ALGEBRAIC TOPOLOGY AND DISTRIBUTED COMPUTING

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ABSTRACT. Algebraic topology has proved to be a useful tool in the study of distributed computing. In this paper, we take a geometric realization problem that arises in distributed computing and formulate it as a more general problem in algebraic topology. We go on to give solutions in some initial cases, show that a modified problem has solutions in a wider range of cases, and relate the solutions back to distributed computing.

1. Introduction

This paper investigates problems distributed computing via algebraic topology. Broadly speaking, a *distributed computing system* consists of multiple software components that are on multiple computers but run as a single system. The goal of distributed computing is to make such a network operate as a single computer [26]. Examples of distributed systems include the world wide web, massive multi-player online games and telecommunications networks. Algebraic topology studies the inherent shape of topological spaces that remains unchanged by continuous deformations. This is done by assigning algebraic objects to spaces, such as homology or homotopy groups, that do not change if the space is changed by a continuous deformation. Algebraic topology has proved to be a very powerful way of reducing complex systems to more manageable ones.

Topology made a dramatic appearance in distributed computing in the fundamental discovery by Fischer, Lynch, and Paterson in 1985 (the FLP impossibility theorem [13]) that demonstrated traditional Turing computability theory [38, 39] is not sufficient for analyzing computability problems in asynchronous distributed systems. This implied that distributed computing is different from standard Turing computing, and led to the creation of highly active research on computability and efficiency in asynchronous distributed systems. The topological methods used involved simplicial complexes and continuous maps, and were more in the realm of point-set topology.

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Algebraic topology appeared in the revolutionary work of Herlihy and Shavit [23] in 1993 as a means of modelling and analyzing computability and complexity problems in asynchronous distributed systems. Herlihy and Rajsbaum [19] used this approach to model loop agreement tasks in terms of 2-dimensional simplicial complexes and simplicial maps, and showed that the notion of one task being harder than another (one implements the other) could be characterized algebraically in terms of homomorphisms between fundamental groups of the corresponding simplicial complexes. A higher dimensional version of this was introduced by Liu, Xu and Pan [28] in which the algebraic invariants used were homology groups instead of the fundamental group, and the homomorphisms that needed to be geometrically realized were for $(n+1)$-dimensional spaces with torsion-free homology concentrated in degree n (that is, $H_n(X;\mathbb{Z})$ is a direct sum of copies of $\mathbb{Z}$ and $H_m(X;\mathbb{Z}) \cong 0$ for $m \notin \{0, n\}$). This was developed further by Yue, Wu and Lei [35] by extending from the torsion-free case to any abelian group, but still with homology concentrated in degree n. In this paper we extend further by allowing for homology in multiple degrees.

The paper aims to generate interest among algebraic topologists for the field of distributed computing and vice-versa. It is organized into three parts. The first focuses on algebraic topology (Sections 2 to 6), the second part focuses on computing science (Sections 7 and 8), and the third part focuses on the connection between the two in Section 9. Some key concepts from algebraic topology are homology and homotopy groups, homology decomposition, mapping cones, the Hurewicz homomorphism and simplicial approximation. Some key concepts from distributed omputing are distributed systems, tasks, particularly rendezvous tasks, protocols, and the computability and complexity of solving tasks.

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2. Motivation in algebraic topology

Prompted by a problem in distributed computing, explained further in Section 8, we consider the following geometric realization problem.

Realization Problem: Let X and X' be path-connected spaces and fix $n \geq 2$. Suppose that there are maps $\varphi\colon S^n \longrightarrow X$ and $\varphi'\colon S^n \longrightarrow X'$ and there is a commutative diagram

$$\begin{array}{ccc} H_*(S^n;\mathbb{Z}) & \xrightarrow{\varphi_*} & H_*(X;\mathbb{Z}) \\ & \searrow^{\varphi'_*} & \downarrow{\scriptstyle\gamma} \\ & & H_*(X';\mathbb{Z}) \end{array}$$

for some $\mathbb{Z}$-module map γ. Find a map $g\colon X \longrightarrow X'$ such that $g_* = \gamma$ and there is a diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\varphi} & X \\ & \searrow^{\varphi'} & \downarrow{\scriptstyle g} \\ & & X' \end{array}$$

that commutes up to homotopy (that is, $\varphi' \simeq g \circ \varphi$).

The problem will be broken down into a Homological Problem and a Homotopical Problem. The Homological Problem looks for any map $\overline{g}$ such that $\overline{g}_* = \gamma$. The Homotopical Problem then aims to choose a map g among all possible maps $\overline{g}$ satisfying the Homological Problem with the property that $g \circ \varphi \simeq \varphi'$. In Theorem 5.7, a criterion is given for when the Realization Problem can be solved that recovers the solutions in [28] and [35]. It is too much to hope that the Realization Problem always has a solution but in Theorem 6.7 the Realization Problem is "normalized" in a manner that shows a solution to the Homological Problem implies a solution to the Homotopical Problem.

Geometric realization problems like this have a long and rich history in Algebraic Topology. Steenrod famously asked which graded algebras could be realized as the cohomology rings of spaces, and this has generated an enormous amount of mathematics. Steenrod [37] showed that the polynomial algebra $\mathbb{Z}[x]$ with x of even degree k can be realized if and only if $k \in \{2, 4\}$. Changing the ground ring to the integers modulo a prime p, a complete classification of those polynomial algebras generated by multiple even degree elements has only recently been solved [1, 32]. Steenrod's problem could be extended to ask, for those graded algebras that can be realized as the cohomology rings of spaces, which algebra maps can also be realized as maps of spaces. Much less has been done on this problem. Our Realization Problem is a homological version of this, which further asks for a "trace" to be preserved in the form of a Hurewicz image (the image of φ_* being sent to the image of φ'_*). As such, it is an interesting problem in its own right.

The Realization Problem could, of course, be extended by changing S^n to some other fixed space Y. This would increase the complexity of the problem. While this would be interesting from the point of view of algebraic topology, it may diverge from distributed computing, where an appropriate link is so far only known for the case of the sphere. So for now we will deal only with the case of a sphere.

Throughout Sections 3 to 6 it will be assumed the reader has background in homology (for example, Hatcher's book [17]) and homotopy theory (for example, the books of Arkowitz [2], Brown [8] or Selick [37]).

3. Preliminary information on the homotopy theory of spheres and Moore spaces

We begin with a brief description of homotopy cofibrations, as this will be an important tool used through Sections 3 to 6. Suppose that $A \xrightarrow{f} B$ is a continuous map between pointed path-connected spaces. Let $I = [0, 1]$ be the unit interval with basepoint 0 and let $a_0 \in A$ be the basepoint. The reduced cone on A is the quotient space $CA = (A \times I)/\sim$ where $(a,1) \sim (a',1)$ and $(a_0,t) \sim (a_0,0)$ for all $a, a' \in A$ and $t \in I$. Intuitively, collapse the top of the cylinder $A \times I$ to a point and collapse the line over the basepoint to a_0. The *mapping cone* of f is the quotient space

$$B \cup_f CA = (B \amalg CA)/\sim$$

where $B \amalg CA$ is the disjoint union of B and CA and the relation is given by $f(a) \sim (a,0)$ for all $a \in A$. There is a map $B \longrightarrow B \cup_f CA$ given by inclusion. This results in a sequence of maps

$$A \xrightarrow{f} B \longrightarrow C$$

where $C = B \cup_f CA$. This is referred to as a *homotopy cofibration*. A key property is that it induces a long exact sequence in homology groups. (This should be thought of as a "working definition" of a homotopy cofibration, and is all we need for now. The actual definition of a cofibration appears in Section 9, which is in terms of an extension property. A homotopy version of this then leads to the precise definition of a homotopy cofibration.)

Going a step further, taking the mapping cone of the inclusion $B \longrightarrow B \cup_f CA$ gives a space homotopy equivalent to the one obtained by identifying the subspace B of $B \cup_f CA$ to the basepoint, which in turn is homotopy equivalent to the suspension ΣA. This lets us extend the maps in a homotopy cofibration to a *homotopy cofibration sequence*

$$A \xrightarrow{f} B \longrightarrow C \xrightarrow{\delta} \Sigma A.$$

The map δ geometrically realizes the boundary map in the long exact sequence of homology groups induces by the homotopy cofibration $A \xrightarrow{f} B \longrightarrow C$. Finally, any homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B \end{array}$$

induces a homotopy commutative diagram of homotopy cofibration sequences

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & C & \xrightarrow{\delta} & \Sigma A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma f} \\ A' & \xrightarrow{f'} & B' & \longrightarrow & C' & \xrightarrow{\delta'} & \Sigma A' \end{array}$$

which could be truncated to a homotopy commutative diagram of homotopy cofibrations.

Now we move on to spheres and Moore spaces. Let k be an integer. For $n \geq 1$, let

$$k \colon S^n \longrightarrow S^n$$

be the map of degree k. Precisely, S^n is homeomorphic to $\Sigma^{n-1}S^1$, the $(n-1)$-fold suspension of S^1, and k is the $(n-1)$-fold suspension of the degree k map on S^1. It induces multiplication by k on $H_n(S^n;\mathbb{Z})$.

Define the *mod-k Moore space* $P^{n+1}(k)$ by the homotopy cofibration

$$S^n \xrightarrow{k} S^n \longrightarrow P^{n+1}(k).$$

This Moore space is characterized by the fact that

$$\widetilde{H}_m(P^{n+1}(k);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Note that if $k = 1$ the mapping cone is contractible and if k is negative then the mapping cone is homotopy equivalent to that for $|k|$. So in what follows it may be assumed that $k \geq 2$. The characterization also implies that $\Sigma P^{n+1}(k) \simeq P^{n+2}(k)$. Further, if p and q are coprime, then the group isomorphism $\mathbb{Z}/pq\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$ induces a homotopy equivalence

$$P^{n+1}(pq) \simeq P^{n+1}(p) \vee P^{n+1}(q). \tag{1}$$

Iterating the homotopy equivalence (1), if $k = p_1^{r_1} \cdots p_\ell^{r_\ell}$ is a factorization of k into a product of prime powers, where $\{p_1, \ldots, p_\ell\}$ are distinct primes, then there is a homotopy equivalence

$$P^{n+1}(k) \simeq \bigvee_{j=1}^{\ell} P^{n+1}(p_j^{r_j}). \tag{2}$$

Therefore, for Moore spaces, we need only focus on those of the form $P^{n+1}(p^r)$ where p is a prime.

The next three lemmas describe building block cases leading to Proposition 3.5.

