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Self-adaptive Technique with Double Inertial Steps for Inclusion Problem on Hadamard Manifolds

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Abstract

In this article, we investigate monotone and Lipschitz continuous variational inclusion problem in the settings of Hadamard manifolds. We propose a forward–backward method with a self-adaptive technique for solving variational inclusion problem. To increase the rate of convergence of our proposed method, we incorporate our iterative method with double inertial steps and establish a convergence result of our iterative method under some mild conditions. Finally, in order to illustrate the computational effectiveness of our method, some numerical examples are also discussed. The result present in this article is new in this space and extends many related results in the literature.

Keywords Variational inclusion problem · Double inertial method · Hadamard manifold · Monotone operator · Riemannian manifold

Mathematics Subject Classification 47H09 · 49J25 · 65K10 · 90C25

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1 Introduction

Let $\mathcal{K}$ be a nonempty, closed geodesic convex subset of a Hadamard manifold $\mathbb{P}$, $T_x\mathbb{P}$ be the tangent space of $\mathbb{P}$ at $x \in \mathbb{P}$ and $T\mathbb{P}$ be the tangent bundle of $\mathbb{P}$. The variational inclusion problem (VIP) is to find $\overline{x} \in \mathbb{P}$ such that

$$\mathbf{0} \in \Phi\overline{x} + \Psi\overline{x}, \tag{1}$$

where $\Phi : \mathcal{K} \to T\mathbb{P}$ is a single-valued vector field, $\Psi : \mathcal{K} \to 2^{T\mathbb{P}}$ is a multi-valued vector field and $\mathbf{0}$ denotes the zero section of $T\mathbb{P}$. We denote the solution set of (1) by Ω. The variational inclusion problem has received much attention due to its various applications in signal processing, image recovery and statistical regression, (see [3, 13, 44, 47, 49]). It is known that several optimization problem such as convex optimization problem can be translated into finding a zero of a maximal monotone operator defined on a Hilbert space $\mathbb{M}$. The problem of finding a zero of the sum of two (maximal) monotone operators is of fundamental importance in convex optimization and variational analysis (see [1, 19, 26, 33, 43, 52]). For solving VIP (1), the forward–backward splitting method (FBM) (see [13, 28, 29, 48]) is usually employed and is defined in the following manner: $q_1 \in \mathbb{M}$ and

$$q_{k+1} = (I + r\Psi)^{-1}(q_k - r\Phi q_k),\ k \geqslant 1, \tag{2}$$

where $r > 0$, $\Psi : \mathbb{M} \to 2^{\mathbb{M}}$ is a set-valued operator and $\Phi : \mathbb{M} \to \mathbb{M}$ is an operator. In this case, each step of iterates involves only with Φ as the forward step and Ψ as the backward step, but not the sum of operators. The FBM defined in (2) above requires one of the operators to be inverse strongly monotone. This assumption imposed on one of the operators is very difficult to meet the practical problems. In order to dispense with the condition, many authors have introduced several iterative methods. For instance, Tseng [48] introduced the following forward–backward–forward method which is a two-step iterative scheme as follows:

$$\begin{cases} w_k = (I + r_k\Psi)^{-1}(I - r_k\Phi)q_k, \\ q_{k+1} = w_k - r_k(\Phi w_k - \Phi q_k), \end{cases} \tag{3}$$

where the step size $\{r_k\}$ can be updated by Armijo linesearch methods. When the mapping Φ is Lipschitz continuous and the mapping Ψ is maximal monotone, (3) converges weakly to a solution of VIP in the settings of real Hilbert spaces.

In 2019, Shehu [41] extended Tseng's splitting method to the settings of real Banach spaces. He proposed the following iterative method for approximating solution of VIP in a 2-uniformly convex Banach space $\mathbb{E}$ which is also uniformly smooth as follows:

$$\begin{cases} q_1 \in \mathbb{E}, \\ w_k = J_{r_k}^{\Psi} J^{-1}(Jq_k - r_k \Phi q_k), \\ q_{k+1} = Jw_k - r_k(\Phi w_k - \Phi q_k), \ \forall\, k \geqslant 1, \end{cases} \tag{4}$$

where $\Phi : \mathbb{E} \to \mathbb{E}$ is monotone and L-Lipschitz continuous, $J_{r_k}^{\Psi} = (J + r_k\Psi)^{-1}J$ is the resolvent of Ψ and J is the duality mapping from $\mathbb{E}$ to $\mathbb{E}^*$ ($\mathbb{E}^*$ is the dual of $\mathbb{E}$). He obtained a weak convergence result.

