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Viscosity extragradient with modified inertial method for solving equilibrium problems and fixed point problem in Hadamard manifold

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Abstract

In this article, we propose a viscosity extragradient algorithm together with an inertial extrapolation method for approximating the solution of pseudomonotone equilibrium and fixed point problem of a nonexpansive mapping in the setting of a Hadamard manifold. We prove that the sequence generated by our iterative method converges to a solution of the above problems under some mild conditions. Finally, we outline some implications of our results and present several numerical examples showing the implementability of our algorithm. The results of this article extend and complement many related results in linear spaces.

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Keywords: Equilibrium problem; Nonexpansive mapping; Fixed point problems; Viscosity; Hadamard manifold and extragradient algorithm

1 Introduction

The Equilibrium Problem (EP) is well known because it has wide application in spatial price equilibrium models, computer and electric networks, market behavior, and economic and financial networks models. For instance, the spatial price equilibrium models arising from EPs have provided a framework for analyzing competitive systems over space and time and have formulated contributions that have stimulated the development of new methodologies and opened up prospects for their applications in the energy sector, minerals economics, finance, and agriculture (see, for example, [1]). It is well known that many interesting and challenging problems in nonlinear analysis, such as complementarity, fixed point, Nash equilibrium, optimization, saddle point and variational inequalities, can be reformulated as EP (see [2]). The EP for a bifunction $g : C \times C \to R$, satisfying the condition $g(x,x) = 0$ for every $x \in C$ is defined as follows:

$$\text{Find } u \in C \text{ such that } g(u,v) \geq 0, \forall v \in C, \tag{1.1}$$

where C is a non-empty subset of a topological space X. We denote by EP(g) the solution set of (1.1). Several iterative methods have been designed to approximate the solution of

EP (1.1) if the bifunction g is monotone (see, for example, [3–5] and reference therein). One of the crucial methods for solving EP when g is pseudomonotone is the extragradient Method (EM), which involves solving two strongly convex optimization problems at each iteration. In 1997, Korpelevich [6] and Antipin [7] employed the EM for solving the saddle point problems. Later, in 2008, Quoc et al. [8] extended this idea to solve the pseudomonotone EP. Since then, many authors have employed EM and other methods to solve EP of pseudomonotone type in Hilbert and Banach spaces (see, for example, [9–11]). It is important for us to consider EP in a more general space, such as a Riemannian manifold. This idea of extending optimization methods from Euclidean space to Riemannian manifolds has some remarkable advantages. From the point of view of the Riemannian manifold, it is possible to convert nonconvex problems to convex ones by endowing the space with an appropriate Riemannian manifold metric (see, for example, [12–14]). Moreover, a constrained problem can be viewed as unconstrained due to the Riemannian geometry. Recently, some results on Hilbert spaces have been generalized to more general settings, such as the Riemannian manifold, to solve nonconvex cases (see, for example, [15–18]). Most extended methods from linear settings, such as Hilbert space to Riemannian manifolds, require the Riemannian manifold to have nonpositive sectional curvature. This is an essential property shared by a large class of Riemannian manifolds, and it is strong enough to imply tough topology restrictions and rigidity phenomena. The algorithm for solving equilibrium (EP) on Hadamard manifolds has received great concentration (see, for example [19–21]). Lately, Cruz Neto et al. [22] extended the work by Van Nguyen et al. [23] and acquired an extragradient method for solving the equilibrium problem on a complete simple connected sectional curvature. They employed the following algorithm:

$$\begin{cases} y_n = \arg\min\{g(x_n, z) + \frac{1}{2\lambda_n} d^2(x_n, z)\} \\ x_{n+1} = \arg\min\{g(y_n, z) + \frac{1}{2\lambda_n} d^2(x_n, z)\} \end{cases} \tag{1.2}$$

where $0 < \lambda_n < \beta < \min\{\alpha_1^{-1}, \alpha_2^{-1}\}$ and α_1, α_2 are Lipschitz constants of the bifunction g. It should be known that Lipschitz-type constants are laborious to approximate even in complex nonlinear problems, and they are generally unknown. In 2020, Junfeng et al. [24] introduced a new extragradient-like method for (EP) on the Hadamard manifold. Their algorithm performed without prior knowledge of the Lipschitz-type constants.

Let C be a non-empty closed and convex subset of a complete Riemannian manifold M. A fixed point set of T is represented by

$$F(T) = \{y \in C : T(y) = y\}, \tag{1.3}$$

and a mapping $T : C \to C$ is said to be

(i) a contraction if there exist $\alpha \in [0, 1]$, such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in C,$$

(ii) nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C.$$

Several researchers use different methods for the approximation of a fixed point of nonexpansive mapping. Approximation methods have received so much attention in fixed point theory because they are very compelling and important tools of nonlinear science. Moreover, the viscosity-type algorithm converges faster than the Halpern-type algorithm (see, for example, [16, 17, 25–33]). In 2000, Moudafi [25] initiated the viscosity for approximation method for nonexpansive mapping in the Hilbert space, he obtained strong convergence results of both implicit and explicit schemes in Hilbert spaces. In 2004, Xu [26] extended Moudafi's results [25] to Banach space. Now, the concept of viscosity was recently extended to more general space such as Riemannian manifold. In 2016, Jeong J.U. [34] demonstrated some results using generalized viscosity approximation methods for mixed equilibrium problems and fixed point. Motivated by the work of Daun and He [27], Renu Chugh and Mandeep Kumari [35] extended the work of Duan and He [27] in the framework of Riemannian manifold as follows.

Theorem 1.1 *Let C be a closed convex subset of Hadamard manifold M, and let $T : C \to C$ be a nonexpansive mappings such that $F(T) \neq \emptyset$. Let $\psi_n : C \to C$ be ρ_n- contraction with $0 \le p_i = \lim_{n\to\infty} \inf \rho_n \le \lim_{n\to\infty} \sup \rho_n = \rho_u \le 1$ suppose that $\{\psi_n(x)\}$ is uniformly convergence for any $x \in A$, where A is any bounded subset of C if the sequence $\{\lambda_n\} \subset (0,1)$ satisfies the following conditions*:

(i) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$,

(ii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| \le \infty$ *and*

(iii) $\lim_{n\to\infty} \frac{\lambda_n - \lambda n-1}{\lambda_n} = 0$.

Then, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \exp_{\psi_n(x_n)}(1-\lambda_n)\exp^{-1}\psi_n(x_n)Tx_n.$$

An inertial term is necessary to improve the iterative sequence to accomplish the desired solution. These methods of inertial term are basically used to accelerate the iterative sequence towards the required solution by speeding up the convergence rate of the iterative scheme. Several analyzes have shown that inertial effects improve the performance of the algorithm in terms of the number of iterations and time of execution. Due to these two advantages, inertial term have attracted more attention in solving different problems. An algorithm with an inertial term was first initiated by Polyak [36], who proposed inertial extrapolation for solving a smooth convex minimization problem (MP). Since then, authors introduced algorithms with inertial term in different spaces (see, [37, 38]). Khamahawong et al. [39] introduced an inertial Mann algorithm for approximating a fixed point of a nonexpansive mapping on a Hadamard manifold. Under suitable assumptions, they proved that their method was also dedicated to solving inclusion and equilibrium problems. They defined their algorithm as follows.

Let M be a Hadamard manifold, and $F : M \to M$ is a mapping. Choose $x_0, x_1, \in M$. Define a sequence $\{x_n\}$ by the following iterative scheme:

$$\begin{cases} y_n = \exp_{x_n}(-\lambda_n)\exp^{-1}_{x_n} x_n \\ x_{n+1} = \exp_{y_n}(1-\gamma_n)\exp_{y_n} F(y_n), \end{cases} \tag{1.4}$$

where $\{\lambda_n\} \subset [0,\infty)$ and $\{\gamma_n\} \subset (0,1)$ satisfying the following conditions

(C_1) $0 \leq \lambda_n < \lambda < 1 \, \forall n \geq 1$,
(C_2) $\sum_{n=1}^{\infty} \lambda_n d^2(x_n, x_{n-1}) \leq \infty$,
(C_3) $0 < \gamma_1 \leq \gamma_n \leq \gamma_2$, $n \geq 1$,
(C_4) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

They proved that the sequence generated by their algorithm converges faster and strongly to an element in the solution set. Motivated and inspired by the following works [22, 35, 38–42], we study the viscosity extragradient with a modified inertial algorithm for solving equilibrium and fixed point problems in the Hadamard manifold. The advantages of our results over existing ones are the following:

(i) Our algorithm converges faster than the existing results due to the inertial term we added to the algorithm.
(ii) Our results is obtained in the more general space Hadamard manifold, in contrast to the results in Hilbert and Banach spaces (see, for example, [43, 44]).

