

An analytic approach to the RTA Boltzmann attractor

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We reformulate the Boltzmann equation in the relaxation time approximation undergoing Bjorken flow in terms of a novel partial differential equation for the generating function of the moments of the distribution function. This is used to obtain a series expansion of this system's far-from-equilibrium attractor. This expansion possesses a finite radius of convergence, which proves the existence of the attractor. Furthermore, the series can be analytically continued to late times and we find that this procedure reproduces the known values of shear viscosity and other transport coefficients to high accuracy. We also provide a simple approximate analytic expression that describes the attractor in the entire domain of interest for studies of quark-gluon plasma dynamics.

I. INTRODUCTION

Studies of Bjorken flow [1] in models of quark-gluon plasma dynamics have led to important insights into the physics of relativistic non-equilibrium evolution. These include the significance of nonhydrodynamic modes and the role of far-from-equilibrium attractors [2] (see also the reviews [3, 4]). An important class of relativistic models describing the onset of hydrodynamic behavior is formulated in the language of kinetic theory. The simplest of these are based on the Boltzmann equation in the relaxation time approximation (RTA), with many studies devoted to the dynamics of Bjorken flow in this system. This is also our focus here.

A very concise way of expressing the dynamics of kinetic theory is in the form of an infinite hierarchy of ordinary differential equations for a set of moments of the distribution function [5]. There are various ways of defining appropriate moments for relativistic gases [6–9]. Since in this work we will focus exclusively on Bjorken flow, it will be convenient to adopt the definitions of Blaizot and Yan [10], which take advantage of the special features of boost-invariant and transversely homogeneous dynamics. The physics of the RTA Boltzmann equation has been explored by truncating this hierarchy and studying the resulting finite set of coupled ODEs. These truncated systems capture many important features very well [10–12], but they provide only an approximate picture of the dynamics of the underlying kinetic theory. For instance, the free streaming at early times is captured approximately, but not exactly. Furthermore, the truncations of the hierarchy necessarily miss the striking features of the large order behavior of the gradient expansion found in Ref. [13]. Furthermore, while there are situations where hydrodynamic truncations appropriate, there are also situations involving out of equilibrium behavior where the much richer structure of kinetic theory becomes important. One clear example is the response to hard probes encoded in the jet quenching parameter where full knowledge about the out of equilibrium distribution function is needed at early times [14, 15].

Here we pursue an alternative route by introducing a generating function for the moments. This quantity satisfies a novel partial differential equation which we derive, providing a compact description of the dynamics of RTA kinetic theory. We study its solutions perturbatively in the early and late time regimes.

An important element of our analysis is a reformulation of the Blaizot-Yan hierarchy in terms of a new set of dimensionless moments and the dimensionless variable $w \equiv \tau/\tau_R$, where τ is the proper time and τ_R is the relaxation time. It is then straightforward to make contact with previous studies of attractors in systems undergoing Bjorken flow [2–4, 16, 17]. In the early time regime, we identify solutions regular at $w = 0$ in the form of power series expansions. In particular, we calculate the expansion of the far-from-equilibrium attractor in the pressure anisotropy of RTA kinetic theory. We find that this series representation of the attractor is convergent and we have determined the singularities of its analytic continuation. Remarkably, the series can be continued all the way into the late-time, hydrodynamic region, accurately reproducing known values of the transport coefficients of the RTA Boltzmann equation. In addition, we find attractor solutions for the higher moments, which supports the claim that the entire distribution function follows a far-from-equilibrium attractor [17]. We also study series solutions valid at late times, corresponding to the

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hydrodynamic gradient expansion. The PDE satisfied by the generating function leads to a very efficient method of computing these series, whose large-order behavior extends the pattern of cuts in the Borel-Padé plane noted in Refs. [13, 18]. Since we generate many more terms of the series, we also identify additional cuts. These features demonstrate the richness of RTA kinetic theory and suggest that its early-time dynamics is very different from the Mueller-Israel-Stewart theory [6, 19], which describes its near-equilibrium regime.

II. THE MOMENT HIERARCHY

The RTA Boltzmann kinetic equation in the case of Bjorken flow [20] may be written in Milne coordinates $x^\mu = (\tau, x, y, \varsigma)$ [21] as follows

$$\frac{\partial f}{\partial \tau} = \frac{f_{\text{eq}} - f}{\tau_R} \quad (1)$$

where $\tau = \sqrt{t^2 - z^2}$ and $\varsigma = \tanh^{-1}(z/t)$, $p_\mu = (p_0, p_x, p_y, p_\varsigma)$ is the 4-momentum of the massless particle (with $p_0 = \sqrt{p_x^2 + p_y^2 + p_\varsigma^2/\tau^2}$), f is the distribution function, f_{eq} is its equilibrium form (taken to be the classical Boltzmann distribution), and τ_R is the relaxation time. In Bjorken flow, the symmetries impose that f may be expressed as a function of τ, p_0, p_ς [21]. We will consider an implicit time dependence for the relaxation time of the form $\tau_R = \gamma T(\tau)^{-\Delta}$, where the effective temperature $T(\tau)$ is related to the energy density via $\mathcal{E} \propto T^4$. Note that $\Delta = 0$ corresponds to the case of constant relaxation time [22], while $\Delta = 1$ corresponds to the conformally invariant fluid [13, 18].