In 1964, Polyak [38] introduced the inertial extrapolation method which is a useful tool for speeding up the rate of convergence of iterative methods. The idea of inertial extrapolation method was inspired by an implicit discretization of a second-order in-time dissipative dynamical system, so-called heavy ball with friction. The heavy ball friction is a simplified version of the differential system describing the motion of a heavy ball that rolls over the graph f and that keep rolling under its own inertia until friction stop it at a critical point of f. This nonlinear oscillation with damping, which is called the "heavy ball with friction" system, has been considered by several authors from the optimization point of view, establishing different convergence results and identifying circumstances under which the rate of convergence is better than the one of the first-order-steepest descent method (see [4, 6, 38]). Alvarez and Attouch [5] introduced and constructed the heavy-ball method with the proximal point algorithm to solve a problem of maximal monotone operator. They defined their method as follows:

$$\begin{cases} q_0, q_1 \in \mathbb{M}, \\ w_k = q_k + \theta_k(q_k - q_{k-1}), \\ q_{k+1} = (I + r_k\Psi)^{-1}w_k, \ \forall\, k \geqslant 1, \end{cases} \tag{5}$$

where $\{\theta_k\} \subset [0, 1)$ and $\{r_k\}$ is nondecreasing with $\sum_{k=1}^{\infty} \theta_k\|q_k - q_{k-1}\| < \infty$. They established that the sequence generated by (5) converges weakly to a zero of the monotone operator Ψ. In 2003, Moudafi and Oliny [32] introduced the following inertial proximal point method for finding the zero of the sum of two monotone operators:

$$\begin{cases} w_k = q_k + \theta_k(q_k - q_{k-1}), \\ q_{k+1} = (I + r_k\Psi)^{-1}(w_k - r_k\Phi q_k), \ k \geqslant 1. \end{cases} \tag{6}$$

They obtained a weak convergence theorem provided that $r_k < \frac{2}{L}$ with L being the Lipschitz constant of Φ and $\sum_{k=1}^{\infty} \theta_k\|q_k - q_{k-1}\| < \infty$ holds. Polyak [37] explored the potential of enhancing the convergence speed of numerical iteration methods for solving optimization problems by incorporating multistep inertial extrapolation steps. However, it is important to note that [37, 39] do not provide an established convergence analysis or rate of convergence for these multi-step inertial methods. Thus, the use of two or more inertial terms could guarantee necessary acceleration (see [30]). For growing interests in this direction (see [1, 2, 24, 51]).

Recently, Dong et al. [16] introduced the double inertial Mann algorithm and proved the convergence of the proposed algorithm under some suitable conditions: the algorithm is given by

$$\begin{cases} z_k = q_k + \lambda_k(q_k - q_{k-1}), \\ y_k = q_k + \theta_k(q_k - q_{k-1}), \\ q_{k+1} = (1-\phi_k)z_k + \phi T(y_k), \end{cases} \tag{7}$$

where T is a nonexpansive mapping, $\lambda, \theta \in [0, 1]$ and $\phi \in (0, 1)$.

Very recently, Suantai et al. [45] also considered a double inertial forward–backward algorithm in the settings of real Hilbert spaces.

Extension of concepts and techniques from linear spaces to Riemannian manifolds has some important advantages (see [17, 22, 40]). For instance, some optimization problems with nonconvex objective functions become convex from the Riemannian geometry point of view, and some constrained optimization problems can be regarded as unconstrained ones with an appropriate Riemannian metric. In addition, the study of convex minimization problems and inclusion problems in nonlinear spaces have proved to be very useful in computing medians and means of trees, which are very important in computational phylogenetics, diffusion tensor imaging, consensus algorithms and modeling of airway systems in human lungs and blood vessels (see [9–11]). Thus, nonlinear spaces are more suitable frameworks for the study of optimization problems from linear to Riemannian manifolds.