The remain sections of the paper are organized as follows: We first give some basic concepts and useful tools in Sect. 2. In Sect. 3, we provide our proposed method and states its convergence analysis. In Sect. 4, we provide some numerical examples. In Sect. 5, we show the outcomes of a computational trial that demonstrate the effectiveness of our approaches.

2 Preliminaries

In this section, we present some basic concepts, definitions, and preliminary results that will be useful in what follows.

Suppose that M is a simply connected n-dimensional manifold. The tangent space of M at x is denoted by T_xM, which is a vector space of the same dimension as M. A tangent bundle of M is given by $TM = \bigcup_{x \in M} T_xM$. A smooth mapping $\langle \cdot, \cdot \rangle \to \mathbb{R}$ is called a Riemannian metric on M if $\langle \cdot, \cdot \rangle_x : T_xM \times T_xM \to \mathbb{R}$ is an inner product for $t \in M$. We denote the norm by $\| \cdot \|_x$ related to the inner product $\langle \cdot, \cdot \rangle$ on T_xM . The length of a piecewise smooth curve $c : [a, b] \to M$ joining x to r defined using the metric $L(c) = \int_a^b \|c'(t)\| \, dt$ where $c(a) = x$ and $c(b) = r$. Then, the Riemannian distance denoted by $d(x, r)$ is defined to be the minimal length over the set of all such curves joining x to r, which induces the topology on M. A geodesic in M joining $x \to r$ is said to be minimal geodesic if its length is equal to $d(x, r)$. A geodesic triangle $\Delta(x_1, x_2, x_3)$ of a Riemannian manifold is defined to be a set consisting of points x_1, x_2, x_3 and three minimal geodesic γ_i joining x_i to x_{i+1} with $i = 1, 2, 3 \mod 3$. A Riemannian manifold is said to be complete if for any $x \in M$ all geodesic emerging from x are defined for all $t \in (-\infty, \infty)$. Let M be a complete Riemannian manifold, any pair in M can be joined by minimizing geodesic (Hopf-Rinow Theorem [45]). Thus, (M, d) is a complete metric space, and closed bounded subset is compact. The exponential map $\exp_x : T_xM \to M$ at $x \in M$ such that $\exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(i) = \gamma_v(1, x)$ is geodesic starting at x with velocity v. Then, $\exp_x tv = \gamma_v(t, x)$ for each real number t, and $\exp_x 0 = \gamma_v(0, x) = x$. It should be noted that the mapping $\exp_x$ exhibits differentiability on T_xM for each x within M. For any x and y belonging to M, the exponential map $\exp_x$ possesses an inverse denoted as $\exp^{-1} M \to T_xM$. Given any $x, y \in M$, we can observe the quantity $d(x, y) = \| \exp_y^{-1} x\| = \| \exp_x^{-1} y\|$ (refer to [46] for additional examples).

Definition 2.1 A complete simple connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold.

Lemma 2.2 [47] *Let $k, p \in \mathbb{R}$ and $\lambda \in [0,1]$. Then, the following holds:*

(i) $\|\lambda k + (1-\lambda)p\|^2 = \lambda\|k\|^2 + (1-\lambda)\|p\|^2 - \lambda(1-\lambda)\|k-p\|^2$,

(ii) $\|k \pm p\|^2 = \|k\|^2 \pm 2\langle k, p\rangle + \|p\|^2$,

(iii) $\|k+p\|^2 \leq \|k\|^2 + 2|\langle p, k+p\rangle|$.

Lemma 2.3 *Let ρ be a lower semi-continuous, proper, and convex function on Hadamard manifold M, and $z, t \in M$, $\lambda > 0$. If $t = \text{prox}_{\lambda\rho}(z)\,\forall y \in M$, then*

$$\langle \exp_t^{-1} y, \exp_t^{-1} z\rangle \leq \lambda\big(\rho(y) - \rho(t)\big).$$

Proposition 2.4 [48] *Assume that M is a Hadamard manifold, and let $d : M \times M \to \mathbb{R}$ represent a metric function. Then, d is a convex function with respect to the product Riemannian metric, which is, given any two of geodesic $\gamma_1 : [0,1] \to M$ and $\gamma : [0,1] \to M$ the following inequality holds for all $t \in [0,1]$*

$$d\big(\gamma_1(t), \gamma_2(t)\big) \leq (1-t)d\big(\gamma_1(0), \gamma_2(0)\big) + td\big(\gamma_1(1), \gamma_2(1)\big).$$

In fact, for each $y \in M$, the function $d(\cdot, y) : M \to \mathbb{R}$ is convex function.

Definition 2.5 [49] Let C be a non-empty, closed and convex subset of M.

A bifunction $f : M \times M \to \mathbb{R}$ is said to be

(i) monotone if $f(x,y) + f(y,x) \leq 0\,\forall x, y \in C$;

(ii) pseudomonotone if

$$f(x,y) \geq 0 \quad \Longrightarrow \quad f(y,x) \leq 0 \quad \forall x, y \in C;$$

(iii) Lipschitz-type continuous if there exist constants $c_1 > 0$ and $c_2 > 0$, such that

$$f(x,y) + f(y,x) \geq f(x,z) - c_1 d^2(x,y) - c_2 d^2(y,z) \quad \forall x, y, z \in C.$$

Lemma 2.6 [48] *Let $x \in M$, $\{x_k\} \subset M$ and $x_k \to x$, then for all $y \in M$*

$$\exp_{x_k} y \to \exp_{x_0} y \quad \textit{and} \quad \exp_y x_k \to \exp_y^{-1} x_0.$$

Proposition 2.7 [48] *For any point $x \in MP_N x$ is a singleton, and the following inequality holds for all $r \in N$*

$$\langle \exp_N x, \exp_N r\rangle \leq 0,$$

where $N \subset M$.

Proposition 2.8 [50] (*Comparison theorem for triangle*) *Let $\Delta(x_, x_2, x_3)$ be a geodesic triangle. Denote each $i = 1,2,3 \pmod 3$ by $\gamma_i : [0, l_i] \to M$ the geodesic joining x_i to x_{i+1}, and set $\alpha_i = \angle(\gamma_i', \gamma_{i-1}(l_{i-1}))$, the angle between the vector $\gamma_i'(0)$ and $-\gamma_{i-1}(l_{i-1})$ and $l_i = L(\gamma_i)$, then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi, \qquad l_i^2 + l_{i+1}^2 - 2l_i l_{i+1}\cos\alpha_{i+1} \leq l_{i+1}^2. \tag{2.1}$$

Using the distance and the exponential map, (2.1) *can be indicated as*

$$d^2(x_i,x_{i+1}) + d^2(x_{i+1},x_{i+2}) - 2\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{i+2}\rangle \le d^2(x_{i-1},x_i). \tag{2.2}$$

Since

$$\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle = d(x_i,x_{i+1})d(x_{i+1},x_{i+2})\cos\alpha_{i+1}. \tag{2.3}$$

Let $x_{i+2} = x_i$*, then in association with* (2.3)*, we get*

$$\|\exp_{x_{i+1}}^{-1} x_i\|^2 = \langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+1}\rangle = d^2(x_i,x_{i+1}).$$

Proposition 2.9 [48] *Let* K *be a non-empty convex subset of a Hadamard manifold* M *and* $g : K \to \mathbb{R}$ *be a convex subdifferentiable and lower semi-continuous function on* K*. Then,* p *is a solution to the following convex problem*

$$\min\{g(x) : x \in K\}.$$

if and only if $0 \in \partial_g(p) + N_k(p)$*.*

Proposition 2.10 [48] *Let* $p \in M$*. The exponential mapping* $\exp_p : T_pM \to M$ *is a diffeomorphism, and for any two points* $p,q \in M$*, there exists a unique normalized geodesic joining* p *to* q*, which can be expressed by the formula*

$$\omega(t) = \exp_p t\exp_p^{-1} q \quad \forall t \in [0,1]. \tag{2.4}$$

A geodesic triangle $\Delta(p_1,p_2,p_3)$ *of a Riemannian manifold* M *is a set consisting of three points* p_1*,* p_2*, and* p_3 *and three minimizing geodesic joining these points.*

