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Optimum Plan and Statistical Inference for Step-Stress Models Based on Censored Samples

by

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Thesis for the degree of Doctor of Philosophy

October, 2023

University of Southampton

Abstract

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In lifetime data analysis, life testing for items under normal use conditions can often take a long time to obtain a reasonable number of failures. In this situation, accelerated life test (ALT) procedures are performed in order to obtain failure time data in a shorter time. In the step-stress accelerated life test (SSALT), which is a special class of ALT, the stress setting for survival units is changed step by step to higher stress levels at predetermined times during the experiment. In this way, information about the parameters of the life distribution is obtained quicker than would be possible under normal operating conditions. The censoring methodology is commonly used in planning the ALTs to reduce the cost and test time.

Statistical inference of model parameters and optimum test plans are two main aspects in the SSALT studies. The main objectives of this thesis are to develop an optimal design for a simple SSALT model and to make statistical inferences for a simple SSALT model based on progressive type-II censoring schemes. It is assumed that the lifetimes at each stress level follow the generalized exponential distribution (GED) and a cumulative exposure model is assumed as a life-stress model. The study also compares the total test time of censored samples with complete samples. The maximum likelihood method is used to estimate three unknown parameters of the cumulative exposure model. Also, the asymptotic confidence intervals (CIs) for the parameters based on the observed Fisher information matrix are derived. Moreover, the bootstrap approach is used to obtain the CIs for the model parameters using two methods, percentile and bias-corrected and accelerated. The steps of obtaining asymptotic and bootstrap CIs based on two bootstrap methods are described.

In ALT design, the optimal stress change time and optimal censoring schemes under the variance (V)-optimality criterion are studied in detail to provide the most precise estimates of percentile lifetime under the GED at the usage stress level. The optimal design is studied based on minimizing the asymptotic variance of the maximum likelihood estimate (MLE) of the $100\wp^{th}$ percentile lifetime under the GED at the usage stress level. Different progressive censoring schemes are compared to find the optimal censoring scheme.

Extensive simulation studies are conducted to investigate and compare the performance of the MLEs and CIs with associated precision for different values of sample size, failure percentage, stress change time and progressive censoring schemes using the Monte Carlo simulation technique. Also, numerical analysis is performed using the golden section search method to estimate the locally optimal stress change time. In addition, the optimal censoring scheme is also obtained and compared with the worst censoring scheme and the complete sample using relative efficiency and relative time.

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Table B.26 The sensitivity of the SSALTs model based on $AVar\big(\hat{t}_{0.95}(x_0)\big)$ and the associated ratio of base scenario and random scenario for $\mathcal{R} = \mathcal{R}1$, $\tau = 0.8$ and $n = 80$ with 80%FP 176

List of Abbreviations

AB	Absolute bias.
AL	Average length.
ALT	Accelerated life test.
$AVar$	Asymptotic variance.
AV-C	Asymptotic variance-covariance.
BCa	Bias corrected and accelerated
CDF	Cumulative distribution function.
CEM	Cumulative exposure model.
CI	Confidence interval.
CP	Coverage probability.
FP	Failure percentage.
GED	Generalized exponential distribution.
ML	Maximum likelihood.
MLE	Maximum likelihood estimate.
MSE	Mean squared error.
PCS	Progressive censoring scheme.
PDF	Probability density function.
RE	Relative efficiency.
RT	Relative expected time.
SSALT	Step-stress accelerated life test.

Research Thesis: Declaration of Authorship

Print name: Mariam Mohammed Mominkhan

Title of thesis: Optimum Plan and Statistical Inference for Step-Stress Models Based on Censored Samples

I declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

I confirm that:

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3. Where I have consulted the published work of others, this is always clearly attributed;
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6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
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Acknowledgements

In the name of Allah, the most gracious and the most merciful. Alhamdulillah, all praises and thanks are to God for the strengths and his blessing and support to completing this thesis by success.

I would like to express my thanks to all people who contributed to the realization of this thesis. Firstly, I would like to express my deep and sincere gratitude to my supervisor *Dr. Alan Kimber* and *Dr. Stefanie Biedermann* for their advice, detailed comments, helpful discussions, offered facilities, patience and for taking a lot of time to support me during the preparation of the thesis. Without their encouragement and effort this thesis would not have been completed. I appreciate them for being continuously supportive and interested, special in the difficult year of Covid-19.

My special much gratitude and cordial appreciation to my parents whom supported me by every way, for their unwavering love, patience, continuous prayers for me. Special thank goes to my mother, who died in 2020, I wish I can be able to give you this moment of PhD done. This thesis would never have been done without their support. Also, I would like to thank my dear two brothers and sister, to stay by my side during my studies. Also, many thanks go in particular to my lovely husband *Basheer Sorour* for his extra patience, support, understanding, continued encouragement, the confidence that kept me going and motivation. I wish to express my love to my son *Fouad*, forgive me for being busy for years of your life, but I promise you that I'll make you proud of your Mom. Big thanks for my angel baby *Sedra,* forgive me my baby for being busy in your first days in this world.

I'm thankful to King Abdulaziz University and Saudi Arabian Cultural Bureau in the UK for the funding opportunities and their continuous support during my academic journey. I am also grateful to all the members of the Mathematical School at the University of Southampton for providing me with all the necessary facilities and resources and have continuously supported me.

My greatest thankful and appreciation goes to my friends, who was always a great support in all my struggles and frustrations in my studies and life.

Mariam Mohammed Mominkhan …

Chapter 1
Introduction

Measuring the reliability characteristics of a product and investigating its lifetime distribution is vital to the design and manufacturing process. Understanding the reliability can help to improve product design to satisfy the consumer's expectations and decrease the number of failures in a guarantee period. The life data analysis aims to quantify the reliability characteristics of the products. Life testing is usually designed to obtain lifetime data sets for components or products. Lifetime testing plays a vital role in many fields, such as industry, engineering and medicine.

In order to cope with increasing market challenges and requirements, today's manufacturers face strong pressure to develop high technology products with improved reliability and better overall quality designed to give long product life. Testing the lifetimes of products under normal use conditions is, however, rarely feasible in a short time because of long test duration and the consequent high cost. The accelerated life test (ALT) model is used to solve this problem by testing the products under accelerated conditions. It is designed to obtain sufficient amounts of information about highly reliable products with noticeable reductions in testing time and cost. In ALT experiments, failures are produced in a shorter time by exposing products to higher than usage stress level, such as a high temperature or pressure.

ALTs are widely used and are an important experimental strategy in reliability and survival analysis. They consist of a variety of test methods for shortening the life of products by testing them at higher than usage stress level. The step-stress accelerated life test (SSALT) model is the most commonly used method of ALTs. It allows the stress level applied to each test unit to be changed step-by-step during the test. SSALTs have a wide application in the fields of science, engineering and medicine. The main advantage of this test strategy is that it can provide a noticeable increase in number of failure observations within a shortened testing period. The problems of designing optimum plans and drawing inferences from SSALTs have been extensively investigated by many researchers. Nelson (1990), for example, has provided a set of guidelines for planning SSALT.

Although an ALT plan can be implemented to shorten the experiment time, waiting for all of the units to fail can still present practical and financial problems. Moreover, in life testing experiments, it is common for units to be either lost or withdrawn before failure. This may happen due to cost or time limitations. In these cases, censoring schemes can be useful strategies, in which the test is terminated before all products have run to failure, or where some items are withdrawn at specific points in the test. Several censoring schemes have been developed and discussed in the literature. One of the versatile types of censoring used is the progressive censoring scheme (PCS) proposed by Herd (1956). Balakrishnan and Aggarwala (2000) provide an exhaustive review of various PCSs. Since then, it has attracted the interest of several researchers, in addition to developing the equipment required for conducting ALT experiments, making it easier to conduct the experiments.

The PCSs can save time, effort and cost, due to its flexibility in removing units at various stages during the test. Using PCSs allows experimenters to be more flexible in the design stage by allowing the removal of test units at different points throughout the test. As a result, PCS is highly effective in utilizing the available resources. Progressive censoring is an umbrella term encompassing two forms known as Type-I and Type-II censoring schemes, in which units are censored only at the termination of the test either at a predetermined time or upon the occurrence of fixed number of failures.

In SSALT experiments, failure time data collected in the accelerated conditions is analysed to predict the lifetime of units under normal operating conditions. SSALT models are generally developed based on determining an underlying life distribution and selecting a model which relates the life distributions of test units at different stress levels to each other. The most common model used in the literature is the cumulative exposure model (CEM) proposed by Sedyakin (1966). The CEM assumes that the remaining lifetime of a test unit depends only on the current cumulative fraction failed and current stress, regardless of how the fraction is accumulated see Nelson (1980, 1990). The next step after conducting the test is to apply statistical inference methods to estimate the unknown parameters of the model.

Statistical methodology for SSALTs has been studied and developed extensively by many authors. In some situations, under a simple SSALT model, the hold time at low stress levels might be relatively short, resulting in few or no failure data and thus affecting the quality of the inference of the maximum likelihood estimate (MLE). Determining an optimal hold time of low stress, then, is important to obtain sufficient information at different stress levels. This will improve the efficiency of the statistical inference. Furthermore, in ALTs under PCS, it is important to choose the optimal censoring scheme that provides the maximum information regarding the parameter of interest. Therefore, the precision of the MLEs will be increased by using the optimal censoring scheme.

Estimating the model parameters with the maximum precision has noticeable effect on the precision of reliability estimation which subsequently has directly impact on its related issues such as determining the warranty and scheduling the maintenance.

1.1 The Research Problem

The problem of designing an optimum plan and making inferences from SSALTs has been the subject of continuous interest and extensive study over the last few decades. However, few studies compared progressive Type-II censored samples with Type-II censored samples and complete samples for life testing, and they did not consider SSALT. Accordingly, there is a need to study the impact of different censoring schemes to better understand SSALT plans based on progressive Type-II censoring scheme and make more effective use of censoring schemes. This study, therefore, is designed to investigate the statistical impact of using different optimum plans on test efficiency and estimation precision of the MLEs.

In designing SSALT under progressive Type-II censoring scheme, the products are tested with respect to the cost of the experiment and test time. Therefore, one should carefully select the set of initial values in the designing stage, such as the sample size with the number of items to be censored and when they will be censored. So, an optimal design is utilized to balance design efficiency and total test time, which strongly relates to the cost of the experiment.

1.2 Main Objectives

In general, there are two major aspects related to the SSALT models: statistical inference of model parameters and the optimal test plan. The main objectives of this thesis are to develop an optimal test plan for a simple SSALT model and to make statistical inferences based on progressive type-II censoring schemes.

To discuss the statistical analysis of the model parameters for SSALT under progressive Type-II censoring schemes, the generalized exponential distribution (GED) is considered as a lifetime distribution. In addition, the cumulative exposure model (CEM) is also assumed to relate the cumulative distribution function of failure times under different stress levels. The maximum likelihood (ML) method is used to estimate three unknown parameters of the CEM; θ_1, θ_2 and α. Also, the asymptotic confidence intervals (CIs) for the parameters based on the observed Fisher information matrix are derived. Moreover, the bootstrap approach is used to obtain CIs for the model parameters using two methods, percentile and bias-corrected and accelerated (BCa), based on bootstrap samples.

In this thesis, the optimum ALT design is determined with respect to the variance(V)-optimality criterion for sets of initial values. The V-optimality criterion minimizes the asymptotic variance ($AVar$) of the 5^{th}, 50^{th} and 95^{th} percentile life estimates at usage stress level. The design of ALT plans is studied with respect to choosing the optimal stress change time and optimal censoring scheme. Furthermore, a sensitivity study is conducted in order to identify the parameters that should be estimated with special care.

Many tasks involving statistical analysis and the optimal plan are carried out in order to accomplish the suggested objectives. In the analysis part, the following tasks will be undertaken:

1- The design of the steps and assumptions for modelling SSALT under progressive Type-II censored samples will be described and discussed in detail. The GED is assumed as a lifetime distribution and the CEM is also assumed as a life-stress model.

2- Under the assumption of the CEM, the ML method will be implemented to obtain the estimators of the SSALT model parameters using numerical methods.

3- Consequently, the observed Fisher information matrix will be derived to obtain the asymptotic variance-covariance (AV-C) matrix of the estimates, which is used to construct the asymptotic CIs.

4- CIs for the unknown parameters will be estimated, based on parametric bootstrap methods. Percentile and BCa bootstrap CIs will be obtained, and their performance will be compared with asymptotic CIs. The steps of obtaining bootstrap CIs based on two bootstrap methods will be described.

5- The simulation study is performed to investigate the performance of the proposed algorithms with associated standard error based on different sets of initial values of model parameters, sample size and failure percentage. The influence of different censoring schemes, the sample size, the failure percentage and the stress change time on the performance of the MLEs and CIs will be investigated via a simulation study. The performance of these estimators will be studied using relevant statistical measures such as relative absolute bias, mean square error, interval length and coverage probability. Moreover, the precision of the performance of the MLEs and CIs will be described by the standard error.

In the design part, the tasks are described as follows:

1- The V-optimality criterion will be explained and utilized to identify an optimal design for sets of initial values, such that the estimates of the percentile under the usage stress level has the most precision.

2- The optimal stress change time will be determined for the 5th, 50th and 95th percentiles using the V-optimality criterion. The golden section search numerical method is used to determine the optimal stress change time based on a selected set of initial values of sample size, failure percentage, model parameters and different censoring schemes.

3- The impact of changing the sample size, failure percentage, model parameters and different censoring schemes on the precision of parameter estimation along with the optimal stress change time will be investigated.

4- A sensitivity analysis will be carried out to examine the robustness of the proposed plans and identify the sensitive parameters that must be estimated carefully.

5- In addition, the optimal censoring scheme will be determined in two scenarios: one in which a large sample size with 11 proposed censoring schemes will be tested, and the other in which a small sample size with all possible censoring schemes will be investigated. The relative efficiency and relative time of the optimal censoring scheme with respect to the worst censoring scheme, the basic Type-II censoring scheme and the complete sample are calculated to compare different censoring schemes and find the nearly optimal censoring scheme in the case of a small sample size.

6- The optimal censoring scheme is compared with the worst censoring scheme, Type-II censoring scheme and complete sample for different values of the sample size, failure percentage, model parameters and stress change time.

7- Extensive Monte-Carlo simulation is carried out to understand the effectiveness of the proposed methods for determining the optimal censoring scheme based on different sets of initial values of the sample size, the failure percentage, the model parameters and the stress change time.

1.3 Thesis Organization

This thesis is presented in four chapters, besides the introduction and discussion chapters. Chapter 1 introduces the research and gives a general description of the ALT and its related design methods. The research problem is stated here, along with the purpose of the study and its objectives.

The basic concepts of ALTs and a literature review form the basis of Chapter 2. We provide a detailed description of the concepts used in this thesis, which includes life testing, ALT, different censoring strategies, SSALT under different censoring strategies, life-stress relationship and life-stress model with the description of the CEM used in this thesis. Furthermore, we provide a brief review of the related literature on MLE, bootstrap CIs and optimum plans of SSALTs model under basic and progressive censoring schemes. In Chapter 3, we describe the baseline distribution used in this thesis. We discuss the characteristics of the GED and the performance of the MLEs of the model parameters. Then, the GED is compared with the Weibull and Gamma distributions. Also, previous studies and applications based on GED are briefly reviewed.

In Chapter 4, we present general design steps and assumptions for modelling SSALTs of GED lifetimes under progressive Type-II censored samples. The ML procedure is then used to derive point and interval estimates of the unknown model parameters. Following that, two methods for constructing CIs for the model parameters are discussed: the asymptotic and bootstrap methods. The asymptotic CIs based on the observed Fisher information matrix are derived. Moreover, we use the bootstrap approach to obtain the CIs for the model parameters using two methods: percentile and BCa. Also, we use the bootstrap and jackknife methods to estimate the bias and variance of the estimators for the model parameters. Then, the performance of the MLEs and CIs will be estimated using Monte Carlo simulation under different sample sizes, percent of failures, stress change time and PCSs. The simulation study results are presented in Appendix A to compare the performance of the MLEs.

We begin Chapter 5 with a description of the V-optimality criterion and how we utilize it to investigate the optimal stress change time and optimal scheme for the SSALT under progressive Type-II censoring scheme under the assumption of the CEM. The V-optimality criterion is used to estimate the $100\wp^{th}$percentile lifetime under the GED at a usage stress level. The optimum plan in this chapter is divided into two main components: optimal stress change time and optimal censoring scheme. The simulation studies are used to find the optimal stress change time and examine the influences of the sample size, the failure percentage, model parameters and different censoring schemes on the optimal stress change time and the corresponding $AVar$ of the $100\wp^{th}$percentile lifetime. Next, the determination of the optimal censoring scheme is discussed. The optimal censoring scheme is compared with the worst, Type-II censoring scheme and the complete sample. The impact of various sets of initial values of the sample size, the failure percentage, the model parameters and the stress change time is discussed in detail. Also, the robustness of the parameters will be investigated by considering the effect of parameter misspecification on the optimal stress change time and the optimal censoring scheme for a small sample size. The simulation results of optimal design of SSALTs model are presented in Appendix B and Appendix C.

The final chapter includes the summarization of the findings of this study and outlines some further problems of interest related to the objective of this thesis.

Chapter 2

Fundamentals of Accelerated Life Tests and Literature Review

This chapter introduces basic concepts and essential definitions used in this thesis, such as reliability and life testing, ALTs, censored samples and simple SSALTs with different censoring methods. In addition, the concept of the life-stress model is reviewed and the CEM is discussed in detail, along with its mathematical derivation. Section 2.8 presents a review of relevant literature for this research and discusses the methodology used in analysing SSALTs with advantages and drawbacks.

2.1 Reliability and Life Testing

The reliability function, also known as the survival function, is the probability that a component or system will perform its required functions under determined operating conditions for a specified period of time. Reliability testing is generally considered for life data analysis in which items are tested to failure in order to obtain failure-time data. The reliability function $R(t)$ complements the cumulative distribution function (CDF) $\mathrm{F}(t)$. In order to model the time to failure, the CDF represents the probability that a failure time variable T is less than or equal t. The reliability function represents the probability that an item is continuously working until a specific time. For a given time t, $R(t)$ is the probability that the time to failure is greater than or equal to t. It is mathematically given as:

$$R(t) = \mathrm{P}(\mathrm{T} > t) = 1 - \mathrm{F}(t).$$

Over the last decades, life-testing has been of continuous interest due to its importance in many fields such as industry, engineering and medical experiments. Life-test experiments focus on recording the failure times or lifetimes of the items being tested. The lifetime of a product or individual is the time until a given event has occurred for the product or individual. It is worth mentioning that the failure times are not only when the product is failing; it is generally a time when an event occurs. This time of an event could be a time of getting recovered from a disease or a time until the product reaches some determined usage level. The lifetime of the item can be measured by metrics such as hours, years, or cycles. The distribution that represents those time points is called a lifetime distribution. In typical life data analysis, the experimenter examines the data collected from life testing, in which a sample of items is operated under normal use conditions, to investigate the life characteristics of the product (Meeker and Escobar, 1998).

2.2 Accelerated Life Tests

ALTs are essential models in reliability and survival analysis. They consist of various test methods for shortening the product life or hastening the degradation of their performance by placing the test units under higher than normal stress conditions. Such testing aims to obtain data faster, thereby saving time and money. Under the ALT design, items fail one-by-one according to their weakness as the test time increases. Thus, the analysis of lifetime data for ALT designs depends on the theory of order statistics (Nelson, 1990). Generally, failure information from tests at high levels of one or more accelerating variables that affect the life of the products (e.g. temperature, voltage, humidity or pressure) is transformed, based on a physically reasonable statistical model, to obtain estimates of product life or performance at normal levels of the accelerating variable(s) (Nelson, 1990).

There are many ways of accelerating tests to speed up the time to failure, such as applying high usage rates and overstressing the product. The stress loading in ALTs can be applied by various methods according to the time dependency of the stress variables, as shown in Figure 2.1, which follows Nelson's (1990) description of these essential types of ALTs. Commonly used methods are constant stress, in which the stress is time-independent (i.e. the stress is kept at a constant higher-than-usual level of stress throughout the test), step-stress, in which the stresses are time-dependent or time-varying (i.e. the product is subjected to stress levels that vary at specified change points) and progressive stress (linearly increasing stress).

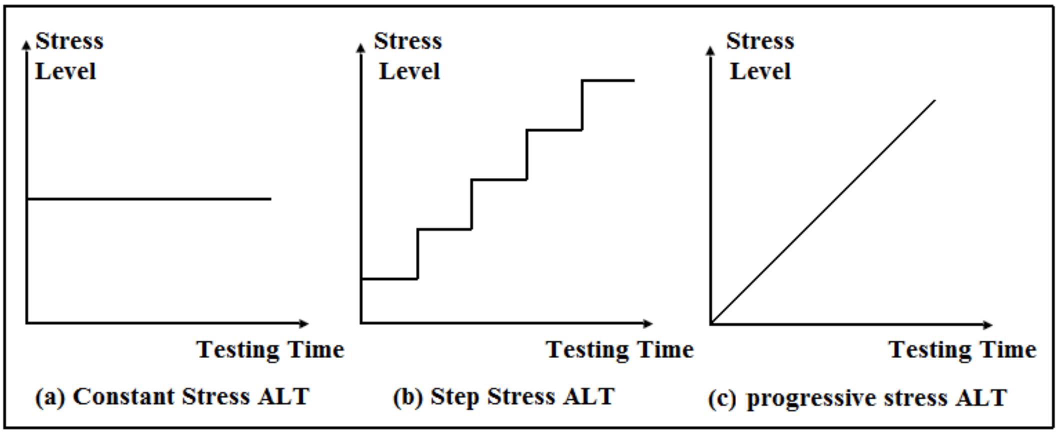


Figure 2.1 Graphical representation of different types of ALT.

In ALTs, the model under constant stress raises the problem of selecting the appropriate stress levels. Suppose the selected stress levels are too low. In that case, there will be few or no product failures during the limited test time, and the effectiveness of the accelerated testing will be consequently reduced. In order to address this problem, SSALTs can be used (Nelson, 1990).

2.3 Censored Samples

In statistical analysis, it is preferable to have information about the complete sample to estimate the parameters accurately. However, removing items from an experiment may be preferable before they fail in a practical life testing experiment, or it may be important to terminate the test before all the failure times have been observed. For many reasons, such as cost reduction, time constraints and facility availability, the censored sample is used. It is a type of incomplete data that is commonly used in reliability and life testing.

The life testing process involves recording the lifetimes of units undergoing the test. In life testing, data is recorded as either failure or censored. If the item is removed from the test before it stops working, it is referred to as censored data. If all units have failed by the end of the test, a complete sample is produced, while if some units survive longer than the test duration, the sample is then said to contain censored data. Figure 2.2 below is based on Nelson (1990, p.14, Figure 2.1).

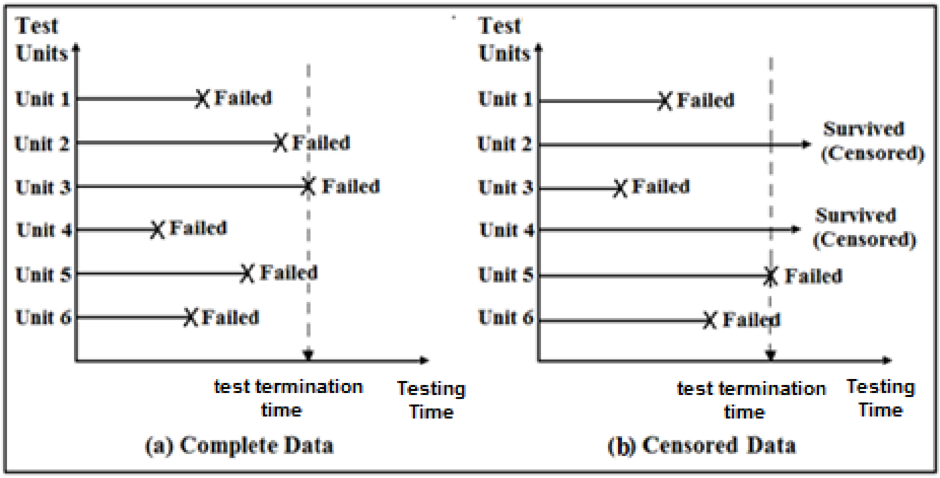


Figure 2.2 Graphical representations of complete and censored data.

Censoring is commonly used in industrial and medical survival analysis; it is often the case that an item is lost or withdrawn before failure occurs. These methods have been linked to a variety of applications in different sciences. As a result, designing ALT models under different censoring schemes, such as Type-I, Type-II and progressive censoring, have received considerable attention over the years. For ALTs based on progressively censored data see for example; Aggarwala (2001), Balakrishnan et al (2011), Cramer et al (2021) and the references in Section 2.8.

2.3.1 Progressive Censoring Schemes

The progressive censoring scheme (PCS) provides a flexible life test design by allowing test units to be removed not only at the termination of the test but also at intermediate stages within the testing time. It is a generalization of the basic Type-I and Type-II censoring schemes where items are removed at single time point. In life testing under Type-I censoring, items are removed after a specified survival time, while under Type-II censoring, items are removed after a predetermined number of observed failures have occurred. To generalize these two types of censoring, items are removed either at predetermined censoring times or upon the occurrence of observed failures; the former is called progressive Type-I censoring and the latter is called progressive Type-II censoring. The test continues for the remaining surviving items, either until they fail or until a further stage of censoring takes place. The concept of PCS was introduced initially by Herd (1956) and was developed by Cohen (1963). PCSs provide more flexibility in the design stage than basic censoring methods, as items are removed in multiple intermediate stages of the experiment based on the PCS. For a detailed explanation of PCS, see Balakrishnan and Cramer (2014).

One of the drawbacks of Type-I and Type-II censoring schemes is that the items cannot be removed early. However, in some life-test experiments, observations are withdrawn from the test at various times during the experiment for reasons that are independent of the experimental factor of interest. For example, in a medical test, some patients may leave the test due to personal reasons before the appearance of the event of interest. Therefore, PCSs can be used to design a life test experiment where items would be removed at different times during the experiment. Furthermore, PCS is one of the preferred censoring methods as it allows the data to be censored in various steps during the test (Balakrishnan and Cramer, 2014). Therefore, PCSs provide the experimenter with more information than the basic Type-I and Type-II censoring, about the whole density curve.

One of the important factors in designing life tests is keeping the experiment's costs as low as possible. Generally, life testing is expensive, not only in terms of production costs but also in terms of test allocation, test time, test equipment, human costs and power costs. Censored items during the test will release the facilities used for testing the items early. Also, removing survival items from the test has benefits as these censored items can be used for other purposes. So, censoring items during the test is a way of reducing the total cost of the life testing. The comparison of the test cost under a complete sample and a progressive Type-II censored sample will be discussed in Section 2.5.

Applying the PCSs in life tests is useful in many situations, for example:

1- In some fields, such as industry, measuring the effect of human usage on the effectiveness of a product is required in order to improve its performance and, sequentially, increase its reliability. Removing products from life tests based on the PCS method can be utilized in order to measure specific characteristics of the product or for each part of it (Aggarwala, 2001).

2- For reliability evaluation during the production prototype process, engineers can use the removed item to perform modifications depending on its reliability at the time of removal. They can determine which parts suffer more impact at certain times and which repairs are required to renovate the product (Balakrishnan and Cramer, 2014).

3- In cases where facilities used for the experiment are limited or available for a short time, PCS is used to release facilities early before the test ends. Moreover, limited funds can lead to a reduction in the usage time of test facilities. Using PCSs may free up some facilities in a shorter time than when using basic censoring schemes (Han, 2015).

4- For the same reason of limited funds, studying the factors contributing to the reliability of items or complete systems is required as well as estimating the mean life time of the items. So, censored items can be used either to measure specific characteristics or as test items in related experiments that do not require the items to be in their original state (Balakrishnan , 2007).

5- If the process of failure of an item has a negative effect, then it is preferable that items be removed during the early stages of the test. Also, in the case where failures cost more than censored items, ALT under PCS is preferable to use.

Under standard environmental conditions for all test items, PCSs can be used in experiments to achieve two purposes: estimating the reliability of an item and determining the growth rate of specific characteristics of that item. To study the functionality of an item, it may be necessary to measure its functions at various times. Thus, the censored items can be used for this purpose. In addition, if the failure of items requires a sequential process, then progressively censoring the items from the experiment is the appropriate solution to reduce the consequent impact of removing all censored items at the end of the test. This may happen, for example, when the life test is performed for items that are a part of a complex system that has to function continuously. The maintenance should be done immediately when the item fails to keep the system working. So, censoring the items progressively will spread the workload of maintenance.

2.4 Step-Stress Accelerated Life Tests

The most common type of ALT is a step-stress model that can substantially shorten the test's duration by allowing the stress on each item to be increased step-by-step over time. This model is often preferred to constant stress ALTs because not only can the test time and expense be considerably decreased, but it could also avoid a high stress start point and possibly unrelated failure modes. If only two stress levels are used in a test, the step-stress test is called a simple step-stress test, while, in the multiple-step SSALT there is more than one change of stress level.

In SSALTs, units usually start at a specified constant low stress level and continue either until a predetermined time or fixed number of failures. If the unit does not fail, the stress is increased and held constant for another specified time or number of failures. The first case is known as time step-stress test and the second case is known as failure step-stress test. Stress is then repeatedly increased and held constant, and the test continues. The test could be terminated when all units fail or at a predetermined time (see Kundu and Ganguly, 2017 for more details).

2.4.1 Simple Step-Stress Accelerated Life Tests

The most used kind of SSALTs in the literature is the simple step-stress test. See for example: Xiong (2003), Nelson (2005), Han and Bai (2020), Ling and Hu (2020) and the references cited there. It has a single change of stress using only two stress levels x_1 and x_2, where $x_1 < x_2$. In a simple step-stress test, units are initially placed on test at a low stress level x_1 and run for a previously specified time, at which point the stress is changed to the higher stress level x_2. The test is continued until all units fail, or until a fixed censoring time or upon occurrence the number of failures.

2.4.1.1 Simple Step-Stress with Type-I Censoring

In a simple SSALT with Type-I censoring, the test is terminated at a specified censoring time τ_c before all n test items have failed. Therefore, the length of the experiment τ_c is fixed but the number of failures r at that fixed time is a random variable. Even though fixing the test duration is generally an advantage, it may result in few failures or possibly none occurring before time τ_c. The test procedure for this type is discussed by Nelson (1990) and can be summarized in the following steps:

Step 1: Firstly, n items are placed under a test at initial low stress level x_1.

Step 2: The test is continued at the low stress level x_1 until a predetermined time τ_1 is reached, then the stress is increased to high stress level x_2 for the survival items.

Step 3: The test is continued until a predetermined censoring time τ_c is reached; see Figure 2.3. If $n = r$, then a complete sample is produced.

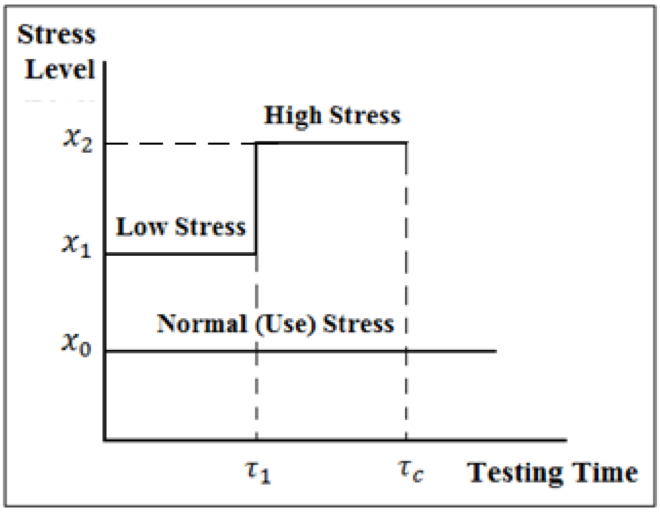


Figure 2.3 Simple SSALT model with Type-I censoring.

2.4.1.2 Simple Step-Stress with Type-II Censoring

In a simple SSALT with Type-II censoring, the test terminates when a specified number of failures $(r), (1 \leq r < n)$ occurs. If $r = n$, then a complete sample is produced. It should be noted that in Type-II censoring, the number of failures r is fixed while the total time of the test is a random variable. Fixing the number of failures makes Type-II censoring preferable to Type-I for many researchers, as a reasonable number of failures is fixed in advance. On the other hand, the randomness of the test period can make Type-II censoring less preferable. The test procedure for this type is the same as simple SSALTs with Type-I censoring, but with Step 3 changed to:

Step 3-1 as shown in Figure 2.4 below, where the test is continued with stress level x_2 until the total r items fail (Nelson, 1990).

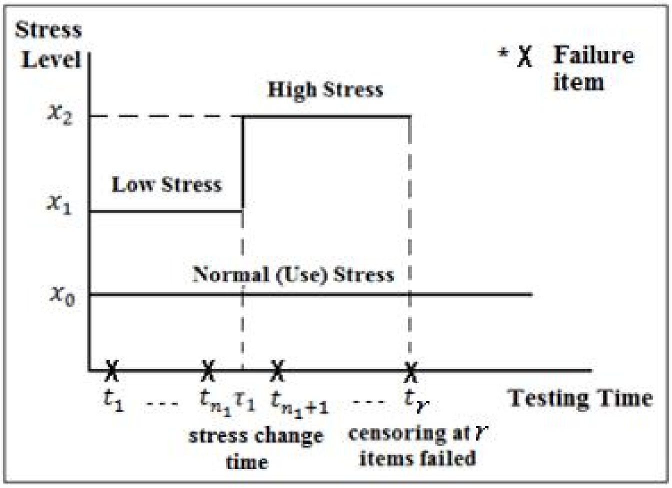


Figure 2.4 Simple SSALTs model with Type-II censoring.

2.4.1.3 Simple Step-Stress with Progressive Type-I Censoring

According to Balakrishnan and Cramer (2014), in simple SSALTs with progressive Type-I censoring, fixed numbers of items are withdrawn at multiple pre-specified times. The test will terminate at the fixed time of the last set of removals. The following steps can be used as a guide for implementing this scheme for SSALT design: (for a simple SSALT model, $r = 2$):

Step 1: determine r censoring times $\tau_1, \tau_2, \dots, \tau_r$ with associated r numbers of censored items $R_1, R_2, \dots,\ R_r$, where r is a specified number of censored stages. Under each stress level x_i, there is n_i failures occur. So, $n = \sum_{i=1}^{r} n_i + R_1 + R_2 + \ \dots + R_r$

Step 2: n items are placed on a test at initial low stress level x_1.

Step 3: When the first censoring time τ_1 occurs, the stress is changed to x_2 and R_1 of survival items are removed from the test. Following that, at the second censoring time τ_2, the stress is changed to x_3 and R_2 surviving are removed from the test, and so on. The number of failures at each stress level x_i is denoted by n_i, where $i = 1, \dots, r$. The observed ordered failure time data $t_{\mathrm{i:n}}$ are:

$$0 < t_{1:\mathrm{n}} < \cdots < t_{\mathrm{n}_1:\mathrm{n}} \leq \tau_1 < t_{\mathrm{n}_1+1:\mathrm{n}} < \cdots < t_{(\mathrm{n}_1+\mathrm{n}_2):\mathrm{n}} \leq \tau_2 < t_{(\mathrm{n}_1+\mathrm{n}_2)+1:\mathrm{n}} < \cdots < \tau_c.$$

Step 4: The test is continued until the time τ_c when the remaining R_r survival items are removed.

Figure 2.5 explains the SSALT model with progressive Type-II censoring.

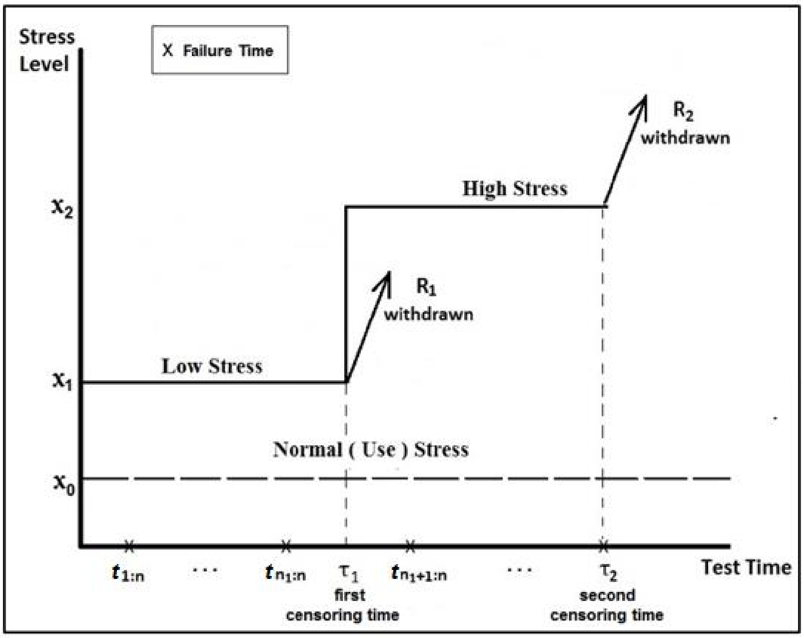


Figure 2.5 Simple SSALT model with progressive Type-I censoring.

It is worth noticing that the number of surviving items at each censored time is random, which may lead to lack of number of units to be removed. Therefore, the planned PCS may differ from the applied PCS (Balakrishnan and Cramer, 2014). The test could be terminated at any of the censoring times $\tau_k, k = 1, \ldots, r-1$ before τ_c, if the survival units at that time are less than the corresponding number of censored items R_k.

2.4.1.4 Simple Step-Stress with Progressive Type-II Censoring

According to Balakrishnan (2009), under the progressive Type-II censoring scheme, fixed numbers of items are removed from the test after each observed failure. The test is terminated when a specified number of failures are observed. Consequently, the time of the test is random, whereas the experimenter determines the number of observed items. Fixing the number of failures in advance makes Type-II censoring preferable to Type-I for many researchers. On the other hand, the randomness of the test period may make it undesirable. The test procedure for this type is:

Step 1: sets r number of censored items $R_1, R_2, \ldots, R_r$, where r is a specified number of failures and $n = r + R_1 + R_2 + \ldots + R_r$

Step 2: n items are placed on a test at initial low stress level x_1.

Step 3: At the time of the first failure occurrence, R_1 of survival items are removed from the test. Following that, when the second item fails, R_2 of survival items are removed from the test, and so on. Simultaneously, the stress level is changed to the higher stress x_2 at the stress change time τ.

Step 4: The test is terminated when the r^{th} failure is observed.

Figure 2.6 presents the test procedures of SSALT model with progressive Type-II censored data.

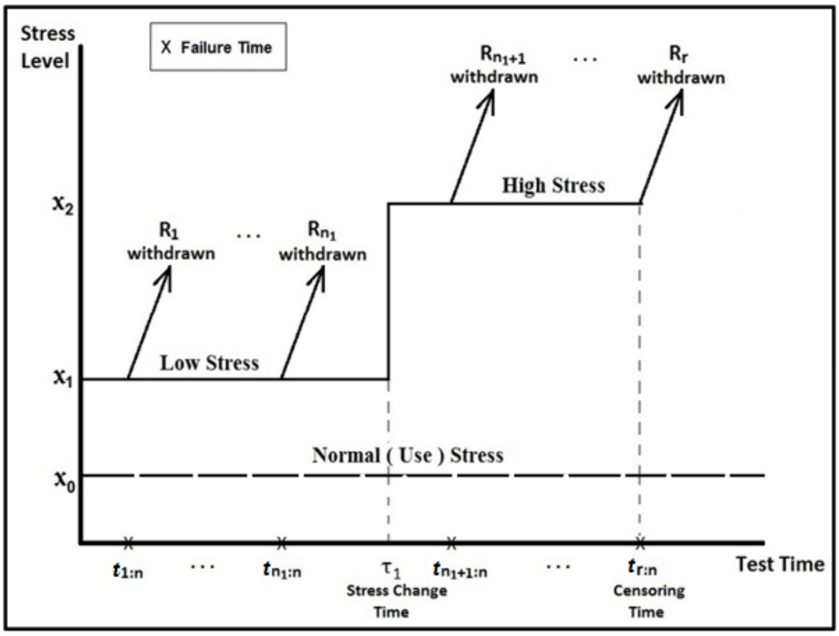


Figure 2.6 Simple SSALT model with progressive Type-II censoring.

2.5 Cost of ALTs

The cost corresponding to a life test is an essential issue to be considered in designing ALT. The total cost of a life test experiment is affected by many factors, such as

1- The cost of the facilities used in the test. This cost is affected by the test duration and the number of products to be tested. However, removing items from the experiments reduces this cost by stopping the usage of the facility, and these items can be used for another purpose.

2- The cost of the products to be tested. However, this cost depends on the number of products under test and the number of observed failures. The cost of a failure is greater than that of a censored item. Removing products from the test before it fails allows the experimenter to use them for another purpose. Also, some parts of censored products could be recycled and used for other purposes.

3- The cost of stress applied to the products. Also, this cost depends on the number of failures and censored data. For example, if the stress was the voltage, then more products mean more voltage units needed to run the ALT.

The total cost is difficult to estimate as it depends on variable features based on specific circumstances. However, Han (2015) assumed that the total cost of a life test consists of 6 main parts. These parts are:

1- The installation cost C_{set}, which is the cost of the experiment set-up. It is the cost of the installation of all items in the ALT experiment to be tested. This cost is constant with respect to n and r.

2- The product cost C_u, which is the cost of each item to be tested.

3- The operation cost $C_{op}(x_s)$ per unit time, which is the cost of running an ALT for each item under the stress level that has been imposed.

4- The inspection cost $C_{ins}(x_s)$ per unit time, which is the cost of the inspection tool used to examine the items under the test. This cost is calculated for each test unit.

5- The failure cost C_f, which is the cost of waste management.

6- The censored items cost C_{cen}, it is the cost of recycling the product or repairing it so that it may be used for another test or purpose.

These components are affected by the total sample size n and the number of observed sample size r, which is the number of the observed sample. If n or r increases, then the total test cost will increase. It can be noticed that $C_{cen} \leq C_f$ as the censored items can be used, unlike the failures. Also, the total operation's cost and the total inspection's cost for all items under the test reduce when items are censored in the early times of the test under Type-II progressive censoring. Under progressive type-II censoring, items are removed at various times during the test, unlike Type-II censoring, where items are removed at the end of the test. Thus, it can be concluded that the test cost of the ALTs based on progressive type-II censoring is less than that of the ALTs based on type-II censoring or a complete sample.

2.6 Life-Stress Model

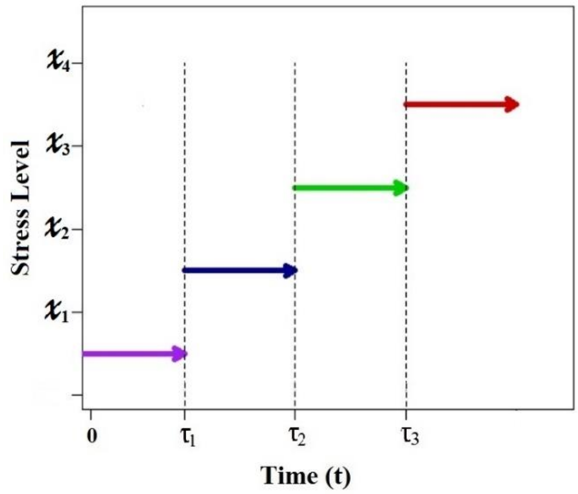
SSALTs are developed based on the underlying life distribution that describes the lifetime of items under each stress level and choosing the appropriate life-stress model that relates the life distribution of test units at different stress levels. It is necessary to estimate the PDF at the usage stress level using data collected at higher stress conditions based on the PDFs at each accelerated stress level. Because the units will be tested under more than one stress level in a SSALT, the cumulative effect of the applied stresses must be considered. To analyse the data from a SSALT, a model is needed to describe the effect of changing stress and to relate the life distribution of the units at a certain stress level to the distribution at the higher stress level. There are several kinds of life-stress model, such as the CEM, which is the most commonly used model proposed by Nelson (1980).

2.6.1 Cumulative Exposure Model

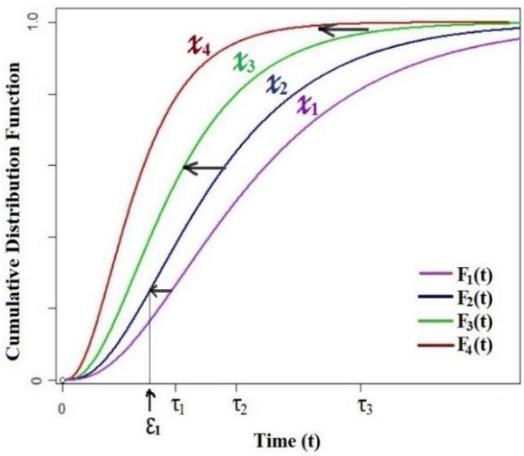
According to Nelson (1980, 1990), the CEM is based on the assumption that the remaining life of the test units depends only on the cumulative exposure the units have seen at the current stress level and that the units do not remember how such exposure was accumulated. Moreover, since the units are held at a constant stress at each step, the surviving units will fail according to the distribution at the current stress level, but with a starting age corresponding to the total accumulated time up to the beginning of the current stress level. However, the CEM relates the distribution under step-stressing to the distribution under constant stress. In other words, the model explains the cumulative effect of the applied stresses. Therefore, the distribution function of a random variable describing the lifetime in a SSALT is obtained from the CEM.

To explain the concept of the CEM for a failure mode, assume there are four prespecified stress levels within a certain life test, x_1, x_2, x_3 and x_4, with $x_1 < x_2 < x_3 < x_4$. The stress levels are increased step by step at the fixed stress change time $\tau_i, i = 1,2,3$. According to Nelson (1980, 1990), Figure 2.7 clarifies the CEM for the four accelerated stress levels. Part (A) presents a step-stress pattern over time with four steps and the stress change times, τ_i , when the stress is increased to a higher level.

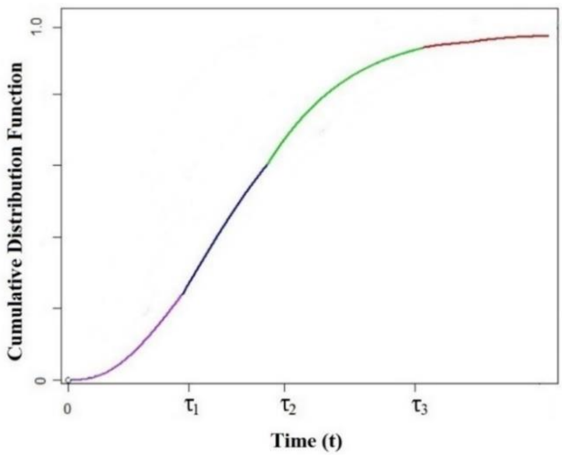
In CEM, it is assumed that the data under test follow the distribution of the stress that they are being subjected. So, there are four different distribution functions $F_1(t), F_2(t), F_3(t)$ and $F_4(t)$, representing the lifetime distribution of items failing under constant stress; x_1, x_2, x_3 and x_4, respectively; see part (B) of Figure 2.7. These four lifetime distribution functions belong to the same family of distributions as it can be seen in part (C) of Figure 2.7.



Part (A): Step-stress pattern



Part (B): CDF of lifetime at each stress level



Part (C): composite CDF of lifetime at SSALTs

Figure 2.7 Cumulative exposure model for a SSALT.

The life experiment starts at time $t = 0$, when all test units are exposed to the stress level x_1 and their failure times follow the CDF:

$$G(t) = F_1(t), 0 \leq t < \tau_1.$$

The failure times of failed units are observed, and the test is run continuously until time τ_1. At that time, the survival units will be exposed to the higher stress level x_2 , where the CDF of the failures will be $\mathrm{F}_2(t)$. Hence, as shown in part (B) of Figure 2.7, the CDF $\mathrm{F}_2(t)$ will start at the previously accumulated fraction failed ε_1 which is equivalent to the ending time (changing time) of the first stress level (Nelson, 1980;1990).

The equivalent time ε_1 represents the exposure the units have seen at stress level x_1. ε_1 can be calculated by equating the probability of failure under x_2 at the time ε_1, with the probability of failure under x_1 at the change time τ_1 as:

$$\mathrm{F}_2(\varepsilon_1) = \mathrm{F}_1(\tau_1).$$

Consequently, the CDF of the units failing under the second level x_2 is

$$\mathrm{G}(t) = \mathrm{F}_2(t - \tau_1 + \varepsilon_1), \tau_1 \leq t < \tau_2.$$

Similarly, the stress level will be changed from x_2 to x_3 at a predetermined time τ_2. The test units under stress level x_3 have the CDF:

$$\mathrm{G}(t) = \mathrm{F}_3(t - \tau_2 + \varepsilon_2), \qquad \tau_2 \leq t < \tau_3$$

where ε_2 is the accumulated fraction failed, and can be found by solving:

$$\mathrm{F}_3(\varepsilon_2) = \mathrm{F}_2(\tau_2 - \tau_1 + \varepsilon_1),$$

where $\mathrm{F}_3(\varepsilon_2)$ is the CDF under stress level x_3 for the equivalent time ε_2 and $\mathrm{F}_2(\tau_2 - \tau_1 + \varepsilon_1)$ is the is the CDF under stress level x_2; see Figure 2.7 part (B). Similarly, the same steps are applied when the stress is increased from x_3 to x_4.

Generally, the CEM defined by Nelson (1990) for k-step SSALTs can be expressed as follows:

$$\mathrm{G}(t) = \begin{cases} \mathrm{F}_1(t), & 0 \leq t < \tau_1, \\ \mathrm{F}_2(t - \tau_1 + \varepsilon_1), & \tau_1 \leq t < \tau_2, \\ \mathrm{F}_3(t - \tau_2 + \varepsilon_2), & \tau_2 \leq t < \tau_3, \\ \vdots & \vdots \\ \mathrm{F}_\mathrm{k}(t - \tau_{\mathrm{k}-1} + \varepsilon_{\mathrm{k}-1}), & \tau_{\mathrm{k}-1} \leq t < \infty, \end{cases}$$

and the corresponding PDF is given by:

$$\mathrm{g}(t) = \begin{cases} \mathrm{f}_1(t), & 0 \leq t < \tau_1, \\ \mathrm{f}_2(t - \tau_1 + \varepsilon_1), & \tau_1 \leq t < \tau_2, \\ \mathrm{f}_3(t - \tau_2 + \varepsilon_2), & \tau_2 \leq t < \tau_3, \\ \vdots & \vdots \\ \mathrm{f}_\mathrm{k}(t - \tau_{\mathrm{k}-1} + \varepsilon_{\mathrm{k}-1}), & \tau_{\mathrm{k}-1} \leq t < \infty, \end{cases}$$

where, $\tau_s, s = 1,2, \dots, k-1, k \geq 2$ is the time of changing the stress from the s^{th} stress level to the $(s+1)^{th}$ stress level, $F_x(t), x = 1, \dots, k$ is the CDF of the lifetime under the x^{th} stress level. Moreover, $\varepsilon_s, s = 1,2, \dots, k-1$ is the equivalent starting time for each step-stress, and can be calculated by solving:

$$F_{s+1}(\varepsilon_s) = F_s(\tau_s - \tau_{s-1} + \varepsilon_{s-1}),$$

such that $\varepsilon_0 = \tau_0 = 0$.

The resulting CDF of the lifetime under the CEM is presented in Figure 2.7 part (C), that consists of the segments of CDFs for four stress levels.

2.7 Life-Stress Relationship

In order to connect product lifetimes with life testing stress levels, Nelson (1990) presented several models for the life-stress relationship. These models describe a life characteristic of the distribution as the relationship between the lifetime and stress levels. These models are based on the type and number of accelerated stresses and how the stress affects the life characteristic.

In this research, the GED is assumed as the underlying lifetime distribution for the ALT models. If T is a random variable has a GED, then the distribution function of T is as follows:

$$F(t; \alpha, \theta) = \begin{cases} [1 - \exp(-(t/\theta))]^{\alpha} & t > 0 \ , \ \alpha, \theta > 0 \\ 0 & \text{otherwise} \end{cases}$$

where α is the shape parameter and θ is the scale parameter.

The log-linear acceleration model is assumed to model the relationship between stress and the scale parameter of the GED. The scale parameters θ_0, θ_1 and θ_2 at the usage stress level, lower stress level and higher stress level are assumed to be a log-linear function of stress levels. The relationship is defined as:

$$ln(\theta_k) = \beta_0 + \beta_1 x_k \ , \quad k = 0,1,2 \ , \tag{2.1}$$

where x_0, x_1 and x_2 are usage, lower and higher stress levels, respectively. The life-stress model parameters β_0 and $\beta_1 (< 0)$ are unknown parameters.

Figure 2.8 shows the relationship between the stress level and the lifetime. The line shows that as the stress level increases, the failure time of the product decreases. Consequently, the product needs longer time to fail when it operates under usage stress level. This relationship provides an effective means for accelerating the test.

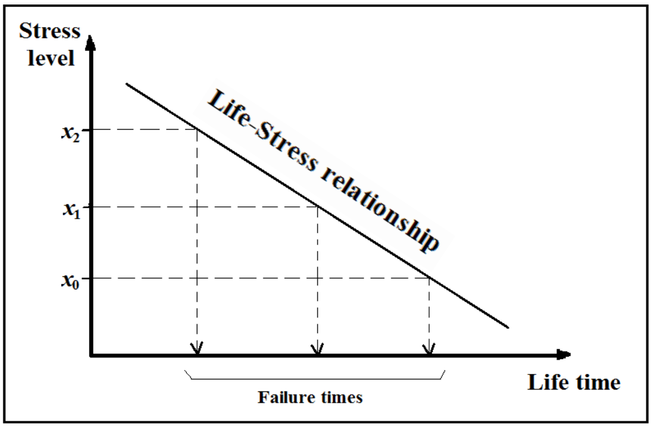


Figure 2.8 The relationship between stress level and lifetime.

After estimating the CEM parameters under lower and higher stress levels, the life-stress relationship is utilized to estimate the model parameters under usage stress level to predict the lifetime under usage stress level; see Chapter 5.

2.8 Review of Relevant Literature

In this section, we will present an overview of relevant research to this thesis. The literature review will be presented regarding two categories: statistical inference of the SSALT model parameter and optimal test design problems based on progressive Type-II censoring scheme.

The ALT is an experimental strategy that is widely used in product life testing by engineers, especially electrical and electronic engineers. It offers a significant reduction in the time and cost of the test. Chernoff (1962) and Bessler et al. (1962) were the first to study the optimum design of ALTs. A substantial number of articles have analyzed product lifetime by using a wide range of ALT mechanisms under different censoring schemes. In this thesis we study the SSALT which is a special case of ALT. There are a few publications on SSALTs with complete samples, but most publications involve censored samples. One reason for this is the cost reduction under censored samples. Also, it is not always possible to observe failure time in practical experiments.

In a SSALT, the CEM describes the life distribution of test units at different stress levels. Most studies relating to the analysis of SSALTs use the CEM. The SS scheme with CEM was proposed by Sedyakin (1966) and was further discussed and generalized by Nelson (1980), who analyzed Weibull distributed lifetimes under the inverse power law as a selected life-stress relationship. The problems of designing an optimum plan and making inferences from the SSALTs have been studied widely, and a guideline for planning SSALTs has been published by Nelson (1990). Recently, Limon et al. (2017) provide a review of relevant research and books on various ALT designs and different optimal test plans. They provide an extensive guidance for researchers in this area and help the researchers to conduct a new study regarding ALTs. Also, Kundu and Ganguly (2017) provided a monograph for reviewing, in detail, the inference and optimum design of SSALT models and related topics. Moreover, a summarized review of ALT models by Chen et al. (2018) focuses on designing an optimal plan for ALT models, and discusses problems and solutions related to ALTs. Using examples of engineering problems arising from ALT applications, the authors provide guidelines on selecting appropriate theories for designing ALTs and also suggest solutions.

2.8.1 Statistical Inference of Simple SSALT

In analysing the SSALT model, the bootstrap method invented by Efron (1979) is utilized for constructing CIs for unknown parameters. It is a well-known computational technique that is often used to estimate the CIs of unknown parameters. The early work by Efron (1982) and Efron and Tibshirani (1993) extensively discussed the Jackknife and the bootstrap technique and its properties. Also, various bootstrap methods have been presented to estimate the CIs of unknown parameters. Also, Efron (1981) used the bootstrap method to calculate point and interval estimates under Type-II censoring. Furthermore, Efron and Stein (1981) discussed using the Jackknife method to estimate the variance of the parameters. They argued that the bootstrap could be used to obtain the most robust estimate of bias and variance of the unknown parameters.

In life testing, it is highly desirable to reduce the cost and time of the experiment. To achieve this, a scheme involving progressively censored items can be used. Analysis of lifetime distributions under PCSs, including generalized censoring schemes for Type-I and Type-II censoring schemes, was first proposed by Herd (1956) who named it 'multi-censoring'. Seven years later, Cohen (1963) officially introduced progressive censoring schemes, which have attracted great interest in the last two decades due to their wide application in the fields of engineering, social sciences and medicine. Balakrishnan and Sandhu (1995) used an independent result regarding a progressive type-II censored sample from the uniform distribution to propose a simple simulation algorithm for generating a progressive type-II censored sample from any continuous distribution. This algorithm requires the generation of only the observed data instead of a complete sample size. The earlier works of Balakrishnan and Aggarwala (2000) and Balakrishnan (2007) provide a thorough overview of various PCSs. Balakrishnan and Cramer (2014) have published the most comprehensive research into various methods of progressive censoring models and their applications, particularly progressive Type-II censoring scheme. Their study provides a detailed introduction to progressive censoring with practical methodology, basic theory and examples, as well as developments in progressive censoring schemes.

Data analysis problems and statistical inferences involve modelling progressive Type-II censoring schemes using various life distributions, have been studied by several authors. See, for example, Krishna and Kumar (2011), Dey and Dey (2014) and Mohie El-Din et al. (2016), Kotb and Raqab (2019), Zhang and Gui (2019) and references cited therein.

Under progressive Type-II censoring, Xie et al. (2008) analyzed the SSALT model for exponential life distribution with a mean that is a log-linear function of the stress, and a CEM assumed. Parameter estimation used the ML method, and CIs were obtained using exact, approximate and bootstrap methods.

Based on a bootstrap sample, three bootstrap methods are used to estimate the CIs of the model parameters: studentized-t, percentile and BCa bootstrap. They study the performance of the proposed methods by Monte Carlo simulation for a small sample size of 20. They concluded that the asymptotic method is not proper to be used for a small sample size. While the exact CIs is the best method to use, unless it is difficult to calculate, then the BCa bootstrap method is advised to be used for estimating the CIs. Balakrishnan (2009) provided a comprehensive review of various studies focusing on exact inferential methods for the model parameters of simple SSALTs under different censoring schemes, including a progressive censoring scheme. The CEM with the exponential distribution as the underlying lifetime distribution was assumed.

Pradhan and Kundu (2009) studied the statistical inference for the shape and scale parameters of the GED under progressively Type-II censored data. Three scenarios of censoring schemes were assumed: Type-II censoring scheme, reverse of Type-II censoring scheme and one random censoring scheme. They computed the MLEs and asymptotic CIs numerically as they realized that the MLEs can not obtained in closed-form expression. They concluded that the average bias and the standard deviation of the MLEs are smallest under reverse of Type-II censoring scheme. However, by comparing the distribution function of the three proposed schemes with the complete sample based on a real data set, they found that Type-II censoring offers an accurate estimate of the distribution function based on the complete sample. Chernick and LaBudde (2011) provided a detailed reference for researchers in the field of bootstrap techniques with applications using the R programming language and a guide for numerically estimating the CIs using bootstrap methods.

In addition, Alkhalfan (2012) studied ML estimation with the optimal design of simple SSALTs under different censoring schemes, including: Type-I, Type-II, progressively Type-I and progressively Type-II censoring based on CEM for different cases of stress levels. In addition, CIs for the unknown parameters were obtained using both the approximate and the parametric bootstrap methods. The Gamma distribution was assumed as a lifetime distribution with a mean that is a log-linear function of stress. A simulation study with illustrative examples was presented to examine the performance of the proposed models. For SSALT based on progressive Type-II censoring, he concluded that the mean squared error for the scale parameter of the lower stress level is larger than the scale parameter of the higher stress level. Based on the coverage probability of the CIs, he recommended using the bootstrap method for estimating the CIs for the model parameters.

A simple SSALT model was studied by Abd El-Monem and Jaheen (2015) for data following the GED distribution based on Type-II censored data. The MLEs and bootstrap CIs of the parameters were obtained, and their performance was investigated. They concluded that the bias and the MSE of the scale parameter at the higher stress level are smaller than the bias and the MSE of the scale parameter at the lower stress level. Also, they noticed that the number of failures in each stress level affects the performance of MLEs. However, they assumed very large values of the sample size (200 and 300) with a large FP$> 75\%$.

In a different method of reliability analysis, Cramer and Jorge (2015) established a relationship between the failure times of components under a coherent system and the progressively Type-II censored data. The aim was to use all information available in the system operation procedure. They presented various detailed examples of many systems, including a bridge system. The ML method was used to obtain the parameter estimates.

In life testing experiments, especially in industrial and medical survival analysis, it is common to encounter censoring. Censoring schemes have been linked to a variety of applications. Simple SSALT models with Type-I and Type-II censoring have been studied extensively in the literature. These can overcome many difficulties in life testing, such as shortening the duration of testing time. Among the many publications on ALTs under Type-I and Type-II censoring, the following are worth mentioning.

Miller and Nelson (1983) studied optimum plans for two types of simple SS: time-step and failure-step SSALT, based on complete samples assuming exponentially distributed lifetimes. The V-optimality criterion is used to minimize the $AVar$ of the MLE of the mean life at a usage stress level. The CEM and a log linear life-stress relationship were assumed. Bai et al. (1989) extended the results of Miller and Nelson (1983) to the case of Type-I censoring. They calculated the MLEs and the Fisher information matrix that is used to calculate optimal stress change time/failure for simple time-step and failure-step SSALTs, respectively. By conducting a sensitivity analysis, they found that the stress change point is not too sensitive to incorrect estimate of the model parameters. Similarly, Xiong (1998) presented three different scenarios of failures occurrence for simple SSALTs Type-II censoring. The MLEs with CIs of model parameters were constructed, and hypothesis tests of model parameters were then discussed. Bai and Kim (1993) provide extensions to both Miller and Nelson (1983) and Bai et al. (1989) to the case of Type-I censoring. They presented an optimum plan of stress change time for simple SSALTs for the Weibull distribution under Type-I censoring. They assumed that a log-linear relationship exists between the Weibull scale parameter and the stress, and the CEM holds. Khamis (1997) made the comparison between simple SSALTs and constant stress for a Weibull lifetime model by measuring the efficiencies using ratios of MSEs of the two suggested models. He concluded that a simple SS test plan is much more efficient than a constant-stress test plan under Type-I censoring for all sample sizes.

One of the background sources for SSALTs is provided by Gouno and Balakrishnan (2001). They presented a brief overview of different methodologies for SSALTs, particularly in the case of Type-II censoring. A procedure to determine the optimum simple SSALT plan under type-I censoring was proposed by Elsayed and Zhang (2004) based on either CEM or a hazard rate function model.

Balakrishnan et al. (2007) analyzed simple SSALTs under Type-II censoring, with lifetimes being exponentially distributed and the log-linear acceleration model assumed. They obtained MLEs of the parameters and their exact distributions assuming a CEM. They suggested three approaches to determining the CIs for the model parameters: exact, asymptotic, and parametric BCa bootstrap methods. They concluded that as the stress change time is increasing, the estimate of the scale parameter under lower stress level will be precise. They argued that the BCa bootstrap can be used as interval estimate for the scale parameter of the higher stress level in all cases, whilst it can be used to estimate the CIs for the scale parameter at the lower stress level for large sample size. Also, it was concluded that a large observed sample size is required to use the asymptotic CIs. Interestingly, they obtained the CIs using a nonparametric bootstrap method, as well, and they found that the coverage probabilities were away from the nominal levels. Moreover, Balakrishnan et al. (2009) analyzed the same simple SSALTs model but under Type-I censoring. They also concluded that the approximate CIs for the model parameter is advised to be used only with large sample size.

Kateri and Balakrishnan (2008) considered the simple SSALT model under Type-II censoring, assuming a Weibull lifetimes distribution. The MLEs of the model parameters and the corresponding observed Fisher information matrix were derived. Closed form MLEs cannot be obtained, therefore they used the Newton-Raphson algorithm to compute the MLEs numerically. The CIs based on asymptotic and bootstrap using percentile and BCa methods were obtained. From the simulation study, they found that for large sample size, the asymptotic CIs are narrower than the bootstrap CIs. They argued that the BCa bootstrap provide better estimate of the CIs in the case of small sample size. Srivastava and Shukla (2008) obtained an optimal simple SSALTs for the case of type-I censoring schemes. They discussed statistical inference including MLEs, CIs and hypotheses tests about model parameters of the life-stress relationship. The model was based on the log-logistic distribution with a median that is a log-linear function of stress.

Based on simple SSALT under Type-I censoring, Abdel-Hamid and AL-Hussaini (2009) obtained the MLEs and the bootstrap CIs for the parameters when the CEM holds, assuming GED for the lifetime of products. Liu (2010) developed ML and Bayesian methods to analyze and plan a simple SSALT with type-I censoring and found the optimal stress change time and the optimal sample size. He assumed a log-linear acceleration model and that the failure times at each stress level follows the Weibull distribution. Yuan et al. (2012) proposed an optimal plan using two optimization algorithms for simple SSALTs of the Weibull lifetime distribution where the characteristic life and the stress have a logarithmically linear relationship. They used the Bayesian approach and assumed the CEM in the presence of type-I censoring. The proposed Bayesian approach is also extended to the design of three-level SSALTs.

2.8.2 Optimum Test Design

Selecting the optimal censoring scheme has received considerable attention in the statistical literature. The ALT model is aimed to be improved by considering the optimal design with respect to different priorities such as minimizing the precision of the parameter estimates. There has been extensive research in the literature concerning the investigation of optimal design for SSALT models based on various methods of censoring. The V-optimality, A-optimality and D-optimality criteria are commonly used criteria for designing optimum SSALTs plans. After reviewing the work that have been done in SSALTs field, it can be noticed that the aim of most of the literature relevant to planning SSALT under a variety of censoring methods is to determine the optimal stress changing time. On the other hand, the research related to planning simple SSALT under PCSs are interesting in selecting the optimal censoring scheme over a range of proposed censoring schemes. Since 2000, the statistical literature in SSALTs based on PCS has been interested in determining the optimal censoring scheme. One of the basic references for the optimal censoring scheme is Chapter 10 of Balakrishnan and Aggarwala (2000). Some literature on optimal design based on progressively censored data is: Han (2008), Salemi et al (2018) and Hakamipour (2020).

Li (2009) studied the problem of estimation with optimal design of SSALTs under type-I censoring based on CEM. He used the exponential and Weibull as lifetime distributions for a simple SSALTs model. He used the V-optimality criterion to obtain the optimal stress change time. Two objective functions are assumed to minimize their variance: the percentile life and the reliability estimate under usage stress level. By using a numerical algorithm to calculate the optimal time, he concluded that the optimal stress change time is slightly sensitive to changing the model parameters.

In addition to studying the statistical inference for the parameters of the GED, Pradhan and Kundu (2009) investigated, also, the optimal censoring scheme under progressive Type-II censoring schemes among all possible censoring schemes for small sample size ≤ 15. They utilized the optimization criterion that measures the information on the parameters that is gained under the censoring scheme. The proposed information measure depends only on the shape parameter. Also, it depends on the weight of percentile rank. They numerically compared the proposed criterion with V-optimality, A-optimality and D-optimality. They concluded that the proposed criterion gives the same optimal scheme as A-optimality criterion.

Xie et al. (2008) discussed an optimal plan where not only the optimal change time, but also, the best and the worst progressive censoring schemes were investigated. They used the V-optimality and MSE-optimality criteria, assuming two objective functions, respectively, as follows:

Objective1= $var(\hat{\theta}_1) + var(\hat{\theta}_2)$, where $var(\hat{\theta}_1)$ and $var(\hat{\theta}_2)$ are the variances of the scale parameter under the lower and higher stress levels, respectively.

Objective2= $MSE(\hat{\theta}_1) + var(\hat{\theta}_2)$, where $MSE(\hat{\theta}_1)$ is the mean squared error of the scale parameter under the lower stress level.

They tested all possible schemes under small sample sizes$= 10,12,16,20$ with failure percentages ranging between 50% and 80% to determine the optimal and the worst scheme. From the results conducted via Monte-Carlo simulation, it was observed that the optimal and worst schemes based on two optimality criteria are different in most cases. Also, the relative efficiency of the worst to the optimal scheme under V-optimality is decreasing, and then it is increasing as the stress change time increases. On the other hand, under the MSE-optimality, the relative efficiency always decreases with increasing the stress change time. These results mean the bias of θ_1 affects the selection of the progressive scheme.

Ng et al (2004) studied the optimal censoring scheme for a progressive Type-II censored sample from a Weibull distribution. The expected Fisher information matrix and the AV-C matrix were obtained. The optimal censoring scheme was investigated based on three methods: A-optimality, D-optimality and maximizing the trace of the Fisher information matrix. They investigated the sensitivity of the censoring scheme by choosing some schemes that are close to the optimal one. They found that the relative efficiency is slightly changed which indicates that the optimal censoring scheme is robust.

Chapter 3

Generalized Exponential Distribution

Adding one or more parameters to a distribution function makes the resulting distribution more flexible for modelling data. So, adding a parameter to a cumulative distribution function, $\mathbb{D}(t;\underline{\theta})$ by exponentiating it, produces the class of exponentiated distributions (Gupta et al, 1998). It has the form

$$F(t;\alpha,\underline{\theta}) = \left[\mathbb{D}(t;\underline{\theta})\right]^{\alpha}.$$

Gupta and Kundu (1999) defined the GED, also known as the exponentiated exponential distribution, as a member of the class of exponentiated distributions by assuming $\mathbb{D}(t;\underline{\theta})$ to be the CDF of the exponential distribution. In addition, it is a particular member of the three-parameter exponentiated Weibull distribution, introduced by Mudholkar and Srivastava (1993). The GED is one of the most commonly used generalizations of the standard exponential distribution. Since the beginning of this century, the GED has been studied quite extensively by many authors in the life testing field, especially in presence of censored samples, some of which will be reviewed in the last section of this chapter. Moreover, Chapter 2 reviewed research that assumes the GED as the underlying lifetime distribution in analysing ALT models. This chapter will concisely review the GED and its properties. Furthermore, the GED is compared to the Weibull and gamma distributions to investigate its similarity and distinction to these distributions.

3.1 Description of the GED

If T is a GED random variable, then the distribution function of the two-parameter GED is as follows:

$$\mathrm{F}(t;\alpha,\theta) = \begin{cases} [1-\exp(-(t/\theta))]^{\alpha} & t>0\ ,\ \ \alpha,\theta>0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

The corresponding density function is:

$$f(t;\alpha,\theta) = \begin{cases} \frac{\alpha}{\theta}\left[1-\exp\left(-\frac{t}{\theta}\right)\right]^{\alpha-1}.\exp\left(-\frac{t}{\theta}\right) & t>0\ ,\ \ \alpha,\theta>0 \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where α is the shape parameter and θ is the scale parameter. If $\alpha = 1$, then the GED coincides with the exponential distribution with mean θ as a special case. This property is shared with the Gamma and Weibull distributions.

The shape of the GED density depends only on α. The scale parameter has an impact on the spread of the distribution. Some of the possible shapes of the GED density functions for different values of the shape parameter α at $\theta = 1$ and for different values of the scale parameter at $\alpha = 3$ are provided in Figure 3.1 and Figure 3.2, respectively.

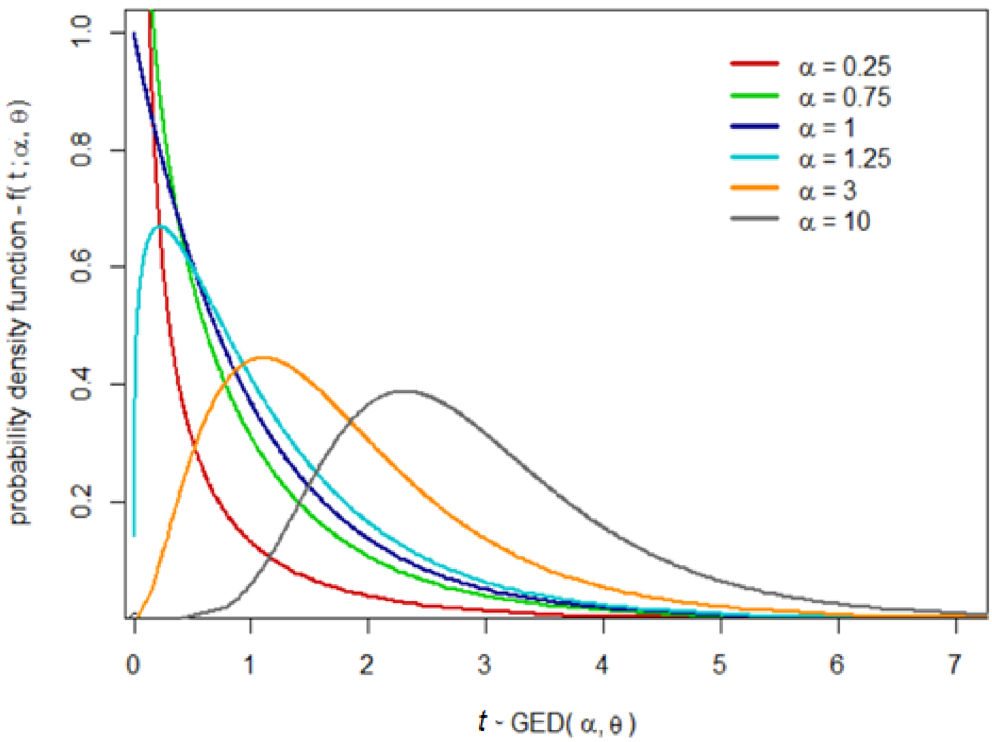


Figure 3.1 A graphical comparison of the PDF of GED for different values of α at $\theta = 1$

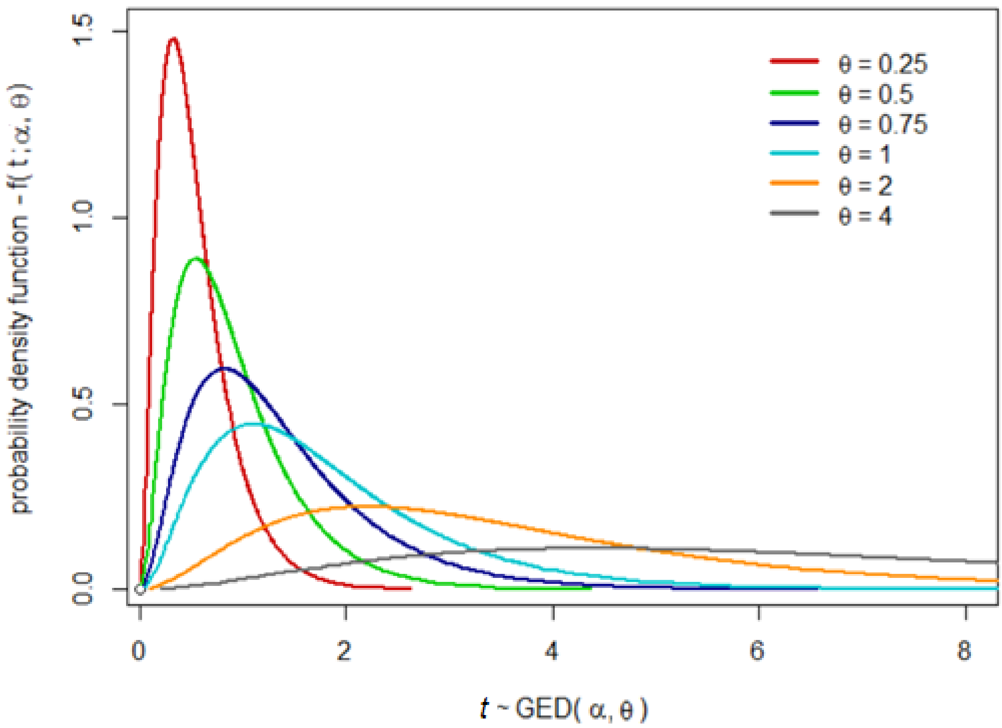


Figure 3.2 A graphical comparison of the PDF of GED for different values of θ at $\alpha = 3$

It can be seen from Figure 3.1, one important property of GED is that its PDF is always right skewed as we will see in Section 3.1.1. As a result, the GED can be considered for situations where a skewed distribution for a positive random variable is needed. For this reason, it can be used effectively to analyze lifetime data, particularly in the presence of censoring; see Gupta and Kundu (2003, 2007). Based on the properties of the GED studied by Gupta and Kundu (1999) and from Figure 3.1, it is obvious that the PDF of the GED is strictly decreasing with a reverse "J" shape for $\alpha \leq 1$, whereas it is unimodal and right skewed for $\alpha > 1$. It becomes more symmetric as the shape parameter increases. In addition, it can be observed that even for a small value of the shape parameter, the distribution is not symmetric (Gupta and Kundu, 2007). It should be noted, see Gupta and Kundu (2007) for numerical details, that the skewness and kurtosis of the GED have an inverse relationship with the shape parameter; see Section 3.1.1 for more details.