Lemma 3.1. *Suppose that there is a homomorphism $H_n(S^n;\mathbb{Z}) \xrightarrow{\gamma} H_n(S^n;\mathbb{Z})$. Then there is a map $g\colon S^n \longrightarrow S^n$ such that $g_* = \gamma$.*

Proof. Let a generate $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. As γ is a homomorphism it is determined by $\gamma(a)$. If $\gamma(a) = t \cdot a$ for some $t \in \mathbb{Z}$ then let $g\colon S^n \longrightarrow S^n$ be the map of degree t. This satisfies $g_*(a) = t \cdot a = \gamma(a)$, so $g_* = \gamma$. □

Fix a prime p and an integer $r \geq 1$. As a CW-complex, $P^{n+1}(p^r)$ has one 0-cell, one n-cell and one $(n+1)$-cell. Let

$$\tau\colon S^n \longrightarrow P^{n+1}(p^r)$$

be the inclusion of the 0 and n-cells. Observe that if a generates $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ and b generates $H_n(P^{n+1}(p^r);\mathbb{Z}) \cong \mathbb{Z}/p^r\mathbb{Z}$ then $\tau_*(a) = b$.

For any path-connected space X and integer t there is a map $\Sigma X \xrightarrow{t} \Sigma X$ of degree t. This follows since there is a homeomorphism $\Sigma X \cong S^1 \wedge X$ and the map t is obtained by mutiplying by t in the S^1-coordinate and doing the identity map in the X-coordinate. Consequently, in homology t_*

is multiplication by t on $H_n(\Sigma X;\mathbb{Z})$ for all $n \geq 1$. In particular, if $n \geq 2$ then $P^{n+1}(p^r)$ is a suspension, implying that it has a degree t map.

Lemma 3.2. *Suppose that there is a homomorphism $H_n(S^n;\mathbb{Z}) \xrightarrow{\gamma} H_n(P^{n+1}(p^r);\mathbb{Z})$ where p is a prime and $r \geq 1$. Then there is a map $g\colon S^n \longrightarrow P^{n+1}(p^r)$ such that $g_* = \gamma$.*

Proof. Let a generate $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ and b generate $H_n(P^{n+1}(p^r);\mathbb{Z}) \cong \mathbb{Z}/p^r\mathbb{Z}$. As γ is a homomorphism it is determined by $\gamma(a)$. Suppose that $\gamma(a) = t \cdot b$ for some $t \in \mathbb{Z}/p^r\mathbb{Z}$. Let g be the composite

$$g\colon S^n \xrightarrow{\tau} P^{n+1}(p^r) \xrightarrow{t} P^{n+1}(p^r).$$

Then $g_*(a) = t_*(\tau_*(a)) = t_*(b) = t \cdot b = \gamma(a)$. Thus $g_* = \gamma$. □

Lemma 3.3. *Suppose that there is a homomorphism $H_n(P^{n+1}(p^r);\mathbb{Z}) \xrightarrow{\gamma} H_n(P^{n+1}(q^s);\mathbb{Z})$ where p and q are primes and $r, s \geq 1$. Then there is a map $g\colon P^{n+1}(p^r) \longrightarrow P^{n+1}(q^s)$ such that $g_* = \gamma$.*

Proof. Let a generate $H_n(P^{n+1}(p^r);\mathbb{Z}) \cong \mathbb{Z}/p^r\mathbb{Z}$ and let b generate $H_n(P^{n+1}(q^s);\mathbb{Z}) \cong \mathbb{Z}/q^s\mathbb{Z}$. As γ is a homomorphism it is determined by $\gamma(a)$. Suppose that $\gamma(a) = t \cdot b$ for some $t \in \mathbb{Z}/q^s\mathbb{Z}$. There are three cases.

Case 1: p and q are different primes. The only homomorphism $\mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}/q^s\mathbb{Z}$ is the trivial one, so if we take $g\colon P^{n+1}(p^r) \longrightarrow P^{n+1}(q^s)$ to be the trivial map, then $g_* = \gamma$.

Case 2: $p = q$ and $s \geq r$. Then $\mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}/p^s\mathbb{Z}$ is determined by $\gamma(a) = t \cdot b$ where $t = p^{s-r} \cdot t'$ for some $t' \geq 1$. Define the map ω_r^s by the homotopy cofibration diagram

$$\begin{array}{ccccc} S^n & \xrightarrow{p^r} & S^n & \longrightarrow & P^{n+1}(p^r) \\ \| & & \downarrow{\scriptstyle p^{s-r}} & & \downarrow{\scriptstyle \omega_r^s} \\ S^n & \xrightarrow{p^s} & S^n & \longrightarrow & P^{n+1}(p^s). \end{array}$$

Notice that ω_r^s is degree p^{s-r} on the n-cell and degree 1 on the $(n+1)$-cell. Therefore, in integral homology, $(\omega_r^s)_*(a) = p^{s-r} \cdot b$. Let g be the composite

$$g\colon P^{n+1}(p^r) \xrightarrow{\omega_r^s} P^{n+1}(p^s) \xrightarrow{t'} P^{n+1}(p^s).$$

Then $g_*(a) = t'_*((\omega_r^s)_*(a)) = t'_*(p^{r-s} \cdot b) = t' \cdot p^{r-s} \cdot b = t \cdot b$. Thus $g_* = \gamma$.

Case 3: $p = q$ and $s < r$. Then $\mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}/p^s\mathbb{Z}$ is determined by $\gamma(a) = t \cdot b$ where $t \geq 1$. Define the map ρ_r^s by the homotopy cofibration diagram

$$\begin{array}{ccccc} S^n & \xrightarrow{p^r} & S^n & \longrightarrow & P^{n+1}(p^r) \\ \downarrow{\scriptstyle p^{r-s}} & & \| & & \downarrow{\scriptstyle \rho_r^s} \\ S^n & \xrightarrow{p^s} & S^n & \longrightarrow & P^{n+1}(p^s). \end{array}$$

Notice that ρ_r^s is degree 1 on the n-cell and degree p^{r-s} on the $(n+1)$-cell. Therefore, in integral homology, $(\rho_r^s)_*(a) = b$. Let g be the composite

$$g\colon P^{n+1}(p^r) \xrightarrow{\rho_r^s} P^{n+1}(p^s) \xrightarrow{t} P^{n+1}(p^s).$$

Then $g_*(a) = t_*((\rho_r^s)_*(a)) = t_*(b) = t \cdot b$. Thus $g_* = \gamma$. □

Remark 3.4. Since there is no nontrivial homomorphism $\mathbb{Z}/p^r\mathbb{Z} \longrightarrow \mathbb{Z}$, any homomorphism $H_n(P^{n+1}(p^r);\mathbb{Z}) \xrightarrow{\gamma} H_n(S^n;\mathbb{Z})$ is geometrically realized by the trivial map $P^{n+1}(p^r) \longrightarrow S^n$.

Next, the previous lemmas are combined. Fix $n \geq 2$ so that all spheres and Moore spaces are suspensions. Let

$$M = \left(\bigvee_{i=1}^{k} S^n\right) \vee \left(\bigvee_{j=1}^{\ell} P^{n+1}(p_j^{r_j})\right) \qquad \text{and} \qquad N = \left(\bigvee_{s=1}^{u} S^n\right) \vee \left(\bigvee_{t=1}^{v} P^{n+1}(p_t^{r_t})\right).$$

Then

$$H_n(M;\mathbb{Z}) \cong \left(\bigoplus_{i=1}^{k} \mathbb{Z}\right) \oplus \left(\bigoplus_{j=1}^{\ell} \mathbb{Z}/p_j^{r_j}\mathbb{Z}\right) \qquad \text{and} \qquad H_n(N;\mathbb{Z}) \cong \left(\bigoplus_{s=1}^{u} \mathbb{Z}\right) \oplus \left(\bigoplus_{t=1}^{v} \mathbb{Z}/p_t^{r_t}\mathbb{Z}\right).$$

Proposition 3.5. *Suppose that there is a homomorphism $H_n(M;\mathbb{Z}) \xrightarrow{\gamma} H_n(N;\mathbb{Z})$. Then there is a map $g\colon M \longrightarrow N$ such that $g_* = \gamma$.*

Proof. Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_\ell\}$ respectively generate the integral and torsion parts of $H_n(M;\mathbb{Z})$, and let $\{c_1, \ldots, c_u\}$ and $\{d_1, \ldots, d_v\}$ respectively generate the integral and torsion parts of $H_n(N;\mathbb{Z})$. As γ is a homomorphism it is determined by its action on the generating set of $H_n(M;\mathbb{Z})$. Suppose that for $1 \leq i \leq k$ we have

$$\gamma(a_i) = \sum_{s=1}^{u} \alpha_{i,s} \cdot c_s + \sum_{t=1}^{v} \beta_{i,t} \cdot d_t, \tag{3}$$

where $\alpha_{i,s} \in \mathbb{Z}$ and $\beta_{i,t} \in \mathbb{Z}/p_t^{r_t}\mathbb{Z}$. Noting that there are no nontrivial homomorphisms from a finite abelian group to $\mathbb{Z}$, suppose that for $1 \leq j \leq \ell$ we have

$$\gamma(b_j) = \sum_{t=1}^{v} \delta_{j,t} \cdot d_t, \tag{4}$$

where $\delta_{j,t} \in \mathbb{Z}/p_t^{r_t}\mathbb{Z}$.

For a space X let $\Sigma X \xrightarrow{\sigma} \Sigma X \vee \Sigma X$ be the standard suspension comultiplication. For any integer $m \geq 2$, let

$$\sigma_m\colon \Sigma X \longrightarrow \bigvee_{i=1}^{m} \Sigma X$$

be an m-fold iteration of σ. As σ is homotopy coassociative, the order in which the iteration forming σ_m takes place is irrelevant. For $1 \leq i \leq k$ define $\epsilon_i\colon S^n \longrightarrow N$ by the composite

$$\epsilon_i\colon S^n \xrightarrow{\sigma_{u+v}} \left(\bigvee_{s=1}^{u} S^n\right) \vee \left(\bigvee_{t=1}^{v} S^n\right) \xrightarrow{(\bigvee_{s=1}^{u} \alpha_{i,s}) \vee (\bigvee_{t=1}^{v} \beta_{i,t} \cdot \tau)} \left(\bigvee_{s=1}^{u} S^n\right) \vee \left(\bigvee_{t=1}^{v} P^{n+1}(p_t^{r_t})\right) = N.$$

Then the definition of ϵ_i, Lemmas 3.1 and 3.2, and (3) imply that $(\epsilon_i)_*(a_j) = \gamma(a_i)$.