Proposition 2.11 [48] *Let* $\Delta(p_1,p_2,p_3)$ *be a geodesic triangle in* M*. Then,*

$$d^2(p_1,p_2) + d^2(p_2,p_3) - 2\langle \exp_{p_2}^{-1} p_1, \exp_{p_2}^{-1} p_3\rangle \le d^2(p_3,p_1),$$

and

$$d^2(p_1,p_2) \le \langle \exp_{p_1}^{-1} p_3, \exp_{p_1}^{-1} p_2\rangle + \langle \exp_{p_2}^{-1} p_3, \exp_{p_2}^{-1} p_1\rangle.$$

Furthermore, if α *is the angle at* p_1*, then we have*

$$\langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3\rangle = d(p_2,p_1)d(p_1,p_3)\cos\alpha.$$

The connection between geodesic triangle in Riemannian manifolds and triangles in $\mathbb{R}^2$ *has been established in* [51]*.*

Lemma 2.12 [51] *Let* $\Delta(p_1,p_2,p_3)$ *be a geodesic triangle in* M*. Then, there exits a comparison triangle* $\Delta(\bar{p_1},\bar{p_2},\bar{p_3})$ *for* $\Delta(p_1,p_2,p_3)$ *such that* $d(p_i,p_{i+1}) = \|\bar{p_i},\bar{p_{i+1}}\|$*, with the indices taken modulo* 3*, it is unique up to an isometry of* $\mathbb{R}^2$*.*

Lemma 2.13 [51] *Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle and $\Delta(\bar{p_1}, \bar{p_2}, \bar{p_3})$ be a comparison triangle.*

(i) *Let α_1, α_2, α_3 be the angle of $\Delta(p_1, p_2, p_3)$, respectively, and let $(\bar{\alpha_1}, \bar{\alpha_2}, \bar{\alpha_3})$ be the angles of $\Delta(\bar{p_1}, \bar{p_2}, \bar{p_3})$, respectively. Then,*

$$\alpha_1 \leq \bar{\alpha_1}, \qquad \alpha_2 \leq \bar{\alpha_2} \quad \text{and} \quad \alpha_3 \leq \bar{\alpha_3}.$$

(ii) *Let q be a point on geodesic joining p_1 to p_2 and $\bar{q}$ its comparison point in interval $[\bar{p_1}, \bar{p_2}]$. If $d(p_1, q) = \|\bar{p_1} - \bar{q}\|$ and $d(p_2, q) = \|\bar{p_2} - \bar{q}\|$ then,*

$$d(p_3, q) \leq \|\bar{p_3} - \bar{q}\|.$$

Proposition 2.14 [51] *Let $x \in M$. Then $\exp_x : T_xM \to M$ is diffeomorphism. For any two points $x, r \in M$, there exists a unique normalized geodesic joining x to r, which is, in fact, a minimal geodesic. This result shows that M has the topology and differential structure similar to $\mathbb{R}^n$. Thus, Hadamard manifolds and Euclidean spaces have some similar geometrical properties.*

Definition 2.15 [51] A subset $K \subset M$ is said to be convex if for any $p, q \in K$, the geodesic connecting p and q is in K.

Proposition 2.16 *Let M be Hadamard manifold and $x \in M$. The map $\psi_x(y) = d^2(x, y)$ satisfying the following*:

(i) *ψ_x is convex. Indeed, for any geodesic, $\gamma : [0, 1] \to M$. The following inequality holds for $t \in [0, 1]$*:

$$d^2\big(x, \gamma(t)\big) \leq (1-t)d^2\big(x, \gamma(0)\big) + td^2\big(x, \gamma(1)\big) - t(1-t)d^2\big(\gamma(0), \gamma(1)\big)$$

(ii) *ψ_x is smooth. Further, $\partial \psi_x(y) = -2\exp_y^{-1} x$*

Lemma 2.17 [45] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1-\alpha_k)a_n + \alpha_n b_n, \quad n \geq 0.$$

If $\lim_{k\to\infty} \sup b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_k\}$ satisfying the condition

$$\lim_{k\to\infty} \inf(a_{n_{n_k+1}} - a_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Let C be a non-empty, closed convex subset of a Hadamard manifold M and $S : C \to C$ be a nonexpansive mapping. Let $\Omega = F(S) \cap EP(f, C)$ be a non-empty solution set. Let $\phi : C \to C$ be a ρ-contraction with the bifunction f satisfying the following conditions:

(D_1) For each $z \in C$, f is pseudomonotone;

(D_2) f satisfies the Lipschitz-type condition on C;

(D_3) $f(x,\cdot)$ is convex and subdifferentiable on C, $\forall$ fixed $x \in C$;
(D_4) $f(\cdot,y)$ is upper semi-continuous $\forall \in C$.
Moreover, we assume that the sequence $\{\beta_n\}$, $\{\alpha_n\} \subset (0,1)$ satisfies
(C_1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(C_2) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C_3) $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

Algorithm 3.1 Initialization algorithm: choose $x_0 \in C$ and a parameter λ satisfying $0 \le \lambda < \min\{\frac{1}{2\theta_1}, \frac{1}{2\theta_2}\}$, where θ_1 and θ_2 are positive constants.
Iterative steps: Given x_{n-1}, x_n, choose $\mu \in [0, \bar{\mu_n}]$, where

$$\bar{\mu_n} = \begin{cases} \min\{\mu, \frac{\epsilon_n}{d(x_n,x_{n-1})}\}, & \text{if } x_n \neq x_{n-1} \\ \mu & \text{otherwise.} \end{cases}$$

Step 1: Compute

$$\begin{cases} w_n = \exp_{x_n}(\mu_n \exp^{-1} x_{n-1}) \\ u_n = \arg\min\{\lambda f(w_n,v) + \frac{1}{2}d^2(w_n,v), v \in C\}. \end{cases}$$

If $u_n = w_n$, then stop. Otherwise, go to the next step.
Step 2: Compute

$$y_n = \arg\min\left\{\lambda f(u_n,v) + \frac{1}{2}d^2(w_n,v) : v \in T_m\right\},$$

where the half-space T_m is defined by

$$T_m = \{v \in M : \langle \exp_{u_n}^{-1} w_n - \lambda z_n, \exp_{u_n}^{-1} v\rangle \le 0\}. \quad \text{and}$$

$$z_n \in \partial_2 f(w_n,u_n).$$

Step 3: Compute

$$t_n = \exp_{y_n}(1-\beta_n)\exp_{y_n}^{-1} S(y_n).$$

Step 4 Calculate

$$x_{n+1} = \exp_{\phi(x_n)}(1-\alpha_n)\exp_{\phi(x_n)}^{-1} t_n.$$

Set $n = n+1$ and return to step 1.

To prove our main results, we first prove the following Lemma.

Lemma 3.2 *For any $t \in C$ and $\lambda > 0$:*
(i) $\lambda[f(w_n,t) - f(w_n,u_n)] \ge \langle \exp_{u_n}^{-1} w_n, \exp^{-1} u_n t\rangle$,
(ii) $\lambda[f(u_n,t) - f(u_n,y_n)] \ge \langle \exp_{y_n}^{-1} w_n, \exp^{-1} y_n t\rangle$.

Proof From the definition of u_n in Algorithm 3.1 and Proposition 2.9, we have

$$0 \in \partial_2\left[\lambda f(w_n,t) + \frac{1}{2}d^2(w_n,t)\right]u_n + N_C(u_n).$$

Hence, there is $\bar{z} \in N_C(u_n)$ and $z \in \partial_2 f(w_n,u_n)$ such that

$$\lambda z - \exp_{u_n}^{-1} w_n + \bar{z} = 0.$$

Thus, for any $t \in C$,

$$\langle \exp_{u_n}^{-1} w_n, \exp_{u_n}^{-1} t\rangle = \lambda\langle z, \exp_{u_n}^{-1} t\rangle + \langle \bar{z}, \exp_{u_n}^{-1} t\rangle. \tag{3.1}$$

Now, as $\bar{z} \in N_k(u_n)$, $\langle \bar{z}, \exp_{u_n}^{-1} t\rangle \leq 0$ for any $t \in M$. Thus, we have

$$\langle \exp_{u_n}^{-1} w_n, \exp_{u_n}^{-1} t\rangle \leq \lambda\langle z, \exp_{u_n}^{-1} t\rangle. \tag{3.2}$$

Furthermore, from the fact that $z \in \partial_2 f(w_n,u_n)$ and the definition of subdifferential, we obtain

$$f(w_n,t) - f(w_n,u_n) \geq \langle z, \exp_{u_n}^{-1} t\rangle \quad \forall t \in M. \tag{3.3}$$