Figure 3.2 demonstrates that the distribution is denser according to the area related to the value of the scale parameter. Also, as the value of the scale parameter increases, the distribution becomes flatter.

Basic functions used in reliability theory are the reliability and the hazard functions. The reliability of GED is:

$$R(t;\alpha,\theta) = 1 - [1 - \exp(-(t/\theta))]^{\alpha}, \tag{3.3}$$

and the hazard function is:

$$h(t;\alpha,\theta) = \frac{\frac{\alpha}{\theta}\left[1 - \exp\left(-\frac{t}{\theta}\right)\right]^{\alpha-1} . \exp\left(-\frac{t}{\theta}\right)}{1 - [1 - \exp(-(t/\theta))]^{\alpha}}, \tag{3.4}$$

To illustrate how the hazard is influenced by the shape parameter, Figure 3.3 shows the hazard function for $\theta = 1$ and various values of α.

Figure 3.3 shows that the hazard function of a GED can be increasing, decreasing or constant depending on the shape parameter. The situation is similar with gamma and Weibull distributions. The hazard function is constant when $\alpha = 1$, and in this case the hazard function of the exponential distribution is achieved. Furthermore, if $\alpha < 1$, the hazard function is decreasing to a finite number (scale parameter θ), as for a gamma distribution. On the other hand, if $\alpha > 1$, it increases from zero to a finite number (scale parameter θ), whereas in the Weibull distribution the hazard function increases from $0 \rightarrow \infty$.

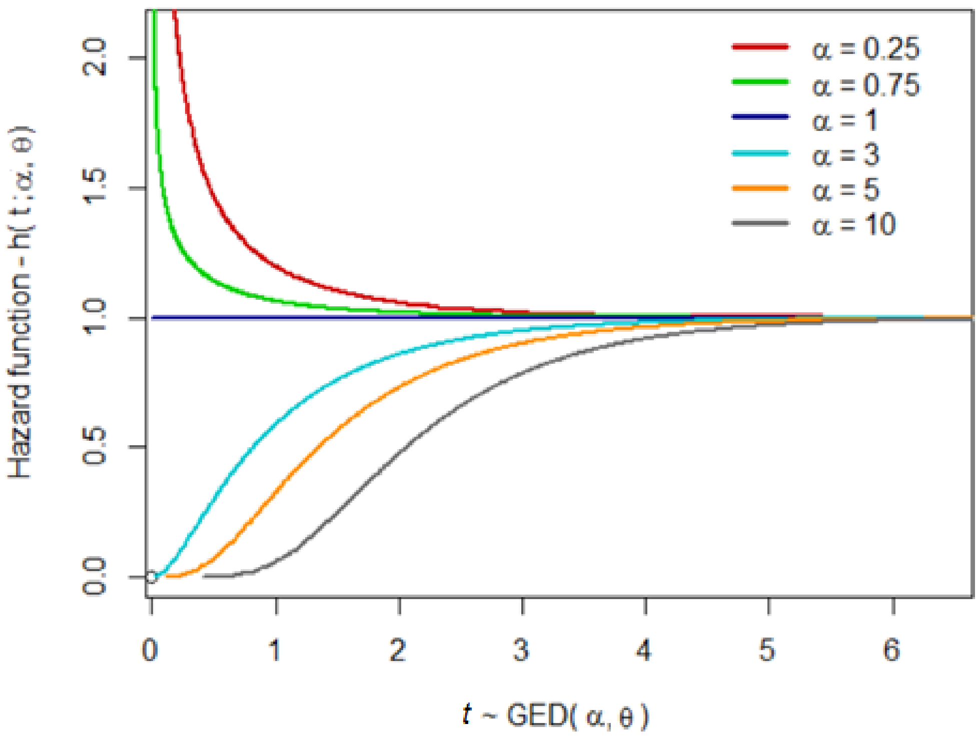


Figure 3.3 A graphical comparison of hazard of GED for different values of α at $\theta = 1$

3.1.1 Properties of the GED

In this section, some of the main properties of the GED are presented.

If T is a random variable that follows the GED in (3.1), then the moment generating function $M_T(y)$ of T can be expressed as:

$$M_T(y) = E(e^{yT}) = \frac{\Gamma(\alpha+1)\Gamma(1-y\theta)}{\Gamma(\alpha-y\theta+1)}, \qquad y < \frac{1}{\theta}, \alpha > 0$$

where $\Gamma(y) = \int_0^\infty e^{-t}\, t^{y-1}\, dt, y > 0$ is a gamma function (Gupta and Kundu, 2007).

Consequently, the mean and the variance of the GED can be obtained based on $M_T(y)$. The mean of a random variable is the first derivative of the moment generating function at $y = 0$. That is

$$Mean(T) = M'_T(0)$$

$$= \theta[\psi(\alpha+1) - \psi(1)]$$

where $\psi(z) = \frac{d}{dz}\ln\big(\Gamma(z)\big) = \frac{\Gamma'(z)}{\Gamma(z)}$ is a digamma function (Gupta and Kundu, 1999).

Similarly, the variance can be obtained by calculating the first and second derivatives of the moment generating function at $y = 0$; which can be given by:

$$Var(T) = M_T''(0) - [M_T'(0)]^2$$

$$= \theta^2[\psi'(1) - \psi'(\alpha + 1)].$$

The quantile function of the GED is:

$$Q(p; \alpha, \theta) = -\theta \log\left(1 - (p)^{\frac{1}{\alpha}}\right) \qquad 0 < p < 1 \ , \ \alpha, \theta > 0 \qquad (3.5)$$

So, the median can be calculated from the quantile function:

$$Median(T) = -\theta \log\left(1 - (0.5)^{\frac{1}{\alpha}}\right)$$

Moreover, the coefficient of skewness of the GED can be calculated as

$$Skew(T) = \frac{\mu_3}{Var(T)^{3/2}},$$

where μ_3 is the third central moment (moment about the mean) can be obtained as

$$\mu_3 = E[(T - E(T))^3] = \mu_3^* - 3\mu_1^*\mu_2^* + 2(\mu_1^*)^3,$$

where μ_1^*, μ_2^*, μ_3^* are the first, second and third raw moments (moment about the zero).

Using the moment generating function, $M_T(y)$, the raw moments can be calculated as

$$\mu_i^* = M_T^{(i)}(0) = \frac{d^i}{dy^i} M_T(y)_{|y=0}$$

Gupta and Kundu (2007) represented $M_T''(0)$ and $M_T'''(0)$ using the digamma and polygamma functions as follows

$$M_T''(0) = \mu_2^* = \theta^2 \left[\psi'(1) - \psi'(\alpha + 1) + \left(\psi(\alpha + 1) - \psi(1)\right)^2\right]$$

$$M_T'''(0) = \mu_3^* = \theta^3 \left[\psi''(\alpha + 1) - \psi''(1) + 3\left(\psi(\alpha + 1) - \psi(1)\right)\left(\psi'(1) - \psi'(\alpha + 1)\right)\right.$$
$$\left. + \left(\psi(\alpha + 1) - \psi(1)\right)^3\right]$$

The skewness of the GED will be numerically calculated and compared with Weibull and gamma distributions in Section 3.2.

Figure 3.4 represents the mean, median, mode, variance and skewness of GED as a function of α, when $\theta = 1$. The mean, mode and median of the GED are functions of the shape and scale parameters of the distribution; see Gupta and Kundu (1999, 2007). As can be seen from Figure 3.4, for fixed θ, the mean, median and mode increase and converge to a limit value $= \theta \log(\alpha)$ as α increases. Similarly, as α

increases, the variance of GED increases and approximates to $\frac{(\theta\pi)^2}{6}$. In contrast, the skewness is a decreasing function of α. Gupta and Kundu (1999) numerically indicated that the skewness has a limiting value $\cong 1.1395$, regardless of the value of θ.

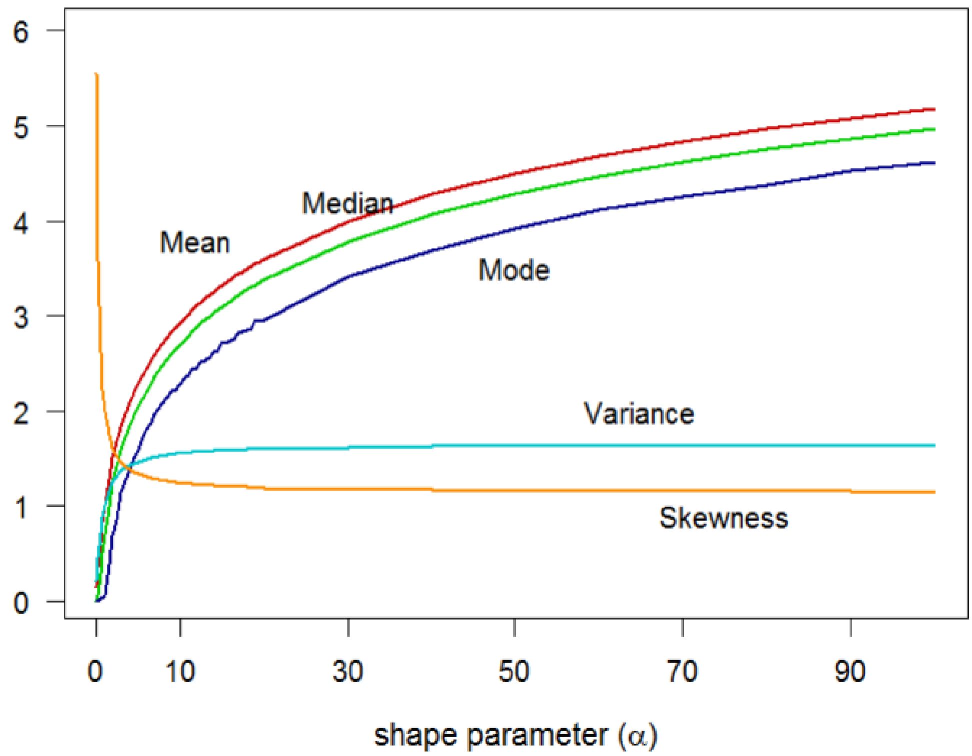


Figure 3.4 The impact of α on the mean, median, mode, variance and skewness of GED at $\theta = 1$

3.2 Comparison with Gamma and Weibull Distributions

In this section the properties of the GED is compared to those of the Weibull and gamma distributions. These three distributions are commonly used in reliability and life testing, due to their distinct properties and preferred physical interpretations. The GED family has some interesting features and certain distinct properties compared to distributions. However, its properties are quite similar to those of the gamma and Weibull families. In fact, it has been observed that in certain situations, the GED has some specific features which make it more desirable to use in lifetime tests than the gamma and Weibull distributions (Gupta and Kundu, 2003; 2004).

To do a comparison of GED, Weibull and gamma distributions, the skewness coefficient will be calculated for the three distribution based on the moment generating function. The skewness will be measured based on the central moments. Skewness is the third central moment of a random variable; it is measured as:

$$Skew(T) = \frac{\mu_3}{Var(T)^{3/2}},$$

where μ_3 is the third central moment (moment about the mean).

The moment coefficient of skewness provides information about the amount and direction of the density curve from symmetry. If $Skew > 0$, the density is positively skewed. In contrast, if $Skew < 0$, the density is negative skewed. As the value of $|Skew|$ increases, it indicates an extremely asymmetric distribution. In contrast, as the value of $|Skew| \to 0$, it indicates that the distribution is symmetric. A symmetric distribution has $skew = 0$. However, the value of the standard skewness is invariant of the location and scale of the random variable.

In the case that the random variable follows the Weibull distribution, its pdf is given as:

$$f_{Weibull}(t; \alpha, \theta) = \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1} . \exp\left(-\frac{t}{\theta}\right)^{\alpha} \qquad t \geq 0$$

where $\alpha, \theta > 0$ are the shape and scale parameter, respectively.

Based on Rinne (2009), by using the central moments, the skewness measure of Weibull distribution is given by

$$Skew_W(T) = \frac{\Gamma\left(1+\frac{3}{\alpha}\right) - 3\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right) + 2\Gamma\left(1+\frac{1}{\alpha}\right)^3}{\left(\Gamma\left(1+\frac{2}{\alpha}\right) - \left(\Gamma\left(1+\frac{1}{\alpha}\right)\right)^2\right)^{3/2}}$$

where $\Gamma(.)$ is the gamma function.

Also, the skewness coefficient can be written as

$$Skew_W(T) = \frac{\Gamma\left(1+\frac{3}{\alpha}\right)\theta^3 - 3\mu_1^*\sigma^2 - (\mu_1^*)^3}{\sigma^3}$$

When $\alpha \approx 3.60$, the Weibull distribution approximates to the normal distribution with $Skew_W \cong 0$. Consequently, Weibull distribution has a positive skewed curve for small values of the shape parameter. On the other hand, the distribution is negatively skewed for large values of the shape parameter (Rinne, 2009).

Let T be a random variable which follows the gamma distribution. The pdf of the gamma distribution is shown below:

$$f_{gamma}(t; \alpha, \theta) = \frac{1}{\theta^{\alpha}\Gamma(\alpha)} t^{\alpha-1} . exp\left(-\frac{t}{\theta}\right), \qquad t > 0\ ,\ \alpha, \theta > 0$$

where α, θ are the shape and scale parameter, respectively.

The coefficient of skewness is:

$$Skew_G(T) = \frac{2}{\sqrt{\alpha}}$$

To compare the skewness of the GED with that of the gamma and Weibull distributions, the standard skewness of each distribution are numerically calculated using "R" program. The results are shown in Figure 3.5 for scale parameter $\theta = 1$ and various values of the shape parameter.

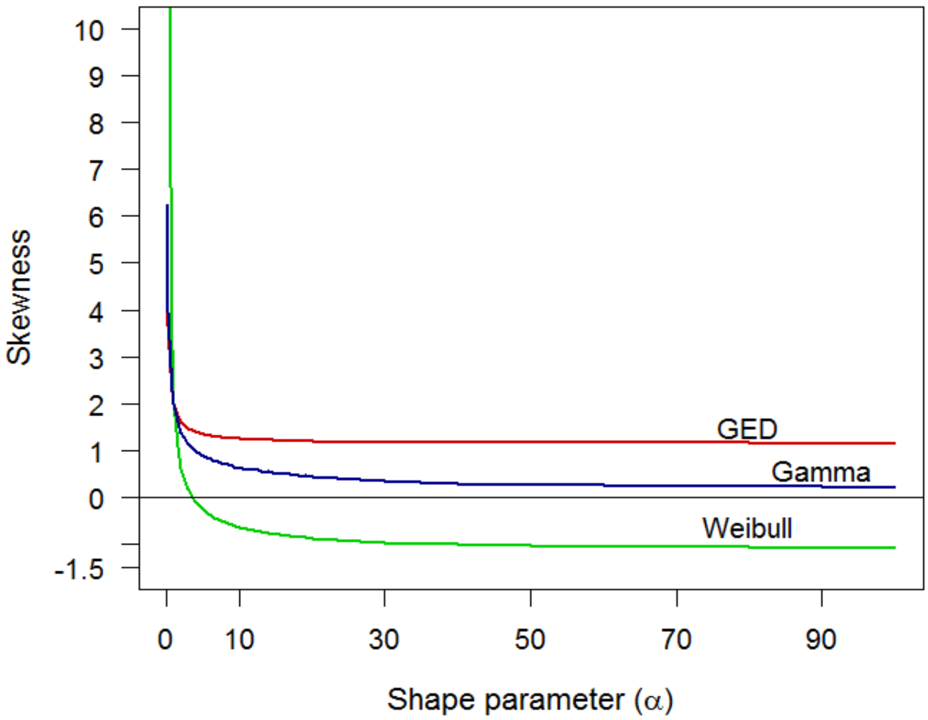


Figure 3.5 The impact of α on the skewness of GED, Weibull and gamma distributions.

From Figure 3.5, it is obvious that the skewness of the three distributions converges to a constant value as $\alpha \longrightarrow \infty$. However, both gamma distribution and GED have positive skewness. On the other hand, the Weibull distribution has positive and negative skewed PDF for approximately $\alpha < 3.6$ and $\alpha > 3.6$, respectively. Furthermore, it can be clearly observed that, among the three distribution, the GED has the largest skewed distribution for all values of shape parameter. Thus, it is observed that the GED fits better than Weibull and gamma distributions for some lifetime data with highly right skewed as it has the largest value of skewness coefficient; (Gupta et al. (2002), Gupta and Kundu (2001a)). However, using the maximum likelihood ratio method, Gupta et al. (2002) concluded that the GED fitted better than Weibull and gamma distributions for two given real-life data sets.

The variance of all three distribution: GED, Weibull and gamma increase as the shape parameter increases. In the case of the GED, Gupta and Kundu (1999, 2007) argued that the variance is increasing to $\frac{(\theta\pi)^2}{6}$ as α increases. Similarly, the variance of the Weibull distribution $\approx \frac{(\theta\pi)^2}{6\alpha^2}$ for large values of α. In contrast, the variance of the gamma distribution $\rightarrow \infty$ for increasing values of α.

The main disadvantage of the gamma distribution arises when the shape parameter is not an integer. In this circumstance, the distribution function or the reliability function cannot be expressed in a closed form (Gupta and Kundu, 2001a). In many survival analyses, the complexity of the gamma distribution leads to intractable analysis, especially in the presence of ALT models with censored data. Therefore, GED can be a proper alternative due to its numerical simplicity and the fact that its distribution function is flexible, which make the numerical generating of samples from GED under different types of censoring, much easier to conduct for a Monte Carlo simulation, compared with gamma distribution.

Moreover, Gupta and Kundu (2001a) discussed the ordering properties between GED, Weibull and gamma distributions. They noticed that the hazard ordering between the three distribution is:

$h_{Weibull}(t;\alpha,\theta) \geq h_{GED}(t;\alpha,\theta) \geq h_{gamma}(t;\alpha,\theta)$, if $\alpha > 1$

$h_{Weibull}(t;\alpha,\theta) \leq h_{GED}(t;\alpha,\theta) \leq h_{gamma}(t;\alpha,\theta)$, if $\alpha < 1$

Unlike the Weibull distribution, which is used to represent the series system, the GED represents a parallel system (Gupta and Kundu, 2007). If each of the n independent identical components in the parallel system follow the exponential distribution with reliability,

$R_{component}(t,\theta) = \exp(-(t_i/\theta))$,

then the system reliability is given by:

$$R_{system}(t;\mathrm{n},\theta) = \prod_{i=1}^{n}\left(R_{component}(t_i,\theta)\right)$$

$$= [1-\exp(-(t/\theta))]^n$$

Obviously, the above equation represents the reliability function of GED in equation (3.3) with $\alpha = \mathrm{n}$. In other words, for the case where α is a positive integer, this can represent the lifetime of a parallel system of n components each has exponential distribution.

To sum up, if a set of data fit for a gamma distribution, then it will fit for the GED as well in most of the cases. However, the GED and Weibull distributions both have closed form expressions for CDF, reliability and hazard functions.

3.3 Parameter Estimation and Simulation

A brief discussion of the MLEs for the GED parameters will be presented in this section.

Suppose $t_1, \ldots, t_n$ are independent identically random variables of size n from GED, then, using the PDF from equation (3.2), the log-likelihood function $\ell(\alpha,\theta;\underline{t})$ is:

$$\ell(\alpha,\theta;\underline{t}) = n\ln(\alpha) - \mathrm{n}\ln(\theta) - \sum_{i=1}^{\mathrm{n}}\left(\frac{t_{\mathrm{i:n}}}{\theta}\right) + (\alpha-1)\sum_{i=1}^{\mathrm{n}}\ln\left[1-\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta}\right)\right] \tag{3.6}$$

Consequently, the MLEs of the scale and shape parameters are obtained by setting the first partial derivatives of $\ell(\alpha,\theta;\underline{t}) = 0$

$$\frac{\partial \ell(\alpha,\theta;\underline{t})}{\partial \theta} = -\frac{n}{\theta} + \sum_{i=1}^{n}\left(\frac{t_{\mathrm{i:n}}}{\theta^2}\right) - \frac{(\alpha-1)}{\theta^2}\sum_{i=1}^{\mathrm{n}}\frac{t_{\mathrm{i:n}}\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta}\right)}{\left[1-\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta}\right)\right]} = 0 \tag{3.7}$$

$$\frac{\partial \ell(\alpha,\theta;\underline{t})}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left[1-\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta}\right)\right] = 0 \tag{3.8}$$

As it can be seen, neither of the above equations has a closed form solution. Thus, the MLEs of θ and α can be obtained by numerically solving the above two equations (3.7) and (3.8).

A simulation study is undertaken to investigate the impact of parameters changes on the performance of the estimation. This provides a starting point for estimating the values of the model parameters in the SSALT model under progressive type-II censoring in the next chapter. The equations (3.7) and (3.8) are solved, using the "maxLik" package (Henningsen and Toomet, 2011) in "R" program, to derive the MLEs of the parameters.

To investigate the impact of the parameter values on the performance of the estimators, the estimated absolute bias (AB) and the estimated mean squared error (MSE) of the MLEs are plotted against various sample sizes; $n = (10, 20, \dots 90, 100)$. The following steps demonstrate how to derive MLEs $\hat{\vartheta} = (\hat{\alpha}, \hat{\theta})$ for GED parameters $\vartheta = (\alpha, \theta)$.

Step (1): The values of the shape parameter are chosen as $\alpha = (0.3, 0.7, 1, 1.2, 1.5, 2)$. These values are tested based on two different values of the scale parameter $\theta = (0.5, 1)$.

Step (2): Generate a random sample from the $\mathrm{Uniform}(0,1)$ distribution.

Step (3): Using the equation (3.5), calculate the random variables $t_{1:\mathrm{n}} < \cdots < t_{\mathrm{n:n}}$

Step (4): The maxBFGS built-in function from the maxLik package (Henningsen and Toomet, 2011) is utilized under constraints ($\alpha > 0$ and $\theta > 0$), for obtaining MLEs of GED parameters. This built-in function is used based on the logarithm of the likelihood function $\ell(\alpha,\theta;\underline{t})$ in equation (3.6) as well as the gradient functions in equations (3.7) and (3.8).

Step (5): Repeat the above steps (2 to 4) $p = 1000$ times.

Step (6): The average of the MLEs of the model parameter $\vartheta = (\alpha, \theta)$ over the number of replications is calculated. Subsequently, estimated AB and estimated MSE for the MLEs are obtained, respectively, as follows

$$AB(\hat{\vartheta}) = |\hat{\vartheta} - \vartheta|/p,$$

$$MSE(\hat{\vartheta}) = \left((\hat{\vartheta} - \vartheta)^2\right)/p.$$

The result of the investigation is shown in the following Figure 3.6 – Figure 3.7.

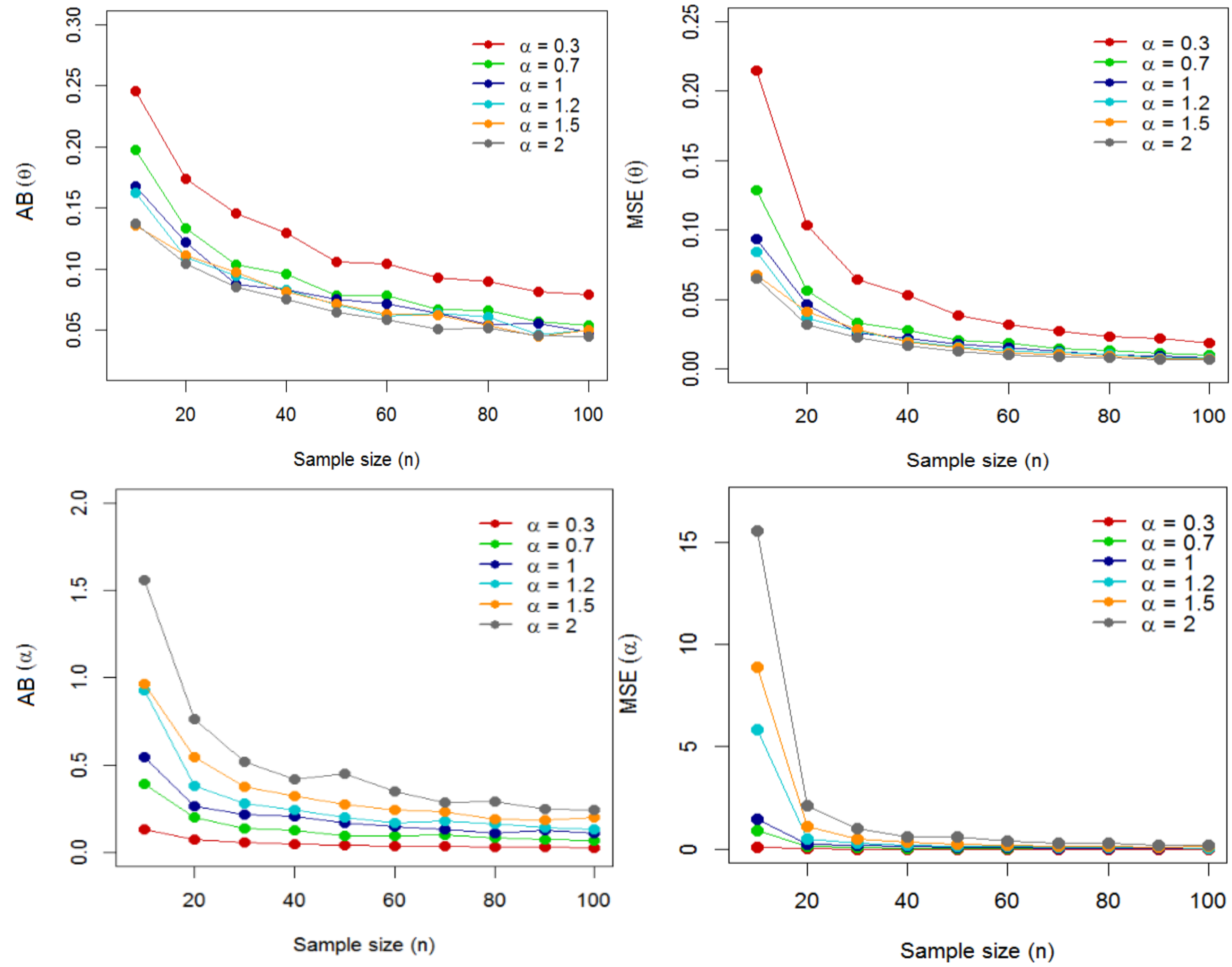


Figure 3.6 Estimated AB and estimated MSE of the MLEs of θ and α obtained for different values of α and $\theta = 0.5$

From Figure 3.6 and Figure 3.7 it can be seen that the estimated AB and estimated MSE of both parameters decrease as the sample size increases. In other words, larger sample size results in more efficient MLEs of GED parameters. Also, as the shape parameter increase, the $MSE(\hat{\vartheta})$ and $AB(\hat{\vartheta})$ decrease. Whereas, the performance of MLE $\hat{\alpha}$ is better for small values of α.

Moreover, the increase in the shape parameter results in more efficient estimates of the scale parameter. Looking further, for $\alpha < 1.5$, the value of $MSE(\hat{\vartheta})$ and $AB(\hat{\vartheta})$ is almost the same. As can be seen, there is a considerable increase of $MSE(\hat{\vartheta})$ and $AB(\hat{\vartheta})$ when $\alpha < 1$. In another words, the MLE $\hat{\vartheta}$ is more robust when $\alpha \geq 1$.

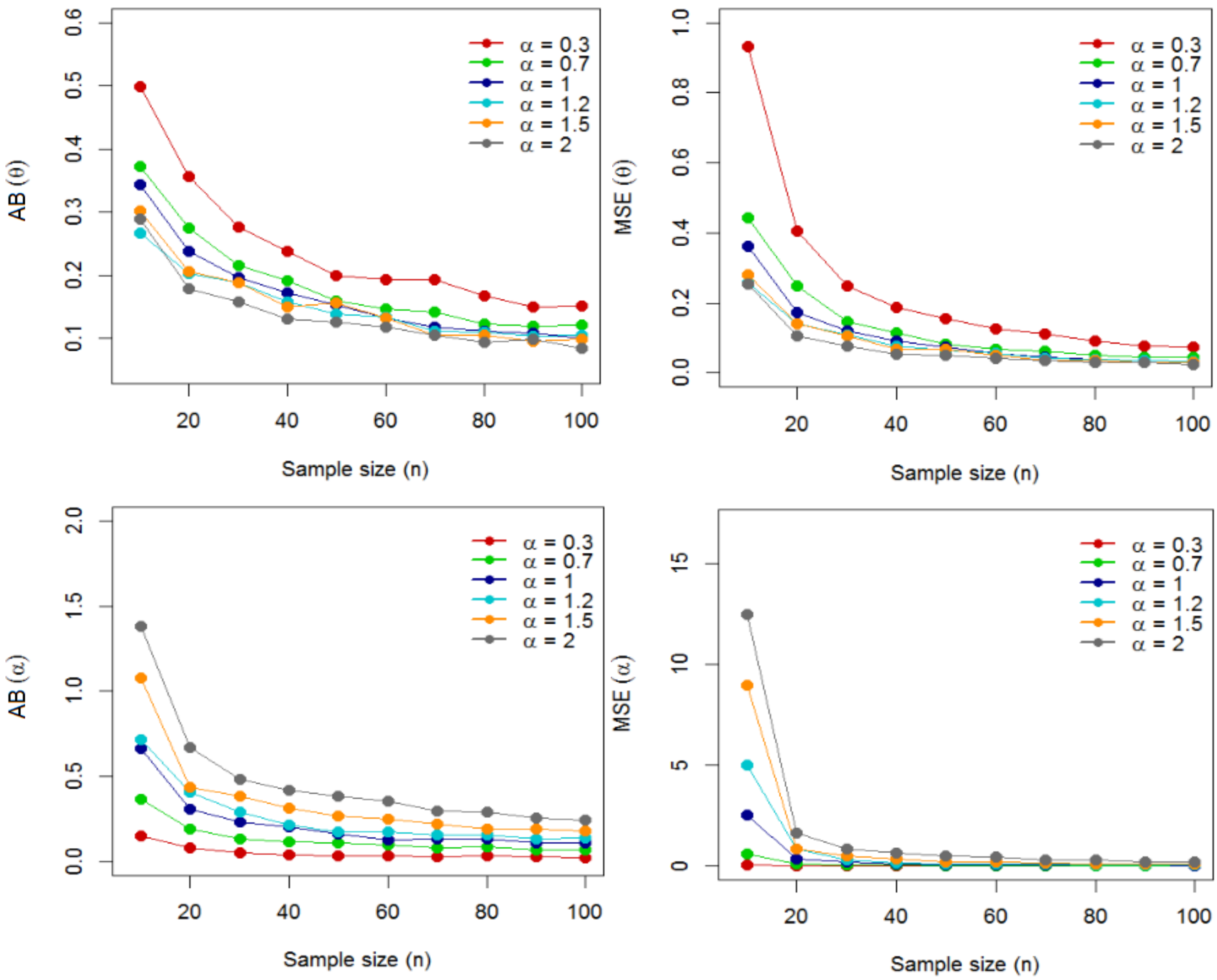


Figure 3.7 Estimated AB and estimated MSE of the MLEs of θ and α obtained for different values of α and $\theta = 1$

3.4 Review of GED in the Literature

The GED has been studied extensively by many authors in different statistical fields. The properties of the GED were first studied by Gupta and Kundu (2001a). As mentioned earlier in this section, they observed that most of the properties of the GED are similar to those of the gamma and Weibull distributions, whilst the GED can fit better than both. Gupta and Kundu (2001b) investigated the effects of six estimating strategies on the performance of estimators for unknown parameters of the GED. From extensive simulation studies they argued that the ML method provides better estimates of the parameters than the other estimation methods; like method of moments estimators and least square estimators. They observed that for large sample sizes the different estimation methods behave in similar manner, while for small sample sizes the ML method performed better. Similarly, a recommendation to use the MLE was made, according to the comparison of three estimation procedures: MLE, estimation of method moments and the estimation based on the probability plot. This comparison has been done by Chen and Lio (2010) for estimating the model parameters under progressive type-I interval censoring. Moreover, they studied a real data set of medical experiment and concluded that GED provides a better fit for this data than the Exponential, Weibull and Exponentiated Weibull distributions.

Also, Gupta and Kundu (2002) presented different inference procedures for the parameters of the GED. They discussed the behaviour of MLEs of the unknown parameters from complete samples, type-I and type-II censored samples. Under GED, the MLEs of the parameters involved when SSALTs with a scale parameter which is a log-linear function of the stress is applied, based on type-I censored data, were obtained by Abdel-Hamid and AL-Hussaini (2009). For a comprehensive review of the mathematical aspects of the GED such as the moment generating function with some moments properties, mean deviation about the mean and the median, moments of order statistics, asymptotic distribution of the extreme order statistics and other statistical distributions, one may refer to Nadarajah (2011). Furthermore, Abd El-Monem and Jaheen (2015) examined the performance of the MLEs for lifetime data, following GED for the simple SSALT model under Type-II censoring for large sample sizes $n = 200{,}300$ with failure percentage $> 75\%$. They concluded that the bias and MSE of the MLE for the scale parameter under the higher stress level is smaller than the bias and MSE of the MLE for the scale parameter under the lower stress level. Moreover, they get the conclusion that the resulting information in each stress level, impacts the scale parameter of that level.

The GED has been widely used for data modelling in different fields, such as biomedical, clinical and engineering. Some researchers find that the GED fits the data better than the other distributions they have considered. One of the applications of GED can be found in medical research by Khan et al (2014) for black Hispanic female breast cancer data. They used different criteria to measure the goodness of fit tests, such as Akaike information criterion, and concluded that GED model provided a better fit than either the Exponentiated Weibull, Beta generalized exponential or Beta inverse Weibull, for the breast cancer survival data. Also, Madi and Raqab (2007) observed that the GED fits well for Los Angeles rainfall data. In another technological application, Cota-Felix et al. (2009) estimated the mean life of power system equipment under censored data. The estimate was calculated based on GED as an alternative to Weibull and Normal distributions. Recently, research on sea clutter in the electronic engineering field evaluated the effectiveness of GED in fitting a data set, using a goodness-of-fit test. Li et al. (2020) conclude that the GED is the most suitable to model the given data set compared with 6 other distributions.

Chapter 4

Statistical Inference for a Simple Step-Stress Model Based on Progressive Type-II Censored Data

4.1 Introduction

The SSALT is an important ALT design that allows the experimenter to increase the stress level at pre-specified times during the experiment to obtain failures quicker than would be possible under normal operating conditions. It provides a remarkable reduction in time and cost of testing as the cost is related to the test time. The simple SSALT uses only two stress levels.

In this chapter, the objective is to address the problem of statistical inference of a SSALT plan under progressive Type-II censoring. It is assumed that the failure times follow the GED as a lifetime distribution. The model and assumptions for the CEM based on progressive Type-II censoring are described in Section 4.2. The MLEs of the scale and shape parameters are derived in Section 4.3. The observed Fisher information matrix is obtained in Section 4.4 in order to construct the AV-C matrix of the MLEs to be used to construct asymptotic CIs of the unknown parameters. Section 4.5 discusses obtaining the interval estimate of the model parameters based on the AV-C matrix along with percentile and bias-corrected and accelerated (BCa) bootstrap methods. The detailed description of simulation studies is provided in Section 4.6 to illustrate all the point and interval estimation methods discussed in this chapter. A summary of this chapter is provided finally in Section 4.7.

4.2 Test Method and Model Description

In this section, the steps for applying the simple SSALT model under progressive Type-II censoring are described in detail for failure time data that follow the GED. Suppose n independent units are initially placed on a test at time $\tau_0 = 0$, and are subjected to an initial lower stress level x_1 until a pre-specified time τ. At that time, the stress level is increased to higher stress level x_2 for the survived units. The test is terminated when a pre-specified number of failures r are observed. Thus, the operation time at higher stress level x_2 depends on the occurrence of the r^{th} failure while the time τ at lower stress level x_1 is fixed. During the test, at the time of each failure $\mathrm{t}_{j:\mathrm{n}}, j = 1, \dots, r$, a predetermined number of censored items R_j are randomly selected and removed from the test. Note that since $n = r + \sum_{j=1}^{r} R_j$, R_j censored items are selected from $\left(n - j - \sum_{k=1}^{j-1} R_k\right)$ survival items.

The Type-II censoring is a special case of progressive Type-II censoring scheme, when all censored items are removed at the time of the last failure, i.e. $R_1 = R_2 = \cdots = R_{r-1} = 0$ and $R_r = n - r$.

Furthermore, if all $R_j = 0$, $j = 1, \dots, r$ i.e. there are no censored items and all n units fail in the experiment, then the complete sample situation from the step-stress test is applying.

Let $n_1, 0 \le n_1 \le r$ denote the number of failures that occur at lower stress level x_1 before τ. On the other hand, $n_2 = r - n_1$ denotes the number of failures that occur at higher stress level x_2 after τ and before the test is terminated.

The simple SSALT based on progressive Type-II censoring scheme is illustrated in the following Figure.

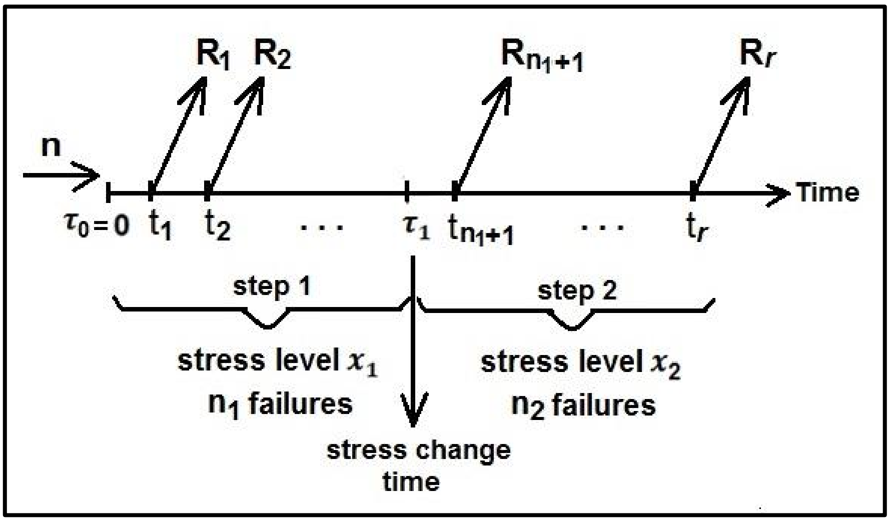


Figure 4.1 Representation of simple SSALT model under progressive Type-II censoring.

The observed ordered failure time data $t_{i:n}$ (under lower and higher stresses) are:

$$0 < t_{1:n} < \cdots < t_{n_1:n} \le \tau < t_{n_1+1:n} < \cdots < t_{r:n}.$$

The basic assumptions of CEM for simple SSALTs under the GED are:

1) For any level of stress $x_s, s = 1,2$, the lifetimes of test units are independent and follow the GED with CDF given by

$$F_s(t; \theta_s, \alpha) = \begin{cases} \left[1 - \exp[-t/\theta_s]\right]^{\alpha}, & t > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{4.1}$$

where $\theta_s > 0$ and α are the scale and shape parameters respectively. The shape parameter α is assumed to be constant for all stress levels.

The corresponding density function is:

$$f_s(t; \theta_s, \alpha) = \begin{cases} \frac{\alpha}{\theta_s}\left[1 - \exp\left(-\frac{t}{\theta_s}\right)\right]^{\alpha-1} . \exp\left(-\frac{t}{\theta_s}\right) & t > 0 \ , \ \theta_s, \alpha > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

2) The CEM holds. That is, the remaining life of a test unit depends only on the current cumulative failure probability and current stress level regardless of how the probability is accumulated (Nelson, 1990).

Based on CEM (see Section 2.6.1), the cumulative exposure distribution of a test unit for a simple SSALT is given by:

$$\mathrm{G}(t) = \begin{cases} \mathrm{F}_1(t), & 0 \le t < \tau, \\ \mathrm{F}_2(t - \tau + \varepsilon_1), & \tau \le t < \infty, \end{cases}$$

where $\varepsilon_1 = \frac{\theta_2}{\theta_1}\tau$, is the solution of $\mathrm{F}_2(\varepsilon_1) = \mathrm{F}_1(\tau)$.

Therefore,

$$\mathrm{G}(t) = \begin{cases} \mathrm{F}_1(t), & 0 \le t < \tau, \\ \mathrm{F}_2\left(t - \tau + \frac{\theta_2}{\theta_1}\tau\right), & \tau \le t < \infty. \end{cases} \tag{4.3}$$

So, from (4.1) and (4.3), the generalized exponential cumulative exposure distribution of time to failure from a simple SSALT is:

$$\mathrm{G}(t) = \begin{cases} G_1(t) = \left[1 - \exp\left(-\frac{t}{\theta_1}\right)\right]^{\alpha}, & 0 \le t < \tau, \\ G_2(t) = \left[1 - \exp\left(-\frac{(t-\tau)}{\theta_2} - \frac{\tau}{\theta_1}\right)\right]^{\alpha}, & \tau \le t < \infty. \end{cases} \tag{4.4}$$

Thus, the corresponding PDF of time to failure of test units is:

$$\mathrm{g}(t) = \begin{cases} g_1(t) = \frac{\alpha}{\theta_1}\left[1 - \exp\left(-\frac{t}{\theta_1}\right)\right]^{\alpha-1} . \exp\left(-\frac{t}{\theta_1}\right), & 0 \le t < \tau, \\ g_2(t) = \frac{\alpha}{\theta_2}\left[1 - \exp\left(-\frac{(t-\tau)}{\theta_2} - \frac{\tau}{\theta_1}\right)\right]^{\alpha-1} . \exp\left(-\frac{(t-\tau)}{\theta_2} - \frac{\tau}{\theta_1}\right), & \tau \le t < \infty. \end{cases} \tag{4.5}$$

4.3 Point Estimates of the Model Parameters

In this section, the MLEs of the model parameters assuming a CEM with lifetimes distributed according to GED are obtained based on a progressively Type-II censored sample. Based on the simple step-stress procedure under progressive Type-II censoring explained in Section 4.2, it can be seen that there exist three different cases of failure occurrence of test units (Balakrishnan, 2009). So, by using the CEM in (4.4) and (4.5), obtaining the likelihood function of each case is discussed.

Case(1): if all failures occur at lower stress level x_1 (i.e. $n_1 = r, n_2 = 0$), then the test will be ALT with progressive Type-II censored data. The likelihood function takes the form:

$$L(\theta_1, \theta_2, \alpha; \underline{t}) = C_p \prod_{i=1}^{r} \{g_1(t_{i:n})[1 - G_1(t_{i:n})]^{R_i}\}$$

$$= C_p \left(\frac{\alpha}{\theta_1}\right)^r \prod_{i=1}^{r} \left\{ \left[1 - \exp\left(-\frac{t_{i:n}}{\theta_1}\right)\right]^{\alpha-1} \exp\left(-\frac{t_{i:n}}{\theta_1}\right) \times \left[1 - \left[1 - \exp\left(-\frac{t_{i:n}}{\theta_1}\right)\right]^{\alpha}\right]^{R_i} \right\},$$

$$0 < t_{1:n} < \cdots < t_{r:n} \leq \tau$$

where, $C_p = n(n-1-R_1)(n-2-R_1-R_2) \ldots \left(n - r + 1 - \sum_{i=1}^{r-1} R_i\right)$.

Hence, the MLE of θ_2 does not exist in this case, since there are no failures observed after τ.

Case(2): all failures occur at higher stress level x_2 (i.e. $n_1 = 0, n_2 = r$). According to Nelson, the CEM can still be applied even though no failures are observed in the first step of the SSALT. The likelihood function is:

$$L(\theta_1, \theta_2, \alpha; \underline{t}) = C_p \prod_{i=1}^{r} \{g_2(t_{i:n})[1 - G_2(t_{i:n})]^{R_i}\}$$

$$= C_p \left(\frac{\alpha}{\theta_2}\right)^r \prod_{i=1}^{r} \left\{ \left[1 - \exp\left(-\frac{(t-\tau)}{\theta_2} - \frac{\tau}{\theta_1}\right)\right]^{\alpha-1} \exp\left(-\frac{(t-\tau)}{\theta_2} - \frac{\tau}{\theta_1}\right) \right.$$

$$\left. \times \left[1 - \left[1 - \exp\left(-\frac{(t-\tau)}{\theta_2} - \frac{\tau}{\theta_1}\right)\right]^{\alpha}\right]^{R_i} \right\}, \qquad \tau < t_{1:n} < \cdots < t_{r:n}$$

According to equations (4.4) and (4.5) and Section 2.6.1, the cumulative exposure distribution is undefined as no failures are observed before τ. The $g_1(t_{i:n})$ and $G_1(t_{i:n})$ are undefined. So, the MLE of θ_1 does not exist as the LH is increasing with θ_1.

Case(3): At least one failure occurs at lower stress level x_1 before τ, and at least one failure occurs at higher stress level x_2 after the times τ_1(i.e. $n_1 \neq 0,\ n_2 \neq 0,\ n_1 + n_2 = r, 0 < n_1 < r$). So, the likelihood function in this case is:

$$L(\theta_1, \theta_2, \alpha; \underline{t}) = C_p \prod_{i=1}^{n_1} \{g_1(t_{i:n})[1 - G_1(t_{i:n})]^{R_i}\} \times \prod_{i=n_1+1}^{r} \{g_2(t_{i:n})[1 - G_2(t_{i:n})]^{R_i}\}.$$

$$\mathrm{L}(\theta_1,\theta_2,\alpha;\underline{t}) = \mathrm{C}_p \prod_{\mathrm{i}=1}^{\mathrm{n}_1}\left\{\frac{\alpha}{\theta_1}\left[1-\exp\left(-\frac{t_{i:\mathrm{n}}}{\theta_1}\right)\right]^{\alpha-1}.\exp\left(-\frac{t_{i:\mathrm{n}}}{\theta_1}\right)\left[1-\left[1-\exp\left(-\frac{t_{i:\mathrm{n}}}{\theta_1}\right)\right]^{\alpha}\right]^{R_\mathrm{i}}\right\}$$

$$\times \prod_{\mathrm{i}=\mathrm{n}_1+1}^{r}\left\{\frac{\alpha}{\theta_2}\left[1-\exp\left(-\frac{(t_{i:\mathrm{n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha-1}.\exp\left(-\frac{(t_{i:\mathrm{n}}-\tau)}{\theta_2}\right.\right.$$

$$\left.\left.-\frac{\tau}{\theta_1}\right)\left[1-\left[1-\exp\left(-\frac{(t_{i:\mathrm{n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha}\right]^{R_\mathrm{i}}\right\},$$

$$0 < t_{1:\mathrm{n}} < \cdots < t_{\mathrm{n}_1:\mathrm{n}} \le \tau < t_{\mathrm{n}_1+1:\mathrm{n}} < \cdots < t_{r:\mathrm{n}}. \tag{4.6}$$

Obviously, the MLEs of θ_1 and θ_2 exist only when $0 < \mathrm{n}_1 < r$.

The log-likelihood function $\ell(\delta_k,\underline{t}) \equiv \ln \mathrm{L}(\theta_1,\theta_2,\alpha;\underline{t})$, is:

$$\ell(\delta_k,\underline{t}) = \ln(\mathrm{C}_p) + \sum_{i=1}^{n_1}\left[\ln\left(\left(\frac{\alpha}{\theta_1}\right)[\mathrm{A}_1(t_{\mathrm{i:n}})]^{(\alpha-1)}[1-[\mathrm{A}_1(t_{\mathrm{i:n}})]^{\alpha}]^{R_i}\right)-\mathrm{A}_3(t_{\mathrm{i:n}})\right]$$

$$+\sum_{i=n_1+1}^{r}\left[\ln\left(\left(\frac{\alpha}{\theta_2}\right)[\mathrm{A}_2(t_{\mathrm{i:n}})]^{(\alpha-1)}[1-[\mathrm{A}_2(t_{\mathrm{i:n}})]^{\alpha}]^{R_i}\right)-\mathrm{A}_4(t_{\mathrm{i:n}})\right],\mathrm{k}=1,2,3 \tag{4.7}$$

where $\delta_1 = \theta_1$, $\delta_2 = \theta_2$ and $\delta_3 = \alpha$,

and

$$\mathrm{A}_\mathrm{p}(t_{\mathrm{i:n}}) = \begin{cases} 1-\exp\left(-\dfrac{(t_{\mathrm{i:n}}-\tau_{p-1})}{\theta_\mathrm{p}}-\dfrac{\tau_{p-1}}{\theta_{\mathrm{p}-1}}\right) & \text{if } \mathrm{p}=1,2 \\ \dfrac{(t_{\mathrm{i:n}}-\tau_{p-3})}{\theta_{\mathrm{p}-2}}+\dfrac{\tau_{p-3}}{\theta_{\mathrm{p}-3}} & \text{if } \mathrm{p}=3,4 \end{cases} \tag{4.8}$$

such that $\tau_0 = 0$ and $\tau_1 = \tau$.

The MLEs $\hat{\delta}_k = (\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha})$ of the model parameters δ_k are the values which maximize the likelihood function $\mathrm{L}(\theta_1,\theta_2,\alpha;\underline{t})$ defined in (4.6) or, equivalently, maximizes the log-likelihood function $\ell(\delta_k,\underline{t})$ obtained in (4.7).

The first partial derivatives of the log-likelihood function (4.7) with respect to each of the parameters are:

$$\frac{\partial \ell(\delta_k, \underline{t})}{\partial \theta_1} = -\frac{\mathrm{n}_1}{\theta_1} + \left(\frac{\sum_{i=1}^{\mathrm{n}_1}(t_{\mathrm{i:n}})}{\theta_1^2}\right) + \left(\frac{\mathrm{n}_2\tau}{\theta_1^2}\right) - \frac{(\alpha-1)}{\theta_1^2}\sum_{i=1}^{\mathrm{n}_1}\left[\frac{t_{\mathrm{i:n}}\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta_1}\right)}{\left[1-\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta_1}\right)\right]}\right]$$

$$-\left(\frac{\tau(\alpha-1)}{\theta_1^2}\right)\sum_{i=\mathrm{n}_1+1}^{r}\left[\frac{\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)}{\left[1-\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]}\right]$$

$$+\left(\frac{\alpha}{\theta_1^2}\right)\sum_{i=1}^{\mathrm{n}_1}\left[R_i t_{\mathrm{i:n}}\frac{\left[1-\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta_1}\right)\right]^{\alpha-1}\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta_1}\right)}{\left[1-\left[1-\exp\left(-\frac{t_{\mathrm{i:n}}}{\theta_1}\right)\right]^{\alpha}\right]}\right]$$

$$+\left(\frac{\alpha\tau}{\theta_1^2}\right)\sum_{i=\mathrm{n}_1+1}^{r}\left[R_i\frac{\left[1-\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha-1}\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)}{\left[1-\left[1-\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha}\right]}\right].$$

For simplicity, let

$$\mathrm{B_p}(t_{\mathrm{i:n}}) = \begin{cases} \dfrac{1-\mathrm{A_p}(t_{\mathrm{i:n}})}{\mathrm{A_p}(t_{\mathrm{i:n}})} & \text{if} \quad \mathrm{p}=1,2 \\ R_i\dfrac{\left[\mathrm{A_{p-2}}(t_{\mathrm{i:n}})\right]^{\alpha}}{1-\left[\mathrm{A_{p-2}}(t_{\mathrm{i:n}})\right]^{\alpha}} & \text{if} \quad \mathrm{p}=3,4 \end{cases} \tag{4.9}$$

Then, the first derivative of the log-likelihood function with respect to θ_1 is:

$$\frac{\partial \ell(\delta_k, \underline{t})}{\partial \theta_1} = \frac{1}{\theta_1^2}\left[\begin{matrix} -\mathrm{n}_1\theta_1 + \mathrm{n}_2\tau + \sum_{i=1}^{\mathrm{n}_1}\left[\theta_1\mathrm{A}_3(t_{\mathrm{i:n}}) - \left[\mathrm{t_{i:n}}\mathrm{B}_1(t_{\mathrm{i:n}})[(\alpha-1)-\alpha\mathrm{B}_3(t_{\mathrm{i:n}})]\right]\right] \\ -\sum_{i=\mathrm{n}_1+1}^{r}\left[\tau_1\mathrm{B}_2(t_{\mathrm{i:n}})[(\alpha-1)-\alpha\mathrm{B}_4(t_{\mathrm{i:n}})]\right] \end{matrix}\right]. \tag{4.10}$$

Also, the first derivative of the log-likelihood function with respect to θ_2 is derived as

$$\frac{\partial \ell(\delta_k, \underline{t})}{\partial \theta_2} = -\frac{\mathrm{n}_2}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=\mathrm{n}_1+1}^{r}(t_{\mathrm{i:n}}-\tau) - \frac{(\alpha-1)}{\theta_2^2}\sum_{i=\mathrm{n}_1+1}^{r}\frac{(t_{\mathrm{i:n}}-\tau)\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)}{\left[1-\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]}$$

$$+\frac{\alpha}{\theta_2^2}\sum_{i=\mathrm{n}_1+1}^{r}\left\{R_i\frac{\left[1-\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha-1}(t_{\mathrm{i:n}}-\tau)\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)}{\left[1-\left[1-\exp\left(-\frac{(t_{\mathrm{i:n}}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha}\right]}\right\}$$

By using (4.8) and (4.9), $\frac{\partial \ell(\delta_k;\underline{t})}{\partial \theta_2}$ can be written as

$$\frac{\partial \ell(\delta_k,\underline{t})}{\partial \theta_2} = \frac{1}{\theta_2^2}\left[-n_2\theta_2 + \sum_{i=n_1+1}^{r}(t_{i:n}-\tau)\left[1-B_2(t_{i:n})[(\alpha-1)-\alpha B_4(t_{i:n})]\right]\right]. \quad (4.11)$$

Moreover, the first derivative of the log-likelihood function with respect to α is:

$$\frac{\partial \ell(\delta_k,\underline{t})}{\partial \alpha} = \frac{n_1+n_2}{\alpha} + \sum_{i=1}^{n_1}\ln\left[1-\exp\left(-\frac{t_{i:n}}{\theta_1}\right)\right] + \sum_{i=n_1+1}^{r}\ln\left[1-\exp\left(-\frac{(t_{i:n}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]$$
$$-\sum_{i=1}^{n_1}\left[\frac{R_i\left[1-\exp\left(-\frac{t_{i:n}}{\theta_1}\right)\right]^{\alpha}\ln\left[1-\exp\left(-\frac{t_{i:n}}{\theta_1}\right)\right]}{\left[1-\left[1-\exp\left(-\frac{t_{i:n}}{\theta_1}\right)\right]^{\alpha}\right]}\right]$$
$$-\sum_{i=n_1+1}^{r}\left[\frac{R_i\left[1-\exp\left(-\frac{(t_{i:n}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha}\ln\left[1-\exp\left(-\frac{(t_{i:n}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]}{\left[1-\left[1-\exp\left(-\frac{(t_{i:n}-\tau)}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]^{\alpha}\right]}\right].$$

According to (4.8) and (4.9), $\frac{\partial \ell(\delta_k,\underline{t})}{\partial \alpha}$ takes the form

$$\frac{\partial \ell(\delta_k,\underline{t})}{\partial \alpha} = \frac{r}{\alpha} + \sum_{i=1}^{n_1}\ln[A_1(t_{i:n})]\,[1-B_3(t_{i:n})] + \sum_{i=n_1+1}^{r}\ln[A_2(t_{i:n})]\,[1-B_4(t_{i:n})]. \quad (4.12)$$

Therefore, the MLEs of δ_k are obtained by setting the first partial derivatives of $\ell(\delta_k;\underline{t})$ to be equal to zero as shown in the following equations:

$$\left.\begin{aligned}
&\frac{\partial \ell(\delta_k,\underline{t})}{\partial \theta_1} = \frac{1}{\theta_1^2}\left[\begin{array}{l}-n_1\theta_1 + n_2\tau_1 + \sum_{i=1}^{n_1}\left[\theta_1 A_3(t_{i:n}) - \left[t_{i:n}B_1(t_{i:n})[(\alpha-1)-\alpha B_3(t_{i:n})]\right]\right] \\ -\sum_{i=n_1+1}^{r}\left[\tau B_2(t_{i:n})[(\alpha-1)-\alpha B_4(t_{i:n})]\right]\end{array}\right] = 0 \\
&\frac{\partial \ell(\delta_k,\underline{t})}{\partial \theta_2} = \frac{1}{\theta_2^2}\left[-n_2\theta_2 + \sum_{i=n_1+1}^{r}(t_{i:n}-\tau)\left[1-B_2(t_{i:n})[(\alpha-1)-\alpha B_4(t_{i:n})]\right]\right] = 0 \\
&\frac{\partial \ell(\delta_k,\underline{t})}{\partial \alpha} = \frac{r}{\alpha} + \sum_{i=1}^{n_1}\ln[A_1(t_{i:n})]\,[1-B_3(t_{i:n})] + \sum_{i=n_1+1}^{r}\ln[A_2(t_{i:n})]\,[1-B_4(t_{i:n})] = 0
\end{aligned}\right\} \quad (4.13)$$

Thus, to obtain the MLEs of δ_k, the system of three non-linear equations (4.13) will be solved. However, they do not have a closed form solution for the unknown parameters. Therefore, an iterative procedure must be applied to obtain the MLEs of the unknown parameters. For this purpose, the R program is used. The maxBFGS built-in function from the maxLik package (Henningsen and Toomet, 2011) is used for obtaining the MLEs of the model parameters.

4.4 Fisher Information Matrix

The Fisher information matrix $F(\delta_k)$ is a measure of the information content of the data relative to the parameters being estimated. It is symmetric matrix obtained by taking the $E\left(-\frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\delta_i\partial\delta_j}\right)$, where $i,j=1,2,3$. Unfortunately, the exact mathematical expression for the expectation is very difficult to obtain. Therefore, the observed Fisher information matrix can be obtained by estimating the expected value using MLEs, (Cohen, 1965). So, the asymptotic Fisher information matrix for observations at $x_s, s=1,2$ is:

$$F(\hat{\delta}_k)=\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}=-\begin{bmatrix} \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\theta_1^2} & \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\theta_1\partial\theta_2} & \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\theta_1\partial\alpha} \\ \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\theta_2\partial\theta_1} & \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\theta_2^2} & \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\theta_2\partial\alpha} \\ \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\alpha\partial\theta_1} & \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\alpha\partial\theta_2} & \frac{\partial^2\ell(\delta_k,\underline{t})}{\partial\alpha^2} \end{bmatrix}|\delta_k=\hat{\delta}_k \tag{4.14}$$

The elements of this matrix are obtained by taking the second and mixed partial derivatives of (4.10), (4.11) and (4.12) with respect to δ_k. By substituting from (4.8) and (4.9), then the second partial derivatives is simplified as:

f_{11}

$$=\frac{-1}{\theta_1^4}\left[\begin{array}{c} \theta_1^2\mathrm{n}_1-(2\theta_1\mathrm{n}_2\tau) \\ -\sum_{i=1}^{\mathrm{n}_1} t_{\mathrm{i:n}}\left[\begin{array}{c} 2\theta_1+\mathrm{D}_1(t_{\mathrm{i:n}})\left[[t_{\mathrm{i:n}}-2\theta_1]+[t_{\mathrm{i:n}}\mathrm{B}_1(t_{\mathrm{i:n}})]\right] \\ -[\alpha\mathrm{B}_1(t_{\mathrm{i:n}})\mathrm{B}_3(t_{\mathrm{i:n}})]\left[\left[\left(-t_{\mathrm{i:n}}\mathrm{D}_1(t_{\mathrm{i:n}})\right)+(t_{\mathrm{i:n}}-2\theta_1)\right]-\left[\alpha\, t_{\mathrm{i:n}}\mathrm{B}_1(t_{\mathrm{i:n}})\frac{\mathrm{B}_3(t_{\mathrm{i:n}})}{R_i}\right]\right] \end{array}\right] \\ -\sum_{i=\mathrm{n}_1+1}^{r} \tau\left[\begin{array}{c} \mathrm{D}_2(t_{\mathrm{i:n}})\left[[\tau-2\theta_1]+[\tau\mathrm{B}_2(t_{\mathrm{i:n}})]\right] \\ -[\alpha\mathrm{B}_2(t_{\mathrm{i:n}})\mathrm{B}_4(t_{\mathrm{i:n}})]\left[\left[\left(-\tau_1\mathrm{D}_2(t_{\mathrm{i:n}})\right)+(\tau-2\theta_1)\right]-\left[\alpha\tau\mathrm{B}_2(t_{\mathrm{i:n}})\frac{\mathrm{B}_4(t_{\mathrm{i:n}})}{R_i}\right]\right] \end{array}\right] \end{array}\right],$$

$$f_{12}=\left(\frac{\tau}{\theta_1^2\theta_2^2}\right)\sum_{i=\mathrm{n}_1+1}^{r}(t_{\mathrm{i:n}}-\tau)\mathrm{B}_2(t_{\mathrm{i:n}})\left[\begin{array}{c} \left[\frac{(\alpha-1)}{\mathrm{A}_2(t_{\mathrm{i:n}})}\right] \\ -\left[\alpha\mathrm{B}_4(t_{\mathrm{i:n}})\left[\left(1-\mathrm{D}_2(t_{\mathrm{i:n}})\right)-\left(\alpha\,\mathrm{B}_2(t_{\mathrm{i:n}})\frac{\mathrm{B}_4(t_{\mathrm{i:n}})}{R_i}\right)\right]\right] \end{array}\right],$$

$$f_{13}=\frac{1}{\theta_1^2}\left[\begin{array}{c} \sum_{i=1}^{\mathrm{n}_1} t_{\mathrm{i:n}}\mathrm{B}_1(t_{\mathrm{i:n}})\left[1-\left[\mathrm{B}_3(t_{\mathrm{i:n}})\left[1+\frac{\alpha\ln\left(\mathrm{A}_1(t_{\mathrm{i:n}})\right)}{[1-[\mathrm{A}_1(t_{\mathrm{i:n}})]^{\alpha}]}\right]\right]\right] \\ +\sum_{i=\mathrm{n}_1+1}^{r} \tau\mathrm{B}_2(t_{\mathrm{i:n}})\left[1-\left[\mathrm{B}_4(t_{\mathrm{i:n}})\left[1+\frac{\alpha\ln\left(\mathrm{A}_2(t_{\mathrm{i:n}})\right)}{[1-[\mathrm{A}_2(t_{\mathrm{i:n}})]^{\alpha}]}\right]\right]\right] \end{array}\right],$$

$$f_{22} = -\frac{n_2}{\theta_2^2} + \sum_{i=n_1+1}^{r} \frac{(t_{i:n}-\tau)}{\theta_2^4} \left[\begin{array}{c} 2\theta_2 + \left[D_2(t_{i:n})\left[\left((t_{i:n}-\tau)-2\theta_2\right)+(t_{i:n}-\tau)B_2(t_{i:n})\right]\right] \\ -\left[\alpha B_2(t_{i:n})B_4(t_{i:n})\left[\begin{array}{c}\left[\left((t_{i:n}-\tau)-2\theta_2\right)-\left((t_{i:n}-\tau)D_2(t_{i:n})\right)\right] \\ -\left[\alpha(t_{i:n}-\tau)B_2(t_{i:n})\frac{B_4(t_{i:n})}{R_i}\right]\end{array}\right]\right] \end{array} \right],$$

$$f_{23} = \sum_{i=n_1+1}^{r} \frac{(t_{i:n}-\tau)}{\theta_2^2} B_2(t_{i:n})\left[1-\left[B_4(t_{i:n})\left((1-[A_2(t_{i:n})]^{\alpha})+\alpha\ln\left(A_2(t_{i:n})\right)\right)\right]\right],$$

$$f_{33} = \frac{(n_1+n_2)}{\alpha^2} + \sum_{i=1}^{n_1}\left[\frac{[\ln[A_1(t_{i:n})]]^2 B_3(t_{i:n})}{[1-[A_1(t_{i:n})]^{\alpha}]}\right] + \sum_{i=n_1+1}^{r}\left[\frac{[\ln[A_2(t_{i:n})]]^2 B_4(t_{i:n})}{[1-[A_2(t_{i:n})]^{\alpha}]}\right],$$

where $D_q(t_{i:n}) = (\alpha-1)B_q(t_{i:n})$, for $q = 1,2$

4.5 Interval Estimates for the Model Parameters

In this section, different methods for obtaining the CIs for the unknown parameters are discussed. The AV-C matrix of the MLEs is presented. Confidence intervals based on the asymptotic normality of the MLEs are obtained. Parametric bootstrap CIs are also obtained.

4.5.1 Asymptotic Confidence Intervals for the Model Parameters

Since the MLEs of θ_1, θ_2 and α are not in closed form expressions, then it is not possible to derive their exact distributions. Thus, the corresponding exact CIs cannot be obtained. Therefore, asymptotic CIs for these parameters based on the asymptotic distributions of the MLEs of θ_1, θ_2 and α are derived.

The $AVar$ of $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$ can be obtained from the AV-C matrix, which is defined as the inverse of the asymptotic Fisher information matrix in (4.14), as follows

$$AV.C = F^{-1}(\delta_k) = \begin{bmatrix} AVar(\hat{\theta}_1) & Cov(\hat{\theta}_1,\hat{\theta}_2) & Cov(\hat{\theta}_1,\hat{\alpha}) \\ Cov(\hat{\theta}_2,\hat{\theta}_1) & AVar(\hat{\theta}_2) & Cov(\hat{\theta}_2,\hat{\alpha}) \\ Cov(\hat{\alpha},\hat{\theta}_1) & Cov(\hat{\alpha},\hat{\theta}_2) & AVar(\hat{\alpha}) \end{bmatrix}, \quad k = 1,2,3 \tag{4.15}$$

For large sample size, the MLEs of δ_k under appropriate regularity conditions, are asymptotically normality distributed with mean δ_k and AV-C $=F^{-1}(\delta_k)$.

Using the asymptotic normality of the MLEs, the asymptotic two-sided $100(1-\gamma)\%$ CIs with an approximate confidence coefficient $(1-\gamma)$ for θ_1, θ_2 and α are constructed such that

$$P\left[\hat{\delta}_k - z_{\frac{\gamma}{2}}\sqrt{AVar(\hat{\delta}_k)} \leq \delta_k \leq \hat{\delta}_k + z_{\frac{\gamma}{2}}\sqrt{AVar(\hat{\delta}_k)}\right] \cong 1-\gamma \quad , k = 1,2,3 \tag{4.16}$$

where $z_{\frac{\gamma}{2}}$ is the upper $(\gamma/_2)$th percentile of a standard normal distribution and $AVar(\hat{\delta}_k)$ represents the asymptotic variances of $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$. These variances are obtained using the AV-C matrix defined in (4.15).

Therefore, the lower limit (LL) and the upper limit (UL) of the CIs for θ_1, θ_2 and α, are given respectively, as follows

$$LL(\underline{\delta}) = \hat{\delta}_k - z_{\frac{\alpha}{2}}\sqrt{AVar(\hat{\delta}_k)} \qquad UL(\underline{\delta}) = \hat{\delta}_k + z_{\frac{\alpha}{2}}\sqrt{AVar(\hat{\delta}_k)}, k = 1,2,3 \tag{4.17}$$

where $\hat{\delta}_1 = \hat{\theta}_1$, $\hat{\delta}_2 = \hat{\theta}_2$ and $\hat{\delta}_3 = \hat{\alpha}$

4.5.2 Bootstrap Confidence Intervals for the Model Parameters

The CIs of the model parameters can be found using the A-VC matrix in Section 4.5.1. However, their expressions are complicated and must be computed numerically. In addition, the large sample approximations may not work well for small samples. Therefore, bootstrap CI approaches will be studied in this section. The bootstrap is a popular random resampling procedure extensively discussed by Efron and Tibshirani (1993).

This subsection presents two methods of constructing the CIs for θ_1, θ_2 and α, percentile and BCa bootstrap methods based on parametric bootstrap samples. In a parametric bootstrap, the data are assumed to follow a known parametric model with unknown parameters. The main idea of the bootstrap method is using the empirical distribution of the resulting samples as the estimates of the sampling distribution of the estimator. Thus, bootstrap data are sampled from a specified distribution where its parameters have been estimated from the original data. In other words, based on parametric bootstrap, the B samples are generated from the parametric estimate of the population. Then, the parameter estimates from these samples are calculated to obtain B bootstrap estimates $\mathfrak{B}(\hat{\underline{\delta}}^*)$; see Efron and Tibshirani (1993) for more details.

$$\mathfrak{B}(\hat{\delta}_k^*) = \left\{\hat{\delta}_k^{*(1)}, \hat{\delta}_k^{*(2)}, \ldots, \hat{\delta}_k^{*(B)}\right\}, \qquad k = 1,2,3$$

where $\hat{\delta}_1^* = \hat{\theta}_1^*$, $\hat{\delta}_2^* = \hat{\theta}_2^*$ and $\hat{\delta}_3^* = \hat{\alpha}^*$.

The t-bootstrap method requires calculating the variance of parameters, which is not in closed form in the proposed model here, and it needs to be estimated. Thus, t-bootstrap method does not work well for this model.

4.5.2.1 Percentile method

The CDF of $\hat{\delta}_k^*$ is $\mathbb{B}(\mathfrak{y}) = P\left(\hat{\delta}_k^* \leq \mathfrak{y}\right)$. Thus, define $\hat{\delta}_{kboot}^* = \mathbb{B}^{-1}(\mathfrak{y})$ for given $\mathfrak{y}$.

Then, the lower and upper limits of $100(1-\gamma)\%$ percentile CIs are defined as

$LL = \mathbb{B}^{-1}\left(\frac{\gamma}{2}\right)$ and $UL = \mathbb{B}^{-1}\left(1-\frac{\gamma}{2}\right)$.

However, the number of estimates in the bootstrap sample is finite B. Thus, a two-sided $100(1-\gamma)\%$ percentile bootstrap CI for a model parameter is

$$\left(\hat{\delta}_k^{*\left[\frac{\gamma}{2}B\right]}, \hat{\delta}_k^{*\left[\left(1-\frac{\gamma}{2}\right)B\right]}\right), \quad k = 1,2,3 \tag{4.18}$$

where $\hat{\delta}_1^* = \hat{\theta}_1^*, \hat{\delta}_2^* = \hat{\theta}_2^*, \hat{\delta}_3^* = \hat{\alpha}^*$, and $[\mathscr{b}]$ denotes the largest integer less than or equal to $\mathscr{b}$.

4.5.2.2 Bias Corrected and Accelerated Method

A two-sided $100(1-\gamma)\%$ BCa bootstrap CIs for model parameters θ_1, θ_2 and α have the form

$$\left(\hat{\delta}_k^{*[\gamma_{1k}B]}, \hat{\delta}_k^{*[\gamma_{2k}B]}\right), k = 1,2,3 \tag{4.19}$$

where

$$\gamma_{1k} = \Phi\left\{\hat{\mathcal{Z}}_{0k} + \frac{\hat{\mathcal{Z}}_{0k} + \mathcal{Z}_{1-\gamma/2}}{1-\hat{\alpha}_k\left(\hat{\mathcal{Z}}_{0k} + \mathcal{Z}_{1-\gamma/2}\right)}\right\},$$

and

$$\gamma_{2k} = \Phi\left\{\hat{\mathcal{Z}}_{0k} + \frac{\hat{\mathcal{Z}}_{0k} + \mathcal{Z}_{\gamma/2}}{1-\hat{\alpha}_k\left(\hat{\mathcal{Z}}_{0k} + \mathcal{Z}_{\gamma/2}\right)}\right\}.$$

Here Φ is the CDF of the standard normal distribution and $\mathcal{Z}_\gamma$ is the upper γ point of the standard normal distribution. The bias correction factor $\hat{\mathcal{Z}}_{0k}$ can be calculated as

$$\hat{\mathcal{Z}}_{0k} = \Phi^{-1}\left\{\frac{\text{number of } \hat{\delta}_k^{*(b)} < \hat{\delta}_k}{B}\right\}. \qquad ,k = 1,2,3 \ \ ,b = 1,\ldots,B,$$

An estimate of the acceleration factor $\hat{\alpha}$ can be obtained based on jackknife resampling method. This involves generating r replicates of the original sample with one observation being omitted, where $(r-1)$ is the number of observations in each sample (Balakrishnan et al., 2007). The first jackknife replicate is obtained by leaving out the first value of the original sample, the second by leaving out the second value, and so on, until r sub-samples of size $r-1$ are generated. For each of the sub-samples, the MLEs $\hat{\delta}_{k(i)}$, $k = 1,2,3$ corresponding to the model parameters are calculated.

The mean of these estimates can be calculated as follows:

$$\hat{\delta}_{k(.)} = \frac{1}{r}\sum_{i=1}^{r} \hat{\delta}_{k(i)} \qquad , k = 1,2,3.$$

Therefore, an estimate of the acceleration factor based on the jackknife technique, can be calculated as:

$$\hat{a}_k = \frac{\sum_{i=1}^{r}\left[\hat{\delta}_{k(.)} - \hat{\delta}_{k(i)}\right]^3}{6\left\{\sum_{i=1}^{r}\left[\hat{\delta}_{k(.)} - \hat{\delta}_{k(i)}\right]^2\right\}^{\frac{3}{2}}} \qquad , k = 1,2,3.$$

where $\hat{\delta}_{k(i)}$ is the MLE of δ_k based on the original sample with the i^{th} observation deleted.

Note that, if $\hat{Z}_{0k} = 0$ and $\hat{a}_k = 0$, then the BCa interval tends to be the same as the percentile interval.

The steps of obtaining CIs based on bootstrap methods will be explain in next section.

4.6 Simulation Studies

Since it is mathematically intractable to obtain a closed form expression for the MLEs, a numerical study is carried out to investigate the performance of the proposed methods. In this section, extensive Monte-Carlo simulation is performed to assess the performance of the MLEs and to estimate the CIs for parameters of the proposed SSALT model based on progressive Type-II censoring, where test items have a GED. The numerical study was carried out using maxLik package in R. The built-in function called maxBFGS is used with the constraint:

$\theta_1 > \theta_2$, $\theta_2 > 0$ and $\alpha > 0$

The simulation studies are carried out and the steps for obtaining the point and interval estimates are described in Section 4.6.1 based on ML and bootstrap methods. In Section 4.6.2, simulation results are presented together with discussion.

4.6.1 Simulation Description

In an ALT experiment, choosing the initial values of $\theta_1, \theta_2, \alpha, n, \tau$ and failure percentage (FP) is one of the complicated challenges for the statistician. Many variables must be chosen in advance to strike a balance between the restricted time and cost of the experiment and the efficiency of the statistical analysis. For example, the censoring percentage must also be relevant to the sample size. Obviously, a considerable increase in the stress level value may accelerate failure times, which reduces the total test time. However, an increase the stress level could result in large variance of the parameter estimates due to the large difference between the accelerated stress levels and the usage stress level. Therefore, to get more reasonable simulation results, it is required to choose a set of initial values along with studying the impact of changing any of these values on the performance of the MLEs. Thus, before starting to get the result of simulation, the steps were tested for a wide range of values of the model parameters along with stress change time and different censoring schemes, to determine the sets of the initial values to be used in this chapter. Also, different values of n, FP, τ, α and PCSs were chosen to investigate their influence on the parameter estimates.