We follow Ref. [10] and consider a set of moments of the distribution function, which in our notation are defined as

$$\mathcal{L}_n \equiv \int \frac{d^3 p}{(2\pi)^3 \tau} p_0 P_{2n}(\cos \psi) f(\tau, p_0, p_\varsigma), \quad \forall n \geq 0 \quad (2)$$

where $\cos \psi = p_\varsigma/(p_0 \tau)$, and P_{2n} are Legendre polynomials of degree $2n$. Note that $\mathcal{E} \equiv \mathcal{L}_0$. The moments $\mathcal{L}_n$ obey a hierarchy of coupled, ordinary differential equations of the form

$$\tau \frac{d\mathcal{L}_n}{d\tau} = -a(n)\mathcal{L}_n - b(n)\mathcal{L}_{n-1} - c(n)\mathcal{L}_{n+1} - \frac{\tau}{\tau_R} \mathcal{L}_n (1 - \delta_{n,0}), \quad n \geq 0. \quad (3)$$

with

$$\begin{aligned} a(n) &= \frac{7}{4} + \frac{5}{16} \frac{1}{4n-1} - \frac{5}{16} \frac{1}{4n+3}; \\ b(n) &= \frac{1}{4} + \frac{n}{2} - \frac{5}{16} \frac{1}{4n-1} - \frac{9}{16} \frac{1}{4n+1}; \\ c(n) &= -\frac{n}{2} + \frac{9}{16} \frac{1}{4n+1} + \frac{5}{16} \frac{1}{4n+3}. \end{aligned} \quad (4)$$

We now introduce the dimensionless moments (similar to [8, 23–25])

$$\mathcal{M}_n \equiv \frac{\mathcal{L}_n}{\mathcal{L}_0}, \quad n \geq 0. \quad (5)$$

Note that $\mathcal{M}_0 = 1$, while $\mathcal{M}_1$ is related to the pressure anisotropy of Bjorken flow $\mathcal{A} = -3\mathcal{M}_1$ (see e.g. Refs. [3, 4]). This definition is motivated by earlier studies of hydrodynamization.

For $n \geq 1$, one then finds the following hierarchy of differential equations:

$$\hat{O}_w \mathcal{M}_n + a(n)\mathcal{M}_n + b(n)\mathcal{M}_{n-1} + c(n)\mathcal{M}_{n+1} = 0, \quad n \geq 1, \quad (6)$$

where $\hat{O}_w$ is independent of n and given by

$$\hat{O}_w = \left(1 - \frac{\Delta}{3} - \frac{\Delta}{6} \mathcal{M}_1\right) w \frac{\partial}{\partial w} + \left(-\frac{4}{3} - \frac{2}{3} \mathcal{M}_1 + w\right). \quad (7)$$

Note that to recover the energy density $\mathcal{E} \equiv \mathcal{L}_0$, one also needs the relation

$$\left(1 - \frac{\Delta}{3} - \frac{\Delta}{6} \mathcal{M}_1\right) \frac{\partial \ln \mathcal{L}_0}{\partial \ln w} + \frac{4}{3} + \frac{2}{3} \mathcal{M}_1 = 0. \quad (8)$$

These equations constitute a reformulation of the original Boltzmann equation in the RTA.

III. GENERATING FUNCTIONS FOR THE MOMENTS

We now introduce a generating function for the moments,

$$G_{\mathcal{M}}(x, w) = \sum_{n=0}^{+\infty} x^n \mathcal{M}_n(w), \quad (9)$$

where x is a formal variable. This generating function satisfies a partial differential equation, which will be the basis of our subsequent investigations. To determine it we follow the general approach of Ref. [26] (see also [23, 24]): we multiply Eq. (6) by x^n and sum over all $n \geq 0$, obtaining

$$\begin{aligned} \hat{O}_w G_{\mathcal{M}}(x, w) + \frac{7+x}{4} G_{\mathcal{M}}(x, w) - \frac{5}{16} \mathcal{M}_0 + \frac{9}{16} \mathcal{M}_1 - w + \frac{5}{16} (1+x) \sum_{n \geq 0} \frac{x^n}{4n+3} (\mathcal{M}_{n+1} - \mathcal{M}_n) + \\ + \frac{9}{16} \sum_{n \geq 1} \frac{x^n}{4n+1} (\mathcal{M}_{n+1} - \mathcal{M}_{n-1}) - \sum_{n \geq 1} x^n \frac{n}{2} (\mathcal{M}_{n+1} - \mathcal{M}_{n-1}) = 0. \end{aligned}$$