Multiplying the both sides of inequality (3.3) by $\lambda > 0$ and using (3.2), we get

$$\lambda\big[f(w_n,t) - f(w_n,u_n)\big] \geq \langle \exp_{u_n}^{-1} w_n, \exp_{u_n}^{-1} t\rangle \quad \forall t \in C. \tag{3.4}$$

Similarly, we can prove (*ii*) using the same idea as in part (*i*). □

Lemma 3.3 *Suppose that* $\{y_n\}$, $\{w_n\}$, $\{u_n\}$ *are generated by Algorithm* 3.1, $p \in \Omega$ *and* $\lambda > 0$,

$$d^2(y_n,p) \leq d^2(w_n,p) - (1-2\lambda\alpha_1)d^2(u_n,w_n) - (1-2\lambda\alpha_2)d^2(y_n,u_n).$$

Proof From the fact that $y_n \in T_m$ and by definition of T_m, it follows that

$$\langle \exp_{u_n}^{-1} w_n - \lambda z_n, \exp_{u_n}^{-1} y_n\rangle \leq 0,$$

for some $z_n \in \partial_2 f(w_n,u_n)$. Hence,

$$\lambda\langle z_n, \exp_{u_n}^{-1} y_n\rangle \geq \langle \exp_{u_n}^{-1} w_n, \exp_{u_n}^{-1} y_n\rangle. \tag{3.5}$$

Subsequently, by the definition of subdifferential, we obtain $z_n \in \partial_2 f(w_n,u_n)$.

$$f(w_n,y) - f(w_n,u_n) \geq \langle z_n, \exp_{u_n}^{-1} y\rangle \quad \forall y \in M. \tag{3.6}$$

Letting $y = y_n$ into (3.6), we deduce

$$f(w_n,y_n) - f(w_n,u_n) \geq \langle v_n, \exp_{u_n}^{-1} y_n\rangle. \tag{3.7}$$

It follows from the last inequality and from (3.5) that

$$\lambda\big[f(w_n,y_n)-f(w_n,u_n)\big] \geq \big\langle \exp_{u_n}^{-1} w_n, \exp_{u_n}^{-1} y_n\big\rangle \tag{3.8}$$

Moreover, by Proposition 2.9 and the definition of y_n in Algorithm 3.1, we have

$$0 \in \partial_2\left[\lambda f(u_n,y) + \frac{1}{2}d^2(w_n,y)\right](y_n) + N_{T_m}(y_n).$$

Thus, there exists $z \in \partial_2 f(u_n y_n)$ and $\bar{z} \in N_{T_m}(y_n)$ such that

$$\lambda z - \exp_{y_n}^{-1} w_n + \bar{z} = 0. \tag{3.9}$$

Note that $\bar{z} \in N_{T_m}(y_n)$ and the definition of the normal cone implies that

$$\big\langle \bar{z}, \exp_{y_n}^{-1} y\big\rangle \leq 0, \tag{3.10}$$

for any $y \in T_m$. Hence, thus from (3.10), it follows that

$$\big\langle \lambda z - \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} y\big\rangle \geq 0 \quad \forall y \in T_m. \tag{3.11}$$

Equivalently,

$$\lambda\big\langle z, \exp_{y_n}^{-1} y\big\rangle \geq \big\langle \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} y\big\rangle \quad \forall y \in T_m. \tag{3.12}$$

Now, using the definition of subdifferential for $z \in \partial_2 f(u_n,y_n)$, we have

$$f(u_n,y) - f(u_n,y_n) \geq \big\langle z, \exp_{y_n}^{-1} y\big\rangle \quad \forall y \in M \tag{3.13}$$

This together with (3.13) gets

$$\lambda\big[f(u_n,y)-f(u_n,y_n)\big] \geq \big\langle \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} y\big\rangle \quad \forall y \in T_m. \tag{3.14}$$

Furthermore, letting $y = p$ in relation (3.14), we obtain

$$\lambda\big[f(u_n,p)-f(u_n,y_n)\big] \geq \big\langle \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} p\big\rangle. \tag{3.15}$$

Now, as $p \in \Omega$, $f(p,u_n) \geq 0$. So, by pseudomonotonicity of f, $f(u_n,p) \leq 0$, and since $\lambda > 0$, we have

$$-\lambda\big(f(u_n,y_n)\big) \geq \big\langle \exp_{y_n}^{-1} w_n, \exp_{y_n} p\big\rangle. \tag{3.16}$$

Applying the Lipschitz-type continuity of f, we have

$$f(u_n,y_n) \geq f(w_n,y_n) - f(w_n,u_n) - \alpha_1 d^2(w_n,u_n) - \alpha_2 d^2(u_n,y_n). \tag{3.17}$$

Multiplying both sides of (3.17) by $\lambda > 0$ and merging (3.8) and (3.16), respectively, we obtain

$$\begin{aligned}
-\langle \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} p \rangle &\geq \lambda f(u_n, y_n) \\
&\geq \lambda\left[f(w_n, y_n) - f(w_n, u_n)\right] - \lambda\alpha_1 d(w_n, u_n) - \lambda\alpha_2 d^2(u_n, y_n) \\
&\geq \langle \exp_{u_n}^{-1} w_n, \exp_{u_n}^{-1} y_n \rangle - \lambda\alpha_1 d^2(w_n, u_n) - \lambda\alpha_2 d^2(u_n, y_n).
\end{aligned}$$

Thus, applying Proposition 2.8, we obtain

$$\begin{aligned}
&d^2(p, w_n) - d^2(w_n, y_n) - d^2(p, y_n) \\
&\quad\geq -2\langle \exp_{y_n}^{-1} w_n, \exp_{y_n}^{-1} p \rangle \\
&\quad\geq 2\langle \exp_{u_n} w_n, \exp_{u_n} y_n \rangle - 2\lambda\alpha_1 d^2(w_n, u_n) - 2\lambda\alpha_2 d^2(u_n, y_n) \\
&\quad\geq -d^2(w_n, y_n) + d^2(w_n, u_n) + d^2(y_n, u_n) - 2\lambda\alpha_1 d^2(w_n, u_n) - 2\lambda\alpha_2 d^2(u_n, y_n).
\end{aligned}$$

Thus, we have

$$d^2(y_n, p) \leq d^2(w_n, p) - (1 - 2\lambda\alpha_1) d^2(u_n, w_n) - (1 - 2\lambda\alpha_2) d^2(y_n, u_n), \tag{3.18}$$

which implies that

$$d(y_n, p) \leq d(w_n, p). \tag{3.19}$$

□

Lemma 3.4 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{t_n\}$ are bounded.*

Proof Let $p \in \Omega$, then consider the geodesic triangle $\Delta(w_n, x_n, p)$ and $\Delta(x_n, x_{n-1}, p)$ with their comparison triangle $\Delta(w_n', x_n', p')$ and $\Delta(x_n', x_{n-1}', p')$, respectively, in $\mathbb{R}^2$ and where $p \in \Omega$. We have $d(w_n, p) = \|w_n' - p'\|$, $d(x_n, p) = \|x_n' - p'\|$ and $d(x_n, x_{n-1}) = \|x_n' - x_{n-1}'\|$. Since $w_n = \exp_{x_n} \mu_n \exp_{x_n}^{-1} x_{n-1}$, then the comparison point of w_n is $w_n' = x_n' + \mu_n(x_{n-1}' - x_n')$. Thus, we have

$$\begin{aligned}
d(w_n, p) &= \|w_n' - p'\| \\
&= \|x_n' + \mu_n(x_{n-1}' - x_n') - p'\| \\
&= \|x_n' + \mu_n x_{n-1}' - \mu_n x_n' - p'\| \\
&\leq \|x_n' - p'\| + \mu_n \|x_n' - x_{n-1}'\| \\
&= \|x_n' - p'\| + \alpha_n \frac{\mu_n}{\alpha_n} \|x_n' - x_{n-1}'\|.
\end{aligned} \tag{3.20}$$

Since $\frac{\mu_n}{\alpha_n}\|x_n' - x_{n-1}'\| = \frac{\mu_n}{\alpha_n} d(x_n, x_{n-1}) \to 0$ as $n \to \infty$, there exists a constant $M_1 > 0$ such that $\frac{\mu_n}{\alpha_n} d(x_n, x_{n-1}) \leq M_1 \; \forall n \geq 1$. Thus, we obtain

$$d(w_n, p) \leq d(x_n, p) + \alpha_n M_1. \tag{3.21}$$

It is obvious from simple computation that

$$d(w_n,p) \le d^2(x_n,p) + 2\mu_n d(x_n,p)d(x_n,x_{n-1}) + \mu^2 d^2(x_n,x_{n-1})). \tag{3.22}$$