The performance of the estimators of simple SSALT model parameters is evaluated. The estimated absolute bias (AB) and the estimated mean squared error (MSE) associated with the estimator are obtained for each model parameter. Also, the average length (AL), which is the mean of the interval length, and an estimate of the actual coverage probabilities (CP) of the intervals are estimated. The bootstrap and jackknife methods are planned to be used in order to estimate bias and variance of the estimators for the model parameters (see Efron and Stein, 1981, Efron and Tibshirani, 1993 and Kisielinska, 2013 for more details)

A numerical study is provided to investigate the influence of the sample size n and stress change time τ on the accuracy of the parameter estimates. The impact of the FP on the precision of the estimators will be investigated. Also, the effect of different PCSs on the performance of the MLEs and on the percentage of failed items in each step.

The Monte Carlo simulation generates different samples by repeatedly running the simulation algorithm. Since the numerical study is carried out to investigate the performance of the proposed methods, it is essential to consider the sampling error of the results. The sampling error results from using samples to estimate the model parameters. So, the overinterpretation of the numerical results is assessed using the standard error. The Monte Carlo standard errors estimate the standard errors of estimated performance, such as estimated AB and estimated MSE. It is used to quantify simulation certainty. The standard error represents the precision of the estimate relative to the estimator. It will be utilized to assess the precision of the performance of the MLEs and the CIs for parameters of the proposed SSALT model. The Monte Carlo standard error is the standard deviation of an estimator over the number of repetitions of the simulation steps. It is the standard deviation of the sampling distribution of an estimator.

For simplicity, we will use the notation of Balakrishnan and Cramer (2014) for describing the censoring schemes. For example, when $n = 20$ and $r = 15$, the scheme $\mathcal{R} = (2, 3, 0^{*13})$ means that, after the first failure of the test items, two items are randomly removed from the remaining 19 survival items. Similarly, after the second failure in the test, three items are randomly removed from the remaining 16 survival items. For the subsequent 13 failures, no items are removed from the test. However, for the case of the complete sample the censoring scheme will be $\mathcal{R} = (0^{*n})$ where no item will be removed after each of the failure items. Also, for the case of Type-II censored data the censoring scheme will be $\mathcal{R} = \left(0^{*(r-1)}, (n-r)\right)$ where the remaining $(n - \mathrm{r})$ items surviving after the last failure will be censored.

The following four different PCSs are considered in this section:

Scheme-1: $\mathcal{R}1 = \left(0^{*(r-1)}, (n-r)\right)$,

where all censored items are removed from the test at the time of occurrence the final observed censored item. This scheme presents the basic Type-II censoring scheme which is a special case of progressive Type-II censoring.

Scheme-2: $\mathcal{R}2 = \left((n-r)\,, 0^{*(r-1)}\right)$

This scheme is the reverse of the previous scheme wherein $(n-r)$ censored items are removed completely when the first item fails. The test then is continuously run until r items fail.

Scheme-3: $\mathcal{R}3 = \left(0^{*\left\lfloor\frac{(r-z)}{2}\right\rfloor}, R_i = \frac{(n-r)}{z}, 0^{*\left\lceil\frac{(r-z)}{2}\right\rceil}\right)$, where: $i = \left(\left\lfloor\frac{(r-z)}{2}\right\rfloor + 1\right), \dots, \left(\left\lfloor\frac{(r-z)}{2}\right\rfloor + z\right)$,

where $\lfloor . \rfloor$ is rounding the number down and $\lceil . \rceil$ is rounding the number up.

In this scheme, censored items are removed at z different middle points of the test, where z is the number of failures by the middle of the test.

Scheme-4: $\mathcal{R}4 = \left(0^{*\left(\frac{2r-n}{2}\right)}, R_i = 1, 0^{*\left(\frac{2r-n}{2}\right)}\right)$; where $i = \left(\left(\frac{2r-n}{2}\right) + 1\right), \dots, \left(\left(\frac{2r-n}{2}\right) + (n-r)\right)$,

For this situation, a single item is randomly removed after each failure in the middle of intervals between failures.

In addition, in the case of the complete sample, there are no removed items throughout the test, and $\mathcal{R}0 = (0^{*n})$.

The steps for generating samples from CEM based on the GED are presented in Section 4.6.1.1 along with MLEs and asymptotic CIs for the model parameters. Also, the algorithm to generate the bootstrap samples and assess the performance of the bootstrap CIs is discussed in Section 4.6.1.2.

4.6.1.1 Simulation Steps for Obtaining the MLEs and Asymptotic Confidence Intervals

The following algorithm is utilized to illustrate the method for obtaining MLEs from GED under SSALT based on progressive Type-II censoring. The initial values of the scale parameter at lower and higher stress levels are chosen to be $\theta_1 = 0.6$, $\theta_2 = 0.3$. The different choices of observed sample size, $n = 40,60,80$, and different censoring schemes are assumed. The number of replications is $\mathrm{p} = 3000$.

Step (1): Generate a random sample from the $\mathrm{Uniform}(0,1)$ distribution and obtained the order statistics $(U_{1:n}, \dots, U_{n:n})$.

Step (2): Based on the 4 censoring schemes that are assumed previously in Section 4.6.1, obtain the observed sample of failures by randomly removing R_{i} items from the test after each failure.

Step (3): Calculate the numbers n_1 and n_2 which present the number of failures at lower and higher stress level, respectively. This can be achieved firstly by transforming the given stress change time $\tau = 0.4$ and 0.6 from the GED to the $\mathrm{Uniform}(0,1)$ distribution by using the transformation $\tau_u = \left(1 - e^{-\tau/\theta_1}\right)^{\alpha}$. Then, determine the number of failures at lower stress levels, such that

$0 \leq U_{n_1:n} \leq \tau_u$, and the value of n_2 can consequently calculated as $n_2 = r - n_1$.

Step (4): Eliminate samples that have no failures in either the first or second step-stress level. i.e. exclude samples with $n_1 = 0$ or $n_2 = 0$. Thus, check if $n_1 = 0$ or $n_2 = 0$, eliminate the sample and go to step(1)

Step (5): From steps (2) and (3), the ordered failure observations $t_{1:\mathrm{n}} < \cdots < t_{\mathrm{n}_1:\mathrm{n}} < t_{\mathrm{n}_1+1:\mathrm{n}} < \cdots < t_{r:\mathrm{n}}$ are calculated as follows

$$t_{\mathrm{i:n}} = \begin{cases} -\theta_1 \ln\left(1 - (\mathrm{U}_{\mathrm{i:n}})^{1/\alpha}\right) & \text{if} \quad 1 \leq \mathrm{i} \leq n_1 \\ -\theta_2 \ln\left(1 - (\mathrm{U}_{\mathrm{i:n}})^{1/\alpha}\right) + \tau - \dfrac{\theta_2}{\theta_1}\tau & \text{if} \quad n_1 < \mathrm{i} \leq r. \end{cases}$$

Step (6): The maxBFGS built-in function is utilized under constraints ($\theta_1 > \theta_2$ and $\theta_1, \theta_1, \alpha > 0$) based on $\ell(\delta_k, \underline{\mathrm{t}})$ in (4.7) and the gradient functions in (4.10) - (4.12) to obtain the MLEs $\hat{\delta}_k$ of parameters δ_k (where δ_{k} is a general notation that can be replaced by $\delta_1 \equiv \theta_1, \delta_2 \equiv \theta_2, \delta_3 \equiv \alpha$). The computations are carried out for FP of 60%, 80% and 100% (complete sample).

Step (7): Repeat the above steps (1-6) 3000 times.

Step (8): The mean value $\hat{\delta}_{\mathrm{k}}, k = 1,2,3$ of the estimate is reported for the number of replications $\mathrm{p} = 3000$ as follows:

$$\hat{\delta}_{\mathrm{k}} = \frac{1}{\mathrm{p}} \sum_{\mathrm{j}=1}^{\mathrm{p}} \hat{\delta}_{\mathrm{kj}} \,,$$

Subsequently, estimated AB and estimated MSE for the MLEs are obtained, respectively, as follows

$$\widehat{AB}(\hat{\delta}_{\mathrm{k}}) = \frac{1}{p} \sum_{j=1}^{p} \left|\hat{\delta}_{\mathrm{kj}} - \delta_{\mathrm{k}}\right|,$$

$$\widehat{MSE}(\hat{\delta}_{\mathrm{k}}) = \frac{1}{p} \sum_{j=1}^{p} \left(\left(\hat{\delta}_{\mathrm{kj}} - \delta_{\mathrm{k}}\right)^2\right),$$

where $k = 1,2,3$ such as $\delta_1 \equiv \theta_1, \delta_2 \equiv \theta_2, \delta_3 \equiv \alpha$

Step (9): As the number of replications $p = 3000$ is large, thus, we can assume the central limit theorem holds, and thus following Koehler, Brown and Haneuse (2009), the Monte Carlo standard errors of the estimated AB and estimated MSE are obtained as:

$$SE\left(\widehat{AB}(\hat{\delta}_k)\right) = \frac{1}{p}\sqrt{\sum_{j=1}^{p}(\widehat{AB}(\hat{\delta}_{kj}) - \widehat{AB}(\hat{\delta}_k))^2},$$

$$SE\left(\widehat{MSE}(\hat{\delta}_k)\right) = \frac{1}{p}\sqrt{\sum_{j=1}^{p}(\widehat{MSE}(\hat{\delta}_{kj}) - \widehat{MSE}(\hat{\delta}_k))^2},$$

where $k = 1,2,3$ such as $\delta_1 \equiv \theta_1, \delta_2 \equiv \theta_2, \delta_3 \equiv \alpha$

Step (10): Compute the mean of the proportions pn_1 and pn_2 of test units that failed at lower and higher stress levels, respectively, and determine the average mean test duration time in each case.

Step (11): Using the AV-C matrix, the approximate two-sided confidence limits can be obtained with confidence levels $(1-\gamma) = 0.90, 0.95$ and 0.99, using (4.17). Also, the interval's average mean length has been calculated along with the estimated actual CP to study the performance of the asymptotic CIs.

Step (12): The Monte Carlo standard errors of the estimated AL and estimated CP are calculated as:

$$SE\left(\widehat{AL}(\hat{\delta}_k)\right) = \frac{1}{p}\sqrt{\sum_{j=1}^{p}(\widehat{AL}(\hat{\delta}_{kj}) - \widehat{AL}(\hat{\delta}_k))^2},$$

$$SE\left(\widehat{CP}(\hat{\delta}_k)\right) = \sqrt{\frac{\widehat{CP}(\hat{\delta}_k)\left(1 - \widehat{CP}(\hat{\delta}_k)\right)}{p}},$$

where $k = 1,2,3$ such as $\delta_1 \equiv \theta_1, \delta_2 \equiv \theta_2, \delta_3 \equiv \alpha$

The simulation steps are summarized in the flowchart in Figure 4.2.

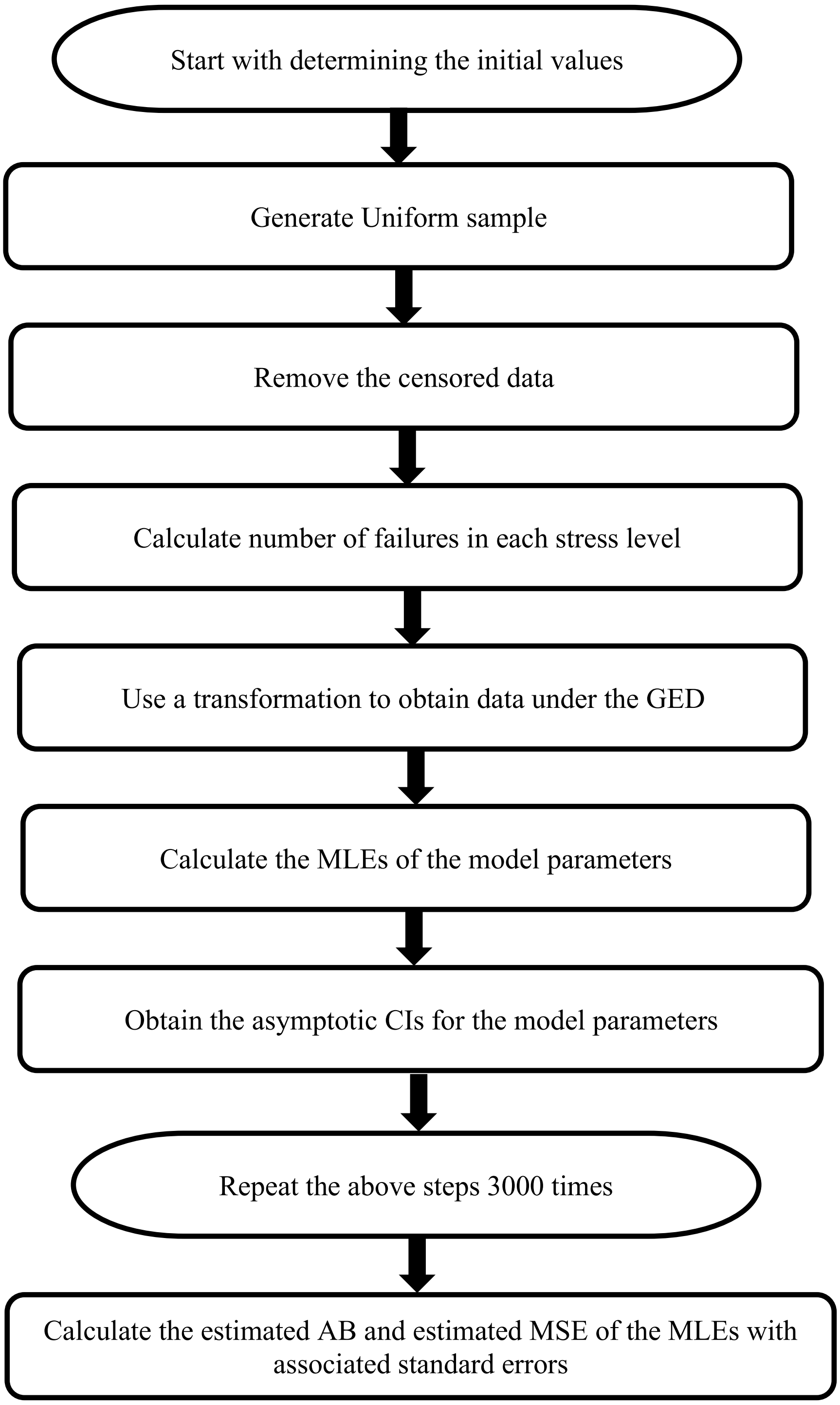


Figure 4.2 Summary of the simulation algorithm.

4.6.1.2 Steps for Obtaining Bootstrap Samples & Confidence Intervals

Within each replication of the Monte-Carlo simulation, the parameter estimates of $B = 1000$ bootstrap samples are obtained and then the CIs for the model parameters are calculated using the methods discussed in Sections 4.5.2.1 and 4.5.2.2 based on the percentile and BCa bootstrap methods, respectively.

The following algorithm is utilized to generate parametric bootstrap samples and to obtain point and interval estimates for each of the 3000 replications. The steps for obtaining the MLEs that were explained in Section 4.6.1.1, are used in the algorithm for obtaining the bootstrap CIs. The steps for obtaining the bootstrap CIs are:

Step (1): Based on the MLEs $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$ obtained previously in Section 4.6.1.1, by using the steps (1-6) in Section 4.6.1.1, the bootstrap estimates $\hat{\theta}_1^*$, $\hat{\theta}_2^*$ and $\hat{\alpha}^*$ are computed numerically based on the parametric bootstrap sample.

Step (2): Repeat the previous step $B = 1000$ times, then the set of B bootstrap estimates $\hat{\delta}_k^*$, $k = 1,2,3$ are presented in ascending order, as follows:

$$\mathfrak{B}(\hat{\delta}_k^*) = \left\{\hat{\delta}_k^{*(1)}, \hat{\delta}_k^{*(2)}, \dots, \hat{\delta}_k^{*(B)}\right\}, k = 1,2,3\,,$$

where $\hat{\delta}_1^* = \hat{\theta}_1^*\,, \hat{\delta}_2^* = \hat{\theta}_2^*$ and $\hat{\delta}_2^* = \hat{\alpha}^*$.

Step (3): Based on the bootstrap resampling method, the bias and the variance of the MLEs are estimated, respectively, (Efron and Tibshirani,1993) as follows

$$\widehat{bias}(\hat{\delta}_k) = \hat{\delta}_k^{*(.)} - \hat{\delta}_k,$$

and $$\widehat{var}(\hat{\delta}_k) = \frac{1}{(B-1)}\sum_{b=1}^{B}\left[\hat{\delta}_k^{*(b)} - \hat{\delta}_k^{*(.)}\right]^2 \,,$$

where $$\hat{\delta}_k^{*(.)} = \frac{1}{B}\sum_{b=1}^{B}\hat{\delta}_k^{*(b)}\,.$$

Step (4): Using the bootstrap estimates generated by the algorithm given above, percentile and BCa bootstrap CIs are derived according to (4.18) and (4.19) in Section 4.5.2.1 and 4.5.2.2, respectively.

4.6.1.3 Steps for Obtaining Jackknife Estimators

The variance and bias of estimators are estimated using two resampling methods: the bootstrap and the jackknife method. Unlike the bootstrap, the jackknife estimate of the variance of the MLE will not change for a given sample (Efron and Stein, 1981). For each sample, delete observations one-by-one at a time and calculate the MLEs of the model parameters. The jackknife estimate $\tilde{\delta}_{k(i)}$ is the estimate of δ_k based on the original sample with the i^{th} observation omitted (see Efron and Stein, 1981 for more details). It can be calculated numerically as follows:

Step (1): For each sample replication with r observations, sequentially omitting each observation from the sample and calculate the MLEs of the model parameter based on the remaining $(r-1)$ observations. Subsequently, update the number of failures at lower and higher stress levels n_1 and n_2.

Step (2): Compute r jackknife estimates for the model parameters $\tilde{\delta}_{k(1)}, \tilde{\delta}_{k(2)}, \dots, \tilde{\delta}_{k(r)}$ where $\tilde{\delta}_1 = \tilde{\theta}_1\,, \tilde{\delta}_2 = \tilde{\theta}_2$ and $\tilde{\delta}_1 = \tilde{\alpha}$

Step (3): Based on the jackknife estimates, the bias and the variance of the MLEs are estimated as follows (Efron, 1982)

$$\widehat{bias}_{jack}(\hat{\delta}_k) = (r-1)(\tilde{\delta}_{k(.)} - \hat{\delta}_k),$$

$$\widehat{var}_{jack}(\hat{\delta}_k) = \frac{(r-1)}{r}\sum_{i=1}^{r}\left[\tilde{\delta}_{k(i)} - \tilde{\delta}_{k(.)}\right]^2 ,$$

where $\tilde{\delta}_{k(.)} = \frac{1}{r}\sum_{i=1}^{r}\tilde{\delta}_{k(i)}$, $\quad k = 1,2,3$

All the above steps for obtaining the CIs based on asymptotic, percentile and BCa bootstrap method are programmed manually in R. Also, the steps for deriving the jackknife estimates for the variance and the bias of the estimators are programmed in R. The Monte-Carlo simulation is done by using the High-Performance Computing facility Iridis-4 with job array using batch files. Different sets of initial values have been sent in each array iteration using (R CMD BATCH) command.

4.6.2 Simulation Results and Discussion

Simulation results are presented in Tables A.1– A,21 in Appendix A for different sample sizes n, FP%, stress change time τ and 4 censoring schemes. Tables (A.1– A.6) give the average mean of the MLEs besides the estimated AB, estimated MSE for the estimators of δ_k based on ML, with associated standard errors. Also, the estimated AB, and the estimated MSE for the estimators of δ_k based on bootstrap and jackknife method are given in Tables (A.1– A.6). The AL and the estimated CP of the asymptotic, bootstrap and Jackknife CIs at confidence levels 95% and 99% for θ_1, θ_2 and α are presented in Tables A.7– A.18. Note that, for the sake of conciseness, CIs with 90% confidence level are not presented here but are available upon request. Table A.19 reports the mean of the proportions of failures at each stress levels, along with the mean of the total test duration T. Incidentally, for the purpose of conciseness, a subset of simulation results is reported.

The following Figure 4.3 – Figure 4.5 summarize the influence of increasing the sample size upon the performance of MLEs of model parameters based on four censoring schemes.

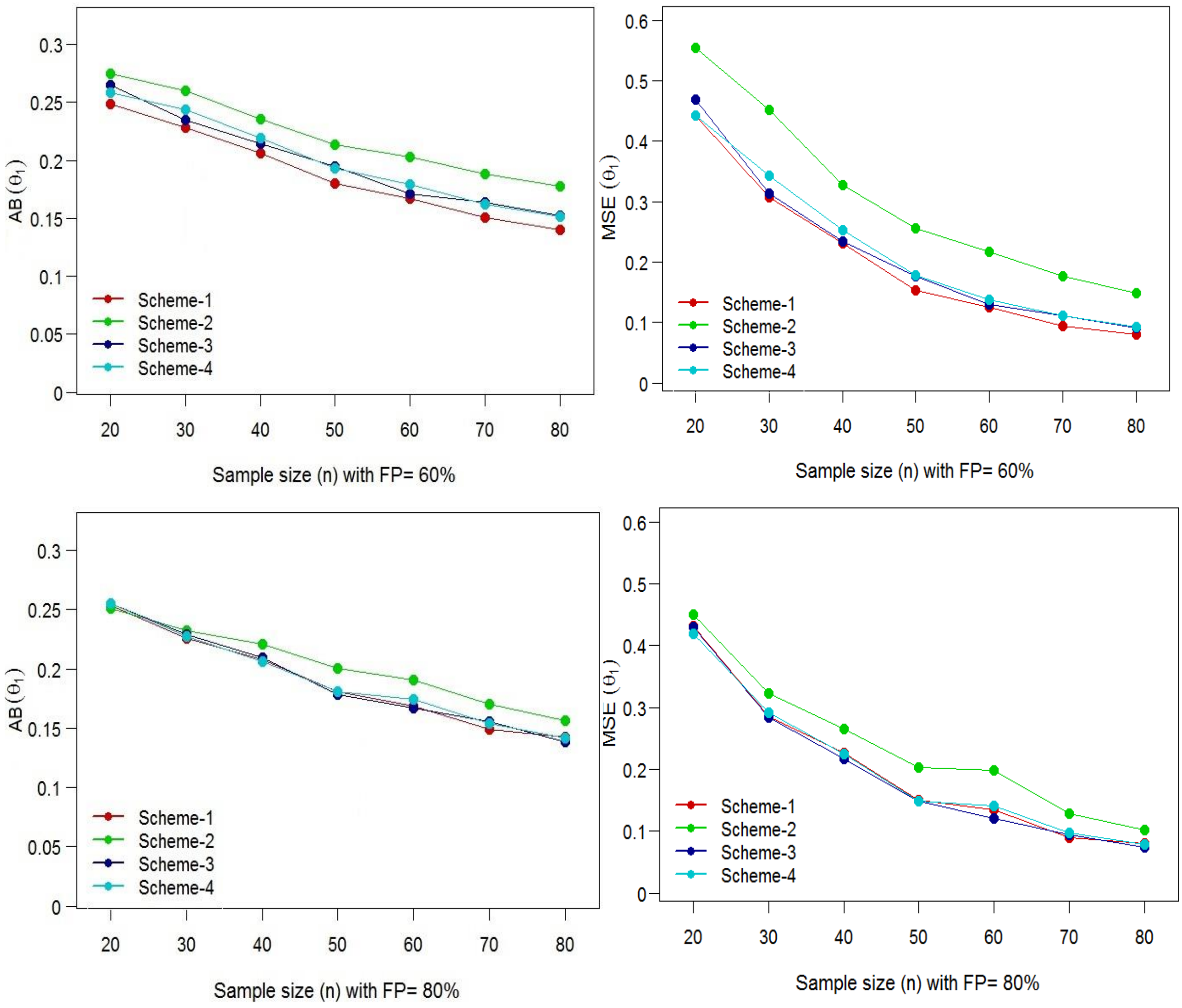


Figure 4.3 The effect of increasing the sample size and failure percentage on the performance of $\hat{\theta}_1$ under different progressive censoring schemes.

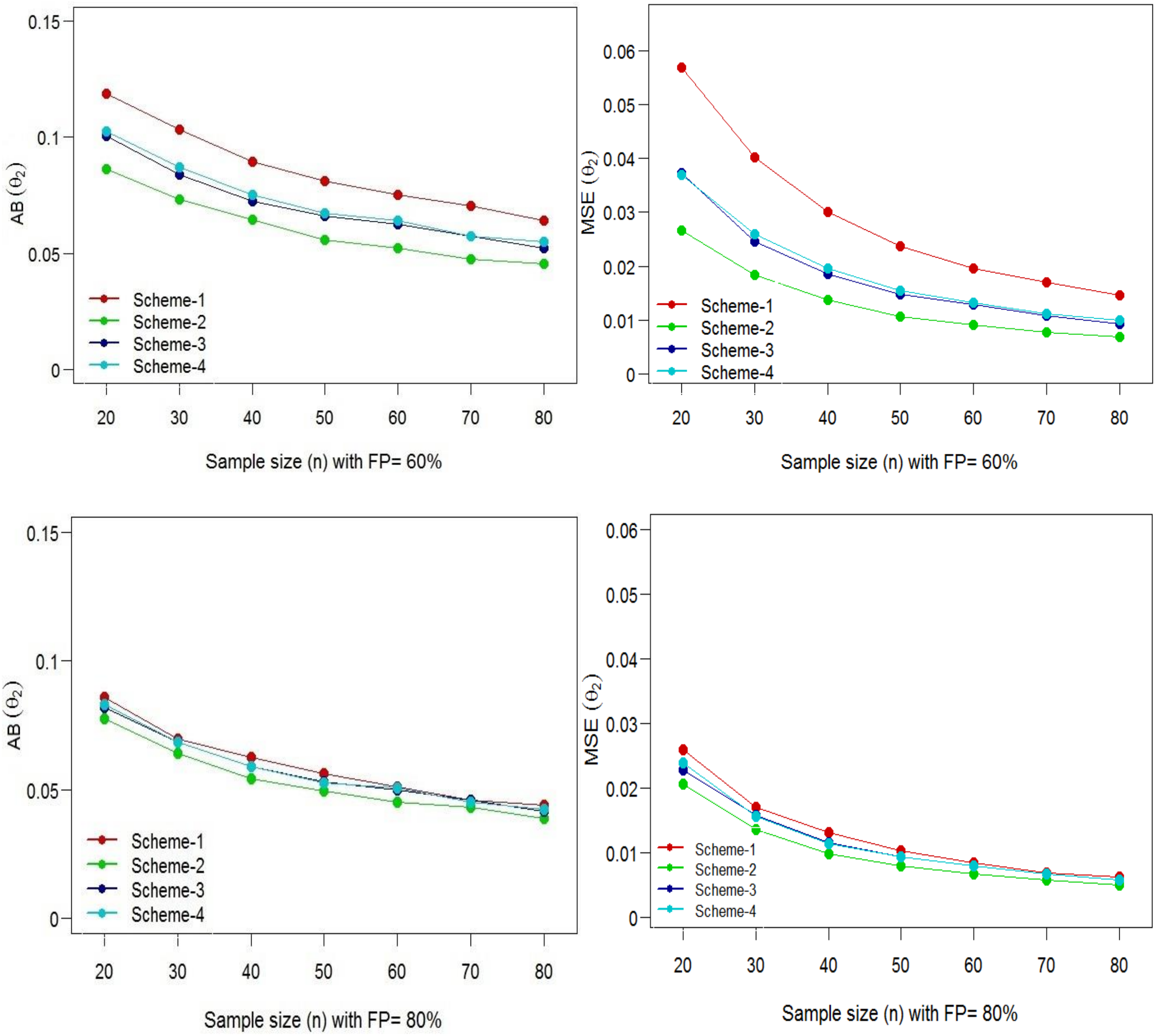


Figure 4.4 The effect of increasing the sample size and failure percentage on the performance of $\hat{\theta}_2$ under different progressive censoring schemes.

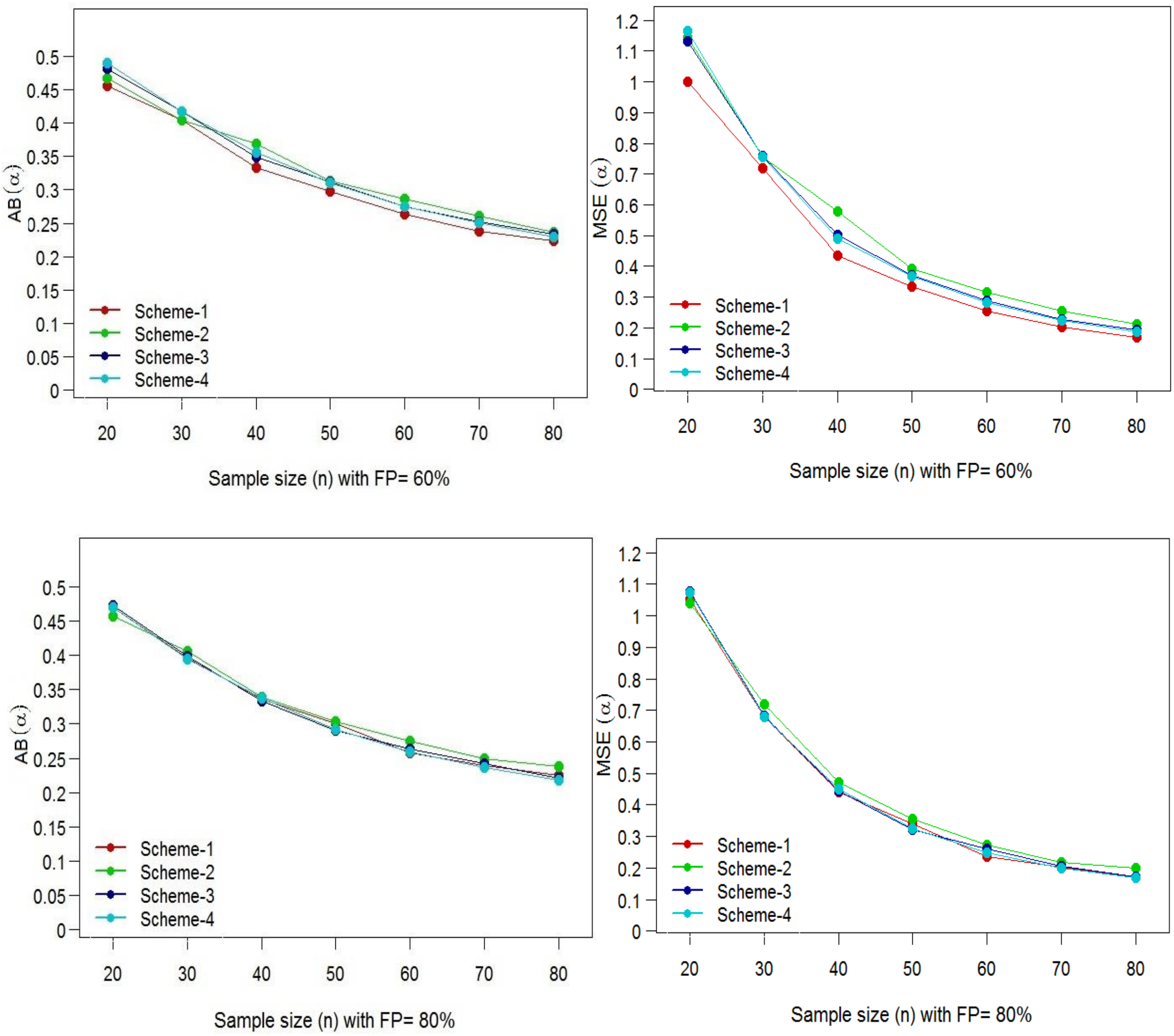


Figure 4.5 The effect of increasing the sample size and failure percentage on the performance of $\widehat{\alpha}$ under different progressive censoring schemes.

From the results shown in Tables A.1– A.13 in Appendix A and Figure 4.3 – Figure 4.5, the following comments are concluded on the performance of parameters estimates for the generalized exponential lifetime distribution for simple SSALT models under progressive Type-II censoring.

1) For fixed values of censoring scheme, stress change time τ and FP, as the sample size n increases, it is noticed that the AB of the MLEs $\hat{\theta}_1$ and $\hat{\alpha}$ decrease considerably, whereas there is a slight decrease in AB and MSE of $\hat{\theta}_2$. As expected, the standard error of the estimated AB, MSE, AL and CP are decreasing as the sample size increases. For confidence levels 0.95% and 0.99%, the CIs of the model parameters become much narrow with larger sample size. Also, there is a noticeable increase in the total test time as the sample size increases, except for few cases under Type-II censoring.

2) For the fixed values of n, censoring scheme and τ, by increasing the FP, it is seen that the AB and MSE of MLEs decrease, except in a few cases where AB and MSE of $\hat{\theta}_1$ and $\hat{\alpha}$ have a slight increase.

Moreover, it is noticed that there are a few cases where the censoring scheme gives smaller AB and MSE than the complete sample. In this situation, running the SSALT for a complete sample does not improve the performance of MLEs of the model parameters. Since increasing the number of failures would result in more precise estimates, the standard errors of the performance measures slightly decrease as the FP increases. The proportion of failures at a lower stress level pn_1 decreases by 7%-20% with increasing FP for $\mathcal{R}1, \mathcal{R}3$ and $\mathcal{R}4$, while it is constant for $\mathcal{R}2$. This means that as the number of observed failures increases, the number of test units that fail at a lower stress level decreases.

3) For a larger value of τ, which results in more failures at a lower stress level, pn_1 increases. This results in providing more information about $\hat{\theta}_1$. Thus, the performance of MLEs of θ_1 and α gets better as the AB and MSE decrease, along with the estimated CP of the CI getting closer to the nominal levels at 95% and 99%. On the contrary, the AB and MSE of $\hat{\theta}_2$ increase and the CP becomes far from nominal levels. However, the test takes more time with increasing τ.

4) Figure 4.3 – Figure 4.5 show that among all censoring schemes, the AB and MSE of $\hat{\theta}_1$ were the largest under $\mathcal{R}2$, where all censored data were removed at the time of the first failure, which probably meant less information about the lower stress level. In contrast, the AB and MSE of $\hat{\theta}_2$were the largest under $\mathcal{R}1$ that presented Type-II censoring, where all censored data is removed at the end of the test, which means less information about the higher stress level. The differences in the AB and MSE of the MLEs, based on different schemes, reduce as the FP increases or n increases. Furthermore, it can be noticed that $\mathcal{R}3$ and $\mathcal{R}4$ have the same impact on the AB and MSE of the MLEs of the model parameters. However, the precision of the performance measures among $\mathcal{R}1, \mathcal{R}3$ and $\mathcal{R}4$ censoring schemes is almost equal and comparable to the precision under a complete sample. Therefore, running the life test under Type-II progressive censoring would not affect the precision of the performance measures of the MLEs of the model parameters.

5) For fixed τ, removing the censored items based on $\mathcal{R}2$ does not affect the failure proportions pn_1 and pn_2 at lower and higher stress levels, respectively. However, pn_1 and pn_2 based on $\mathcal{R}2$ are the same as pn_1 and pn_2 based on the complete sample.

6) Comparing the results for $\mathcal{R}3$ and $\mathcal{R}4$, where the censored data is removed in the middle stages of the test, it was observed that all results have almost the same values under the two schemes. This means that while removing items in the middle stages of the test, removing the items one by one or as a pre-determined number in the middle of the test have the same impact on the MLEs.

7) In general, it can be concluded that the jackknife method provides better estimates of AB and MSE of the MLEs, except for a few cases where the bootstrap method provides smaller values of the MSE estimators. Also, the values of AB under bootstrap and asymptotic MLEs are relatively the same.

8) It is observed that the approximate CIs for α provides good CP (close to the nominal level) regardless of the sample size. This was observed in all cases.

9) In general, the precision of the performance measures is high as the Monte Carlo standard errors for the performance measures ≤ 0.0175 for all different scenarios when $\tau = 0.6$ and it is ≤ 0.0275 for all different scenarios when $\tau = 0.4$.

10) The failure proportion pn_1 at the lower stress level is shown to be more than pn_2 at the higher stress level for all censoring schemes, except for $\mathcal{R}2$ and $\tau = 0.4$. For scheme $\mathcal{R}2$ and $\tau = 0.4$, it is noticed that the AB and the MSE of $\hat{\theta}_1$ and $\hat{\alpha}$ are the highest, whereas they are the lowest for $\hat{\theta}_2$ for all assumed values of n and FP. However, it may require studying the case where $pn_1 < pn_2$ for other PCSs.

11) For the scale parameter θ_1, it is noticed that for confidence levels 0.95% and 0.99% asymptotic CIs become considerably narrower than bootstrap CIs for all different values of n, PCSs and τ. To the contrary, the CPs of the asymptotic CIs are less than nominal levels by 5% - 6%. In contrast, the estimated CPs of the percentile bootstrap CIs at 95% and the estimated CPs of the BCa bootstrap CIs at 99% are getting closer to the nominal levels.

12) For the scale parameter θ_2, the AL of the CIs under the three methods is approximately the same. However, at 95%, the AL of asymptotic CIs is slightly shorter than that of bootstrap CIs. The BCa bootstrap method provides good CP (close to the nominal level) of the CI estimates.

13) For the shape parameter α, asymptotic CIs have the shortest estimated AL among the three methods. The estimated CP of the asymptotic CIs is getting closer to the nominal levels at 95% and 99%. However, the estimated level of CP of the asymptotic CIs is more than the nominal level for almost all cases at 95% and some cases at 99%.

14) Time duration for life test depends on number of observed data as well as actual sample size. It is clear from Table A.13 that Type-II censoring provides the shortest test with respect to total test time. The reason for this is that in Type-II censoring, items with highest lifetimes are removed from the test. Therefore, the longer test duration among different censoring schemes was under scheme-2. However, it was observed that as the data are more likely to be removed at later stages of the test, the time required for running the test reduces.

The following Table 4.1 – Table 4.2 summarize the values of standard errors as a percentage of the estimated value of the performance measurements of the point and interval estimates of θ_1, θ_2 and α.

Table 4.1 Summary of the standard errors as a percentage of the estimated AB and estimated MSE.

τ		$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\alpha}$	
		SEP(AB)	**SEP(MSE)**	**SEP(AB)**	**SEP(MSE)**	**SEP(AB)**	**SEP(MSE)**
0.4	**min**	1.64	3	1.3	1.05	1.59	1.54
	max	3.25	10.6	1.55	2.5	2.08	2.34
0.6	**min**	1.48	2.43	1.29	1.07	2.33	2.19
	max	2.67	6.88	1.78	3.34	5.66	5.34

Table 4.2 Summary of the standard error as a percentage of the estimated AL and estimated CP.

τ	confidence level		$\hat{\theta}_1$		$\hat{\theta}_2$		$\hat{\alpha}$	
			SEP(AL)	**SEP(CP)**	**SEP(AL)**	**SEP(CP)**	**SEP(AL)**	**SEP(CP)**
0.4	95%	min	0.87	0.005	0.29	0.004	0.5	0.003
		max	2.49	0.006	0.88	0.007	0.97	0.004
	99%	min	0.97	0.004	0.52	0.003	0.5	0.002
		max	1.66	0.005	0.99	0.005	0.97	0.003
0.6	95%	min	0.67	0.002	0.34	0.009	0.43	0.003
		max	1.44	0.006	1.71	0.012	0.94	0.006
	99%	min	0.81	0.001	0.56	0.003	0.43	0.002
		max	1.51	0.005	1.8	0.009	0.94	0.003

Despite the complexity of the proposed SSALTs model, it can be seen from the above table, the standard errors as a percentage of the estimated value are generally small except a few cases when the sample size is small $n = 40$. Thus, the estimated measures are precise for $n > 40$.

The standard error of the MSE of $\hat{\theta}_1$ as a percentage of the estimated MSE when $\tau = 0.4$ is relatively large in some cases, particularly when $n = 40$ and $\tau = 0.04$ where there is a small number of failures at the lower stress level. Thus, in this scenario, we should take care not to overinterpret the estimated MSE values, e.g. in comparisons.

On the other hand, for all assumed initial values, the standard errors in estimating the AL and CP are at most 2.5% of the estimated value of the performance measure. Thus, the AL and the CP are precisely estimated and can be used to assess the performance of the MLEs of the model parameters. Furthermore, the percentage of the standard errors of the CP is less than 0.009%, indicating that the estimated CPs of the asymptotic CIs are precise for all scenarios of the initial values.

4.7 Summary

This chapter has studied statistical inference for the SSALT model based on progressive Type-II censoring where the failures follow the GED. The CEM was described in detail based on progressive Type-II censoring. The MLEs of the model parameters were derived and their performance was assessed using the estimated AB and MSE. The bootstrap and the Jackknife methods were utilized to estimate the AB and the MSE of the estimates. In addition, CIs of the model parameters were obtained using three methods; the asymptotic CIs, percentile and BCa bootstrap methods. Simulation studies with detailed descriptions were conducted under variable sets of the sample sizes, FP, stress change time and 4 censoring schemes. Results of simulation studies were utilized to illustrate the impact of a set of initial values on both the point and interval estimation methods and to assess the performance of the estimates.

Generally, it can be seen that the impact of the suggested initial values on the performance of the MLEs is the same for the three used methods. Also, the initial values have the same impact on both asymptotic

CIs and bootstrap CIs. Thus, the following Table 4.1 summarizes the impact of different initial values on the point and interval estimates of θ_1, θ_2 and α.

Table 4.3 The impact of n, FP and τ on the point and interval estimates of the model parameters.

Measure	**Parameter**	**Initial value**		
		$n\uparrow$	**FP** ↑	$\tau\uparrow$
AB	θ_1	↓	↓ except few cases	↓
	θ_2		↝ fluctuated	
	α		↓ except few cases	
MSE	θ_1	↓	↓ except few cases	↓
	θ_2		↝ fluctuated	
	α		↓ except few cases	
AL	θ_1	↓	↓	↓
	θ_2			↑
	α			↓
CP	θ_1	≈	≈	≈
	θ_2			≉
	α		≉	≈

Simulation results indicate that estimates of θ_1 and α have improved by decreasing the estimated AB and estimated MSE of the model parameters and decreasing the estimated AL of the CIs when the number of failures at a low stress level increases. In addition, the precision of the estimated performance measures is improved when the number of failures at the lower stress level increases by increasing the sample size or increasing the stress change time to get more failures at the lower stress level. This is a consequence of the lower stress level being closer to the usage stress level. Consequently, failures under lower stress level may behave similarly to failures under usage stress level. In other words, the density function for items under a lower stress level is close to that of items under a usage stress level.

On the other hand, the number of observed items on the higher stress level slightly affects the performance of $\hat{\theta}_2$. Thus, we can suggest increasing the FP under a lower stress level to get better estimates of the model parameters with less AB and MSE.

In addition, the simulation results in this chapter indicate that scheme-1, which is the basic Type-II censoring, leads to an increase in the AB and MSE of $\hat{\theta}_2$, besides increasing pn_1 compared with other examined schemes. On the other hand, scheme-2 increases the AB and MSE of $\hat{\theta}_1$ and $\hat{\alpha}$ and decreases pn_1 among other examined schemes. In addition, scheme-3 and scheme-4 almost lead to the same results.

The numerical results show that progressive Type-II censoring could reduce the number of failures and decrease the total testing time without losing much precision of the estimates. However, it is crucial to identify the optimal stress change time that yields an appropriate number of failures at each stress level to get more precise estimations of model parameters. Moreover, in designing the SSALTs based on Progressive Type-II censoring, it is essential to choose the optimal censoring scheme that improves the statistical inference of the model parameters. So, the next chapter studies the optimal design for obtaining the optimal stress change time and optimal censoring scheme based on variable set of test settings

Chapter 5
Optimal Step-Stress Model for Progressive Type-II Censored Data

5.1 Introduction

When designing ALT experiments, the number of items to be tested is usually dependent upon the available budget, the cost of the test equipment and the availability of the test facilities. Furthermore, the number of items allowed to fail under the test may be determined before setting the test for the same reasons. Consequently, the question may also arise when each item could be censored from the test with the least impact on the precision of parameter estimation. In addition, a further question that may be considered is whether the optimal value of the censoring scheme is fixed or whether there is a flexible range of values with similar efficiency to choose from.

The general purpose of reliability studies is often to maximize the amount of information about a product gathered within certain restrictions, such as the number of items to be examined. Another consideration about the best time to move to the higher stress level may arise, subject to having sufficient information from both stress levels. Therefore, the criteria that result in optimal design are required. An optimal design provides the most precise estimates of the model parameters; it minimizes the $AVar$ of the MLEs of the parameters of interest at the usage stress level.

In a non-optimal design, a large number of items must be tested to achieve the same level of accuracy in parameter estimation as an optimal design. Therefore, an optimal design can reduce the number of items under test, which reduces the total cost of the test.

This chapter will study the optimal design to determine the best time to increase the stress level from low level to high level. It should be noted that the optimal design studied in this chapter is locally based on different sets of initial values. Furthermore, this chapter aims to investigate censoring schemes to choose the appropriate number of items to be removed from the test after each failure under a progressive Type-II censoring scheme. All these optimal studies will be done using the V-optimality criterion, considering minimizing the $AVar$ of the MLE of the $100\wp^{th}$percentile lifetime under the GED at usage stress level.

Numerical analysis using the golden section search method, which will be explained in Section 5.3.1, is presented to illustrate the derivation of the optimal design for a simple SSALT. After determining the optimal design in terms of the optimal stress change time and optimal censoring scheme, a sensitivity analysis is carried out to identify the model parameters that need to be estimated with particular attention. The chapter concludes with a review of the main points.

5.2 Optimality Criterion

When conducting an ALT experiment, researchers aim to estimate the parameters of interest with the highest degree of precision and minimum dispersion. In this thesis, under the step-stress setting, we are interested in the life estimate of a product at the usage stress level.

The most commonly used criteria for designing optimum SSALT are V-optimality, determinant (D)-optimality and average variance (A)-optimality (Kundu and Ganguly, 2017). All three of these criteria are based on the Fisher information matrix. From the practitioner's perspective, the optimality criterion will be chosen regarding the objective of the experiment. Under the circumstances when the experimenter is concerned with the estimate's precision of the mean time to failure under usage stress level, V-optimality is the best criterion to use. On the other hand, if the life-stress relationship parameters are the most important to estimate with high precision, then D-optimality should be used (Ng et al., 2004).

In this chapter, the optimal test design under the V-optimality criterion is studied in detail to get the most efficient MLEs of the model parameters. The objective of V-optimality is to identify an optimal design such that the quantile failure estimate under the usage stress level has the least variability. V-optimality is minimizing $AVar\left(\hat{t}_{\wp}(x_0)\right)$ (Li, 2009).

Depending on the practitioner/producer, the value of $\wp$ is specified based on the purpose of the life test experiment. Suppose a producer is interested in providing a product warranty for his/her clients. In the manufacturing field, the manufacturer's budget may cover the repair of 5% of the products. Thus, this manufacturer will be interested in estimating the largest product lifetime such that 5% of the total number of manufactured products have a lifetime equal to or less than the determined product lifetime. Therefore, in this scenario, the optimization criterion is to minimize the $AVar(\hat{t}_{0.05}(x_0))$ (Balakrishnan and Aggarwala, 2000). It is essential to accurately estimate $t_{0.05}(x_0)$, as it plays an important role in reducing the repairing cost under guarantee, since the warranty cost may significantly increase with early failures.

From another point of view, a manufacturer may be interested in testing the reliability of products manufactured by a new production line. In this situation, it is suggested to estimate the median of the lifetime or estimate the lifetime of the product such that 95% of products fail before it. Thus, this chapter studies the estimation of the optimal stress change time for $5^{th}, 50^{th}$ and 95^{th} percentile lifetime under the GED at the usage stress level.

As it is mentioned in (4.1) in Section 4.2, the CDF of GED at a time t is:

$$F_s(t;\theta,\alpha) = \left[1-\exp[-t/\theta]\right]^{\alpha}. \qquad t > 0$$

The estimated $100\wp^{th}$ percentile life of the GED with specified probability $\wp$ under usage stress level x_0 is:

$$\hat{t}_{\wp}(x_0) = -\hat{\theta}_0 \ln\left(1 - \wp^{\frac{1}{\hat{\alpha}}}\right), \qquad \wp = 0.05, 0.50, 0.95, \qquad (5.1)$$

where $\hat{\theta}_0$ is the MLE of the scale parameter under the usage stress level x_0, and $\hat{\alpha}$ is the MLE of the shape parameter. When $\wp = 0.5$, $\hat{t}_{\wp}(x_0)$ is the MLE of the median life at the usage stress level.

In order to calculate $\hat{\theta}_0$, the life-stress relationship is used to relate the product lifetimes with life testing stress levels (see Miller and Nelson, 1983 for more details). This thesis assumes that the scale parameter θ_k, which is the characteristic life of a test unit at any stress level x_k, is a log-linear function of the stress. According to (2.1) in Chapter 2, the life-stress relationship is defined as follows:

$$ln(\theta_k) = \beta_0 + \beta_1 x_k\,, \qquad k = 0,1,2\,,$$

where x_0, x_1 and x_2 are usage, lower and higher stress levels, respectively. The life-stress model parameters β_0 and $\beta_1 (< 0)$ are unknown parameters.

By solving the equations of life-stress relationship regarding lower and higher stress level to find the value of β_0 and β_1, we get

$$\beta_0 = ln(\theta_1) - \beta_1 x_1\,,$$

$$\beta_1 = \frac{ln(\theta_1) - ln(\theta_2)}{(x_1 - x_2)}.$$

After estimating θ_1 and θ_2, the parameters β_0 and β_1 can be estimated by using the invariance property of MLEs. Thus, the life-stress relationship could be used to find the estimated value of θ_0 at the usage stress level x_0. By using the invariance property of MLEs, the MLE of θ_0 can be calculated as:

$$\hat{\theta}_0 = e^{(\hat{\beta}_0 + \hat{\beta}_1 x_0)}\,.$$

Now, according to Li (2009), the standardized stress level is defined as

$$x_s = \frac{x_1 - x_0}{x_2 - x_0}.$$

Then

$$x_0 = \frac{x_1 - x_2 x_s}{(1 - x_s)}.$$

Next, by substituting the value of x_0 in the life-stress relationship and using the invariance property of the ML method, $\hat{\theta}_0$ is obtained as follows:

$$\hat{\theta}_0 = \exp\left(\frac{ln(\hat{\theta}_1) - x_s\, ln(\hat{\theta}_2)}{(1 - x_s)}\right).$$

To estimate the asymptotic variance of $100\wp^{th}$percentile lifetime, the Fisher information matrix $F(\underline{\delta})$ is used. According to Li (2009), the delta method is used to obtain the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ at probabilities $\wp = 0.05, 0.50, 0.95$ as

$$AVar\left(\hat{t}_{\wp}(x_0)\right) = AVar\left(-exp\left(\frac{\ln(\hat{\theta}_1) - x_s \ln(\hat{\theta}_2)}{(1 - x_s)}\right) ln\left(1 - \wp^{\frac{1}{\hat{\alpha}}}\right)\right)$$

$$= H\hat{F}^{-1}H^T, \tag{5.2}$$

where $\hat{F}^{-1}$ denotes the inverse of the observed Fisher information matrix in (4.14) evaluated at the MLEs $\hat{\delta}_k = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}\}$. Furthermore, H is a row vector of the first derivative of the $\hat{t}_{\wp}(x_0)$ and H^T is the column vector with respect to θ_1, θ_2 and α.

$$H = \left[\frac{\partial t_{\wp}(x_0)}{\partial \theta_1}, \frac{\partial t_{\wp}(x_0)}{\partial \theta_2}, \frac{\partial t_{\wp}(x_0)}{\partial \alpha}\right]_{|\left(\underline{\delta}=\hat{\underline{\delta}},\, t_{\wp}(x_0)=\hat{t}_{\wp}(x_0)\right)}$$

$$= \begin{bmatrix} -\left(\dfrac{1}{\hat{\theta}_1(1 - x_s)}\right) exp\left(\dfrac{\ln(\hat{\theta}_1) - x_s \ln(\hat{\theta}_2)}{(1 - x_s)}\right) \ln\left(1 - \wp^{\frac{1}{\hat{\alpha}}}\right), \\ \left(\dfrac{x_s}{\hat{\theta}_2(1 - x_s)}\right) exp\left(\dfrac{\ln(\hat{\theta}_1) - x_s \ln(\hat{\theta}_2)}{(1 - x_s)}\right) \ln\left(1 - \wp^{\frac{1}{\hat{\alpha}}}\right), \\ -exp\left(\dfrac{\ln(\hat{\theta}_1) - x_s \ln(\hat{\theta}_2)}{(1 - x_s)}\right) \cdot \dfrac{\wp^{\frac{1}{\hat{\alpha}}} \ln(\wp)}{\hat{\alpha}^2\left(1 - \wp^{\frac{1}{\hat{\alpha}}}\right)} \end{bmatrix}.$$

The above $AVar\left(\hat{t}_{\wp}(x_0)\right)$ in (5.2) is complicated to calculate in closed form. Therefore, a numerical method is used in Section 5.3.1 to estimate the optimal stress change time and in Section 5.4.1 to investigate the optimal censoring scheme. The optimal test design in terms of stress change time and censoring scheme will be discussed in Section 5.3 and 5.4, respectively, for a given set of initial values of the model parameters $\underline{\delta} = (\theta_1, \theta_2, \alpha)$, failure percentage and sample size. The advantage of investigating the optimal stress change time and the optimal censoring schemes for a range of n and FP is not only examining their effects on the optimal design, but also providing the experimenter with the flexibility to choose them regarding the corresponding objective function.

5.3 Optimal Stress Change Time

In some situations, under a simple step-stress model, the hold time at low stress level might be relatively short, resulting in few or no failure data at low stress level and thus affecting the quality of the MLEs. Also, if all items under the test fail at the lower stress level before increasing the stress level, then no information will be obtained from the experiment about the lifetimes at the higher stress level. Moreover, given that the sample size in a reliability test is often small, choosing the optimal stress time is essential to obtaining observed failures in both stress levels. However, the precision of the MLE $\hat{\theta}_0$ depends on the run time at each stress level. For all those reasons, determining an optimal stress change time to change the stress level is essential to ensure the availability of sufficient items at each stress level. This will improve the efficiency of statistical inference of the model parameters and the $100\mathcal{p}^{th}$percentile under usage stress level.

Furthermore, in Section 4.6.2, the simulation results have shown that increasing τ results in getting better MLEs of θ_1 and α. In fact, the total time of the test increases with increasing the stress change time as increasing the stress level would accelerate the failure rate. So, it is required to determine the optimal stress change time τ^*, which balances between accelerating the failure rate and resulting in precise MLEs for the model parameters. The optimal stress change time τ^* is estimated by minimizing the $AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$. The ALT design is developed considering minimization of the $AVar$ of the median and 5^{th} and 95^{th} percentile life estimates of GED under SSALTs based on progressive Type-II censoring schemes. Therefore, the next section will explain the steps for numerically determining the optimal stress change time, using (5.2) with $\mathcal{p} = 0.05, 0.50, 0.95$ for different sample sizes, failure percentages and model parameters.

5.3.1 Numerical Study

The computation of the $AVar$ of the MLEs would be based on the Fisher information matrix. However, due to the complexity of the proposed model, $AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$ cannot be found theoretically. Therefore, numerical analysis is required.

The numerical method that is used in this chapter to solve the optimization problem is the golden section search method. The golden section search strategy was first introduced by Kiefer (1953) to solve an optimisation problem. Braun and Murdoch (2021) discussed the golden section method as a straightforward technique for determining the unique minimiser/maximiser of a univariate function $f(.)$ over a closed interval.

The golden section search is a modification of the bisection method, used to find the root of differentiable and non-differentiable functions. The golden section search requires less calculation than bisection method since the objective function $f(.)$ has to be evaluated once at one new point for each iteration. However, it reuses specific values from the previous iteration, as seen later in this section.

The general idea of golden section search is summarised as repeatedly shrinking the closed interval $[s_L, s_U]$ that contains the minimiser of the objective function $f(.)$. The shrinkage will be stopped when the interval length is less than or equal to a predetermined small tolerance value ε. After that, the midpoint of the smallest interval is obtained as the minimiser. However, the interval $[s_L, s_U]$ that contains the minimiser can be determined using the density plot of the objective function $f(.)$.

In each repetition of the golden section search algorithm, two test points $s_1, s_2 \in [s_L, s_U]$ are determined using the golden ratio, such that $s_1 < s_2$. The golden ratio is defined as:

$$\varphi = \frac{\sqrt{5}-1}{2} \cong 0.618$$

Therefore, the value of the test points s_1, s_2 can be calculated as:

$$s_1 = s_U - \ell \,. \tag{5.3}$$

$$s_2 = s_L + \ell \,, \tag{5.4}$$

where ℓ is the distance from s_1 and s_2 to the boundaries of the search interval $[s_L, s_U]$, such that:

$$\ell = (s_U - s_L) \times \varphi.$$

The test points s_1, s_2 divide the interval into three parts. The function $f(.)$ is evaluated at s_1 and s_2 to determine which part should be discarded. At each iteration, the interval length is reduced by a factor of $\varphi \cong 0.618$; see Chong and Zak (2013). However, the golden ratio is the convergence rate in the golden section search method. The golden section search method is terminated when

$$s_U - s_L \leq \varepsilon \,,$$

where ε is the specified tolerance that is used to compare the closeness of s_U and s_L. In this thesis, the tolerance is assumed to be $\varepsilon = e^{-10}$.

Then, the minimiser can be calculated as

$$s_{optimal} = \frac{s_L + s_U}{2}.$$

The algorithm of the golden section search is summarised in Figure 5.1.

The golden section search method is used to find the minimum $AVar\left(\hat{t}_{\wp}(x_0)\right)$ of GED under SSALT over a predetermined range of the stress change time, which gives the corresponding optimal stress change time.

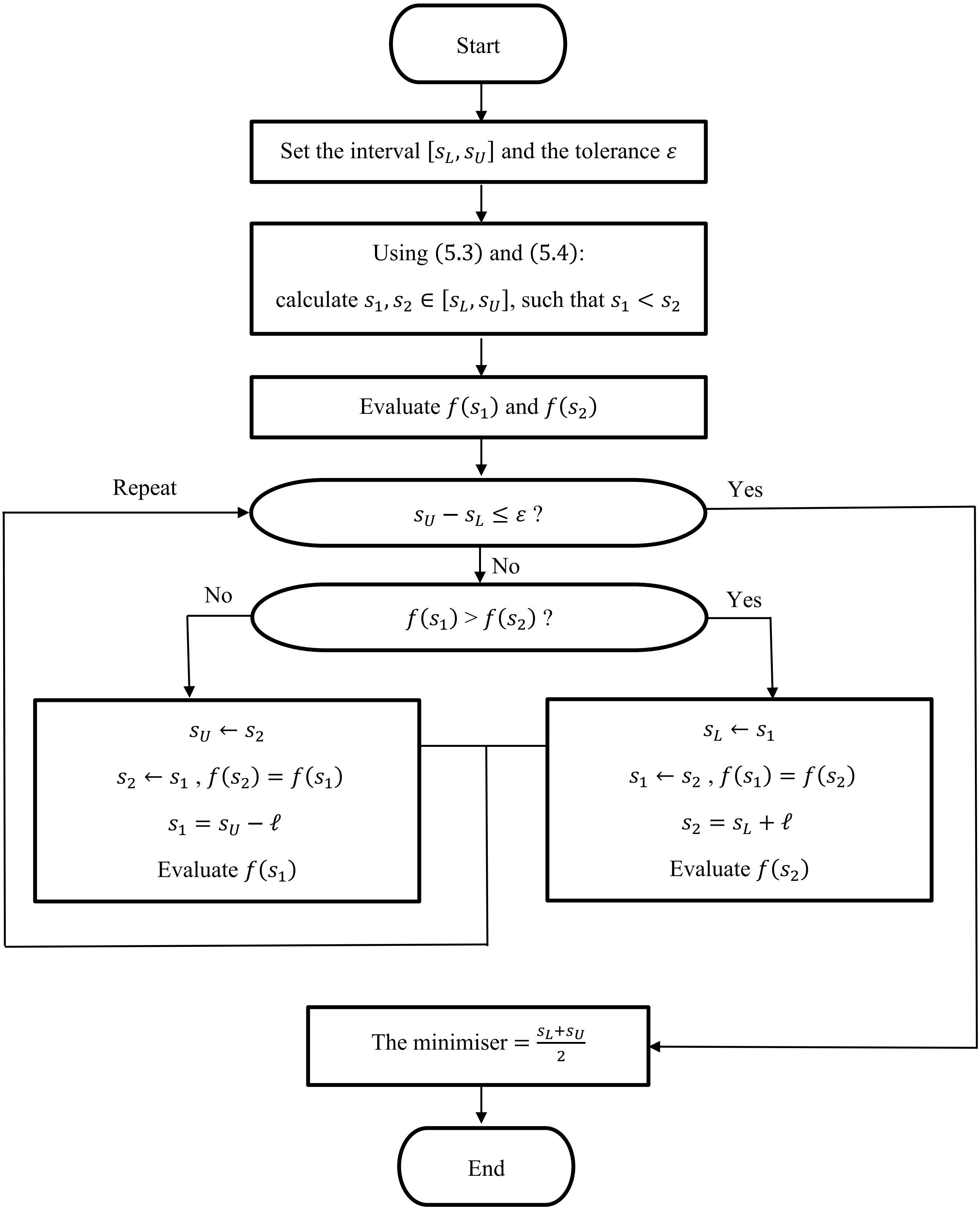


Figure 5.1 A summary of the golden section search algorithm.

5.3.1.1 Numerical Explanation and Simulation Steps

The problem of minimising the $AVar\left(\hat{t}_{p}(x_0)\right)$ of GED under SSALTs is solved numerically. A numerical study is provided in this section to investigate the optimal design as a function of various parameters. The impact of changing n, FP, $\theta_1, \theta_2, \alpha$ and several censoring schemes on the optimal stress change time are studied.

For performing the algorithm for estimating the optimal stress change time τ^*, an initial estimate of the model parameters is required to estimate $t_{p}(x_0)$ of the life distribution. The model parameters are estimated from a previous experiment or based on a pilot study. So, the MLEs of the model parameters are obtained and used to determine the optimal τ^* that minimize the $AVar\left(\hat{t}_{p}(x_0)\right)$.

However, before applying the numerical method to estimate the optimal stress change time τ^*, the estimated value of $AVar\left(\hat{t}_{p}(x_0)\right)$ is graphically presented against τ. The plot is then used to identify the interval containing the minimum value of τ. This interval is then used in the golden section search method to estimate τ^*.

The following algorithm illustrates the method for estimating the optimal stress change time τ^* at which $AVar\left(\hat{t}_{p}(x_0)\right)$, achieves its minimum value with different sets of initial values. The algorithm is performed under the choice of lower and higher stress-levels from an example from Nelson (1990) of 76 times (in minutes) to oil breakdown of an insulating fluid at constant voltage stress (KV). The design voltage stress level is $x_0 = 20$ KV, the lower voltage stress is $x_1 = 26$ KV and the higher voltage stress level is $x_2 = 38$ KV. However, Guan et al. (2014) showed that the GED fits the data at each stress level. They used the Kolmogorov–Smirnov distance between the empirical distribution function and the fitted distribution function based on the MLEs of the parameters.

Step (1): For different choices of n, FP, $\theta_1, \theta_2, \alpha$ and censoring schemes, we generate progressively Type-II censored samples from GED and calculate the MLEs $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$ of the model parameters, using the algorithm presented in Section 4.6.1.1 .

Step (2): From the life-stress relationship, we have:

$$ln(\theta_k) = \beta_0 + \beta_1 x_k\,, \qquad k = 0,1,2\,,$$

$$\hat{\beta}_1 = \frac{ln(\hat{\theta}_1) - ln(\hat{\theta}_2)}{(x_1 - x_2)}\ .$$

$$\hat{\beta}_0 = ln(\hat{\theta}_1) - \hat{\beta}_1 x_1.$$

Then, $\hat{\theta}_0$ can be evaluated as

$\hat{\theta}_0 = e^{(\hat{\beta}_0+\hat{\beta}_1 x_0)}$.

Step (3): Using (5.1), the MLE of the $100\wp^{th}$ percentile of the life distribution at usage stress level is obtained as follows:

$$\hat{t}_{\wp}(x_0) = -\hat{\theta}_0 \ln\left(1 - \wp^{\frac{1}{\hat{\alpha}}}\right).$$

Step (4): Using a loop, repeat the above steps (1 – 3), 3000 times to get a sample of the MLE $\hat{t}_{\wp}(x_0)$.

Step (5): Find the empirical variance of the $100\wp^{th}$ percentile.

Step (6): For each value of τ, repeat the above 5 steps to calculate the corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$.

Step (7): Plot τ for the values $\tau \in [0.01,2.5]$ with its corresponding variance of the MLEs of the $100\wp^{th}$ percentile.

Step (8): Using the golden section search algorithm discussed in the previous Section 5.3.1, find the optimal stress change time, where the objective function $f(.) = AVar\left(\hat{t}_{\wp}(x_0)\right)$ and the search interval $[s_L, s_U]$ that include the optimal τ^* is determined from the plot of the relationship between τ and $AVar\left(\hat{t}_{\wp}(x_0)\right)$.

Step (9): Repeat the above steps (1 – 8), 50 times to obtain a sample of τ^* with its corresponding minimum value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$. Calculate the mean, median, minimum and maximum for the optimal sample. It is worth noting that the replication is chosen to be 50 due to the limited time to run a single job on Iridis, which is 60 hours. After running many jobs on the Iridis to obtain a sample of τ^* based on different sets of initial values, it was noticed that if the replication was > 50, then the run time would exceed 60 hours. Thus, it was found that repeating steps (1 – 8) 50 times takes run time < 60 hours to obtain the optimal τ^*. So, we can get the results based on all the suggested initial values.

Simulation studies have been performed using the above algorithm for the three percentiles with probability $\wp = 0.05, 0.50, 0.95$.

The Monte Carlo standard error of the estimated $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is obtained, for some scenario of the initial values, to assess the precision of the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for the $5^{th}, 50^{th}$ and 95^{th} percentile lifetime of the GED at the usage stress level. By assuming the central limit theorem holds, and thus following Koehler, Brown and Haneuse (2009), the Monte Carlo standard errors of the estimated $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is obtained as:

$$SE\left(\widehat{AVar}\left(\hat{t}_{\wp}(x_0)\right)\right) = \sqrt{\frac{\text{var}\left(\widehat{AVar}\left(\hat{t}_{\wp}(x_0)\right)\right)}{50}},$$

where $\wp = 0.05, 0.50, 0.95$

Hence, both $AVar\left(\hat{t}_{\wp}(x_0)\right)$ and $MSE\left(\hat{t}_{\wp}(x_0)\right)$ were estimated for different sets of the initial values. It was concluded that the $MSE\left(\hat{t}_{\wp}(x_0)\right)$ has the same behaviour as $AVar\left(\hat{t}_{\wp}(x_0)\right)$. Thus, $MSE\left(\hat{t}_{\wp}(x_0)\right)$ gives the same corresponding optimal τ^* as $AVar\left(\hat{t}_{\wp}(x_0)\right)$. Therefore, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is only used and presented to calculate the optimal τ^*.

Simulation studies in this chapter are performed in R program version 3.5.1 to numerically solve the optimal design problem. The maxBFGS built-in function from the maxLik package (Henningsen and Toomet, 2011) is used for calculating the MLEs of the model parameters. The maxBFGS function applies quasi-Newton algorithms to estimate the minimum/maximum for the objective function (Henningsen and Toomet, 2011). Moreover, the data generation, the progressive algorithm, and the golden section search method are manually programmed along with functions and loops.

The program for finding the optimal value under different sets of initial values is computationally intensive. Thus, it is vital to speed up these intensive computations using high performance computing. The simulation steps are performed utilising the Iridis4 and Iridis5 facilities on the University of Southampton high performance computing facility. The array jobs are submitted to Iridis4 using PBS commands and for Iridis5 using SLURM commands for execution. The batch file on Iridis runs the script of R using the command (R CMD BATCH) according to a variable set of initial values for varied cases. The output contains two files: the first file contains the results, which are saved as a word document file using the "rtf" package (Michael E. S., 2021) in R. The second file is the output file which resembles the console screen of R and includes the code with its execution results and any relevant program' messages. The batch files are run in serial on Iridis4 and in both serial and parallel on Iridis5 to run more cases simultaneously and get the results faster.

5.3.2 Numerical Results and Concluding Remarks

For an illustration of the results of the optimal design problem, a numerical study based on a selected set of the initial values: n, FP, θ_1, θ_2, α and several censoring schemes has been conducted. The results are reported in this section and in the Appendix B. The R codes and Iridis commands are available upon request.

The observed sample under SSALT is obtained from the complete sample based on the following 4 censoring schemes:

PSC1: The first PCS, when $(n-r)$ items are removed from the test at the time of the first failure. This scheme is the reverse of Type-II censoring scheme.

$$\mathcal{R}1 = \left((n-r), 0^{*(r-1)}\right)$$

PSC2: The second PCS, where several items are removed at multiple consecutive times of the failure occurrence in the middle of the test.

$$\mathcal{R}2 = \begin{cases} \left(0^{*\left(\frac{(r-z)}{2}\right)}, R_i = \frac{(n-r)}{z}, 0^{*\left(\frac{(r-z)}{2}\right)}\right), \text{where: } i = \left(\frac{(r-z)}{2}+1\right), \dots, \left(\frac{(r-z)}{2}+z\right) & \text{if } z \text{ is even} \\ \left(0^{*\left(\frac{(r-z-1)}{2}\right)}, R_i = \frac{(n-r)}{z}, 0^{*\left(\frac{(r-z-1)}{2}+1\right)}\right), \text{where: } i = \left(\frac{(r-z+1)}{2}+1\right), \dots, \left(\frac{(r-z+1)}{2}+z\right) & \text{if } z \text{ is odd} \end{cases}$$

where z is the number of failures that the censored items are removed from the test at their occurrence.

PSC3: The third PCS, where a single item is randomly removed after each failure in the middle interval of the test.

$$\mathcal{R}3 = \left(0^{*\left(\frac{2r-n}{2}\right)}, 1^{*(n-r)}, 0^{*\left(\frac{2r-n}{2}\right)}\right)$$

PSC4: The fourth PCS, in which $(n-r)$ items survive at the end of the test. This scheme is Type-II censoring scheme.

$$\mathcal{R}4 = \left(0^{*(r-1)}, (n-r)\right).$$

The program for plotting the relationship between the stress change time τ and $AVar\left(\hat{t}_{\wp}(x_0)\right)$ has been executed multiple times for various combinations of initial values, such as n, the FP, the model parameters and the 4 censoring schemes PSC1- PSC4. The phase of graphically determining the relationship between τ and $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is the starting stage of investigating the optimal stress change time. The plots are used to determine the interval that includes the value of τ^* that results in the minimal value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$. This interval is used as the search interval $[s_L, s_U]$ in the golden section search method, as discussed in Section 5.3.1.

Figure 5.2 describes the $AVar\left(\hat{t}_{0.5}(x_0)\right)$ of the GED as a function of τ. The model parameters $\theta_1 = 0.6$, $\theta_2 = 0.3$ and $\alpha = 1.2$ are assumed. The 3 censoring schemes PSC1- PSC3 are used to study the relationship between τ and $AVar\left(\hat{t}_{0.5}(x_0)\right)$. The PCSs for different combinations of n and FP are described as follows

n	FP	r	$\mathcal{R}1$	$\mathcal{R}2$	$\mathcal{R}3$
40	60%	24	$(16\,, 0^{*23})$	$(0^{*10}, 4^{*4}, 0^{*10})$	$(0^{*4}, 1^{*16}, 0^{*4})$
40	80%	32	$(8\,, 0^{*31})$	$(0^{*14}, 2^{*4}, 0^{*14})$	$(0^{*12}, 1^{*8}, 0^{*12})$
60	60%	36	$(24\,, 0^{*35})$	$(0^{*15}, 4^{*6}, 0^{*15})$	$(0^{*6}, 1^{*24}, 0^{*6})$
60	80%	48	$(12\,, 0^{*47})$	$(0^{*22}, 3^{*4}, 0^{*22})$	$(0^{*18}, 1^{*12}, 0^{*18})$
80	60%	48	$(32\,, 0^{*47})$	$(0^{*20}, 4^{*8}, 0^{*20})$	$(0^{*8}, 1^{*32}, 0^{*8})$
80	80%	64	$(16\,, 0^{*63})$	$(0^{*30}, 4^{*4}, 0^{*30})$	$(0^{*24}, 1^{*16}, 0^{*24})$

By applying steps 1 to 7 of the algorithm for estimating τ^* in Section 5.3.1.1, we obtained a plot of τ with its corresponding $AVar(\hat{t}_{0.5}(x_0))$ as illustrated in Figure 5.2.

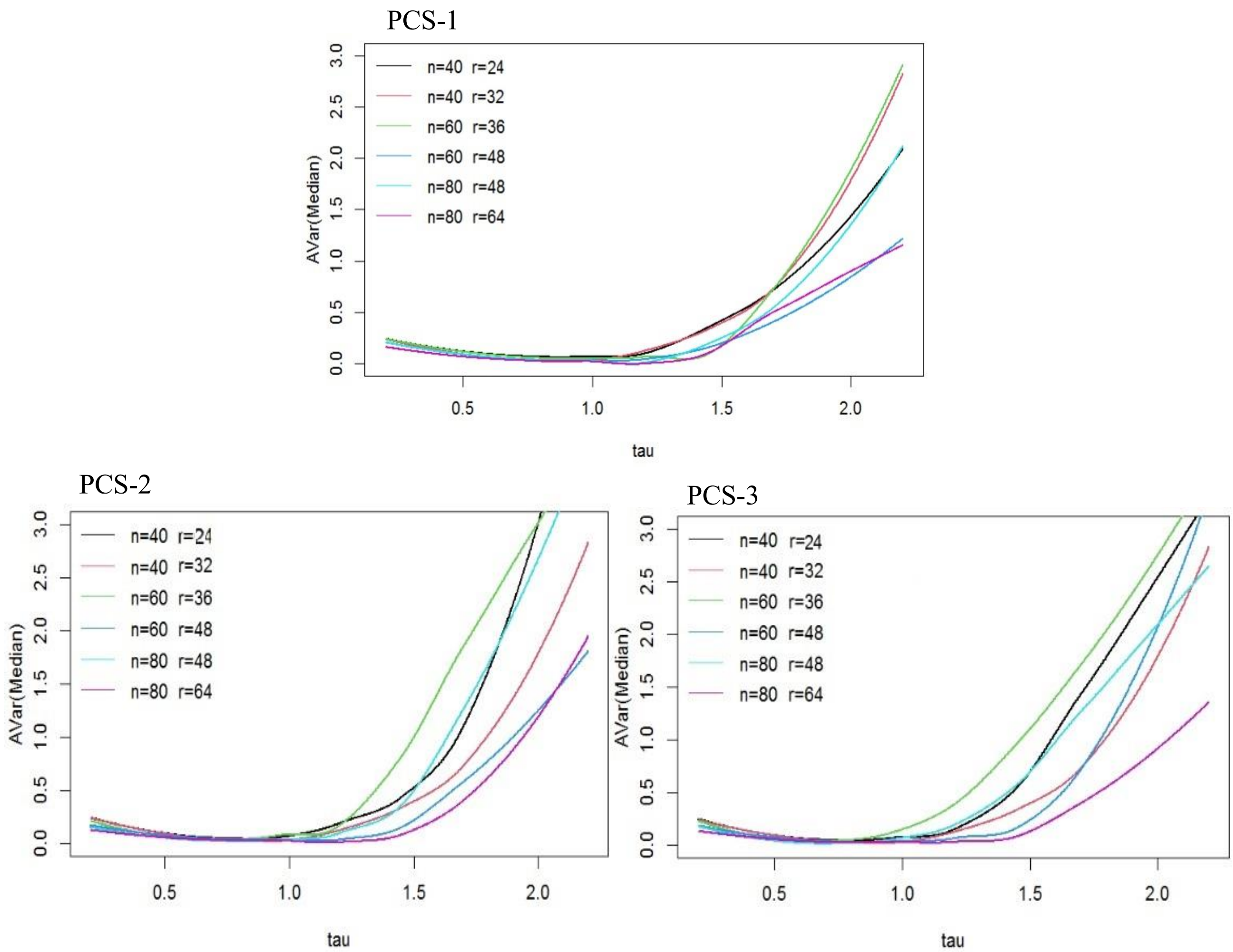


Figure 5.2 Plot of time τ vs. $AVar(\hat{t}_{0.5}(x_0))$ for three censoring schemes and different combinations of n, r and FP.