For $1 \leq j \leq \ell$ define $\varepsilon_j : P^{n+1}(p_j^{r_j}) \longrightarrow N$ by the composite

$$\varepsilon_j : P^{n+1}(p_j^{r_j}) \xrightarrow{\sigma_v} \bigvee_{t=1}^{v} P^{n+1}(p_j^{r_j}) \xrightarrow{\bigvee_{t=1}^{v} \delta_{j,t} \cdot f_{j,t}} \bigvee_{t=1}^{v} P^{n+1}(p_t^{r_t}) \xrightarrow{I} \left(\bigvee_{s=1}^{u} S^n \right) \vee \left(\bigvee_{t=1}^{v} P^{n+1}(p_t^{r_t}) \right) = N,$$

where I is the inclusion of the right wedge summand and each map $f_{j,t}$ is obtained by applying Lemma 3.3 to the homomorphism $\gamma_{j,t} : H_n(P^{n+1}(p_j^{r_j}); \mathbb{Z}) \longrightarrow H_n(P^{n+1}(p_t^{r_t}); \mathbb{Z})$ determined by $\gamma_{j,t}(b_j) = d_t$. Then the definition of ε_j and (4) imply that $(\varepsilon_j)_*(b_j) = \gamma(b_j)$. Finally, let

$$g : M = \left(\bigvee_{i=1}^{k} S^n \right) \vee \left(\bigvee_{j=1}^{\ell} P^{n+1}(p_j^{r_j}) \right) \longrightarrow N$$

be the wedge sum of the maps ϵ_i for $1 \leq i \leq k$ and ε_j for $1 \leq j \leq \ell$. Then $(g_*)(a_i) = (\epsilon_i)_*(a_j) = \gamma(a_i)$ for all $1 \leq i \leq k$ and $(g_*)(b_j) = (\varepsilon_j)_*(b_j) = \gamma(b_j)$ for all $1 \leq j \leq \ell$. Thus $g_* = \gamma$. $\square$

4. Homology decompositions

Define $M(\mathbb{Z}/k\mathbb{Z}, n) = P^{n+1}(k)$ and $M(\mathbb{Z}, n) = S^n$, and for any finitely generated abelian groups A and B define $M(A \oplus B, n) = M(A, n) \vee M(B, n)$. Then

$$\widetilde{H}_m(M(A, n); \mathbb{Z}) \cong \begin{cases} A & m = n, \\ 0 & m \neq n. \end{cases}$$

Observe that if A is torsion-free then $M(A, n)$ is a wedge of n-dimensional spheres, implying that it is a CW-complex of dimension n, while if A has torsion then $M(A, n)$ is a wedge of n-dimensional spheres and $(n+1)$-dimensional Moore spaces, implying that it is a CW-complex of dimension $n+1$. Thus there is a distinction between the dimension of a CW-complex and the largest degree in homology that is nonzero. This motivates the following definition.

Definition 4.1. A topological space X has *homological dimension* n if $H_n(X; \mathbb{Z}) \neq 0$ and $H_m(X; \mathbb{Z}) = 0$ for all $m > n$.

In particular, $M(A, n)$ has homological dimension n regardless of whether A is torsion-free or not. The following theorem (see, for example, [17, Theorem 4H.3]), describes the *homology decomposition* of a simply-connected CW-complex.

Theorem 4.2. *Let X be a simply-connected CW-complex of homological dimension n. For $2 \leq m \leq n$, let $H_m = H_m(X; \mathbb{Z})$. Then there is a sequence of subcomplexes $\{X_m\}_{m=1}^{n}$ such that:*

(a) $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \simeq X$;
(b) *X_1 is the basepoint of X;*

(c) *for $2 \leq m \leq n$ there is a homotopy cofibration*

$$M(H_m, m-1) \xrightarrow{f_{m-1}} X_{m-1} \longrightarrow X_m,$$

where $(f_{m-1})_*$ is the zero homomorphism in integral homology;

(d) $H_i(X_m;\mathbb{Z}) \cong H_i(X;\mathbb{Z})$ *for $i \leq m$ and $H_i(X_m;\mathbb{Z}) = 0$ for $i > m$.* □

Extending the homotopy cofibration in part (b) of Theorem 4.2 to the right and recalling that $\Sigma M(H_m, m-1) \simeq M(H_m, m)$, we obtain for $2 \leq m \leq n$ a homotopy cofibration

$$X_{m-1} \longrightarrow X_m \longrightarrow M(H_m, m).$$

Parts (b) and (c) of Theorem 4.2 may be rephrased as follows.

Corollary 4.3. *For $2 \leq m \leq n$ the homotopy cofibration $X_{m-1} \longrightarrow X_m \longrightarrow M(H_m, m)$ induces isomorphisms*

$$H_i(X_m;\mathbb{Z}) \cong \begin{cases} H_i(X_{m-1};\mathbb{Z}) & \text{if } i < m, \\ H_m(M(H_m, m);\mathbb{Z}) & \text{if } i = m, \\ 0 & \text{if } i > m. \end{cases}$$ □

We give two examples of homology decompositions determining homotopy types.

Lemma 4.4. *Let X be a simply-connected CW-complex with the property that*

$$\widetilde{H}_m(X;\mathbb{Z}) \cong \begin{cases} G & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

where G is a finitely generated abelian group. Then X is homotopy equivalent to a wedge of n-dimensional spheres and $(n+1)$-dimensional Moore spaces.

Proof. By Theorem 4.2, the description of $H_*(X;\mathbb{Z})$ implies that the homology decomposition $\{X_m\}_{m=1}^n$ of X has the following properties: (i) $X_1, \ldots, X_{n-1}$ are all contractible and (ii) if $G \cong (\bigoplus_{j=1}^s \mathbb{Z}) \oplus (\bigoplus_{k=1}^t \mathbb{Z}/p_k^{r_k}\mathbb{Z})$ then $X_n \simeq (\bigvee_{j=1}^s S^n) \vee (\bigvee_{k=1}^t P^{n+1}(p_k^{r_k}))$. Since $X \simeq X_n$, the lemma follows. □

In general, the Hurewicz homomorphism is a map $\pi_m(Y) \longrightarrow H_m(Y;\mathbb{Z})$ given by sending the homotopy class of a map $f\colon S^m \longrightarrow Y$ to $f_*(\iota_m)$, where $\iota_m \in H_m(S^m;\mathbb{Z})$ is a generator. A proof of the following result can be found in [17, Theorem 4.37].

Theorem 4.5 (Hurewicz)**.** *Let $n \geq 2$. If Y is $(n-1)$-connected then the Hurewicz homorphism induces an isomorphism $\pi_n(Y) \cong H_n(Y;\mathbb{Z})$.* □

In particular, if Y is $(n-1)$-connected then any map $S^n \xrightarrow{f} Y$ that has zero image in homology is null homotopic.

Lemma 4.6. *Let X be a space with the property that*

$$\widetilde{H}_m(X;\mathbb{Z}) \cong \begin{cases} T & \text{if } m = n, \\ G & \text{if } m = n-1, \\ 0 & \text{if } m \notin \{n-1, n\}, \end{cases}$$

where G is a finitely generated abelian group and T is a finitely generated torsion-free abelian group. Then X is homotopy equivalent to a wedge of $(n-1)$ and n-dimensional spheres and n-dimensional Moore spaces.

Proof. By Theorem 4.2, the description of $H_*(X;\mathbb{Z})$ implies that the homology decomposition $\{X_m\}_{m=1}^n$ of X has the following properties: (i) $X_1, \ldots, X_{n-2}$ are all contractible, (ii) if $G \cong (\bigoplus_{j=1}^s \mathbb{Z}) \oplus (\bigoplus_{k=1}^t \mathbb{Z}/p_k^{r_k}\mathbb{Z})$ then $X_{n-1} \simeq (\bigvee_{j=1}^s S^{n-1}) \vee (\bigvee_{k=1}^t P^n(p_k^{r_k}))$, (iii) if $T \cong \bigoplus_{\ell=1}^u \mathbb{Z}$, then there is a homotopy cofibration

$$\bigvee_{\ell=1}^{u} S^{n-1} \xrightarrow{f_{n-1}} \left(\bigvee_{j=1}^{s} S^{n-1}\right) \vee \left(\bigvee_{k=1}^{t} P^n(p_k^{r_k})\right) \longrightarrow X_n, \tag{5}$$

where $(f_{n-1})_* = 0$, and (iv) $X_n \simeq X$.

Since X_{n-1} is $(n-2)$-connected, Theorem 4.5 implies that $\pi_{n-1}(X_n) \cong H_{n-1}(X_{n-1};\mathbb{Z})$. Therefore, as $(f_{n-1})_* = 0$, the restriction of f_{n-1} to any of the wedge summands in $\bigvee_{\ell=1}^u S^{n-1}$ has zero image in homology and so must be null homotopic. Hence f_{n-1} is null homotopic. Thus the homotopy cofibration (5) implies that there is a homotopy equivalence

$$X_n \simeq \left(\bigvee_{j=1}^{s} S^{n-1}\right) \vee \left(\bigvee_{k=1}^{t} P^n(p_k^{r_k})\right) \vee \left(\bigvee_{\ell=1}^{u} S^{n}\right).$$

Since $X \simeq X_n$, the Lemma is proved. □

5. A method for solving the Realization Problem

The Realization Problem can be broken into two separate problems. The hypotheses are that there are maps $\varphi\colon S^n \longrightarrow X$ and $\varphi'\colon S^n \longrightarrow X'$ and a commutative diagram

$$\begin{array}{ccc} H_*(S^n;\mathbb{Z}) & \xrightarrow{\varphi_*} & H_*(X;\mathbb{Z}) \\ & \searrow^{\varphi'_*} & \downarrow \gamma \\ & & H_*(X';\mathbb{Z}) \end{array}$$

for some $\mathbb{Z}$-module map γ.

The Homological Problem: Find a map $\overline{g}\colon X \longrightarrow X'$ such that $\overline{g}_*$ equals γ in degree n homology.

Note that having $\overline{g}_*$ equal γ in degree n homology implies that $\overline{g}_* \circ \varphi_* = \varphi'_*$.

The Homotopical Problem: Given a map $X \xrightarrow{\overline{g}} X'$ such that $\overline{g}_* \circ \varphi_* = \varphi'_*$, find a map $g\colon X \longrightarrow X'$ such that $g \circ \varphi \simeq \varphi'$.

Solutions to both the Homological and Homotopical Problems solve the Realization Problem.