Note that we have added and subtracted the case $n = 0$ from Eq. (6) – adding terms independent of x will not change the results below. We now change variables by setting $x = \xi^4$. A simple calculation reveals

$$\begin{aligned} \frac{5}{16} (1 + \xi^4) \sum_{n \geq 0} \frac{\xi^{4n}}{4n+3} (\mathcal{M}_{n+1} - \mathcal{M}_n) &= \frac{5}{16} \frac{(1 + \xi^4)}{\xi^3} \left[\int^\xi d\eta (\eta^{-2} - \eta^2) G_{\mathcal{M}}(\eta, w) - \mathcal{M}_0 \int^\xi d\eta \eta^{-2} \right]; \\ \frac{9}{16} \sum_{n \geq 1} \frac{\xi^{4n}}{4n+1} (\mathcal{M}_{n+1} - \mathcal{M}_{n-1}) &= \frac{9}{16} \xi^{-1} \left[\int^\xi d\eta (\eta^{-4} - \eta^4) G_{\mathcal{M}}(\eta, w) - \int^\xi d\eta (\eta^{-4} \mathcal{M}_0 + \mathcal{M}_1) \right]; \\ - \sum_{n \geq 1} \xi^{4n} \frac{n}{2} (\mathcal{M}_{n+1} - \mathcal{M}_{n-1}) &= -\frac{\xi}{8} \frac{d}{d\xi} [(\xi^{-4} - \xi^4) G_{\mathcal{M}}(\xi, w) - \xi^{-4} \mathcal{M}_0 - \mathcal{M}_1]. \end{aligned}$$

To turn these differential/integral equations into a single partial differential equation, we need to multiply the full equation by ξ^3 and then differentiate it 5 times with respect to ξ , so that the indefinite integrals no longer appear. Proceeding this way we find that Eq. (6) can be rewritten as the following PDE

$$(\hat{O}_w \hat{O}_x^{(5)} + \hat{O}_x^{(6)}) G_{\mathcal{M}}(x, w) = 0, \quad (10)$$

where

$$\begin{aligned} \hat{O}_x^{(5)} &= \sum_{\ell=1}^5 k_\ell \left(4 \frac{\partial}{\partial \ln x} \right)^\ell, \\ \hat{O}_x^{(6)} &= \sum_{\ell=0}^6 \left(a_{-1}^{(\ell)} \frac{1}{x} + a_0^{(\ell)} + a_1^{(\ell)} x \right) \left(4 \frac{\partial}{\partial \ln x} \right)^\ell. \end{aligned} \quad (11)$$

Here k_ℓ and $a_i^{(\ell)}$ are numerical coefficients given by:

$$\{k_\ell\} = \{0, -6, -5, 5, 5, 1, 0\}; \quad (12)$$

$$\{a_{-1}^{(\ell)}\} = \left\{ 0, 60, -97, \frac{119}{2}, -\frac{69}{4}, \frac{19}{8}, -\frac{1}{8} \right\}; \quad (13)$$

$$\{a_0^{(\ell)}\} = \left\{ 0, -8, -5, 10, \frac{35}{4}, \frac{7}{4}, 0 \right\}; \quad (14)$$

$$\{a_1^{(\ell)}\} = \left\{ 1344, 1928, 1088, \frac{625}{2}, \frac{97}{2}, \frac{31}{8}, \frac{1}{8} \right\}, \quad (15)$$

with $\ell \in \{0, 1, \dots, 6\}$. Note that the operator $\hat{O}_w$ appearing in Eq. (10) depends on $\mathcal{M}_1 = \partial_x G_{\mathcal{M}}(x, w)|_{x=0}$.

IV. EARLY TIME SOLUTIONS

Our basic observation is that Eq. (10) can be used to calculate the expansion of all the moments in powers of w . We will see that there are exactly two such solutions regular at $w = 0$, one of which is an attractor, while the other is repulsive — this is similar to what is known from studies of MIS theory [2, 27]. The system also possesses solutions which are divergent at $w = 0$. Their series representation starts with $\mathcal{M}_n = c_n/w^4 + \dots$, where the c_n 's are integration constants. These are the generic solutions seen in numerical studies [17], which quickly approach the attractor already in the far-from-equilibrium regime, again in a way reminiscent of what is seen in MIS theory.

We look for regular solutions by expanding the moments in powers of w :

$$\mathcal{M}_n = \sum_{k=0}^{+\infty} w^k m_k^{(n)} \quad , \quad n \geq 1. \quad (16)$$

where $m_0^{(0)} = 1$ and $m_k^{(0)} = 0$ for $k \geq 1$. The generating function can then be written as

$$G_{\mathcal{M}}(x, w) = \sum_{k \geq 0} w^k h_k(x) \quad , \quad \text{where} \quad h_k(x) \equiv \sum_{n=0}^{+\infty} x^n m_k^{(n)}. \quad (17)$$