From Figure 5.2, it can be seen that for all cases, the value of τ^* that corresponds to the minimal value of $AVar(\hat{t}_{0.5}(x_0))$ is in the interval $[0.5, 1.5]$. However, it is obvious that for $0.3 < \tau < 1$, there is only a slight change in $AVar(\hat{t}_{0.5}(x_0))$. That may be because changing the stress change time in this interval has a small effect on the number of failures at the lower and higher stress levels. Therefore, the amount of information gained from the experiment regarding both stress levels will be stable. Consequently, this reduces the impact on the $AVar(\hat{t}_{0.5}(x_0))$.

Before studying the optimal τ^* under different sets of initial values, let us investigate the behaviour of the optimal τ^* sample. Steps 1 to 8 of the simulation algorithm described in Section 5.3.1.1 are implemented to generate a sample of 500 optimal τ^*. Figure 5.3 presents a histogram, the kernel density estimates and a boxplot of a sample of 500 optimal τ^* under two values of $n = 20{,}60$ with FP= 80%. The initial values of the model parameters are $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and the items are censored in the middle of the test.

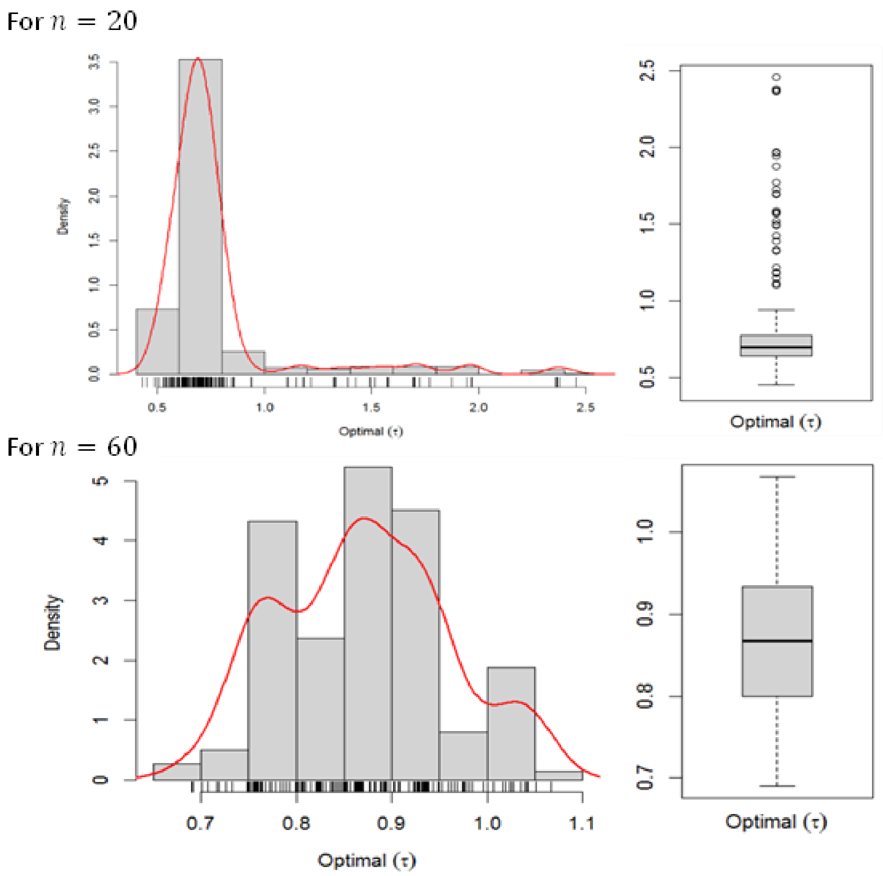


Figure 5.3 Density plot and boxplot of optimal τ^*.

It can be seen that for the small sample size $n = 20$, the optimal τ^* is right skewed with outliers. For $n = 20$, the mean of the τ^* sample= 0.7761 and the median= 0.6939. On the other hand, for $n = 60$, the mean $= 0.8709$ and the median= 0.8666. However, as would be seen later from the simulation results under different values, the optimal τ^* sample is skewed only for a few cases, such as when $n \leq 40$. Thus, the optimal τ^* median is obtained because it is more robust and less sensitive to the skewness of the distribution and the existence of outliers.

Moreover, a sample of the MLE of the percentile $t_{0.5}(x_0)$ is calculated at the optimal τ^* to understand the behaviour of $\hat{t}_{0.5}$. The steps 1 to 4 of the simulation algorithm described in Section 5.3.1.1 are implemented to generate a sample of 3000 MLE $\hat{t}_{0.5}(x_0)$ at the optimal τ^* value. The parameter values are assumed to be $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$. The simulation study is done for $n = 100$ with 40% observed failures and the data are censored in the middle of the test based on PCS2. Figure 5.4 shows the kernel density estimates and the boxplot of $\hat{t}_{0.5}(x_0)$ of the GED under SSALT based on Type-II progressive censored data.

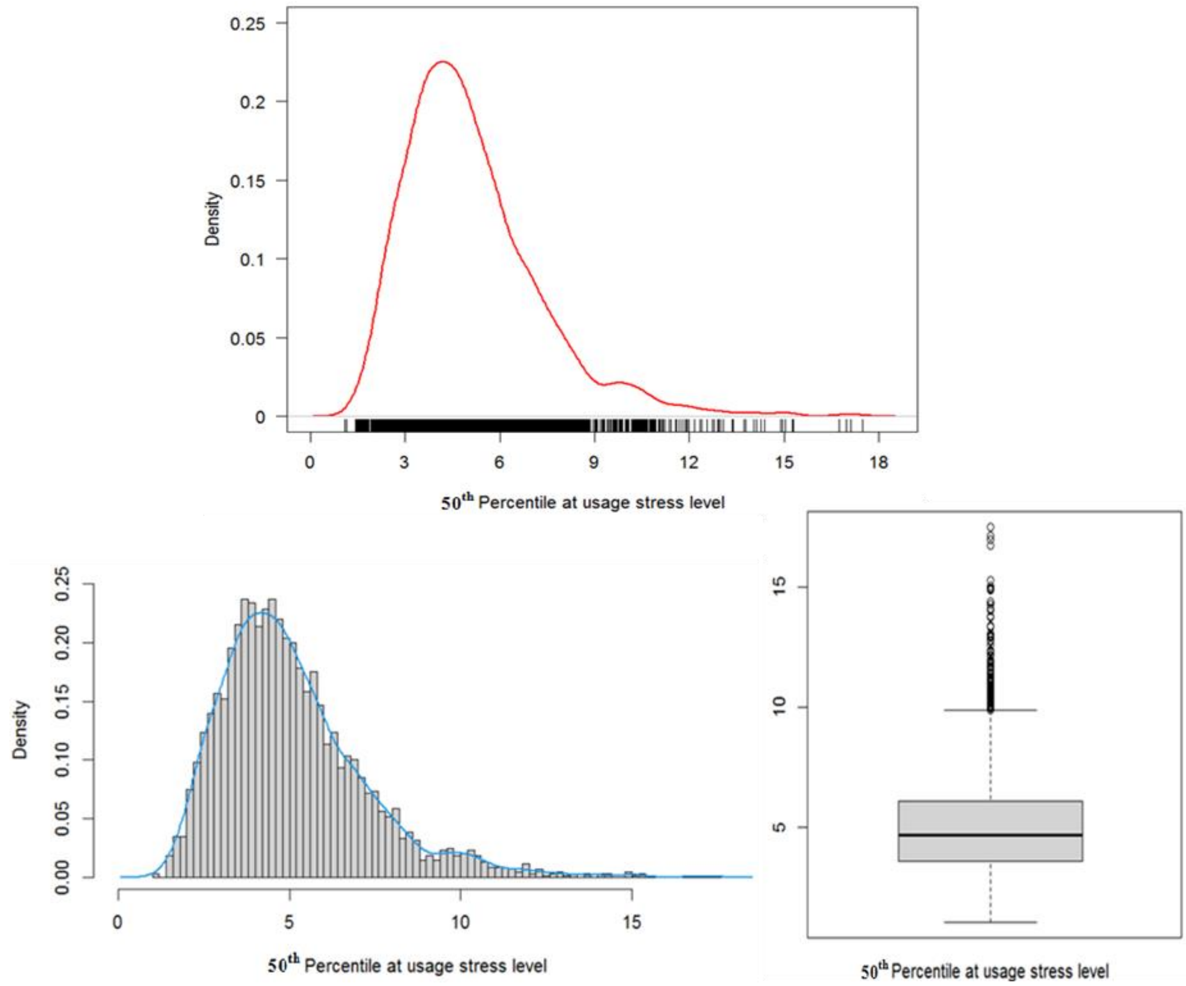


Figure 5.4 Density plot and boxplot of $\hat{t}_{0.5}(x_0)$.

It can be seen from Figure 5.4 that the distribution of $\hat{t}_{0.5}(x_0)$ is right skewed with outliers. However, outliers occur in only a few cases. The density curve is right skewed because the data follows the GED which is a skewed distribution.

Multiple sets of the model parameters, n, FP and censoring schemes are examined to investigate the impact of changing them on the optimal τ^*. Simulation results are presented in Tables B.1 – B.9 in Appendix B. These tables present 4 measures, which are the mean, the median, the minimum and the maximum of both the optimal τ^* and $AVar\left(\hat{t}_{p}(x_0)\right)$.

To study the impact of the sample size, $n = (20,40,60,80,100,150,200)$ is considered with 80% of items observed under the test. Also, 4 schemes PSC1 - PSC4 of removing censored items from the test are suggested. The parameter values are assumed to be $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$. The results are presented in Figure 5.5 – Figure 5.7 and are presented in Tables B.1 – B.5 in Appendix B.

The purpose of the following plot is to show the impact of censoring items in the middle of the test, based on PSC2 and PSC3, on the optimal τ^*. Figure 5.5 shows the boxplot of optimal τ^* for two censoring scenarios in which items are removed in the middle of the test based on PSC2 and PSC3.

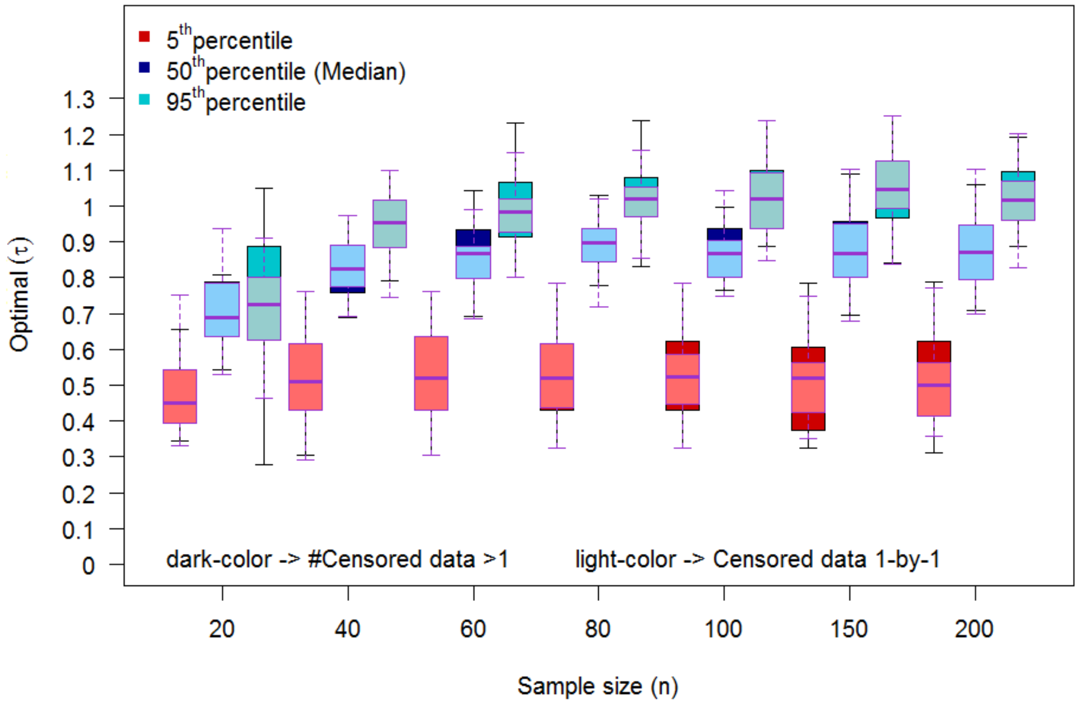


Figure 5.5 Boxplot of optimal τ^* vs. sample size stratified by $\wp$, for two censoring schemes.

From Tables B.1, B.2 and B.3, both censoring schemes have the same impact on the results i.e. both schemes result in essentially the same average values of $AVar\left(\hat{t}_{\wp}(x_0)\right)$. Thus, none of the two scenarios is preferable to the other, and the choice between the two schemes is dependent on the experimenter's point of view. From one point of view, PSC3 allows items to be censored early in the test. As a result, these items are probably more reliable as they are run for a shorter time under the test, and they can be reused in another experiment for another purpose, or their components could be recycled or reused. On the other hand, if it is not desirable to remove items frequently, the experimenter may remove several items at once based on PSC2. However, from Tables B.1, B.2 and B.3, it can be noticed that the mean and the median of the optimal τ^* slightly decrease in most cases when items are removed based on PSC3.

Figure 5.6 presents the boxplot of optimal τ^* for $5^{th}, 50^{th}$ and 95^{th} percentiles, under 3 censoring schemes PSC1, PSC2 and PSC4.

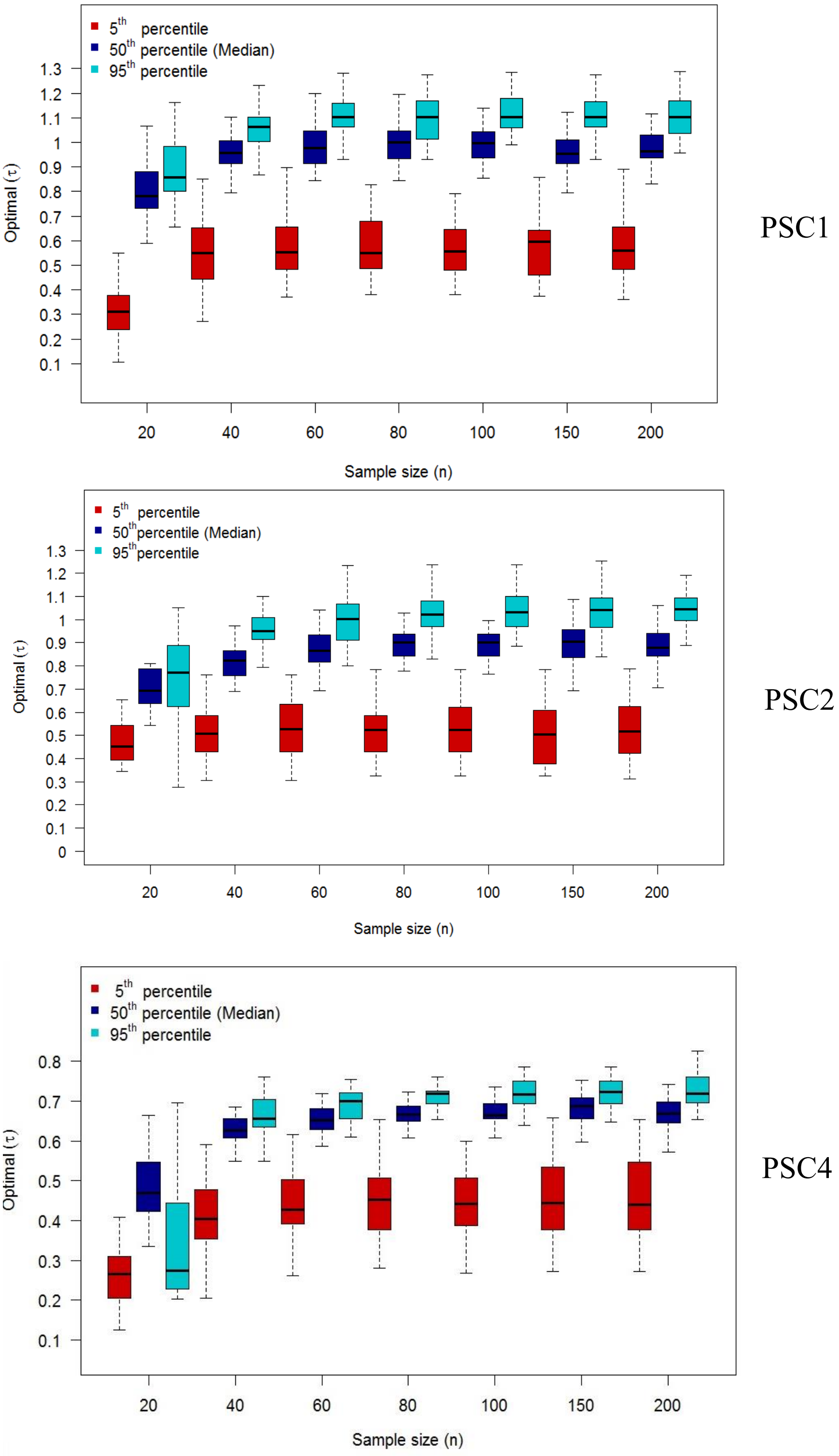


Figure 5.6 Boxplot of optimal τ^* vs. sample size stratified by $\wp$ for three censoring schemes.

Figure 5.6 and Tables B.1, B.2 and B.3, show that as n increases, the mean and median of the optimal stress change time also increases just for $n < 60$. However, the mean and median of optimal τ^* are more stable as n increases; $n \geq 60$. Also, the optimal τ^* variability is modest when the objective of the V-optimality is to minimize the $AVar\left(\hat{t}_{0.5}(x_0)\right)$. In addition, SSALTs based on PSC1 and PSC2 censoring schemes require almost the same duration of hold time under x_1 to get the minimum $AVar\left(\hat{t}_{\wp}(x_0)\right)$.

Furthermore, that the optimal τ^* is smaller if estimating $t_{0.05}(x_0)$ under the SSALTs is of interest. This means it is not required to run the experiment under a low stress level for a long time. As a result, the stress level could be increased faster, and the total time of the experiment would be reduced. On the other hand, if estimating $t_{0.5}(x_0)$ or $t_{0.95}(x_0)$ of the GED under the SSALTs is of interest, then, the experiment must run longer under x_1 to get the most precise estimate of the percentile with the smallest value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$. In fact, increasing the value of $\wp$, results in increasing the value of optimal change time. This may be because as $\wp$ increases, more failure information is required under x_1 regarding to the right tail of the distribution curve.

Figure 5.7 shows the impact of the sample size on $AVar\left(\hat{t}_{\wp}(x_0)\right)$ that corresponds to the optimal τ^* under three censoring schemes. The value of the $AVar$ decreases, and its range slightly decreases as n increases (see Tables B.1 - B.5 in Appendix B). Obviously, increasing n yields more information about the data, which probably increases the precision of estimation $t_{\wp}(x_0)$. In particular, as n increases, $AVar\left(\hat{t}_{\wp}(x_0)\right) \to 0$. In addition, it can be concluded that the time of removing the items from the test has no obvious effect on $AVar\left(\hat{t}_{\wp}(x_0)\right)$. Thus, increasing n reduces the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ regardless of which censoring scheme is implemented from the given three schemes PSC1, PSC2 and PSC4.

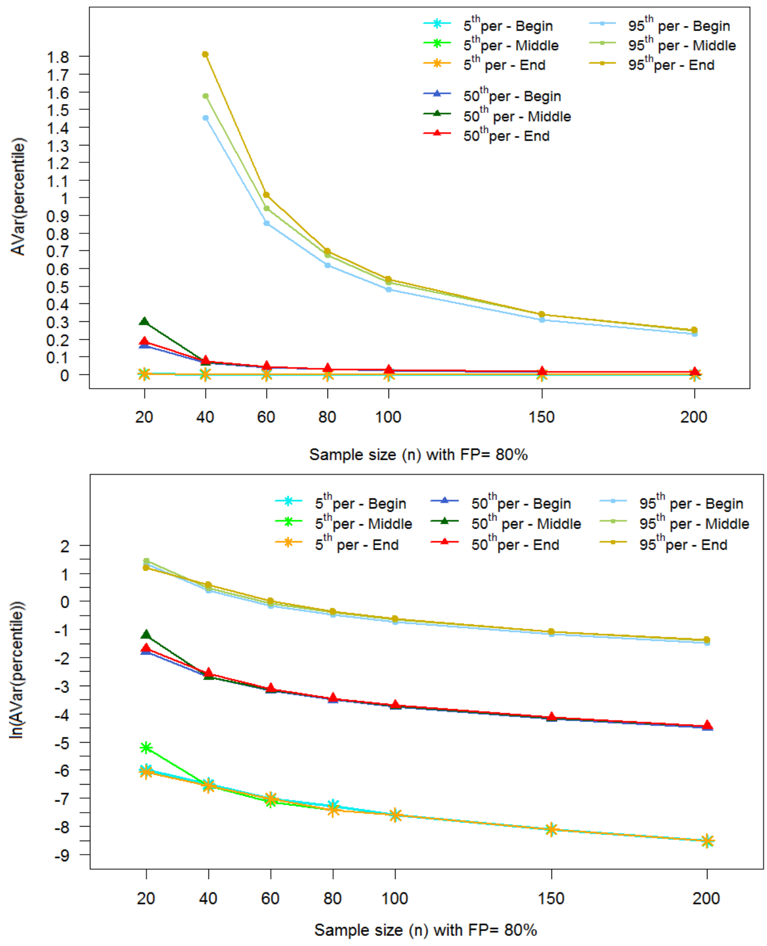


Figure 5.7 $AVar\left(\hat{t}_{\wp}(x_0)\right)$ and $\ln\left(AVar\left(\hat{t}_{\wp}(x_0)\right)\right)$ vs. sample size stratified by $\wp$ for three censoring schemes.

The standard error of the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is presented in Tables B.4 and B.5 in Appendix B to assess the precision of the estimated $AVar\left(\hat{t}_{\wp}(x_0)\right)$. The standard error of the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ are decreasing as the sample size increases for estimating the $5^{th}, 50^{th}$ and 95^{th} percentile. The standard error as a percentage of the estimated $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is less than 0.08 for all cases. The estimated $AVar\left(\hat{t}_{\wp}(x_0)\right)$ values are precise for $n \geq 40$ as the percentage the standard errors as a percentage of the estimated value are generally small < 0.022. When the sample size is small $n = 20$, the standard error percentage $= 0.087$. Thus, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is precisely estimated, consequently, the optimal values of the stress change time are precisely estimated and the precision of the optimal τ^* is improved when the sample size increases.

After studying the impact of increasing n, the impact of the observed sample size is of interest. The FP determines the number r of failures under the test. The value of $n = 40, 100$ with the FP ranged between 20 and 100 with increments of 20 is assumed. The parameter values are assumed to be $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and the data are censored in the middle of the test based on PCS2. Figure 5.8 illustrates the relationship between the FP and optimal τ^*. For more details, the results are tabulated in Table B.6 in Appendix B.

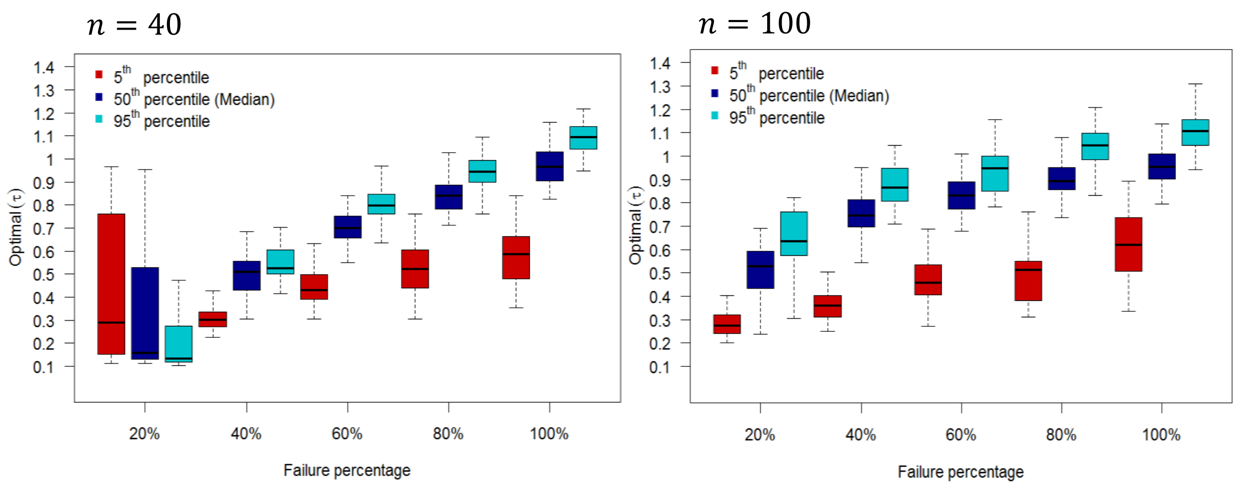


Figure 5.8 Boxplot of optimal τ^* vs. FP stratified by $\wp$ for $n = 40{,}100$.

As shown in Figure 5.8, for $n = 40$, the optimal τ^* increases as the FP increases. However, that the optimal τ^* only slightly increases if estimating $t_{0.05}(x_0)$ under the SSALTs is of interest. From Table B.6, it can be seen that the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is slightly changed with increasing FP, except for 20% FP, especially when $\wp = 0.05$. So, $FP = 40\%$ does not remarkably increase the value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$, whilst it reduces the cost of the experiment by censoring 60% of the items from the test. The impact of the FP on the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is illustrated in Figure 5.9. If estimating $t_{0.05}(x_0)$ of the GED under the SSALTs is of interest, the impact of $FP = 40\%$ on $AVar\left(\hat{t}_{0.05}(x_0)\right)$ is the same as the impact of the complete sample, with no censoring, on $AVar(\hat{t}_{0.05}(x_0))$. In this situation, to reduce the cost of the SSALT, FP= 40% is better to be suggested than FP= 100% of the complete sample case.

It is concluded that, under a specific censoring scheme, the precision of $\hat{t}_{\wp}(x_0)$ may be improved by increasing either the sample size or the FP.

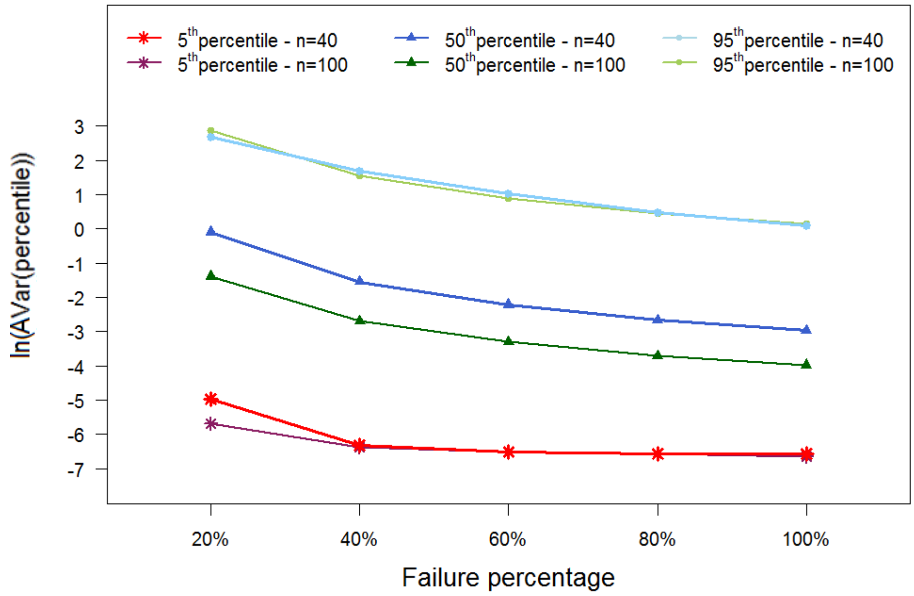


Figure 5.9 $\ln\left(AVar\left(\hat{t}_{\wp}(x_0)\right)\right)$ vs. FP stratified by $\wp$.

One of the interesting questions is "What is the impact of changing the censoring scheme on the optimal τ^*? ". So, Table 5.1 – Table 5.3 represent the optimal τ^* and its corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for $5^{th}, 50^{th}$ and 95^{th} percentiles under 10 various censoring schemes. The simulation study is done for $n = 20, 40, 60$ with 60% observed failures and the model parameters $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ are assumed.

What is interesting to note from Table 5.1 – Table 5.3 is that, as the censoring time shifts to the end of the test, the optimal τ^* decreases, and the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ increases, except for a few cases where items are censored at the beginning of the test. Similarly, the optimal τ^* decreases and the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ increases as items are censored at both the beginning and end of the test. These results hold whether $\frac{(n-r)}{2}$ items are censored after the first failure and $\frac{(n-r)}{2}$ items are censored after the last failure, or if $\frac{(n-r)}{2}$ items are censored one by one after the first $\frac{r}{3}$ failures and $\frac{(n-r)}{2}$ items are censored one by one after the last $\frac{r}{3}$ failures.

It is concluded that as several items are censored at the end of the test, the optimal τ^* decreases as the more reliable items are removed. However, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ increases because of the lack of information regarding the upper tail of the distribution under SSALT. This conclusion holds in all cases where either all items or a number of censored items are removed at the end of the test.

Table 5.1 The impact of changing the censoring scheme on optimal τ^* and its corresponding $AVar(\hat{t}_{0.05}(x_0))$ when FP=60%, and $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$.

			Optimal τ^*				**$AVar(\hat{t}_{0.05}(x_0))$**			
n	r	$\mathcal{R}$	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
20	12	$(8, 0^{*11})$	0.4328	0.4321	0.2060	0.6328	0.0041	0.0040	0.0037	0.0046
		$(0^{*5}, 8, 0^{*6})$	0.3986	0.3820	0.2264	0.7151	0.0038	0.0034	0.0032	0.0187
		$(0^{*11}, 8)$	0.3281	0.2812	0.2117	0.7921	0.0065	0.0036	0.0031	0.0536
		$(4, 0^{*10}, 4)$	0.3373	0.3179	0.2218	0.8055	0.0044	0.0038	0.0033	0.0216
		$(0^{*2}, 4, 0^{*6}, 4, 0^{*2})$	0.3531	0.3334	0.2353	0.8334	0.0039	0.0034	0.0031	0.0141
		$(1^{*8}, 0^{*4})$	0.3889	0.3755	0.2667	0.7312	0.0036	0.0035	0.0032	0.0055
		$(0^{*2}, 1^{*8}, 0^{*2})$	0.3768	0.3606	0.2413	0.9196	0.0042	0.0035	0.0031	0.0196
		$(0^{*4}, 1^{*8})$	0.3252	0.3082	0.2258	0.9421	0.0047	0.0036	0.0031	0.0452
		$(1^{*4}, 0^{*4}, 1^{*4})$	0.3712	0.3413	0.2450	1.2249	0.0051	0.0036	0.0032	0.0452
		$(0, 1^{*4}, 0^{*2}, 1^{*4}, 0)$	0.3849	0.3375	0.2390	0.9413	0.0046	0.0035	0.0032	0.0288
40	24	$(16, 0^{*23})$	0.5974	0.5906	0.3750	0.7671	0.0019	0.0019	0.0018	0.0020
		$(0^{*11}, 16, 0^{*12})$	0.4521	0.4545	0.3082	0.5816	0.0015	0.0015	0.0014	0.0016
		$(0^{*23}, 16)$	0.3598	0.3351	0.2654	0.8529	0.0033	0.0017	0.0016	0.0446
		$(8, 0^{*22}, 8)$	0.4183	0.4189	0.2662	0.7786	0.0019	0.0018	0.0017	0.0085
		$(0^{*5}, 8, 0^{*12}, 8, 0^{*5})$	0.4188	0.4073	0.2927	0.7906	0.0016	0.0016	0.0015	0.0027
		$(1^{*16}, 0^{*8})$	0.4944	0.4838	0.3472	0.6657	0.0016	0.0015	0.0015	0.0017
		$(0^{*4}, 1^{*16}, 0^{*4})$	0.4230	0.4161	0.3082	0.5501	0.0015	0.0015	0.0015	0.0017
		$(0^{*8}, 1^{*16})$	0.3749	0.3702	0.2631	0.5345	0.0016	0.0016	0.0015	0.0018
		$(1^{*8}, 0^{*8}, 1^{*8})$	0.4050	0.3927	0.2826	0.9259	0.0018	0.0016	0.0016	0.0122
		$(0^{*2}, 1^{*8}, 0^{*4}, 1^{*8}, 0^{*2})$	0.4189	0.4159	0.2974	0.7679	0.0016	0.0016	0.0015	0.0027
60	36	$(24, 0^{*35})$	0.6158	0.6260	0.4772	0.9003	0.0012	0.0012	0.0011	0.0012
		$(0^{*17}, 24, 0^{*18})$	0.4591	0.4674	0.3082	0.6973	0.0009	0.0009	0.0008	0.0010
		$(0^{*35}, 24)$	0.3876	0.3532	0.2505	0.9418	0.0032	0.0010	0.0010	0.0392
		$(12, 0^{*34}, 12)$	0.4339	0.4321	0.3081	0.5501	0.0011	0.0011	0.0010	0.0012
		$(0^{*9}, 12, 0^{*16}, 12, 0^{*9})$	0.4385	0.4443	0.305	0.6606	0.0009	0.0009	0.0009	0.0010
		$(1^{*24}, 0^{*12})$	0.5176	0.4870	0.3082	0.7665	0.0009	0.0009	0.0009	0.0010
		$(0^{*6}, 1^{*24}, 0^{*6})$	0.4518	0.4321	0.3021	0.6583	0.0009	0.0009	0.0009	0.0010
		$(0^{*12}, 1^{*24})$	0.3839	0.3746	0.2668	0.7320	0.0010	0.0009	0.0009	0.0017
		$(1^{*12}, 0^{*12}, 1^{*12})$	0.4240	0.4291	0.3096	0.5501	0.0010	0.0010	0.0009	0.0011
		$(0^{*3}, 1^{*12}, 0^{*6}, 1^{*12}, 0^{*3})$	0.4337	0.4320	0.3021	0.5561	0.0009	0.0009	0.0009	0.0010

Table 5.2 The impact of changing the censoring scheme on optimal τ^* and its corresponding $AVar(\hat{t}_{0.5}(x_0))$ when FP=60%, and $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$.

			Optimal τ^*				**$AVar(\hat{t}_{0.5}(x_0))$**			
n	r	$\mathcal{R}$	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
20	12	$(8, 0^{*11})$	0.6172	0.6212	0.3750	0.9416	0.2638	0.2627	0.2325	0.3462
		$(0^{*5}, 8, 0^{*6})$	0.5052	0.5021	0.3750	0.6583	0.2422	0.2411	0.2261	0.2801
		$(0^{*11}, 8)$	0.3264	0.3084	0.2257	0.6583	0.2834	0.2478	0.2274	1.2323
		$(4, 0^{*10}, 4)$	0.3959	0.3839	0.2404	0.7427	0.2759	0.2580	0.2377	0.6728
		$(0^{*2}, 4, 0^{*6}, 4, 0^{*2})$	0.4174	0.4261	0.2361	0.7094	0.2600	0.2520	0.2264	0.4350
		$(1^{*8}, 0^{*4})$	0.5612	0.5501	0.3337	0.9416	0.2662	0.2499	0.2238	0.5032
		$(0^{*2}, 1^{*8}, 0^{*2})$	0.4610	0.4776	0.2924	0.7409	0.2655	0.2579	0.2288	0.4389
		$(0^{*4}, 1^{*8})$	0.3863	0.3750	0.2255	0.8483	0.2998	0.2547	0.2366	1.3931
		$(1^{*4}, 0^{*4}, 1^{*4})$	0.3807	0.3641	0.2413	0.7920	0.2650	0.2531	0.2392	0.6634
		$(0, 1^{*4}, 0^{*2}, 1^{*4}, 0)$	0.4119	0.3878	0.2668	0.8589	0.2733	0.2486	0.2312	0.8765
40	24	$(16, 0^{*23})$	0.8304	0.8347	0.7642	1.1167	0.1036	0.1016	0.0960	0.1356
		$(0^{*11}, 16, 0^{*12})$	0.7217	0.7252	0.4871	0.9416	0.1119	0.1100	0.1025	0.1578
		$(0^{*23}, 16)$	0.4089	0.4024	0.3495	0.5246	0.1544	0.1401	0.1304	1.5972
		$(8, 0^{*22}, 8)$	0.5552	0.5501	0.4419	0.6592	0.1343	0.1227	0.1072	0.2975
		$(0^{*5}, 8, 0^{*12}, 8, 0^{*5})$	0.6473	0.6582	0.4953	0.7665	0.1174	0.1156	0.1071	0.1510
		$(1^{*16}, 0^{*8})$	0.7746	0.7672	0.6528	1.0085	0.1054	0.1047	0.0991	0.1215
		$(0^{*4}, 1^{*16}, 0^{*4})$	0.6559	0.6510	0.5599	0.7667	0.1110	0.1100	0.1022	0.1414
		$(0^{*8}, 1^{*16})$	0.5257	0.5295	0.4419	0.6012	0.1254	0.1181	0.1072	0.4257
		$(1^{*8}, 0^{*8}, 1^{*8})$	0.5841	0.5914	0.4675	0.6996	0.1239	0.1209	0.1092	0.1633
		$(0^{*2}, 1^{*8}, 0^{*4}, 1^{*8}, 0^{*2})$	0.6379	0.6349	0.5501	0.7665	0.1162	0.1148	0.1070	0.1391
60	36	$(24, 0^{*35})$	0.9811	0.9850	0.7665	1.1167	0.0586	0.0584	0.0550	0.0658
		$(0^{*17}, 24, 0^{*18})$	0.8129	0.8176	0.6328	0.9737	0.0656	0.0651	0.0616	0.0770
		$(0^{*35}, 24)$	0.4324	0.4332	0.3509	0.5185	0.0662	0.0792	0.0707	0.2345
		$(12, 0^{*34}, 12)$	0.5773	0.5829	0.4832	0.6592	0.0724	0.0684	0.0626	0.1595
		$(0^{*9}, 12, 0^{*16}, 12, 0^{*9})$	0.7193	0.7168	0.5501	0.8334	0.0648	0.0647	0.0587	0.0723
		$(1^{*24}, 0^{*12})$	0.8097	0.8055	0.6569	0.9514	0.0623	0.0623	0.0580	0.0674
		$(0^{*6}, 1^{*24}, 0^{*6})$	0.7202	0.7153	0.6071	0.8328	0.0650	0.0648	0.0619	0.0729
		$(0^{*12}, 1^{*24})$	0.5592	0.5561	0.4773	0.6583	0.0687	0.0680	0.0645	0.0953
		$(1^{*12}, 0^{*12}, 1^{*12})$	0.6220	0.6238	0.5381	0.6945	0.0689	0.0672	0.0633	0.1348
		$(0^{*3}, 1^{*12}, 0^{*6}, 1^{*12}, 0^{*3})$	0.6807	0.6837	0.5766	0.7679	0.0657	0.0656	0.0608	0.0722

Table 5.3 The impact of changing the censoring scheme on optimal τ^* and its corresponding $AVar(\hat{t}_{0.95}(x_0))$ when FP=60%, and $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$.

			Optimal τ^*				**$AVar(\hat{t}_{0.95}(x_0))$**			
n	r	$\mathcal{R}$	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
20	12	$(8, 0^{*11})$	0.6746	0.6692	0.2254	1.0085	6.2699	6.3073	5.0220	7.2551
		$(0^{*5}, 8, 0^{*6})$	0.4962	0.5060	0.2191	0.7832	6.2773	6.2000	5.7703	8.2750
		$(0^{*11}, 8)$	0.1616	0.1263	0.1013	0.5430	4.4938	4.1028	3.5148	12.0111
		$(4, 0^{*10}, 4)$	0.3378	0.3300	0.2111	0.6606	5.9219	5.7483	4.9540	12.2591
		$(0^{*2}, 4, 0^{*6}, 4, 0^{*2})$	0.3476	0.3082	0.2157	0.9259	6.1516	5.9623	5.2415	12.8614
		$(1^{*8}, 0^{*4})$	0.5477	0.5513	0.2413	0.8431	6.3360	6.2984	5.7069	7.3004
		$(0^{*2}, 1^{*8}, 0^{*2})$	0.4347	0.4230	0.2418	0.7665	6.3941	6.2794	5.8502	8.0077
		$(0^{*4}, 1^{*8})$	0.1907	0.1366	0.1034	0.6427	5.0562	4.6462	3.9332	12.6422
		$(1^{*4}, 0^{*4}, 1^{*4})$	0.2320	0.1568	0.1054	0.5201	5.2408	4.6801	3.9120	12.4179
		$(0, 1^{*4}, 0^{*2}, 1^{*4}, 0)$	0.2826	0.1568	0.1034	0.6564	5.2013	4.3391	3.9786	11.5821
40	24	$(16, 0^{*23})$	0.9744	0.9784	0.7798	1.1008	2.2630	2.2317	2.0258	2.8711
		$(0^{*11}, 16, 0^{*12})$	0.8408	0.8502	0.5256	1.0394	2.7560	2.7059	2.4046	3.5990
		$(0^{*23}, 16)$	0.4389	0.4444	0.3596	0.5580	3.9818	3.7211	3.3303	7.1570
		$(8, 0^{*22}, 8)$	0.5887	0.5818	0.4588	0.7007	3.2518	3.0956	2.7208	4.9375
		$(0^{*5}, 8, 0^{*12}, 8, 0^{*5})$	0.7348	0.7261	0.6267	0.9393	2.7810	2.7122	2.4167	3.8198
		$(1^{*16}, 0^{*8})$	0.8548	0.8522	0.6583	1.0085	2.5047	2.4703	2.3214	2.9223
		$(0^{*4}, 1^{*16}, 0^{*4})$	0.7273	0.7124	0.5849	0.9416	2.7526	2.7238	2.3479	3.3596
		$(0^{*8}, 1^{*16})$	0.5735	0.5758	0.4832	0.6583	3.0460	2.9693	2.6660	5.2682
		$(1^{*8}, 0^{*8}, 1^{*8})$	0.6235	0.6193	0.5167	0.7811	3.0350	2.8921	2.5971	5.0669
		$(0^{*2}, 1^{*8}, 0^{*4}, 1^{*8}, 0^{*2})$	0.6887	0.6772	0.5904	0.8334	2.7340	2.7194	2.4390	3.2460
60	36	$(24, 0^{*35})$	1.0315	1.0356	0.8640	1.1583	1.2364	1.2297	1.1655	1.4954
		$(0^{*17}, 24, 0^{*18})$	0.8902	0.8862	0.7214	1.0629	1.5090	1.5032	1.3882	1.7107
		$(0^{*35}, 24)$	0.4493	0.4464	0.4006	0.5088	1.9672	1.9571	1.7909	2.2383
		$(12, 0^{*34}, 12)$	0.6110	0.6165	0.5203	0.6861	1.6062	1.5867	1.4624	1.9975
		$(0^{*9}, 12, 0^{*16}, 12, 0^{*9})$	0.7837	0.7800	0.6491	0.9417	1.4486	1.4339	1.3387	1.6693
		$(1^{*24}, 0^{*12})$	0.9123	0.9081	0.7604	1.1167	1.3940	1.3867	1.3246	1.4956
		$(0^{*6}, 1^{*24}, 0^{*6})$	0.7956	0.8000	0.6425	0.9516	1.4785	1.4712	1.4064	1.6303
		$(0^{*12}, 1^{*24})$	0.6185	0.6194	0.5256	0.6819	1.6046	1.5722	1.4594	2.2917
		$(1^{*12}, 0^{*12}, 1^{*12})$	0.6725	0.6740	0.5725	0.7790	1.5831	1.5255	1.3846	2.3705
		$(0^{*3}, 1^{*12}, 0^{*6}, 1^{*12}, 0^{*3})$	0.7225	0.7252	0.6486	0.8334	1.4788	1.4657	1.3733	1.6473

Next, the effect of changing θ_1 and θ_2 on the optimal τ^* is studied. The fixed initial values of $n = 80$ with FP= 80% are assumed. The censored items are removed at the middle points of the test, subject to the censoring scheme $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$. Also, $\alpha = 1.2$ is assumed. In the case of testing the impact of θ_1, $\theta_2 = 0.3$ is assumed and θ_1 is tested for the values $\theta_1 = 0.6, 0.8, 1.0, 1.2, 1.5, 1.8, 2.0, 2.5$. On the other hand, in the case of testing the impact of θ_2, $\theta_1 = 0.6$ is assumed and θ_2 is tested for the values $\theta_2 = 0.1, 0.2, 0.3, 0.4, 0.5$.

The results are presented in Figure 5.10 and Figure 5.11 and in Tables B.7 and B.8 in Appendix B.

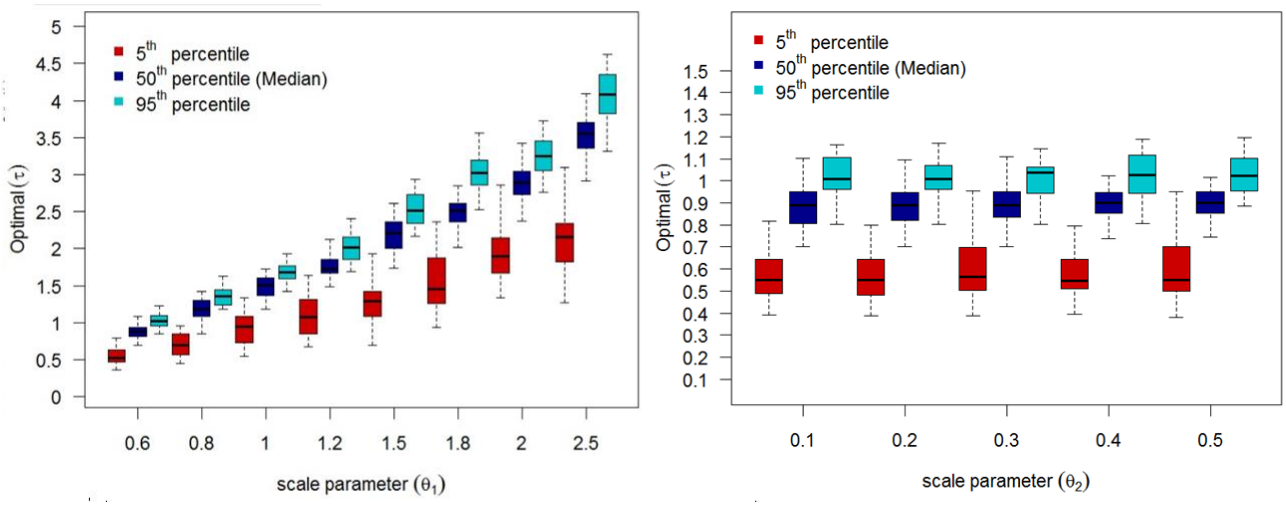


Figure 5.10 Boxplot of optimal τ^* vs. scale parameters θ_1 and θ_2 stratified by $\wp$.

Figure 5.10 shows the value of the optimal τ^* as a function of θ_1 and θ_2 for three different percentiles of interest: $5^{th}, 50^{th}$ and 95^{th}. From Figure 5.10 and Tables B.7 and B.8, it is obvious that the value of θ_1 has a noticeable effect on the optimal τ^*. For estimating the $5^{th}, 50^{th}$ and 95^{th} percentile, it is observed that as θ_1 increases, the estimated optimal τ^* and the optimal τ^* sample range are gradually increasing. In contrast, θ_2 has a negligible effect on the optimal τ^*. This result may relate to the conclusion obtained from Section 4.6.2 that the number of failures in each stress level only slightly impacts the estimate of θ_2. So, changing θ_2 does not require changing the value of τ to get the most precise estimate of θ_2.

The scale parameter represents the characteristic life of a test unit under any stress level (Nelson 1990). The scale parameter has a negative relationship with the stress level. So, as θ_1 increases, the stress level decreases and becomes closer to the usage stress level. Therefore, the items take a longer time to fail. As a result, the stress change time τ should be increased when θ_1 increases to get a large number of failures at a low stress level.

Regarding the impact of changing θ_1 and θ_2 on $AVar\left(\hat{t}_\wp(x_0)\right)$ calculated at the optimal τ^*, Figure 5.11 shows $\ln\left(AVar\left(\hat{t}_\wp(x_0)\right)\right)$ versus θ_1 and θ_2. The impact of increasing the value of θ_1 and θ_2 on the value of $AVar\left(\hat{t}_\wp(x_0)\right)$ is the same for the three percentile values.

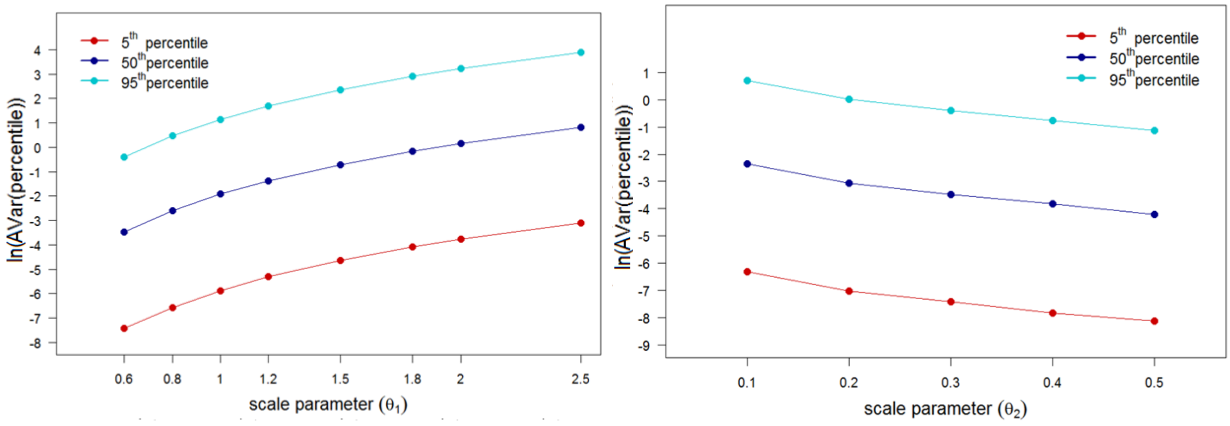


Figure 5.11 $\ln\left(AVar\left(\hat{t}_{\wp}(x_0)\right)\right)$ vs. θ_1 and θ_2 stratified by $\wp$.

For a fixed value of $\theta_2 = 0.3$, the value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$ increases, and its range slightly increases as θ_1 is increasing; see Tables B.7 and B.8 in Appendix B:

$$AVar\big(\hat{t}_{\wp}(x_0)|\theta_1 = 0.6\big) < AVar\big(\hat{t}_{\wp}(x_0)|\theta_1 = 0.8\big) < \cdots < AVar\big(\hat{t}_{\wp}(x_0)|\theta_1 = 2.5\big).$$

This increase in $AVar\left(\hat{t}_{\wp}(x_0)\right)$ may be due to increasing the difference between θ_1 and θ_2, which indicates increasing the difference between lower and higher stress levels.

On the other hand, for a fixed value of $\theta_1 = 0.6$, the value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$ decreases, and its range slightly decreases as θ_2 increases:

$$AVar\big(\hat{t}_{\wp}(x_0)|\theta_2 = 0.1\big) > AVar\big(\hat{t}_{\wp}(x_0)|\theta_2 = 0.2\big) > \cdots > AVar\big(\hat{t}_{\wp}(x_0)|\theta_2 = 0.5\big).$$

For both θ_1 and θ_2, the increasing of $AVar\left(\hat{t}_{\wp}(x_0)\right)$ and its range is greater as $\wp$ increases.

$$AVar\big(\hat{t}_{0.95}(x_0)\big) > AVar\big(\hat{t}_{0.5}(x_0)\big) > AVar\big(\hat{t}_{0.05}(x_0)\big)\,.$$

From these results, the question of the impact of changing θ_1 and θ_2 on the optimal τ^* is of interest. As the scale parameters have a log-linear relationship with the stress level, so the impact of their values is of interest. The following table includes the results of studying the impact of different sets of scale parameters to study their effects on optimal τ^*. The simulation study is done for $n = 80$ with 80% observed failures and the censoring scheme $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$. Also, $\alpha = 1.2$ is assumed.

Table 5.4 The impact of changing θ_1 and θ_2 on optimal τ^* and its corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for $100\wp^{th}$percentile when $n = 80$, FP=80%, $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$ at $\alpha = 1.2$

				Optimal τ^*				**$AVar\left(\hat{t}_{\wp}(x_0)\right)$**			
$\wp$	**θ_1**	**θ_2**	**θ_2/θ_1**	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
0.05	0.5	0.4	0.8	0.5347	0.5411	0.3927	0.7151	0.0002	0.0002	0.0002	0.0002
	0.6	0.3	0.5	0.5806	0.5533	0.3980	0.9493	0.0006	0.0006	0.0005	0.0007
	0.7	0.2	0.29	0.6630	0.6720	0.4447	0.9493	0.0014	0.0014	0.0014	0.0015
	0.8	0.7	0.88	0.7762	0.7377	0.5118	1.3506	0.0006	0.0006	0.0006	0.0007
	0.9	0.6	0.67	0.8533	0.8403	0.5472	1.3506	0.0010	0.0010	0.0010	0.0011
	1.0	0.5	0.5	0.8943	0.8816	0.5427	1.3559	0.0017	0.0017	0.0016	0.0018
	1.2	0.8	0.67	1.0643	1.0595	0.6927	1.5986	0.0018	0.0018	0.0017	0.0019
	1.5	0.5	0.33	1.2300	1.2266	0.7013	1.7507	0.0058	0.0058	0.0056	0.0060
0.50	0.5	0.4	0.8	0.7628	0.7609	0.5533	0.9493	0.0112	0.0112	0.0106	0.0125
	0.6	0.3	0.5	0.8750	0.8658	0.7013	1.1026	0.0311	0.0310	0.0295	0.0338
	0.7	0.2	0.29	1.0310	1.0345	0.8184	1.1973	0.0749	0.0746	0.0719	0.0795
	0.8	0.7	0.88	1.2085	1.2249	0.9485	1.3870	0.0241	0.0241	0.0222	0.0254
	0.9	0.6	0.67	1.3434	1.3331	1.0248	1.5782	0.0492	0.0492	0.0472	0.0529
	1.0	0.5	0.5	1.4586	1.4536	1.2059	1.7527	0.0863	0.0864	0.0831	0.0901
	1.2	0.8	0.67	1.7692	1.7706	1.4787	2.1965	0.0879	0.0877	0.0828	0.0924
	1.5	0.5	0.33	2.0676	2.0496	1.7097	2.3885	0.2953	0.2946	0.2844	0.3121
0.95	0.5	0.4	0.8	0.8851	0.8917	0.7012	1.0440	0.2439	0.2423	0.2336	0.2684
	0.6	0.3	0.5	1.0075	1.0141	0.7918	1.2635	0.6732	0.6733	0.6408	0.7175
	0.7	0.2	0.29	1.1675	1.1665	0.9501	1.3858	1.6135	1.6112	1.5368	1.7109
	0.8	0.7	0.88	1.3732	1.3672	1.0485	1.6124	0.5259	0.5226	0.4983	0.5722
	0.9	0.6	0.67	1.5527	1.5738	1.2855	1.8986	1.0619	1.0621	0.9898	1.1288
	1.0	0.5	0.5	1.7000	1.7003	1.3689	1.9888	1.8664	1.8647	1.7569	1.9721
	1.2	0.8	0.67	2.0461	2.0861	1.6644	2.3239	1.8801	1.8820	1.7974	1.9845
	1.5	0.5	0.33	2.5456	2.5884	2.1249	2.8124	6.3424	6.3357	6.1135	6.6366

From the above table, it can be noticed that as the difference between the values of θ_1 and θ_2 increase, the value of optimal stress change τ^* and its corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ increase for different $100\wp^{th}$percentiles. In fact, increasing the difference between θ_1 and θ_2 indicates increasing the difference between the two stress levels under the test. However, for the same ratio of θ_1 and θ_2, as the two values of θ_1 and θ_2 increase, the value of optimal τ^* and its corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ increase.

Similar to the effects of changing the scale parameters on the optimal τ^*, the effect of changing the shape parameter α is presented in Figure 5.12. The set of initial values is assumed as $n = 80$, FP=80%, $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$ with $\theta_1 = 0.6$ and $\theta_2 = 0.3$. The results are numerically presented in Table B.9 in Appendix B.

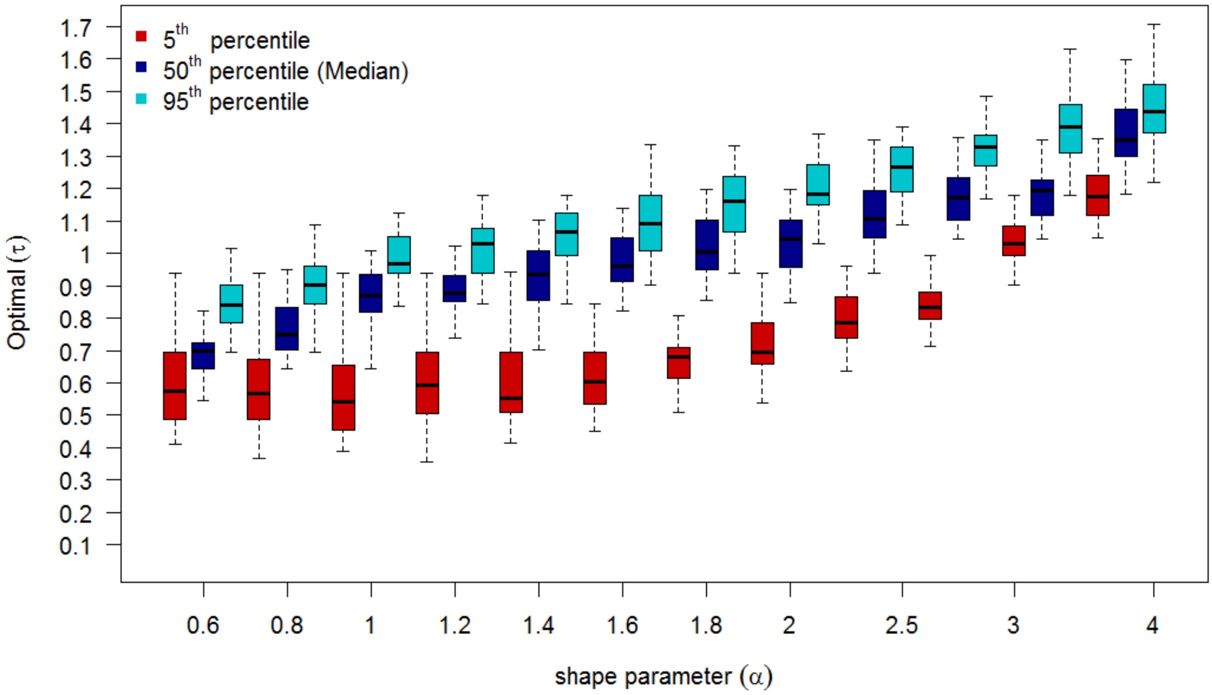


Figure 5.12 Boxplot of optimal τ^* vs. shape parameter α stratified by $\wp$.

From Figure 5.12 and Table B.9, it can be seen that for $0.6 < \alpha \leq 1.0$, the optimal τ^* that corresponds to the minimum $AVar(\hat{t}_{0.05}(x_0))$ slightly decreases. Whereas the optimal τ^* slightly increases when estimating the $50^{th}$and 95^{th}percentile. For $1.0 < \alpha \leq 1.6$, the optimal τ^* is reasonably constant with respect to α. Moreover, when $\alpha > 1.6$, the optimal τ^* that minimizes $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is increasing with increasing values of α. From Figure 3.1, it can be seen that the density curve becomes more skewed for increasing α, so the value of the median would increase. Thus, the time under the lower stress level is increasing to get a large number of failures.

The following Figure 5.13 shows the impact of α on $AVar\left(\hat{t}_{\wp}(x_0)\right)$ that corresponds to optimal τ^*. The values of $AVar(\hat{t}_{0.05}(x_0))$ and $AVar(\hat{t}_{0.5}(x_0))$ slightly increase as α increases. Whereas the $AVar(\hat{t}_{0.95}(x_0))$ and its range are stable for all different values of α. In other words, if the objective of the V-optimality is to minimize $AVar(\hat{t}_{0.95}(x_0))$, then the choice of the stress change time would not affect the precision of $\hat{t}_{\wp}(x_0)$.

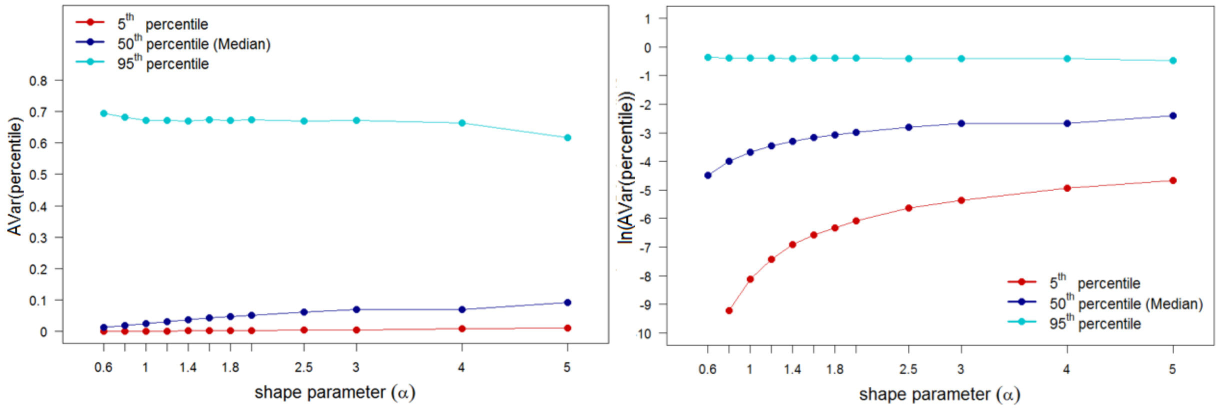


Figure 5.13 Variance and $\ln\left(AVar\left(\hat{t}_{\wp}(x_0)\right)\right)$ vs. shape parameter α stratified by $\wp$.

5.3.3 Sensitivity Analysis

In a SSALT design, estimation of the optimal τ^* depends on the initial estimates of the model parameters θ_1, θ_2 and α. Incorrect estimation of the model parameters gives a poor estimate of $t_{\wp}(x_0)$ under the GED. A sensitivity analysis is performed to investigate the robustness of the optimal τ^* to the model parameters. Consequently, a parameter that has a large impact on optimal τ^* must be estimated with special care. The importance of the sensitivity analysis is to avoid erroneously estimating the optimal τ^*. Robustness will be investigated by considering the effect of parameter misspecification. However, the proposed SSALTs plan is robust if a small change in θ_1, θ_2 and α has no or small impact on the optimal stress change time τ^*.

From the results in Tables B.1 – B.5 and Figure 5.6, it can be seen that the range of changing $AVar\left(\hat{t}_{\wp}(x_0)\right)$ has become smaller as the sample size increases, whilst the range of optimal τ^* is unchanged. Thus, the optimal τ^* is robust for all given sample size values except $n = 20$.

To examine the robustness of the optimal τ^* against the change of the initial estimates of model parameters θ_1, θ_2 and α, a sensitivity analysis is performed. For fixed $n = 80$, FP = 80%, $\alpha = 1.2$ and the censoring scheme $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$, the following Figure 5.14 – Figure 5.16 show the result of the sensitivity analysis for the model parameters θ_1, θ_2 and α.

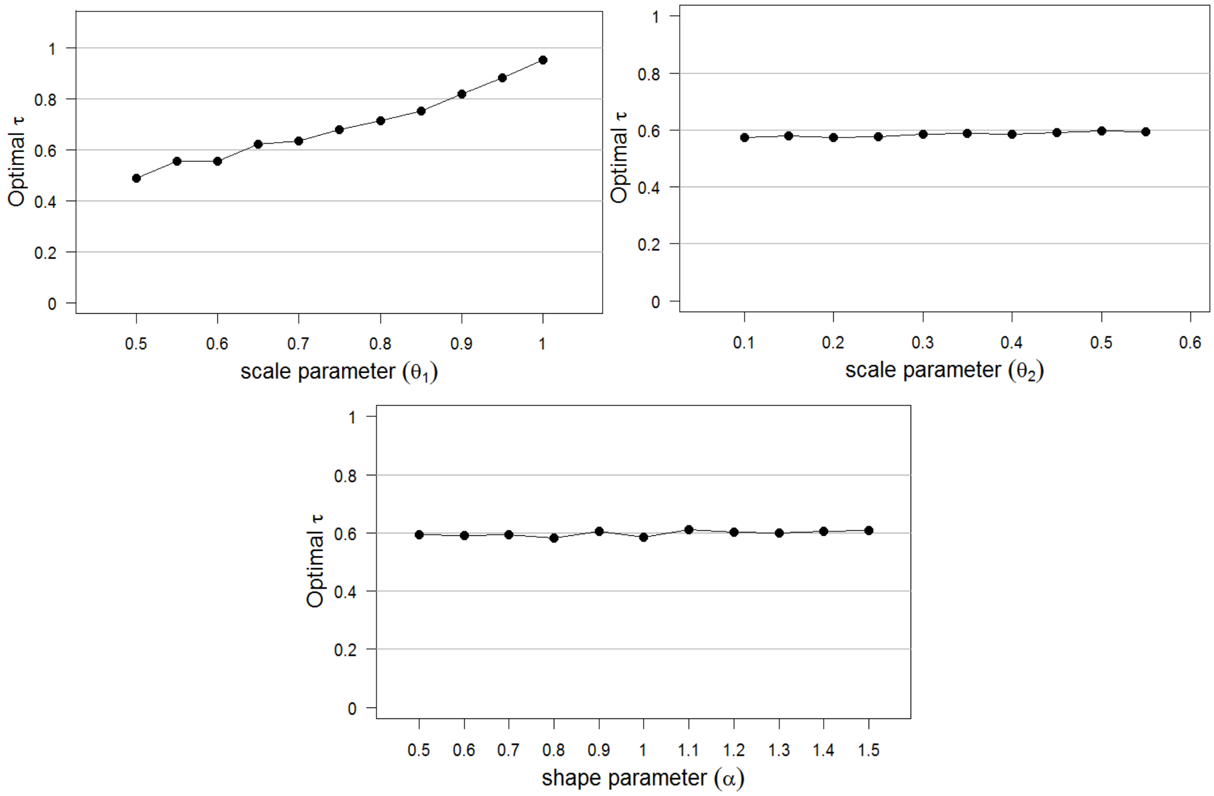


Figure 5.14 Optimal τ^* vs. changes in θ_1, θ_2, α for 5^{th} percentile.

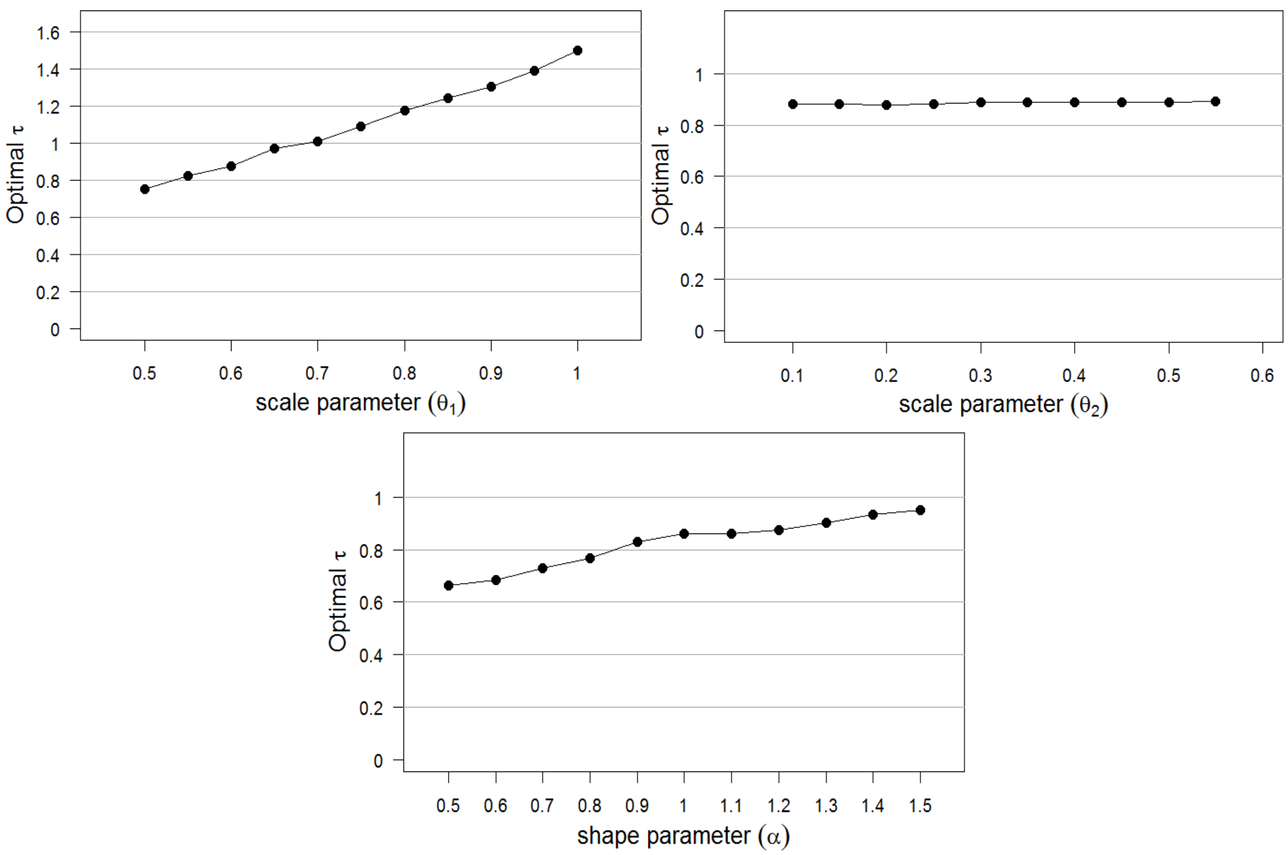


Figure 5.15 Optimal τ^* vs. changes in θ_1, θ_2, α for 50^{th} percentile.

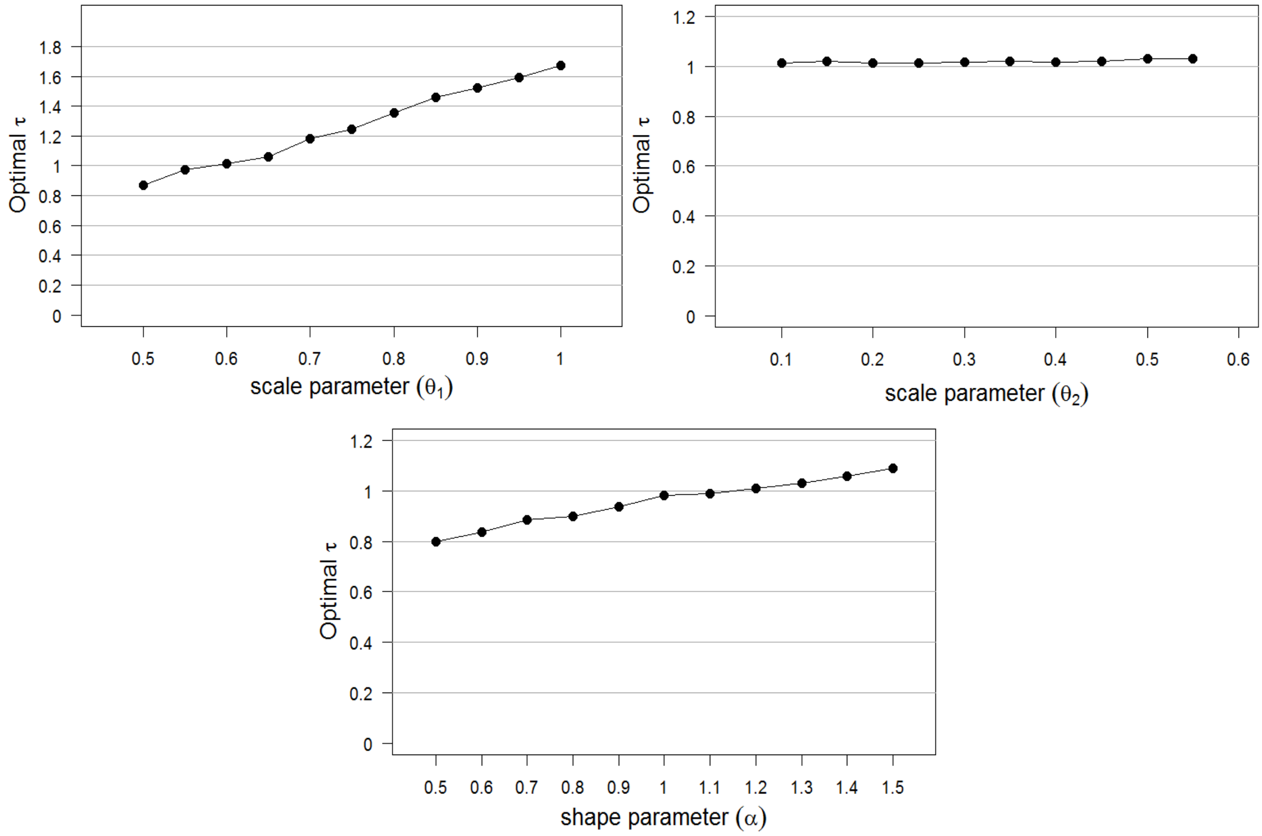


Figure 5.16 Optimal τ^* vs. changes in θ_1, θ_2, α for 95^{th} percentile.

From Figure 5.14 – Figure 5.16, it is observed that:

1- The value of θ_2 has no impact on the estimate of optimal τ^*. Thus, optimal τ^* is robust for the value of θ_2.

2- For changing the value of α, the optimal τ^* becomes more robust for $1 \leq \alpha \leq 1.5$ in the case of estimating the 50^{th} and 95^{th} percentile. On the other hand, it is obvious that in the case of estimating $t_{0.05}(x_0)$ the optimal τ^* is robust for all given range of $0.5 \leq \alpha \leq 1.5$.

3- The optimal τ^* is more sensitive to changing the value of θ_1 than to changing α. Noticeably, when the objective function is to minimise the $AVar\left(\hat{t}_{0.05}(x_0)\right)$, the optimal τ^* is less sensitive to θ_1 when θ_1 is close to θ_2. This implies that the optimal τ^* becomes robust when the two stress levels are close.

These observations indicate that the MLE of θ_2 does not affect the value of optimal τ^*. In contrast, θ_1 should be estimated with special care to obtain the precise optimal τ^* that minimizes the $AVar\left(\hat{t}_{p}(x_0)\right)$.