We first address the Homological Problem. Suppose that X and X' are simply-connected CW-complexes of homological dimension n such that

$$H_n(X;\mathbb{Z}) \cong \Big(\bigoplus_{i=1}^{k}\mathbb{Z}\Big) \oplus \Big(\bigoplus_{j=1}^{\ell}\mathbb{Z}/p_j^{r_j}\mathbb{Z}\Big) \qquad \text{and} \qquad H_n(X';\mathbb{Z}) \cong \Big(\bigoplus_{s=1}^{u}\mathbb{Z}\Big) \oplus \Big(\bigoplus_{t=1}^{v}\mathbb{Z}/p_t^{r_t}\mathbb{Z}\Big).$$

Here, we allow the possibility that $k = 0$, in which case $H_n(X;\mathbb{Z})$ is a finite group, as well as the corresponding possibilities that one of ℓ, u or v equals zero. Let

$$M = \Big(\bigvee_{i=1}^{k} S^{n-1}\Big) \vee \Big(\bigvee_{j=1}^{\ell} P^n(p_j^{r_j})\Big) \qquad \text{and} \qquad M' = \Big(\bigvee_{s=1}^{u} S^{n-1}\Big) \vee \Big(\bigvee_{t=1}^{v} P^n(p_t^{r_t})\Big).$$

If $k = 0$, regard $\bigvee_{i=1}^{k} S^{n-1}$ as a point, and similarly for one of ℓ, u or v being zero. Then $H_n(\Sigma M;\mathbb{Z}) \cong H_n(X;\mathbb{Z})$ and $H_n(\Sigma M';\mathbb{Z}) \cong H_n(X';\mathbb{Z})$. By Theorem 4.2, X and X' have homology decompositions $\{X_m\}_{m=1}^{n}$ and $\{X'_m\}_{m=1}^{n}$ respectively with $X_n \simeq X$ and $X'_n \simeq X'$. In particular, there are homotopy cofibration sequences

$$M \xrightarrow{f} X_{n-1} \longrightarrow X \xrightarrow{\delta} \Sigma M \qquad \text{and} \qquad M' \xrightarrow{f'} X'_{n-1} \longrightarrow X' \xrightarrow{\delta'} \Sigma M'$$

with $f_* = f'_* = 0$ and both δ_* and δ'_* inducing isomorphisms in degree n homology.

One approach for obtaining a map $X \xrightarrow{\overline{g}} X'$ such that $\overline{g}_* = \gamma$ in degree n is as follows. Let

$$\gamma_n \colon H_n(X;\mathbb{Z}) \longrightarrow H_n(X';\mathbb{Z})$$

be the restriction of γ to degree n. Since δ_* and δ'_* are isomorphisms in degree n homology, γ_n can equivalently be regarded as a map $H_n(\Sigma M;\mathbb{Z}) \xrightarrow{\gamma'_n} H_n(\Sigma M';\mathbb{Z})$ where $\gamma'_n = \delta'_* \circ \gamma_n \circ \delta_*^{-1}$. As the homology suspension is an isomorphism, γ'_n can then equivalently be regarded as a map $H_{n-1}(M;\mathbb{Z}) \xrightarrow{\gamma''_n} H_{n-1}(M';\mathbb{Z})$ where $\Sigma\gamma''_n = \gamma'_n$. Since M and M' are wedges of spheres and Moore spaces, by Proposition 3.5, there is a map

$$g'' \colon M \longrightarrow M'$$

such that $g''_* = \gamma''_n$. The goal is to find a map ϵ that gives a homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & X_{n-1} \\ \big\downarrow{\scriptstyle g''} & & \big\downarrow{\scriptstyle \epsilon} \\ M' & \xrightarrow{f'} & X'_{n-1}. \end{array} \tag{6}$$

Proposition 5.1. *In the Realization Problem, suppose that X and X' are simply-connected CW-complexes of homological dimension n. If there is a map ϵ making the diagram (6) homotopy commute then there is a map $\overline{g} \colon X \longrightarrow X'$ with the property that $\overline{g}_* = \gamma$ in degree n homology.*

Proof. Given an ϵ such that (6) homotopy commutes, we obtain a homotopy cofibration diagram

$$\begin{array}{ccccccc} M & \xrightarrow{f} & X_{n-1} & \longrightarrow & X & \xrightarrow{\delta} & \Sigma M \\ \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \Sigma g''} \\ M' & \xrightarrow{f'} & X'_{n-1} & \longrightarrow & X & \xrightarrow{\delta'} & \Sigma M' \end{array}$$

for some map $\overline{g}$. Since g'' geometrically realizes γ''_n, we have $(\Sigma g'')_* = \Sigma\gamma''_n = \gamma'_n$. Since δ, δ' and γ'_n are isomorphisms in degree n homology, from the right square we obtain in degree n homology an isomorphism $\overline{g}_* = (\delta')^{-1}_* \circ \gamma'_n \circ \delta_*$. By definition, $\gamma'_n = \delta'_* \circ \gamma_n \circ \delta^{-1}_*$, so we obtain $\overline{g}_* = \gamma_n$ in degree n homology. By definition, γ_n is the restriction of γ to degree n, so we obtain $\overline{g}_* = \gamma$ in degree n homology. □

A straightforward but meaningful example of Proposition 5.1 is the following.

Corollary 5.2. *Suppose that the map f' in (6) is null homotopic. Then there is a map $\overline{g}\colon X \longrightarrow X'$ with the property that $\overline{g}_* = \gamma$ in degree n homology.*

Proof. In (6), take ϵ to be the trivial map. Then $f' \circ g'' \simeq \epsilon \circ f$ since both f' and ϵ are null homotopic. Now apply Proposition 5.1. □

Remark 5.3. In general, the homotopy commutativity of (6) is a challenging problem. The homology decomposition implies that the maps f and f' in (6) induce the zero map in homology, however, their homotopy classes may be nontrivial. The triviality condition in Corollary 5.2 is designed to make the homology and homotopy properties align. In the general case, one might try to proceed in two steps: first, show the composition $X \xrightarrow{\delta} \Sigma M \xrightarrow{\Sigma g''} \Sigma M' \xrightarrow{\Sigma f'} \Sigma X'_{n-1}$ is null homotopic, implying that $\Sigma f' \circ \Sigma g''$ extends across Σf to a map $\epsilon'\colon \Sigma X_{n-1} \longrightarrow \Sigma X'_{n-1}$, and second, show that ϵ' is the suspension of a map ϵ, and in such a way that the homotopy $\epsilon' \circ \Sigma f \simeq \Sigma f' \circ \Sigma g''$ "de-suspends". It would be interesting to have concrete conditions for when a map ϵ exists that makes (6) homotopy commute.

For example, suppose that

$$\widetilde{H}_m(X'; \mathbb{Z}) \cong \begin{cases} G & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

where G is a finitely generated abelian group. We say that the homology of X' is *concentrated in degree n*. The homological decomposition of X' implies that X'_{n-1} is contractible while Lemma 4.4 implies that X' is homotopy equivalent to a wedge of n-dimensional spheres and $(n+1)$-dimensional Moore spaces. The contractibility of X'_{n-1} implies that in the homotopy cofibration sequence $M \xrightarrow{f'} X'_{n-1} \longrightarrow X' \xrightarrow{\delta} \Sigma M$ the map f' is trivial. By Corollary 5.2 we obtain the following.

Proposition 5.4. *In the Realization Problem, suppose that X is a simply-connected CW-complex of homological dimension n and the homology of X' is concentrated in degree n. Then the Homological Problem has a solution. That is, there is a map $\overline{g}\colon X \longrightarrow X'$ with the property that $\overline{g}_* = \gamma$.* □

As another example, suppose that

$$\widetilde{H}_m(X';\mathbb{Z}) \cong \begin{cases} T & \text{if } m = n, \\ G & \text{if } m = n-1, \\ 0 & \text{if } m \notin \{n-1, n\}, \end{cases} \tag{7}$$

where G is a finitely generated abelian group and T is a finitely generated torsion-free abelian group. By Lemma 4.6, X' is homotopy equivalent to a wedge of spheres and Moore spaces. In particular, as in the proof of that Lemma, the homotopy cofibration sequence $M \xrightarrow{f'} X'_{n-1} \longrightarrow X' \xrightarrow{\delta} \Sigma M$ has the map f' being null homotopic. By Corollary 5.2 we obtain the following.

Proposition 5.5. *In the Realization Problem, suppose that X is a simply-connected CW-complex of homological dimension n and the homology of X' satisfies (7). Then the Homological Problem has a solution. That is, there is a map $\overline{g}\colon X \longrightarrow X'$ with the property that $\overline{g}_* = \gamma$ in degree n homology.* □

We next address the Homotopical Problem. In this section this is done briefly in a special case through an appeal to the Hurewicz Theorem. In Section 6 it is done more systematically.

Proposition 5.6. *In the Realization Problem, suppose that the Homological Problem has a solution: there is a map $\overline{g}\colon X \longrightarrow X'$ such that $\overline{g}_* = \gamma$ in degree n homology. If the homology of X' is concentrated in degree n then there is a map $g\colon X \longrightarrow X'$ such that $g \circ \varphi \simeq \varphi'$.*

Proof. Since X' has its homology concentrated in degree n, it is $(n-1)$-connected. Theorem 4.5 therefore implies that any two maps $S^n \longrightarrow X'$ which induce the same map in homology are homotopic. In our case, by hypothesis, $\overline{g} \circ \varphi$ and φ' induce the same map in homology and hence are homotopic. Now take $g = \overline{g}$. □

Combining Propositions 5.4 and 5.6 we immediately obtain the following.

Theorem 5.7. *In the Realization Problem, suppose that X is a simply-connected CW-complex of homological dimension n and the homology of X' is concentrated in degree n. Then there is a map $g\colon X \longrightarrow X'$ such that $g \circ \varphi \simeq \varphi'$.* □

Theorem 5.7 generalizes [35], which considered the case when the homology of X is also concentrated in degree n. They did not state it in these terms, but their Lemma 9 is the analogue of our Proposition 3.5 and the proof of their Lemma 10 includes the corresponding version our Theorem 5.7. The result in [35] in turn generalized [28], which considered the case when the homology of X is concentrated in degree n and both the homology of X and X' are torsion-free. Again, they did not state their result in our terms but their Lemma 4 is the torsion-free analogue of our Lemma 3.5 and the proof of their Lemma 5 includes the corresponding version of our Theorem 5.7.

6. A normalization procecdure for the Homotopical Problem

In this section we introduce additional methods to prove a further case of the Realization Problem in Corollary 6.5 that will, in particular, result in a solution when X and X' both have homology of the form (7). We then extend the methods to show in Theorem 6.7 that when the Homology Problem has a solution, while the Homotopical Problem may not always have a solution, a certain "normalized" problem will always have a solution.

In general, for path-connected spaces A and B, let

$$A \vee B \xrightarrow{j} A \times B$$

be the inclusion of the wedge into the product.