The PDE given in Eq. (6) can be used to determine the expansion coefficients $m_k^{(n)}$ appearing above. Using it, one obtains the following relations:

$$\frac{4}{3} \left(1 + \frac{m_0^{(1)}}{2} \right) \hat{O}_x^{(5)} h_0(x) - \hat{O}_x^{(6)} h_0(x) = 0; \quad (18)$$

$$\left(\left(\frac{4}{3} + \Delta \frac{k}{3} \right) \left(1 + \frac{m_0^{(1)}}{2} \right) - k \right) \hat{O}_x^{(5)} h_k(x) - \hat{O}_x^{(6)} h_k(x) = \hat{O}_x^{(5)} h_{k-1}(x) - \sum_{\ell=0}^{k-1} \left(\frac{2}{3} + \Delta \frac{\ell}{6} \right) m_{k-\ell}^{(1)} \hat{O}_x^{(5)} h_\ell(x). \quad (19)$$

One first needs to find the initial condition $G_{\mathcal{M}}(x, 0) = h_0(x)$, which amounts to finding the initial values for the moments $m_0^{(n)}$ for $n \geq 0$. This is determined by Eq. (18), which leads directly to the recursion relation

$$-K^{(n)} \left(\frac{4}{3} + \frac{2}{3} m_0^{(1)} \right) m_0^{(n)} + A_{-1}^{(n+1)} m_0^{(n+1)} + A_0^{(n)} m_0^{(n)} + A_1^{(n-1)} m_0^{(n-1)} = 0, \quad (20)$$

where

$$K^{(n)} \equiv \sum_{\ell=1}^5 k_\ell (4n)^\ell; \quad (21)$$

$$A_i^{(n)} \equiv \sum_{\ell=0}^6 a_i^{(\ell)} (4n)^\ell, \quad i = -1, 0, 1. \quad (22)$$

The recursion relations Eq. (20) possess two solutions; they can be obtained numerically by truncating the sequence at some level L by setting $m_0^{(n)} = 0$ for $n > L$ (and to accelerate the convergence one can use methods such as Richardson extrapolation [28]). Here, we focus on the one that corresponds to the attractor. This way one finds that $m_0^{(1)} = -1/2$. Given this value, all the remaining $m_0^{(n)}$ can be generated from the recursion relation Eq. (20). The resulting sequence turns out to be

$$m_0^{(n)} = \frac{(-1)^n (2n-1)!}{2^{2n-1} n! (n-1)!}, \quad n \geq 1, \quad (23)$$

which translates into an exact form for the $w = 0$ contribution to the generating function:

$$h_0(x) = \frac{1}{\sqrt{1+x}}. \quad (24)$$

One can verify directly that this solves Eq. (18) exactly. This is a very important result: it shows that there is a regular solution for *all* the moments $\mathcal{M}_n$. This demonstrates, that the entire distribution function follows a far-from-equilibrium

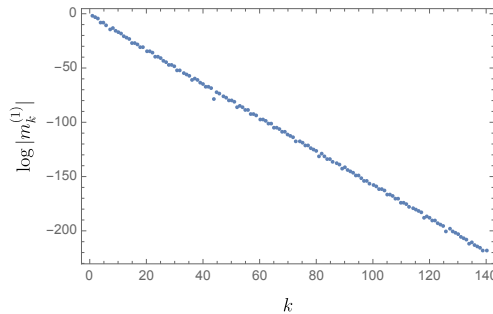


FIG. 1. Convergence of the early time expansion of the pressure anisotropy: this plot shows the exponential growth of the coefficients $m_k^{(1)}$ as a function of k .

attractor already at early times, corroborating the observation made by Strickland in Ref. [17] on the basis of numerical simulations. Note that Eq. (23) describes the free-streaming distribution function [10].

The higher contributions $h_k(x)$, $k > 0$ can be obtained by solving the ODEs Eq. (19) using a decisive simplification: we define $\tilde{h}_k$ as the solutions of the homogeneous equation:

$$\left(\frac{4-\Delta}{4}k-1\right)\hat{O}_x^{(5)}\tilde{h}_k + \hat{O}_x^{(6)}\tilde{h}_k = 0, \quad k \geq 0, \quad (25)$$

solved for a generic value of k as a matrix equation of the same rank as the the number of moments we want to calculate. The final solutions to Eq. (19) are then given by $h_k(x) = \sum_{\ell=0}^k \alpha_{k\ell} \tilde{h}_\ell(x)$, $k \geq 0$, where the coefficients α are fixed in closed form such that the inhomogeneous part of Eq. (19) is canceled. The details of this calculation are given in Appendix A.

We now focus on the conformal case, $\Delta = 1$. We have calculated the first 220 moments, which enables us to estimate the radius of convergence of the early-time expansion. In the case of the first moment (which is proportional to the pressure anisotropy), one can easily see that the radius of convergence of the small w series of $\mathcal{M}_1$ follows from $m_k^{(1)} \sim e^{-a k}$ with $a \approx 1.55$. This can be seen in Fig. 1. To go beyond the radius of convergence (given by e^{-a}) one can use analytic continuation by means of an off-diagonal Padé approximant (one order higher in the denominator than in the numerator, to account for expected behavior at large w).

