + K^{(n)}\sum_{\ell=0}^{k-1}\left(\frac{2}{3}+\Delta\frac{\ell}{6}\right)m_{k-\ell}^{(1)}\,m_\ell^{(n)}\,. \tag{25}$$

The form of the ODE solved by the function $h_k(x)$, Eq. (19), suggests a decisive simplification. We set

$$h_k(x) = \sum_{\ell=0}^{k}\alpha_{k\ell}\,\tilde{h}_\ell(x)\,, \quad k \geq 0 \tag{26}$$

where $\alpha_{kk} = 1$, and $\tilde{h}_0 = h_0$. By choosing the coefficients α such that the inhomogeneous part of Eq. (19) is canceled, we find that the functions $\tilde{h}_k$ obey the same ODE for each k:

$$\left(\frac{4-\Delta}{4}\,k-1\right)\hat{O}_x^{(5)}\tilde{h}_k + \hat{O}_x^{(6)}\tilde{h}_k = 0, \quad k \geq 0. \tag{27}$$

In order that this happens, the coefficients $\alpha_{k\ell}$ must obey

$$\alpha_{k\ell}\,\frac{4-\Delta}{4}\,(k-\ell) = -\alpha_{k-1,\ell} + \frac{2}{3}\sum_{m=\ell}^{k-1}\frac{4+m\Delta}{4}\,m_{k-m}^{(1)}\,\alpha_{m\ell}\,, \tag{28}$$

for $\ell = 0, \cdots, k-1$, and $\alpha_{kk} = 1$. We can now write

$$\tilde{h}_k = C_k\sum_{n=0}^{+\infty}x^n\,\beta_k^{(n)}\,, \quad \beta_k^{(0)} = 1\,. \tag{29}$$

in terms of new coefficients $\beta_k^{(n)}$, which can be determined recursively using Eq. (27). This amounts to solving

$$\left(K^{(n)}\left(k\,\frac{4-\Delta}{4}-1\right)+A_0^{(n)}\right)\beta_k^{(n)} + A_{-1}^{(n+1)}\,\beta_k^{(n+1)} + A_1^{(n-1)}\,\beta_k^{(n-1)} = 0\,, \quad n \geq 1; \tag{30}$$

$$\beta_k^{(0)} = 1\,. \tag{31}$$

We proceed by truncating these relations at some level L by setting $\beta_k^{(m)} = 0$ for $m > L$. This leads to a a system of linear equations of the form $Q\,\boldsymbol{x} + \boldsymbol{b} = 0$ with a tridiagonal $L \times L$ matrix Q, a vector of unknowns $\boldsymbol{x} = \left(\beta_k^{(1)}, \cdots, \beta_k^{(N)}\right)$

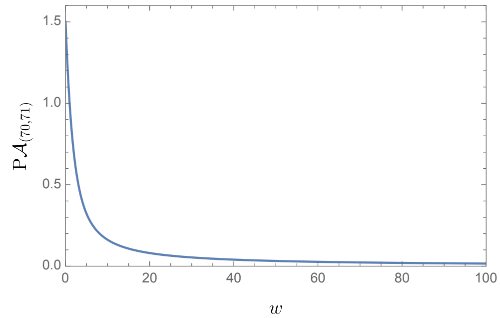

FIG. 3. Analytic continuation on early time expansion of the pressure anisotropy $\mathcal{A}$ as a function of w, given by a Padé approximant of order $(70, 71)$.

and a constant vector $\boldsymbol{b} = (A_1(0), 0, \cdots, 0)$, which can be solved very efficiently for any value of k. The results we show below were obtained for $L = 220$. Increasing the number of moments included in the truncation will increase the accuracy of the coefficients $m_k^{(1)}$ of the pressure anisotropy (for any k). This calculation is very efficient: determining the coefficients for $k \leq 160$ of the first $L = 220$ moments takes less than 2 minutes in Mathematica. The large k behavior of the coefficient $\beta_k^{(1)}$, which is related to the pressure anisotropy, is given by

$$\beta_k^{(1)} \sim \frac{32}{45}\frac{1}{k} + \mathcal{O}\left(k^{-2}\right), \tag{32}$$

which does not change with truncation level for $L > 200$. This is already a hint suggesting that the small w expansion of the pressure anisotropy possesses a finite radius of convergence.