Lemma 6.1. *Let Y be a CW-complex of dimension $\leq n$. If X is simply-connected then the map $X \vee S^n \xrightarrow{j} X \times S^n$ induces a monomorphism $[Y, X \vee S^n] \xrightarrow{j_*} [Y, X \times S^n]$.*

Proof. In general, for a path-connected pointed space A, the based loop space ΩA is the space of all continuous pointed maps $S^1 \longrightarrow A$. Suppose that A and B are path-connected pointed spaces. Using methods developed by Ganea [15], it can be shown that the homotopy fibre of the inclusion $A \vee B \xrightarrow{j} A \times B$ is homotopy equivalent to $\Sigma\Omega A \wedge \Omega B$ (see, for example, [37, Theorem 7.7.4]). In our case, the homotopy fibre of the inclusion $X \vee S^n \xrightarrow{j} X \times S^n$ is homotopy equivalent to $\Sigma\Omega X \wedge \Omega S^n$. Since X is simply-connected, the space $\Sigma\Omega X \wedge \Omega S^n$ is n-connected. Thus if Y is any CW-complex of dimension $\leq n$ then the map j induces a monomorphism $[Y, X \vee S^n] \xrightarrow{j_*} [Y, X \times S^n]$. □

For path-connected spaces A and B, let

$$p_1 : A \vee B \longrightarrow A \qquad \text{and} \qquad p_2 : A \vee B \longrightarrow B$$

be the pinch maps onto the left and right wedge summand respectively. Given a map $f : Y \longrightarrow A \vee B$ let f_1 and f_2 be the composites

$$f_1 : Y \xrightarrow{f} A \vee B \xrightarrow{p_1} A \qquad \text{and} \qquad f_2 : Y \xrightarrow{f} A \vee B \xrightarrow{p_2} B.$$

If Y is a co-H-space, let

$$\sigma : Y \longrightarrow Y \vee Y$$

be the comultiplication.

Lemma 6.2. *Let Y be a CW-complex of dimension $\leq n$ that is a co-H-space. If X is simply-connected then any map $Y \xrightarrow{f} X \vee S^n$ is homotopic to the composite $Y \xrightarrow{\sigma} Y \vee Y \xrightarrow{f_1 \vee f_2} X \vee S^n$.*

Proof. Since Y is an n-dimensional CW-complex, Lemma 6.1 implies that the homotopy class of $Y \xrightarrow{f} X \vee S^n$ is determined by the composite

$$\overline{f} \colon Y \xrightarrow{f} X \vee S^n \xrightarrow{j} X \times S^n.$$

Any map into a product is determined by its projections to the factors. Explicitly, if

$$\pi_1 \colon X \times S^n \longrightarrow X \qquad \text{and} \qquad \pi_2 \colon X \times S^n \longrightarrow S^n$$

are the projections onto the left and right factors respectively, and $\overline{f}_1$ and $\overline{f}_2$ are the composites

$$\overline{f}_1 \colon Y \xrightarrow{\overline{f}} X \times S^n \xrightarrow{\pi_1} X \qquad \text{and} \qquad \overline{f}_2 \colon Y \xrightarrow{\overline{f}} X \times S^n \xrightarrow{\pi_2} S^n,$$

then $\overline{f}$ is homotopic to the composite

$$Y \xrightarrow{\ \Delta\ } Y \times Y \xrightarrow{\overline{f}_1 \times \overline{f}_2} X \times S^n,$$

where Δ is the diagonal map. Consider the diagram

$$\begin{array}{ccccc} & & Y \vee Y & \xrightarrow{f_1 \vee f_2} & X \vee S^n \\ & \overset{\sigma}{\nearrow} & \big\downarrow j & & \big\downarrow j \\ Y & \xrightarrow{\ \Delta\ } & Y \times Y & \xrightarrow{\overline{f}_1 \times \overline{f}_2} & X \times S^n. \end{array}$$

The left triangle homotopy commutes since σ is a comultiplication. Observe that for $i \in \{1, 2\}$ we have $p_i = \pi_i \circ j$ and therefore $\overline{f}_i = \pi_i \circ \overline{f} = \pi_i \circ j \circ f = p_i \circ f = f_i$. Thus the right square commutes by the naturality of j. The bottom row of the diagram is homotopic to $\overline{f}$, so we obtain $\overline{f} \simeq j \circ (f_1 \vee f_2) \circ \sigma$. On the other hand, by definition, $\overline{f} = j \circ f$. By Lemma 6.1, j induces a monomorphism $[Y, X \vee S^n] \longrightarrow [Y, X \times S^n]$. Thus, as $j \circ f \simeq j \circ (f_1 \vee f_2) \circ \sigma$, we have $f \simeq (f_1 \vee f_2) \circ \sigma$. □

In general, a homotopy cofibration sequence $A \longrightarrow B \longrightarrow C \xrightarrow{\delta} \Sigma A$ of path-connected spaces comes equipped with a *homotopy coaction*

$$\theta \colon C \longrightarrow C \vee \Sigma A.$$

This has the property that $p_1 \circ \theta$ is homotopic to the identity map on C and $p_2 \circ \theta$ is homotopic to δ.

Lemma 6.3. *Let X be a simply-connected CW-complex of homological dimension n. Suppose that there is a map $\varphi \colon S^n \longrightarrow X$ such that the induced map in cohomology $H^*(X; \mathbb{Z}) \xrightarrow{\varphi^*} H^*(S^n; \mathbb{Z})$ is a surjection. Then there is a map $\vartheta \colon X \longrightarrow X \vee S^n$ such that $p_1 \circ \vartheta$ is homotopic to the identity map on X and $p_2 \circ \vartheta$ is a left homotopy inverse for φ.*

Proof. First observe that φ^* being a surjection implies that $H^n(X; \mathbb{Z})$ must have a $\mathbb{Z}$-summand, since all homomorphisms from a torsion abelian group to $\mathbb{Z}$ are trivial. The universal coefficient theorem therefore implies that $H_n(X;\mathbb{Z})$ also has a $\mathbb{Z}$-summand. Thus $H_n(X;\mathbb{Z}) \cong (\bigoplus_{j=1}^{s}\mathbb{Z}) \oplus (\bigoplus_{k=1}^{t}\mathbb{Z}/p_k^{r_k}\mathbb{Z})$ where $s \geq 1$ and possibly $t = 0$. Let

$$M = \Big(\bigvee_{j=1}^{s} S^{n-1}\Big) \vee \Big(\bigvee_{k=1}^{t} P^n(p_k^{r_k})\Big).$$

Then $H_n(X;\mathbb{Z}) \cong H_n(\Sigma M;\mathbb{Z})$. As X is simply-connected and has homological dimension n, by Theorem 4.2 it has a homology decomposition $\{X_m\}_{m=1}^{n}$ with $X_n \simeq X$. The description of $H_n(X;\mathbb{Z})$ implies that there is a homotopy cofibration sequence

$$M \xrightarrow{f} X_{n-1} \longrightarrow X \xrightarrow{\delta} \Sigma M,$$

where δ^* is an isomorphism in degree n homology. Let $\overline{\delta}$ be the composite

$$\overline{\delta}\colon X \xrightarrow{\delta} \Sigma M \xrightarrow{p} \bigvee_{j=1}^{s} S^n,$$

where p is the pinch map. Then $\overline{\delta}_*$ is an isomorphism when restricted to the torsion-free subgroup of $H_n(X;\mathbb{Z})$. Therefore, by the universal coefficient theorem, $\overline{\delta}^*$ is an isomorphism onto the torsion-free subgroup of $H^n(X;\mathbb{Z})$.

Next consider the composite

$$S^n \xrightarrow{\varphi} X \xrightarrow{\overline{\delta}} \bigvee_{j=1}^{s} S^n.$$

Since $\overline{\delta}^*$ is a surjection and, by hypothesis, φ^* is a surjection, the composite $\varphi^* \circ \overline{\delta}^*$ is also a surjection. Let $v \in H^n(S^n;\mathbb{Z})$ be a generator and suppose that $H^n(\bigvee_{j=1}^{s} S^n;\mathbb{Z}) \cong \bigoplus_{j=1}^{s} H^n(S^n;\mathbb{Z})$ is generated by elements $v_1,\ldots,v_s$. As $\varphi^* \circ \overline{\delta}^*$ is a surjection, we must have $(\varphi^* \circ \overline{\delta}^*)(v_\ell) = v$ for some $1 \leq \ell \leq s$. Thus, if $q_\ell\colon \bigvee_{j=1}^{s} S^n \longrightarrow S^n$ is the pinch map onto the ℓ^{th}-wedge summand then the composite

$$S^n \xrightarrow{\varphi} X \xrightarrow{\overline{\delta}} \bigvee_{j=1}^{s} S^n \xrightarrow{q_\ell} S^n$$

induces an isomorphism in cohomology, and hence in homology. The Hurewicz Theorem therefore implies that $q_\ell \circ \overline{\delta} \circ \varphi$ is homotopic to ± 1. Composing q_ℓ with $S^n \xrightarrow{-1} S^n$ if necessary, we may assume that $q_\ell \circ \overline{\delta} \circ \varphi$ is homotopic to the identity map. Hence $q_\ell \circ \overline{\delta}$ is a left homotopy inverse for φ.

Finally, let

$$\theta\colon X \longrightarrow X \vee \Sigma M$$

be the homotopy coaction corresponding to δ and let ϑ be the composite

$$\vartheta\colon X \xrightarrow{\theta} X \vee \Sigma M \xrightarrow{1\vee p} X \vee \Big(\bigvee_{j=1}^{s} S^n\Big) \xrightarrow{1\vee q_\ell} X \vee S^n.$$

Then as $p_1 \circ \theta$ is homotopic to the identity map on X and $p_2 \circ \theta$ homotopic to δ, the naturality of the pinch maps p_1 and p_2 implies that $p_1 \circ \vartheta$ homotopic to the identity map on X and $p_2 \circ \vartheta$ homotopic to $q_\ell \circ \overline{\delta}$, which is a left homotopy inverse for φ. □

Proposition 6.4. *In the Realization Problem, suppose that X is a simply-connected CW-complex of homological dimension n. Suppose also that the Homological Problem has a solution: there is a map $\overline{g}\colon X \longrightarrow X'$ such that $\overline{g}_* = \gamma$ in degree n homology. If φ^* is a surjection then there is a map $g\colon X \longrightarrow X'$ such that $g \circ \varphi \simeq \varphi'$.*

Proof. Since X is a simply-connected CW-complex of homological dimension n and φ^* is a surjection, Lemma 6.3 implies that there is a map $\vartheta\colon X \longrightarrow X \vee S^n$ such that $p_1 \circ \vartheta$ is homotopic to the identity map on X and $p_2 \circ \vartheta$ is a left homotopy inverse for φ. Define the map

$$d\colon S^n \longrightarrow X'$$

by the difference $d = (\overline{g} \circ \varphi) - \varphi'$ and define the map g by the composite

$$g\colon X \xrightarrow{\vartheta} X \vee S^n \xrightarrow{\overline{g} \vee -d} X' \vee X' \xrightarrow{\nabla} X',$$

where ∇ is the fold map. We will show that $g_* = \gamma$ in degree n homology and $g \circ \varphi \simeq \varphi'$.