While this series solution was obtained at early times, the large radius of convergence makes it possible for this approximation of the far-from-equilibrium, pre-hydrodynamic attractor to be extended *all the way* into the near-equilibrium domain. It is, therefore, very interesting to compare it to expectations based on hydrodynamics. At $w \gg 1$, hydrodynamics predicts the asymptotic behavior [13, 29]

$$\mathcal{A} \sim \frac{8/5}{w} + \frac{32/105}{w^2}. \quad (26)$$

The leading term above reflects the known value of the shear viscosity for RTA kinetic theory, $\eta/s = \gamma/5$ (see e.g. [3]). Our early-time series solution analytically continued to $w = 100$ reproduces this value of the shear viscosity to three decimal places. This can be seen by re-expanding the Padé approximant of the pressure anisotropy $\text{PA}_{(70,71)}$ for $w \gg 1$:

$$\text{PA}_{(70,71)} \sim \frac{1.60004}{w} + \frac{0.27348}{w^2}, \quad (27)$$

which demonstrates the excellent agreement with the hydrodynamic behavior at large w .

The validity of the analytic continuation over such a large domain is due to the large radius of convergence of the early-time expansion, along with the fact that singularities of the Padé approximant appear in the second and third quadrants of the complex plane. We note that the same is not true for the generic solutions divergent at the origin.

We close this section with the following approximate analytic formula for the attractor in the form of an off-diagonal Padé approximant

$$\mathcal{A} = \frac{1500 - 12w + 30w^2}{1000 + 434w + 95w^2 + 9w^3} \quad (28)$$

The values of the Padé coefficients have been approximated to obtain the above expression. This formula works very well for the physically interesting range $0 \leq w \leq 5$, i.e., from the earliest times all the way into the near-equilibrium

regime. In particular, it reproduces the leading two orders of the early-time expansion, including the free-streaming behavior at $w = 0$, and gives an accurate approximation up to $w \approx 5$ with a relative error of about 2%.

We note finally that Eq. (28) can be used in conjunction with the recursion relation Eq. (6) to give approximate analytic formulae for all the higher moments.

V. THE LATE TIME EXPANSION

The generating function approach that we have developed here can also be applied to investigate the late-time expansion — the asymptotic series for large values of w :

$$\mathcal{M}_n = w^{-n} \sum_{k \geq 0} w^{-k} M_k^{(n)}. \quad (29)$$

The PDE satisfied by the generating function, Eq. (10), can be used to establish recursion relations for the coefficients $M_k^{(n)}$, in a very similar way to what was presented in the previous Section. The main difference is that changing the truncation at a particular order w^{-k} does not change the accuracy of the lower orders already calculated.

The $w \gg 1$ expansion of the generating function then takes the form:

$$G_{\mathcal{M}}(x, w) = \sum_{k=0}^{+\infty} w^{-k} g_k(x), \quad \text{where} \quad g_k(x) \equiv \sum_{n=0}^k x^n M_{k-n}^{(n)}. \quad (30)$$

Using the form of the generating function in the PDE Eq. (10) returns a set of recursive ODEs for the $g_k(x)$: then obtain :

$$\hat{O}_x^{(5)} g_0 = 0; \quad (31)$$

$$\hat{O}_x^{(5)} g_1 = -\hat{O}_x^{(6)} g_0; \quad (32)$$

$$\hat{O}_x^{(5)} g_2 = \left(\frac{4}{3} + \left(1 - \frac{\Delta}{3} \right) \right) \hat{O}_x^{(5)} g_1 - \hat{O}_x^{(6)} g_1; \quad (33)$$

$$\hat{O}_x^{(5)} g_{k+1} = \left(\frac{4}{3} + \left(1 - \frac{\Delta}{3} \right) k \right) \hat{O}_x^{(5)} g_k - \hat{O}_x^{(6)} g_k + \sum_{\ell=1}^{k-1} \left(\frac{2}{3} - \frac{\ell \Delta}{6} \right) M_{k-1-\ell}^{(1)} \hat{O}_x^{(5)} g_\ell, \quad k \geq 2. \quad (34)$$

Using the polynomial expansion of the functions $g_k(x)$ (30) in the above differential equations, we directly obtain recursion relations for the expansion coefficients of these functions. These are

$$M_0^{(0)} = 1; \quad (35)$$

$$K^{(1)} M_0^{(1)} = -A_1^{(0)}; \quad (36)$$

$$M_1^{(0)} = 0; \quad (37)$$

$$K^{(2)} M_0^{(2)} = -A_1^{(1)} M_0^{(1)}; \quad (38)$$

$$K^{(1)} M_1^{(1)} = - \left(\frac{4}{3} + \left(1 - \frac{\Delta}{3} \right) \right) A_1^{(0)} - A_0^{(1)} M_0^{(1)}; \quad (39)$$