Very recently, Khammahawong et al. [20] proposed the following forward–backward splitting method for solving variational inclusion problem (1) in the settings of a Hadamard manifold:

$$\begin{cases} \mathbf{0} \in \Gamma_{q_k,p_k}\Phi(p_k) + \Psi(q_k) - \frac{1}{\tau_k}\exp^{-1}_{q_k} p_k, \\ p_{k+1} = \exp_{q_k}(\tau_k(\Gamma_{q_k,p_k}\Phi(p_k) - \Psi(q_k))), \end{cases} \tag{8}$$

where

$$\tau_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu d(p_k, q_k)}{\|\Gamma_{q_k,p_k}\Phi(p_k) - \Phi(q_k)\|}, \tau_k \right\}, & \text{if } \Gamma_{q_k,p_k}\Phi(p_k) - \Phi(q_k) \neq 0, \\ \tau_k, & \text{otherwise}, \end{cases} \tag{9}$$

and $\mu > 0$. They proved that the sequence by their proposed method converges to an element in Ω.

Furthermore, it will be crucial to expand the idea of the double inertial method to the Hadamard manifold because of the significance of our space of interest and the importance of the inertial method in dynamical systems.

Motivated by the aforementioned results in linear and nonlinear spaces, we proposed a forward–backward method together with a double step inertial method for solving variational inclusion problem in the settings of a Hadamard manifold. We prove that the sequence generated by our method converges to a solution of VIP (1) without the prior knowledge of the Lipschitz constant via a self-adaptive technique. In order to fasten the rate of convergence of our proposed method, we introduce a double inertial steps. Lastly, we compare our results with some related results in the literature to show the performance of our method. To the best of our knowledge, no result on double inertial steps have been discussed in the settings of nonlinear spaces. Our result extends and generalizes many related results in the literature.

2 Preliminaries

Let $\mathbb{P}$ be an m-dimensional manifold, let $x \in \mathbb{P}$ and let $T_x\mathbb{P}$ be the tangent space of $\mathbb{P}$ at $x \in \mathbb{P}$. We denote by $T\mathbb{P} = \bigcup_{x\in\mathbb{P}} T_x\mathbb{P}$ the tangent bundle of $\mathbb{P}$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian metric on $\mathbb{P}$ if $\langle\cdot,\cdot\rangle_x : T_x\mathbb{P} \times T_x\mathbb{P} \to \mathbb{R}$ is an inner product for all $x \in \mathbb{P}$. The corresponding norm induced by the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x\mathbb{P}$ is denoted by $\|\cdot\|_x$. We will drop the subscript x and adopt $\|\cdot\|$ for the corresponding norm induced by the inner product. A differentiable manifold $\mathbb{P}$ endowed with a Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. In what follows, we denote the Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle$ when no confusion arises. Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{P}$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$), we define the length $l(\gamma)$ of γ by $l(\gamma) := \int_a^b \|\gamma'(t)\|\mathrm{d}t$. The Riemannian distance $d(x, y)$ is the minimal length over the set of all such curves joining x to y. The metric topology induced by d coincides with the original topology on $\mathbb{P}$. We denote by ∇ the Levi-Civita connection associated with the Riemannian metric [40].

Let γ be a smooth curve in $\mathbb{P}$. A vector field X along γ is said to be parallel if $\nabla_{\gamma'} X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' itself is parallel along γ, then we say that γ is a geodesic and $\|\gamma'\|$ is a constant. If $\|\gamma'\| = 1$, then the geodesic γ is said to be normalized. A geodesic joining x to y in $\mathbb{P}$ is called a minimizing geodesic if its length equals $d(x, y)$. A Riemannian manifold $\mathbb{P}$ equipped with a Riemannian distance d is a metric space $(\mathbb{P}, d)$. A Riemannian manifold $\mathbb{P}$ is said to be complete if for all $x \in \mathbb{P}$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$. The Hopf–Rinow theorem [40] posits that if $\mathbb{P}$ is complete, then any pair of points in $\mathbb{P}$ can be joined by a minimizing geodesic. Moreover, if $(\mathbb{P}, d)$ is a complete metric space, then every bounded and closed subset of $\mathbb{P}$ is compact. If $\mathbb{P}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{P} \to \mathbb{P}$ at $x \in \mathbb{P}$ is defined by