For the homological assertion, by hypothesis $\overline{g}_* = \gamma$ in degree n homology. By the hypothesis of the Realization Problem, $\gamma \circ \varphi_* = \varphi'_*$, implying that $\overline{g}_* \circ \varphi_* = \varphi'_*$. By definition, $d = (\overline{g} \circ \varphi) - \varphi'$, so $d_* = 0$. Therefore, the definition of g implies that $g_* = \overline{g}_*$. Thus $g_* = \gamma$ in degree n homology.

For the homotopy, consider the diagram

$$\begin{array}{ccccccc} S^n & \xrightarrow{\sigma} & S^n \vee S^n & & & & \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi \vee 1} & & & & \\ X & \xrightarrow{\vartheta} & X \vee S^n & \xrightarrow{\overline{g} \vee (-d)} & X' \vee X' & \xrightarrow{\nabla} & X'. \end{array} \tag{8}$$

Since S^n is a co-H-space and X is simply-connected, Lemma 6.2 implies that $f = \vartheta \circ \varphi$ is homotopic to the composite $S^n \xrightarrow{\sigma} S^n \vee S^n \xrightarrow{f_1 \vee f_2} X \vee S^n$, where $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$. As $f = \vartheta \circ \varphi$ and $p_1 \circ \vartheta$ is homotopic to the identity map on X we obtain $f_1 \simeq \varphi$. As $f = \vartheta \circ \varphi$ and $p_2 \circ \vartheta$ is a left homotopy inverse for φ, we obtain that f_2 is homotopic to the identity map on S^n. Thus $\vartheta \circ \varphi \simeq (\varphi \vee 1) \circ \sigma$, showing that the square in (8) homotopy commutes. The upper direction around (8) is the definition of $(\overline{g} \circ \varphi) - d$. Since the lower row of (8) is the definition of g, the lower direction around that diagram is $g \circ \varphi$. Thus we obtain $g \circ \varphi \simeq (\overline{g} \circ \varphi) - d$. By definition, $d = (\overline{g} \circ \varphi) - \varphi'$. Hence $g \circ \varphi \simeq \varphi'$. □

Proposition 6.4 allows for a solution to the Realization Problem in certain cases. The following corollary should be viewed as an extension of Theorem 5.7.

Corollary 6.5. *In the Realization Problem, suppose that X is a simply-connected CW-complex of homological dimension n and X' has its homology of the form (7). If φ^* is a surjection then there is a map $g\colon X \longrightarrow X'$ such that $g \circ \varphi \simeq \varphi'$.*

Proof. By Proposition 5.5 there is a map $\overline{g}\colon X \longrightarrow X'$ with the property that $\overline{g}_* = \gamma$ in degree n homology. Since φ^* is a surjection, Proposition 6.4 implies that there is a map $g\colon X \longrightarrow X'$ such that $g \circ \varphi \simeq \varphi'$. □

Example 6.6. Suppose that both X and X' are simply-connected CW-complexes having homology of the form (7). If φ^* is a surjection then the Realization Problem has a solution.

Another use of Proposition 6.4 is to produce a normalizing procedure for the Homotopical Problem. Suppose that there is a commutative diagram

$$\begin{array}{ccc} H_*(S^n;\mathbb{Z}) & \xrightarrow{\varphi_*} & H_*(X;\mathbb{Z}) \\ & \searrow^{\varphi'_*} & \downarrow{\gamma} \\ & & H_*(X';\mathbb{Z}) \end{array}$$

for some $\mathbb{Z}$-module map γ and the Homological Problem has a solution: there is a map $\overline{g}\colon X \longrightarrow X'$ such that $\overline{g}_* \circ \varphi_* = \varphi'_*$. Usually it is not the case that φ^* is a surjection, so Proposition 6.4 does not apply. To obtain a solution we normalize by altering X to $X \vee S^n$ as follows.

Define the map $\overline{\varphi}$ by the composite

$$\overline{\varphi}\colon S^n \xrightarrow{\sigma} S^n \vee S^n \xrightarrow{\varphi \vee 1} X \vee S^n,$$

define $\overline{\gamma}$ by the composite

$$\overline{\gamma}\colon H_*(X \vee S^n;\mathbb{Z}) \xrightarrow{(p_1)_*} H_*(X;\mathbb{Z}) \xrightarrow{\gamma} H_*(X';\mathbb{Z}),$$

and define $\overline{h}$ by the composite

$$\overline{h}\colon X \vee S^n \xrightarrow{p_1} X \xrightarrow{\overline{g}} X'.$$

Observe that $p_1 \circ \overline{\varphi} = \varphi$. Therefore $\overline{\gamma} \circ \overline{\varphi}_* = \gamma \circ (p_1)_* \circ \overline{\varphi}_* = \gamma \circ \varphi_*$ and $\overline{h} \circ \overline{\varphi} = \overline{g} \circ p_1 \circ \overline{\varphi} = \overline{g} \circ \varphi$. Thus $\overline{\gamma} \circ \overline{\varphi}_* = \varphi'_*$ and $\overline{h}_* \circ \overline{\varphi}_* = \overline{g}_* \circ \varphi_* = \varphi'_*$. Therefore the original problem has been *normalized* to a commutative diagram

$$\begin{array}{ccc} H_*(S^n;\mathbb{Z}) & \xrightarrow{\overline{\varphi}_*} & H_*(X \vee S^n;\mathbb{Z}) \\ & \searrow^{\varphi'_*} & \downarrow{\overline{\gamma}} \\ & & H_*(X';\mathbb{Z}) \end{array}$$

and the solution to the original Homological Problem, that is, the existence of the map $\overline{g}$ such that $\overline{g}_* \circ \varphi_* = \varphi'_*$, implies the existence of a homological solution to the Normalized Problem via the existence of the map $\overline{h}$ satisfying $\overline{h}_* \circ \overline{\varphi}_* = \varphi'_*$.

Theorem 6.7. *In the Realization Problem, suppose that X is a simply-connected CW-complex of homological dimension n and the Homological Problem has a solution: that is, there is a map $\overline{g}\colon X \longrightarrow X'$ such that $\overline{g} = \gamma$ in degree n homology. Then the Normalized Problem has a solution: there is a map $h\colon X \vee S^n \longrightarrow X'$ such that $h \circ \overline{\varphi} \simeq \varphi'$.*

Proof. Since the composite $S^n \xrightarrow{\overline{\varphi}} X \vee S^n \xrightarrow{p_2} S^n$ is homotopic to the identity map, the map $(\overline{\varphi})^*$ is a surjection and Proposition 6.4 applies. □

Example 6.8. Suppose that X is a simply-connected CW-complex of homological dimension n and X' has its homology of the form (7). By Proposition 5.5 there is a map $\overline{g}\colon X \longrightarrow X'$ with the property that $\overline{g}_* = \gamma$ in degree n homology. Therefore, by Theorem 6.7, the Normalized Problem has a solution: there is a map $h\colon X \vee S^n \longrightarrow X'$ such that $h \circ \overline{\varphi} \simeq \varphi'$. Note that, unlike Example 6.5, this does not require φ^* to be a surjection.

Remark 6.9. The Normalized Problem is perhaps more of a mathematical construction, it is not clear whether there is any connection to distributed computing.

7. Background and motivation in distributed computing

A *distributed computing system* consists of finitely many sequential processes that communicate via some facilities, for example, shared read/write memory with possible augmentations [29]. The processes are usually considered to be synchronous, semi-synchronous, or asynchronous, each of which corresponds to a specific assumption of a bound on relative process speed. In this paper, the processes are asynchronous, which means there is no bound on relative process speed. They can also fail by stopping, so it is indistinguishable whether an irresponsive process has failed or is only running slowly.

A *task* is a distributed coordination problem involving multiple computing processes [19], each of which starts with a private input taken from a finite set, communicates with other processes, and eventually decides on a private output, taken from a possibly different finite set. Examples of tasks include consensus [13], renaming [4] and set agreement [11]. A *protocol* is a distributed program that solves a task. A protocol is said to be *wait-free* if it tolerates halting failures by n out of $n+1$ ($n \geq 1$) processes.

Computability and complexity are important topics in distributed computational theory. *Computability* means the solvability of tasks, while *complexity* measures how many resources are needed to solve the task. Examples of resources in distributed computing include communication rounds, shared memory size, and so on. There is a long line of work that deals with computability and complexity for numerous tasks in different systems under various failure models [12, 21, 22, 23, 24, 25, 28, 30, 33, 34, 35]. Such efforts date back to 1988, when Biran et al. [6] established a graph-theoretical necessary and sufficient condition for the wait-free solvability of distributed tasks in message-passing systems. However, their framework proved hard to extend to fewer failing processes and even the problem of characterizing the solvability of specific tasks such as k-consensus and renaming for any number of processes remained unsolved for a long time.

A game-changing framework for modelling and analysis based on algebraic topology was introduced by Herlihy and Shavit in 1993 [23] to understand computability and complexity problems in asynchronous distributed systems. In that framework, a task in an asynchronous distributed system is modelled by a triple $(\mathcal{I}, \mathcal{O}, \mathcal{S})$, in which $\mathcal{I}$ is the set of inputs, $\mathcal{O}$ is the set of outputs, and $\mathcal{S}$ is the task specification which describes all allowable (or legal) outputs for an input. A particular execution of the distributed computing system may yield an allowable outcome or it may yield a non-allowable outcome. A task is solvable if there is a protocol (that is, a distributed program) which always produces allowable outcomes.

Herlihy and Shavit's work [23] presented a topological characterization of asynchronous computability. It also implied a topological characterization of the complexity of general tasks with $t \geq 1$ crash failures in a share-memory model. Later, their work was extended to a complete topological characterization of the solvability of wait-free tasks in shared-memory models [24].

Subsequently, their approach was generalized in three directions. First, it was generalized to systems with arbitrary communication objects, to arbitrary synchrony, or to arbitrary resilience (rather than wait-freedom), including crash failures and Byzantine failures [14, 21, 22, 30, 31, 33]. Second, it was generalized to explore the complexity of decision tasks in some communicating model by finding upper and/or lower bounds for complexity or giving a theoretical estimate of the cost of time/space/etc [5, 7, 9, 10, 16, 18, 25]. Third, it was generalized to classify tasks in asynchronous distributed systems: two tasks are equivalent if and only if they can implement each other in some specified manner [20, 28, 35].

The classification of tasks is a difficult problem in distributed computing theory. Special cases with solutions are loop agreement tasks. These can be defined in terms of an edge loop in a decision space (modelled as a 2-dimensional simplicial complex) with three distinguished points on the loop. They form a large family of tasks including set agreement and approximate agreement. Herlihy and Rajsbaum showed that a loop agreement task is solvable in certain models if and only if the loop is contractible in the 2-complex [19]. They also showed that two loop agreement tasks are equivalent, that is, each implements the other, if and only if there are simplicial maps between the 2-complexes in both directions (the composition of the maps in either order need not be the identity). Further, they were able to express this algebraically by defining the *signature* of the task as the fundamental group of the 2-complex and a distinguished element in it, and showing that two loop agreement tasks are equivalent if and only if there is an isomorphism between their signatures [20]. This equated a problem in computing to a problem in algebraic topology. The benefit is that the algebraic topology form of the problem is accessibly calculated.