$$M_2^{(0)} = 0; \quad (40)$$

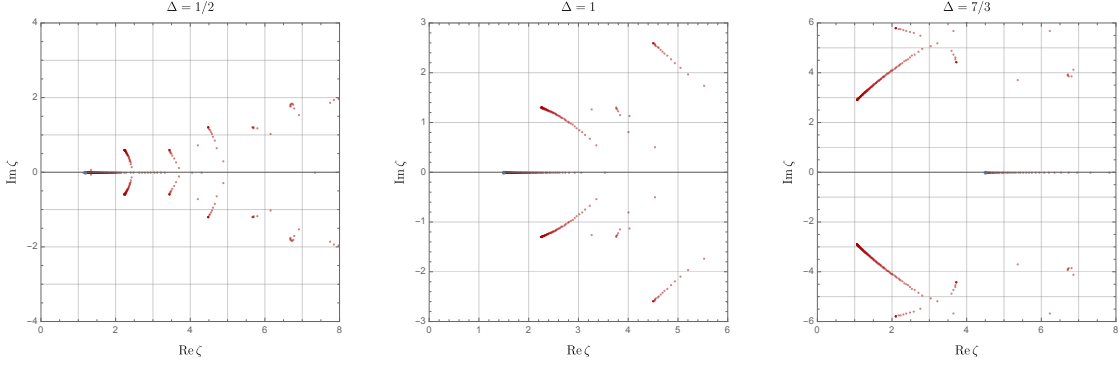


FIG. 2. Poles of the Borel-Padé approximant of order 500 of the late time expansion of the pressure anisotropy $\mathcal{A}$, $\text{BP}_{500}(\mathcal{A})(\zeta)$, for different values of Δ . In blue: start of the leading cut in the real axis, at position ζ_Δ : $\zeta_{1/2} = 6/5$, $\zeta_1 = 3/2$ and $\zeta_{7/3} = 9/2$.

and for $k \geq 3$ we have

$$K^{(k)} M_0^{(k)} = -A_1^{(k-1)} M_0^{(k-1)}; \quad (41)$$

$$K^{(k-1)} M_1^{(k-1)} = K^{(k-1)} \left(\frac{4}{3} + \left(1 - \frac{\Delta}{3}\right) (k-1) \right) M_0^{(k-1)} - A_0^{(k-1)} M_0^{(k-1)} - A_1^{(k-2)} M_1^{(k-2)}; \quad (42)$$

$$K^{(n)} M_{k-n}^{(n)} = K^{(n)} \left(\frac{4}{3} + \left(1 - \frac{\Delta}{3}\right) (k-1) \right) M_{k-1-n}^{(n)} - A_0^{(n)} M_{k-1-n}^{(n)} - A_1^{(n-1)} M_{k-1-n+1}^{(n-1)} - A_{-1}^{(n+1)} M_{k-2-n}^{(n+1)} + K^{(n)} \sum_{\ell=n}^{k-2} M_{k-2-\ell}^{(1)} \left(\frac{2}{3} - \frac{\Delta}{6} \ell \right) M_{\ell-n}^{(n)}, \quad (2 \leq n \leq k-2); \quad (43)$$

$$K^{(1)} M_{k-1}^{(1)} = K^{(1)} \left(\frac{4}{3} + \left(1 - \frac{\Delta}{3}\right) (k-1) \right) M_{k-2}^{(1)} - A_0^{(1)} M_{k-2}^{(1)} - A_{-1}^{(2)} M_{k-3}^{(2)} + K^{(1)} \sum_{\ell=1}^{k-2} M_{k-2-\ell}^{(1)} \left(\frac{2}{3} - \frac{\Delta}{6} \ell \right) M_{\ell-1}^{(1)}; \quad (44)$$

$$M_k^{(0)} = 0. \quad (45)$$

In the above equations $K^{(n)}$ and $A_i^{(n)}$ are the coefficients defined in Eq. (21) and Eq. (22). With these recursion equations we can determine the coefficients $M_{k-n}^{(n)}$, $0 \leq n \leq k$ in terms of the coefficients $M_{k-1-n}^{(n)}$ where now $n \leq k-1$. This allows us to determine the coefficients to any arbitrary order. Note that to obtain the expansion coefficient $M_k^{(1)}$ of the pressure anisotropy, we need to solve these equations up to some order $k+1$.

These recursion relations can be used to generate these coefficients in a vastly more efficient way than by using hitherto existing methods: we have been able to calculate 1000 of them in about 10 minutes. In contrast to the early-time solutions, the series appearing here have zero radius of convergence and need to be interpreted in the sense of asymptotic analysis. The series coefficients grow factorially, so it is natural to use a Borel transform to give meaning to the divergent sum. The analytic continuation of the Borel transform, which can be carried out by means of a Padé approximant, reveals a pattern of singularities whose physical significance is of great interest. Here we present some results concerning the pressure anisotropy $\mathcal{A}$, related to the moment $\mathcal{M}_1$, shown in Fig. 2. The singularity structure is strongly dependent on the value of the parameter Δ , as noted in Ref. [18].

In the conformal case ($\Delta = 1$), we reproduce the pattern found in Ref. [13], consisting of the cut on the real axis and a symmetric pair of branch points with nonvanishing imaginary parts. Since we can easily generate many more terms of the gradient expansion than were hitherto available, we can also discern further branch points at complex-conjugate locations off the real axis. It is natural to conjecture that there is an infinite number of such cuts. While the cut on the real axis clearly corresponds to the nonhydrodynamic mode of the RTA kinetic theory, identifying the physical meaning of the singularities with nonvanishing imaginary parts remains a challenge for the future.