Besides, by the convexity of Remannian manifold $t_n = \gamma_n^1(1-\beta_n)$, where $\gamma_n^1 : [0,1] \to M$ is a sequence of geodesic joining y_n to $S(y_n)$, we obtain

$$\begin{aligned} d(t_n,p) &= d\big(\gamma_n^1(1-\beta_n),p\big) \\ &\le \beta_n d\big(\gamma_n^1(0),p\big) + (1-\beta_n)d\big(\gamma_n^1(1),p\big) \\ &= \beta_n d(y_n,p) + (1-\beta_n)d\big(S(y_n),p\big) \\ &\le \beta_n d(y_n,p) + (1-\beta_n)d(y_n,p) \\ &= d(y_n,p). \end{aligned} \tag{3.23}$$

Furthermore, by the convexity of Remannian manifold $x_{n+1} = \gamma_n^2(1-\beta_n)$, where $\gamma_n^2 : [0,1] \to M$ is a sequence of geodesic joining $\phi(x_n)$ to t_n, and from 3.1, (3.19), and (3.21), we obtain

$$\begin{aligned} d(x_{n+1},p) &\le d\big(\gamma_n^2(1-\alpha_n),p\big) \\ &\le \alpha_n d\big(\gamma_n^2(0),p\big) + (1-\alpha_n)d\big(\gamma_n^2(1),p\big)) \\ &= \alpha_n d\big(\phi(x_n),p\big) + (1-\alpha_n)d(t_n,p) \\ &\le \alpha_n d\big(\phi(x_n),p\big) + (1-\alpha_n)d(y_n,p) \\ &\le \alpha_n d\big(\phi(x_n),p\big) + (1-\alpha_n)d(w_n,p) \\ &\le \alpha_n\big[d\big(\phi(x_n),\phi(p)\big) + d\big(\phi(p),p\big)\big] + (1-\alpha_n)d(w_n,p) \\ &\le \alpha_n \rho d(x_n,p) + (1-\alpha_n)\big[d(x_n,p) + \alpha_n M_1\big] + \alpha_n d\big(\phi(p),p\big) \\ &\le \big(1-\alpha_n(1-\rho)\big)d(x_n,p) + (1-\alpha_n)\alpha_n\alpha_n(1-\rho)\frac{d(\phi(p),p)}{1-\rho} + M_1 \\ &\le \max\left\{d(x_n,p), \frac{M_1 + d(\phi(p),p)}{(1-\rho)}\right\}. \end{aligned} \tag{3.24}$$

Using mathematical induction, we get

$$d(x_{n+1},p) \le \max\left\{d(x_0,p), \frac{M_1 + d(\phi(p),p)}{(1-\rho)} + M_1\right\}.$$

Thus, the sequence $\{x_n\}$ is bounded. Consequently, the sequence $\{w_n\}$, $\{u_n\}$, $\{t_n\}$ and $\{\phi(x_n)\}$ are also bounded. □

Lemma 3.5 *Let $\{x_{n_k}\}$ be the sequence generated by the Algorithm 3.1. Then, the following conclusion holds*:

(i) $\lim_{k\to\infty} d(u_{n_k}, w_{n_k}) = 0$,
(ii) $\lim_{k\to\infty} d(y_{n_k}, u_{n_k}) = 0$,
(iii) $\lim_{k\to\infty} d(y_{n_k}, S(y_{n_k})) = 0$,

(iv) $\lim_{k\to\infty} d(w_{n_k}, x_{n_k}) = 0$,

(v) $\lim_{k\to\infty} d(t_{n_k}, y_{n_k}) = 0$.

Proof Let $p \in \Omega$ and satisfy $p \in P_\Omega f(p)$. Observe that this fixed point equation has a unique solution by Boyd-Wong fixed point theorem. Now, fix $n \geq 1$, and let $q = \phi(x_n)$, $r = t_n$ and $s = \phi(p)$. We consider the following geodesic triangle with their respective comparison triangle in $\mathbb{R}^2$ $\Delta(q,r,s)$ and $\Delta(q',r',s')$, $\Delta(s,r,q)$ and $\Delta(s',r',q')$; $\Delta(s,r,p)$ and $\Delta(s,r,p)$ and $\Delta(s',r',p')$. By Lemma 2.12, we get

$$d(q,r) = \|q' - r'\|, \qquad d(q,s) = \|q' - s'\| \quad \text{and} \quad d(q,p) = \|q' - p'\|. \tag{3.25}$$

Now, using the definition in Algorithm 3.1, we get

$$x_{n+1} = \exp_q^{-1}(1-\alpha_n)\exp_q^{-1} r.$$

The comparison point of x_{n+1} in $\mathbb{R}^2$ is $x'_{n+1} = \alpha_n q' + (1-\alpha_n)r'$. Let a and a' denote the angles at p and p' in the triangle $\Delta(s, x_{n+1}, p)$ and $\Delta(s', x'_{n+1}, p')$, respectively. Then, we obtain $a' \leq a$ and $\cos a' \leq \cos a$. By applying Lemma 2.2, we have

$$\begin{aligned}
d(x_{n+1},p) &= \|x'_{n+1} - p'\|^2 \\
&= \|\alpha_n(q'-s') + (1-\alpha_n)(r'-s')\|^2 \\
&\leq \|\alpha_n(q'-s') + (1-\alpha_n)(r'-p')\|^2 + 2\alpha_n\langle x'_{n+1}-p', s'-p'\rangle \\
&\leq (1-\alpha_n)\|r'-p'\|^2 + \alpha_n\|q'-s'\|^2 + 2\alpha_n\|x'_{n+1}-p'\|\|s'-p'\|\cos a' \\
&\leq (1-\alpha_n)d^2(r,p) + \alpha_n d^2(q,s) + 2\alpha d(x_{n+1},p)d(s,p) \\
&= (1-\alpha_n)d^2(t_n,p) + \alpha_n d^2(\phi(x_n),\phi(p)) \\
&\quad + 2\alpha_n d(x_{n+1},p)d(\phi(p),p)\cos a.
\end{aligned} \tag{3.26}$$

Now, since $d(x_{n+1},p)d(\phi(p),p)\cos a = \langle \exp_p^{-1}\phi(p), \exp_p^{-1}x_{n+1}\rangle$.

Then, we can rewrite (3.26) as

$$d^2(x_{n+1},p) \leq (1-\alpha_n)d^2(t_n,p) + \alpha_n\rho d^2(x_n,p) + 2\alpha_n\langle \exp_p^{-1}\phi(p), \exp_p^{-1}x_{n+1}\rangle. \tag{3.27}$$

However, from Lemma 2.16, we obtain

$$\begin{aligned}
d^2(t_n,p) &= d^2(\gamma'_n(1-\beta_n),p) \\
&\leq \beta_n d^2(\gamma'_n(0),p') + (1-\beta_n)d^2(\gamma'_n(1),p) - \beta_n(1-\beta_n)d^2(\gamma'_n(0),\gamma'_n(1)) \\
&\leq \beta_n d^2(y_n,p) + (1-\beta_n)d^2(y_n,p) - \beta_n(1-\beta_n)d^2(y_n,S(y_n)) \\
&\leq \beta_n d^2(y_n,p) + (1-\beta_n)d^2(y_n,p) - \beta_n(1-\beta_n)d^2(y_n,S(y_n)) \\
&= d^2(y_n,p) - \beta_n(1-\beta_n)d^2(y_n,S(y_n)).
\end{aligned} \tag{3.28}$$

By substituting Lemma 3.3 and (3.22) into (3.28), we get

$$d^2(t_n,p) \leq d^2(w_n,p) - (1-2\lambda\alpha_1)d^2(u_n,w_n) - (1-2\lambda\alpha_2)d^2(y_n,u_n)$$

$$
\begin{aligned}
&- \beta_n(1-\beta_n)d^2\big(y_n, S(y_n)\big) \\
&\le d^2(x_n,p) + 2\mu_n d(x_n,p)d(x_n,x_{n-1}) + \mu_n^2 d^2(x_n,x_{n-1}) - (1-2\lambda\alpha_1)d^2(u_n,w_n) \\
&- (1-2\lambda\alpha_2)d^2(y_n,u_n) - \beta_n(1-\beta_n)d^2\big(y_n,S(y_n)\big).
\end{aligned}
$$