5.4 Optimal Censoring Scheme

In the ALT plan, the sample size and FP are usually fixed and can be determined by the experimenter based on a budget of the ALT and the availability of the test facilities. However, under progressive Type-II censoring, one may wonder when each item should be censored from the test with minimal effect on the precision of the parameter estimation. In fact, some questions exist in the design of the

ALTs, such as "What is the best scheme to be tested amongst all the various schemes with given sample size and failure percentage of items?". Another question is, "What is the efficiency of the optimal censoring scheme $\mathcal{R}^*$ for SSALTs under progressive Type-II censoring schemes compared with the basic Type-II censoring scheme and a complete sample situation?". This part of the thesis is organized to answer these questions. However, the issue of determining the optimal $\mathcal{R}^*$ is substantial in designing the ALT experiment as there are an enormous number of distinct censoring schemes. For example, given the sample size $n = 20$ and the FP $= 60\%$ (number of failures $r = 12$), there are 75,582 ways to remove data from the test under the progressive Type-II censoring scheme. This number considerably increases as the sample size or FP increases. Obviously, different censoring schemes have varying impacts on the precision of the MLEs of the model parameters. Therefore, it is crucial to choose the optimal $\mathcal{R}^*$ that achieves the purpose of using the progressive Type-II censoring strategy, which aims to reduce the time, effort and cost of the ALT. One of the benefits of obtaining the optimal $\mathcal{R}^*$ over all censoring schemes for small sample size is that one can do a pilot study with the optimal initials to get the most accurate estimates. These estimates will be used as an initial value in the main experiment. So, this results in producing the most efficient estimates for the parameters of interest. A detailed test design, including determining the optimal stress change time and the optimal $\mathcal{R}^*$, should be designed before conducting the ALTs to accurately estimate the reliability measures at the usage stress level.

Removing items at the beginning of the test is preferable as it guarantees the experimenter will get the most reliable products that can be used for other purposes. On the other hand, removing all the items after the first failure may lead to a lack of information about the lower or higher stress level, particularly with small sample size. Also, removing all censored items at the end of the test may lead to a lack of information about the higher stress level. Furthermore, the censored items can be used in many ways, such as to examine the mechanism of the product's internal parts after working for some time, or the internal parts can be used as a replacement in other products. Although removing the items early from the life test is beneficial in terms of cost, it may reduce the information about the products obtained from the experiment. So, determining the optimal $\mathcal{R}^*$ helps the experimenter balance between design efficiency, time spent by testing facilities and total test time, which strongly relates to the cost of the experiment. Thus, an optimization criterion is used in this thesis to obtain the optimal $\mathcal{R}^*$. The optimal censoring scheme is investigated for $100\wp^{th}$percentiles of the GED lifetime distribution under progressive Type-II censoring. In Section 5.4.1, the detailed steps of the numerical analysis are presented. Then, extensive simulation studies are carried out in Section 5.4.2 to obtain the optimal $\mathcal{R}^*$ and compare different censoring schemes for different sample sizes, FP, τ and model parameters. Also, a comparison between the optimal $\mathcal{R}^*$ and the other schemes will be studied in Section 5.4.2.1 to examine the sensitivity of the censoring scheme regarding an erroneous estimate of model parameters for small sample size.

5.4.1 Simulation Study

In ALTs based on progressive Type-II censored data, one may wonder when each item should be censored from the test with the least impact on the precision of the estimation of the parameters. On the other hand, are the optimal values inflexible to choose from, or is there a flexible range of values with similar efficiency. Thus, this section focuses on numerically investigating the issue of selecting the optimal $\mathcal{R}^*$ that satisfies a considerable decrease in the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ of the GED. The V-optimality criterion discussed in Section 5.2 is used.

The optimization technique minimizes the objective function $AVar\left(\hat{t}_{\wp}(x_0)\right)$ with respect to every possible censoring scheme $\mathcal{R} = (R_1, R_2, \dots R_r)$. Note that the optimization is a discrete optimization issue (Pradhan and Kundu, 2009). Thus, the concept of obtaining the optimal $\mathcal{R}^*$ relies on the selection of the best set of censoring schemes $\mathcal{R} = (R_1, R_2, \dots R_r)$ within all alternative censoring schemes, such that:

$$\sum_{i=1}^{r} R_i = n - r$$

The optimal scheme is the one that minimizes the objective function by providing the maximum information about the product from the observed sample. Even though the total number of all possible censoring schemes is finite, it might be reasonably numerous, especially with increasing n and FP. There are $\binom{n-1}{r-1}$ censoring schemes for fixed n and a fixed number of failures r (Pradhan and Kundu, 2009). For example, for $n = 25$ and FP $= 60\%$, there are 1,961,256 different ways to remove data from the test under the progressive Type-II censoring scheme. Choosing one of these schemes that minimize the objective function $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is essential. However, this number is considerably large. Thus, it is impossible to test and compare the impact of all these schemes as it requires an extremely long time to calculate the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ under each of $\binom{n-1}{r-1}$ censoring scheme numerically. Therefore, in this thesis, two cases regarding sample size are investigated. In the first case, for large $n > 20$, 11 different censoring schemes are suggested, which describe 11 different scenarios of removing the items from the ALT. The results and discussion are presented in Section 5.4.2.1. In the second case, all possible censoring schemes are tested for small $n \leq 20$ to obtain the optimal $\mathcal{R}^*$ and compare all schemes with the optimal one. The results and discussion are presented in Section 5.4.2.2.

The 11 different censoring schemes that have been tested in this thesis are given below:

$$\mathcal{R}1 = \left((n-r), 0^{*(r-1)}\right),$$

where all censored items $(n - r)$ are removed at the time of the first failure. This scheme is the reverse of the Type-II censoring scheme. In this thesis, this censoring scheme will be called the Rev-TII censoring scheme.

$$\mathcal{R}2 = \begin{cases} \left(0^{*\left(\frac{r}{2}\right)}, R_{\left[\left(\frac{r}{2}\right)+1\right]} = (n-r), 0^{\left(\left(\frac{r}{2}\right)-1\right)}\right), & \text{if } r \text{ is even} \\ \left(0^{*\left(\frac{(r-1)}{2}\right)}, R_{\left[\frac{(r+1)}{2}\right]} = (n-r), 0^{*\left(\frac{(r-1)}{2}\right)}\right), & \text{if } r \text{ is odd} \end{cases},$$

where all censored items $(n-r)$ are removed at the time of the failure occurrence in the middle of the test.

$$\mathcal{R}3 = \left(0^{*(r-1)}, (n-r)\right),$$

where all censored items $(n-r)$ are removed from the test at the time of the occurrence of the final observed failure. This scheme presents the basic Type-II censoring scheme.

$$\mathcal{R}4 = \begin{cases} \left(R_1 = \frac{(n-r)}{2}, 0^{*(r-2)}, R_r = \frac{(n-r)}{2}\right) & \text{if } r \text{ is even} \\ \left(R_1 = \frac{(n-r-1)}{2}, 0^{*(r-2)}, R_r = \frac{(n-r+1)}{2}\right) & \text{if } r \text{ is odd} \end{cases}.$$

This scheme is the combination of type-II $\mathcal{R}3$ and its opposite $\mathcal{R}1$. Half of censored items $\left(\frac{n-r}{2}\right)$ are removed at the time of the first failure and the other half are removed at the end of the test after the last failure. This scheme achieves both benefit of reducing the test time with removing the censored items while reliability is good.

$$\mathcal{R}5 = \begin{cases} \left(0^{*\approx q1}, R_{r*25/100} = \frac{(n-r)}{2}, 0^{*\approx q2}, R_{r*75/100} = \frac{(n-r)}{2}, 0^{*\approx q1}\right) & \text{if } r \text{ is even} \\ \left(0^{*\approx q1}, R_{r*25/100} = \frac{(n-r-1)}{2}, 0^{*\approx q2}, R_{r*75/100} = \frac{(n-r+1)}{2}, 0^{*\approx q1}\right) & \text{if } r \text{ is odd} \end{cases},$$

where $q1 = (r-2) \times 25/100$ and $q2 = (r-2) \times 50/100$.

This scheme is like scheme 4, but the two removing times are located after 25% of failures and after 75% of failures.

The subsequent 5 censoring schemes $\mathcal{R}6 - \mathcal{R}10$ follow the same scenario as the above 5 schemes $\mathcal{R}1 - \mathcal{R}5$, but the items are removed one by one instead of removing all items at once.

$$\mathcal{R}6 = \left(1^{*(n-r)}, 0^{*(2r-n)}\right).$$

$$\mathcal{R}7 = 0^{*\left(\frac{2r-n}{2}\right)}, 1^{*(n-r)}, 0^{*\left(\frac{2r-n}{2}\right)}$$

$$\mathcal{R}8 = \left(0^{*(2r-n)}, 1^{*(n-r)}\right),$$

$$\mathcal{R}9 = \begin{cases} \left(1^{*\frac{(n-r)}{2}}, 0^{*(2r-n)}, 1^{*\frac{(n-r)}{2}}\right) & \text{if } r \text{ is even} \\ \left(1^{*\frac{(n-r-1)}{2}}, 0^{*(2r-n)}, 1^{*\frac{(n-r+1)}{2}}\right) & \text{if } r \text{ is odd} \end{cases}.$$

$$\mathcal{R}10 = \begin{cases} \left(0^{*\approx q1}, 1^{*\frac{(n-r)}{2}}, 0^{*\approx q2}, 1^{*\frac{(n-r)}{2}}, 0^{*\approx q1}\right) & \text{if } r \text{ is even} \\ \left(0^{*\approx q1}, 1^{*\frac{(n-r-1)}{2}}, 0^{*\approx q2}, 1^{*\frac{(n-r+1)}{2}}, 0^{*\approx q1}\right) & \text{if } r \text{ is odd} \end{cases}.$$

where $q1 = (2r-n) \times 25/100$ and $q2 = (2r-n) \times 50/100$.

The last censoring scheme, $\mathcal{R}11 = (0^{*r})$, is for the complete sample case with no censored items. All n items are observed in the test $n = r$.

Obviously, as the observed sample size increases (with reducing FP or increasing n or both), the precision of the MLE will increase. However, the experiment will be run for a longer time. In fact, both $AVar\left(\hat{t}_{\wp}(x_0)\right)$ and T depend on the selected censoring scheme. Therefore, it is required to compare the total test time with $AVar$ to determine the proper censoring scheme that balances minimizing the precision of the estimator and reducing the total test time.

A numerical study provided in this section is to investigate the optimal $\mathcal{R}^*$ among $\mathcal{R}1 - \mathcal{R}10$ schemes. The information obtained under various schemes can be measured by the precision of the MLEs of $t_{\wp}(x_0)$ of GED. So, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is compared based on the suggested 11 censoring schemes for the 5^{th}, 50^{th} and 95^{th} percentile life estimates of GED. Considering the V-optimality criterion, the worst censoring scheme corresponding to the maximum $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is determined. The relative efficiency (RE) of the optimal $\mathcal{R}^*$ with respect to the worst censoring scheme, the basic Type-II censoring scheme and the complete sample is calculated. In addition, the relative expected time (RT) of the expected total test time is also calculated to compare the test time under the optimal $\mathcal{R}^*$ and the other censoring schemes. It may be noted that for fixed n and r, the $\mathcal{R}3$ scheme which is the Type-II censoring scheme has the shortest T among all schemes considered. In contrast, the $\mathcal{R}1$ scheme has the longest T. The other censoring schemes are likely to have a T between these two extreme values (Pareek et al., 2009). The RE and RT are used to investigate the differences between the censoring schemes and the optimal ones. This comparison may help the experimenter choose between the optimal $\mathcal{R}^*$ or a scheme that is practically more convenient for ALT design.

Moreover, the impact of changing n, FP, τ and the model parameters $\theta_1, \theta_2, \alpha$ on the censoring schemes will be studied. The algorithm for figuring out the optimal $\mathcal{R}^*$ is similar to the algorithm for estimating the optimal stress change time τ^* discussed in Section 5.3.1.1. Based on an example from Nelson (1990) of 76 times (in minutes) to oil breakdown of an insulating fluid, the values of stress levels are $x_0 = 20$ KV, $x_1 = 26$ KV, $x_2 = 38$ KV for the usage, lower and higher stress, respectively.

The following steps describe the simulation procedure for obtaining the optimal $\mathcal{R}^*$ that minimizes the $AVar\left(\hat{t}_{\wp}(x_0)\right)$, with different sets of initial values.

Step (1): Based on a selected censoring scheme, we generate progressively Type-II censored samples from GED for different choices of n, FP, τ, $\theta_1, \theta_2, \alpha$. The MLEs $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$ are calculated using the algorithm presented in Section 4.6.1.1.

Step (2): the MLE of the scale parameter under the usage stress level is

$\hat{\theta}_0 = e^{(\hat{\beta}_0 + \hat{\beta}_1 x_0)}$, where

$$\hat{\beta}_1 = \frac{ln(\hat{\theta}_1) - ln(\hat{\theta}_2)}{(x_1 - x_2)} \ .$$

$$\hat{\beta}_0 = ln(\hat{\theta}_1) - \hat{\beta}_1 x_1.$$

Step (3): Using (5.1), the MLE of the $100\wp^{th}$ percentile of the life distribution at usage stress level is obtained as follows:

$$\hat{t}_{\wp}(x_0) = -\hat{\theta}_0 \ln\left(1 - \wp^{\frac{1}{\hat{\alpha}}}\right), \qquad \wp = 0.05, 0.50, 0.95.$$

In addition, the expected total test time is obtained as $T = \mathrm{t_r}$.

Step (4): Using a loop, repeat the above 3 steps, 3000 times to get a sample of the MLE $\hat{t}_{\wp}(x_0)$.

Step (5): Find the empirical variance of the $100\wp^{th}$ percentile.

Step (6): Repeat the above 5 steps 11 times to calculate the corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for each of the 11 censoring schemes $\mathcal{R}1 - \mathcal{R}11$.

Step (7): Find the optimal and the worst censoring schemes, with their corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ and T. In addition, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ and T that correspond to the Type-II censoring scheme and complete sample will be calculated.

Step (8): Calculate the RE and RT, as percentages, of the optimal $\mathcal{R}^*$ with respect to other censoring schemes as follows

$$\mathrm{RE}(\mathcal{R}^*, R_{suggested}) = \frac{AVar(\hat{t}_{\wp}(x_0)_{|\mathcal{R}^*})}{AVar\left(\hat{t}_{\wp}(x_0)_{\ |R_{suggested}}\right)} \times 100, \tag{5.5}$$

$$\mathrm{RT}(\mathcal{R}^*, R_{suggested}) = \frac{T_{|\mathcal{R}^*}}{T_{|R_{suggested}}} \times 100\ . \tag{5.6}$$

Simulation studies have been performed using the above algorithm for calculating the MLEs of three percentiles with probability $\wp = 0.05, 0.50, 0.95$.

Similarly to the simulation studies performed in the R program for estimating the optimal τ^*, simulation studies for determining the optimal $\mathcal{R}^*$ are performed in the R program version 3.5.1 to solve the optimal design problem numerically. Also, all the computations are performed at the Iridis5 facilities on the University of Southampton high performance computing.

5.4.2 Numerical Results and Concluding Remarks

In this section, the optimal censoring scheme is numerically provided based upon selected initial values. The optimal $\mathcal{R}^*$ is obtained as it minimizes the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for different $\wp = 0.05, 0.50, 0.95$. For a given n, FP, τ, θ_1, θ_2 and α, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$, T, RE and RT are calculated based on progressive Type-II censoring schemes. For the first case where $n > 20$, a Monte Carlo simulation study is carried out to obtain the optimal $\mathcal{R}^*$ based on the 11 different censoring schemes $\mathcal{R}1 - \mathcal{R}11$. The results are presented in Section 5.4.2.1 via plots and tables for each optimal $\mathcal{R}^*$, the worst censoring scheme, the Type-II censoring scheme and the complete sample. For the second case where $n < 20$, a Monte Carlo simulation study is carried out to obtain the optimal $\mathcal{R}^*$ based on all possible censoring schemes $\mathcal{R} = (R_1, R_2, \dots R_r)$, such that $\sum_{i=1}^{r} R_i = n - r$. The results are presented in Section 5.4.2.2 via plots and tables for each of the optimal $\mathcal{R}^*$, the worst censoring scheme, the Type-II censoring scheme and the complete sample. The remaining tables for other various schemes are available upon request. The results of the first and second cases are reported in this section and Appendix B. The R codes and Iridis commands are available upon request.

5.4.2.1 Large Sample Size

Multiple sets of n, FP, τ and the model parameters are examined to investigate the impact of changing them on the optimal $\mathcal{R}^*$. Simulation results are presented in Tables B.10 – B.24 in Appendix B.

First, we look at the impact of the sample size on the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for $\wp = 0.05, 0.50, 0.95$ based on 11 censoring schemes described in Section 5.4.1. The sample size values are $n = (20,40,60,80,100,150,200)$, with 80% of the observed failures upon the test. The model parameter values $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$ are assumed. The results are presented in Figure 5.17, Figure C.1 and Figure C.2 in Appendix C, and they are also presented in Tables B.10 – B.12 in Appendix B.

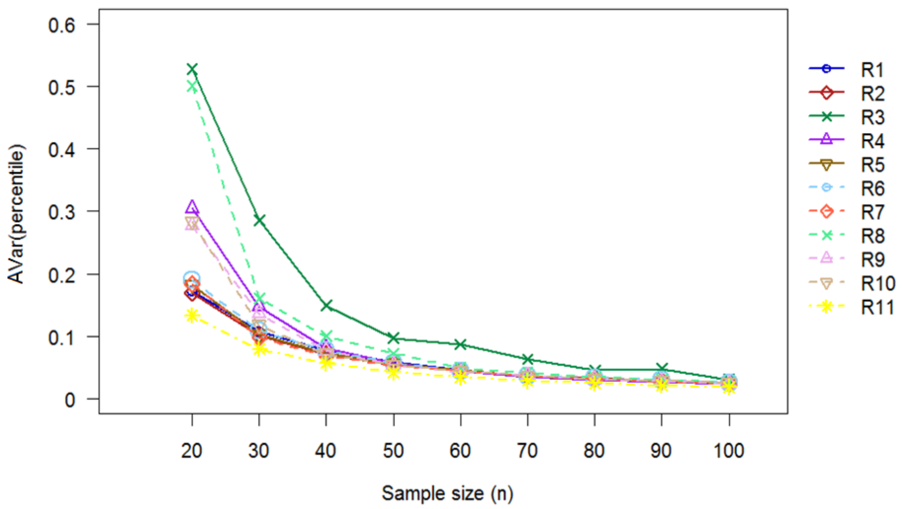


Figure 5.17 $AVar\left(\hat{t}_{0.5}(x_0)\right)$ vs. sample size for 11 censoring schemes.

Figure 5.17 shows the impact of increasing n on the $AVar(\hat{t}_{0.50}(x_0))$ based on 11 different censoring schemes. Although the type-II censoring scheme reduces the test time, it is the worst scheme that provides the least RE and has the largest $AVar(\hat{t}_{0.50}(x_0))$ for all n and $\wp$. As $n \to \infty$, RE of the worst scheme $\to 100\%$, which means the schemes become robust and the experimenter could have the flexibility to choose the most appropriate scheme to be applied for the SSALT experiment. These conclusions also hold for estimating the 5^{th} and 95^{th} percentiles. For $n \leq 50$, the optimal $\mathcal{R}^*$ for estimating the $5^{th}, 50^{th}$ or 95^{th} percentile is where $(n-r)$ items are removed in the middle of the test; either at one time or one by one, based on scheme $\mathcal{R}2$ or $\mathcal{R}7$, respectively. This will emphasise having enough information (enough failures) at lower and higher stress levels. However, $\mathcal{R}2$ is the optimal $\mathcal{R}^*$ for all given sample sizes, when estimating the 5^{th} percentile of the GED under the SSALTs is of interest. On the other hand, for estimating the 50^{th} and 95^{th} percentile, $\mathcal{R}^* = \mathcal{R}4$ when $n > 50$. Also, from Tables B.10 – B.12 in Appendix B, it can be noticed that the RE for $n > 50$, the RT is 50% of the total time under the complete sample, whilst the RE increased 20%, which is relative to FP. Thus, using progressive type-II censoring reduces the total time of the test by 50%, which leads to a reduction in the cost of running the experiment.

After studying the impact of increasing n, the impact of the FP is of interest. We take $n = 40, 100$ and FP= $50\%, 60\%, 70\%, 80\%$. The parameter values are assumed to be $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$. The results are tabulated in the following Table 5.5 – Table 5.7. Also, Figures C.3 – C.5 in Appendix C illustrate the effect of the FP on the $AVar(\hat{t}_{0.50}(x_0))$ of 5^{th}, 50^{th} and 95^{th}percentile under various censoring schemes.

Table 5.5 – Table 5.7 show that the Type-II censoring scheme is the worst scheme that provides the least RE for all FP and $\wp$. As expected, the efficiency of the worst scheme against the optimal $\mathcal{R}^*$ decreases as FP decreases. Furthermore, the efficiency of the complete sample scheme $\mathcal{R}11$ against the optimal $\mathcal{R}^*$ increases as FP decreases. For estimating $t_{0.05}(x_0)$, the RT for the optimal $\mathcal{R}^*$ decreases as FP decreases, which means the optimal $\mathcal{R}^*$ not only reduces the number of failures but also reduces the total test time. This implies a reduction in the ALT cost. Also, for estimating $t_{0.05}(x_0)$, the optimal $\mathcal{R}^*$ is $\mathcal{R}2$ for all given FP when $n = 100$. From Table 5.5 – Table 5.7 and Tables B.10 – B.12, it can be seen that the observed sample size does not determine the optimal $\mathcal{R}^*$. For example, when $n = 40$ and $r = 24$ with 60%FP, the optimal $\mathcal{R}^*$ is scheme $\mathcal{R}6$ for all $\wp = 0.05, 0.50, 0.95$. While the optimal scheme for $n = 30$ and $r = 24$ with 80%FP is $\mathcal{R}2$ or $\mathcal{R}7$ for $\wp = 0.05, 0.50, 0.95$. So, for the same r, the optimal $\mathcal{R}^*$ is variable according to determined values of n and FP. It is concluded that the optimal $\mathcal{R}^*$ is varies depending on the FP when $n = 100$; this means the optimal $\mathcal{R}^*$ is sensitive to the FP.

Table 5.5 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on RE and RT, and the corresponding $AVar(\hat{t}_{0.05}(x_0))$ when changing FP for $n = 40{,}100$, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$

n	FP	Scheme		$AVar$	T	RE(%)	RT(%)
40	80%	Optimal	$\mathcal{R}2$	0.0015	1.6216	100	100
		Worst	$\mathcal{R}3$	0.0062	0.9313	24.83	174.12
		Type-II	$\mathcal{R}3$	0.0062	0.9313	24.83	174.12
		Complete	$\mathcal{R}11$	0.0014	1.7426	110.35	93.06
40	70%	Optimal	$\mathcal{R}6$	0.0017	1.5925	100	100
		Worst	$\mathcal{R}3$	0.0182	0.8602	9.35	185.13
		Type-II	$\mathcal{R}3$	0.0182	0.8602	9.35	185.13
		Complete	$\mathcal{R}11$	0.0014	1.7398	119.17	91.53
40	60%	Optimal	$\mathcal{R}6$	0.0019	1.4969	100	100
		Worst	$\mathcal{R}3$	0.0338	0.8340	5.65	179.48
		Type-II	$\mathcal{R}3$	0.0338	0.8340	5.65	179.48
		Complete	$\mathcal{R}11$	0.0014	1.7378	137.05	86.13
40	50%	Optimal	$\mathcal{R}1$	0.0026	1.5219	100	100
		Worst	$\mathcal{R}3$	0.1125	0.8217	2.30	185.22
		Type-II	$\mathcal{R}3$	0.1125	0.8217	2.30	185.22
		Complete	$\mathcal{R}11$	0.0014	1.7388	183.62	87.52
100	80%	Optimal	$\mathcal{R}2$	0.0005	1.8891	100	100
		Worst	$\mathcal{R}3$	0.0008	0.9276	65.77	203.65
		Type-II	$\mathcal{R}3$	0.0008	0.9276	65.77	203.65
		Complete	$\mathcal{R}11$	0.0005	2.0032	106.30	94.30
100	70%	Optimal	$\mathcal{R}2$	0.0005	1.8305	100	100
		Worst	$\mathcal{R}3$	0.0127	0.8399	4.17	217.95
		Type-II	$\mathcal{R}3$	0.0127	0.8399	4.17	217.95
		Complete	$\mathcal{R}11$	0.0005	2.0154	111.11	90.82
100	60%	Optimal	$\mathcal{R}2$	0.0006	1.7604	100	100
		Worst	$\mathcal{R}3$	0.0322	0.8172	1.79	215.40
		Type-II	$\mathcal{R}3$	0.0322	0.8172	1.79	215.40
		Complete	$\mathcal{R}11$	0.0005	2.0097	120.94	87.59
100	50%	Optimal	$\mathcal{R}2$	0.0007	1.6862	100	100
		Worst	$\mathcal{R}3$	0.0827	0.8099	0.80	208.20
		Type-II	$\mathcal{R}3$	0.0827	0.8099	0.80	208.20
		Complete	$\mathcal{R}11$	0.0005	2.0068	134.31	84.02

Table 5.6 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on RE and RT, and the corresponding $AVar(\hat{t}_{0.50}(x_0))$ when changing FP for $n = 40{,}100$, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$

$\boldsymbol{n}$	FP	Scheme		$\boldsymbol{AVar}$	$\boldsymbol{T}$	RE(%)	RT(%)
40	80%	Optimal	$\mathcal{R}2$	0.0701	1.6182	100	100
		Worst	$\mathcal{R}3$	0.1994	0.9307	35.17	173.86
		Type-II	$\mathcal{R}3$	0.1994	0.9307	35.17	173.86
		Complete	$\mathcal{R}11$	0.0553	1.7365	126.89	93.18
40	70%	Optimal	$\mathcal{R}10$	0.0837	1.5206	100	100
		Worst	$\mathcal{R}3$	1.4552	0.8603	5.74	176.75
		Type-II	$\mathcal{R}3$	1.4552	0.8603	5.74	176.75
		Complete	$\mathcal{R}11$	0.0577	1.7379	145.02	87.49
40	60%	Optimal	$\mathcal{R}6$	0.1062	1.4987	100	100
		Worst	$\mathcal{R}3$	2.1273	0.8339	4.99	179.72
		Type-II	$\mathcal{R}3$	2.1273	0.8339	4.99	179.72
		Complete	$\mathcal{R}11$	0.0590	1.7397	179.96	86.14
40	50%	Optimal	$\mathcal{R}1$	0.1283	1.5270	100	100
		Worst	$\mathcal{R}3$	8.9497	0.8223	1.43	185.70
		Type-II	$\mathcal{R}3$	8.9497	0.8223	1.43	185.70
		Complete	$\mathcal{R}11$	0.0599	1.7397	214.33	87.77
100	80%	Optimal	$\mathcal{R}4$	0.0248	1.0979	100	100
		Worst	$\mathcal{R}3$	0.0338	0.9277	73.22	118.34
		Type-II	$\mathcal{R}3$	0.0338	0.9277	73.22	118.34
		Complete	$\mathcal{R}11$	0.0198	2.0118	125.42	54.57
100	70%	Optimal	$\mathcal{R}1$	0.0288	1.9067	100	100
		Worst	$\mathcal{R}3$	1.2013	0.8397	2.40	227.07
		Type-II	$\mathcal{R}3$	1.2013	0.8397	2.40	227.07
		Complete	$\mathcal{R}11$	0.0199	2.0140	145.00	94.67
100	60%	Optimal	$\mathcal{R}1$	0.0348	1.8554	100	100
		Worst	$\mathcal{R}3$	2.2544	0.8169	1.54	227.11
		Type-II	$\mathcal{R}3$	2.2544	0.8169	1.54	227.11
		Complete	$\mathcal{R}11$	0.0191	2.0166	181.77	92.00
100	50%	Optimal	$\mathcal{R}1$	0.0428	1.7993	100	100
		Worst	$\mathcal{R}3$	6.4913	0.8100	0.65	222.12
		Type-II	$\mathcal{R}3$	6.4913	0.8100	0.65	222.12
		Complete	$\mathcal{R}11$	0.0200	2.0172	214.37	89.19

Table 5.7 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on RE and RT, and the corresponding $AVar(\hat{t}_{0.95}(x_0))$ when changing FP for $n = 40,100$, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$

n	FP	Scheme		$AVar$	T	RE(%)	RT(%)
40	80%	Optimal	$\mathcal{R}7$	1.6762	1.6117	100	100
		Worst	$\mathcal{R}3$	2.7453	0.9331	61.05	172.71
		Type-II	$\mathcal{R}3$	2.7453	0.9331	61.05	172.71
		Complete	$\mathcal{R}11$	1.3902	1.7324	120.57	93.03
40	70%	Optimal	$\mathcal{R}10$	1.9722	1.5060	100	100
		Worst	$\mathcal{R}3$	26.5742	0.8610	7.42	174.90
		Type-II	$\mathcal{R}3$	26.5742	0.8610	7.42	174.90
		Complete	$\mathcal{R}11$	1.3592	1.7326	145.10	86.92
40	60%	Optimal	$\mathcal{R}6$	2.5628	1.4971	100	100
		Worst	$\mathcal{R}3$	70.9390	0.8342	3.61	179.46
		Type-II	$\mathcal{R}3$	70.9390	0.8342	3.61	179.46
		Complete	$\mathcal{R}11$	1.3890	1.7329	184.50	86.39
40	50%	Optimal	$\mathcal{R}1$	3.4603	1.5206	100	100
		Worst	$\mathcal{R}3$	167.8368	0.8222	2.06	184.94
		Type-II	$\mathcal{R}3$	167.8368	0.8222	2.06	184.94
		Complete	$\mathcal{R}11$	1.4086	1.7404	245.65	87.37
100	80%	Optimal	$\mathcal{R}4$	0.5267	1.0992	100	100
		Worst	$\mathcal{R}3$	0.6183	0.9279	85.18	118.45
		Type-II	$\mathcal{R}3$	0.6183	0.9279	85.18	118.45
		Complete	$\mathcal{R}11$	0.4412	2.0068	119.37	54.77
100	70%	Optimal	$\mathcal{R}5$	0.6298	1.7721	100	100
		Worst	$\mathcal{R}3$	9.4559	0.8397	6.66	211.05
		Type-II	$\mathcal{R}3$	9.4559	0.8397	6.66	211.05
		Complete	$\mathcal{R}11$	0.4357	2.0036	144.53	88.44
100	60%	Optimal	$\mathcal{R}5$	0.7696	1.6861	100	100
		Worst	$\mathcal{R}3$	38.5470	0.8169	1.99	206.39
		Type-II	$\mathcal{R}3$	38.5470	0.8169	1.99	206.39
		Complete	$\mathcal{R}11$	0.4240	2.0102	181.49	83.87
100	50%	Optimal	$\mathcal{R}1$	0.9693	1.8030	100	100
		Worst	$\mathcal{R}3$	89.7060	0.8101	1.08	222.57
		Type-II	$\mathcal{R}3$	89.7060	0.8101	1.08	222.57
		Complete	$\mathcal{R}11$	0.4528	2.0128	214.05	89.57

Next, the effect of changing θ_1 and θ_2 on the optimal $\mathcal{R}^*$ is studied. The fixed initial values of $n = 80$ with FP= 80% and $\tau = 0.8$ are assumed. Also, $\alpha = 1.2$ is assumed. In the case of testing the impact of θ_1, $\theta_2 = 0.3$ is assumed, and θ_1 is tested for the values $\theta_1 = 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.5, 3.0$. On the other hand, in the case of testing the impact of θ_2, $\theta_1 = 0.6$ is assumed, and θ_2 is tested for the values $\theta_2 = 0.1, 0.2, 0.3, 0.4, 0.5$. The results are presented in Tables B.13 – B.18 in Appendix B. In addition, Figures C.6 – C.7 in Appendix C and Figure 5.18 illustrate the effect of changing θ_1 and θ_2 on the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ of 5^{th}, 95^{th} and 50^{th} percentile, respectively, under various censoring schemes.

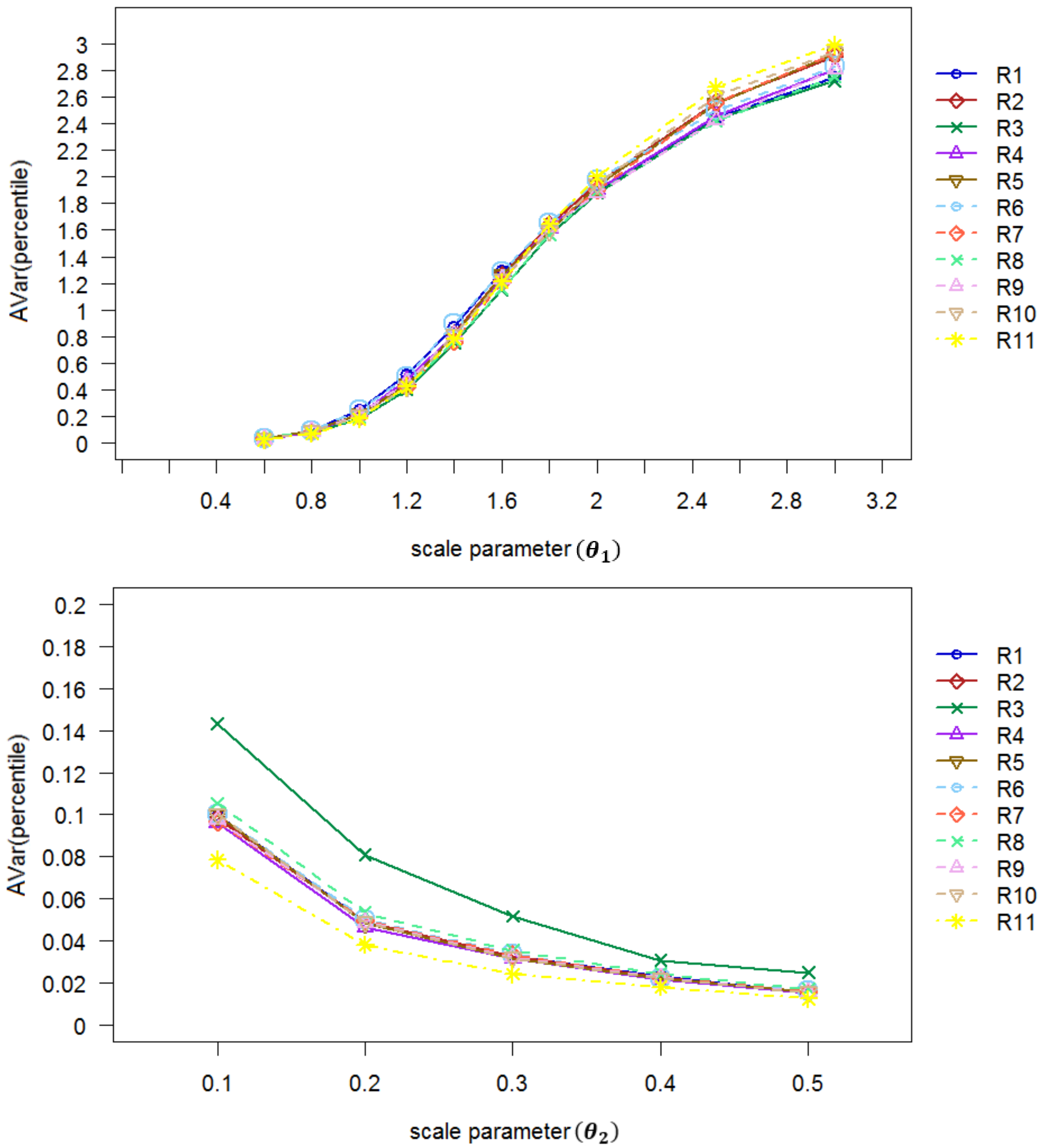


Figure 5.18 $AVar(\hat{t}_{0.5}(x_0))$ vs. scale parameters θ_1 and θ_2 for 11 censoring schemes.

Figure 5.18 and Figures C.6 – C.7 in Appendix C show that the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ increases as θ_1 increases and the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ decreases as θ_2 increases for three different percentiles of interest: 5^{th}, 50^{th} and 95^{th}. That means, as the difference between the values of θ_1 and θ_2 decrease, the value of

$AVar\left(\hat{t}_{p}(x_0)\right)$ decreases. This conclusion agrees with the conclusion that is obtained in Section 5.3.2. By comparing Figure 5.18 with Figure 5.11 it can be seen that $AVar\left(\hat{t}_{p}(x_0)\right)$ increases as θ_1 is increasing and decreases as θ_2 increases. This behaviour holds not only for the supposed scheme $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$ used for studying optimal τ^*, but holds also for all 11 censoring schemes. Furthermore, the value of θ_2 has no impact on determining the optimal $\mathcal{R}^*$. Thus, both optimal τ^* and optimal $\mathcal{R}^*$ are robust for the value of θ_2.

From Tables B.13 – B.18 in Appendix B it can be seen that for all values of θ_1 and θ_2 the optimal $\mathcal{R}^* = \mathcal{R}2$ or $\mathcal{R}7$ where items are removed in the middle of the test, for estimating the 5^{th} percentile. Whilst unexpectedly the optimal $\mathcal{R}^* = \mathcal{R}3$, which is Type-II censoring, for estimating the 50^{th} and 95^{th} percentile. In contrast, $\mathcal{R}3$ is the worst scheme for all given values of θ_2. The worst scheme is $\mathcal{R}1$ or $\mathcal{R}6$ where items are removed in beginning of the test for most of the cases for changing θ_1, especially when $\theta_1 \leq 1.6$ for $5^{th}, 50^{th}$ and 95^{th}percentile.

In addition, it can be noticed that the $RE \approx 100\%$ for the 5^{th} percentile for all θ_1 and θ_2. This means the efficiency of $\hat{t}_{0.05}(x_0)$ under PCS is almost equal to the efficiency of $\hat{t}_{0.05}(x_0)$ under the complete sample. However, ALTs under PCSs reduce the time and cost of the experiment. Surprisingly, $RE < 100\%$ for $\theta_1 > 1$ for 50^{th} and 95^{th}percentile, which means the precision of the MLE of $t_{p}(x_0)$ based on PCS is better than the precision of the MLE of $t_{p}(x_0)$ under the complete sample. The reason for that may be the skewness of the GED, which may result in extreme values in the sample that affect the efficiency of the MLEs. So, removing items at the end of the test is desirable. It is noticed that the RE decreases as θ_1 increases. Also, for large scale parameter values, the GED density becomes flatter, which may lead to the occurrence of extreme values on both sides that affect the precision of the MLEs. In fact, this is a noticeable advantage of using progressive Type-II censoring scheme as it increases the precision of an estimator as well as reducing the time and cost.

After studying the impact of changing the scale parameters on various censoring schemes, the impact of the shape parameter is of interest. The effect of changing α on $AVar\left(\hat{t}_{p}(x_0)\right)$ based on 11 censoring schemes is presented in Tables B.19 – B,21 in Appendix B. Also, Figure 5.19 and C.8 and C.9 in Appendix C illustrate the behaviour of the objective function of V-optimality with respect to α for estimating the $50^{th}, 5^{th}$ and 95^{th} percentile. The set of initial values is assumed as $n = 80$, FP=80%, $\tau = 0.8$ with $\theta_1 = 0.6$ and $\theta_2 = 0.3$.

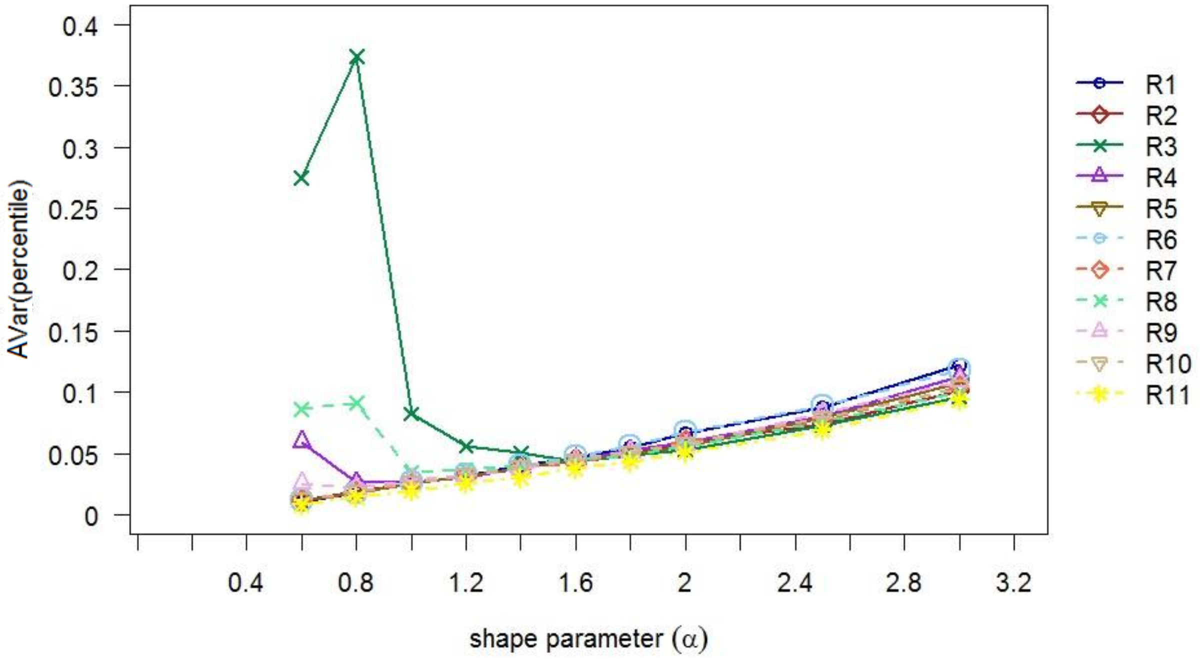


Figure 5.19 $AVar(\hat{t}_{0.5}(x_0))$ vs. α for 11 censoring schemes.

From Figure 5.19 and Table B.9, it can be seen that the difference in the optimal $\mathcal{R}^*$ with respect to α is large, as α affect the the shapes of the GED density functions as discussed in Section 3.1. The density curve becomes right skewed for increasing α, so for $\alpha \geq 1.6$ the worst scheme is $\mathcal{R}1$ and $\mathcal{R}6$ where items are removed at the first failure, whilst the optimal $\mathcal{R}^* = \mathcal{R}3$ where the most skewed items are removed. In contrast, when $\alpha \leq 1$, the optimal $\mathcal{R}^* = \mathcal{R}1$ for estimation of the 50^{th} and 95^{th}percentiles, whilst the worst scheme is $\mathcal{R}3$. Moreover, the RE→ 100% as α increases as the skewness of the distribution may leads to the existence of the extreme values that increases the $AVar\left(\hat{t}_{\wp}(x_0)\right)$. Thus, for large values of α, it is preferable to use the progressive censored samples rather than the complete sample, as censoring schemes reduce up to 60% of the total time of the test while holding the efficiency of an estimator.

Next, it is of interest to study the impact of the stress change time on the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ at probabilities $\wp = 0.05, 0.50, 0.95$ based on 11 suggested censoring schemes when $n = 80$ with FP=80%, and the model parameters $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$. The results are numerically presented in Tables B.22-B.24 in Appendix B. The impact of changing τ on the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for all schemes considered are presented in Figure 5.20, Figure C.10 and Figure C.11 in Appendix C for 50^{th}, 5^{th} and 95^{th} percentile, respectively.

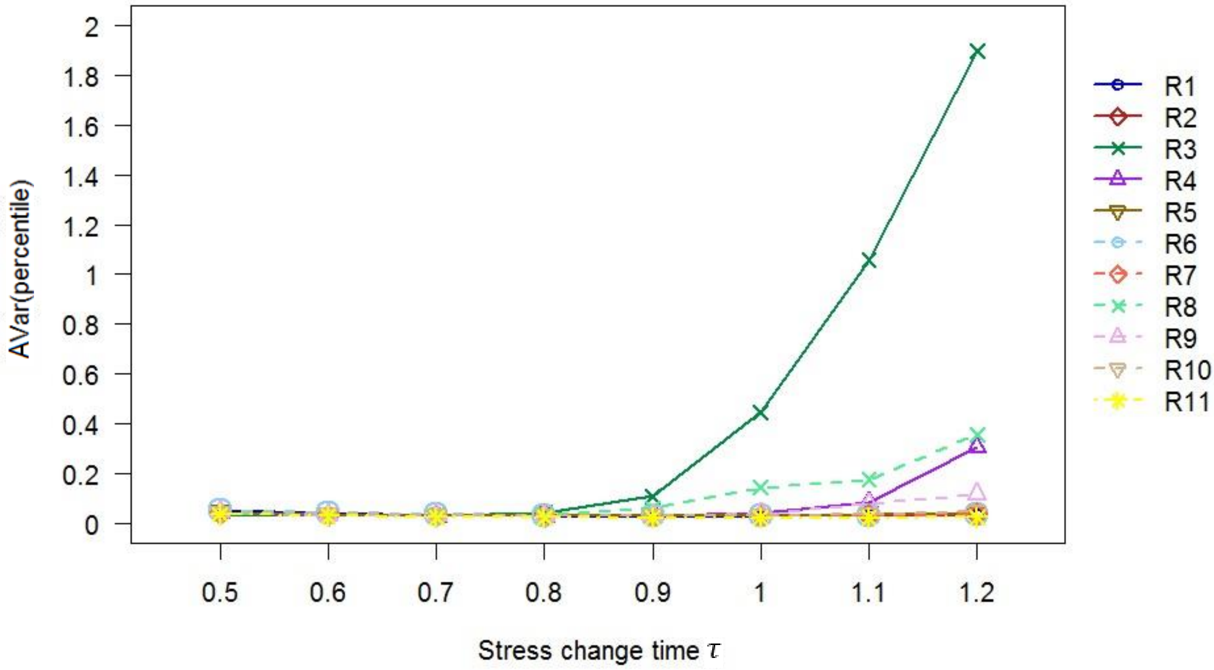


Figure 5.20 $AVar(\hat{t}_{0.5}(x_0))$ vs. τ for 11 censoring schemes.

The schemes $\mathcal{R}1$ and $\mathcal{R}6$ where removing the items at the beginning of the test are the worst schemes when $\tau \leq 0.7$, and they become the optimal $\mathcal{R}^*$ when $\tau > 0.9$. This result emphasizes the importance of choosing the optimal $\mathcal{R}^*$ with respect to the initial values, as a censoring scheme could improve the precision of MLEs of the parameters or it could negatively affect the precision of an estimator. For $\tau \leq 0.7$, the test needs more information regarding the lower stress level before the stress level is increased. Thus, the optimal $\mathcal{R}^* = \mathcal{R}3,\ \mathcal{R}8$ and the worst schemes are $\mathcal{R}1, \mathcal{R}6$. In contrast, when $\tau \geq 0.9$, the test is run longer on the lower stress level, so, it is not preferable to remove items in the last stage of the test. Therefore, $\mathcal{R}3$ is the worst scheme and the optimal $\mathcal{R}^* = \mathcal{R}1$, $\mathcal{R}6$.

Also, it can be seen that as τ increases, the efficiency of the optimal $\mathcal{R}^*$ is almost double the efficiency of the worst scheme. The optimal $\mathcal{R}^*$ is $\mathcal{R}1$ and $\mathcal{R}6$, where items are removed at the beginning of the test. In contrast, the worst scheme is $\mathcal{R}3$, where items are removed at the end of the test. Although the worst scheme reduces the total test time by more than 75%, however, it decreases the efficiency up to only 10% by increasing the stress change time. It can be seen that the RE of the worst scheme suddenly drops off when τ increases from 0.8 to 0.9. The RE> 66% for $\tau \leq 0.8$, whilst it drops off to less than 28% for lower τ. However, as τ increases, the RE is continually decreasing down to 2%. To investigate the reason for that, the proportion of failures at each stress level pn_1 and pn_2 were calculated. It was concluded that pn_2 rapidly decreases as $\tau > 0.8$.

The standard error of the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is presented in Tables B.22 - B.24 in Appendix B to assess the precision of the estimated $AVar\left(\hat{t}_{\wp}(x_0)\right)$. The following table summarize the values of standard errors as a percentage of the estimated value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$.

$\wp^{th}$percentile	$\widehat{AVar}\left(\hat{t}_{\wp}(x_0)\right)$	
	min	max
5^{th}	0.0019	0.2210
50^{th}	0.0018	0.4541
95^{th}	0.0019	0.4690

From the above table, it can be seen that the maximum standard errors percentages are large. For $5^{th}, 50^{th}$ and 95^{th} percentiles, the standard error percentages > 0.11 and less than 0.4690 when $\tau \geq 0.8$ and estimating $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for the worst censoring scheme, which is Type-II censoring. This happens due to a decreasing number of failures at the upper stress level as the stress change time τ increases. Also, the censored data are removed at the time of the last failure, which leads to the loss of information about the right tail of the right-skewed life distribution. Thus, when $\tau \geq 0.8$, we should take care not to overinterpret the results regarding the worst censoring schemes.

5.4.2.1.1 Sensitivity Analysis

Since the resulting optimal censoring scheme is local, the sensitivity of the scheme will be investigated by considering the effect of parameter misspecification. This section discusses the sensitivity of the $AVar\left(\hat{t}_p(x_0)\right)$ measure (based on which optimal censoring schemes are selected) to changes in the model parameters.

For the 'base' scenario, the initial values are chosen to be $n = 80$, FP= 80%, $\tau = 0.8$, $\theta_1 = 0.6$, $\theta_2 = 0.3$ and $\alpha = 1.2$. While censoring scheme $\mathcal{R}2$ was optimal for this base scenario for estimating the 5^{th} percentile, censoring scheme $\mathcal{R}4$ was optimal for estimating the 50^{th} and the 95^{th} percentile, respectively. However, in all these 3 cases, censoring scheme $\mathcal{R}1$ was close to the optimal scheme in terms of $AVar\left(\hat{t}_p(x_0)\right)$ and was therefore chosen for this study. In particular, the values of $AVar\left(\hat{t}_p(x_0)\right)$ in the base scenario for $\mathcal{R}1$ are 0.0008, 0.0317 and 0.7024, respectively, whereas the corresponding optimal values are 0.0007, 0.0313 and 0.6956. The worst censoring scheme in each of these cases would have $AVar\left(\hat{t}_p(x_0)\right)$ between 1.5 and 2 times the optimal value of $AVar\left(\hat{t}_p(x_0)\right)$.

The selected range of values for α was 0.8 to double this value at 1.6. The range of values for θ_1 was 0.4 to double this value at 0.8. The range of θ_2 was from 0.1 to $\theta_1 - 0.1$. According to the life-stress relationship in (2.1) in Chapter 2, the scale parameter has an inverse relation to the stress level. Thus, as $x_1 < x_2$, then $\theta_1 > \theta_2$. Simulation studies have been performed using the above initial values ranges for calculating the MLEs of the 5^{th}, 50^{th} and 95^{th} percentile.

The ratio of each selected scenario with respect to the base scenario is used to compare $AVar\left(\hat{t}_{\wp}(x_0)\right)$ among different sets of initial values of model parameters and to determine the loss in estimation efficiency. The ratio was calculated as follows

$$\text{Ratio}\left(AVar\left(\hat{t}_{\wp}(x_0)\right)\right) = \frac{AVar\left(\hat{t}_{\wp}(x_0)_{\,|\theta_1,\theta_2,\alpha}\right)}{AVar\left(\hat{t}_{\wp}(x_0)_{|\theta_1=0.6,\theta_2=0.3,\alpha=1.2}\right)}$$

Note that smaller Ratio $\left(AVar\left(\hat{t}_{\wp}(x_0)\right)\right)$ values indicate that the current scenario of θ_1, θ_2 and α gives smaller value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$ than the base scenario when $\theta_1 = 0.6, \theta_2 = 0.3$ and $\alpha = 1.2$. In other words, when Ratio $\left(AVar\left(\hat{t}_{\wp}(x_0)\right)\right) < 1$, the current scenario of θ_1, θ_2 and α results in a more precise estimate of $t_{\wp}(x_0)$. The results of the sensitivity analysis are presented in the following Table 5.8 and in Tables B.25 and B.26 in Appendix B.

Table 5.8 The sensitivity of the SSALTs model based on $AVar(\hat{t}_{0.50}(x_0))$ and the associated ratio of base scenario and random scenario for $\mathcal{R} = \mathcal{R}1$, $\tau = 0.8$ and $n = 80$ with 80%FP.

		$\alpha = 0.8$		$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$	
θ_1	θ_2	$AVar$	Ratio	$AVar$	Ratio	$AVar$	Ratio	$AVar$	Ratio
0.4	0.1	0.0209	0.66	0.0299	0.94	0.0340	1.07	0.0374	1.18
	0.2	0.0098	0.31	0.0142	0.45	0.0173	0.55	0.0185	0.58
	0.3	0.0055	0.17	0.0083	0.26	0.0098	0.31	0.0108	0.34
0.5	0.1	0.0318	1.00	0.0526	1.66	0.0634	2.00	0.0729	2.30
	0.2	0.0159	0.50	0.0263	0.83	0.0317	1.00	0.0369	1.16
	0.3	0.0104	0.33	0.0171	0.54	0.0204	0.64	0.0239	0.75
	0.4	0.0068	0.21	0.0112	0.35	0.0134	0.42	0.0155	0.49
0.6	0.1	0.0538	1.70	0.0954	3.01	0.1105	3.49	0.1391	4.39
	0.2	0.0272	0.86	0.0484	1.53	0.0558	1.76	0.0712	2.25
	0.3	0.0182	0.57	0.0317	1.00	0.0384	1.21	0.0429	1.35
	0.4	0.0127	0.40	0.0226	0.71	0.0259	0.82	0.0330	1.04
	0.5	0.0087	0.27	0.0156	0.49	0.0201	0.63	0.0230	0.73
0.7	0.1	0.0875	2.76	0.1557	5.03	0.2121	6.69	0.2668	8.42
	0.2	0.0453	1.43	0.0836	2.64	0.1042	3.29	0.1343	4.24
	0.3	0.0305	0.96	0.0558	1.76	0.0712	2.25	0.0884	2.79
	0.4	0.0221	0.70	0.0421	1.33	0.0519	1.64	0.0660	2.08
	0.5	0.0167	0.53	0.0302	0.95	0.0387	1.22	0.0471	1.49
	0.6	0.0122	0.38	0.0228	0.72	0.0287	0.91	0.0351	1.11
0.8	0.1	0.1407	4.44	0.2856	9.01	0.3790	11.96	0.4813	15.18
	0.2	0.0717	2.26	0.1427	4.50	0.1935	6.10	0.2330	7.35
	0.3	0.0475	1.50	0.0747	2.79	0.1219	3.85	0.1582	4.99
	0.4	0.0364	1.15	0.0697	2.20	0.0946	2.98	0.1159	3.66
	0.5	0.0272	0.86	0.0556	1.75	0.0739	2.33	0.0915	2.89
	0.6	0.0214	0.68	0.0429	1.35	0.0566	1.79	0.0712	2.25
	0.7	0.0167	0.53	0.0323	1.02	0.0428	1.35	0.0542	1.71

Some general observations are made below. For each percentile:

1- For fixed values of θ_1 and θ_2, $AVar\left(\hat{t}_p(x_0)\right)$ appears to increase as α increases.

2- For fixed values of α, $AVar\left(\hat{t}_p(x_0)\right)$ appears to decrease as θ_1 and θ_2 get closer together and to increase as θ_1 gets larger. This result is expected, as the scale parameter has a relationship to the stress level according to the life-stress relationship in (2.1) in Chapter 2. As θ_1 and θ_2 get closer together, this means the stress levels x_1 and x_2 get closer to each other. Consequently, items under the SSALT are exposed to two stress levels with minimal difference in their level. Thus, the estimated quantile $\hat{t}_p(x_0)$ values have the least variability in this scenario.

3- The largest value of $AVar\left(\hat{t}_p(x_0)\right)$ is observed for the largest values of α and θ_1 and the smallest value of θ_2.

4- The smallest value of $AVar\left(\hat{t}_p(x_0)\right)$ is observed for the smallest values of α and θ_1 and the largest possible value of θ_2 for this choice of θ_1.

To assess the sensitivity of $AVar\left(\hat{t}_p(x_0)\right)$ for our chosen censoring scheme and value of τ, we compare the size of $AVar\left(\hat{t}_p(x_0)\right)$ at the base scenario with the smallest and the largest $AVar\left(\hat{t}_p(x_0)\right)$ within each scenario regarding percentile estimation as follows:

1- For the 5^{th} percentile, $AVar\left(\hat{t}_p(x_0)\right)$ for the base scenario is approximately 8 times larger than the smallest $AVar\left(\hat{t}_p(x_0)\right)$, and the largest $AVar\left(\hat{t}_p(x_0)\right)$ is approximately 15 times larger than $AVar\left(\hat{t}_p(x_0)\right)$ for the base scenario.

2- For the 50^{th} percentile, $AVar\left(\hat{t}_p(x_0)\right)$ for the base scenario is approximately 6 times larger than the smallest $AVar\left(\hat{t}_p(x_0)\right)$, and the largest $AVar\left(\hat{t}_p(x_0)\right)$ is approximately 15 times larger than $AVar\left(\hat{t}_p(x_0)\right)$ for the base scenario.

3- For the 95^{th} percentile, $AVar\left(\hat{t}_p(x_0)\right)$ for the base scenario is approximately 4 times larger than the smallest $AVar\left(\hat{t}_p(x_0)\right)$, and the largest $AVar\left(\hat{t}_p(x_0)\right)$ is approximately 13 times larger than $AVar\left(\hat{t}_p(x_0)\right)$ for the base scenario.

Moreover, $AVar\left(\hat{t}_p(x_0)\right)$ for estimating higher percentiles is larger than for lower percentiles. Comparing the values of $AVar\left(\hat{t}_p(x_0)\right)$ for the base scenario, we find that $AVar\left(\hat{t}_p(x_0)\right)$ for estimating the median is approximately 40 times the value of $AVar\left(\hat{t}_{0.05}(x_0)\right)$, with $AVar\left(\hat{t}_{0.95}(x_0)\right)$ being 22 times higher again than $AVar\left(\hat{t}_p(x_0)\right)$ for estimating the median.

It is recommended to examine the sensitivity analysis of all censoring schemes to determine if they have the same order regarding the minimum and maximum $AVar\left(\hat{t}_p(x_0)\right)$. Then, evaluate the impact of misspecification of the initial parameter values on the order of the schemes based on $AVar\left(\hat{t}_p(x_0)\right)$. So, if, for example, the scheme $\mathcal{R}1$ has the smallest or second-smallest value of $AVar\left(\hat{t}_p(x_0)\right)$ across all evaluated scenarios of initial values, then the scheme $\mathcal{R}1$ is considered robust.

Based on the sensitivity analysis of the optimal censoring scheme in this section, it is concluded that incorrect estimation of the unknown model parameters results in an inaccurate estimation of the failure quantile under the usage stress level. Therefore, the experimenter should have information about the unknown model parameters from prior experiments or employ a pilot study to estimate the unknown parameters and choose the optimal censoring scheme that minimizes the associated $AVar\left(\hat{t}_p(x_0)\right)$.

5.4.2.2 Small Sample Size

The concept of obtaining the optimal $\mathcal{R}^*$ relies on the selection of the best combination of censoring scheme $\mathcal{R} = (R_1, R_2, \dots R_r)$ within all alternative censoring schemes, such that $\sum_{i=1}^{r} R_i = n - r$. Thus, in this section, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ is calculated for all possible censoring schemes $\mathcal{R} = (R_1, R_2, \dots R_r)$. Note that $\mathcal{R} = (R_1, R_2, \dots R_r)$ consists of fixed constants specifying the number of surviving items to be censored at each failure time. However, the number of possible censoring schemes $= \binom{n-1}{r-1}$ is enormous and the programme must be run for a considerable amount of time in order to obtain the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ based on each of the $\binom{n-1}{r-1}$ censoring schemes. The maximum time available on Iridis5 is 60 hours, However, for $n > 20$ it takes more than 60 hours to obtain the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for all $\mathcal{R}$. Therefore, in this case only small sample size $n \leq 20$ is tested. The optimal $\mathcal{R}^*$ is selected as it minimizes the $AVar\left(\hat{t}_{\wp}(x_0)\right)$.

Testing all possible schemes allows the experimenter to decide which scheme is more efficient given some initial conditions. For example, if the experimenter has a constraint that a fixed number of items must be removed from the test after the 3^{rd} failure, while the times for removing other censored items $R_{\mathrm{i}}, 3 < i \leq r$ are random. So, testing all possible censoring schemes gives the experimenter an idea about the best scheme with respect to the given constraint of removing a fixed number of items from the test after the 3^{rd} failure.

Another advantage is when the total time to apply the experiment based on optimal $\mathcal{R}^*$ is long, but the experimenter prefers to reduce the test time. So, he/she can choose a censoring scheme that has a RE close to 100% and a less total time T. Moreover, in the situation where it is preferable to remove items at fewer times, but the optimal $\mathcal{R}^*$ was the scheme where items are removed from the test at multiple times. Then the experimenter has a flexibility to choose the best censoring scheme that is more convenient to apply for ALTs that are at least 95% as efficient as the optimal $\mathcal{R}^*$ (Balakrishnan and Aggarwala, 2000).

In this section, $AVar\left(\hat{t}_{p}(x_0)\right)$ is calculated for all possible schemes. Then, the optimal $\mathcal{R}^*$ is determined as the scheme that provides the minimum value of $AVar\left(\hat{t}_{p}(x_0)\right)$. After that, the RE and RT are calculated using (5.5) and (5.6), respectively, of the optimal $\mathcal{R}^*$ with respect to all other censoring schemes. By using the RE with RT, the experimenter will have the flexibility to choose the most convenient scheme to use under the ALT. It is clear that when RE is close to 100%, then the scheme has the same impact as the optimal one on the precision of $\hat{t}_{p}(x_0)$ (Balakrishnan and Aggarwala, 2000). So, the experimenter will have a flexibility of choosing the most practically effective scheme or a range of schemes that reduce the time of the test based on comparing RT. Thus, the balance between reduction of the cost with the most precision of the $\hat{t}_{p}(x_0)$ is attained. For example, maybe the scheme $\mathcal{R}$x is 95% as efficient as the optimal $\mathcal{R}^*$, and it reduces the test time by 40% and reduces the number of times items are removed from the test. So, the experimenter may prefer to use $\mathcal{R}$x rather than the optimal $\mathcal{R}^*$.

In this section, different sets of n and FP are assumed as follows:

n	FP	r	Number of schemes
10	60%	6	126
	70%	7	84
	80%	8	36
15	60%	9	3003
	70%	10	2002
	80%	12	364
20	80%	16	3876

The above sets are run with the initial values of the model parameter $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$ are assumed. For the set of $\binom{n-1}{r-1}$ schemes, the first scheme is $\mathcal{R}1 = \left((n-r)\,, 0^{*(r-1)}\right)$ and the last scheme is $\mathcal{R}\binom{n-1}{r-1} = \left(0^{*(r-1)}, (n-r)\right)$. The other schemes are between these two end schemes. As the space of the thesis is limited, we tabulate only a part of the complete results. Table 5.9 contains the results for the optimal $\mathcal{R}^*$, the worst scheme, $\mathcal{R}1$, $\mathcal{R}\binom{n-1}{r-1}$ and the complete sample case. The remaining results for other various schemes are available upon request. Also, Figure 5.21 and Figure 5.22 show the $AVar(\hat{t}_{0.5}(x_0))$ for all $\binom{n-1}{r-1}$ various censoring schemes and the complete sample case for $n = 10{,}15{,}20$ and FP= $60\%, 70\%, 80\%$.

Table 5.9 shows that the total test time is reduced by $\approx 50\%$ as the items tend to be removed at the end of the test. As FP increases for fixed n, the RT decreases as increasing the number of observed data leads to an increase in the test time. However, the difference in RE is considerable with respect to different censoring schemes. The efficiency of the worst censoring scheme is $< 20\%$ of the efficiency of the optimal $\mathcal{R}^*$ in most cases. Therefore, selecting a censoring scheme that improves the precision of the MLE of $t_{\wp}(x_0)$ is vital. It is interesting to note that the optimal $\mathcal{R}^*$ under $n = 10{,}15$ are compatible with the results obtained by Pradhan and Kundu (2009), who studied statistical inference and optimal schemes for the GED under progressive Type-II censored data.

Table 5.9 The comparison of $\mathcal{R}^*$ with the worst, Type-II and the complete sample, and corresponding $AVar\big(\hat{t}_{0.50}(x_0)\big)$ when changing FP and n for $\theta_1 = 0.6, \theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$.

n	FP	Scheme		$AVar$	T	RE(%)	RT(%)
10	60%	Optimal	(4 , 0 , 0 , 0 , 0 , 0)	0.3549	1.0522	100	100
		Worst	(0 , 0 , 1 , 0 ,1 , 2)	27.2998	0.7265	1.30	144.83
		$\mathcal{R}1$	(4 , 0 , 0 , 0 , 0 , 0)	0.3549	1.0522	100	100
		$\mathcal{R}\binom{n-1}{r-1}$	(0 , 0 , 0 , 0 , 0 , 4)	4.3179	0.6842	8.22	153.78
		Complete	(0 , 0 , 0 , 0 , 0 , 0)	0.2452	1.2086	144.75	87.06
	70%	Optimal	(2 , 1 , 0 , 0 , 0 , 0 , 0)	0.3303	1.0870	100	100
		Worst	(0 , 0 , 0 , 0 , 0 , 1 , 2)	5.2405	0.7459	6.30	145.72
		$\mathcal{R}1$	(3 , 0 , 0 , 0 , 0 , 0 , 0)	0.3357	1.0910	98.38	99.62
		$\mathcal{R}\binom{n-1}{r-1}$	(0 , 0 , 0 , 0 , 0 , 0 , 3)	1.0946	0.7265	30.17	149.61
		Complete	(0 , 0 , 0 , 0 , 0 , 0 , 0)	0.2506	1.2128	131.76	89.62
	80%	Optimal	(2 , 0 , 0 , 0 , 0 , 0 ,0 , 0)	0.2787	1.1391	100	100
		Worst	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 2)	0.9600	0.7958	29.03	143.13
		$\mathcal{R}1$	(2 , 0 , 0 , 0 , 0 , 0 ,0 , 0)	0.2787	1.1391	100	100
		$\mathcal{R}\binom{n-1}{r-1}$	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 2)	0.9600	0.7958	29.03	143.13
		Complete	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 0)	0.2435	1.2122	114.48	93.96
15	60%	Optimal	(2 , 3 , 0 , 1 , 0 , 0 ,0 , 0, 0)	0.2560	1.1524	100	100
		Worst	(0 , 0 , 0 , 1 , 0 , 1 ,3 , 0 , 1)	23.0547	0.8330	1.11	138.34
		$\mathcal{R}1$	(6 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0)	0.2599	1.1719	98.49	98.33
		$\mathcal{R}\binom{n-1}{r-1}$	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 6)	2.3326	0.6780	10.97	169.97
		Complete	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0)	0.1972	1.3377	129.81	86.14
	70%	Optimal	(0 , 4 , 0 , 1 , 0 , 0 ,0 , 0 , 0, 0)	0.2336	1.1843	100	100
		Worst	(0 , 0 , 0 , 0 , 0 , 1 ,0 , 0 , 2, 2)	5.4106	0.7583	4.31	156.17
		$\mathcal{R}1$	(5 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0, 0)	0.2429	1.2119	96.17	97.72
		$\mathcal{R}\binom{n-1}{r-1}$	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0, 5)	0.9437	0.7056	24.75	167.84
		Complete	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0, 0)	0.1927	1.3364	121.22	88.61
	80%	Optimal	(0 , 0 , 2 , 0 , 0 , 1 ,0 , 0 , 0, 0 , 0)	0.2029	1.2375	100	100
		Worst	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 1 , 0, 1 , 1)	0.7452	0.9001	27.22	137.47
		$\mathcal{R}1$	(3 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0, 0 , 0)	0.2278	1.2651	89.07	97.81
		$\mathcal{R}\binom{n-1}{r-1}$	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0, 0 , 3)	0.2961	0.3414	68.53	153.75
		Complete	(0 , 0 , 0 , 0 , 0 , 0 ,0 , 0 , 0, 0 , 0)	0.1903	1.3450	106.60	92.00
20	80%	Optimal	(0,0,1,0,0,1,1,1,0,0,0,0,0,0,0,0)	0.1627	1.3348	100	100
		Worst	(0,0,0,0,0,0,0,0,1,0,0,0,0,0,3,0)	0.9264	1.0797	17.56	123.62
		$\mathcal{R}1$	(4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	0.1832	1.3571	88.81	98.35
		$\mathcal{R}\binom{n-1}{r-1}$	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4)	0.2074	0.8086	78.44	165.07
		Complete	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	0.1624	1.4249	100.18	93.67

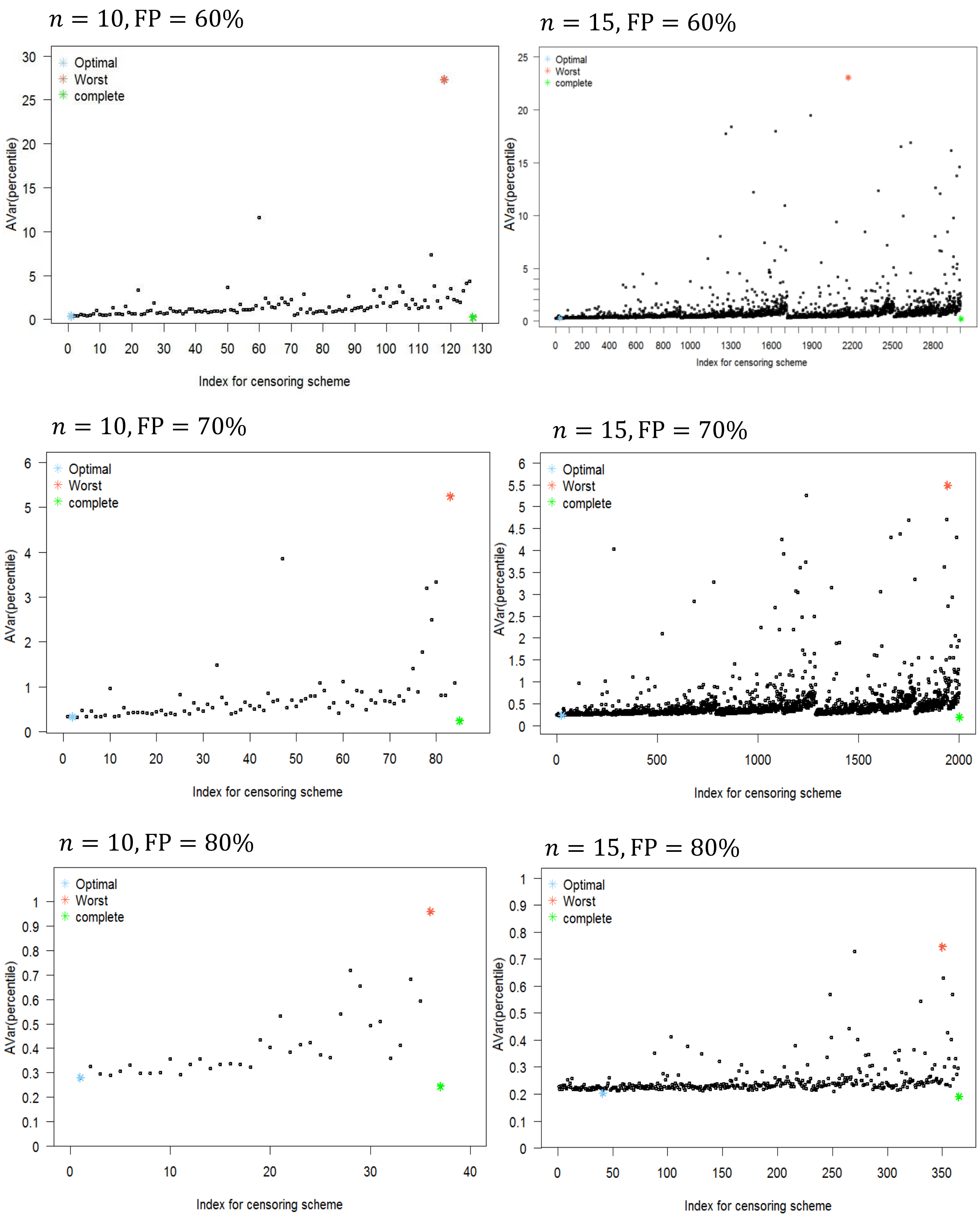


Figure 5.21 Plot of $AVar(\hat{t}_{0.5}(x_0))$ based on various censoring schemes.

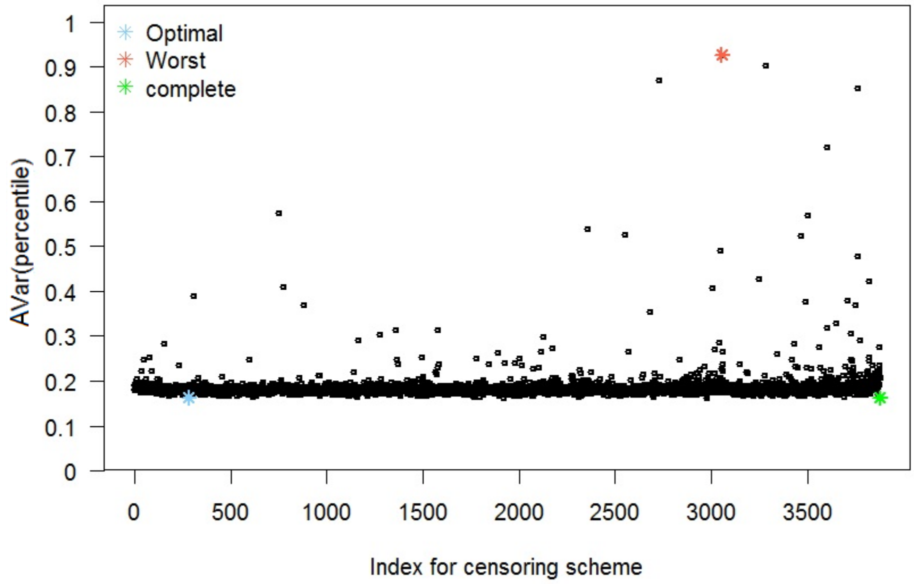


Figure 5.22 Plot of $AVar(\hat{t}_{0.5}(x_0))$ based on various censoring schemes for $n = 20$ and FP=80%.

For $n = 10$ with 60% FP, less than 2% of all schemes, only 3 schemes have RE $\geq 80\%$, which can be assumed the best schemes that improve the precision of the MLE of $t_{0.5}(x_0)$. All 3 best schemes involve removing items at the time of the first two failures, similar to the optimal $\mathcal{R}^*$. On the other hand, 25% of schemes have RE $\leq 20\%$, which can be assumed to be the worst schemes that leads to an increase the $AVar(\hat{t}_{0.5}(x_0))$. All the worst schemes involve removing items at the time of the last two failures. To sum up, the set of schemes that have similar structure of removing items as the optimal scheme can be assumed to be similar to the optimal scheme. So, the experiment can be set up using the most relevant scheme from this set of schemes. Also, it is concluded that $0.35 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 3$ for 90% of censoring schemes.

For $n = 10$ with 70% FP, 15% of schemes have RE $\geq 80\%$, which can be assumed as the best schemes that improve the precision of the MLE of $t_{0.5}(x_0)$. All 15% best schemes have the same structure of removing items at times of first failures like the optimal $\mathcal{R}^*$. On the other hand, 7% of schemes have RE $\leq 20\%$, which can be assumed to be the worst schemes that leads to increase the $AVar(\hat{t}_{0.5}(x_0))$. All worst schemes involve removing items at the time of the last two failures. Also, it is concluded that $0.33 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 3$ for 96% of censoring schemes, however, 96% of censoring schemes have $0.33 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 1.5$.

For $n = 10$ with 80% FP, 50% of schemes have RE $\geq 80\%$ and 12% have RE $\geq 95\%$. On the other hand, only 2 schemes (5%) have RE $\leq 40\%$. It can be seen for fixed n as FP increases, the censoring schemes become more efficient. Also, it is concluded that $0.27 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 1$ for all possible censoring schemes. So, the percent of best censoring schemes, that reduce the $AVar(\hat{t}_{0.5}(x_0))$, increases as the FP increases for fixed small sample size.