A series of papers have since generalized this approach, equating the solution of a family of tasks in distributed computing to a problem in algebraic topology. One such family that has been considered is *rendezvous tasks* [27, 20], which are used in many applications, for example web-crawling, peer-to-peer lookup, and meeting scheduling. A rendezvous task intuitively models the scenarios where autonomous agents move around in a specific space (the decision space) to meet one another. A loop agreement task is a one-dimensional rendezvous task. An n-dimenional rendezvous task is called *nice* if the reduced homology groups of the decision space are trivial in all dimensions except n and the n^{th}-homology group is free abelian. In this case the algebraic signature of a rendezvous task is defined to be the pair consisting of the n^{th}-homology group of the decision space and a distinguished element in that group. Liu, Xu and Pan [28] proved that the classification of nice rendezvous tasks is completely characterized by their algebraic signatures. Later, Yue, Wu and Lei [35] generalized this result by assuming the reduced homology groups of the decision space are trivial in all dimensions except n and the n^{th}-homology group can be any abelian group.

One would like to go further and consider rendezvous tasks with decision spaces that have more complicated homology occuring in many dimensions. That is one of the objectives of this paper.

8. Modelling rendezvous tasks

Let $D_{\mathbb{I}}$ and $D_{\mathbb{O}}$ be the input and output data types. These are finite sets, which are possibly the same or possibly different. Suppose there are m processes, each of which takes a private input value from $\mathbb{D}_I$ and produces an output value in $\mathbb{D}_O$. The processes collectively may be regarded as taking an input vector $\vec{I}$ with m components and producing an output vector with m components. It may be the case that some processes do not participate in the execution, in which case a distinguished element $\perp$ is used. Here, if the i^{th} process is not used then $\perp$ appears in position i of $\vec{I}$ and similarly $\perp$ could appear as a component in $\vec{O}$. At least one process must be executed, so the vector $(\perp, \ldots, \perp)$ must be excluded.

An m-process task T is a triple $(\mathcal{I}, \mathcal{O}, \mathcal{S})$, where $\mathcal{I} \subseteq (D_{\mathbb{I}} \cup \{\perp\})^m - \{(\perp, \ldots, \perp)\}$ is the set of input vectors, $\mathcal{O} \subseteq (D_{\mathbb{O}} \cup \{\perp\})^m - \{(\perp, \ldots, \perp)\}$ is the set of output vectors, and $\mathcal{S} \subseteq \mathcal{I} \times \mathcal{O}$ is the task specification. Here, for each input vector in $\mathcal{I}$, the task specification describes all allowable output vectors. In particular, a given input vector $\vec{I}$ may have multiple possible output vectors, some of which are allowed by the task specification and some of which are not. Those that are allowed form a set $\mathcal{S}(\vec{I})$.

The set $\mathcal{I}$ is *prefix-closed* [20] if all the "prefixes", except $(\perp, \ldots, \perp)$, of any $\vec{I} \in \mathcal{I}$ remain in $\mathcal{I}$. Here, a prefix of $\vec{I}$ is a vector obtained by replacing any components of $\vec{I}$ with $\perp$'s. Similarly for $\mathcal{O}$. An input vector $\vec{I}$ *matches* an output vector $\vec{O}$ if, when $\perp$ appears in component i of $\vec{I}$, then $\perp$ appears in component i of $\vec{O}$. It will be assumed that the task specification $\mathcal{S}$ sends each input vector to a prefix-closed nonempty set of matching output vectors. The matching condition means that only processes participating in the execution may produce an output. An execution of a protocal sends an input vector to an output vector. A protocol solves a task if it always sends any $\vec{I} \in \mathcal{I}$ into $\mathcal{S}(\vec{I})$, that is, the outputs are always allowable.

To define rendezvous tasks, we follow [28]. Let Δ^{n+1} be the $(n+1)$-dimensional simplex spanned by $\{v_i\}_{i=0}^{n+1}$ in which $v_0 = (1, 0, \ldots, 0)$, $v_1 = (0, 1, 0, \ldots, 0)$, …, $v_n = (0, \ldots, 0, 1)$ for $0 \le i \le n$, and $v_{n+1} = (-1, -1, \ldots, -1)$. Let Σ^n be the boundary of Δ^{n+1}.

Definition 8.1. (Definition 1 in [28]) An n-rendezvous task T is defined as a triple $(K, \delta(\Sigma^n), f)$, where the decision space K is an $(n+1)$-dimensional simplicial complex, $\delta(\Sigma^n)$ is a simplicial subdivision of the simplicial complex Σ^n, and $f : \delta(\Sigma^n) \longrightarrow K$ is a simplicial embedding (that is, f is a simplicial map sending $\delta(\Sigma^n)$ isomorphically onto a subcomplex of K). It is assumed that $|K|$ is simply-connected.

Now we interpret a rendezvous task in terms of m-process tasks. Let $D_{\mathbb{I}} = \{0, \ldots, n+1\}$ and let $D_{\mathbb{O}} = V(K)$ be the set of vertices of K. For any $U \subsetneq \{0, 1, \ldots, n+1\}$, let $\alpha_U \in \Sigma^n$ be the simplex spanned by $\{v_i : i \in U\}$. Let $\delta(\alpha_U)$ be the subdivision of α_U determined by restricting the subdivision δ of Σ^n to α and let $K_U = f(\delta(\alpha_U)) \subseteq K$. Then K_U is a subcomplex of K.

For any positive integer m, the m-process task represented by the rendezvous task $(K, \delta(\Sigma^n), f)$ is given by $T = (\mathcal{I}, \mathcal{O}, \mathcal{S})$, where $\mathcal{I} = (D_{\mathbb{I}} \bigcup \{\perp\})^m \setminus \{(\perp, \ldots, \perp)\}$ is the set of input vectors, $\mathcal{O} = (D_{\mathbb{O}} \bigcup \{\perp\})^m \setminus \{(\perp, \ldots, \perp)\}$ is the set of output vectors, and the task specification $\mathcal{S}$ satisfies

$$\mathcal{S}(\vec{I}) = \begin{cases} \left\{\vec{O} \in \mathcal{O} : \vec{O} \text{ matches } \vec{I} \text{ and } val(\vec{O}) \text{ spans a simplex in } K_{val(\vec{I})}\right\} & \text{if } val(\vec{I}) \subsetneq D_{\mathbb{I}}\,; \\ \left\{\vec{O} \in \mathcal{O} : \vec{O} \text{ matches } \vec{I} \text{ and } val(\vec{O}) \text{ spans a simplex in } K\right\} & \text{if } val(\vec{I}) = D_{\mathbb{I}}\,. \end{cases}$$

Here, $val(u_1, u_2, \ldots, u_m) = \{u_i : 1 \le i \le m\} \setminus \{\perp\}$ for any vector $(u_1, u_2, \cdots, u_m)$. That is, $val(u_1, u_2, \ldots, u_m)$ is the set of distinct values among the elements $u_1, \ldots, u_m$. Intuitively, imagine the input of a process is the starting position of an agent while the output is the ending position; the process may move the agent around. A rendezvous task requires agents that start close together always end close together.

Example 8.2. Consider the set agreement task of an $(n+2, n+1)$-agreement, which means the agreement whose set of input values is $\{0, 1, \ldots, n+1\}$ and each of whose executions produces at most $n+1$ distinct values. This is described by the rendezvous task (Σ^n, Σ^n, f), where there is no further subdivision and f is the identity map on Σ^n.

As in [28], algebraic data is assigned to each task T that encodes essential features of T. Let $\iota_n \in H_n(S^n) \cong \mathbb{Z}$ be a generator. For any subdivision $\delta(\Sigma^n)$ of Σ^n, there is a homeomorphism $S^n \cong |\delta(\Sigma^n)|$. We may then regard ι_n as generating $H^n(|\delta(\Sigma^n)|)$.

Definition 8.3. Let $T = (K, \delta(\Sigma^n), \psi)$ be a rendezvous task. The *signature* $sig(T)$ is defined by $sig(T) = (H_n(|K|), |\psi|_*(\iota_n))$.

Let $T = (K, \delta(\Sigma^n), \psi)$ and $T' = (K', \delta'(\Sigma^n), \psi')$ be two rendezvous tasks. Rendezvous task T *implements* rendezvous task T' if the output complex of T, or some subdivision of it, can be used as a protocal for solving T'. Two rendezvous tasks are equivalent if each implements the other. A topological interpretation of this is the following.

Lemma 8.4. *([28, Lemma 4.1]) Rendezvous task $T = (K, \delta(\Sigma^n), \psi)$ implements rendezvous task $T' = (K', \delta'(\Sigma^n), \psi')$ if and only if there is a commutative diagram*

$$\begin{array}{ccc} |\delta(\Sigma^n)| & \xrightarrow{|\psi|} & |K| \\ {\scriptstyle g_|}\downarrow & & \downarrow{\scriptstyle g} \\ |\delta'(\Sigma^n)| & \xrightarrow{|\psi'|} & |K'| \end{array}$$

where g is a map of polyhedra, $g_|$ is the restriction of g to $|\delta(\Sigma^n)|$, and $g_|$ induces the identity map in homology. □

The statement of Lemma 8.4 requires some explanation. First consider $g_|$. Since $|\psi'|$ is a bijection onto its image it has an inverse on that image. So if $g \circ |\psi|$ is contained in the image of $|\psi'|$, then it lifts through $|\psi'|$, giving a commutative diagram. The lift may be identified with the restriction of g to the image of $|\psi|$, hence its name $g_|$. Second, consider what is meant by $g_|$ inducing the identity map in homology. Since Σ^n is the boundary of Δ^{n+1} and $\delta(\Sigma^n)$ is a subdivision, then $|\delta(\Sigma^n)| \simeq S^n$, and similarly for $|\delta'(\Sigma^n)|$. Thus $g_|$ may be regarded as a self-map of S^n, in which case it makes sense to say it could induce the identity map in homology.

It is useful to also observe that $g_|$ inducing the identity map in homology implies by the Hurewicz Theorem (Theorem 4.5) that it induces the identity map on S^n. Therefore the diagram in the statement of Lemma 8.4 can be rewritten as a commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{|\psi|} & |K| \\ & {\scriptstyle |\psi'|}\searrow & \downarrow{\scriptstyle g} \\ & & |K'|. \end{array}$$

This is reminiscent of the Realization Problem, more of which will be said in Section 9.