The relation between the coefficients $m_k^{(n)}$ of the moments and the $\beta_k^{(n)}$ is given by

$$m_k^{(n)} = \sum_{\ell=0}^{k} C_\ell \, \beta_\ell^{(n)} \, \alpha_{k\ell} \,. \tag{33}$$

Recalling that $\mathcal{M}_0 = 1$, and thus $m_0^{(0)} = 1$, $m_{k\geq 1}^{(0)} = 0$, we can determine the C_k using

$$C_k = -\sum_{\ell=0}^{k-1} \alpha_{k\ell} \, C_\ell \,. \tag{34}$$

The equations that determine the $\alpha_{k\ell}$ directly are given in (28) for $\ell = 1, \cdots, k-1$, recalling that $\alpha_{kk} = 1$. For $\ell = 0$, we use (28) together with Eq. (33), which yields

$$\alpha_{k0}\left(\frac{4-\Delta}{4}k + \frac{2}{3}C_0\left(\beta_k^{(1)} - \beta_0^{(1)}\right)\right) = -\alpha_{k-1,0} + \frac{2}{3}\sum_{r=1}^{k-1}\alpha_{kr}\,C_r\left(\beta_r^{(1)} - \beta_k^{(1)}\right) + \\ + \frac{2}{3}\sum_{m=1}^{k-1}\frac{4+m\Delta}{4}\alpha_{m0}\sum_{r=0}^{k-m} C_r\,\beta_r^{(1)}\,\alpha_{k-m,r} \,. \tag{35}$$

Note that the $\alpha_{k\ell}$ depend on no other coefficients of the functions $\tilde{h}_k$ except $\beta_k^{(1)}$. To solve them for each k, we need to start with the $\alpha_{k\ell}$ with ℓ decreasing from k to 0. We can then find C_k from Eq. (34) and determine the moment coefficients in Eq. (33).

We now focus on the conformal case, $\Delta = 1$. We have calculated the first 220 moments, which enables us to estimate the radius of convergence of the early-time expansion. In the case of the first moment (which is proportional to the pressure anisotropy), one can easily see that the radius of convergence of the small w series of $\mathcal{M}_1$ follows from $m_k^{(1)} \sim \mathrm{e}^{-a\,k}$ with $a \approx 1.55$. This can be seen in Fig. 2.

We can now plot the pressure anisotropy $\mathcal{A} = -3\mathcal{M}_1$ using the convergent small w expansion. To go beyond the radius of convergence (given by e^{-a}) we use analytic continuation by means of an off-diagonal Padé approximant (one order higher in the denominator than in the numerator, to account for expected behavior at large w). The result can be seen in Fig. 3, using a Padé approximant of order $(70, 71)$, $\mathrm{P}\mathcal{A}_{70,71}$. While this series solution was obtained at early times, the large radius of convergence makes it possible for this approximation of the far-from-equilibrium, pre-hydrodynamic attractor to be extended *all the way* into the near-equilibrium domain. It is, therefore, very interesting to compare it to expectations based on hydrodynamics. At $w \gg 1$, hydrodynamics predicts the asymptotic behavior [11, 23]

$$\mathcal{A} \sim \frac{8/5}{w} + \frac{32/105}{w^2}\,. \tag{36}$$

The leading term above reflects the known value of the shear viscosity for RTA kinetic theory, $\eta/s = \gamma/5$ (see e.g. [3]). Our early-time series solution analytically continued to $w = 100$ reproduces this value of the shear viscosity to three decimal places. This can be seen by re-expanding the Padé approximant of the pressure anisotropy $\mathrm{P}\mathcal{A}_{(70,71)}$ for $w \gg 1$:

$$\mathrm{P}\mathcal{A}_{(70,71)} \sim \frac{1.60004}{w} + \frac{0.27348}{w^2}, \tag{37}$$

which demonstrates the excellent agreement with the hydrodynamic behavior at large w.

The validity of the analytic continuation over such a large domain is due to the large radius of convergence of the early-time expansion, along with the fact that singularities of the Padé approximant appear in the second and third quadrants of the complex plane. We note that the same is not true for the generic solutions divergent at the origin.

We close this section with the following approximate analytic formula for the attractor in the form of an off-diagonal Padé approximant

$$\mathcal{A} = \frac{1500 - 12w + 30w^2}{1000 + 434w + 95w^2 + 9w^3} \tag{38}$$

The values of the Padé coefficients have been approximated to obtain the above expression. This formula works very well for the physically interesting range $0 \le w \le 5$, i.e., from the earliest times all the way into the near-equilibrium regime. In particular, it reproduces the leading two orders of the early-time expansion, including the free-streaming behavior at $w = 0$, and gives an accurate approximation up to $w \approx 5$ with a relative error of about 2%.