$$\exp_x v := \gamma_v(1, x), \quad \forall\, v \in T_x\mathbb{P},$$

where $\gamma_v(\cdot, x)$ is the geodesic starting from x with velocity v (that is, $\gamma_v(0, x) = x$ and $\gamma_v'(0, x) = v$). Then, for any t, we have $\exp_x tv = \gamma_v(t, x)$ and $\exp_x \mathbf{0} = \gamma_v(0, x) = x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{P}$ for every $x \in \mathbb{P}$. The exponential map $\exp_x$ has an inverse $\exp_x^{-1} : \mathbb{P} \to T_x\mathbb{P}$. For any $x, y \in \mathbb{P}$, we have $d(x, y) = \|\exp_y^{-1} x\| = \|\exp_x^{-1} y\|$ (see [40] for more details). The parallel transport $\Gamma_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{P} \to T_{\gamma(b)}\mathbb{P}$ on the tangent bundle $T\mathbb{P}$ along $\gamma : [a, b] \to \mathbb{R}$ with respect to ∇ is defined by

$$\Gamma_{\gamma,\gamma(b),\gamma(a)} v = F(\gamma(b)), \;\; \forall\, a, b \in \mathbb{R} \;\text{ and } v \in T_{\gamma(a)}\mathbb{P},$$

where F is the unique vector field such that $\nabla_{\gamma'(t)} v = \mathbf{0}$ for all $t \in [a,b]$ and $F(\gamma(a)) = v$. If γ is a minimizing geodesic joining x to y, then we write $\Gamma_{y,x}$ instead of $\Gamma_{\gamma,y,x}$. Note that for every $a, b, r, s \in \mathbb{R}$, we have

$$\Gamma_{\gamma(s),\gamma(r)} \circ \Gamma_{\gamma(r),\gamma(a)} = \Gamma_{\gamma(s),\gamma(a)} \text{ and } \Gamma^{-1}_{\gamma(b),\gamma(a)} = \Gamma_{\gamma(a),\gamma(b)}.$$

Also, $\Gamma_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{P}$ to $T_{\gamma(b)}\mathbb{P}$, that is, the parallel transport preserves the inner product

$$\langle \Gamma_{\gamma(b),\gamma(a)}(u), \Gamma_{\gamma(b),\gamma(a)}(v)\rangle_{\gamma(b)} = \langle u, v\rangle_{\gamma(a)}, \quad \forall\, u, v \in T_{\gamma(a)}\mathbb{P}. \tag{10}$$

Below is an example of a Hadamard manifold.

Space 1: Let $\mathbb{R}^m_{++}$ be the product space $\mathbb{R}^m_{++} := \{(x_1, x_2, \cdots, x_m) : x_i \in \mathbb{R}_{++},\ i = 1, 2, \cdots, m\}$. Let $\mathbb{P} = (\mathbb{R}^m_{++}, \langle\cdot,\cdot\rangle)$ be the m-dimensional Hadamard manifold with the Riemannian metric $\langle p, q\rangle = p^{\mathrm{T}} q$ and the distance $d(x, y) = |\ln \frac{x}{y}| = |\ln \sum_{i=1}^{m} \frac{x_i}{y_i}|$, where $x, y \in \mathbb{P}$ with $x = \{x_i\}_{i=1}^m$ and $y = \{y_i\}_{i=1}^m$.

A subset $\mathcal{K} \subset \mathbb{P}$ is said to be convex if for any two points $x, y \in \mathcal{K}$, the geodesic γ joining x to y is contained in $\mathcal{K}$. That is, if $\gamma : [a,b] \to \mathbb{P}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in \mathcal{K}$ for all $t \in [0,1]$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard manifold. We denote by $\mathbb{P}$ a finite dimensional Hadamard manifold. Henceforth, unless otherwise stated, we represent by $\mathcal{K}$ a nonempty, closed and convex subset of $\mathbb{P}$.