By substituting (3.29) into (3.28), we have

$$
\begin{aligned}
d^2(x_{n+1},p) &\le (1-\alpha_n)d^2(x_n,p) + (1-\alpha_n)\big[2\mu_n d(x_n,p)d(x_n,x_{n-1}) \\
&\quad + \mu_n^2 d^2(x_n,x_{n-1}) - (1-2\lambda\alpha_1)d^2(u_n,w_n) \\
&\quad - (1-2\lambda\alpha_2)d^2(y_n,u_n) - \beta_n(1-\beta_n)d^2(y_n,S(y_n))\big] \\
&\quad + \alpha_n\rho d(x_n,p) + 2\alpha_n\langle \exp_p^{-1}\phi(p), \exp_p^{-1}x_{n+1}\rangle \\
&= \big[1-\alpha_n(1-\rho)\big]d^2(x_n,p) + \alpha_n(1-\rho)H_n - (1-\alpha_n)\big[(1-2\lambda\alpha_1)d^2(u_n,w_n) \\
&\quad + (1-2\lambda\alpha_2)d^2(y_n,u_n) - \beta_n(1-\beta_n)d^2\big(y_n,S(y_n)\big)\big],
\end{aligned}
\tag{3.29}
$$

where

$$
H_n = \frac{1}{1-\rho}\left[2\langle \exp_p^{-1}\phi(p), \exp_p^{-1}x_{n+1}\rangle + \frac{2\mu_n}{\alpha_n}d(x_n,p)d(x_n,x_{n-1}) + \frac{\mu_n^2}{\alpha_n}d(x_n,x_{n-1})\right]. \tag{3.30}
$$

equation (3.29) can be rewritten as

$$
\big[1-\alpha_n(1-\rho)\big]d^2(x_n,p) + \alpha_n(1-\rho)M_2, \tag{3.31}
$$

where

$$
M_2 = \sup_{n\in\mathbb{N}} H_n. \tag{3.32}
$$

Thus, it is not difficult to see from (3.32) that if we let $h_n = \alpha_n(1-\rho)$, then the sequence $\{a_n\}$ satisfies

$$
a_{n+1} \le (1-h_n)\alpha_n + h_n H_n, \tag{3.33}
$$

where

$$
a_{n+1} = d^2(x_n,p).
$$

Next, we claim that $\limsup_{k\to\infty} H_{n_k} \le 0$. Suppose that there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$, which satisfies $\liminf_{k\to\infty}(a_{n_k} - a_{n_k}) \ge 0$. Now, from (3.29), we have

$$
\begin{aligned}
&\limsup_{k\to\infty}(1-\alpha_{n_k})\big[(1-2\lambda\alpha_1)d^2(u_{n_k},w_{n_k}) + (1-2\lambda\alpha_2)d^2(y_{n_k},u_{n_k}) \\
&\quad - \beta_{n_k}(1-\beta_{n_k})d^2\big(y_{n_k},S(y_{n_k})\big)\big] \\
&\le \limsup_{k\to\infty}\big[(1-\alpha_{n_k})a_{n_k} - a_{n_k+1}\big] \\
&\quad + (1-\rho)M_2 \limsup_{k\to\infty}\alpha_{n_k}
\end{aligned}
$$

$$
\begin{aligned}
&= - \lim_{k\to\infty} (a_{n_{k+1}-a_{n_k}}) \\
&\leq 0. \qquad (3.34)
\end{aligned}
$$

Thus, using (C_1) and the fact that $(1 - 2\lambda\alpha_i) > 0$, for $i = 1, 2$, we obtain

$$
\lim_{k\to\infty} d(u_{n_k}, w_{n_k}) = 0, \quad \text{and} \quad \lim_{k\to\infty} d(y_{n_k}, u_{n_k}) = 0, \qquad \lim_{k\to\infty} d\big(y_{n_k}, S(y_{n_k})\big) = 0. \qquad (3.35)
$$

Also, from (3.35), we have that $\lim_{k\to\infty} d(y_{n_k}, w_{n_k}) = 0$. Furthermore, from (3.20) and (C_2), we have that

$$
\begin{aligned}
\lim_{k\to\infty} d(w_{n_k}, x_{n_k}) &\leq \lim_{k\to\infty} \alpha_{n_k} \frac{\mu_{n_k}}{\alpha_{n_k}} d(x_{n_k}, x_{n_{k-1}}) \\
&= 0. \qquad (3.36)
\end{aligned}
$$

Using step 2 of Algorithm 3.1 and (3.34), we get

$$
\begin{aligned}
\lim_{k\to\infty} d(t_{n_k}, y_{n_k}) &\leq \lim_{k\to\infty} (1 - \beta_{n_k}) d\big(y_{n_k}, S(y_{n_k})\big) \\
&= 0.
\end{aligned}
$$

From step 4 of Algorithm 3.1 and (C_2), we get

$$
\begin{aligned}
d(x_{n_{k+1}}, t_{n_k}) &\leq \alpha_n d\big(\phi(x_n), t_{n_k}\big) + (1 - \alpha_{n_k}) d(t_{n_k}, t_{n_k}) \\
&= 0. \qquad (3.37)
\end{aligned}
$$

Finally, from (3.33), (3.36), and (3.37), we get

$$
\lim_{k\to\infty} d(y_{n_k}, x_{n_k}) = 0, \qquad \lim_{k\to\infty} d(t_{n_k}, x_{n_k}) = 0, \qquad \lim_{k\to\infty} d(u_{n_k}, x_{n_k}) = 0, \qquad (3.38)
$$

□

Theorem 3.6 *Suppose that $(D_1) - (D_4)$ holds and $EP(f, E) \neq \emptyset$, then the sequence $\{x_n\}$ generated by algorithm (3.1) converges to a solution of Ω. Let $\{x_{n_{k_i}}\}$ be a subsequence of $\{x_{n_k}\}$ that converges to $x^* \in C$.*

Proof Let p be in a limit point of the sequence $\{x_{n_k}\}$. There exists subsequence $\{x_{n_{k_i}}\}$ that converges to x^*. So, let $t \in M$ be an arbitrary element. From Lemma 3.2 (*ii*), we obtain

$$
\lambda f(u_{n_{k_i}}, t) \geq \lambda f(u_{n_{k_i}}, y_{n_{k_i}}) + \big\langle \exp^{-1}_{y_{n_{k_i}}} w_{n_{k_i}}, \exp^{-1}_{y_{n_{k_i}}} t \big\rangle. \qquad (3.39)
$$

Using Lipschitz-type continuity of f, we have

$$
f(u_{n_{k_i}}, y_{n_{k_i}}) \geq f(w_{n_{k_i}}, y_{n_{k_i}}) - f(w_{n_{k_i}}, u_{n_{k_i}}) - \alpha_1 d^2(w_{n_{k_i}}, u_{n_{k_i}}) - \alpha_2 d^2(u_{n_{k_i}}, y_{n_{k_i}}). \qquad (3.40)
$$

When we let $t = y_{n_{k_i}}$ in Lemma 3.2 (*i*), we obtain the following

$$
\lambda \big[f(w_{n_{k_i}}, y_{n_{k_i}}) - f(w_{n_{k_i}}, u_{n_{k_i}}) \big] \geq \big\langle \exp^{-1}_{u_{n_{k_i}}} w_{n_{k_i}}, \exp^{-1}_{u_{n_{k_i}}} y_{n_{k_i}} \big\rangle. \qquad (3.41)
$$

Thus, we obtain the following from (3.41) and (3.40)

$$\lambda f(u_{n_{k_i}}, y_{n_{k_i}}) \geq \langle \exp^{-1}_{u_{n_{k_i}}} w_{n_{k_i}}, \exp^{-1}_{u_{n_{k_i}}} y_{n_{k_i}} \rangle - \lambda\alpha_1 d^2(w_{n_{k_i}}, u_{n_{k_i}}) - \lambda\alpha_2 d^2(u_{n_{k_i}}, y_{n_k}) \tag{3.42}$$

Merging (3.42) and (3.39), we obtain

$$\begin{aligned} \lambda f(u_{n_{k_i}}, t) \geq {} & \langle \exp^{-1}_{u_{n_{k_i}}} w_{n_{k_i}}, \exp^{-1}_{u_{n_{k_i}}} y_{n_{k_i}} \rangle + \langle \exp^{-1}_{y_{n_{k_i}}} w_{n_{k_i}}, \exp^{-1}_{y_{n_k}} t \rangle \\ & - \lambda\alpha_1 d^2(w_{n_k}, u_{n_{k_i}}) - \lambda\alpha_2 d^2(u_{n_{k_i}}, y_{n_{k_i}}). \end{aligned} \tag{3.43}$$