The late time expansion– The generating function approach that we have developed here can also be applied to investigate the late-time expansion — the asymptotic series for large values of w:

$$\mathcal{M}_n = w^{-n} \sum_{k \ge 0} w^{-k} M_k^{(n)}. \tag{39}$$

The PDE satisfied by the generating function, Eq. (10), can be used to establish recursion relations for the coefficients $M_k^{(n)}$, in a very similar way to what was presented in the previous Section. The main difference is that changing the truncation at a particular order w^{-k} does not change the accuracy of the lower orders already calculated. The recursion relations can be used to generate these coefficients in a vastly more efficient way than by using hitherto existing methods: we have been able to calculate 1000 of them in about 10 minutes. In contrast to the early-time solutions, the series appearing here have zero radius of convergence and need to be interpreted in the sense of asymptotic analysis. The series coefficients grow factorially, so it is natural to use a Borel transform to give meaning to the divergent sum. The analytic continuation of the Borel transform, which can be carried out by means of a Padé approximant, reveals a pattern of singularities whose physical significance is of great interest. Here we present some results concerning the pressure anisotropy $\mathcal{A}$, related to the moment $\mathcal{M}_1$, shown in Fig. 4. The singularity structure is strongly dependent on the value of the parameter Δ, as noted in Ref. [15].

In the conformal case ($\Delta = 1$), we reproduce the pattern found in Ref. [11], consisting of the cut on the real axis and a symmetric pair of branch points with nonvanishing imaginary parts. Since we can easily generate many more terms of the gradient expansion than were hitherto available, we can also discern further branch points at complex-conjugate locations off the real axis. It is natural to conjecture that there is an infinite number of such cuts. While the cut on the real axis clearly corresponds to the nonhydrodynamic mode of the RTA kinetic theory, to identify the physical meaning of the singularities with nonvanishing imaginary parts remains a challenge for the future.

Outlook– In this Letter, we studied the dynamics of the RTA Boltzmann equation by applying generating function techniques to the hierarchy of moment equations. This yields a new reformulation of this theory in terms of a partial differential equation in two variables: the formal variable x, whose power tags each of the moments, and w, whose

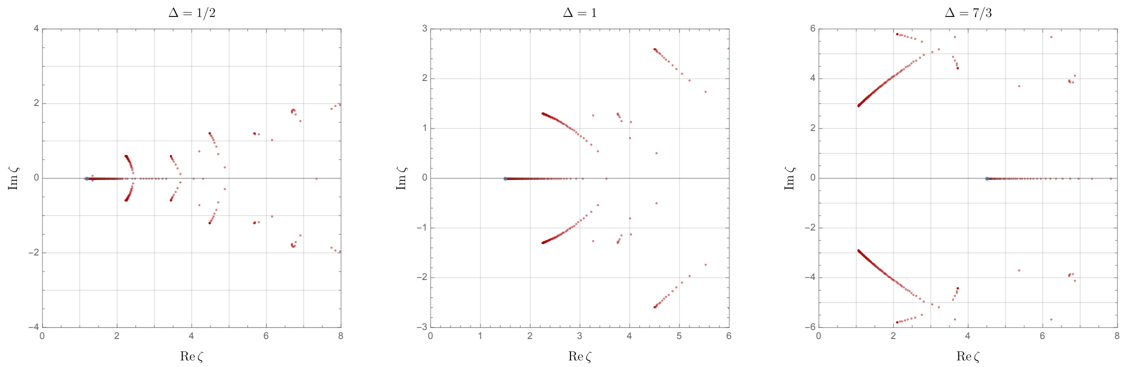

FIG. 4. Poles of the Borel-Padé approximant of order 500 of the late time expansion of the pressure anisotropy $\mathcal{A}$, $\mathrm{BP}_{500}(\mathcal{A})(\zeta)$, for different values of Δ. In blue: start of the leading cut in the real axis, at position ζ_Δ: $\zeta_{1/2} = 6/5$, $\zeta_1 = 3/2$ and $\zeta_{7/3} = 9/2$.

power at large times encodes the order of the gradient expansion. We expect that several further insights can be obtained from this approach.

Here, we have focused mainly on the early-time behavior of RTA kinetic theory. Earlier works showed that the dynamics of this model reaches the hydrodynamic domain following a far-from-equilibrium attractor, which has previously been obtained by numerical solutions of the Boltzmann equation. Using the generating function technique presented here, we have provided the first solid analytic evidence for this attractor. We have, in particular, obtained a series solution with a finite radius of convergence, which provides an accurate account of the attractor in a large domain, fully describing the passage from the far-from-equilibrium regime to the near-equilibrium domain.

We have also shown that the formulation of RTA kinetic theory presented here opens the door to calculations of the gradient expansion to large orders at a qualitatively new level of efficiency, allowing for hundreds of coefficients to be computed in minutes (as opposed to weeks). This may eventually lead to a better understanding of the peculiar features hinted at by the results of earlier studies of this problem. It would be interesting to see how this generating function technique can lead to new insight into attractors in other kinetic systems, such as $\lambda\phi^4$ theory [24] and high-temperature QCD plasmas [25].

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* I.Aniceto@soton.ac.uk
† jn0508@illinois.edu
‡ michal.spalinski@ncbj.gov.pl

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