For $n = 15$ with 60% FP, 9% of schemes have RE $\geq 80\%$, whilst, 11% schemes have RE $\leq 20\%$. Also, it is concluded that $0.26 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 3$ for 98% of censoring schemes. Also, for $n = 15$ with 80% FP, 80% of schemes have RE $\geq 80\%$, however, 28% of schemes have RE $\geq 90\%$. In contrast, only 2% of total schemes have RE $\leq 40\%$. Also, it is concluded that $0.20 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 0.5$ for 99% of censoring schemes. These results mean that the schemes are robust and lead to high precision of the MLE. Thus, in this case, the experimenter would have a flexibility to choose the most practical scheme for the ALT. Also, for $n = 15$ with 70% FP, 22% of schemes have RE $\geq 80\%$. On the other hand, only 3% of all schemes have RE $\leq 20\%$. Also, it is concluded that $0.23 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 3$ for 99% of censoring schemes. In addition, for $n = 15$ with 80% FP, 80% of schemes have RE $\geq 80\%$, however, 28% of schemes have RE $\geq 90\%$. In contrast, only 2% of total schemes have RE $\leq 40\%$. Also, it is concluded that $0.20 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 0.5$ for 99% of censoring schemes. These results mean that the schemes are robust and lead to high precision of the MLE. Thus, in this case, the experimenter would have a flexibility to choose the most practical scheme for the ALT.

Moreover, for $n = 20$ with 80% FP, 96% of schemes have RE $\geq 80\%$, however, 48% of schemes have RE $\geq 90\%$. In contrast, only 1% of total schemes have RE $\leq 60\%$. Also, it is concluded that $0.16 \leq AVar(\hat{t}_{0.5}(x_0)) \leq 0.5$ for 99.7% of censoring schemes. These results mean that the schemes are robust and lead to high precision of the MLE. Thus, in this case, the experimenter would have a flexibility to choose the most practical scheme for the ALT. Also, from Figure 5.21 and Figure 5.22, it can be realized that the range of $AVar(\hat{t}_{0.5}(x_0))$ become narrow as the FP increases. That means the efficiency of censoring schemes $RE \to 100\%$ as FP increases.

In conclusion, from the above cases of small sample size with different FP and as it can be noticed from Figure 5.21 and Figure 5.22, the optimal scheme and the best schemes with respect to minimizing $AVar(\hat{t}_{0.5}(x_0))$ are located when items are removed at the time of early failures. On the other hand, the worst schemes that lead to increase the $AVar(\hat{t}_{0.5}(x_0))$ are located when items are censored at late failure times.

5.5 Summary

This chapter has studied the optimal design for the ALT based on progressive Type-II censoring from two main views: the optimal stress change time and the optimal censoring scheme. The V-optimality criterion is considered to obtain the optimal test design with respect to minimize the $AVar\left(\hat{t}_{\wp}(x_0)\right)$. The optimal test design was studied for the $5^{th}, 50^{th}$ and 95^{th} percentile lifetime of the GED at the usage stress level. The golden section search method was used to determine the optimal τ^* that minimizes the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ of GED under SSALT. Simulation studies were carried out using R to illustrate the theoretical results of the optimization problem. The impact of different sets of initial values n, FP, $\theta_1, \theta_2, \alpha$ and 9 censoring schemes on the optimal τ^* was studied. In addition, 11 different censoring schemes are compared, and the optimal $\mathcal{R}^*$ among them is determined for different set of initial values. Also, all possible censoring schemes are calculated and compared for small sample size to provide the optimal $\mathcal{R}^*$. The Monte Carlo standard errors of the estimated $AVar\left(\hat{t}_{\wp}(x_0)\right)$ are obtained, in some cases, to assess the precision of the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for the $5^{th}, 50^{th}$ and 95^{th} percentile lifetime of the GED at the usage stress level.

The simulation studies in this chapter indicate that as $\wp$ increases, more information must be collected from the experiment to get a precise MLE of the $t_{\wp}(x_0)$. This means the sample size and the FP are required to be large. Also, it was concluded that the choice of the censoring scheme to be applied under the test is crucial. The gain in efficiency of the MLEs under applying the optimal $\mathcal{R}^*$ compared with other schemes is remarkable in certain cases. Furthermore, it was noticed that the place of removing the items has almost the same results, whether the items are censored at one time or one-by-one, with respect to the $AVar\left(\hat{t}_{\wp}(x_0)\right)$. However, the difference is in the total test time. The test time when removing a set of items at one time will be shorter than the total test time when removing the items one by one.

The results in Section 5.4.2 show that designing the ALT based on choosing the optimal $\mathcal{R}^*$ will considerably increase efficiency. The RE of the worst scheme is <50% in many cases. Therefore, it is crucial to select the optimal $\mathcal{R}^*$.

The following Table 5.10 summarizes the impact of different initial values on the optimal τ^*.

Table 5.10 The impact of n, FP, $\mathcal{R}$, θ_1, θ_2, α on the optimal τ^*.

<table>
<tr><th>Initial variable</th><th>5^{th} percentile</th><th>50^{th} percentile</th><th>95^{th} percentile</th></tr>
<tr><td>n ↑</td><td colspan="3">$n < 60$ → slightly ↑
$n \geq 60$ → constant</td></tr>
<tr><td>FP ↑</td><td colspan="3">↓</td></tr>
<tr><td>$\mathcal{R}$ →</td><td colspan="3">↓</td></tr>
<tr><td>θ_1 ↑</td><td colspan="3">considerable ↑</td></tr>
<tr><td>θ_2 ↑</td><td colspan="3">constant</td></tr>
<tr><td rowspan="2">α ↑</td><td>$0.6 < \alpha \leq 1.0$ →
slightly ↓</td><td colspan="2">$0.6 < \alpha \leq 1.0$ →
slightly ↓</td></tr>
<tr><td colspan="3">$1.0 < \alpha \leq 1.6$ → constant
$\alpha > 1.6$ → ↑</td></tr>
</table>

From the simulation results, it can be noticed that the optimal τ^* is sensitive to small sample size or small FP. So, the case for a small observed sample size must be investigated. Furthermore, it is concluded that the optimal τ^* is not sensitive to θ_2 value. Therefore, it is interesting to investigate the relationship between the FP under each stress level and the optimal τ^*.

From Section 5.3, it was observed that the behaviour of optimal τ^* is the same with respect to all sets of initial values when estimating any of $5^{th}, 50^{th}$ and 95^{th} percentile of the GED under the usage stress level.

Surprisingly, the RE of the optimal $\mathcal{R}^*$ with respect to the complete sample can be below 100% whilst the RT <60%. This means that the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ based on the optimal $\mathcal{R}^*$ is less than when the complete sample case is used; besides, the optimal $\mathcal{R}^*$ yields reduction in the T of more than 40%. This happens when estimating 50^{th} and 95^{th} percentile with respect to increasing θ_1 which corresponds to decreasing the lower stress level to be close to the normal stress level. In this case, optimal $\mathcal{R}^* = \mathcal{R}3$ when removing items at the end of the test. Thus, removing the items at the end of the test with extreme timing yields reduces the $AVar\left(\hat{t}_{\wp}(x_0)\right)$.

The following Table 5.11 summarizes the impact of different initial values on the optimal $\mathcal{R}^*$. As a reminder, censoring scheme $\mathcal{R}2$ removes all censored items at the time of the failure occurrence in the middle of the test, whereas censoring schemes $\mathcal{R}1$ and $\mathcal{R}3$ remove all censored items at the first and the last observed failure, respectively.

Table 5.11 The impact of n, FP, θ_1, θ_2, α, τ on the optimal $\mathcal{R}^*$.

Initial variable	**scheme**	**5^{th} percentile**	**50^{th} percentile**	**95^{th} percentile**
n	optimal	$\mathcal{R}2$	$n \leq 50 \rightarrow \mathcal{R}7$ $n > 50 \rightarrow \mathcal{R}4$	
	worst		$\mathcal{R}3$	
FP - 40	optimal	$\mathcal{R}6$, $\mathcal{R}1$	Not specified	Not specified
	worst		$\mathcal{R}3$	
FP - 100	optimal	$\mathcal{R}2$	$\mathcal{R}1$	Not specified
	worst		$\mathcal{R}3$	
θ_1	optimal	$\mathcal{R}2$, $\mathcal{R}7$	$\theta_1 = 0.6 \rightarrow \mathcal{R}4$ $\theta_1 > 0.6 \rightarrow \mathcal{R}3, \mathcal{R}8$	$\theta_1 = 0.6 \rightarrow \mathcal{R}4$ $0.6 < \theta_1 \leq 1.8 \rightarrow \mathcal{R}3$ $\theta_1 > 1.8 \rightarrow \mathcal{R}1$
	worst		$\theta_1 = 0.6 \rightarrow \mathcal{R}3$	
		$\theta_1 > 0.6 \rightarrow \mathcal{R}1$	$0.6 < \theta_1 \leq 2 \rightarrow \mathcal{R}1, \mathcal{R}6$ $\theta_1 > 2 \rightarrow \mathcal{R}10$	$0.6 < \theta_1 \leq 1.6 \rightarrow \mathcal{R}6$ $\theta_1 > 1.6 \rightarrow \mathcal{R}2, \mathcal{R}7$
θ_2	optimal	$\mathcal{R}2$	$\mathcal{R}4$	$\mathcal{R}4$, $\mathcal{R}9$
	worst		$\mathcal{R}3$	
α	optimal	$\mathcal{R}2$	$\alpha \leq 1 \rightarrow \mathcal{R}1$	
			$1.2 \leq \alpha \leq 1.4 \rightarrow \mathcal{R}4, \mathcal{R}9$	
			$\alpha > 1.4 \rightarrow \mathcal{R}3$	

<table>
<tr><td rowspan="2"></td><td rowspan="2">worst</td><td colspan="2">$\alpha \leq 1.4 \rightarrow \mathcal{R}3$</td></tr>
<tr><td>$\alpha > 1.4 \rightarrow \mathcal{R}1$</td><td>$\alpha > 1.4 \rightarrow \mathcal{R}6$</td></tr>
<tr><td rowspan="3">τ</td><td rowspan="2">optimal</td><td>$\tau \leq 0.8 \rightarrow \mathcal{R}2$</td><td>$\tau \leq 0.6 \rightarrow \mathcal{R}3,\ \mathcal{R}8$</td></tr>
<tr><td colspan="2">$\tau \geq 0.9 \rightarrow \mathcal{R}1$, $\mathcal{R}6$</td></tr>
<tr><td>worst</td><td colspan="2">$\tau \leq 0.7 \rightarrow \mathcal{R}1, \mathcal{R}6$
$\tau > 0.8 \rightarrow \mathcal{R}3$</td></tr>
</table>

From the above table, some general observations are made below.

1- It can be seen that although it is not easy to determine the optimal $\mathcal{R}^*$ that minimizes the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for the three given $\wp$ simultaneously, it is possible to determine the optimal $\mathcal{R}^*$ that reduces the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for both the 50^{th} percentile and the 95^{th} percentile simultaneously.

2- For estimating the 5^{th} percentile, the optimal $\mathcal{R}^*$ is often when censored items are removed at the time of the failure occurrence in the middle of the test ($\mathcal{R}2$).

3- for estimating the 50^{th} and 95^{th} percentile, the optimal $\mathcal{R}^*$ is when removing the censored items at the time of first failure ($\mathcal{R}1$) or last failure ($\mathcal{R}3$), or at the time of both first and last failures ($\mathcal{R}4$). In other words, the optimal scheme removes the censored items that are located in the tails of the life distribution.

4- The censoring scheme $\mathcal{R}3$ which is Type-II censoring, is often the worst scheme, which leads to maximizing the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for 5^{th}, 50^{th} and 95^{th} percentile for many cases. Nevertheless, $\mathcal{R}3$ is the optimal scheme when the GED density becomes more skewed or if the difference between the stress levels increases.

5- The optimal censoring scheme $\mathcal{R}^*$ is affected by changing the initial values, especially θ_1 and α.

However, from the above results, it is concluded that the optimal scheme is sensitive to the change in any of the initial values: of n, FP, θ_1, θ_2, α, τ. There is no scheme that would be optimal for all scenarios of the initial values. Thus, more studies should be conducted by testing more schemes which are closed to the optimal and worst scheme in each case. Furthermore, it is suggested using an optimality criterion for obtaining robust test plans based on optimal censoring schemes that results in minimizing the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ under different sets of initial values.

As discussed at the beginning of Section 5.4, the number of schemes based on a given n and FP is $\binom{n-1}{r-1}$. This number is enormous and requires considerable time to compute the objective function based on each of these schemes. So, finding a method to obtain the optimal $\mathcal{R}^*$ without testing each of the $\binom{n-1}{r-1}$ censoring schemes is essential.

Although the given simulation studies in this chapter provide a basic reference for designing ALT with progressive Type-II censored data, it is not easy to generalize the result for all values of the initial value.

However, this chapter shows the importance of using the progressive Type-II censoring scheme with respect to getting the most precise estimate of the percentile with the smallest value of $AVar\left(\hat{t}_{\wp}(x_0)\right)$ for 5^{th}, 50^{th} and 95^{th} percentiles. Also, it is essential to design the ALT with the optimal $\mathcal{R}^*$ and optimal τ^* in order to obtain the most precise estimates of the model parameters.

Chapter 6

Conclusion and Future Work

6.1 General Discussion and Conclusion

In lifetime data analysis, life testing for items under normal use conditions can often take a long time to obtain a reasonable number of failures. Thus, there can be high cost to obtain information about the lifetime of products under normal condition as it takes a long time to observe failures. In this situation, ALT procedures are performed in order to obtain failure time data in a shorter time. The SSALT model is a commonly used method in life testing. It allows the stress level applied to each test unit to be changed step-by-step during the test. In this way information about the parameters of the life distribution is obtained more quickly than under normal operating conditions.

There are two major aspects in the SSALT studies: statistical inference of model parameters and optimal test design. The main objectives of this thesis were to develop an optimal test plan for a simple SSALT model and to make statistical inferences for a simple SSALT model based on progressive type-II censoring schemes for a determined set of initial values. General design steps and assumptions for modelling SSALT under progressive Type-II censored samples have been described and discussed in detail. The GED is assumed as a lifetime distribution and the CEM is also assumed for the CDF of failure times under different stress levels.

6.1.1 Analysis of SSALT Under Progressive Type-II Censoring

Statistical inference for model parameters was discussed. The ML method has been used to estimate the three unknown parameters of the CEM; θ_1, θ_2 and α based on different sets of initial values. Also, the asymptotic CIs for the parameters based on the observed Fisher information matrix have been derived. Moreover, the bootstrap approach has been utilized to obtain the CIs for the model parameters using two methods, percentile and BCa based on parametric bootstrap samples.

The MLEs and the CIs based on three methods were obtained numerically because they are not in closed-form expressions. Extensive Monte-Carlo simulation was performed, using R and the High-Performance Computing facility Iridis-4 and Iridis-5 with PBS and SLURM commands. To assess the performance of the MLEs and CIs, simulation studies have been carried out under different sample sizes, FP, stress change time and PCSs. The performance of the MLEs for the model parameters θ_1, θ_2 and α have been evaluated by using AB and MSE. The bootstrap and jackknife resampling methods have been used to estimate the bias and MSE of the MLEs. Also, the CPs of the intervals as well as the AL of CIs have been determined at nominal confidence levels 90%, 95% and 99% to study the performance of the CIs for the parameters.

It is noticed that the MLE of θ_1 is remarkably influenced by the number of failures under lower stress level. So, to get precise estimates of θ_1 with small AB and MSE, the number of observed data under the lower stress level should be increased. That could be done by increasing the value of the stress change time, increasing the sample size, avoiding removing items in the early stages of the test or by a balance between these three factors. On the other hand, the number of failures at each stress level has less impact on the estimate of θ_2. The MLE of the scale parameter θ_2 is less sensitive than MLE of θ_1.

Based on the assumed sets of initial values, it is concluded that the performance of the CIs under asymptotic approach is close to the performance of the CIs based on BCa bootstrap method. However, the asymptotic CIs are shorter in all cases than the other two methods, whereas, the BCa bootstrap CIs provide better CP that is close to the nominal level.

6.1.2 Design of SSALT Under Progressive Type-II Censoring

In some situations, under a simple SS model, the hold time at low stress levels might be relatively short, resulting in few or no failure data and thus affecting the performance of the MLEs. Thus, determining an optimal hold time at low stress is essential to obtain sufficient information at different stress levels. Moreover, choosing the optimal censoring scheme is crucial for censoring items at the best time to assure having enough failures in each stress level. This will improve the efficiency of statistical inference.

In ALT, determining the optimal design is essential to maximize the precision of the estimates of percentile lifetime under the GED at usage stress level. Therefore, optimal test design under V-optimality criterion has been studied in detail based on determining the optimal stress change time and optimal censoring scheme. Determining the optimal design has been studied for $5^{th}, 50^{th}$ and 95^{th} percentiles under usage stress level. The impact of a different set of initial values n, FP, $\theta_1, \theta_2, \alpha$ and nine censoring schemes on the optimal τ^* has been studied. In addition, 11 different censoring schemes are compared, and the optimal $\mathcal{R}^*$ among them is determined for a different set of initial values. Also, all possible censoring schemes are calculated and compared for small sample size to provide the optimal $\mathcal{R}^*$.

It is observed that the behaviour (increase/decrease) of optimal τ^* is the same with respect to all assumed sets of initial values when estimating any of $5^{th}, 50^{th}$ and 95^{th} percentile of the GED under the usage stress level. Moreover, for the suggested sets of initial values, it was noticed that the range of optimal τ^* for estimating the 50^{th} percentile intersects with the range of optimal τ^* for estimating the 95^{th} percentile. So, the 50^{th} percentile and the 95^{th} percentile lifetime of the GED under the usage stress level can be estimated with reducing the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ based on the same value of optimal τ^*. Also, it is concluded that the choice of the censoring scheme is crucial to decreases the $AVar\left(\hat{t}_{\wp}(x_0)\right)$. Also, it is noticed that the place of removing the items has almost the same results, whether the items are censored at one time or one-by-one, with respect to the $AVar\left(\hat{t}_{\wp}(x_0)\right)$. However, the difference is in the total test time. The test time when removing a set of items at one time will be shorter than the total test time when removing the items one by one.

6.2 Limitation of Study and Related Suggestion

In Chapter 4, the range of the initial values is limited. However, it could be suggested to consider a broader range of the initial values to investigate their impact on the MLEs and the sensitivity of the model parameters. This should be studied in future work.

Also, it should be noted that the results in Chapter 5 are limited to the set of fixed initial values. In other words, we investigate the impact of changing one initial value at a time, except by changing the scale parameters θ_1 and θ_2 together. However, assuming the stress levels are fixed, there are 7 initial values: θ_1, θ_2, α, n, FP, τ and censoring scheme. So, if only 3 values are chosen for each variable, then there are 2187 possible sets of initial values, which makes it impossible to test all of them. Now, these $3^7 = 2187$ combinations can be viewed as a full factorial design with 7 factors, each at 3 levels. Therefore, in future work, a fractional factorial design with a more manageable number of combinations could be chosen to enable us to study how changing variable values simultaneously affects the optimal value of τ.

It is noticed that the run time of the simulation is varied according different proposed methods. The computational time to run the program and get the result is between 5 minutes for deriving the MLEs and 8-10 hours for constructing the bootstrap CIs in the statistical inference part in Chapter 4. While in the design part at Chapter 5, the computational run time ranges between 6 hours and 56 hours using the high-performance computing facilities (Iridis) at the University of Southampton. As we know that Iridis speeds up the computational process to get the results faster, then, the computational process may take a longer time with a normal CPU. Therefore, considering the cost of the computational process is of interest in terms of the cost of using computational facilities and the time required to get the results. However, the maximum time allowable in Iridis (60 hours) is a limitation for my calculation in the design chapter. I was not able to take more than 50 replications for the optimal τ^* as in some cases I found that the run time exceeded 60 hours and the jobs were terminated before completion. Also, I could not test all censoring schemes for larger values of n and FP as the number of censoring schemes will be > 4000 and would take more than 60 hours.

The suggestion to solve this problem of limited running time is to use parallel computing. As the censoring schemes are parallel, so parallel jobs can be used to calculate the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ under the various censoring schemes. Then, using a series job to find the optimal $\mathcal{R}^*$ that corresponds to the minimum $AVar\left(\hat{t}_{\wp}(x_0)\right)$ besides the near optimal $\mathcal{R}^*$ that have RE ≥ 95%. To take advantage of parallel computing using (mpirun) in Iridis, a parallel computing R script should be written using a parallel package, such as (doParallel) or (Rmpi) to write R commands using the message passing Interface with multi-core or Open Multiprocessing with multi-node. So, multiple tasks will be running in parallel. It is more efficient than series jobs with respect to reducing the running time of total computation.

It is important to note that some of the suggestions I already started to study, but due to limited time of the PhD and difficulties of programming the strategy, I was not able to get or discuss results.

6.3 Extensions and Generalization of Current Model

In this thesis, the parametric bootstrap approach has been used to construct the CIs of the model parameter. The parametric bootstrap distribution is smoother than the nonparametric one, because the samples are drawn from a continuous distribution. Therefore, the parametric bootstrap method may be more accurate than nonparametric bootstrap, just if the original distribution is estimated efficiently. Due to the complexity of the SSALT model and CEM, it may difficult to estimate the distribution of the population. Therefore, it may be beneficial to use the nonparametric bootstrap method to calculate the CIs of the model parameter. So, the original sample is used to represent an unknown population. In a nonparametric bootstrap, B samples are re-sampled with replacement from the original sample. For each of these B samples, the MLEs for the model parameters are calculated to obtain a sample of B bootstrap estimates. Then, the nonparametric bootstrap CIs for the parameters are derived and their performance are compared with parametric bootstrap CIs and asymptotic CIs.

However, it should be mentioned that the computational time for calculating the bootstrap CIs increases as the sample size or FP increases. So, it is of interest to compare the computational time for obtaining the CIs based on different methods that are used to estimate the CIs for model parameters. As reducing the analysis time leads to reduce the total time requires for life test analysis and associated costs.

One of the interesting aspects of designing SSALT under progressive Type-II censoring is the impact of optimal τ^* with the optimal $\mathcal{R}^*$ on the MLE of the $100\wp^{th}$percentile lifetime under the GED at usage stress level. So, the optimal τ^* could be determined under the 11 suggested schemes. Then, the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ could be calculated based on each of the 11 schemes with their related optimal τ^* to compare their effects on the $AVar\left(\hat{t}_{\wp}(x_0)\right)$ and determine the optimal $\mathcal{R}^*$. In this case, the experimenter will have the optimal censoring scheme to set for the SSALT with the optimal time to change the stress level.

From Chapter 4 and Chapter 5, we concluded that the number of failures under each stress level has a considerable effect on the estimation of the parameter of interest. The stress level is changed from lower stress level to higher stress level at a pre-determined stress change time. So, the number of failures under lower and higher stress levels are random. For this reason, there may no or a small number of failures at either lower or higher stress levels. Therefore, it may worth to change the stress level after predetermined number of failures occured. This model is called a failure-stress SSALT model in which items are tested under lower stress level until a specified proportion of failures under lower stress level (see Nelson, 1990 for more details).

In planning SSALTs, different factors affect the cost of the test, such as the sample size, the number of observed failures and the cost of equipment and facilities to apply the experiment. However, the experimental budget is usually predetermined and fixed. So, it is crucial to consider designing the SSALTs under cost constraints.

For further study, it is planned to examine different optimality criteria, such as A-optimality and D-optimality criteria. Then, the results of optimal τ^* and optimal $\mathcal{R}^*$ can be compared with results obtained based on V-optimality criteria. The A-optimality and the D-optimality criteria are used to measure the overall variability. However, A-optimality is used when the correlation between the MLEs of the model parameters is low (Ng, et al., 2004). On the other hand, D-optimality is used when there is a correlation between the MLEs. In A-optimality, the objective function is based on the sum of $AVar$ of the MLEs of the model parameters. Thus, it is the sum of the diagonal elements of the AV-C matrix. The optimal τ^* can be calculated by minimizing the trace function of the AV-C matrix $Tr\left(F^{-1}(\hat{\delta}_k)\right)$, where

$$Tr\left(F^{-1}(\hat{\delta}_k)\right) = AVar(\hat{\theta}_1) + AVar(\hat{\theta}_2) + AVar(\hat{\alpha}), \qquad k = 1,2,3$$

If the life-stress relationship parameters are the most important to estimate with high precision, then D-optimality should be used (Gouno et al., 2004). In D-optimality, the optimal τ^* is obtained by minimizing the determinant of the Fisher information matrix of the MLEs of the model parameters. The determinant of the Fisher information matrix can be calculated using equation (4.14) in Chapter 4, as follows

$$\begin{aligned}|F^{-1}(\hat{\delta}_k)| &= AVar(\hat{\theta}_1)[AVar(\hat{\theta}_2) \times AVar(\hat{\alpha}) - \mathrm{Cov}(\hat{\theta}_2, \hat{\alpha}) \times \mathrm{Cov}(\hat{\alpha}, \hat{\theta}_2)] \\ &\quad - \mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)[\mathrm{Cov}(\hat{\theta}_2, \hat{\theta}_1) \times AVar(\hat{\alpha}) - \mathrm{Cov}(\hat{\theta}_2, \hat{\alpha}) \times \mathrm{Cov}(\hat{\alpha}, \hat{\theta}_1)] \\ &\quad + \mathrm{Cov}(\hat{\theta}_1, \hat{\alpha})[\mathrm{Cov}(\hat{\theta}_2, \hat{\theta}_1) \times \mathrm{Cov}(\hat{\alpha}, \hat{\theta}_2) - AVar(\hat{\theta}_2) \times \mathrm{Cov}(\hat{\alpha}, \hat{\theta}_1)]\end{aligned}$$

The simple SSALT model has been assumed in this thesis based on progressively Type-II censored data. In simple SSALT, two higher stress levels are assumed to accelerate the failure times of the products. Thus, extending the results to a more general model, a 3-step SSALT model is of natural interest. In the 3-step SSALTs model, three higher stress levels are assumed. In simple SSALTs, it is noticed that as more failures occur at lower stress, the precision of the estimates is increased. So, it is interesting to investigate the impact of testing items on three stress levels close to the usage stress level. The results can be compared with simple SSALT in two aspects, the time of the total test and the precision of the parameter estimators.

6.4 Related Issues of Interest

This thesis studies the statistical analysis and the design of the SSALT model based on progressive Type-II censoring. In the future plan, it would be interesting to suppose the SSALT model based on other censoring schemes, such as Type-I and Type-II hybrid censoring schemes.

In reliability and life testing, two accelerated models can be used to accelerate failure time: fully, or simply, ALTs and partially-ALTs. In the proposed SSALT model, items are tested under two higher stress levels. In partially-ALTs, the units are tested first under a usage stress level to observe failure times, and then non-failed units are tested further under a higher stress level. So, the partially-ALT model allows the experimenter to collect some of the observed failure times under a usage stress level. The partially-ALT model contains an acceleration factor, which describes the relationship between usage and accelerated stress level. The partially-ALT design based on GED could be illustrated under progressive Type-II censoring schemes. Results could be compared with the ALT under PCSs with respect to the estimator precision of the model parameters.

Appendix A

Statistical Inference Results

Table A.1 The AB and MSE for the ML, bootstrap and jackknife estimates of the scale parameter θ_1 at $(\theta_1 = 0.6, \theta_2 = 0.3, \alpha = 1.2, \tau = 0.4)$ for different n, FP and censoring schemes (C.Sc). The standard error of estimated AB and MSE are presented alongside the ML measures.

n	FP% $= r$	C.Sc	ML					Bootstrap		Jackknife	
			MLE	AB	SE(AB)	MSE	SE(MSE)	AB	MSE	AB	MSE
40	60% = 24	$\mathcal{R}1$	0.6473	0.2036	0.0042	0.2307	0.0132	0.2800	1.4595	0.0783	0.3232
		$\mathcal{R}2$	0.6638	0.2522	0.0082	0.3000	0.0249	0.4661	7.9451	0.1195	1.1642
		$\mathcal{R}3$	0.6176	0.2127	0.0043	0.2475	0.0204	0.2634	0.7325	0.0419	0.1945
		$\mathcal{R}4$	0.6330	0.2185	0.0045	0.2819	0.0201	0.2780	1.0023	0.0614	0.243
	80% = 32	$\mathcal{R}1$	0.6353	0.2046	0.0046	0.2574	0.0273	0.2570	0.5556	0.0711	0.3668
		$\mathcal{R}2$	0.6543	0.2336	0.0051	0.3352	0.0239	0.3238	1.3875	0.0741	0.4039
		$\mathcal{R}3$	0.6339	0.2032	0.0043	0.2310	0.0184	0.2569	0.5073	0.0503	0.1591
		$\mathcal{R}4$	0.6296	0.2042	0.0043	0.2392	0.0224	0.2630	0.7341	0.057	0.2238
	100% = 40	$\mathcal{R}0$	0.6399	0.2050	0.0045	0.2416	0.0200	0.2494	0.4337	0.0641	0.2383
60	60% = 36	$\mathcal{R}1$	0.6166	0.1637	0.0030	0.1159	0.0048	0.1894	0.1692	0.0438	0.2061
		$\mathcal{R}2$	0.6588	0.2119	0.0045	0.2495	0.0159	0.2720	0.567	0.0514	0.2402
		$\mathcal{R}3$	0.6120	0.1692	0.0029	0.1265	0.0048	0.1907	0.1293	0.0200	0.0964
		$\mathcal{R}4$	0.6189	0.1760	0.0032	0.1403	0.0065	0.2043	0.1642	0.0314	0.0866
	80% = 48	$\mathcal{R}1$	0.6235	0.1655	0.0031	0.1203	0.0052	0.1870	0.132	0.0345	0.1152
		$\mathcal{R}2$	0.6334	0.1808	0.0036	0.1621	0.0099	0.2206	0.245	0.0344	0.1022
		$\mathcal{R}3$	0.6199	0.1696	0.0030	0.1217	0.0053	0.1863	0.1188	0.0314	0.1525
		$\mathcal{R}4$	0.6206	0.1676	0.0030	0.1213	0.0060	0.1857	0.1346	0.0273	0.0736
	100% = 60	$\mathcal{R}0$	0.6228	0.1641	0.0030	0.1175	0.0048	0.1865	0.1274	0.0317	0.0793
80	60% = 48	$\mathcal{R}1$	0.6172	0.1399	0.0023	0.0763	0.0023	0.1594	0.0852	0.0330	0.1759
		$\mathcal{R}2$	0.6387	0.1767	0.0035	0.1548	0.0096	0.2108	0.2073	0.0273	0.0852
		$\mathcal{R}3$	0.6023	0.1505	0.0026	0.0911	0.0037	0.1609	0.0670	0.0095	0.0433
		$\mathcal{R}4$	0.6158	0.1528	0.0026	0.0951	0.0042	0.1644	0.0758	0.0198	0.0529
	80% = 64	$\mathcal{R}1$	0.6171	0.1397	0.0025	0.0803	0.0032	0.1523	0.0625	0.0247	0.0874
		$\mathcal{R}2$	0.6186	0.1528	0.0027	0.0981	0.0037	0.1730	0.0923	0.0216	0.0555
		$\mathcal{R}3$	0.6108	0.1434	0.0024	0.0786	0.0027	0.1539	0.0616	0.0169	0.0464
		$\mathcal{R}4$	0.6130	0.1394	0.0024	0.0784	0.0036	0.1529	0.0617	0.0171	0.0445
	100% = 80	$\mathcal{R}0$	0.6152	0.1408	0.0025	0.0792	0.0034	0.1533	0.0609	0.0212	0.0439

Table A.2 The AB and MSE for the ML, bootstrap and jackknife estimates of the scale parameter θ_1 at ($\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.6$) for different n, FP and censoring schemes. The standard error of estimated AB and MSE are presented alongside the ML measures.

n	FP% $= r$	C.Sc	ML					Bootstrap		Jackknife	
			MLE	AB	SE(AB)	MSE	SE(MSE)	AB	MSE	AB	MSE
40	60% = 24	$\mathcal{R}1$	0.7299	0.1833	0.0049	0.1618	0.0084	0.2331	0.2126	0.0721	0.3625
		$\mathcal{R}2$	0.6403	0.1979	0.0041	0.2108	0.0145	0.2445	0.7339	0.0379	0.1541
		$\mathcal{R}3$	0.6099	0.1815	0.0031	0.1441	0.0053	0.1925	0.1327	0.0154	0.0605
		$\mathcal{R}4$	0.6170	0.1802	0.0034	0.1499	0.0085	0.2007	0.1784	0.0237	0.0766
	80% = 32	$\mathcal{R}1$	0.6162	0.1513	0.0027	0.1005	0.0049	0.1695	0.0949	0.0309	0.1190
		$\mathcal{R}2$	0.6240	0.1720	0.0033	0.1388	0.0066	0.1982	0.2417	0.0255	0.0829
		$\mathcal{R}3$	0.6160	0.1623	0.0027	0.1084	0.0037	0.1783	0.1060	0.0146	0.0532
		$\mathcal{R}4$	0.6079	0.1602	0.0028	0.1084	0.0049	0.1745	0.1045	0.0178	0.0592
	100% = 40	$\mathcal{R}0$	0.6183	0.1562	0.0028	0.1025	0.0040	0.1699	0.0971	0.0230	0.0609
60	60% = 36	$\mathcal{R}1$	0.6853	0.1289	0.0029	0.0724	0.0023	0.1311	0.0582	0.0492	0.2579
		$\mathcal{R}2$	0.6255	0.1584	0.0030	0.1124	0.0062	0.1783	0.1108	0.0192	0.0662
		$\mathcal{R}3$	0.6015	0.1514	0.0025	0.0897	0.0035	0.1567	0.0625	0.0096	0.0402
		$\mathcal{R}4$	0.6076	0.1443	0.0024	0.0821	0.0029	0.1518	0.0584	0.0120	0.0382
	80% = 48	$\mathcal{R}1$	0.6095	0.1226	0.0020	0.0574	0.0019	0.1306	0.0399	0.0202	0.0809
		$\mathcal{R}2$	0.6177	0.1379	0.0024	0.0762	0.0025	0.1463	0.0573	0.0136	0.0422
		$\mathcal{R}3$	0.6075	0.1332	0.0022	0.0677	0.0024	0.1366	0.0428	0.0077	0.0302
		$\mathcal{R}4$	0.6030	0.1298	0.0020	0.0621	0.0017	0.1356	0.0422	0.0088	0.0299
	100% = 60	$\mathcal{R}0$	0.6121	0.1214	0.0020	0.0567	0.0017	0.1315	0.0403	0.0126	0.0311
80	60% = 48	$\mathcal{R}1$	0.6645	0.1104	0.0024	0.0497	0.0016	0.1189	0.3312	0.0410	0.2241
		$\mathcal{R}2$	0.6175	0.1338	0.0023	0.0713	0.0025	0.1488	0.0610	0.0109	0.0381
		$\mathcal{R}3$	0.6066	0.1326	0.0021	0.0653	0.0019	0.1351	0.0389	0.0064	0.0306
		$\mathcal{R}4$	0.6010	0.1213	0.0018	0.0532	0.0014	0.1252	0.0336	0.0084	0.0266
	80% = 64	$\mathcal{R}1$	0.6053	0.1096	0.0017	0.0415	0.0011	0.1122	0.0260	0.0157	0.0669
		$\mathcal{R}2$	0.6135	0.1187	0.0019	0.0524	0.0015	0.1272	0.0367	0.0090	0.0273
		$\mathcal{R}3$	0.6018	0.1123	0.0017	0.0452	0.0011	0.1169	0.0277	0.0054	0.0224
		$\mathcal{R}4$	0.6057	0.1117	0.0018	0.0458	0.0013	0.1177	0.0287	0.0062	0.0203
	100% = 80	$\mathcal{R}0$	0.6059	0.1081	0.0017	0.0417	0.0012	0.1120	0.0256	0.0091	0.0221

Table A.3 The AB and MSE for the ML, bootstrap and jackknife estimates of the scale parameter θ_2 at ($\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.4$) for different n, FP and censoring schemes. The standard error of estimated AB and MSE are presented alongside the ML measures.

n	FP% $= r$	C.Sc		ML				Bootstrap		Jackknife	
			MLE	AB	SE(AB)	MSE	SE(MSE)	AB	MSE	AB	MSE
40	60% = 24	$\mathcal{R}1$	0.2929	0.0921	0.0013	0.0307	6e-04	0.0878	0.0142	0.0275	0.0784
		$\mathcal{R}2$	0.2968	0.0637	9e-04	0.0136	2e-04	0.0634	0.0069	0.0096	0.0073
		$\mathcal{R}3$	0.2935	0.0750	0.0011	0.0193	4e-04	0.0733	0.0093	0.0089	0.0101
		$\mathcal{R}4$	0.2982	0.0761	0.0011	0.0198	3e-04	0.07311	0.0093	0.0105	0.0089
	80% = 32	$\mathcal{R}1$	0.2964	0.0626	9e-04	0.0131	2e-04	0.0617	0.0065	0.0104	0.0358
		$\mathcal{R}2$	0.2996	0.0550	8e-04	0.0102	2e-04	0.0555	0.0052	0.0053	0.0055
		$\mathcal{R}3$	0.3005	0.0602	8e-04	0.0121	2e-04	0.0588	0.0059	0.0052	0.0060
		$\mathcal{R}4$	0.2994	0.0597	8e-04	0.0120	2e-04	0.0586	0.0058	0.0056	0.0060
	100% = 40	$\mathcal{R}0$	0.2989	0.0495	7e-04	0.0082	1e-04	0.0497	0.0041	0.0034	0.0043
60	60% = 36	$\mathcal{R}1$	0.2953	0.0770	0.0011	0.0205	4e-04	0.0738	0.0096	0.0199	0.0743
		$\mathcal{R}2$	0.2979	0.0532	7e-04	0.0091	1e-04	0.0524	0.0046	0.0050	0.0048
		$\mathcal{R}3$	0.2971	0.0632	9e-04	0.0131	2e-04	0.0598	0.0060	0.0050	0.0067
		$\mathcal{R}4$	0.2949	0.0630	9e-04	0.0129	2e-04	0.0609	0.0063	0.0047	0.0058
	80% = 48	$\mathcal{R}1$	0.2990	0.0504	7e-04	0.0084	1e-04	0.0507	0.0043	0.0057	0.0326
		$\mathcal{R}2$	0.3006	0.0455	6e-04	0.0068	1e-04	0.0454	0.0034	0.0029	0.0035
		$\mathcal{R}3$	0.2991	0.0493	7e-04	0.0079	1e-04	0.0488	0.0039	0.0023	0.0040
		$\mathcal{R}4$	0.2995	0.0491	7e-04	0.0079	1e-04	0.0486	0.0039	0.0025	0.0039
	100% = 60	$\mathcal{R}0$	0.2981	0.0406	6e-04	0.0053	1e-04	0.0408	0.0027	0.0013	0.0027
80	60% = 48	$\mathcal{R}1$	0.2972	0.0650	9e-04	0.0144	2e-04	0.0671	0.0739	0.0139	0.0698
		$\mathcal{R}2$	0.3005	0.0460	6e-04	0.0069	1e-04	0.0454	0.0034	0.0031	0.0035
		$\mathcal{R}3$	0.2985	0.0525	7e-04	0.0092	1e-04	0.0525	0.0046	0.0028	0.0049
		$\mathcal{R}4$	0.2976	0.0535	7e-04	0.0095	1e-04	0.0537	0.0048	0.0028	0.0044
	80% = 64	$\mathcal{R}1$	0.2985	0.0444	6e-04	0.0064	1e-04	0.0438	0.0032	0.0041	0.0317
		$\mathcal{R}2$	0.3003	0.0386	6e-04	0.0050	1e-04	0.0398	0.0026	0.0016	0.0026
		$\mathcal{R}3$	0.3011	0.0426	6e-04	0.0059	1e-04	0.0422	0.0029	0.0014	0.0030
		$\mathcal{R}4$	0.2999	0.0411	6e-04	0.0057	1e-04	0.0422	0.0029	0.0010	0.0029
	100% = 80	$\mathcal{R}0$	0.2997	0.0352	5e-04	0.0040	1e-04	0.0355	0.0020	0.0010	0.0021

Table A.4 The AB and MSE for the ML, bootstrap and jackknife estimates of the scale parameter θ_2 at ($\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.6$) for different n, FP and censoring schemes. The standard error of estimated AB and MSE are presented alongside the ML measures.

n	FP% $= r$	C.Sc	ML					Bootstrap		Jackknife	
			MLE	AB	SE(AB)	MSE	SE(MSE)	AB	MSE	AB	MSE
40	60% = 24	$\mathcal{R}1$	0.2820	0.1346	0.0024	0.0717	0.0024	0.1090	0.0242	0.0420	0.1196
		$\mathcal{R}2$	0.2970	0.0731	0.0010	0.0183	3e-04	0.0717	0.0088	0.0100	0.0101
		$\mathcal{R}3$	0.2911	0.0868	0.0012	0.0262	4e-04	0.0839	0.0122	0.0138	0.0151
		$\mathcal{R}4$	0.2920	0.0905	0.0012	0.0293	5e-04	0.0869	0.0131	0.0178	0.0166
	80% = 32	$\mathcal{R}1$	0.2948	0.0794	0.0011	0.0219	4e-04	0.0772	0.0103	0.0178	0.0529
		$\mathcal{R}2$	0.2989	0.0663	9e-04	0.0142	2e-04	0.0624	0.0065	0.0067	0.0075
		$\mathcal{R}3$	0.2969	0.0695	9e-04	0.0163	3e-04	0.0674	0.0076	0.0069	0.0089
		$\mathcal{R}4$	0.2979	0.0709	0.001	0.0168	3e-04	0.0677	0.0077	0.0069	0.0092
	100% = 40	$\mathcal{R}0$	0.2976	0.0571	8e-04	0.0108	2e-04	0.0564	0.0053	0.0037	0.0058
60	60% = 36	$\mathcal{R}1$	0.2874	0.1276	0.0022	0.0653	0.0020	0.0936	0.1003	0.0386	0.1223
		$\mathcal{R}2$	0.2975	0.0611	8e-04	0.0123	2e-04	0.0590	0.0058	0.0055	0.0067
		$\mathcal{R}3$	0.3003	0.0747	0.0010	0.0184	3e-04	0.0692	0.0081	0.0070	0.0097
		$\mathcal{R}4$	0.2973	0.0769	0.0010	0.0202	3e-04	0.0732	0.0090	0.0085	0.0103
	80% = 48	$\mathcal{R}1$	0.2973	0.0665	9e-04	0.0146	2e-04	0.0643	0.0070	0.0104	0.0493
		$\mathcal{R}2$	0.3005	0.0537	7e-04	0.0093	1e-04	0.0520	0.0044	0.0023	0.0047
		$\mathcal{R}3$	0.3003	0.0579	8e-04	0.0111	2e-04	0.0565	0.0052	0.0033	0.0060
		$\mathcal{R}4$	0.2991	0.0586	8e-04	0.0111	2e-04	0.0563	0.0052	0.0033	0.0058
	100% = 60	$\mathcal{R}0$	0.2985	0.0480	7e-04	0.0073	1e-04	0.0464	0.0035	0.0013	0.0037
80	60% = 48	$\mathcal{R}1$	0.2928	0.1180	0.0021	0.0580	0.0016	0..1052	0.1832	0.0364	0.1195
		$\mathcal{R}2$	0.2992	0.0533	7e-04	0.0092	1e-04	0.0520	0.0044	0.0026	0.0047
		$\mathcal{R}3$	0.299	0.0634	9e-04	0.0133	2e-04	0.0610	0.0062	0.0039	0.0070
		$\mathcal{R}4$	0.2971	0.0674	9e-04	0.0154	3e-04	0.0657	0.0072	0.0049	0.0078
	80% = 64	$\mathcal{R}1$	0.3010	0.0578	8e-04	0.0110	2e-04	0.0563	0.0053	0.0068	0.0472
		$\mathcal{R}2$	0.3002	0.0466	6e-04	0.0069	1e-04	0.0451	0.0033	0.0015	0.0036
		$\mathcal{R}3$	0.2984	0.0491	7e-04	0.0080	1e-04	0.0494	0.0040	0.0015	0.0043
		$\mathcal{R}4$	0.2998	0.0508	7e-04	0.0084	1e-04	0.0495	0.0040	0.0019	0.0043
	100% = 80	$\mathcal{R}0$	0.3002	0.0416	6e-04	0.0055	1e-04	0.0409	0.0027	0.0008	0.0027

Table A.5 The AB and MSE for the ML, bootstrap and jackknife estimates of the shape parameter α at ($\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.4$) for different n, FP and censoring schemes. The standard error of estimated AB and MSE are presented alongside the ML measures.

n	FP% $= r$	C.Sc	ML					Bootstrap		Jackknife	
			MLE	AB	SE(AB)	MSE	SE(MSE)	AB	MSE	AB	MSE
40	60% = 24	$\mathcal{R}1$	1.3338	0.3226	0.0067	0.4485	0.0254	0.4000	0.4885	0.4523	1.0711
		$\mathcal{R}2$	1.3567	0.3671	0.0073	0.5896	0.0274	0.4377	0.6025	0.3276	0.9309
		$\mathcal{R}3$	1.3798	0.3583	0.0068	0.5265	0.0194	0.4277	0.5387	0.2082	0.4934
		$\mathcal{R}4$	1.3674	0.3503	0.0069	0.5175	0.0218	0.4258	0.5517	0.2049	0.4427
	80% = 32	$\mathcal{R}1$	1.3562	0.3348	0.0069	0.4746	0.0244	0.3881	0.4275	0.2341	0.4282
		$\mathcal{R}2$	1.3563	0.3510	0.0071	0.5261	0.0229	0.3927	0.4145	0.2420	0.6122
		$\mathcal{R}3$	1.3567	0.3338	0.0065	0.4733	0.0212	0.3920	0.4160	0.1884	0.3728
		$\mathcal{R}4$	1.3557	0.3369	0.0068	0.4764	0.0260	0.3940	0.4213	0.1927	0.4002
	100% = 40	$\mathcal{R}0$	1.3476	0.3352	0.0065	0.4524	0.0176	0.3724	0.3575	0.1816	0.3491
60	60% = 36	$\mathcal{R}1$	1.3074	0.2670	0.0047	0.2555	0.0082	0.2873	0.1826	0.3320	0.5384
		$\mathcal{R}2$	1.2957	0.2849	0.0051	0.3192	0.0117	0.3167	0.2203	0.2006	0.3605
		$\mathcal{R}3$	1.3152	0.2718	0.0048	0.2780	0.0088	0.3047	0.2014	0.1260	0.1964
		$\mathcal{R}4$	1.3149	0.2765	0.0049	0.2796	0.0087	0.2988	0.1957	0.1225	0.1854
	80% = 48	$\mathcal{R}1$	1.2982	0.2607	0.0045	0.2448	0.0071	0.2857	0.1795	0.1605	0.2154
		$\mathcal{R}2$	1.3066	0.2755	0.0050	0.2886	0.0098	0.2993	0.1943	0.1479	0.2256
		$\mathcal{R}3$	1.3043	0.2629	0.0046	0.2514	0.0071	0.2857	0.1753	0.1162	0.1773
		$\mathcal{R}4$	1.3055	0.2654	0.0047	0.2569	0.0078	0.2891	0.1829	0.1160	0.1756
	100% = 60	$\mathcal{R}0$	1.2973	0.2597	0.0046	0.2459	0.0076	0.2802	0.1665	0.1123	0.1697
80	60% = 48	$\mathcal{R}1$	1.2709	0.2180	0.0036	0.1663	0.0041	0.2629	0.1901	0.2777	0.3539
		$\mathcal{R}2$	1.2739	0.2491	0.0041	0.2237	0.0059	0.2669	0.1428	0.1379	0.1878
		$\mathcal{R}3$	1.3030	0.2373	0.0041	0.2021	0.0059	0.2522	0.1270	0.0918	0.1279
		$\mathcal{R}4$	1.2771	0.2299	0.0039	0.1845	0.0048	0.2454	0.1206	0.0887	0.1197
	80% = 64	$\mathcal{R}1$	1.2712	0.2154	0.0035	0.1629	0.0038	0.2355	0.1099	0.1263	0.1381
		$\mathcal{R}2$	1.2836	0.2307	0.0040	0.1942	0.0056	0.2473	0.1211	0.1009	0.1241
		$\mathcal{R}3$	1.2827	0.2263	0.0036	0.1717	0.0041	0.2372	0.1112	0.0831	0.1065
		$\mathcal{R}4$	1.2738	0.2150	0.0036	0.1647	0.0040	0.2365	0.1101	0.0805	0.1060
	100% = 80	$\mathcal{R}0$	1.2706	0.2136	0.0035	0.1616	0.0040	0.2297	0.1028	0.0778	0.1014

Table A.6 The AB and MSE for the ML, bootstrap and jackknife estimates of the shape parameter α at ($\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.6$) for different n, FP and censoring schemes. The standard error of estimated AB and MSE are presented alongside the ML measures.

n	FP% $= r$	C.Sc	ML					Bootstrap		Jackknife	
			MLE	AB	SE(AB)	MSE	SE(MSE)	AB	MSE	AB	MSE
40	60% = 24	$\mathcal{R}1$	1.2682	0.2858	0.0067	0.3026	0.0128	0.3204	0.2463	0.5261	1.3903
		$\mathcal{R}2$	1.3277	0.3270	0.0062	0.4237	0.0162	0.3724	0.3437	0.2609	0.6168
		$\mathcal{R}3$	1.3575	0.3279	0.0064	0.4249	0.0200	0.3812	0.3735	0.1588	0.3153
		$\mathcal{R}4$	1.3409	0.3136	0.0057	0.3761	0.0137	0.3635	0.3326	0.1607	0.2914
	80% = 32	$\mathcal{R}1$	1.3153	0.2825	0.0051	0.3003	0.0092	0.3205	0.2355	0.2159	0.3517
		$\mathcal{R}2$	1.3338	0.3081	0.0057	0.3664	0.0165	0.3427	0.2735	0.1785	0.3202
		$\mathcal{R}3$	1.3250	0.2975	0.0054	0.3349	0.0113	0.3341	0.2626	0.1448	0.2469
		$\mathcal{R}4$	1.3426	0.3083	0.0061	0.3761	0.0201	0.3366	0.2630	0.1419	0.2405
	100% = 40	$\mathcal{R}0$	1.3132	0.2906	0.0053	0.3139	0.0100	0.3176	0.2267	0.1357	0.2234
60	60% = 36	$\mathcal{R}1$	1.2297	0.2058	0.0041	0.1542	0.0045	0.2294	0.3634	0.4327	0.7606
		$\mathcal{R}2$	1.2832	0.2587	0.0042	0.2380	0.0061	0.2802	0.1608	0.1467	0.2076
		$\mathcal{R}3$	1.3020	0.2506	0.0045	0.2306	0.0068	0.2735	0.1553	0.0989	0.1573
		$\mathcal{R}4$	1.2944	0.2447	0.0042	0.2140	0.0061	0.2688	0.1508	0.0986	0.1386
	80% = 48	$\mathcal{R}1$	1.2778	0.2255	0.0038	0.1796	0.0050	0.2421	0.1165	0.1525	0.1813
		$\mathcal{R}2$	1.2749	0.2365	0.0040	0.2009	0.0052	0.2588	0.1340	0.1098	0.1418
		$\mathcal{R}3$	1.2857	0.2340	0.0039	0.1922	0.0056	0.2523	0.1278	0.0885	0.1219
		$\mathcal{R}4$	1.2908	0.2346	0.0041	0.1968	0.0055	0.2528	0.1285	0.0884	0.1210
	100% = 60	$\mathcal{R}0$	1.2707	0.2204	0.0036	0.1705	0.0044	0.2405	0.1144	0.0843	0.1127
80	60% = 48	$\mathcal{R}1$	1.2258	0.1877	0.0037	0.1189	0.0034	0.2943	0.1062	0.4108	0.6466
		$\mathcal{R}2$	1.2653	0.2262	0.0037	0.1772	0.0046	0.2366	0.1077	0.1076	0.1307
		$\mathcal{R}3$	1.2691	0.2101	0.0036	0.1556	0.0043	0.2256	0.0980	0.0688	0.0997
		$\mathcal{R}4$	1.2714	0.2019	0.0033	0.1428	0.0035	0.2217	0.0943	0.0700	0.0866
	80% = 64	$\mathcal{R}1$	1.2576	0.1948	0.0030	0.1234	0.0027	0.2010	0.0759	0.1236	0.1272
		$\mathcal{R}2$	1.2539	0.2022	0.0032	0.1399	0.0032	0.2148	0.0872	0.0790	0.0884
		$\mathcal{R}3$	1.2701	0.1984	0.0032	0.1341	0.0031	0.2101	0.0836	0.0633	0.0795
		$\mathcal{R}4$	1.2598	0.1944	0.0032	0.1321	0.0032	0.2094	0.0830	0.0639	0.0782
	100% = 80	$\mathcal{R}0$	1.2530	0.1917	0.0030	0.1229	0.0028	0.1987	0.0734	0.0597	0.0729

Table A.7 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_1 at confidence level = 95%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.4$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	1.1398	0.0167	90.86	0.0053	1.5957	95.97	1.8962	94.89
		$\mathcal{R}2$	1.6607	0.0413	89.33	0.0056	2.8506	95.56	3.7520	94.53
		$\mathcal{R}3$	1.1574	0.0176	87.33	0.0061	1.4732	92.13	2.1690	92.96
		$\mathcal{R}4$	1.2157	0.0200	87.90	0.0060	1.6562	92.32	2.3857	93.64
	80% = 32	$\mathcal{R}1$	1.1137	0.0192	88.83	0.0058	1.4459	94.63	2.1339	95.00
		$\mathcal{R}2$	1.2748	0.0221	88.10	0.0059	1.8979	95.46	2.3911	93.96
		$\mathcal{R}3$	1.1112	0.0167	89.26	0.0057	1.4435	94.06	1.9527	93.33
		$\mathcal{R}4$	1.1084	0.0178	88.46	0.0058	1.4860	93.76	1.9465	93.53
	100% = 40	$\mathcal{R}0$	1.1060	0.0172	89.10	0.0057	1.4037	94.20	1.8785	92.80
60	60% = 36	$\mathcal{R}1$	0.8396	0.0094	89.88	0.0055	1.3910	94.91	1.7196	94.01
		$\mathcal{R}2$	1.1579	0.0172	90.53	0.0053	1.5512	95.80	1.8204	93.93
		$\mathcal{R}3$	0.9123	0.0096	90.10	0.0055	1.0143	92.66	1.3282	92.66
		$\mathcal{R}4$	0.9235	0.0108	88.96	0.0057	1.1038	94.00	1.3479	93.60
	80% = 48	$\mathcal{R}1$	0.8516	0.0095	88.93	0.0057	1.0057	94.46	1.4243	94.56
		$\mathcal{R}2$	0.9590	0.0126	90.20	0.0054	1.2175	94.36	1.3765	92.53
		$\mathcal{R}3$	0.8535	0.0096	88.56	0.0058	0.9989	94.26	1.2050	92.10
		$\mathcal{R}4$	0.8581	0.0096	89.20	0.0057	0.9945	94.46	1.2386	93.30
	100% = 60	$\mathcal{R}0$	0.8457	0.0093	89.96	0.0055	1.0032	93.76	1.1406	94.06
80	60% = 48	$\mathcal{R}1$	0.7182	0.0063	90.89	0.0053	0.9295	94.19	1.2194	93.91
		$\mathcal{R}2$	0.9429	0.0120	90.33	0.0054	1.1595	94.76	1.3206	93.70
		$\mathcal{R}3$	0.7691	0.0072	89.86	0.0055	0.8348	93.76	0.9980	92.56
		$\mathcal{R}4$	0.7805	0.0076	89.66	0.0056	0.8646	93.73	1.0521	93.75
	80% = 64	$\mathcal{R}1$	0.7164	0.0067	91.30	0.0051	0.7994	94.50	1.0651	95.10
		$\mathcal{R}2$	0.7903	0.0080	89.76	0.0055	0.9227	94.76	1.0408	93.73
		$\mathcal{R}3$	0.7127	0.0065	90.43	0.0054	0.8059	93.23	0.9217	93.50
		$\mathcal{R}4$	0.7185	0.0065	90.36	0.0054	0.7996	94.4	0.9176	93.96
	100% = 80	$\mathcal{R}0$	0.7096	0.0066	91.26	0.0052	0.8029	94.50	0.8862	94.60

Table A.8 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_1 at confidence level = 95%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	1.0320	0.0134	98.42	0.0031	1.2381	96.17	1.5926	95.05
		$\mathcal{R}2$	1.0666	0.0154	88.80	0.0058	1.3765	94.55	1.8808	95.20
		$\mathcal{R}3$	0.9596	0.0108	88.76	0.0058	1.0190	91.43	1.3159	94.33
		$\mathcal{R}4$	0.9375	0.0114	89.23	0.0057	1.0776	92.83	1.2510	94.50
	80% = 32	$\mathcal{R}1$	0.7950	0.0084	90.94	0.0052	0.9049	95.06	1.2982	93.91
		$\mathcal{R}2$	0.8988	0.0111	89.83	0.0055	1.0833	94.06	1.3966	93.76
		$\mathcal{R}3$	0.8432	0.0084	89.50	0.0056	0.9430	93.70	1.1549	93.50
		$\mathcal{R}4$	0.8320	0.0088	89.33	0.0056	0.9246	93.10	1.1977	93.80
	100% = 40	$\mathcal{R}0$	0.7985	0.0084	89.96	0.0055	0.9070	94.33	1.1125	93.60
60	60% = 36	$\mathcal{R}1$	0.7489	0.0062	99.33	0.0019	0.9928	94.29	1.1019	94.17
		$\mathcal{R}2$	0.8225	0.0092	90.30	0.0054	0.9584	95.16	1.1294	94.80
		$\mathcal{R}3$	0.7640	0.0072	89.66	0.0056	0.8093	93.40	0.9351	94.36
		$\mathcal{R}4$	0.7323	0.0067	90.80	0.0053	0.7717	93.36	0.9770	94.20
	80% = 48	$\mathcal{R}1$	0.6268	0.0050	91.76	0.0050	0.6773	93.80	0.7974	97.50
		$\mathcal{R}2$	0.7058	0.0063	91.20	0.0052	0.7678	94.23	0.8971	93.66
		$\mathcal{R}3$	0.6694	0.0056	90.03	0.0055	0.7029	93.90	0.8139	94.20
		$\mathcal{R}4$	0.6581	0.0051	90.47	0.0054	0.6985	93.70	0.8230	94.66
	100% = 60	$\mathcal{R}0$	0.6289	0.0049	92.37	0.0048	0.6801	93.96	0.7603	94.53
80	60% = 48	$\mathcal{R}1$	0.6154	0.0046	98.77	0.0026	0.8195	92.98	0.8819	93.15
		$\mathcal{R}2$	0.6896	0.0060	91.37	0.0051	0.7839	93.73	0.8661	93.84
		$\mathcal{R}3$	0.6658	0.0053	90.90	0.0053	0.6890	93.73	0.7741	93.68
		$\mathcal{R}4$	0.6168	0.0044	91.53	0.0051	0.6400	92.43	0.7608	94.72
	80% = 64	$\mathcal{R}1$	0.5328	0.0037	91.26	0.0052	0.5735	93.86	0.6999	94.80
		$\mathcal{R}2$	0.5997	0.0045	91.87	0.0050	0.6563	94.30	0.7102	94.00
		$\mathcal{R}3$	0.5662	0.0038	91.50	0.0051	0.5932	93.86	0.6533	93.56
		$\mathcal{R}4$	0.5697	0.0039	92.00	0.0050	0.5987	93.03	0.6670	93.56
	100% = 80	$\mathcal{R}0$	0.5326	0.0037	91.63	0.0051	0.5712	94.90	0.6193	94.50

Table A.9 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_1 at confidence level = 99%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.4$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	1.4980	0.0229	94.79	0.0041	3.0851	99.14	4.0285	98.82
		$\mathcal{R}2$	2.1825	0.0250	94.03	0.0043	6.6394	99.04	7.7749	98.86
		$\mathcal{R}3$	1.5211	0.0240	92.90	0.0047	2.6793	97.13	3.7151	98.24
		$\mathcal{R}4$	1.5977	0.0252	93.03	0.0046	3.0872	97.84	4.2177	98.68
	80% = 32	$\mathcal{R}1$	1.4636	0.0230	94.00	0.0043	2.6275	98.43	3.4706	99.25
		$\mathcal{R}2$	1.6754	0.0260	93.36	0.0045	3.7073	99.26	4.4696	98.20
		$\mathcal{R}3$	1.4603	0.0229	93.90	0.0044	2.5818	98.70	3.3443	98.30
		$\mathcal{R}4$	1.4567	0.0242	93.36	0.0045	2.6648	98.30	3.3418	98.70
	100% = 40	$\mathcal{R}0$	1.4535	0.0235	94.13	0.0043	2.5056	98.96	3.1774	98.56
60	60% = 36	$\mathcal{R}1$	1.1034	0.0132	94.50	0.0042	1.4982	97.70	1.960	98.29
		$\mathcal{R}2$	1.5217	0.0236	94.93	0.0040	2.8461	99.43	3.1620	99.10
		$\mathcal{R}3$	1.1990	0.0137	94.40	0.0042	1.5897	97.86	2.0083	98.16
		$\mathcal{R}4$	1.2137	0.0151	94.03	0.0043	1.7628	98.10	2.0808	98.60
	80% = 48	$\mathcal{R}1$	1.1192	0.0134	94.10	0.0043	1.5938	98.66	2.1973	98.53
		$\mathcal{R}2$	1.2604	0.0174	94.86	0.0040	2.0163	98.86	2.1946	97.90
		$\mathcal{R}3$	1.1216	0.0135	94.36	0.0042	1.5664	98.53	1.8414	97.86
		$\mathcal{R}4$	1.1277	0.0136	94.76	0.0041	1.5580	98.10	1.8879	98.13
	100% = 60	$\mathcal{R}0$	1.1114	0.0132	94.46	0.0042	1.5730	98.46	1.7160	98.33
80	60% = 48	$\mathcal{R}1$	0.9439	0.0092	95.33	0.0039	1.1839	98.26	1.2943	98.19
		$\mathcal{R}2$	1.2392	0.0167	95.60	0.0037	1.9103	98.83	2.0832	98.53
		$\mathcal{R}3$	1.0108	0.0105	94.66	0.0041	1.2335	98.23	1.4301	98.20
		$\mathcal{R}4$	1.0258	0.0109	94.33	0.0042	1.2982	98.03	1.5453	98.35
	80% = 64	$\mathcal{R}1$	0.9416	0.0097	95.53	0.0038	1.1867	99.03	1.5006	98.63
		$\mathcal{R}2$	1.0387	0.0114	94.73	0.0041	1.4116	99.30	1.5573	98.53
		$\mathcal{R}3$	0.9366	0.0094	95.16	0.0039	1.193	98.03	1.3351	98.10
		$\mathcal{R}4$	0.9443	0.0095	95.13	0.0039	1.1822	98.93	1.3216	98.16
	100% = 80	$\mathcal{R}0$	0.9325	0.0096	95.93	0.0036	1.1861	98.66	1.2742	98.23

Table A.10 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_1 at confidence level = 99%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	1.3562	0.0173	99.33	0.0020	1.9544	99.68	2.6194	97.91
		$\mathcal{R}2$	1.4017	0.0212	93.93	0.0044	2.5595	98.30	3.3111	99.00
		$\mathcal{R}3$	1.2611	0.0152	93.60	0.0045	1.5877	97.30	1.6279	97.66
		$\mathcal{R}4$	1.2321	0.0159	93.67	0.0044	1.7180	97.20	1.4757	96.50
	80% = 32	$\mathcal{R}1$	1.0448	0.0120	95.29	0.0039	1.4015	99.13	1.7204	98.94
		$\mathcal{R}2$	1.1812	0.0154	94.77	0.0041	1.7912	98.26	2.2643	99.16
		$\mathcal{R}3$	1.1082	0.0121	94.20	0.0043	1.4659	98.33	1.7180	98.60
		$\mathcal{R}4$	1.0934	0.0125	93.70	0.0044	1.4326	97.90	1.8043	98.56
	100% = 40	$\mathcal{R}0$	1.0495	0.0120	94.33	0.0042	1.4098	98.70	1.6982	98.43
60	60% = 36	$\mathcal{R}1$	0.9842	0.0098	99.67	0.0013	1.2951	97.72	1.3559	97.93
		$\mathcal{R}2$	1.0809	0.0130	94.77	0.0041	1.5114	98.83	1.7532	99.06
		$\mathcal{R}3$	1.0040	0.0104	95.20	0.0039	1.1810	98.13	1.3045	98.6
		$\mathcal{R}4$	0.9624	0.0097	95.50	0.0038	1.1278	98.21	1.3853	99.00
	80% = 48	$\mathcal{R}1$	0.8238	0.0075	95.83	0.0036	0.9851	98.46	0.9792	97.98
		$\mathcal{R}2$	0.9276	0.0093	95.87	0.0036	1.1437	98.66	1.2989	98.30
		$\mathcal{R}3$	0.8798	0.0084	95.20	0.0039	1.0100	98.20	1.1474	98.53
		$\mathcal{R}4$	0.8648	0.0077	94.63	0.0041	1.0064	98.13	1.1598	98.23
	100% = 60	$\mathcal{R}0$	0.8265	0.0073	95.83	0.0036	0.9806	98.46	1.0769	98.80
80	60% = 48	$\mathcal{R}1$	0.8088	0.0076	99.83	0.0010	0.9932	98.24	1.2021	97.85
		$\mathcal{R}2$	0.9063	0.0088	95.97	0.0036	1.1608	98.43	1.2625	98.40
		$\mathcal{R}3$	0.8750	0.0080	94.83	0.0040	0.9723	98.20	1.0595	98.08
		$\mathcal{R}4$	0.8106	0.0068	96.40	0.0034	0.9083	97.83	1.0514	98.68
	80% = 64	$\mathcal{R}1$	0.7003	0.0057	96.30	0.0034	0.8083	98.43	0.7608	98.40
		$\mathcal{R}2$	0.7882	0.0068	95.63	0.0037	0.9405	98.53	0.9994	98.30
		$\mathcal{R}3$	0.7441	0.0060	95.50	0.0038	0.8280	98.43	0.8912	98.60
		$\mathcal{R}4$	0.7487	0.0061	96.20	0.0035	0.8403	98.06	0.9151	98.30
	100% = 80	$\mathcal{R}0$	0.6999	0.0057	96.27	0.0035	0.8013	98.66	0.8544	98.53

Table A.11 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_2 at confidence level = 95%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.4$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	0.4633	0.0041	87.28	0.0061	0.4301	90.13	0.4691	92.97
		$\mathcal{R}2$	0.3202	0.0017	91.16	0.0052	0.3093	92.28	0.3430	93.03
		$\mathcal{R}3$	0.3773	0.0023	89.53	0.0056	0.3531	90.20	0.3996	91.88
		$\mathcal{R}4$	0.3845	0.0024	90.10	0.0055	0.3533	91.08	0.4017	91.84
	80% = 32	$\mathcal{R}1$	0.3092	0.0017	90.16	0.0054	0.3014	92.50	0.3518	91.13
		$\mathcal{R}2$	0.2792	0.0013	92.80	0.0047	0.2712	93.00	0.2940	93.56
		$\mathcal{R}3$	0.3017	0.0015	91.83	0.0050	0.2862	92.26	0.3181	93.36
		$\mathcal{R}4$	0.3010	0.0015	91.96	0.0050	0.285	92.33	0.3159	93.13
	100% = 40	$\mathcal{R}0$	0.2484	0.0011	92.23	0.0049	0.2432	92.90	0.2616	94.26
60	60% = 36	$\mathcal{R}1$	0.3798	0.0027	89.24	0.0057	0.3381	93.18	0.3495	92.74
		$\mathcal{R}2$	0.2610	0.0011	92.13	0.0049	0.2566	93.36	0.2737	94.16
		$\mathcal{R}3$	0.3087	0.0016	90.90	0.0053	0.2904	92.46	0.3197	92.63
		$\mathcal{R}4$	0.3097	0.0016	91.03	0.0052	0.2965	92.53	0.3224	92.23
	80% = 48	$\mathcal{R}1$	0.2536	0.0011	92.93	0.0047	0.2484	93.93	0.2992	93.23
		$\mathcal{R}2$	0.2273	9e-04	93.13	0.0046	0.2227	93.86	0.2357	94.33
		$\mathcal{R}3$	0.2459	0.0010	92.83	0.0047	0.2385	93.86	0.2537	93.30
		$\mathcal{R}4$	0.2451	0.0010	92.63	0.0048	0.2373	92.96	0.2567	93.20
	100% = 60	$\mathcal{R}0$	0.2019	7e-04	93.33	0.0046	0.2000	94.03	0.2101	94.26
80	60% = 48	$\mathcal{R}1$	0.3261	0.0020	91.26	0.0052	0.2391	93.77	0.4193	93.97
		$\mathcal{R}2$	0.2277	9e-04	93.63	0.0045	0.2226	93.70	0.2376	93.56
		$\mathcal{R}3$	0.2669	0.0011	92.60	0.0048	0.2562	93.26	0.2744	93.28
		$\mathcal{R}4$	0.2714	0.0012	92.50	0.0048	0.2618	93.46	0.2792	93.30
	80% = 64	$\mathcal{R}1$	0.2181	8e-04	93.16	0.0046	0.2149	94.00	0.2574	93.76
		$\mathcal{R}2$	0.1965	6e-04	93.70	0.0044	0.1955	94.26	0.2028	94.06
		$\mathcal{R}3$	0.2136	7e-04	94.03	0.0043	0.2069	92.60	0.2187	93.73
		$\mathcal{R}4$	0.2123	7e-04	94.13	0.0043	0.2069	93.6	0.2199	93.70
	100% = 80	$\mathcal{R}0$	0.1754	5e-04	93.70	0.0044	0.1743	94.43	0.1809	94.50

Table A.12 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_2 at confidence level = 95%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	0.6795	0.0116	81.37	0.0096	0.5303	84.46	0.7914	93.05
		$\mathcal{R}2$	0.3714	0.0022	90.23	0.0054	0.3457	90.85	0.3762	91.50
		$\mathcal{R}3$	0.4424	0.0030	87.40	0.0061	0.3972	90.30	0.4186	92.00
		$\mathcal{R}4$	0.4685	0.0033	87.96	0.0059	0.4111	90.10	0.4093	87.00
	80% = 32	$\mathcal{R}1$	0.3999	0.0028	88.53	0.0058	0.3736	92.18	0.4291	92.72
		$\mathcal{R}2$	0.3212	0.0017	90.33	0.0054	0.3024	93.33	0.3249	94.36
		$\mathcal{R}3$	0.3515	0.0020	90.80	0.0053	0.3248	92.53	0.3559	92.10
		$\mathcal{R}4$	0.3521	0.0020	90.40	0.0054	0.3261	91.96	0.3545	92.53
	100% = 40	$\mathcal{R}0$	0.2858	0.0013	91.80	0.0050	0.2744	93.66	0.2921	92.73
60	60% = 36	$\mathcal{R}1$	0.6477	0.0104	82.10	0.0090	0.5718	90.92	0.4917	94.17
		$\mathcal{R}2$	0.3027	0.0015	91.56	0.0051	0.2865	92.33	0.3062	92.46
		$\mathcal{R}3$	0.3682	0.0021	89.90	0.0055	0.3324	91.16	0.3651	92.00
		$\mathcal{R}4$	0.3909	0.0024	90.23	0.0054	0.3521	91.94	0.3764	93.50
	80% = 48	$\mathcal{R}1$	0.3276	0.0019	91.10	0.0052	0.3133	93.50	0.3138	91.25
		$\mathcal{R}2$	0.2636	0.0012	92.33	0.0049	0.2534	93.80	0.2668	93.60
		$\mathcal{R}3$	0.2893	0.0014	91.93	0.0050	0.2745	94.23	0.2925	92.67
		$\mathcal{R}4$	0.2891	0.0013	92.60	0.0048	0.2737	93.03	0.2913	92.76
	100% = 60	$\mathcal{R}0$	0.2329	9e-04	92.50	0.0048	0.2267	94.10	0.2380	94.30
80	60% = 48	$\mathcal{R}1$	0.6293	0.0096	84.37	0.0086	0.4913	93.10	0.2839	92.93
		$\mathcal{R}2$	0.2627	0.0012	92.37	0.0048	0.2536	94.20	0.2675	93.68
		$\mathcal{R}3$	0.3148	0.0016	91.37	0.0051	0.2954	92.86	0.3145	92.00
		$\mathcal{R}4$	0.3386	0.0019	90.87	0.0053	0.3163	92.66	0.3345	92.88
	80% = 64	$\mathcal{R}1$	0.2870	0.0015	92.20	0.0049	0.2756	93.86	0.2875	92.40
		$\mathcal{R}2$	0.2274	9e-04	92.70	0.0047	0.2207	94.06	0.2307	94.16
		$\mathcal{R}3$	0.2480	0.0010	92.73	0.0047	0.2410	93.60	0.2526	93.93
		$\mathcal{R}4$	0.2504	0.0010	92.80	0.0047	0.2412	93.90	0.2531	94.16
	100% = 80	$\mathcal{R}0$	0.2032	7e-04	93.40	0.0045	0.2004	94.16	0.2064	94.36

Table A.13 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_2 at confidence level = 99%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.4$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	0.6089	0.0060	91.81	0.0050	0.5796	95.97	0.7719	97.41
		$\mathcal{R}2$	0.4208	0.0029	95.30	0.0039	0.4096	96.72	0.4262	97.96
		$\mathcal{R}3$	0.4958	0.0037	94.03	0.0043	0.4667	95.73	0.4813	96.64
		$\mathcal{R}4$	0.5054	0.0038	94.60	0.0041	0.4677	95.88	0.4870	96.60
	80% = 32	$\mathcal{R}1$	0.4063	0.0028	94.66	0.0041	0.4018	96.90	0.3757	95.63
		$\mathcal{R}2$	0.3670	0.0023	96.00	0.0036	0.3591	97.20	0.3678	98.00
		$\mathcal{R}3$	0.3965	0.0026	96.13	0.0035	0.3790	97.10	0.3937	97.73
		$\mathcal{R}4$	0.3956	0.0026	96.00	0.0036	0.3775	97.13	0.3915	97.86
	100% = 40	$\mathcal{R}0$	0.3264	0.0020	96.80	0.0032	0.3217	97.56	0.3300	98.20
60	60% = 36	$\mathcal{R}1$	0.4992	0.0042	93.70	0.0044	0.3391	97.66	0.3918	97.54
		$\mathcal{R}2$	0.3430	0.0021	96.46	0.0034	0.3821	96.73	0.3457	98.33
		$\mathcal{R}3$	0.4057	0.0027	95.13	0.0039	0.3916	96.93	0.3954	97.50
		$\mathcal{R}4$	0.4071	0.0027	94.96	0.0040	0.3290	98.13	0.3988	97.50
	80% = 48	$\mathcal{R}1$	0.3333	0.0020	96.56	0.0033	0.2943	97.80	0.3248	96.63
		$\mathcal{R}2$	0.2988	0.0017	96.70	0.0033	0.3149	97.70	0.3009	98.50
		$\mathcal{R}3$	0.3232	0.0019	96.56	0.0033	0.3131	97.53	0.3208	98.06
		$\mathcal{R}4$	0.3221	0.0019	96.76	0.0032	0.2635	98.03	0.3242	97.90
	100% = 60	$\mathcal{R}0$	0.2653	0.0015	97.10	0.0031	0.3391	97.66	0.2696	98.33
80	60% = 48	$\mathcal{R}1$	0.4286	0.0032	95.43	0.0038	0.3951	97.50	0.4451	98.42
		$\mathcal{R}2$	0.2993	0.0017	97.06	0.0031	0.294	98.23	0.3029	98.43
		$\mathcal{R}3$	0.3508	0.0021	96.53	0.0033	0.3372	97.63	0.3451	97.92
		$\mathcal{R}4$	0.3567	0.0022	96.30	0.0034	0.3446	97.60	0.3503	98.25
	80% = 64	$\mathcal{R}1$	0.2867	0.0017	97.03	0.0031	0.2838	98.16	0.2805	97.16
		$\mathcal{R}2$	0.2583	0.0014	97.86	0.0026	0.2576	98.30	0.2603	98.20
		$\mathcal{R}3$	0.2807	0.0016	97.80	0.0027	0.2725	97.50	0.2795	98.36
		$\mathcal{R}4$	0.2790	0.0015	97.80	0.0027	0.2724	97.9	0.2809	98.00
	100% = 80	$\mathcal{R}0$	0.2305	0.0012	97.93	0.0026	0.2297	98.60	0.2334	98.53