VI. THE DISTRIBUTION FUNCTION

Having a series representation for the moments can be leveraged to obtain corresponding expansions for the full distribution function. This can be achieved by turning the well-known integral equation originally due to Baym [20] and extended in Ref. [30] into an explicit formula by replacing the temperature with its series representation. This is especially useful when applied to the convergent early-time series, since it provides a convergent procedure for computing the full distribution function corresponding to the attractor.

The integral equation for the distribution function appearing in Ref. [30] can be recast in the form

$$f(w, \mathbf{p}) = \frac{1}{D(w, 0)} \left(f_0(\mathbf{p}) + \int_0^w dw' f_{\text{eq}}(w', \mathbf{p}) \partial_{w'} D(w', 0) \right). \quad (46)$$

where f_0 is an initial distribution function

$$D(w, w_0) = \exp \left(\int_{w_0}^w dw' \frac{6}{4 - \mathcal{M}_1} \right). \quad (47)$$

and

$$f_{\text{eq}}(w, \mathbf{p}) = \exp \left(-\frac{1}{w} \sqrt{\frac{p_x^2}{\gamma^2} + (p_x^2 + p_y^2) \frac{w^2}{T^2}} \right). \quad (48)$$

Instead of imposing the Landau condition, we can now use our convergent series for the first moment in Eq. (47). Likewise, starting from Eq. (8) we can obtain the effective temperature

$$T(w) = T_0 w^{-1/3} \exp \left(\frac{2}{3} \int dw' \frac{2\mathcal{M}_1 + 1}{w'(\mathcal{M}_1 - 4)} \right), \quad (49)$$

and use it to eliminate T from Eq. (48). This procedure leads to a series representation for the distribution function valid for small w . One thing worth noting is that generically the first term in Eq. (46) is dominant at small w , with the second term exponentially suppressed, while at large values of w the situation is reversed.

VII. OUTLOOK

In this work, we studied the dynamics of the RTA Boltzmann equation by applying generating function techniques to the hierarchy of moment equations. This yields a new reformulation of this theory in terms of a partial differential equation in two variables: the formal variable x , whose power tags each of the moments, and w , whose power at large times encodes the order of the gradient expansion. Our new approach provides a convergent series representation of the attractor which proves its existence directly in kinetic theory. This goes beyond existing ways of characterizing it, such as numerical approaches, slow-roll approximations, or resummations of the late-time expansion. In particular, truncations of the moment hierarchy lead to ODEs with attractors, but their series representations in positive powers of w or τ – while useful – are guaranteed to differ from the actual RTA series at sufficiently high orders. In contrast, the series we calculate is the first systematic characterization of the attractor of the RTA Boltzmann equation.

We have focused mainly on the early-time behavior of RTA kinetic theory. Earlier works showed that the dynamics of this model reaches the hydrodynamic domain following a far-from-equilibrium attractor, which has previously been obtained by numerical solutions of the Boltzmann equation. Using the generating function technique presented here, we have provided the first solid analytic evidence for this attractor. We have, in particular, obtained a series solution with a finite radius of convergence, which provides an accurate account of the attractor in a large domain, fully describing the passage from the far-from-equilibrium regime to the near-equilibrium domain.

We have also shown that the formulation of RTA kinetic theory presented here opens the door to calculations of the gradient expansion to large orders at a qualitatively new level of efficiency, allowing for hundreds of coefficients with minimal effort. This may eventually lead to a better understanding of the peculiar features hinted at by the results of earlier studies of this problem. It would be interesting to see how this generating function technique can lead to new insight into attractors in other kinetic systems, such as $\lambda\phi^4$ theory [31] and high-temperature QCD plasmas [32]. Potential applications also include calculations of the jet transport coefficient out of equilibrium, following [14, 15].

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Appendix A: Early time moments

The higher contributions $h_k(x)$, $k > 0$ can be obtained by solving the remaining recursion relations which follow from Eq. (19). Using the fact that $m_0^{(1)} = -1/2$ (obtained from the solution at $w = 0$), we obtain

$$\begin{aligned} K^{(n)} \left(k \left(1 - \frac{\Delta}{4} \right) - 1 \right) m_k^{(n)} + A_{-1}^{(n+1)} m_k^{(n+1)} + A_0^{(n)} m_k^{(n)} + A_1^{(n-1)} m_k^{(n-1)} = \\ = -K^{(n)} m_{k-1}^{(n)} + K^{(n)} \sum_{\ell=0}^{k-1} \left(\frac{2}{3} + \Delta \frac{\ell}{6} \right) m_{k-\ell}^{(1)} m_\ell^{(n)}. \end{aligned} \quad (\text{A1})$$