Thus, from the boundedness of $\{x_{n_k}\}$, we obtain

$$f(x^*, t) \geq 0, \quad \forall t \in C. \tag{3.44}$$

Thus, $x^* \in EP(f, C)$. Furthermore, using (3.35), we obtain that $x^* \in F(S)$. Hence, we conclude that $x^* \in \Omega$ □

Next, we show that $\{x_n\}$ converges to $x^* \in \Omega$. To estimate that, we claim that $\lim \lim_{k\to\infty} \sup H_{n_k} \leq 0$. To prove this, we only need to show that

$$\lim_{k\to\infty} \langle \exp^{-1}_p \phi(p), \exp^{-1}_p x_{n_{k+1}} \rangle \leq 0. \tag{3.45}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ that converges to x^* such that

$$\begin{aligned} \lim_{k\to\infty} \langle \exp^{-1}_p \phi(p), \exp^{-1}_p x_{n_{k_i}} \rangle &= \lim_{k\to\infty} \langle \exp^{-1}_p \phi(p), \exp^{-1}_p x_{n_k} \rangle \\ &= \langle \exp^{-1}_p \phi(p), \exp_p x^* \rangle \\ &\leq 0. \end{aligned} \tag{3.46}$$

Hence, by substituting (3.46) into (3.33) and applying Lemma 2.17, we conclude that $\{x_n\}$ converges to $x^* \in \Omega$. Thus, we complete the proof.

If we change the mapping S in Algorithm 3.1 to a contraction mapping, we obtain the following.

Corollary 3.7 *Let C be a non-empty, closed convex subset of a Hadamard manifold M and $S : C \to C$ be a contraction mapping. Let $\Omega = F(S) \cap EP(f, C)$ be the solution set. Let $\phi : C \to C$ be a ρ-contraction with the bifunction f satisfying the following conditions $(D_1 \to D_4)$.*

Algorithm 3.8 Initialization algorithm: choose $x_0 \in C$ and a parameter λ which satisfies $0 \leq \lambda < \min\{\frac{1}{2\theta_1}, \frac{1}{2\theta_2}\}$ iterative steps: Given x_{n-1}, x_n, choose $\mu \in [0, \bar{\mu_n}]$, where

$$\bar{\mu_n} = \begin{cases} \min\{\mu, \frac{\epsilon_n}{d(x_n, x_{n-1})}\}, & \text{if } x_n \neq x_{n-1} \\ \mu & \text{otherwise.} \end{cases}$$

Step 1: Compute

$$\begin{cases} w_n = \exp_{x_n}(\mu_n \exp^{-1} x_{n-1}) \\ u_n = \arg\min\{\lambda f(w_n, v) + \frac{1}{2}d^2(w_n, v), v \in C\}. \end{cases}$$

If $u_n = w_n$, then stop. Otherwise, go to the next step.

Step 2: Compute

$$y_n = \arg\min\left\{\lambda f(u_n, v) + \frac{1}{2}d^2(w_n, v) : v \in T_m\right\}.$$

where the half-space T_m is defined by

$$T_m = \left\{v \in M : \left\langle \exp_{u_n}^{-1} w_n - \lambda z_n, \exp_{u_n}^{-1} v\right\rangle \leq 0\right\} \quad \text{and}$$

$$z_n \in \partial_2 f(w_n, u_n)$$

Step 3: Compute

$$t_n = \exp_{y_n}(1 - \beta_n) \exp_{y_n}^{-1} S(y_n)$$

Step 4 Calculate

$$x_{n+1} = \exp_{\phi(x_n)}(1 - \alpha_n) \exp_{\phi(x_n)}^{-1} t_n$$

Set $n = n + 1$ and return to step 1.

Then our sequence converges strongly to an element in the $p \in \Omega$.

4 Applications

4.1 An application to solving Variational inequality problems

.

Suppose

$$f(x, y) = \begin{cases} \langle Gx, \exp_x^{-1} y\rangle, & \text{if } x, y \in C, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $G : C \to M$ is a mapping. Subsequently, the equilibrium problem aligns with the subsequent variational inequality (VIP) (see [52]):

$$\text{Find } x \in C \text{ such that } \left\langle Gx, \exp_x^{-1} y\right\rangle \geq 0, \forall y \in C. \tag{4.2}$$

Now, the set of solutions of (4.2) is denoted by $VIP(G,C)$. The mapping $G : C \to M$ is said to be pseudomonotone if

$$\langle Gx, \exp_x^{-1} y \rangle \geq 0 \quad \Longrightarrow \quad \langle Gy, \exp^{-1} y \rangle \geq 0, \quad x, y \in C. \tag{4.3}$$

Let us suppose that the function G is pseudomonotone and fulfills the following conditions:

(V1) The function G is pseudomonotone on C with $VIP(G,C) \neq \emptyset$

(V2) G is L-Lipschitz continuous, which is,

$$\|P_{y,x}Gx - G_y\| \leq \|x - y\|, \quad x, y \in C, \tag{4.4}$$

where $P_{y,x}$ is a parallel transport (see [50, 53]).

(V3) $\lim_{n\to\infty} \langle Gx_n, \exp_{x_n}^{-1} y \rangle \leq \langle G_p, \exp_p^{-1} y \rangle$ for every $y \in C$ and $\{x_n\} \subset C$ such that $x_n \to p$.

By substituting the proximal term $\arg\min\{f(x,y) + \frac{1}{2\lambda} d(x,y) y \in M\}$ with $P_C(\exp_x(-\lambda_n G(x)))$, where P_C is metric projection of M onto C in Algorithm 3.1, we have the following method for approximating a point in $VIP(G,C)$.

In this context, we can establish the subsequent convergence theorem for the approximation of a solution to the VIP (4.2)

Theorem 4.1 *Let $g : C \to C$ be a contraction and $G : C \to M$ be a pseudomonotone operator satisfying condition V1-V3. If $0 < k = \sup\{\frac{\psi d(x_n,q)}{d(x_n,q)} : x_n \neq q, n \geq 0, q \in VIP(G,C)\} < 1$, then the sequence $\{x_n\}$ generated by Algorithm 2 converges to an element $p \in VIP(G,K)$ which satisfies $p = P_{VIP(G,C))} g(p)$*

Algorithm 4.2 Initialization algorithm: choose $x_0 \in C$ and a parameter λ which satisfies $0 \leq \lambda < \min\{\frac{1}{2\theta_1}, \frac{1}{2\theta_2}\}$ iterative steps: Given x_{n-1}, x_n, choose $\mu \in [0, \bar{\mu_n}]$, where

$$\bar{\mu_n} = \begin{cases} \min\{\mu, \frac{\epsilon_n}{d(x_n,x_{n-1})}\}, & \text{if } x_n \neq x_{n-1} \\ \mu & \text{otherwise.} \end{cases}$$

Step 1: Compute

$$\begin{cases} w_n = \exp_{x_n}(\mu_n \exp^{-1} x_{n-1}) \\ u_n = P_C(\exp_{w_n} - \lambda_n) G(w_n). \end{cases}$$

If $u_n = w_n$, then stop. Otherwise, go to the next step.

Step 2: Compute

$$y_n = P_{T_m}(\exp_{w_n}(-\lambda_n G(w_n))).$$

where the half-space T_m is defined by

$$T_m = \{v \in M : \langle \exp_{u_n}^{-1} w_n - \lambda z_n, \exp_{u_n}^{-1} v \rangle \leq 0\}. \quad \text{and}$$

$$z_n \in \partial_2 g(w_n, u_n).$$

Step 3: Compute

$$t_n = \exp_{y_n}(1-\beta_n)\exp^{-1}_{y_n} S(y_n).$$

Step 4: Calculate

$$x_{n+1} = \exp_{\phi(x_n)}(1-\alpha_n)\exp^{-1}_{\phi(x_n)} t_n.$$

Set $n = n+1$ and return to step 1.

---

5 Numerical example

In this section, we provide a numerical example to show the performance of our iterative scheme and compare it with results existing in the literature. We use MATLAB programming for our numerical experiment.