Table A.14 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for θ_2 at confidence level = 99%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	0.8930	0.0161	86.22	0.0085	0.6927	91.16	0.7194	96.99
		$\mathcal{R}2$	0.4881	0.0036	94.70	0.0041	0.4558	96.50	0.4558	97.00
		$\mathcal{R}3$	0.5814	0.0047	91.73	0.0050	0.5217	94.90	0.4806	96.00
		$\mathcal{R}4$	0.6158	0.0051	92.70	0.0047	0.5385	95.16	0.4660	90.50
	80% = 32	$\mathcal{R}1$	0.5256	0.0043	93.42	0.0045	0.4952	96.52	0.5490	97.69
		$\mathcal{R}2$	0.4221	0.0029	94.87	0.0040	0.3979	97.00	0.4043	98.06
		$\mathcal{R}3$	0.4620	0.0033	94.90	0.0040	0.4270	97.06	0.4370	97.46
		$\mathcal{R}4$	0.4627	0.0033	94.13	0.0043	0.4280	96.50	0.4358	97.23
	100% = 40	$\mathcal{R}0$	0.3757	0.0024	96.00	0.0036	0.3604	97.56	0.3662	97.93
60	60% = 36	$\mathcal{R}1$	0.8513	0.0145	86.67	0.0080	0.7231	97.14	0.7194	97.50
		$\mathcal{R}2$	0.3978	0.0026	95.70	0.0037	0.3767	97.30	0.3841	97.43
		$\mathcal{R}3$	0.4838	0.0035	94.47	0.0042	0.4357	96.73	0.4474	97.32
		$\mathcal{R}4$	0.5137	0.0038	94.37	0.0042	0.4604	96.89	0.4579	97.50
	80% = 48	$\mathcal{R}1$	0.4306	0.0032	95.67	0.0037	0.4166	97.66	0.3424	96.25
		$\mathcal{R}2$	0.3464	0.0021	96.77	0.0032	0.3330	97.63	0.3375	97.80
		$\mathcal{R}3$	0.3802	0.0024	96.13	0.0035	0.3597	98.16	0.3667	97.67
		$\mathcal{R}4$	0.3799	0.0024	95.67	0.0037	0.3586	97.30	0.3662	97.63
	100% = 60	$\mathcal{R}0$	0.3061	0.0018	96.33	0.0034	0.2975	97.83	0.3035	97.73
80	60% = 48	$\mathcal{R}1$	0.8271	0.0135	88.01	0.0077	0.5926	97.85	0.4910	97.70
		$\mathcal{R}2$	0.3453	0.0021	96.33	0.0034	0.3332	98.16	0.3386	98.08
		$\mathcal{R}3$	0.4138	0.0028	95.43	0.0038	0.3867	97.16	0.3928	97.12
		$\mathcal{R}4$	0.4450	0.0032	94.67	0.0041	0.4129	97.36	0.4132	97.60
	80% = 64	$\mathcal{R}1$	0.3772	0.0025	95.96	0.0036	0.3648	97.56	0.3108	95.60
		$\mathcal{R}2$	0.2988	0.0017	97.20	0.0030	0.2893	98.10	0.2949	98.36
		$\mathcal{R}3$	0.3260	0.0019	96.50	0.0034	0.3156	98.13	0.3207	98.23
		$\mathcal{R}4$	0.3291	0.0020	96.27	0.0035	0.3162	97.80	0.3217	98.10
	100% = 80	$\mathcal{R}0$	0.2671	0.0015	97.47	0.0029	0.2631	98.20	0.2653	98.36

Table A.15 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for α at confidence level = 95%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.4$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	**SE(AL)**	**CP**	**SE(CP)**	**AL**	**CP**	**AL**	**CP**
40	60% = 24	$\mathcal{R}1$	1.6344	0.0142	96.44	0.0036	2.1367	93.01	1.6802	94.49
		$\mathcal{R}2$	1.8880	0.0183	95.13	0.0039	2.3313	92.84	1.9010	94.03
		$\mathcal{R}3$	1.7841	0.0162	96.83	0.0032	2.2619	90.73	1.7712	94.36
		$\mathcal{R}4$	1.7547	0.0165	96.56	0.0033	2.2895	89.88	1.7717	93.76
	80% = 32	$\mathcal{R}1$	1.6579	0.0145	96.40	0.0034	2.0536	92.66	1.6994	94.37
		$\mathcal{R}2$	1.7562	0.0165	95.33	0.0039	2.0756	94.23	1.7686	94.04
		$\mathcal{R}3$	1.6817	0.0160	96.56	0.0033	2.0659	92.76	1.6577	94.80
		$\mathcal{R}4$	1.6709	0.0148	96.63	0.0033	2.0796	91.70	1.6646	94.60
	100% = 40	$\mathcal{R}0$	1.6385	0.0143	95.83	0.0036	1.9596	92.93	1.6428	93.76
60	60% = 36	$\mathcal{R}1$	1.2802	0.0080	95.98	0.0034	1.4924	92.70	1.1973	94.95
		$\mathcal{R}2$	1.4526	0.0105	95.56	0.0038	1.6358	94.16	1.4674	94.30
		$\mathcal{R}3$	1.3523	0.0088	96.63	0.0033	1.5607	92.43	1.3446	94.23
		$\mathcal{R}4$	1.3374	0.0088	96.76	0.0032	1.5388	93.30	1.3427	93.93
	80% = 48	$\mathcal{R}1$	1.2640	0.0077	96.20	0.0035	1.4715	92.56	1.2607	94.50
		$\mathcal{R}2$	1.3585	0.0094	96.73	0.0032	1.5422	92.86	1.3663	93.76
		$\mathcal{R}3$	1.2757	0.0080	96.23	0.0035	1.4646	92.73	1.2883	93.66
		$\mathcal{R}4$	1.2821	0.0081	96.00	0.0036	1.4849	93.10	1.2637	94.06
	100% = 60	$\mathcal{R}0$	1.2579	0.0080	95.86	0.0036	1.4384	92.20	1.2632	94.70
80	60% = 48	$\mathcal{R}1$	1.0692	0.0054	95.76	0.0035	1.2993	93.62	1.1014	93.88
		$\mathcal{R}2$	1.2374	0.0074	95.66	0.0034	1.3595	94.36	1.2441	93.96
		$\mathcal{R}3$	1.1547	0.0063	96.23	0.0035	1.2744	92.50	1.1289	93.80
		$\mathcal{R}4$	1.1082	0.0060	96.00	0.0036	1.2466	93.20	1.1212	93.90
	80% = 64	$\mathcal{R}1$	1.0642	0.0054	95.96	0.0036	1.1956	93.06	1.0742	95.33
		$\mathcal{R}2$	1.1468	0.0063	96.10	0.0035	1.2554	94.20	1.1439	94.40
		$\mathcal{R}3$	1.0782	0.0055	95.96	0.0036	1.2017	92.43	1.0777	94.80
		$\mathcal{R}4$	1.0711	0.0054	96.56	0.0033	1.197	92.86	1.0691	94.86
	100% = 80	$\mathcal{R}0$	1.0556	0.0054	95.60	0.0037	1.1641	93.56	1.0639	94.36

Table A.16 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for α at confidence level = 95%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	1.4010	0.0132	94.49	0.0056	1.6803	94.79	1.4081	93.88
		$\mathcal{R}2$	1.6229	0.0126	95.40	0.0038	1.9500	92.50	1.6574	93.70
		$\mathcal{R}3$	1.5923	0.0126	96.67	0.0033	1.9910	91.10	1.5966	93.33
		$\mathcal{R}4$	1.5313	0.0118	96.26	0.0035	1.9020	91.16	1.5044	94.50
	80% = 32	$\mathcal{R}1$	1.3990	0.0091	96.05	0.0036	1.6662	93.35	1.3917	94.51
		$\mathcal{R}2$	1.5151	0.0110	96.66	0.0033	1.7835	92.66	1.5141	94.10
		$\mathcal{R}3$	1.4535	0.0102	96.60	0.0033	1.7317	92.93	1.4718	94.43
		$\mathcal{R}4$	1.4794	0.0115	96.70	0.0033	1.7456	92.33	1.4668	94.70
	100% = 40	$\mathcal{R}0$	1.3935	0.0096	96.33	0.0034	1.6456	93.13	1.4145	94.00
60	60% = 36	$\mathcal{R}1$	1.0806	0.0068	95.87	0.0047	1.3826	92.39	1.0179	94.70
		$\mathcal{R}2$	1.2794	0.0074	95.67	0.0037	1.4323	94.26	1.2954	94.86
		$\mathcal{R}3$	1.2276	0.0072	96.57	0.0033	1.3926	92.36	1.2380	94.64
		$\mathcal{R}4$	1.1849	0.0066	95.80	0.0037	1.3679	91.63	1.1867	95.50
	80% = 48	$\mathcal{R}1$	1.0967	0.0057	95.93	0.0036	1.2325	93.60	1.0923	93.75
		$\mathcal{R}2$	1.1676	0.0063	95.63	0.0037	1.3174	93.53	1.1905	94.06
		$\mathcal{R}3$	1.1367	0.0060	96.07	0.0035	1.2807	92.63	1.1334	95.13
		$\mathcal{R}4$	1.1394	0.0061	95.80	0.0037	1.2831	92.80	1.1366	95.16
	100% = 60	$\mathcal{R}0$	1.0860	0.0055	95.90	0.0036	1.2218	93.80	1.1009	95.10
80	60% = 48	$\mathcal{R}1$	0.9239	0.0053	95.35	0.0050	1.2120	91.69	1.0029	94.71
		$\mathcal{R}2$	1.0993	0.0057	95.37	0.0038	1.1972	94.10	1.1074	94.56
		$\mathcal{R}3$	1.0275	0.0050	95.73	0.0037	1.1370	92.70	1.0365	94.80
		$\mathcal{R}4$	0.9997	0.0045	96.70	0.0033	1.1188	91.93	1.0057	94.60
	80% = 64	$\mathcal{R}1$	0.9276	0.0040	95.43	0.0038	1.0134	93.56	0.9236	95.20
		$\mathcal{R}2$	0.9934	0.0045	95.47	0.0038	1.0833	94.43	1.0058	94.50
		$\mathcal{R}3$	0.9675	0.0043	96.13	0.0035	1.0568	93.26	0.9702	93.96
		$\mathcal{R}4$	0.9561	0.0043	95.50	0.0038	1.0540	92.50	0.9664	94.50
	100% = 80	$\mathcal{R}0$	0.9212	0.0040	95.07	0.0040	0.9997	94.30	0.9366	94.10

Table A.17 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for α at confidence level = 99%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.4$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	2.1479	0.0186	98.91	0.0022	3.3166	97.56	2.8046	98.66
		$\mathcal{R}2$	2.4813	0.0240	98.00	0.0026	3.6132	98.44	2.6422	98.70
		$\mathcal{R}3$	2.3447	0.0213	98.83	0.0020	3.4738	96.26	2.4225	98.28
		$\mathcal{R}4$	2.3060	0.0216	99.16	0.0017	3.5351	96.64	2.4526	98.56
	80% = 32	$\mathcal{R}1$	2.1789	0.0191	98.73	0.0020	3.1359	97.06	2.2890	99.00
		$\mathcal{R}2$	2.3080	0.0216	98.23	0.0024	3.1465	98.43	2.4239	98.68
		$\mathcal{R}3$	2.2101	0.0210	98.70	0.0021	3.138	97.96	2.2523	98.63
		$\mathcal{R}4$	2.1960	0.0194	98.96	0.0018	3.1557	97.53	2.2614	98.76
	100% = 40	$\mathcal{R}0$	2.1534	0.0188	98.16	0.0024	2.9369	97.93	2.2299	98.50
60	60% = 36	$\mathcal{R}1$	1.6825	0.0105	98.59	0.0019	2.1934	97.28	1.7492	98.85
		$\mathcal{R}2$	1.9091	0.0137	98.23	0.0024	2.3408	98.53	1.9577	98.70
		$\mathcal{R}3$	1.7772	0.0115	99.20	0.0016	2.2366	97.53	1.7847	98.86
		$\mathcal{R}4$	1.7577	0.0116	98.80	0.0020	2.2117	97.63	1.7981	98.86
	80% = 48	$\mathcal{R}1$	1.6611	0.0101	98.66	0.0021	2.1044	97.36	1.6782	98.40
		$\mathcal{R}2$	1.7853	0.0123	98.93	0.0019	2.2001	97.63	1.8235	98.40
		$\mathcal{R}3$	1.6766	0.0105	98.83	0.0020	2.0849	97.53	1.7121	98.33
		$\mathcal{R}4$	1.6850	0.0107	98.73	0.0020	2.1180	97.66	1.6776	98.43
	100% = 60	$\mathcal{R}0$	1.6532	0.0105	98.56	0.0022	2.0410	97.50	1.6803	98.80
80	60% = 48	$\mathcal{R}1$	1.4052	0.0071	98.66	0.0021	1.8513	96.95	1.5291	98.65
		$\mathcal{R}2$	1.6262	0.0097	98.53	0.0022	1.8963	98.40	1.6396	98.76
		$\mathcal{R}3$	1.5176	0.0083	98.73	0.0020	1.7776	97.56	1.4842	98.24
		$\mathcal{R}4$	1.4564	0.0079	98.73	0.0020	1.7424	97.70	1.4906	98.55
	80% = 64	$\mathcal{R}1$	1.3986	0.0070	98.76	0.0020	1.6705	97.80	1.4210	98.83
		$\mathcal{R}2$	1.5072	0.0083	98.83	0.0020	1.7461	98.23	1.5103	98.80
		$\mathcal{R}3$	1.4170	0.0072	98.73	0.0020	1.6722	97.83	1.4249	98.73
		$\mathcal{R}4$	1.4077	0.0071	98.90	0.0019	1.6671	97.90	1.4136	98.76
	100% = 80	$\mathcal{R}0$	1.3873	0.0071	98.86	0.0019	1.6198	98.00	1.4106	98.36

Table A.18 Estimated CP (in %) and AL of asymptotic, BS and Jackknife CIs for α at confidence level = 99%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.6$ for different n, FP, τ and censoring schemes. The standard error of estimated AL and CP are presented alongside the Asymptotic CI performance measures.

n	FP% $= r$	C.Sc.	Asymptotic CI				Percentile BS CI		BCa BS CI	
			AL	SE(AL)	CP	SE(CP)	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	1.8413	0.0173	98.00	0.0034	2.4534	98.99	2.3329	96.29
		$\mathcal{R}2$	2.1329	0.0166	98.20	0.0024	2.8789	98.00	2.2329	98.10
		$\mathcal{R}3$	2.0927	0.0166	98.90	0.0019	2.9729	95.90	2.1015	96.66
		$\mathcal{R}4$	2.0125	0.0155	98.57	0.0022	2.8325	97.26	1.9189	98.00
	80% = 32	$\mathcal{R}1$	1.8386	0.0119	98.76	0.0020	2.4325	98.23	1.9939	98.82
		$\mathcal{R}2$	1.9911	0.0145	98.67	0.0021	2.5989	97.43	2.0430	98.10
		$\mathcal{R}3$	1.9102	0.0134	98.70	0.0021	2.5305	97.30	1.9723	98.40
		$\mathcal{R}4$	1.9442	0.0151	99.30	0.0015	2.5504	97.03	1.9638	98.41
	100% = 40	$\mathcal{R}0$	1.8314	0.0126	98.73	0.0020	2.3840	97.93	1.8938	98.40
60	60% = 36	$\mathcal{R}1$	1.4201	0.0089	98.79	0.0026	1.8429	97.13	1.7291	98.94
		$\mathcal{R}2$	1.6814	0.0097	98.43	0.0023	2.0091	98.60	1.7143	98.66
		$\mathcal{R}3$	1.6134	0.0094	99.07	0.0018	1.9684	97.10	1.6575	99.08
		$\mathcal{R}4$	1.5572	0.0087	98.80	0.0020	1.9293	97.15	1.5593	98.50
	80% = 48	$\mathcal{R}1$	1.4413	0.0075	98.87	0.0019	1.7321	98.13	1.3258	97.50
		$\mathcal{R}2$	1.5345	0.0083	98.57	0.0022	1.8436	98.30	1.5808	98.20
		$\mathcal{R}3$	1.4939	0.0079	98.57	0.0022	1.7944	97.43	1.4942	99.03
		$\mathcal{R}4$	1.4974	0.0081	98.97	0.0018	1.7971	97.43	1.5027	98.93
	100% = 60	$\mathcal{R}0$	1.4272	0.0072	98.90	0.0019	1.7054	98.23	1.4600	99.33
80	60% = 48	$\mathcal{R}1$	1.2142	0.0069	99.27	0.0020	1.6921	98.15	1.5827	98.11
		$\mathcal{R}2$	1.4448	0.0075	98.53	0.0022	1.6515	98.16	1.4591	98.48
		$\mathcal{R}3$	1.3504	0.0065	98.97	0.0018	1.5690	97.70	1.3773	98.56
		$\mathcal{R}4$	1.3139	0.0060	99.23	0.0016	1.5420	97.30	1.3301	98.64
	80% = 64	$\mathcal{R}1$	1.2190	0.0052	98.90	0.0019	1.3931	98.00	1.1237	98.40
		$\mathcal{R}2$	1.3055	0.0060	98.77	0.0020	1.4884	98.30	1.3297	98.66
		$\mathcal{R}3$	1.2715	0.0056	99.07	0.0018	1.4531	97.70	1.2824	98.53
		$\mathcal{R}4$	1.2565	0.0056	98.63	0.0021	1.4483	97.53	1.2767	98.53
	100% = 80	$\mathcal{R}0$	1.2106	0.0053	98.77	0.0020	1.3687	98.13	1.2408	98.76

Table A.19 The proportion pn_1 and pn_2 and average test duration (T) when $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ for different n, r, τ and censoring schemes.

n	FP% $= r$	C.Sc.	$\tau_1 = 0.4$			$\tau_1 = 0.6$		
			pn_1%	pn_2%	T	pn_1%	pn_2%	T
40	60% = 24	$\mathcal{R}1$	69.88	30.12	0.5136	86.62	13.38	0.6546
		$\mathcal{R}2$	42.51	57.49	1.3731	58.55	41.45	1.4802
		$\mathcal{R}3$	60.95	39.05	1.2782	72.63	27.37	1.3487
		$\mathcal{R}4$	60.35	39.65	1.2056	74.89	25.11	1.3084
	80% = 32	$\mathcal{R}1$	52.49	47.51	0.7162	72.09	27.91	0.8183
		$\mathcal{R}2$	42.59	57.41	1.4691	58.12	41.88	1.5675
		$\mathcal{R}3$	51.48	48.52	1.4053	65.50	34.50	1.4998
		$\mathcal{R}4$	51.25	48.75	1.4038	65.44	34.56	1.5146
	100% = 40	$\mathcal{R}0$	42.05	57.95	1.5324	58.02	41.98	1.6453
60	60% = 36	$\mathcal{R}1$	70.22	29.78	0.5118	89.35	10.65	0.6450
		$\mathcal{R}2$	42.99	57.01	1.4939	58.18	41.82	1.6042
		$\mathcal{R}3$	60.50	39.50	1.3944	71.47	28.52	1.5015
		$\mathcal{R}4$	61.02	38.98	1.3068	75.05	24.95	1.4317
	80% = 48	$\mathcal{R}1$	52.65	47.35	0.7220	72.09	27.91	0.8206
		$\mathcal{R}2$	42.33	57.67	1.5849	57.85	42.15	1.6783
		$\mathcal{R}3$	51.43	48.57	1.5319	65.42	34.58	1.6296
		$\mathcal{R}4$	50.99	49.01	1.5318	65.58	34.42	1.6287
	100% = 60	$\mathcal{R}0$	42.11	57.89	1.6483	57.75	42.25	1.7544
80	60% = 48	$\mathcal{R}1$	70.14	29.86	0.5151	90.35	09.65	0.6407
		$\mathcal{R}2$	42.46	57.54	1.5927	58.15	41.85	1.6883
		$\mathcal{R}3$	60.27	39.73	1.4799	71.04	28.96	1.5758
		$\mathcal{R}4$	60.68	39.32	1.4156	75.12	24.88	1.5163
	80% = 64	$\mathcal{R}1$	52.67	47.33	0.7235	72.06	27.94	0.8236
		$\mathcal{R}2$	42.26	57.74	1.6815	57.86	42.14	1.7773
		$\mathcal{R}3$	51.61	48.39	1.6234	65.12	34.88	1.7276
		$\mathcal{R}4$	51.24	48.76	1.6058	65.47	34.53	1.7216
	100% = 80	$\mathcal{R}0$	42.28	57.72	1.7490	57.65	42.35	1.8444

Table A.20 The AB and MSE for the ML of the model parameters at ($\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1$, $\tau = 0.6$) for different n, FP and censoring schemes (C.Sc).

n	FP% $= r$	C.Sc	θ_1			θ_2			α		
			MLE	**AB**	**MSE**	**MLE**	**AB**	**MSE**	**MLE**	**AB**	**MSE**
40	60% = 24	$\mathcal{R}1$	0.8174	0.2395	0.2237	0.2977	0.1578	0.1078	1.0337	0.2180	0.1711
		$\mathcal{R}2$	0.6264	0.1921	0.1813	0.2956	0.0780	0.0214	1.1012	0.2540	0.2547
		$\mathcal{R}3$	0.6199	0.1920	0.1663	0.2930	0.0911	0.0297	1.1018	0.2459	0.2199
		$\mathcal{R}4$	0.6090	0.1817	0.1470	0.2859	0.1003	0.0352	1.1124	0.2455	0.2260
	80% = 32	$\mathcal{R}1$	0.6162	0.1492	0.0951	0.2953	0.0924	0.0308	1.0890	0.2208	0.1774
		$\mathcal{R}2$	0.6223	0.1689	0.1288	0.2999	0.0675	0.0160	1.0877	0.2281	0.1948
		$\mathcal{R}3$	0.6106	0.1648	0.1122	0.2984	0.0748	0.0194	1.1044	0.2353	0.2016
		$\mathcal{R}4$	0.6142	0.1655	0.1155	0.2977	0.0752	0.0193	1.0931	0.2324	0.1963
	100% = 40	$\mathcal{R}0$	0.6140	0.1515	0.0932	0.2970	0.0608	0.0124	1.0888	0.2187	0.1726
60	60% = 36	$\mathcal{R}1$	0.7759	0.1907	0.1228	0.2937	0.1508	0.0969	1.0079	0.1686	0.0989
		$\mathcal{R}2$	0.6312	0.1605	0.1149	0.2987	0.0650	0.0144	1.0580	0.2013	0.1411
		$\mathcal{R}3$	0.6108	0.1507	0.0955	0.2957	0.0776	0.0204	1.0710	0.1916	0.1285
		$\mathcal{R}4$	0.6060	0.1472	0.0840	0.2983	0.0873	0.0256	1.0712	0.1883	0.1230
	80% = 48	$\mathcal{R}1$	0.6119	0.1232	0.0571	0.2961	0.0780	0.0208	1.0527	0.1731	0.1036
		$\mathcal{R}2$	0.6185	0.1392	0.0775	0.2976	0.0567	0.0106	1.0553	0.1850	0.1199
		$\mathcal{R}3$	0.6024	0.1309	0.0637	0.3015	0.0632	0.0132	1.0677	0.1832	0.1162
		$\mathcal{R}4$	0.6036	0.1305	0.0646	0.2986	0.0653	0.0135	1.0668	0.1840	0.1178
	100% = 60	$\mathcal{R}0$	0.6079	0.1223	0.0563	0.3011	0.0511	0.0087	1.0544	0.1753	0.1038
80	60% = 48	$\mathcal{R}1$	0.7547	0.1654	0.0838	0.2983	0.1478	0.0918	0.9975	0.1428	0.0683
		$\mathcal{R}2$	0.6114	0.1353	0.0699	0.2987	0.0577	0.0108	1.0549	0.1746	0.1045
		$\mathcal{R}3$	0.6134	0.1359	0.0709	0.3010	0.0674	0.0155	1.0441	0.1585	0.0857
		$\mathcal{R}4$	0.6070	0.1307	0.0606	0.2969	0.0746	0.0188	1.0520	0.1583	0.0856
	80% = 64	$\mathcal{R}1$	0.6068	0.1049	0.0393	0.2974	0.0678	0.0154	1.0430	0.1472	0.0728
		$\mathcal{R}2$	0.6127	0.1209	0.0523	0.2991	0.0485	0.0078	1.0465	0.1607	0.087
		$\mathcal{R}3$	0.6075	0.1177	0.0483	0.3002	0.0542	0.0097	1.0457	0.1561	0.0804
		$\mathcal{R}4$	0.6046	0.1147	0.0481	0.2989	0.054	0.0097	1.0443	0.1519	0.0771
	100% = 80	$\mathcal{R}0$	0.6066	0.1028	0.0387	0.3006	0.0449	0.0065	1.0413	0.1453	0.0714

Table A.21 Estimated CP (in %) and AL of asymptotic CIs for model parameters at confidence level = 95% and 99%, $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1$, $\tau = 0.6$ for different n, FP, and censoring schemes.

n	FP% $= r$	C.Sc.	θ_1				θ_2				α			
			95%		99%		95%		99%		95%		99%	
			AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
40	60% = 24	$\mathcal{R}1$	1.2053	99.80	1.5840	99.96	0.8255	79.33	1.0849	84.00	1.0800	94.60	1.4193	98.06
		$\mathcal{R}2$	1.0250	89.10	1.3471	93.86	0.4032	89.53	0.5299	93.73	1.2527	96.03	1.6463	98.90
		$\mathcal{R}3$	1.0229	88.40	1.3443	93.20	0.4763	89.10	0.6260	92.23	1.2021	96.20	1.5798	98.63
		$\mathcal{R}4$	0.9404	88.36	1.2359	93.03	0.5084	85.83	0.6682	90.16	1.1873	96.20	1.5604	98.83
	80% = 32	$\mathcal{R}1$	0.7867	90.70	1.0339	95.86	0.4747	87.90	0.6239	92.26	1.0775	96.46	1.4161	98.90
		$\mathcal{R}2$	0.8864	91.00	1.1649	95.30	0.3507	91.53	0.4608	95.30	1.1432	96.13	1.5024	98.96
		$\mathcal{R}3$	0.8506	88.80	1.1179	93.66	0.3817	90.00	0.5016	93.83	1.1343	96.33	1.4907	99.23
		$\mathcal{R}4$	0.8545	89.60	1.1230	94.66	0.3832	90.96	0.5037	95.16	1.1176	96.20	1.4688	98.53
	100% = 40	$\mathcal{R}0$	0.7804	90.43	1.0256	95.56	0.3087	91.73	0.4056	95.96	1.0730	96.86	1.4102	99.16
60	60% = 36	$\mathcal{R}1$	0.8959	99.80	1.1774	99.96	0.7817	81.20	1.0274	84.56	0.8417	94.63	1.1062	98.26
		$\mathcal{R}2$	0.8314	90.70	1.0927	95.60	0.3295	91.06	0.4330	95.13	0.9868	95.26	1.2969	98.83
		$\mathcal{R}3$	0.8012	90.40	1.0530	95.43	0.3907	90.03	0.5135	94.53	0.9414	96.73	1.2373	99.20
		$\mathcal{R}4$	0.7477	90.53	0.9827	95.33	0.4323	88.70	0.5681	92.93	0.9198	96.43	1.2088	98.90
	80% = 48	$\mathcal{R}1$	0.6262	91.66	0.8229	96.06	0.3872	89.10	0.5089	93.16	0.8395	95.36	1.1034	99.06
		$\mathcal{R}2$	0.7047	91.26	0.9261	95.66	0.2835	91.60	0.3726	96.30	0.9018	95.33	1.1852	98.66
		$\mathcal{R}3$	0.6707	91.43	0.8814	96.10	0.3146	91.63	0.4134	96.03	0.8835	96.23	1.1612	99.20
		$\mathcal{R}4$	0.6679	91.13	0.8778	96.03	0.3140	90.83	0.4127	95.23	0.8803	96.10	1.1569	99.00
	100% = 60	$\mathcal{R}0$	0.6184	90.83	0.8127	95.73	0.2558	92.70	0.3362	96.46	0.8379	95.26	1.1012	98.83
80	60% = 48	$\mathcal{R}1$	0.7387	99.86	0.9708	100	0.7611	80.83	1.0003	86.00	0.7142	94.36	0.9387	98.23
		$\mathcal{R}2$	0.6795	90.70	0.893	95.80	0.2854	91.60	0.3751	95.50	0.8569	96.26	1.1262	98.86
		$\mathcal{R}3$	0.6936	91.73	0.9115	96.03	0.3441	91.60	0.4522	95.20	0.789	95.86	1.0369	99.03
		$\mathcal{R}4$	0.6437	91.06	0.846	95.70	0.372	89.80	0.4889	93.86	0.7773	96.26	1.0215	99.00
	80% = 64	$\mathcal{R}1$	0.5311	92.70	0.698	96.76	0.3332	90.86	0.438	94.83	0.7179	96.06	0.9435	99.26
		$\mathcal{R}2$	0.5966	91.90	0.7841	95.80	0.2455	92.96	0.3226	97.30	0.7757	95.53	1.0195	99.06
		$\mathcal{R}3$	0.5842	92.13	0.7678	96.13	0.2703	93.26	0.3553	96.56	0.7454	95.06	0.9796	99.13
		$\mathcal{R}4$	0.5765	91.00	0.7577	96.06	0.2719	92.73	0.3573	96.16	0.7412	96.03	0.9741	98.93
	100% = 80	$\mathcal{R}0$	0.5294	92.03	0.6957	96.36	0.2201	93.26	0.2893	97.06	0.7145	95.80	0.939	99.00

Appendix B

Optimum Plan Results

Table B.1 The impact of changing the sample size on optimal τ^* and the corresponding $AVar(\hat{t}_{0.05}(x_0))$ with two censoring schemes where censored data are removed in the middle of the test and $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and FP=80%.

			Optimal τ^*				**$AVar(\hat{t}_{0.05}(x_0))$**			
n	r	$\mathcal{R}$	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
20	16	$(0^{*7}, 2^{*2}, 0^{*7})$	0.5752	0.4503	0.3439	1.9686	0.0055	0.0029	0.0026	0.0537
		$(0^{*6}, 1^{*4}, 0^{*6})$	0.5646	0.4504	0.3324	1.9679	0.0051	0.0029	0.0026	0.0518
40	32	$(0^{*14}, 2^{*4}, 0^{*14})$	0.5042	0.5068	0.3042	0.7609	0.0014	0.0014	0.0013	0.0015
		$(0^{*12}, 1^{*8}, 0^{*12})$	0.5195	0.5084	0.2928	0.7609	0.0014	0.0014	0.0013	0.0014
60	48	$(0^{*22}, 3^{*4}, 0^{*22})$	0.5328	0.5276	0.3042	0.7609	0.0008	0.0008	0.0008	0.0009
		$(0^{*18}, 1^{*12}, 0^{*18})$	0.5414	0.5207	0.3042	0.7626	0.0008	0.0008	0.0008	0.0009
80	64	$(0^{*30}, 4^{*4}, 0^{*30})$	0.5172	0.5224	0.3233	0.7846	0.0006	0.0006	0.0005	0.0006
		$(0^{*24}, 1^{*16}, 0^{*24})$	0.5282	0.5212	0.3233	0.7846	0.0006	0.0006	0.0006	0.0006
100	80	$(0^{*37}, 4^{*5}, 0^{*38})$	0.5277	0.5231	0.3241	0.7846	0.0005	0.0005	0.0005	0.0005
		$(0^{*30}, 1^{*20}, 0^{*30})$	0.5125	0.5230	0.3233	0.7846	0.0005	0.0005	0.0005	0.0005
150	120	$(0^{*57}, 5^{*6}, 0^{*57})$	0.5110	0.5046	0.3233	0.7846	0.0003	0.0003	0.0003	0.0003
		$(0^{*45}, 1^{*30}, 0^{*45})$	0.5120	0.5193	0.3493	0.7901	0.0003	0.0003	0.0003	0.0003
200	160	$(0^{*75}, 4^{*10}, 0^{*75})$	0.5278	0.5185	0.3109	0.7880	0.0002	0.0002	0.0002	0.0002
		$(0^{*60}, 1^{*40}, 0^{*60})$	0.5156	0.5015	0.3561	0.7846	0.0002	0.0002	0.0002	0.0002

Table B.2 The impact of changing the sample size on optimal τ^* and the corresponding $AVar(\hat{t}_{0.5}(x_0))$ with two censoring schemes where censored data are removed in the middle of the test and $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and FP=80%.

			Optimal τ^*				**$AVar(\hat{t}_{0.50}(x_0))$**			
n	r	$\mathcal{R}$	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
20	16	$(0^{*7}, 2^{*2}, 0^{*7})$	0.8087	0.6945	0.5434	1.9688	0.2953	0.1672	0.1529	1.6520
		$(0^{*6}, 1^{*4}, 0^{*6})$	0.7739	0.6881	0.5298	1.9686	0.2875	0.1674	0.1560	2.0027
40	32	$(0^{*14}, 2^{*4}, 0^{*14})$	0.8276	0.8236	0.6891	1.0416	0.0685	0.0682	0.0647	0.0775
		$(0^{*12}, 1^{*8}, 0^{*12})$	0.8305	0.8251	0.6906	0.9747	0.0688	0.0687	0.0655	0.0761
60	48	$(0^{*22}, 3^{*4}, 0^{*22})$	0.8654	0.8665	0.6913	1.0415	0.0429	0.0429	0.0407	0.0467
		$(0^{*18}, 1^{*12}, 0^{*18})$	0.8548	0.8665	0.6854	1.0671	0.0426	0.0423	0.0405	0.0458
80	64	$(0^{*30}, 4^{*4}, 0^{*30})$	0.8992	0.9008	0.6939	1.0878	0.0312	0.0311	0.0298	0.0339
		$(0^{*24}, 1^{*16}, 0^{*24})$	0.8797	0.8960	0.6803	1.0878	0.0310	0.0309	0.0298	0.0337
100	80	$(0^{*37}, 4^{*5}, 0^{*38})$	0.8938	0.9001	0.6939	1.0898	0.0245	0.0245	0.0235	0.0258
		$(0^{*30}, 1^{*20}, 0^{*30})$	0.8600	0.8665	0.7486	1.0447	0.0245	0.0244	0.0235	0.0261
150	120	$(0^{*57}, 5^{*6}, 0^{*57})$	0.8949	0.9050	0.6939	1.0678	0.0159	0.0159	0.0153	0.0169
		$(0^{*45}, 1^{*30}, 0^{*45})$	0.8715	0.8684	0.6789	1.1038	0.0160	0.0160	0.0155	0.0174
200	160	$(0^{*75}, 4^{*10}, 0^{*75})$	0.8845	0.8799	0.6939	1.0878	0.0118	0.0118	0.0112	0.0123
		$(0^{*60}, 1^{*40}, 0^{*60})$	0.8660	0.8700	0.7002	1.1026	0.0118	0.0118	0.0114	0.0125

Table B.3 The impact of changing the sample size on optimal τ^* and the corresponding $AVar(\hat{t}_{0.95}(x_0))$ with two censoring schemes where censored data are removed in the middle of the test and $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and FP=80%.

			Optimal $\boldsymbol{\tau}^*$				**$\boldsymbol{AVar(\hat{t}_{0.95}(x_0))}$**			
$\boldsymbol{n}$	$\boldsymbol{r}$	$\boldsymbol{\mathcal{R}}$	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
20	16	$(0^{*7}, 2^{*2}, 0^{*7})$	0.7533	0.7708	0.2767	1.0499	4.2218	4.1536	3.8183	4.9184
		$(0^{*6}, 1^{*4}, 0^{*6})$	0.7157	0.7252	0.4626	0.9112	4.1741	4.1200	3.8369	4.7021
40	32	$(0^{*14}, 2^{*4}, 0^{*14})$	0.9570	0.9507	0.7492	1.2210	1.5882	1.5643	1.4820	1.9572
		$(0^{*12}, 1^{*8}, 0^{*12})$	0.9433	0.9535	0.7446	1.0997	1.5779	1.5660	1.4517	1.8262
60	48	$(0^{*22}, 3^{*4}, 0^{*22})$	0.9958	1.0031	0.8003	1.1931	0.9397	0.9337	0.8884	1.0480
		$(0^{*18}, 1^{*12}, 0^{*18})$	0.9796	0.9840	0.8003	1.1504	0.9289	0.9245	0.8780	0.9970
80	64	$(0^{*30}, 4^{*4}, 0^{*30})$	1.0227	1.0229	0.7997	1.2380	0.6741	0.6687	0.6424	0.7331
		$(0^{*24}, 1^{*16}, 0^{*24})$	1.0063	1.0190	0.8033	1.2380	0.6689	0.6672	0.6391	0.7056
100	80	$(0^{*37}, 4^{*5}, 0^{*38})$	1.0345	1.0328	0.8862	1.2380	0.5225	0.5236	0.4874	0.5613
		$(0^{*30}, 1^{*20}, 0^{*30})$	1.0104	1.0204	0.8471	1.2380	0.5237	0.5218	0.5004	0.5478
150	120	$(0^{*57}, 5^{*6}, 0^{*57})$	1.0521	1.0402	0.8398	1.2522	0.3379	0.3379	0.3233	0.3524
		$(0^{*45}, 1^{*30}, 0^{*45})$	1.0497	1.0460	0.8381	1.2505	0.3382	0.3374	0.3233	0.3550
200	160	$(0^{*75}, 4^{*10}, 0^{*75})$	1.0512	1.0460	0.8395	1.2532	0.2480	0.2473	0.2365	0.2594
		$(0^{*60}, 1^{*40}, 0^{*60})$	1.0130	1.0167	0.8283	1.2018	0.2491	0.2484	0.2402	0.2644

Table B.4 The impact of changing the sample size on optimal τ^* and the corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ when FP=80%, $\mathcal{R} = \left((n-r), 0^{*(r-1)},\right)$ and $\theta_1 = 0.6, \theta_2 = 0.3, \alpha = 1.2$.

			Optimal τ^*				**$AVar\left(\hat{t}_{\wp}(x_0)\right)$**				
$\wp$	n	r	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**	**SE**
0.05	20	16	0.3200	0.3114	0.1047	0.5507	0.0025	0.0025	0.0019	0.0030	0.0001
	40	32	0.5497	0.5485	0.2713	0.8526	0.0015	0.0015	0.0014	0.0016	1e-05
	60	48	0.5858	0.5521	0.3699	0.8970	0.0009	0.0009	0.0009	0.0010	5e-06
	80	64	0.5881	0.5495	0.3808	0.8262	0.0007	0.0007	0.0006	0.0007	2e-06
	100	80	0.5928	0.5567	0.3822	0.9652	0.0005	0.0005	0.0005	0.0006	1e-06
	150	120	0.5866	0.5957	0.3752	0.9654	0.0003	0.0003	0.0003	0.0003	8e-07
	200	160	0.5856	0.5580	0.3616	0.8911	0.0002	0.0002	0.0002	0.0002	5e-07
0.50	20	16	0.7922	0.7823	0.5901	1.0649	0.1647	0.1594	0.1474	0.2851	0.0035
	40	32	0.9542	0.9564	0.7960	1.1026	0.0676	0.0671	0.0649	0.0718	0.0009
	60	48	0.9845	0.9759	0.8459	1.1973	0.0415	0.0415	0.0390	0.0448	0.0006
	80	64	0.9893	0.9802	0.7952	1.1973	0.0301	0.0301	0.0284	0.0322	7e-05
	100	80	0.9894	0.9973	0.8546	1.1388	0.0236	0.0236	0.0224	0.0247	6e-05
	150	120	0.9654	0.9541	0.7957	1.1973	0.0153	0.0153	0.0148	0.0160	3e-05
	200	160	0.9784	0.9621	0.8326	1.1166	0.0113	0.0113	0.0109	0.0116	2e-05
0.95	20	16	0.8800	0.8573	0.6557	1.1615	3.8050	3.6830	3.4959	5.2395	0.0318
	40	32	1.0573	1.0643	0.8257	1.2745	1.4519	1.4236	1.3492	1.6574	0.0113
	60	48	1.1025	1.1029	0.9316	1.2801	0.8565	0.8533	0.8115	0.9297	0.0027
	80	64	1.1010	1.1018	0.9316	1.2742	0.6179	0.6183	0.5910	0.6470	0.0016
	100	80	1.1141	1.1029	0.9911	1.2833	0.4823	0.4824	0.4607	0.5052	0.0012
	150	120	1.1157	1.1029	0.9316	1.2742	0.3098	0.3101	0.2956	0.3236	0.0006
	200	160	1.1108	1.1029	0.9566	1.2875	0.2287	0.2284	0.2210	0.2379	0.0005

Table B.5 The impact of changing the sample size on optimal τ^* and the corresponding $AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$ when FP=80%, $\mathcal{R} = \left(0^{*(r-1)}, (n-r)\right)$ and $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$. (Type-II censoring).

			Optimal τ^*				**$AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$**				
$\mathcal{p}$	**n**	**r**	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**	**SE**
0.05	20	16	0.2634	0.2669	0.1250	0.4676	0.0023	0.0023	0.0021	0.0029	2e-04
	40	32	0.4054	0.4052	0.2058	0.5906	0.0014	0.0014	0.0013	0.0015	3e-05
	60	48	0.4403	0.4282	0.2615	0.6156	0.0009	0.0008	0.0008	0.0009	2e-06
	80	64	0.4537	0.4532	0.2808	0.6544	0.0006	0.0006	0.0006	0.0006	1e-06
	100	80	0.4414	0.4426	0.2676	0.5985	0.0005	0.0005	0.0004	0.0005	8e-06
	150	120	0.4481	0.4444	0.2713	0.6580	0.0003	0.0003	0.0003	0.0003	4e-07
	200	160	0.4432	0.4409	0.2713	0.6544	0.0002	0.0002	0.0002	0.0002	6e-07
0.50	20	16	0.4913	0.4687	0.3361	1.1026	0.1849	0.1527	0.1457	1.2904	0.0011
	40	32	0.6289	0.6269	0.5232	0.7347	0.0767	0.0755	0.0696	0.1051	0.0004
	60	48	0.6525	0.6515	0.5869	0.7185	0.0448	0.0445	0.0418	0.0512	0.0004
	80	64	0.6646	0.6672	0.5804	0.7233	0.0318	0.0316	0.0305	0.0351	0.0007
	100	80	0.6702	0.6637	0.6085	0.7349	0.0248	0.0247	0.0237	0.0262	7e-05
	150	120	0.6819	0.6865	0.5971	0.7521	0.0161	0.0161	0.0152	0.0170	6e-05
	200	160	0.6710	0.6681	0.5716	0.7420	0.0119	0.0118	0.0116	0.0125	4e-05
0.95	20	16	0.3482	0.3054	0.2023	0.7136	3.2616	3.1813	2.6766	4.9369	4e-05
	40	32	0.6641	0.6555	0.5485	0.7603	1.8083	1.7957	1.6334	2.0616	0.0175
	60	48	0.6912	0.6991	0.6094	0.7542	1.0162	1.0067	0.9375	1.2027	0.0062
	80	64	0.7106	0.7194	0.6544	0.7603	0.6964	0.6972	0.6520	0.7589	0.0029
	100	80	0.7170	0.7169	0.6387	0.7852	0.5394	0.5367	0.5061	0.5806	0.0022
	150	120	0.7216	0.7239	0.6485	0.7861	0.3421	0.3416	0.3248	0.3656	0.0008
	200	160	0.7259	0.7198	0.6544	0.8257	0.2524	0.2521	0.2410	0.2663	0.0002

Table B.6 The impact of increasing the FP on the optimal τ^* and the corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ when $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$.

				Optimal τ^*				$AVar\left(\hat{t}_{\wp}(x_0)\right)$			
$\wp$	n	FP%	$\mathcal{R}$	Mean	Median	Min	Max	Mean	Median	Min	Max
0.05	40	20	$(0^{*2}, 8^{*4}, 0^{*2})$	0.4488	0.2908	0.1113	1.3439	0.0069	0.0060	0.0028	0.0516
	40	40	$(0^{*5}, 4^{*6}, 0^{*5})$	0.3018	0.3012	0.2262	0.4278	0.0018	0.0018	0.0017	0.0020
	40	60	$(0^{*10}, 4^{*4}, 0^{*10})$	0.4391	0.4304	0.3042	0.6330	0.0015	0.0015	0.0014	0.0016
	40	80	$(0^{*14}, 2^{*4}, 0^{*14})$	0.5266	0.5212	0.3042	0.7609	0.0014	0.0014	0.0013	0.0014
	40	100	(0^{*40})	0.5967	0.5864	0.3524	0.9652	0.0014	0.0013	0.0013	0.0014
	100	20	$(0^{*7}, 16^{*5}, 0^{*8})$	0.3039	0.2735	0.2022	1.0125	0.0034	0.0030	0.0028	0.0183
	100	40	$(0^{*17}, 12^{*5}, 0^{*18})$	0.3629	0.3610	0.2499	0.5485	0.0017	0.0017	0.0017	0.0018
	100	60	$(0^{*27}, 8^{*5}, 0^{*28})$	0.4630	0.4587	0.2713	0.6889	0.0015	0.0015	0.0014	0.0016
	100	80	$(0^{*37}, 4^{*5}, 0^{*38})$	0.5034	0.5117	0.3095	0.8254	0.0014	0.0014	0.0013	0.0015
	100	100	(0^{*100})	0.6193	0.6187	0.3361	0.8911	0.0013	0.0013	0.0013	0.0014
0.5	40	20	$(0^{*2}, 8^{*4}, 0^{*2})$	0.3216	0.1592	0.1113	1.0915	0.9089	0.3286	0.2835	5.8375
	40	40	$(0^{*5}, 4^{*6}, 0^{*5})$	0.5008	0.5085	0.3042	0.6829	0.2127	0.2073	0.1935	0.2995
	40	60	$(0^{*10}, 4^{*4}, 0^{*10})$	0.7025	0.6983	0.5497	0.8393	0.1084	0.1072	0.0983	0.1198
	40	80	$(0^{*14}, 2^{*4}, 0^{*14})$	0.8408	0.8390	0.7127	1.0265	0.0689	0.0686	0.0649	0.0762
	40	100	(0^{*40})	0.9625	0.9652	0.8243	1.1587	0.0511	0.0510	0.0476	0.0553
	100	20	$(0^{*7}, 16^{*5}, 0^{*8})$	0.5404	0.5293	0.2380	1.2346	0.2472	0.2291	0.2081	0.7930
	100	40	$(0^{*17}, 12^{*5}, 0^{*18})$	0.7442	0.7444	0.5430	0.9493	0.0674	0.0670	0.0624	0.0806
	100	60	$(0^{*27}, 8^{*5}, 0^{*28})$	0.8302	0.8326	0.6789	1.0079	0.0366	0.0365	0.0348	0.0388
	100	80	$(0^{*37}, 4^{*5}, 0^{*38})$	0.8951	0.8935	0.7013	1.0802	0.0243	0.0243	0.0235	0.0264
	100	100	(0^{*100})	0.9628	0.9526	0.7952	1.1388	0.0186	0.0186	0.0177	0.0193
0.95	40	20	$(0^{*2}, 8^{*4}, 0^{*2})$	0.2564	0.1319	0.1036	0.9652	14.4831	6.2388	5.4752	74.2034
	40	40	$(0^{*5}, 4^{*6}, 0^{*5})$	0.5385	0.5255	0.4147	0.7013	5.3915	5.3655	4.7911	6.4886
	40	60	$(0^{*10}, 4^{*4}, 0^{*10})$	0.8055	0.7972	0.6049	0.9950	2.7818	2.7034	2.4955	4.0026
	40	80	$(0^{*14}, 2^{*4}, 0^{*14})$	0.9456	0.9460	0.7609	1.0931	1.5890	1.5721	1.4752	2.0946
	40	100	(0^{*40})	1.0911	1.0950	0.8801	1.2177	1.0812	1.0780	0.9687	1.1893
	100	20	$(0^{*7}, 16^{*5}, 0^{*8})$	0.6363	0.6347	0.3035	0.8207	5.6624	5.6091	5.0131	6.9856
	100	40	$(0^{*17}, 12^{*5}, 0^{*18})$	0.8776	0.8654	0.7082	1.0460	4.6939	4.6885	4.1960	5.4571
	100	60	$(0^{*27}, 8^{*5}, 0^{*28})$	0.9388	0.9472	0.7837	1.1549	2.4514	2.4552	2.2804	2.5876
	100	80	$(0^{*37}, 4^{*5}, 0^{*38})$	1.0361	1.0442	0.7846	1.2077	1.5823	1.5743	1.5099	1.7009
	100	100	(0^{*100})	1.1069	1.1078	0.9427	1.3089	1.1440	1.1421	1.0968	1.2080

Table B.7 The impact of changing θ_1 on optimal τ^* and the corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ when $n = 80$, FP=80%, $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$ and $\theta_2 = 0.3$, $\alpha = 1.2$.

		Optimal τ^*				**$AVar\left(\hat{t}_{\wp}(x_0)\right)$**			
$\wp$	θ_1	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
0.05	0.6	0.5534	0.5245	0.3623	0.8889	0.0006	0.0006	0.0006	0.0006
	0.8	0.7135	0.6961	0.4500	0.9593	0.0014	0.0014	0.0014	0.0015
	1.0	0.9533	0.9416	0.5466	1.3364	0.0028	0.0028	0.0027	0.0030
	1.2	1.0703	1.0742	0.6719	1.6342	0.0050	0.0049	0.0048	0.0053
	1.5	1.2587	1.2899	0.6939	1.9635	0.0096	0.0096	0.0092	0.0103
	1.8	1.5452	1.4572	0.9373	2.3633	0.0167	0.0167	0.0160	0.0173
	2.0	1.9934	1.8951	1.3352	2.8786	0.0231	0.0231	0.0220	0.0251
	2.5	2.1620	2.1506	1.2695	3.1633	0.0445	0.0445	0.0423	0.0471
0.50	0.6	0.8757	0.8767	0.6939	1.0878	0.0312	0.0312	0.0300	0.0334
	0.8	1.1753	1.1808	0.8443	1.4243	0.0751	0.0750	0.0704	0.0793
	1.0	1.4980	1.5108	1.1800	1.7259	0.1472	0.1480	0.1399	0.1535
	1.2	1.7507	1.7264	1.3313	2.1191	0.2528	0.2529	0.2393	0.2699
	1.5	2.1855	2.2121	1.7304	2.6060	0.4929	0.4906	0.4745	0.5174
	1.8	2.4590	2.5089	1.8673	2.8515	0.8544	0.8476	0.8097	0.9630
	2.0	2.8947	2.8955	2.3687	3.4223	1.1678	1.1681	1.1133	1.2180
	2.5	3.5204	3.5527	2.6311	4.3035	2.2823	2.2740	2.1578	2.4515
0.95	0.6	1.0161	1.0233	0.8444	1.2306	0.6677	0.6681	0.6356	0.7028
	0.8	1.3572	1.3548	1.1770	1.6229	1.6122	1.6121	1.5485	1.7108
	1.0	1.6721	1.6803	1.4220	2.0450	3.1533	3.1437	3.0264	3.3791
	1.2	2.0255	2.0155	1.6880	2.4098	5.4686	5.4748	5.2272	5.7062
	1.5	2.5162	2.5131	2.1663	2.9325	10.5840	10.6099	10.0046	11.0241
	1.8	3.0154	3.0163	2.5232	3.5623	18.3005	18.3037	17.2486	19.2245
	2.0	3.2577	3.2496	2.7638	3.7250	25.0155	24.9822	23.7461	26.1288
	2.5	4.0585	4.0822	3.3156	4.6226	48.8046	48.6820	46.5784	51.5443

Table B.8 The impact of changing θ_2 on optimal τ^* and the corresponding $AVar\left(\hat{t}_{\wp}(x_0)\right)$ when $n = 80$, FP=80%, $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$ and $\theta_1 = 0.6$, $\alpha = 1.2$.

		Optimal τ^*				**$AVar\left(\hat{t}_{\wp}(x_0)\right)$**			
$\wp$	θ_2	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
0.05	0.1	0.5749	0.5484	0.3927	0.9578	0.0018	0.0018	0.0017	0.0020
	0.2	0.5726	0.5492	0.3862	0.9494	0.0009	0.0009	0.0008	0.0010
	0.3	0.5878	0.5634	0.3894	0.9526	0.0006	0.0006	0.0006	0.0006
	0.4	0.5804	0.5480	0.3947	0.9493	0.0004	0.0004	0.0004	0.0005
	0.5	0.5977	0.5506	0.3804	0.9491	0.0003	0.0003	0.0003	0.0004
0.50	0.1	0.8819	0.8908	0.7013	1.1026	0.0946	0.0945	0.0885	0.1024
	0.2	0.8789	0.8908	0.7013	1.0940	0.0472	0.0472	0.0441	0.0495
	0.3	0.8886	0.8907	0.7013	1.1079	0.0312	0.0312	0.0296	0.0331
	0.4	0.8881	0.8988	0.7007	1.0217	0.0219	0.0219	0.0210	0.0231
	0.5	0.8909	0.8997	0.7013	1.0146	0.0150	0.0149	0.0145	0.0159
0.95	0.1	1.0225	1.0087	0.8042	1.1635	2.0254	2.0225	1.9398	2.1262
	0.2	1.0136	1.0082	0.8042	1.1715	1.0161	1.0170	0.9501	1.1018
	0.3	1.0164	1.0383	0.8042	1.1458	0.6716	0.6705	0.6351	0.7163
	0.4	1.0260	1.0272	0.8052	1.1873	0.4687	0.4661	0.4495	0.4959
	0.5	1.0315	1.0217	0.8862	1.1973	0.3257	0.3248	0.3083	0.3466

Table B.9 The impact of changing α on optimal τ^* and the corresponding $AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$ when $n = 80$, FP=80%, $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$ and $\theta_1 = 0.6$, $\theta_2 = 0.3$.

		Optimal τ^*				**$AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$**			
$\mathcal{p}$	α	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
0.05	0.6	0.6516	0.5731	0.4098	1.8046	0.0000	0.0000	0.0000	0.0001
	0.8	0.5833	0.5657	0.3658	0.9373	0.0001	0.0001	0.0001	0.0001
	1.0	0.5643	0.5412	0.3875	0.9373	0.0003	0.0003	0.0003	0.0003
	1.2	0.6036	0.5941	0.3574	0.9378	0.0006	0.0006	0.0006	0.0006
	1.4	0.6063	0.5533	0.4149	0.9405	0.0010	0.0010	0.0009	0.0010
	1.6	0.6198	0.6051	0.4504	0.9378	0.0014	0.0014	0.0013	0.0015
	1.8	0.6716	0.6793	0.5080	0.9373	0.0018	0.0018	0.0018	0.0020
	2.0	0.7104	0.6939	0.5383	0.9374	0.0023	0.0023	0.0022	0.0025
	2.5	0.7991	0.7869	0.6363	0.9593	0.0036	0.0036	0.0034	0.0037
	3.0	0.8413	0.8332	0.7128	0.9944	0.0047	0.0047	0.0045	0.0049
	4.0	1.0351	1.0306	0.9019	1.1809	0.0071	0.0071	0.0068	0.0074
	5.0	1.1807	1.1738	1.0489	1.3541	0.0094	0.0094	0.0090	0.0097
0.50	0.6	0.6845	0.6979	0.5460	0.8237	0.0113	0.0113	0.0105	0.0119
	0.8	0.7687	0.7475	0.6427	0.9493	0.0185	0.0185	0.0175	0.0194
	1.0	0.8629	0.8684	0.6447	1.0081	0.0252	0.0251	0.0244	0.0268
	1.2	0.8752	0.8752	0.6427	1.0217	0.0313	0.0312	0.0298	0.0337
	1.4	0.9342	0.9334	0.7013	1.1026	0.0370	0.0368	0.0354	0.0395
	1.6	0.9836	0.9621	0.8204	1.1388	0.0419	0.0418	0.0404	0.0433
	1.8	1.0132	1.0046	0.8541	1.1974	0.0466	0.0464	0.0447	0.0486
	2.0	1.0368	1.0440	0.8493	1.1973	0.0509	0.0509	0.0489	0.0533
	2.5	1.1144	1.1058	0.9387	1.3506	0.0609	0.0605	0.0577	0.0657
	3.0	1.1786	1.1726	1.0433	1.3561	0.0685	0.0686	0.0652	0.0723
	4.0	1.1857	1.1920	1.0440	1.3506	0.0686	0.0686	0.0658	0.0714
	5.0	1.3651	1.3516	1.1835	1.5986	0.0913	0.0907	0.0878	0.0968

Table B.9 (continue): The impact of changing α on optimal $\tau^*$and the corresponding $AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$ when $n = 80$, FP=80%, $\mathcal{R} = (0^{*30}, 4^{*4}, 0^{*30})$ and $\theta_1 = 0.6$, $\theta_2 = 0.3$.

		Optimal τ^*				**$AVar\left(\hat{t}_{\mathcal{p}}(x_0)\right)$**			
$\mathcal{p}$	**α**	**Mean**	**Median**	**Min**	**Max**	**Mean**	**Median**	**Min**	**Max**
095	0.6	0.8369	0.8387	0.6939	1.0154	0.6939	0.6922	0.6618	0.7313
	0.8	0.8996	0.9018	0.6719	1.0878	0.6831	0.6806	0.6547	0.7340
	1.0	0.9819	0.9661	0.6939	1.1233	0.6721	0.6687	0.6426	0.7370
	1.2	1.0105	1.0293	0.6939	1.1808	0.6730	0.6695	0.6429	0.7400
	1.4	1.0604	1.0647	0.8443	1.3313	0.6699	0.6652	0.6403	0.7448
	1.6	1.0939	1.0904	0.9018	1.3364	0.6737	0.6729	0.6400	0.7141
	1.8	1.1488	1.1618	0.9373	1.3313	0.6730	0.6744	0.6316	0.7071
	2.0	1.1951	1.1811	0.9373	1.3668	0.6740	0.6726	0.6437	0.7116
	2.5	1.2578	1.2670	1.0878	1.3888	0.6702	0.6671	0.6435	0.7048
	3.0	1.3206	1.3298	1.1680	1.5740	0.6712	0.6716	0.6401	0.7025
	4.0	1.3859	1.3887	1.1808	1.6322	0.6650	0.6605	0.6366	0.7090
	5.0	1.4522	1.4385	1.2188	1.7080	0.6165	0.6169	0.5893	0.6617

Table B.10 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 5^{th} percentile when changing n with 80%FP for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$.

n	r	Scheme		$AVar$	T	RE(%)	RT(%)
20	16	Optimal	$\mathcal{R}1$	0.0040	1.4581	100	100
		Worst	$\mathcal{R}3$	0.0116	0.9443	34.51	154.41
		Type-II	$\mathcal{R}3$	0.0116	0.9443	34.51	154.41
		Complete	$\mathcal{R}11$	0.0033	1.5316	119.14	95.20
30	24	Optimal	$\mathcal{R}2$	0.0023	1.5421	100	100
		Worst	$\mathcal{R}3$	0.0063	0.9371	35.83	164.55
		Type-II	$\mathcal{R}3$	0.0063	0.9371	35.83	164.55
		Complete	$\mathcal{R}11$	0.0020	1.6507	111.03	93.41
40	32	Optimal	$\mathcal{R}2$	0.0015	1.6184	100	100
		Worst	$\mathcal{R}3$	0.0081	0.9337	18.73	173.32
		Type-II	$\mathcal{R}3$	0.0081	0.9337	18.73	173.32
		Complete	$\mathcal{R}11$	0.0014	1.7377	110.06	93.13
50	40	Optimal	$\mathcal{R}2$	0.0011	1.6911	100	100
		Worst	$\mathcal{R}3$	0.0035	0.9298	32.90	181.87
		Type-II	$\mathcal{R}3$	0.0035	0.9298	32.90	181.87
		Complete	$\mathcal{R}11$	0.0011	1.8040	106.65	93.74
60	48	Optimal	$\mathcal{R}2$	0.0009	1.7441	100	100
		Worst	$\mathcal{R}3$	0.0021	0.9282	43.90	187.89
		Type-II	$\mathcal{R}3$	0.0021	0.9282	43.90	187.89
		Complete	$\mathcal{R}11$	0.0009	1.8539	106.07	94.07
70	56	Optimal	$\mathcal{R}2$	0.0007	1.7911	100	100
		Worst	$\mathcal{R}3$	0.0013	0.9269	57.15	193.23
		Type-II	$\mathcal{R}3$	0.0013	0.9269	57.15	193.23
		Complete	$\mathcal{R}11$	0.0007	1.8988	104.22	94.32
80	64	Optimal	$\mathcal{R}2$	0.0007	1.8207	100	100
		Worst	$\mathcal{R}3$	0.0014	0.9276	47.40	196.28
		Type-II	$\mathcal{R}3$	0.0014	0.9276	47.40	196.28
		Complete	$\mathcal{R}11$	0.0006	1.9437	104.65	93.67
100	80	Optimal	$\mathcal{R}2$	0.0005	1.8883	100	100
		Worst	$\mathcal{R}3$	0.0007	0.9270	71.32	203.70
		Type-II	$\mathcal{R}3$	0.0007	0.9270	71.32	203.70
		Complete	$\mathcal{R}11$	0.0005	2.0098	104.69	93.95
150	120	Optimal	$\mathcal{R}7$	0.0003	2.0033	100	100
		Worst	$\mathcal{R}3$	0.0004	0.9284	77.66	215.77
		Type-II	$\mathcal{R}3$	0.0004	0.9284	77.66	215.77
		Complete	$\mathcal{R}11$	0.0003	2.1290	102.61	94.09
200	160	Optimal	$\mathcal{R}7$	0.0002	2.0970	100	100
		Worst	$\mathcal{R}3$	0.0003	0.9289	81.41	225.75
		Type-II	$\mathcal{R}3$	0.0003	0.9289	81.41	225.75
		Complete	$\mathcal{R}11$	0.0002	2.2202	102.34	94.44

Table B.11 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 50^{th} percentile when changing n with 80%FP for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$.

n	r	Scheme		$AVar$	T	RE(%)	RT(%)
20	16	Optimal	$\mathcal{R}2$	0.1692	1.4179	100	100
		Worst	$\mathcal{R}3$	0.5278	0.9447	32.06	150.09
		Type-II	$\mathcal{R}3$	0.5278	0.9447	32.06	150.09
		Complete	$\mathcal{R}11$	0.1335	1.5310	126.78	92.61
30	24	Optimal	$\mathcal{R}7$	0.0980	1.5239	100	100
		Worst	$\mathcal{R}3$	0.2859	0.9367	34.29	162.69
		Type-II	$\mathcal{R}3$	0.2859	0.9367	34.29	162.69
		Complete	$\mathcal{R}11$	0.0800	1.6513	122.47	92.28
40	32	Optimal	$\mathcal{R}7$	0.0686	1.6091	100	100
		Worst	$\mathcal{R}3$	0.1498	0.9339	45.76	172.29
		Type-II	$\mathcal{R}3$	0.1498	0.9339	45.76	172.29
		Complete	$\mathcal{R}11$	0.0575	1.7308	119.19	92.96
50	40	Optimal	$\mathcal{R}7$	0.0539	1.6774	100	100
		Worst	$\mathcal{R}3$	0.0973	0.9296	55.36	180.44
		Type-II	$\mathcal{R}3$	0.0973	0.9296	55.36	180.44
		Complete	$\mathcal{R}9$	0.0424	1.8035	127.18	93.00
60	48	Optimal	$\mathcal{R}2$	0.0431	1.2595	100	100
		Worst	$\mathcal{R}3$	0.0873	0.9282	49.31	135.68
		Type-II	$\mathcal{R}3$	0.0873	0.9282	49.31	135.68
		Complete	$\mathcal{R}11$	0.0350	1.8555	122.90	67.87
70	56	Optimal	$\mathcal{R}4$	0.0352	1.0921	100	100
		Worst	$\mathcal{R}3$	0.0636	0.9278	55.30	117.71
		Type-II	$\mathcal{R}3$	0.0636	0.9278	55.30	117.71
		Complete	$\mathcal{R}11$	0.0291	1.9028	120.72	57.39
80	64	Optimal	$\mathcal{R}4$	0.0313	1.0947	100	100
		Worst	$\mathcal{R}3$	0.0457	0.9286	68.48	117.88
		Type-II	$\mathcal{R}3$	0.0457	0.9286	68.48	117.88
		Complete	$\mathcal{R}11$	0.0251	1.9460	124.78	56.25
100	80	Optimal	$\mathcal{R}4$	0.0240	1.0963	100	100
		Worst	$\mathcal{R}3$	0.0307	0.9273	78.06	118.22
		Type-II	$\mathcal{R}3$	0.0307	0.9273	78.06	118.22
		Complete	$\mathcal{R}11$	0.0193	2.0121	124.18	54.48
150	120	Optimal	$\mathcal{R}4$	0.0157	1.1019	100	100
		Worst	$\mathcal{R}3$	0.0187	0.9288	84.19	118.62
		Type-II	$\mathcal{R}3$	0.0187	0.9288	84.19	118.62
		Complete	$\mathcal{R}11$	0.0129	2.1266	121.82	51.81
200	160	Optimal	$\mathcal{R}4$	0.0116	1.1033	100	100
		Worst	$\mathcal{R}3$	0.0137	0.9290	84.76	118.77
		Type-II	$\mathcal{R}3$	0.0137	0.9290	84.76	118.77
		Complete	$\mathcal{R}11$	0.0095	2.2222	122.63	49.65

Table B.12 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 95^{th} percentile when changing n with 80%FP for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $\tau = 0.8$.

n	r	Scheme		$AVar$	T	RE(%)	RT(%)
20	16	Optimal	$\mathcal{R}7$	4.2261	1.4086	100	100
		Worst	$\mathcal{R}3$	7.7586	0.9442	54.46	149.17
		Type-II	$\mathcal{R}3$	7.7586	0.9442	54.46	149.17
		Complete	$\mathcal{R}11$	3.2938	1.5288	128.30	92.13
30	24	Optimal	$\mathcal{R}7$	2.5041	1.5238	100	100
		Worst	$\mathcal{R}3$	4.8987	0.9366	51.11	162.68
		Type-II	$\mathcal{R}3$	4.8987	0.9366	51.11	162.68
		Complete	$\mathcal{R}11$	2.0708	1.6560	120.92	92.01
40	32	Optimal	$\mathcal{R}7$	1.6471	1.6078	100	100
		Worst	$\mathcal{R}3$	3.1529	0.9314	52.24	172.61
		Type-II	$\mathcal{R}3$	3.1529	0.9314	52.24	172.61
		Complete	$\mathcal{R}11$	1.4674	1.7381	112.24	92.50
50	40	Optimal	$\mathcal{R}4$	1.1836	1.0849	100	100
		Worst	$\mathcal{R}3$	1.7939	0.9306	65.97	116.57
		Type-II	$\mathcal{R}3$	1.7939	0.9306	65.97	116.57
		Complete	$\mathcal{R}9$	1.0225	1.8036	115.75	60.15
60	48	Optimal	$\mathcal{R}4$	0.9615	1.0885	100	100
		Worst	$\mathcal{R}3$	1.2562	0.9287	76.54	117.20
		Type-II	$\mathcal{R}3$	1.2562	0.9287	76.54	117.20
		Complete	$\mathcal{R}11$	0.8315	1.8625	115.63	58.44
70	56	Optimal	$\mathcal{R}4$	0.7893	1.0891	100	100
		Worst	$\mathcal{R}3$	0.9994	0.9272	78.97	117.46
		Type-II	$\mathcal{R}3$	0.9994	0.9272	78.97	117.46
		Complete	$\mathcal{R}11$	0.6624	1.9068	119.15	57.11
80	64	Optimal	$\mathcal{R}4$	0.6956	1.0943	100	100
		Worst	$\mathcal{R}3$	0.9180	0.9274	75.77	117.99
		Type-II	$\mathcal{R}3$	0.9180	0.9274	75.77	117.99
		Complete	$\mathcal{R}11$	0.5826	1.9530	119.40	56.03
100	80	Optimal	$\mathcal{R}4$	0.5204	1.0977	100	100
		Worst	$\mathcal{R}3$	0.6232	0.9279	83.49	118.30
		Type-II	$\mathcal{R}3$	0.6232	0.9279	83.49	118.30
		Complete	$\mathcal{R}11$	0.4534	2.0134	114.77	54.52
150	120	Optimal	$\mathcal{R}4$	0.3330	1.1028	100	100
		Worst	$\mathcal{R}3$	0.3733	0.9273	89.20	118.92
		Type-II	$\mathcal{R}3$	0.3733	0.9273	89.20	118.92
		Complete	$\mathcal{R}11$	0.2889	2.1266	115.26	51.85
200	160	Optimal	$\mathcal{R}4$	0.2437	1.1033	100	100
		Worst	$\mathcal{R}7$	0.2763	2.0915	88.19	52.75
		Type-II	$\mathcal{R}3$	0.2690	0.9294	90.61	118.71
		Complete	$\mathcal{R}11$	0.2130	2.2186	114.39	49.73

Table B.13 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 5^{th} percentile when changing θ_1, for $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.8$ and $n = 80$ with 80%FP.

$\boldsymbol{\theta_1}$	**Scheme**		$\boldsymbol{AVar}$	$\boldsymbol{T}$	**RE(%)**	**RT(%)**
0.6	Optimal	$\mathcal{R}2$	0.0007	1.8236	100	100
	Worst	$\mathcal{R}3$	0.0010	0.9273	65.47	196.65
	Type-II	$\mathcal{R}3$	0.0010	0.9273	65.47	196.65
	Complete	$\mathcal{R}11$	0.0006	1.9490	106.77	93.56
0.8	Optimal	$\mathcal{R}2$	0.0015	1.9294	100	100
	Worst	$\mathcal{R}1$	0.0017	1.9752	87.99	97.68
	Type-II	$\mathcal{R}3$	0.0017	1.0242	89.62	188.38
	Complete	$\mathcal{R}11$	0.0014	2.0448	103.95	94.35
1.0	Optimal	$\mathcal{R}2$	0.0029	1.9875	100	100
	Worst	$\mathcal{R}1$	0.0033	2.0365	86.40	97.59
	Type-II	$\mathcal{R}3$	0.0030	1.0850	94.81	183.17
	Complete	$\mathcal{R}11$	0.0028	2.1047	102.47	94.42
1.2	Optimal	$\mathcal{R}7$	0.0051	2.0146	100	100
	Worst	$\mathcal{R}1$	0.0059	2.0735	85.95	97.15
	Type-II	$\mathcal{R}3$	0.0053	1.1249	96.14	179.08
	Complete	$\mathcal{R}9$	0.0049	2.1417	103.28	94.06
1.4	Optimal	$\mathcal{R}2$	0.0081	2.0454	100	100
	Worst	$\mathcal{R}1$	0.0095	2.1062	85.74	97.11
	Type-II	$\mathcal{R}3$	0.0085	1.1525	95.80	177.47
	Complete	$\mathcal{R}11$	0.0080	2.1775	101.84	93.93
1.6	Optimal	$\mathcal{R}2$	0.0122	2.0722	100	100
	Worst	$\mathcal{R}1$	0.0143	2.1199	85.35	97.75
	Type-II	$\mathcal{R}3$	0.0129	1.1737	94.64	176.55
	Complete	$\mathcal{R}11$	0.0120	2.1915	101.52	94.55
1.8	Optimal	$\mathcal{R}2$	0.0175	2.0791	100	100
	Worst	$\mathcal{R}1$	0.0197	2.1267	88.86	97.76
	Type-II	$\mathcal{R}3$	0.0181	1.1884	96.30	174.95
	Complete	$\mathcal{R}11$	0.0172	2.2010	101.57	94.46
2.0	Optimal	$\mathcal{R}7$	0.0232	2.0797	100	100
	Worst	$\mathcal{R}1$	0.0262	2.1369	88.41	97.32
	Type-II	$\mathcal{R}3$	0.0246	1.1998	94.14	173.34
	Complete	$\mathcal{R}11$	0.0223	2.2020	103.93	94.44
2.5	Optimal	$\mathcal{R}2$	0.0396	2.0898	100	100
	Worst	$\mathcal{R}1$	0.0444	2.1418	89.20	97.57
	Type-II	$\mathcal{R}3$	0.0410	1.2201	96.59	171.27
	Complete	$\mathcal{R}11$	0.0390	2.2118	101.60	94.48
3.0	Optimal	$\mathcal{R}2$	0.0579	2.0794	100	100
	Worst	$\mathcal{R}1$	0.0614	2.1448	94.31	96.94
	Type-II	$\mathcal{R}3$	0.0579	1.2321	99.92	168.76
	Complete	$\mathcal{R}11$	0.0576	2.2067	100.51	94.22

Table B.14 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 50^{th} percentile when changing θ_1, for $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.8$ and $n = 80$ with 80%FP.

$\boldsymbol{\theta_1}$	Scheme		$\boldsymbol{AVar}$	$\boldsymbol{T}$	RE(%)	RT(%)
0.6	Optimal	$\mathcal{R}4$	0.0314	1.0929	100	100
	Worst	$\mathcal{R}3$	0.0446	0.9276	70.43	117.82
	Type-II	$\mathcal{R}3$	0.0446	0.9276	70.43	117.82
	Complete	$\mathcal{R}11$	0.0249	1.9443	126.10	56.20
0.8	Optimal	$\mathcal{R}3$	0.0780	1.0239	100	100
	Worst	$\mathcal{R}6$	0.0967	1.9688	80.63	52.00
	Type-II	$\mathcal{R}3$	0.0780	1.0239	100	100
	Complete	$\mathcal{R}11$	0.0721	2.0354	108.26	50.30
1.0	Optimal	$\mathcal{R}3$	0.1888	1.0857	100	100
	Worst	$\mathcal{R}6$	0.2524	2.0335	74.80	53.38
	Type-II	$\mathcal{R}3$	0.1888	1.0857	100	100
	Complete	$\mathcal{R}11$	0.1837	2.1072	102.79	51.52
1.2	Optimal	$\mathcal{R}3$	0.4039	1.1234	100	100
	Worst	$\mathcal{R}1$	0.5201	2.0807	77.66	53.98
	Type-II	$\mathcal{R}3$	0.4039	1.1234	100	100
	Complete	$\mathcal{R}11$	0.4247	2.1463	95.09	52.33
1.4	Optimal	$\mathcal{R}3$	0.7524	1.1528	100	100
	Worst	$\mathcal{R}6$	0.9025	2.0918	83.37	55.11
	Type-II	$\mathcal{R}3$	0.7524	1.1528	100	100
	Complete	$\mathcal{R}11$	0.7831	2.1740	96.08	53.02
1.6	Optimal	$\mathcal{R}3$	1.1547	1.1737	100	100
	Worst	$\mathcal{R}1$	1.2987	2.1169	88.91	55.44
	Type-II	$\mathcal{R}3$	1.1547	1.1737	100	100
	Complete	$\mathcal{R}11$	1.2102	2.1876	95.41	53.65
1.8	Optimal	$\mathcal{R}3$	1.5618	1.1881	100	100
	Worst	$\mathcal{R}6$	1.6576	2.1246	94.21	55.92
	Type-II	$\mathcal{R}3$	1.5618	1.1881	100	100
	Complete	$\mathcal{R}11$	1.6425	2.2013	95.08	53.97
2.0	Optimal	$\mathcal{R}3$	1.8772	1.2008	100	100
	Worst	$\mathcal{R}6$	1.9742	2.1250	95.08	56.50
	Type-II	$\mathcal{R}3$	1.8772	1.2008	100	100
	Complete	$\mathcal{R}11$	1.9992	2.2107	93.89	54.31
2.5	Optimal	$\mathcal{R}8$	2.4170	1.5178	100	100
	Worst	$\mathcal{R}10$	2.6044	2.0175	92.80	75.23
	Type-II	$\mathcal{R}3$	2.4310	1.2203	99.42	124.37
	Complete	$\mathcal{R}11$	2.6733	2.2120	90.41	68.61
3.0	Optimal	$\mathcal{R}3$	2.7220	1.2323	100	100
	Worst	$\mathcal{R}10$	2.9368	2.0099	92.68	61.31
	Type-II	$\mathcal{R}3$	2.7220	1.2323	100	100
	Complete	$\mathcal{R}11$	2.9894	2.2081	91.05	55.80

Table B.15 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 95^{th} percentile when changing θ_1, for $\theta_2 = 0.3$, $\alpha = 1.2$, $\tau = 0.8$ and $n = 80$ with 80%FP.