The form of the ODE solved by the function $h_k(x)$, Eq. (19), suggests a decisive simplification. We set

$$h_k(x) = \sum_{\ell=0}^k \alpha_{k\ell} \tilde{h}_\ell(x), \quad k \geq 0 \quad (\text{A2})$$

where $\alpha_{kk} = 1$, and $\tilde{h}_0 = h_0$. By choosing the coefficients α such that the inhomogeneous part of Eq. (19) is canceled, we find that the functions $\tilde{h}_k$ obey the same ODE for each k :

$$\left(\frac{4-\Delta}{4} k - 1 \right) \hat{O}_x^{(5)} \tilde{h}_k + \hat{O}_x^{(6)} \tilde{h}_k = 0, \quad k \geq 0. \quad (\text{A3})$$

In order that this happens, the coefficients $\alpha_{k\ell}$ must obey

$$\alpha_{k\ell} \frac{4-\Delta}{4} (k-\ell) = -\alpha_{k-1,\ell} + \frac{2}{3} \sum_{m=\ell}^{k-1} \frac{4+m\Delta}{4} m_{k-m}^{(1)} \alpha_{m\ell}, \quad (\text{A4})$$

for $\ell = 0, \dots, k-1$, and $\alpha_{kk} = 1$. We can now write

$$\tilde{h}_k = C_k \sum_{n=0}^{+\infty} x^n \beta_k^{(n)}, \quad \beta_k^{(0)} = 1. \quad (\text{A5})$$

in terms of new coefficients $\beta_k^{(n)}$, which can be determined recursively using Eq. (A3). This amounts to solving

$$\left(K^{(n)} \left(k \frac{4-\Delta}{4} - 1 \right) + A_0^{(n)} \right) \beta_k^{(n)} + A_{-1}^{(n+1)} \beta_k^{(n+1)} + A_1^{(n-1)} \beta_k^{(n-1)} = 0, \quad n \geq 1; \quad (\text{A6})$$

$$\beta_k^{(0)} = 1. \quad (\text{A7})$$

We proceed by truncating these relations at some level L by setting $\beta_k^{(m)} = 0$ for $m > L$. This leads to a system of linear equations of the form $Q \mathbf{x} + \mathbf{b} = 0$ with a tridiagonal $L \times L$ matrix Q , a vector of unknowns $\mathbf{x} = (\beta_k^{(1)}, \dots, \beta_k^{(N)})$ and a constant vector $\mathbf{b} = (A_1(0), 0, \dots, 0)$, which can be solved very efficiently for any value of k . The results we show below were obtained for $L = 220$. Increasing the number of moments included in the truncation will increase the accuracy of the coefficients $m_k^{(1)}$ of the pressure anisotropy (for any k). This calculation is very efficient: determining

the coefficients for $k \leq 160$ of the first $L = 220$ moments takes less than 2 minutes in Mathematica. The large k behavior of the coefficient $\beta_k^{(1)}$, which is related to the pressure anisotropy, is given by

$$\beta_k^{(1)} \sim \frac{32}{45} \frac{1}{k} + \mathcal{O}(k^{-2}), \quad (\text{A8})$$

which does not change with truncation level for $L > 200$. This is already a hint suggesting that the small w expansion of the pressure anisotropy possesses a finite radius of convergence.

The relation between the coefficients $m_k^{(n)}$ of the moments and the $\beta_k^{(n)}$ is given by

$$m_k^{(n)} = \sum_{\ell=0}^k C_\ell \beta_\ell^{(n)} \alpha_{k\ell}. \quad (\text{A9})$$

Recalling that $\mathcal{M}_0 = 1$, and thus $m_0^{(0)} = 1$, $m_{k \geq 1}^{(0)} = 0$, we can determine the C_k using

$$C_k = - \sum_{\ell=0}^{k-1} \alpha_{k\ell} C_\ell. \quad (\text{A10})$$

The equations that determine the $\alpha_{k\ell}$ directly are given in (A4) for $\ell = 1, \dots, k-1$, recalling that $\alpha_{kk} = 1$. For $\ell = 0$, we use (A4) together with Eq. (A9), which yields

$$\begin{aligned} \alpha_{k0} \left(\frac{4-\Delta}{4} k + \frac{2}{3} C_0 \left(\beta_k^{(1)} - \beta_0^{(1)} \right) \right) &= -\alpha_{k-1,0} + \frac{2}{3} \sum_{r=1}^{k-1} \alpha_{kr} C_r \left(\beta_r^{(1)} - \beta_k^{(1)} \right) + \\ &+ \frac{2}{3} \sum_{m=1}^{k-1} \frac{4+m\Delta}{4} \alpha_{m0} \sum_{r=0}^{k-m} C_r \beta_r^{(1)} \alpha_{k-m,r}. \end{aligned} \quad (\text{A11})$$

Note that the $\alpha_{k\ell}$ depend on no other coefficients of the functions $\tilde{h}_k$ except $\beta_k^{(1)}$. To solve them for each k , we need to start with the $\alpha_{k\ell}$ with ℓ decreasing from k to 0. We can then find C_k from Eq. (A10) and determine the moment coefficients in Eq. (A9).

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