Example 5.1 Let $M = \mathbb{R}^2$ be a Hadamard manifold, and let $G : C \to M$ be defined by

$$G(x) = \begin{bmatrix} (x_1^2 + (x_2-1)^2)(1+x_2) \\ -x_1^3 - x_1(x_2-1)^2 \end{bmatrix}$$

where $C = \{x \in \mathbb{R}^2 : -10 \leq x_i < 10, i = 1,2\}$ and $x = (x_1, x_2)$. By employing the Monte-Carlo approach, it can be shown that G is pseudomonotone on C (see [54]). Define $f(x,y) : \langle G(x), \exp^{-1}_x y\rangle$ for all $y \in C$. It is easy to see that f satisfies conditions (D1)-(D4). To implement our algorithm, we choose $\alpha_n = \frac{1}{n+1}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{2n}{5n+8}$, $\mu = 0.8$ and $\lambda = 0.01$. We use $\|x_{n+1} - x_n\| < 10^{-6}$ as stopping criterion. We use the following as starting point in our implementation:

Case I: $x_0 = [-1,4]$ and $x_1 = [2,5]$
Case II: $x_0 = [8,9]$ and $x_1 = [-3,-6]$
Case III: $x_0 = [1/4, 1/8]$ and $x_1 = [0,4]$
Case IV: $x_0 = [6,1]$ and $x_1 = [-1,-5]$.

We test the algorithm with the non-inertial version of the proposed Algorithm 3.1 by setting $\mu_n = 0$ in the algorithm. The numerical results are shown in Table 1 and Fig. 1.

Example 5.2 Consider the Nash equilibrium model initiated in [55]. Let $f : C \times C \to \mathbb{R}$ be a bifunction given by

$$f(x,y) = \langle Kx + Ry + t, y - x\rangle.$$

Table 1 Computation result for Example 5.1

	Algorithm 3.1		Non-inertial algorithm	
	Iter	CPU time	Iter	CPU time
Case I	9	1.76E–4	23	3.10E–4
Case II	9	1.78E–4	24	4.27E–4
Case III	9	1.96E–4	24	2.37E–4
Case IV	10	1.48E–4	25	4.07E–4

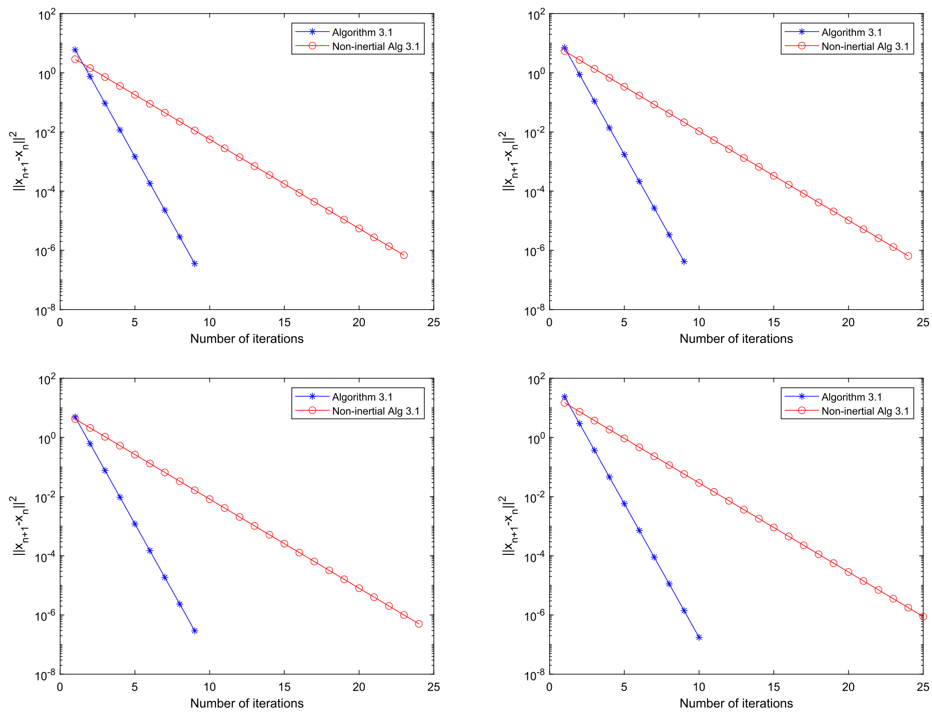

Figure 1 Example 5.1, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV

Table 2 Computation result for Example 5.1

	Algorithm 3.1		Non-inertial algorithm	
	Iter	CPU time	Iter	CPU time
Case I	17	6.54E-3	31	9.46E-3
Case II	14	6.19E-4	26	1.6E-3
Case III	15	6.05E-4	27	5.46E-3
Case IV	16	9.0E-4	30	5.67E-3

Where $C = \{x = (x_1, x_2, \dots x_m) : 1 \leq x_i \leq 100, i = 1, 2, \dots, m\}$. Let $x, y \in C$, and let $t = (t_1, t_2 \dots t_m) \in \mathbb{R}$ be chosen randomly. Besides, K and R are matrices of order $m \times m$ such that R is symmetric positive semidefinete and $R - K$ is negative semidefinite. It was shown in [55] that f is pseudomonotone and satisfies conditions (D1)-(D4) with Lipschitz constants $c_1 = c_2 = \frac{1}{2}\|R - K\|$. We choose the following parameters: $\alpha_n = \frac{1}{2n+14}$, $\beta_n = \frac{4n}{9n+4}$, $\epsilon_n = \frac{1}{n^{1.7}}$, $\mu_1 = 0.5$ and $\lambda = 10^{-2}$. Furthermore, we choose our stopping criterion to $\|x_{n+1} - x_n\|^2 = 10^{-6}$. The starting points x_0 and x_1 are generated randomly in $\mathbb{R}^m$ and consider the following values of m

$$\text{Case I} : m = 100, \qquad \text{Case II} : m = 500, \qquad \text{Case III} : m = 1000, \quad \text{and}$$

$$\text{Case IV} : m = 2000.$$

The numerical results are shown in Table 2 and Fig. 2.

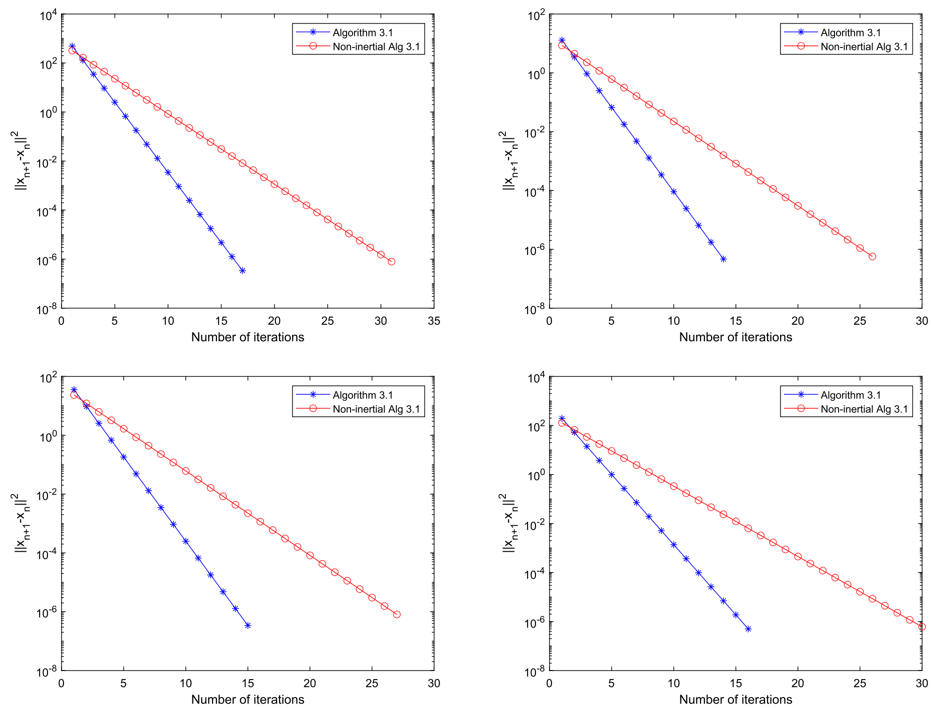

Figure 2 Example 5.2, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV

6 Conclusion

In this paper, we proposed a viscosity extragradient with a modified inertial method for solving the equilibrium problem and fixed point problem within the Hadamard manifold. A strong convergence result was obtained using viscosity technique and inertial method with conditions on the parameters required for generating the sequence of approximation. Moreover, we provide a numerical example to demonstrate the convergence behavior of the proposed algorithm.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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