$\boldsymbol{\theta_1}$	**Scheme**		$\boldsymbol{AVar}$	$\boldsymbol{T}$	**RE(%)**	**RT(%)**
0.6	Optimal	$\mathcal{R}4$	0.6591	1.0936	100	100
	Worst	$\mathcal{R}3$	0.8128	0.9277	81.08	117.8
	Type-II	$\mathcal{R}3$	0.8128	0.9277	81.08	117.8
	Complete	$\mathcal{R}11$	0.5700	1.9421	115.62	56.30
0.8	Optimal	$\mathcal{R}3$	1.9326	1.0258	100	100
	Worst	$\mathcal{R}6$	2.4188	1.9720	79.89	52.01
	Type-II	$\mathcal{R}3$	1.9326	1.0258	100	100
	Complete	$\mathcal{R}11$	1.8138	2.0388	106.54	50.31
1.0	Optimal	$\mathcal{R}3$	4.9440	1.0842	100	100
	Worst	$\mathcal{R}6$	6.4662	2.028	76.45	53.46
	Type-II	$\mathcal{R}3$	4.9440	1.0842	100	100
	Complete	$\mathcal{R}11$	4.7782	2.0977	103.46	51.68
1.2	Optimal	$\mathcal{R}3$	10.9569	1.1232	100	100
	Worst	$\mathcal{R}6$	13.9939	2.0651	78.29	54.38
	Type-II	$\mathcal{R}3$	10.9569	1.1232	100	100
	Complete	$\mathcal{R}11$	11.5778	2.1459	94.63	52.33
1.4	Optimal	$\mathcal{R}3$	19.9918	1.1532	100	100
	Worst	$\mathcal{R}6$	22.0947	2.0943	90.48	55.06
	Type-II	$\mathcal{R}3$	19.9918	1.1532	100	100
	Complete	$\mathcal{R}11$	20.5157	2.1783	97.44	52.93
1.6	Optimal	$\mathcal{R}3$	29.0553	1.1727	100	100
	Worst	$\mathcal{R}6$	31.2474	2.1039	92.98	55.74
	Type-II	$\mathcal{R}3$	29.0553	1.1727	100	100
	Complete	$\mathcal{R}11$	30.2695	2.1844	95.98	53.68
1.8	Optimal	$\mathcal{R}3$	36.1174	1.1895	100	100
	Worst	$\mathcal{R}7$	39.8482	2.0806	90.63	57.17
	Type-II	$\mathcal{R}3$	36.1174	1.1895	100	100
	Complete	$\mathcal{R}11$	38.2206	2.1989	94.49	54.09
2.0	Optimal	$\mathcal{R}1$	41.1437	2.1401	100	100
	Worst	$\mathcal{R}2$	44.6236	2.0860	92.20	102.59
	Type-II	$\mathcal{R}3$	42.5113	1.1998	96.78	178.37
	Complete	$\mathcal{R}11$	45.3612	2.2035	90.70	97.12
2.5	Optimal	$\mathcal{R}1$	47.3938	2.1394	100	100
	Worst	$\mathcal{R}2$	52.5781	2.0902	90.13	102.35
	Type-II	$\mathcal{R}3$	49.7898	1.2196	95.18	175.41
	Complete	$\mathcal{R}11$	53.4715	2.2125	88.63	96.69
3.0	Optimal	$\mathcal{R}1$	49.0296	2.1381	100	100
	Worst	$\mathcal{R}2$	54.5146	2.0780	89.93	102.89
	Type-II	$\mathcal{R}3$	51.8609	1.2316	94.54	173.60
	Complete	$\mathcal{R}11$	55.8334	2.2055	87.81	96.94

Table B.16 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 5^{th} percentile when changing θ_2, for $\theta_1 = 0.6$, $\alpha = 1.2$, $\tau = 0.8$ and $n = 80$ with 80%FP.

$\boldsymbol{\theta_2}$	**Scheme**		$\boldsymbol{AVar}$	$\boldsymbol{T}$	**RE(%)**	**RT(%)**
0.1	Optimal	$\mathcal{R}2$	0.0020	1.1423	100	100
	Worst	$\mathcal{R}3$	0.0036	0.8423	54.41	135.60
	Type-II	$\mathcal{R}3$	0.0036	0.8423	54.41	135.60
	Complete	$\mathcal{R}11$	0.0019	1.1833	106.71	96.52
0.2	Optimal	$\mathcal{R}2$	0.0010	1.4825	100	100
	Worst	$\mathcal{R}3$	0.0020	0.8856	49.08	167.39
	Type-II	$\mathcal{R}3$	0.0020	0.8856	49.08	167.39
	Complete	$\mathcal{R}11$	0.0010	1.5605	103.88	94.99
0.3	Optimal	$\mathcal{R}2$	0.0007	1.8215	100	100
	Worst	$\mathcal{R}3$	0.0012	0.9276	55.48	196.36
	Type-II	$\mathcal{R}3$	0.0012	0.9276	55.48	196.36
	Complete	$\mathcal{R}11$	0.0006	1.9437	106.20	93.71
0.4	Optimal	$\mathcal{R}2$	0.0005	2.1597	100	100
	Worst	$\mathcal{R}3$	0.0009	0.9695	53.72	222.77
	Type-II	$\mathcal{R}3$	0.0009	0.9695	53.72	222.77
	Complete	$\mathcal{R}11$	0.0005	2.3199	105.17	93.09
0.5	Optimal	$\mathcal{R}2$	0.0004	2.5159	100	100
	Worst	$\mathcal{R}3$	0.0007	1.0116	58.50	248.71
	Type-II	$\mathcal{R}3$	0.0007	1.0116	58.50	248.71
	Complete	$\mathcal{R}11$	0.0004	2.7065	107.58	92.95

Table B.17 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 50^{th} percentile when changing θ_2, for $\theta_1 = 0.6$, $\alpha = 1.2$, $\tau = 0.8$ and $n = 80$ with 80%FP.

$\boldsymbol{\theta_2}$	**Scheme**		$\boldsymbol{AVar}$	$\boldsymbol{T}$	**RE(%)**	**RT(%)**
0.1	Optimal	$\mathcal{R}4$	0.0963	0.8984	100	100
	Worst	$\mathcal{R}3$	0.1431	0.8425	67.28	106.63
	Type-II	$\mathcal{R}3$	0.1431	0.8425	67.28	106.63
	Complete	$\mathcal{R}11$	0.0786	1.1824	122.54	75.97
0.2	Optimal	$\mathcal{R}4$	0.0467	0.9969	100	100
	Worst	$\mathcal{R}3$	0.0809	0.8856	57.68	112.56
	Type-II	$\mathcal{R}3$	0.0809	0.8856	57.68	112.56
	Complete	$\mathcal{R}11$	0.0383	1.5600	121.97	63.89
0.3	Optimal	$\mathcal{R}4$	0.0317	1.0921	100	100
	Worst	$\mathcal{R}3$	0.0518	0.9285	61.12	117.61
	Type-II	$\mathcal{R}3$	0.0518	0.9285	61.12	117.61
	Complete	$\mathcal{R}11$	0.0244	1.9455	129.63	56.13
0.4	Optimal	$\mathcal{R}4$	0.0219	1.1920	100	100
	Worst	$\mathcal{R}3$	0.0309	0.9723	71.06	122.59
	Type-II	$\mathcal{R}3$	0.0309	0.9723	71.06	122.59
	Complete	$\mathcal{R}11$	0.0182	2.3229	120.26	51.31
0.5	Optimal	$\mathcal{R}4$	0.0152	1.2894	100	100
	Worst	$\mathcal{R}3$	0.0251	1.0127	60.29	127.31
	Type-II	$\mathcal{R}3$	0.0251	1.0127	60.29	127.31
	Complete	$\mathcal{R}11$	0.0127	2.7085	118.95	47.60

Table B.18 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 95^{th} percentile when changing θ_2, for $\theta_1 = 0.6$, $\alpha = 1.2$, $\tau = 0.8$ and $n = 80$ with 80%FP.

$\boldsymbol{\theta_2}$	**Scheme**		$\boldsymbol{AVar}$	$\boldsymbol{T}$	**RE(%)**	**RT(%)**
0.1	Optimal	$\mathcal{R}9$	2.0790	0.9679	100	100
	Worst	$\mathcal{R}3$	2.6495	0.8424	78.46	114.88
	Type-II	$\mathcal{R}3$	2.6495	0.8424	78.46	114.88
	Complete	$\mathcal{R}11$	1.7548	1.1833	118.47	81.79
0.2	Optimal	$\mathcal{R}9$	1.0412	1.1340	100	100
	Worst	$\mathcal{R}3$	1.2935	0.8855	80.49	128.06
	Type-II	$\mathcal{R}3$	1.2935	0.8855	80.49	128.06
	Complete	$\mathcal{R}11$	0.8874	1.5617	117.33	72.61
0.3	Optimal	$\mathcal{R}4$	0.6894	1.0950	100	100
	Worst	$\mathcal{R}3$	0.9689	0.9279	71.15	118.00
	Type-II	$\mathcal{R}3$	0.9689	0.9279	71.15	118.00
	Complete	$\mathcal{R}11$	0.5911	1.9401	116.62	56.44
0.4	Optimal	$\mathcal{R}9$	0.4812	1.4701	100	100
	Worst	$\mathcal{R}3$	0.5733	0.9718	83.93	151.28
	Type-II	$\mathcal{R}3$	0.5733	0.9718	83.93	151.28
	Complete	$\mathcal{R}11$	0.4148	2.3171	115.99	63.44
0.5	Optimal	$\mathcal{R}9$	0.3238	1.6370	100	100
	Worst	$\mathcal{R}3$	0.5148	1.0138	62.89	161.46
	Type-II	$\mathcal{R}3$	0.5148	1.0138	62.89	161.46
	Complete	$\mathcal{R}11$	0.2885	2.6976	112.22	60.68

Table B.19 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 5^{th} percentile when changing α, for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\tau = 0.8$ and $n = 80$ with 80%FP.

α	Scheme		$AVar$	T	RE(%)	RT(%)
0.6	Optimal	$\mathcal{R}2$	0.0000	1.6184	100	100
	Worst	$\mathcal{R}3$	0.0002	0.8388	8.80	192.95
	Type-II	$\mathcal{R}3$	0.0002	0.8388	8.80	192.95
	Complete	$\mathcal{R}11$	0.0000	1.7386	108.16	93.08
0.8	Optimal	$\mathcal{R}2$	0.0001	1.7104	100	100
	Worst	$\mathcal{R}3$	0.0009	0.8592	12.71	199.07
	Type-II	$\mathcal{R}3$	0.0009	0.8592	12.71	199.07
	Complete	$\mathcal{R}11$	0.0001	1.8206	107.88	93.94
1.0	Optimal	$\mathcal{R}2$	0.0003	1.7725	100	100
	Worst	$\mathcal{R}3$	0.0013	0.8897	25.71	199.22
	Type-II	$\mathcal{R}3$	0.0013	0.8897	25.71	199.22
	Complete	$\mathcal{R}11$	0.0003	1.8939	105.40	93.58
1.2	Optimal	$\mathcal{R}2$	0.0007	1.8252	100	100
	Worst	$\mathcal{R}3$	0.0014	0.9276	47.82	196.76
	Type-II	$\mathcal{R}3$	0.0014	0.9276	47.82	196.76
	Complete	$\mathcal{R}11$	0.0006	1.9456	105.75	93.81
1.4	Optimal	$\mathcal{R}2$	0.0011	1.8754	100	100
	Worst	$\mathcal{R}3$	0.0013	0.9668	79.33	193.98
	Type-II	$\mathcal{R}3$	0.0013	0.9668	79.33	193.98
	Complete	$\mathcal{R}11$	0.0010	1.9895	106.41	94.26
1.6	Optimal	$\mathcal{R}2$	0.0016	1.9088	100	100
	Worst	$\mathcal{R}1$	0.0017	1.9673	89.10	97.02
	Type-II	$\mathcal{R}3$	0.0017	1.0024	91.75	190.43
	Complete	$\mathcal{R}11$	0.0014	2.0302	110.77	94.02
1.8	Optimal	$\mathcal{R}2$	0.0020	1.9443	100	100
	Worst	$\mathcal{R}1$	0.0022	1.9966	87.17	97.37
	Type-II	$\mathcal{R}3$	0.0022	1.0367	90.20	187.54
	Complete	$\mathcal{R}11$	0.0019	2.0663	104.05	94.09
2.0	Optimal	$\mathcal{R}2$	0.0024	1.9679	100	100
	Worst	$\mathcal{R}1$	0.0028	2.0332	88.57	96.78
	Type-II	$\mathcal{R}3$	0.0027	1.0642	91.90	184.91
	Complete	$\mathcal{R}11$	0.0023	2.1025	106.37	93.59
2.5	Optimal	$\mathcal{R}2$	0.0037	2.0411	100	100
	Worst	$\mathcal{R}1$	0.0042	2.0918	88.38	97.57
	Type-II	$\mathcal{R}3$	0.0038	1.1290	95.94	180.78
	Complete	$\mathcal{R}11$	0.0035	2.1693	106.39	94.08
3.0	Optimal	$\mathcal{R}7$	0.0049	2.0920	100	100
	Worst	$\mathcal{R}1$	0.0057	2.1450	86.87	97.52
	Type-II	$\mathcal{R}3$	0.0051	1.1817	97.17	177.03
	Complete	$\mathcal{R}11$	0.0046	2.2125	107.07	94.55

Table B.20 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 50^{th} percentile when changing α, for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\tau = 0.8$ and $n = 80$ with 80%FP.

α	Scheme		$AVar$	T	RE(%)	RT(%)
0.6	Optimal	$\mathcal{R}1$	0.0111	1.6771	100	100
	Worst	$\mathcal{R}3$	0.2751	0.8389	4.03	199.90
	Type-II	$\mathcal{R}3$	0.2751	0.8389	4.03	199.90
	Complete	$\mathcal{R}11$	0.0084	1.7351	132.94	96.65
0.8	Optimal	$\mathcal{R}1$	0.0182	1.7515	100	100
	Worst	$\mathcal{R}3$	0.1738	0.8587	10.48	203.97
	Type-II	$\mathcal{R}3$	0.1738	0.8587	10.48	203.97
	Complete	$\mathcal{R}11$	0.0146	1.8220	124.65	96.13
1.0	Optimal	$\mathcal{R}1$	0.0257	1.8295	100	100
	Worst	$\mathcal{R}3$	0.0825	0.8890	31.13	205.79
	Type-II	$\mathcal{R}3$	0.0825	0.8890	31.13	205.79
	Complete	$\mathcal{R}11$	0.0195	1.8917	131.84	96.71
1.2	Optimal	$\mathcal{R}4$	0.0312	1.0935	100	100
	Worst	$\mathcal{R}3$	0.0562	0.9270	55.43	117.95
	Type-II	$\mathcal{R}3$	0.0562	0.9270	55.43	117.95
	Complete	$\mathcal{R}11$	0.0259	1.9443	120.43	56.24
1.4	Optimal	$\mathcal{R}9$	0.0372	1.3463	100	100
	Worst	$\mathcal{R}3$	0.0508	0.9671	73.31	139.20
	Type-II	$\mathcal{R}3$	0.0508	0.9671	73.31	139.20
	Complete	$\mathcal{R}11$	0.0306	1.9890	121.66	67.68
1.6	Optimal	$\mathcal{R}7$	0.0429	1.8985	100	100
	Worst	$\mathcal{R}6$	0.0482	1.9562	89.03	97.05
	Type-II	$\mathcal{R}3$	0.0436	1.0041	98.36	189.07
	Complete	$\mathcal{R}11$	0.0380	2.0308	112.97	93.48
1.8	Optimal	$\mathcal{R}3$	0.0489	1.0359	100	100
	Worst	$\mathcal{R}6$	0.0560	1.9909	87.39	52.03
	Type-II	$\mathcal{R}3$	0.0489	1.0359	100	100
	Complete	$\mathcal{R}11$	0.0435	2.0739	112.33	49.94
2.0	Optimal	$\mathcal{R}3$	0.0533	1.0670	100	100
	Worst	$\mathcal{R}6$	0.0679	2.0260	78.45	52.66
	Type-II	$\mathcal{R}3$	0.0533	1.0670	100	100
	Complete	$\mathcal{R}11$	0.0509	2.0977	104.74	50.86
2.5	Optimal	$\mathcal{R}3$	0.0731	1.1297	100	100
	Worst	$\mathcal{R}6$	0.0892	2.0822	81.95	54.25
	Type-II	$\mathcal{R}3$	0.0731	1.1297	100	100
	Complete	$\mathcal{R}11$	0.0691	2.1674	105.83	52.12
3.0	Optimal	$\mathcal{R}3$	0.0963	1.1839	100	100
	Worst	$\mathcal{R}1$	0.1230	2.1515	78.29	55.02
	Type-II	$\mathcal{R}3$	0.0963	1.1839	100	100
	Complete	$\mathcal{R}11$	0.0939	2.2156	102.59	53.43

Table B.21 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 95^{th} percentile when changing α, for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\tau = 0.8$ and $n = 80$ with 80%FP.

α	Scheme		$AVar$	T	RE(%)	RT(%)
0.6	Optimal	$\mathcal{R}1$	0.6169	1.6683	100	100
	Worst	$\mathcal{R}3$	14.6333	0.8388	4.21	198.89
	Type-II	$\mathcal{R}3$	14.6333	0.8388	4.21	198.89
	Complete	$\mathcal{R}11$	0.4942	1.7347	124.83	96.17
0.8	Optimal	$\mathcal{R}1$	0.6542	1.7522	100	100
	Worst	$\mathcal{R}3$	6.2095	0.8590	10.53	203.98
	Type-II	$\mathcal{R}3$	6.2095	0.8590	10.53	203.98
	Complete	$\mathcal{R}11$	0.5172	1.8234	126.48	96.09
1.0	Optimal	$\mathcal{R}1$	0.6658	1.8234	100	100
	Worst	$\mathcal{R}3$	2.6600	0.8885	25.03	205.22
	Type-II	$\mathcal{R}3$	2.6600	0.8885	25.03	205.22
	Complete	$\mathcal{R}11$	0.5463	1.8930	121.88	96.31
1.2	Optimal	$\mathcal{R}4$	0.6783	1.0922	100	100
	Worst	$\mathcal{R}3$	1.0331	0.9278	65.65	117.72
	Type-II	$\mathcal{R}3$	1.0331	0.9278	65.65	117.72
	Complete	$\mathcal{R}11$	0.5856	1.9453	115.83	56.14
1.4	Optimal	$\mathcal{R}9$	0.7141	1.3497	100	100
	Worst	$\mathcal{R}6$	0.7890	1.9099	90.50	70.66
	Type-II	$\mathcal{R}3$	0.7322	0.9680	97.52	139.43
	Complete	$\mathcal{R}11$	0.6232	1.9971	114.58	67.58
1.6	Optimal	$\mathcal{R}3$	0.7106	1.0037	100	100
	Worst	$\mathcal{R}6$	0.8832	1.9502	80.45	51.46
	Type-II	$\mathcal{R}3$	0.7106	1.0037	100	100
	Complete	$\mathcal{R}11$	0.6407	2.0285	110.90	49.48
1.8	Optimal	$\mathcal{R}8$	0.7270	1.3493	100	100
	Worst	$\mathcal{R}6$	0.9475	1.9952	76.72	67.62
	Type-II	$\mathcal{R}3$	0.7562	1.0346	96.12	130.41
	Complete	$\mathcal{R}11$	0.6863	2.0699	105.93	65.18
2.0	Optimal	$\mathcal{R}3$	0.7788	1.0649	100	100
	Worst	$\mathcal{R}6$	0.9539	2.0241	81.64	52.61
	Type-II	$\mathcal{R}3$	0.7788	1.0649	100	100
	Complete	$\mathcal{R}11$	0.7689	2.0949	101.29	50.83
2.5	Optimal	$\mathcal{R}3$	0.9363	1.1303	100	100
	Worst	$\mathcal{R}6$	1.1496	2.0856	81.44	54.19
	Type-II	$\mathcal{R}3$	0.9363	1.1303	100	100
	Complete	$\mathcal{R}11$	0.9299	2.1632	100.68	52.25
3.0	Optimal	$\mathcal{R}3$	1.0830	1.1828	100	100
	Worst	$\mathcal{R}6$	1.4187	2.1394	76.33	55.28
	Type-II	$\mathcal{R}3$	1.0830	1.1828	100	100
	Complete	$\mathcal{R}11$	1.0905	2.2145	99.30	53.41

Table B.22 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 5^{th} percentile when changing τ, for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $n = 80$ with 80%FP.

τ	Scheme		$AVar$	SE	T	RE(%)	RT(%)
0.5	Optimal	$\mathcal{R}2$	0.0006	1.7e-6	1.6800	100	100
	Worst	$\mathcal{R}1$	0.0007	2.2e-6	1.7314	85.44	97.02
	Type-II	$\mathcal{R}3$	0.0007	1.7e-6	0.7737	92.02	217.12
	Complete	$\mathcal{R}11$	0.0006	1.5e-6	1.7873	100.33	93.99
0.6	Optimal	$\mathcal{R}2$	0.0006	1.6e-6	1.7205	100	100
	Worst	$\mathcal{R}1$	0.0007	1.9e-6	1.7812	87.53	96.59
	Type-II	$\mathcal{R}3$	0.0007	1.7e-6	0.8245	92.29	208.68
	Complete	$\mathcal{R}11$	0.0006	1.3e-6	1.8403	104.89	93.49
0.7	Optimal	$\mathcal{R}2$	0.0006	1.6e-6	1.7752	100	100
	Worst	$\mathcal{R}3$	0.0008	1.7e-5	0.8742	84.34	203.06
	Type-II	$\mathcal{R}3$	0.0008	1.7e-5	0.8742	84.34	203.06
	Complete	$\mathcal{R}11$	0.0006	1.6e-6	1.8983	105.11	93.51
0.8	Optimal	$\mathcal{R}2$	0.0006	1.4e-6	1.8235	100	100
	Worst	$\mathcal{R}3$	0.0010	2.2e-4	0.9283	66.44	196.43
	Type-II	$\mathcal{R}3$	0.0010	2.2e-4	0.9283	66.44	196.43
	Complete	$\mathcal{R}11$	0.0006	1.6e-6	1.9514	102.02	93.44
0.9	Optimal	$\mathcal{R}6$	0.0007	1.9e-6	1.9157	100	100
	Worst	$\mathcal{R}3$	0.0029	3.9e-4	0.9893	23.28	193.65
	Type-II	$\mathcal{R}3$	0.0029	3.4e-4	0.9893	23.28	193.65
	Complete	$\mathcal{R}11$	0.0006	1.5e-6	1.9902	107.53	96.25
1.0	Optimal	$\mathcal{R}6$	0.0007	2e-6	1.9663	100	100
	Worst	$\mathcal{R}3$	0.0066	8.2e-4	1.0640	10.69	184.79
	Type-II	$\mathcal{R}3$	0.0066	8.2e-4	1.0640	10.69	184.79
	Complete	$\mathcal{R}11$	0.0007	1.3e-6	2.0486	106.59	95.98
1.1	Optimal	$\mathcal{R}6$	0.0007	3.7e-5	2.0202	100	100
	Worst	$\mathcal{R}3$	0.0122	2e-3	1.1492	6.01	175.78
	Type-II	$\mathcal{R}3$	0.0122	2e-3	1.1492	6.01	175.78
	Complete	$\mathcal{R}11$	0.0007	1.8e-6	2.0951	108.04	96.42
1.2	Optimal	$\mathcal{R}6$	0.0008	2.4e-6	2.0708	100	100
	Worst	$\mathcal{R}3$	0.0327	2.7e-3	1.2389	2.45	167.14
	Type-II	$\mathcal{R}3$	0.0327	2.7e-3	1.2389	2.45	167.14
	Complete	$\mathcal{R}11$	0.0007	1.9e-6	2.1422	112.85	96.66

Table B.23 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 50^{th} percentile when changing τ, for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $n = 80$ with 80%FP.

τ	Scheme		$AVar$	SE	T	RE(%)	RT(%)
0.5	Optimal	$\mathcal{R}8$	0.0389	1.2e-4	1.0828	100	100
	Worst	$\mathcal{R}1$	0.0560	1.4e-4	1.7246	69.38	62.78
	Type-II	$\mathcal{R}3$	0.0407	1.3e-4	0.7747	95.47	139.77
	Complete	$\mathcal{R}11$	0.0384	1.3e-4	1.7942	101.19	60.35
0.6	Optimal	$\mathcal{R}8$	0.0340	9.7e-5	1.1314	100	100
	Worst	$\mathcal{R}1$	0.0426	1.7e-4	1.7741	79.64	63.76
	Type-II	$\mathcal{R}3$	0.0344	1.4e-4	0.8233	98.70	137.41
	Complete	$\mathcal{R}11$	0.0314	1.2e-4	1.8453	108.12	61.31
0.7	Optimal	$\mathcal{R}4$	0.0316	1.1e-4	1.0433	100	100
	Worst	$\mathcal{R}6$	0.0362	1.2e-4	1.8114	87.42	57.59
	Type-II	$\mathcal{R}3$	0.0331	5.4e-4	0.8745	95.45	119.31
	Complete	$\mathcal{R}11$	0.0272	6.2e-5	1.9019	116.19	54.85
0.8	Optimal	$\mathcal{R}4$	0.0314	7.7e-5	1.0937	100	100
	Worst	$\mathcal{R}3$	0.0463	2.1e-2	0.9290	67.91	117.72
	Type-II	$\mathcal{R}3$	0.0463	2.1e-2	0.9290	67.91	117.72
	Complete	$\mathcal{R}11$	0.0256	5.5e-5	1.9508	122.91	56.06
0.9	Optimal	$\mathcal{R}1$	0.0303	8.6e-5	1.9316	100	100
	Worst	$\mathcal{R}3$	0.1098	1.9e-2	0.9896	27.61	195.19
	Type-II	$\mathcal{R}3$	0.1098	1.9e-2	0.9896	27.61	195.19
	Complete	$\mathcal{R}11$	0.0246	4.5e-5	1.9966	123.35	96.74
1.0	Optimal	$\mathcal{R}1$	0.0315	7.7e-5	1.9700	100	100
	Worst	$\mathcal{R}3$	0.4469	8.7e-2	1.0643	7.05	185.09
	Type-II	$\mathcal{R}3$	0.4469	8.7e-2	1.0643	7.05	185.09
	Complete	$\mathcal{R}11$	0.0244	4.9e-5	2.0479	129.21	96.19
1.1	Optimal	$\mathcal{R}1$	0.0319	7.7e-5	2.0229	100	100
	Worst	$\mathcal{R}3$	1.0574	2.3e-1	1.1488	3.01	176.09
	Type-II	$\mathcal{R}3$	1.0574	2.3e-1	1.1488	3.01	176.09
	Complete	$\mathcal{R}11$	0.0249	5.8e-5	2.0938	128.13	96.61
1.2	Optimal	$\mathcal{R}1$	0.0321	2.8e-4	2.0833	100	100
	Worst	$\mathcal{R}3$	1.8943	2.3e-1	1.2394	1.76	168.09
	Type-II	$\mathcal{R}3$	1.8943	2.3e-1	1.2394	1.76	168.09
	Complete	$\mathcal{R}11$	0.0263	5.7e-5	2.1421	127.13	97.25

Table B.24 The comparison of optimal $\mathcal{R}^*$ with the worst, Type-II and the complete sample based on estimating the 95^{th} percentile when changing τ, for $\theta_1 = 0.6$, $\theta_2 = 0.3$, $\alpha = 1.2$ and $n = 80$ with 80%FP.

τ	Scheme		$AVar$	SE	T	RE(%)	RT(%)
0.5	Optimal	$\mathcal{R}3$	1.0329	0.0044	0.7739	100	100
	Worst	$\mathcal{R}6$	1.3200	0.0065	1.7214	78.24	44.95
	Type-II	$\mathcal{R}3$	1.0329	0.0044	0.7739	100	100
	Complete	$\mathcal{R}11$	1.0026	0.0044	1.7912	103.02	43.20
0.6	Optimal	$\mathcal{R}3$	0.8110	0.0023	0.8238	100	100
	Worst	$\mathcal{R}6$	1.0254	0.0039	1.7660	79.08	46.64
	Type-II	$\mathcal{R}3$	0.8110	0.0023	0.8238	100	100
	Complete	$\mathcal{R}11$	0.7822	0.0026	1.8478	103.68	44.58
0.7	Optimal	$\mathcal{R}3$	0.7029	0.0182	0.8749	100	100
	Worst	$\mathcal{R}6$	0.8570	0.0027	1.8223	82.02	48.01
	Type-II	$\mathcal{R}3$	0.7029	0.0182	0.8749	100	100
	Complete	$\mathcal{R}11$	0.6604	0.0021	1.8917	106.43	46.24
0.8	Optimal	$\mathcal{R}4$	0.6822	0.0022	1.0951	100	100
	Worst	$\mathcal{R}3$	0.8393	0.3923	0.9269	81.28	118.14
	Type-II	$\mathcal{R}3$	0.8393	0.3923	0.9269	81.28	118.14
	Complete	$\mathcal{R}11$	0.5787	0.0017	1.9450	117.89	56.30
0.9	Optimal	$\mathcal{R}1$	0.6578	0.0021	1.9328	100	100
	Worst	$\mathcal{R}3$	2.4561	1.1520	0.9903	26.78	195.16
	Type-II	$\mathcal{R}3$	2.4561	1.1520	0.9903	26.78	195.16
	Complete	$\mathcal{R}11$	0.5223	0.0015	1.9950	125.95	96.88
1.0	Optimal	$\mathcal{R}1$	0.6428	0.0019	1.9757	100	100
	Worst	$\mathcal{R}3$	3.9247	1.4108	1.0638	16.37	185.71
	Type-II	$\mathcal{R}3$	3.9247	1.4108	1.0638	16.37	185.71
	Complete	$\mathcal{R}11$	0.5037	0.0012	2.0454	127.62	96.59
1.1	Optimal	$\mathcal{R}1$	0.6312	0.0017	2.0337	100	100
	Worst	$\mathcal{R}3$	13.4307	2.0907	1.1490	4.69	176.99
	Type-II	$\mathcal{R}3$	13.4307	2.0907	1.1490	4.69	176.99
	Complete	$\mathcal{R}11$	0.6312	0.0012	2.0337	100	100
1.2	Optimal	$\mathcal{R}1$	0.6264	0.0017	2.0801	100	100
	Worst	$\mathcal{R}3$	30.8444	6.8235	1.2394	2.03	167.82
	Type-II	$\mathcal{R}3$	30.8444	6.8235	1.2394	2.03	167.82
	Complete	$\mathcal{R}11$	0.5019	0.0012	2.1447	124.78	96.98

Table B.25 The sensitivity of the SSALTs model based on $AVar(\hat{t}_{0.05}(x_0))$ and the associated ratio of base scenario and random scenario for $\mathcal{R} = \mathcal{R}1$, $\tau = 0.8$ and $n = 80$ with 80%FP.

		$\alpha = 0.8$		$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$	
θ_1	θ_2	$AVar$	**Ratio**	$AVar$	**Ratio**	$AVar$	**Ratio**	$AVar$	**Ratio**
0.4	0.1	0.0002	0.25	0.0008	1.00	0.0012	1.50	0.0017	2.13
	0.2	0.0001	0.13	0.0004	0.50	0.0006	0.75	0.0009	1.13
	0.3	0.0001	0.13	0.0003	0.38	0.0004	0.50	0.0006	0.75
0.5	0.1	0.0002	0.25	0.0013	1.63	0.0022	2.75	0.0031	3.88
	0.2	0.0001	0.13	0.0007	0.88	0.0011	1.38	0.0015	1.88
	0.3	0.0001	0.13	0.0004	0.50	0.0007	0.88	0.0010	1.25
	0.4	0.0001	0.13	0.0003	0.38	0.0005	0.63	0.0007	0.88
0.6	0.1	0.0004	0.50	0.0019	2.38	0.0036	4.50	0.0051	6.38
	0.2	0.0002	0.25	0.0010	1.25	0.0018	2.25	0.0025	3.13
	0.3	0.0001	0.13	0.0008	1.00	0.0011	1.38	0.0014	1.75
	0.4	0.0001	0.13	0.0005	0.63	0.0009	1.13	0.0012	1.50
	0.5	0.0001	0.13	0.0003	0.38	0.0007	0.88	0.0010	1.25
0.7	0.1	0.0006	0.75	0.0035	4.38	0.0055	6.88	0.0080	10.00
	0.2	0.0003	0.38	0.0017	2.13	0.0028	3.50	0.0041	5.13
	0.3	0.0002	0.25	0.0011	1.38	0.0019	2.38	0.0026	3.25
	0.4	0.0002	0.25	0.0009	1.13	0.0014	1.75	0.0020	2.50
	0.5	0.0001	0.13	0.0007	0.88	0.0011	1.38	0.0016	2.00
	0.6	0.0001	0.13	0.0006	0.75	0.0009	1.13	0.0013	1.63
0.8	0.1	0.0009	1.13	0.0051	6.38	0.0082	10.25	0.0119	14.88
	0.2	0.0005	0.63	0.0026	3.25	0.0041	5.13	0.0060	7.50
	0.3	0.0003	0.38	0.0016	2.13	0.0027	3.38	0.0040	5.00
	0.4	0.0002	0.25	0.0013	1.63	0.0021	2.63	0.0029	3.63
	0.5	0.0002	0.25	0.0010	1.25	0.0016	2.00	0.0024	3.00
	0.6	0.0002	0.25	0.0009	1.13	0.0014	1.75	0.0020	2.50
	0.7	0.0001	0.13	0.0007	0.88	0.0012	1.50	0.0017	2.13

Table B.26 The sensitivity of the SSALTs model based on $AVar(\hat{t}_{0.95}(x_0))$ and the associated ratio of base scenario and random scenario for $\mathcal{R} = \mathcal{R}1$, $\tau = 0.8$ and $n = 80$ with 80%FP.

		$\alpha = 0.8$		$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$	
θ_1	θ_2	$AVar$	**Ratio**	$AVar$	**Ratio**	$AVar$	**Ratio**	$AVar$	**Ratio**
0.4	0.1	0.6206	0.88	0.5661	0.81	0.5749	0.82	0.5491	0.78
	0.2	0.3101	0.44	0.2853	0.41	0.2819	0.40	0.2705	0.39
	0.3	0.1699	0.24	0.1644	0.23	0.1658	0.24	0.1673	0.24
0.5	0.1	1.0547	1.50	1.1258	1.60	1.1735	1.67	1.2103	1.72
	0.2	0.5466	0.78	0.5512	0.78	0.5946	0.85	0.6017	0.86
	0.3	0.3425	0.49	0.3651	0.52	0.3738	0.53	0.3950	0.56
	0.4	0.2261	0.32	0.2362	0.34	0.2504	0.36	0.2623	0.37
0.6	0.1	2.0060	2.86	2.1181	3.02	2.2772	3.24	2.4928	3.55
	0.2	0.9745	1.39	1.0436	1.49	1.2097	1.72	1.2745	1.81
	0.3	0.6555	0.93	0.7024	1.00	0.6673	0.95	0.6788	0.97
	0.4	0.4782	0.68	0.4954	0.71	0.5398	0.77	0.5791	0.82
	0.5	0.3156	0.45	0.3420	0.49	0.3867	0.55	0.4166	0.59
0.7	0.1	3.4095	4.85	3.8820	5.53	4.2585	6.06	4.9352	7.03
	0.2	1.7183	2.45	2.0168	2.87	2.1847	3.11	2.3299	3.32
	0.3	1.1867	1.69	1.3379	1.90	1.4712	2.09	1.6469	2.34
	0.4	0.8306	1.18	0.9958	1.42	1.0788	1.54	1.2017	1.71
	0.5	0.6324	0.90	0.7808	1.11	0.8230	1.17	0.9154	1.30
	0.6	0.4615	0.66	0.5373	0.76	0.6043	0.86	0.6601	0.94
0.8	0.1	5.5593	7.91	6.6198	9.42	7.8499	11.18	8.8922	12.66
	0.2	2.5851	3.68	3.6261	5.16	4.0558	5.77	4.4285	6.3
	0.3	1.9485	2.77	1.9195	2.90	2.6526	3.78	2.9108	4.14
	0.4	1.4563	2.07	1.7890	2.55	1.9602	2.79	2.3051	3.28
	0.5	1.1170	1.59	1.3589	1.93	1.5512	2.21	1.6744	2.38
	0.6	0.8519	1.21	1.0828	1.54	1.2489	1.78	1.4195	2.02
	0.7	0.6678	0.95	0.8190	1.17	0.8862	1.26	1.0732	1.53

Appendix C

Optimum Plan Results – Figures of Censoring Schemes

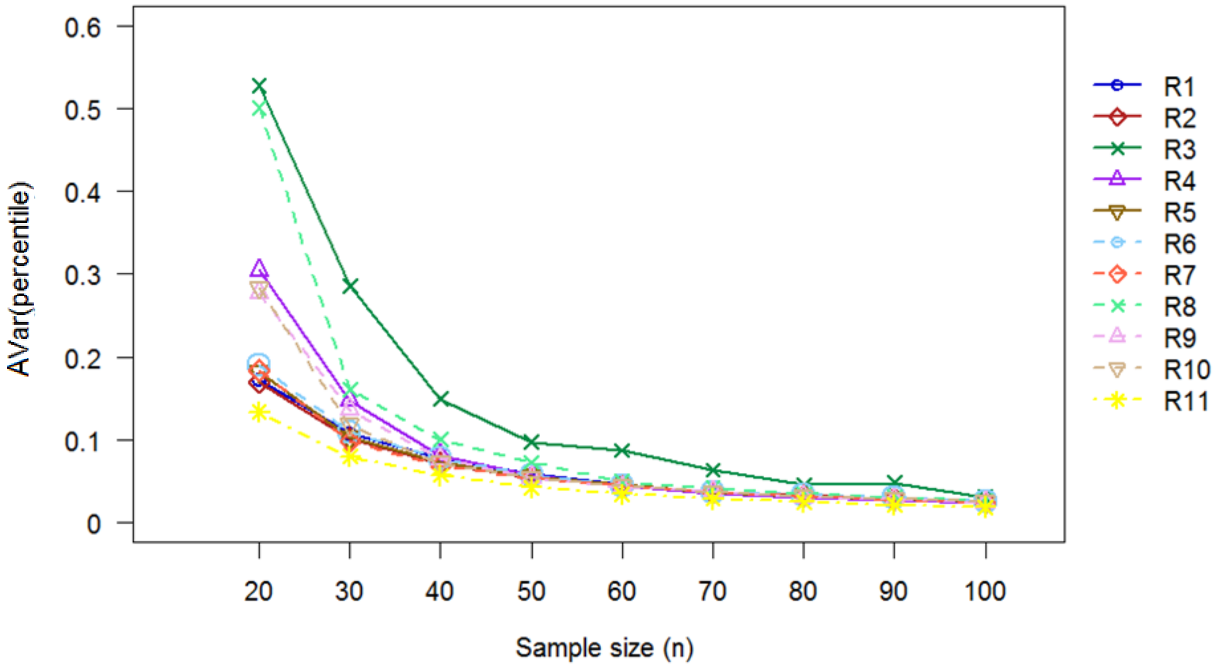


Figure C.1 $AVar(\hat{t}_{0.05}(x_0))$ vs. sample size for 11 censoring schemes.

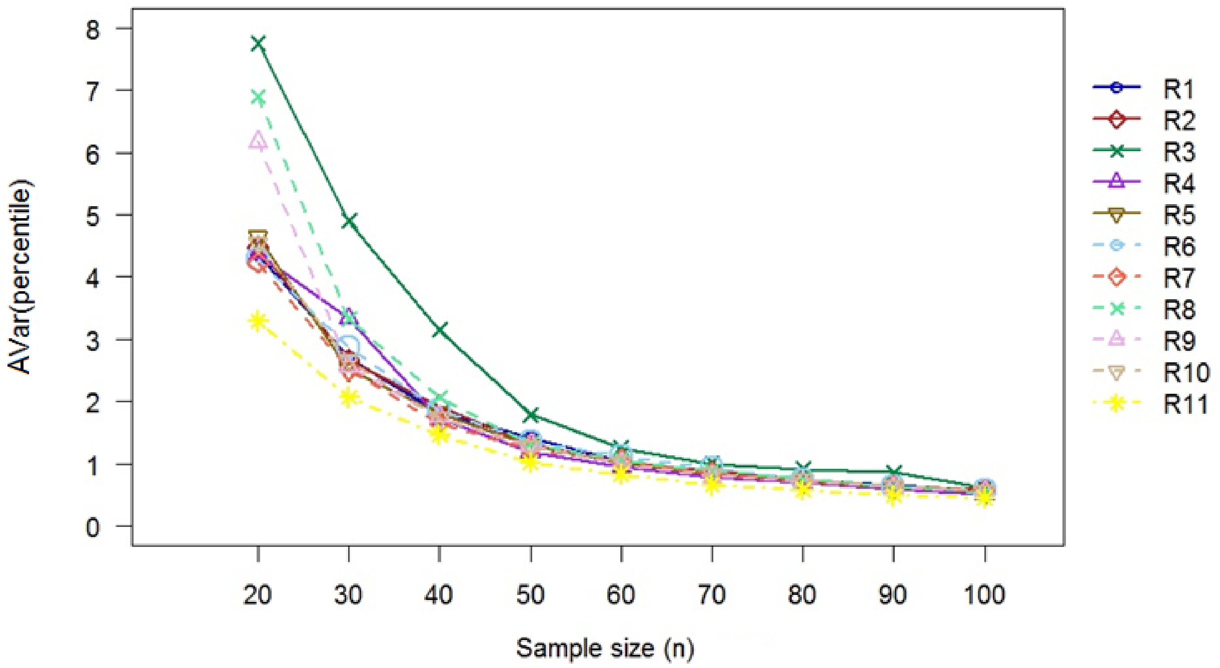


Figure C.2 $AVar(\hat{t}_{0.95}(x_0))$ vs. sample size for 11 censoring schemes.

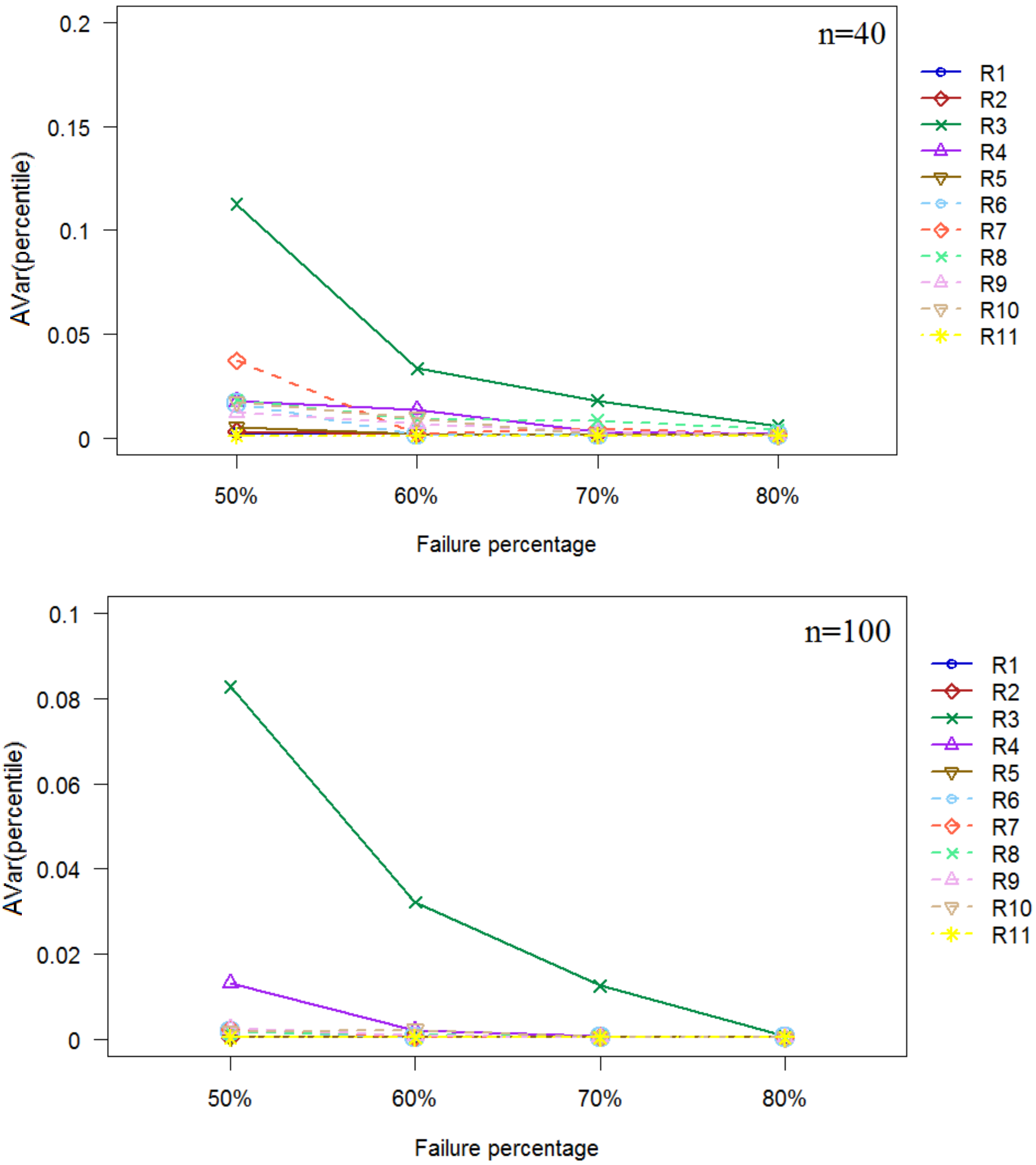


Figure C.3 The impact of increasing the FP on the $AVar\left(\hat{t}_{0.05}(x_0)\right)$ for 11 censoring schemes.

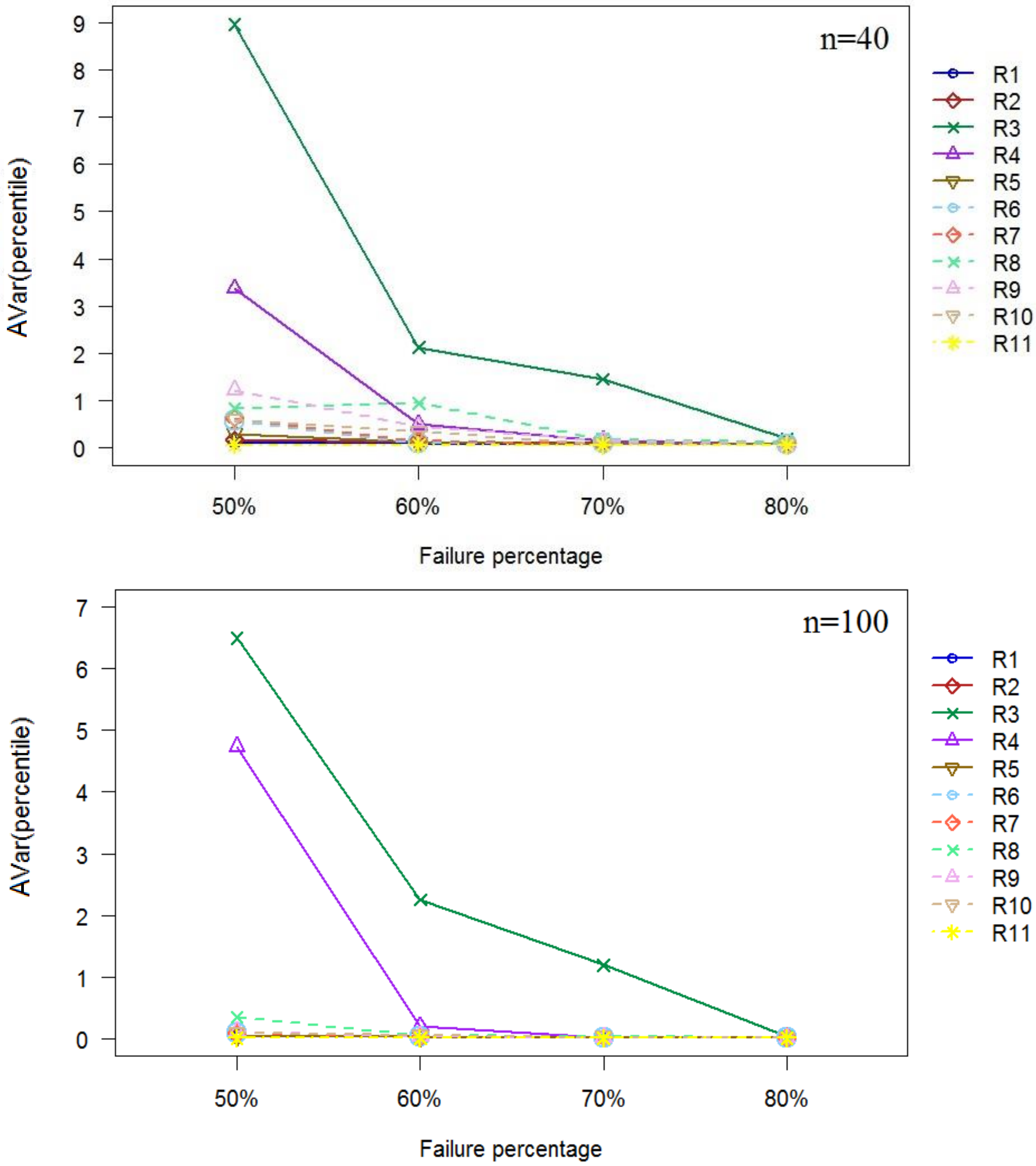


Figure C.4 The impact of increasing the FP on the $AVar\left(\hat{t}_{0.5}(x_0)\right)$ for 11 censoring schemes.

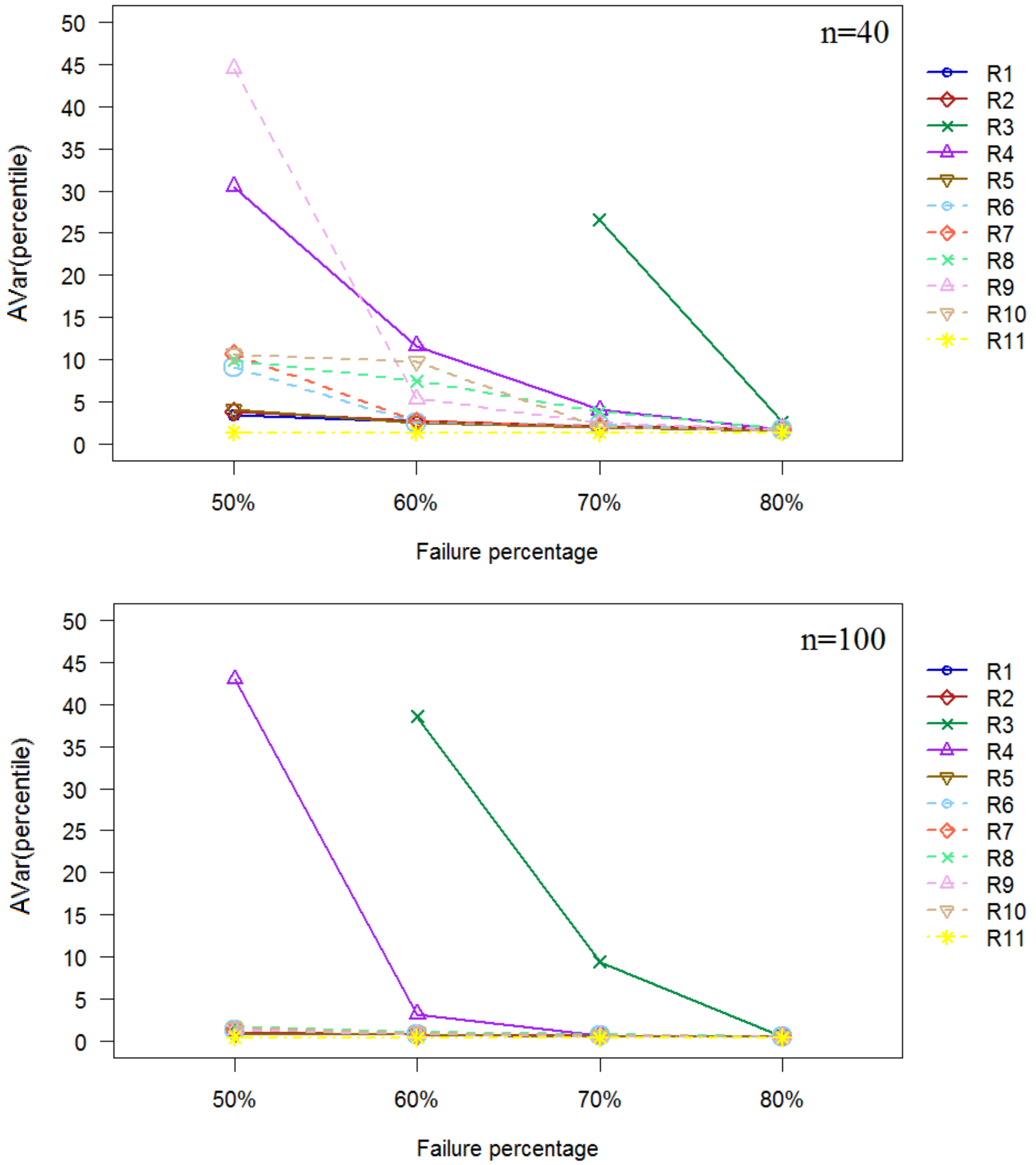


Figure C.5 The impact of increasing the FP on the $AVar(\hat{t}_{0.95}(x_0))$ for 11 censoring schemes.

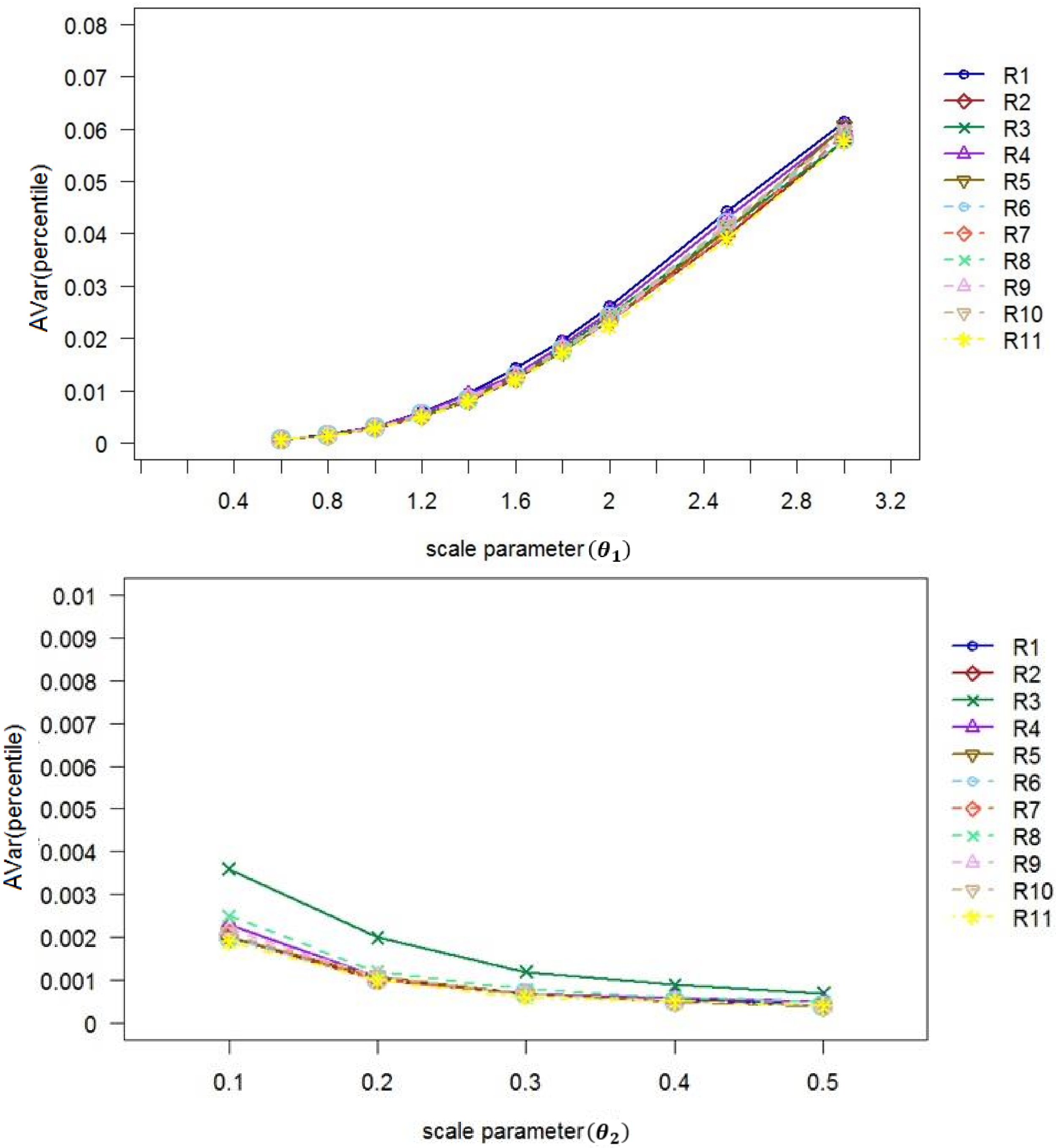
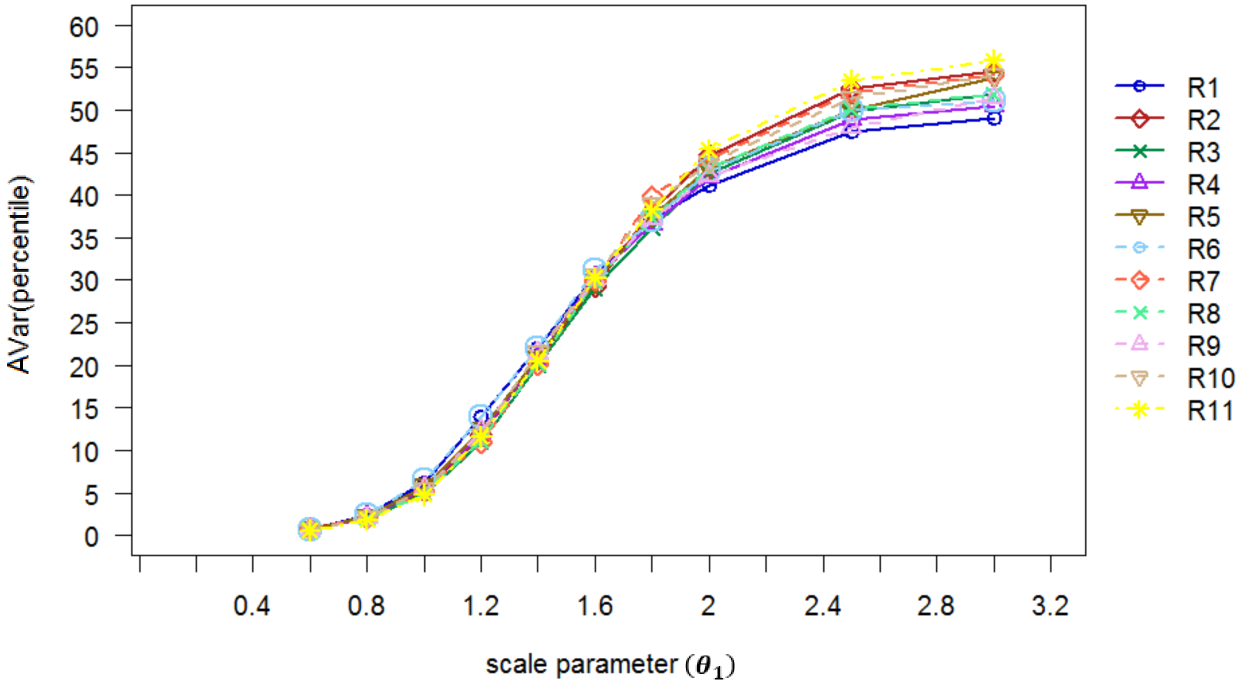


Figure C.6 $AVar(\hat{t}_{0.05}(x_0))$ vs. scale parameters θ_1 and θ_2 for 11 censoring schemes.



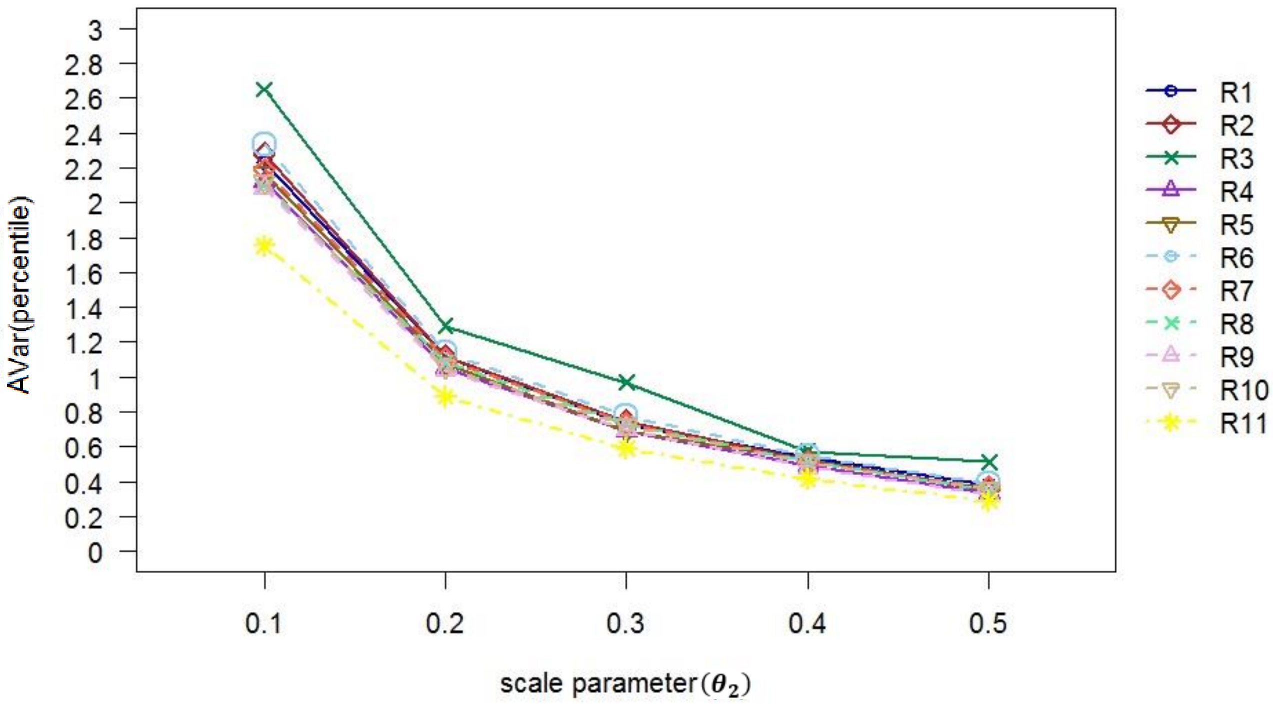


Figure C.7 $AVar(\hat{t}_{0.95}(x_0))$ vs. scale parameters θ_1 and θ_2 for 11 censoring schemes.

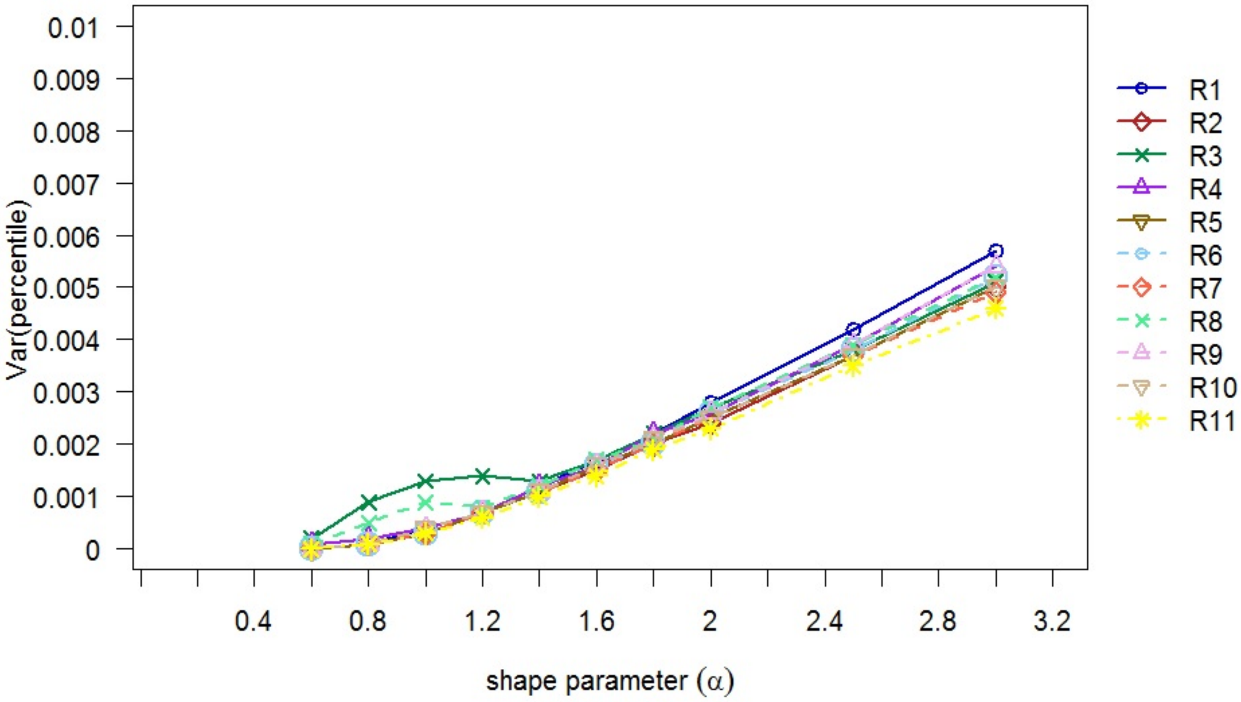


Figure C.8 $AVar(\hat{t}_{0.05}(x_0))$ vs. α for 11 censoring schemes.

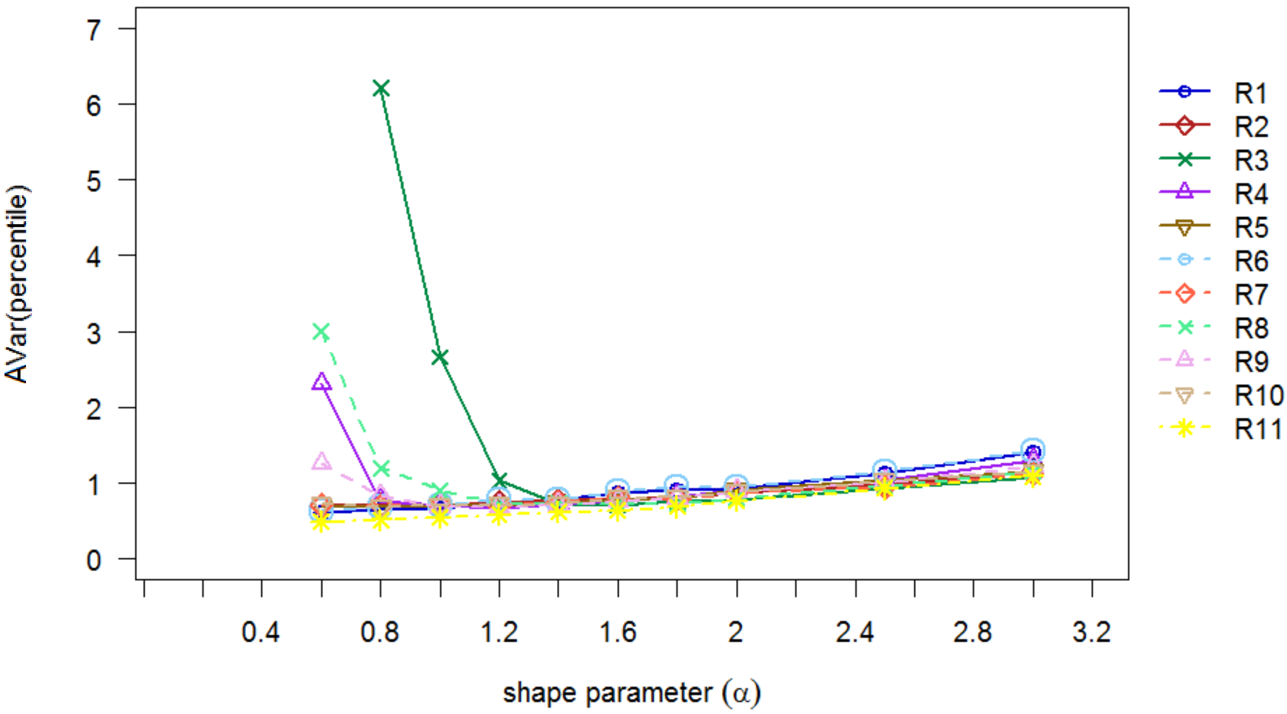


Figure C.9 $AVar(\hat{t}_{0.95}(x_0))$ vs. α for 11 censoring schemes.

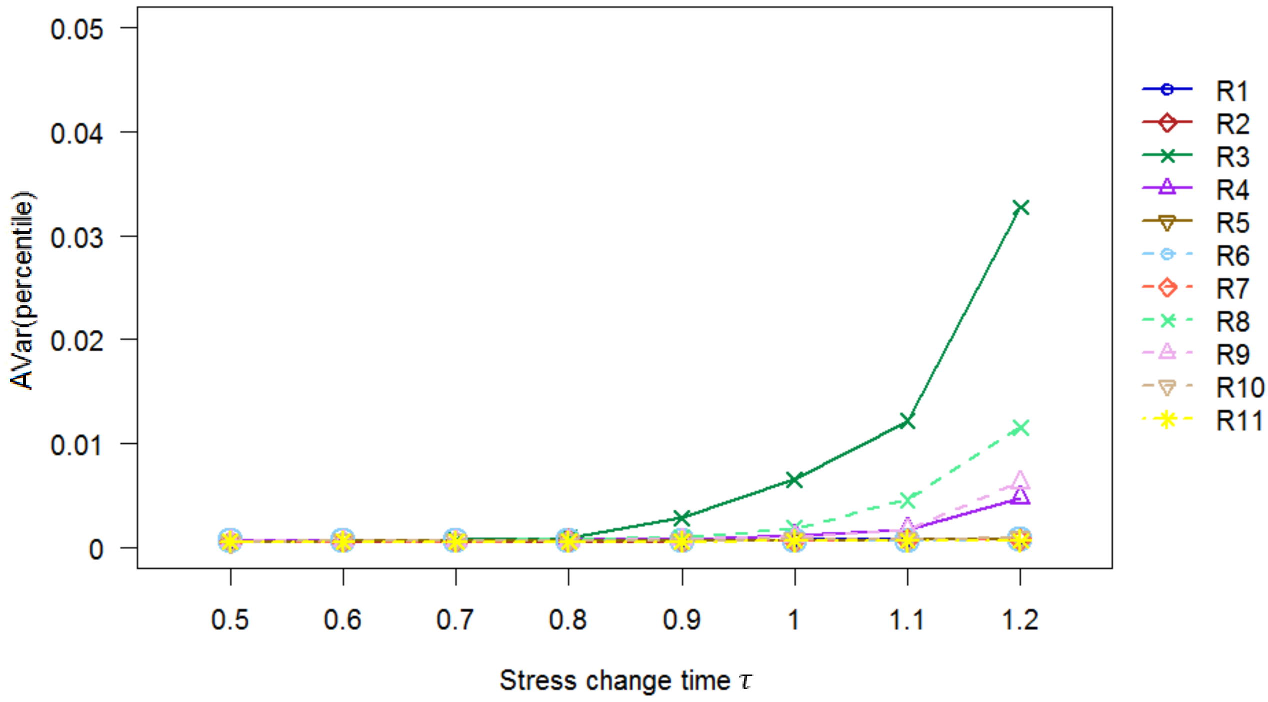


Figure C.10 $AVar(\hat{t}_{0.05}(x_0))$ vs. τ for 11 censoring schemes.

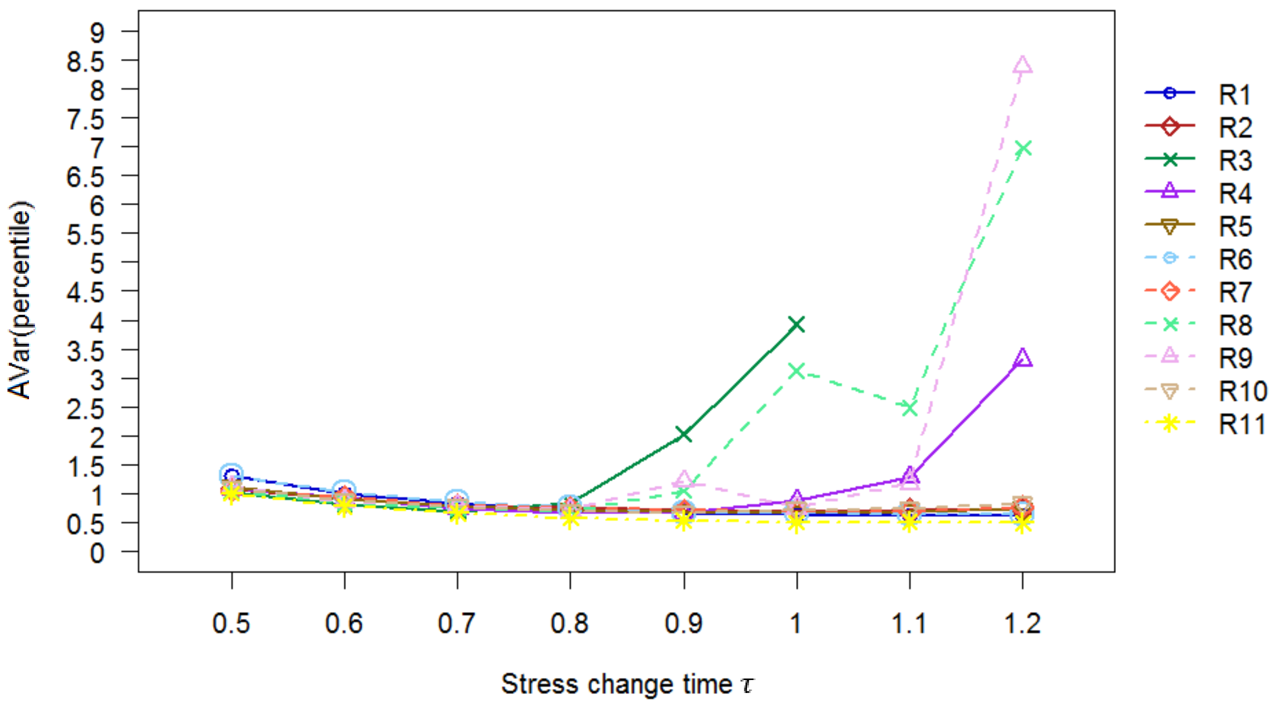


Figure C.11 $AVar(\hat{t}_{0.95}(x_0))$ vs. τ for 11 censoring schemes.

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