A *map of signatures $sig(T) \longrightarrow sig(T')$* is a commutative diagram

$$\begin{array}{ccc} H_n(|\delta(\Sigma^n)|) & \xrightarrow{|\psi|_*} & H_n(|K|) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \gamma} \\ H_n(|\delta'(\Sigma^n)|) & \xrightarrow{|\psi'|_*} & H_n(|K'|) \end{array} \tag{9}$$

for some group homomorphism γ. Note that the equality in the left column should be interpreted as an identity map from $\mathbb{Z}$ to itself. Since any continuous map of spaces induces a morphism in homology, we obtain the following.

Corollary 8.5. *If rendezvous task $T = (K, \delta(\Sigma^n), \psi)$ implements rendezvous task $T = (K', \delta'(\Sigma^n), \psi')$ then there is a map of signatures $sig(T) \longrightarrow sig(T')$.*

Proof. Since T implements T', by Lemma 8.4 there is a commutative diagram

$$\begin{array}{ccc} |\delta(\Sigma^n)| & \xrightarrow{|\psi|} & |K| \\ {\scriptstyle g_|}\downarrow & & \downarrow{\scriptstyle g} \\ |\delta'(\Sigma^n)| & \xrightarrow{|\psi'|} & |K'| \end{array}$$

for some map g with $g_|$ having the property that it induces the identity map in homology. Taking homology and letting $\gamma = g_*$, we obtain a map of signatures as in (9). □

9. Algebraic topology and distributed computing

This section links the Realization Problem in the algebraic topology part of the paper with the distributed computing problem of characterizing rendezvous tasks. This will make use of two important theorems in algebraic topology.

A subspace inclusion $i\colon A \longrightarrow X$ has the *homotopy extension property* if, for any space Y, given a homotopy $H\colon A \times I \longrightarrow Y$ and an extension of H_0 to a map $\overline{f}\colon X \longrightarrow Y$, there is an extension of H to a homotopy $\overline{H}\colon X \times I \longrightarrow Y$ such that $\overline{H}_0 = \overline{f}$. Diagrammatically, this says that there is a commutative diagram

$$\begin{array}{ccccc} A \times \{0\} & \longrightarrow & A \times I & & \\ {\scriptstyle i\times 1}\downarrow & & \downarrow & \searrow{\scriptstyle H} & \\ X \times \{0\} & \longrightarrow & X \times I & \overset{\overline{H}}{\dashrightarrow} & \\ & & \xrightarrow{\overline{f}} & & Y. \end{array}$$

If the map i has the homotopy extension property then it is called a *cofibration*. For example, a subspace inclusion $A \xrightarrow{i} X$ is a cofibration if and only if A is a neighborhood deformation retract of X, that is, there is an open neighborhood of A in X that strong deformation retracts to A (see, for example, [37, Theorem 7.1.10]. The following result is well known.

Theorem 9.1. *Suppose that $A \xrightarrow{i} X$ is a cofibration and there is a homotopy commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & Y. \end{array}$$

Then there is a map g' that is homotopic to g for which there is a strictly commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g'} \\ & & Y. \end{array}$$

Proof. Let $H\colon A \times I \longrightarrow Y$ be a homotopy with $H_0 = g \circ i$ and $H_1 = h$. Note that H_0 extends to $X \xrightarrow{g} Y$. Since i is a cofibration, the homotopy extension property implies that there is a homotopy $\overline{H}\colon X \times I \longrightarrow Y$ such that $\overline{H}_0 = g$ and $\overline{H}(A \times I) = H$. Let $g' = \overline{H}_1$. Then $g' \circ i = \overline{H}_1 \circ i = \overline{H}(A \times \{1\}) = H(A \times \{1\}) = h$, giving the asserted commutative diagram. □

Recall that if K is a simplicial complex then the *barycentric subdivision* $\mathrm{bary}(K)$ is the simplicial complex obtained by inductively adding a new vertex at the barycentre of each k-simplex of K and adding all the simplices of dimension less than or equal to k determined by the extra vertices. Let $\mathrm{bary}^1(K) = \mathrm{bary}(K)$ and for $N \geq 2$ recursively define $\mathrm{bary}^N(K)$ by $\mathrm{bary}(\mathrm{bary}^{N-1}(K))$.

A continuous map $f\colon |K| \longrightarrow |L|$ has a *simplicial approximation* $\widehat{f}\colon K \longrightarrow L$ if for every point $x \in |K|$, $|\widehat{f}|(x)$ lies in the smallest simplex $\tau \in L$ such that $f(x) \in \tau$. A proof of the following result can be found in [3, Theorem 6.7].

Theorem 9.2. *Any continuous map $f\colon |K| \longrightarrow |L|$ has a simplicial approximation $\widehat{f}\colon \mathit{bary}^N(K) \longrightarrow L$ for some sufficiently large N. Further, if K_0 and L_0 are subcomplexes of K and L respectively, then $|\widehat{f}|(|K_0|) \subseteq |L_0|$.* □

We are now ready to link the algebraic topology in the Realization Problem of Section 2 to the problem of implementing rendezvous tasks. The following theorem can be seen as a partial converse to Corollary 8.5.

Theorem 9.3. *Let $T = (K, \delta(\Sigma^n), \psi)$ and $T' = (K', \delta'(\Sigma^n), \psi')$ be rendezvous tasks. Suppose that there is a map of signatures $sig(T) \longrightarrow sig(T')$, that is, suppose there is a group homomorphism $\gamma\colon H_n(|K|) \longrightarrow H_n(|K'|)$ such that $\gamma \circ |\psi|_* = |\psi'|_*$. Let $e\colon S^n \longrightarrow |\delta(\Sigma^n)|$ and $e'\colon S^n \longrightarrow |\delta'(\Sigma^n)|$ be homeomorphisms inducing degree 1 maps on H_n. If the Realization Problem applied to*

$$\begin{array}{ccc} H_*(S^n) & \xrightarrow{(|\psi|\circ e)_*} & H_*(|K|) \\ & \searrow^{(|\psi'|\circ e')_*} & \downarrow \gamma \\ & & H_*(|K'|). \end{array}$$

has a solution then T implements T'.

Proof. By definition, the map of signatures $sig(T) \longrightarrow sig(T')$ means there is a commutative diagram

$$\begin{array}{ccc} H_n(|\delta(\Sigma^n)|) & \xrightarrow{|\psi|_*} & H_n(|K|) \\ \downarrow = & & \downarrow \gamma \\ H_n(|\delta'(\Sigma^n)|) & \xrightarrow{|\psi'|_*} & H_n(|K'|) \end{array}$$

for some group homomorphism γ. Let $e\colon S^n \longrightarrow |\delta(\Sigma^n)|$ and $e'\colon S^n \longrightarrow |\delta'(\Sigma^n)|$ be homeomorphisms inducing degree 1 maps on H_n. Then there is a commutative diagram

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{e_*} & H_n(|\delta(\Sigma^n)|) & \xrightarrow{|\psi|_*} & H_n(|K|) \\ \downarrow = & & \downarrow = & & \downarrow \gamma \\ H_n(S^n) & \xrightarrow{e'_*} & H_n(|\delta'(\Sigma^n)|) & \xrightarrow{|\psi'|_*} & H_n(|K'|) \end{array}$$

which we re-write as a commutative diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{(|\psi|\circ e)_*} & H_n(|K|) \\ & \overset{(|\psi'|\circ e')_*}{\searrow} & \downarrow \gamma \\ & & H_n(|K'|). \end{array}$$

Since the reduced homology of S^n is concentrated in degree n, and both $|K|$ and $|K'|$ are assuming to be simply-connected, this is the same as having a commutative diagram

$$\begin{array}{ccc} H_*(S^n) & \xrightarrow{(|\psi|\circ e)_*} & H_*(|K|) \\ & \overset{(|\psi'|\circ e')_*}{\searrow} & \downarrow \gamma \\ & & H_*(|K'|). \end{array}$$

As there is a solution to the Realization Problem, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{|\psi|\circ e} & |K| \\ & \overset{|\psi'|\circ e'}{\searrow} & \downarrow g \\ & & |K'| \end{array}$$

for some map g with the property that $g_* = \gamma$. The map $|\psi|$ is an embedding, so as e is a homeomorphism the composite $|\psi| \circ e$ is also an embedding, and hence a cofibration. Thus the homotopy extension property (Theorem 9.1) implies that there is a continuous map $g'\colon |K| \longrightarrow |K'|$ such that the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{|\psi|\circ e} & |K| \\ & \overset{|\psi'|\circ e'}{\searrow} & \downarrow g' \\ & & |K'| \end{array}$$

strictly commutes. Precomposing with e^{-1} then gives a strictly commutative diagram

$$\begin{array}{ccc} |\delta(\Sigma^n)| & \xrightarrow{|\psi|} & |K| \\ & \overset{|\psi'|\circ(e'\circ e^{-1})}{\searrow} & \downarrow g' \\ & & |K'|. \end{array}$$

Note that as $|\psi|$, $|\psi'|$, e and e' are all injections, the restriction of g' (in the sense of Lemma 8.4) to $|\delta(\Sigma^n)|$ is $e' \circ e^{-1}$, that is, $g'_| = e' \circ e^{-1}$. This lets us re-write the previous diagram as a commutative diagram

$$\begin{array}{ccc} |\delta(\Sigma^n)| & \xrightarrow{|\psi|} & |K| \\ \downarrow{\scriptstyle g'_|} & & \downarrow{\scriptstyle g'} \\ |\delta'(\Sigma^n)| & \xrightarrow{|\psi'|} & |K'|. \end{array}$$

The simplicial approximation theorem (Theorem 9.2) then implies that g' has a simplicial approximation $\widehat{g}\colon \text{bary}^N(K) \longrightarrow K'$ for some sufficiently large N, and there is a commutative diagram

$$\begin{array}{ccc} |\delta(\Sigma^n)| & \xrightarrow{|\psi|} & |K| \\ \downarrow{\scriptstyle \widehat{g}_|} & & \downarrow{\scriptstyle \widehat{g}} \\ |\delta'(\Sigma^n)| & \xrightarrow{|\psi'|} & |K'|. \end{array}$$

Thus by Lemma 8.4, T implements T'. □

Combining Lemma 8.5 and Theorem 9.3 then implies there is the following algebraic characterization of rendezvous problems.

Corollary 9.4. *Let $T = (K, \delta(\Sigma^n), \psi)$ and $T' = (K', \delta'(\Sigma^n), \psi')$ be rendezvous tasks and suppose the Realization Problem applied to the map of signatures holds. Then T implements T' if and only if there is a map of signatures $sig(T) \longrightarrow sig(T')$.* □

In Theorem 5.7 we showed that a solution to the Realization Problem holds. Therefore, rendezvous problems in those contexts are completely characterized by their signatures.

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