Next, let $\mathcal{H}(\mathcal{K})$ denote the set of all single-valued vector fields $U : \mathcal{K} \to T\mathbb{P}$ such that $U(p) \in T_p\mathbb{P}$, for each $p \in \mathcal{K}$. Let $\mathcal{X}(\mathcal{K})$ denote to the set of all multivalued vector fields $V : \mathcal{K} \to 2^{T\mathbb{P}}$ such that $V(p) \subseteq T_p\mathbb{P}$ for each $p \in \mathcal{K}$, and the denote $\mathrm{Dom}(V)$ the domain of V defined by $\mathrm{Dom}(V) = \{p \in \mathcal{K} : V(p) \neq \varnothing\}$.

We state some results and definitions which are needed in the next section.

Definition 1 [50] A vector field $U \in \mathcal{H}(\mathcal{K})$ is said to be

(i) monotone, if

$$\langle U(p), \exp_p^{-1} q\rangle \leqslant \langle U(q), -\exp_q^{-1} p\rangle, \ \forall\, p, q \in \mathcal{K},$$

(ii) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|\Gamma_{p,q} U(q) - U(p)\| \leqslant L d(p,q), \ \forall\, p, q \in \mathcal{K}.$$

Definition 2 [14] A vector field $V \in \mathcal{X}(\mathcal{K})$ is said to be

(i) monotone, if for all $p, q \in \mathrm{Dom}(V)$,

$$\langle u, \exp_p^{-1} q\rangle \leqslant \langle v, -\exp_q^{-1} p\rangle, \ \forall\, u \in V(p) \text{ and } \forall\, v \in V(q),$$

(ii) maximal monotone if it is monotone and $\forall\, p \in \mathcal{K}$ and $u \in T_p\mathcal{K}$, the condition

$$\langle u, \exp_p^{-1} q\rangle \leqslant \langle v, -\exp_q^{-1} p\rangle,\ \forall\, q \in \mathrm{Dom}(V) \text{ and } \forall\, v \in V(q) \text{ implies that } u \in V(p).$$

Definition 3 [17] Let $\mathcal{K}$ be a nonempty, closed and subset of $\mathbb{P}$ and $\{x_n\}$ be a sequence in $\mathbb{P}$. Then, $\{x_n\}$ is said to be Fejèr convergent with respect to $\mathcal{K}$ if for all $p \in \mathcal{K}$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, p) \leqslant d(x_n, p).$$

Definition 4 [25] Let $V \in \mathcal{X}(\mathcal{K})$ be a vector field and $x_0 \in \mathcal{K}$. Then, V is said to be upper Kuratowski semicontinuous at x_0 if for any sequences $\{x_n\} \subseteq \mathcal{K}$ and $\{v_n\} \subset T\mathbb{P}$ with each $v_n \in V(x_n)$, the relations $\lim\limits_{n\to\infty} v_n = v_0$ imply that $v_0 \in V(x_0)$. Moreover, V is said to be upper Kuratowski semicontinuous on $\mathcal{K}$ if it is upper Kuratowski semicontinuous for each $x \in \mathcal{K}$.

Lemma 1 [17] *Let $\mathcal{K}$ be a nonempty, closed and closed subset of $\mathbb{P}$ and $\{x_n\} \subset \mathbb{P}$ be a sequence such that $\{x_n\}$ be a Fejér convergent with respect to $\mathcal{K}$. Then, the following hold*:

(i) *For every $p \in \mathcal{K}$, $d(x_n, p)$ converges.*
(ii) *$\{x_n\}$ is bounded.*
(iii) *Assume that every cluster point of $\{x_n\}$ belongs to $\mathcal{K}$, then $\{x_n\}$ converges to a point in $\mathcal{K}$.*

Proposition 1 [40]. *Let $x \in \mathbb{P}$. The exponential mapping $\exp_x : T_x\mathbb{P} \to \mathbb{P}$ is a diffeomorphism. For any two points $x, y \in \mathbb{P}$, there exists a unique normalized geodesic joining x to y, which is given by*

$$\gamma(t) = \exp_x t \exp_x^{-1} y, \quad \forall\, t \in [0, 1].$$

A geodesic triangle $\Delta(p, q, r)$ of a Riemannian manifold $\mathbb{P}$ is a set containing three points p, q, r and three minimizing geodesics joining these points.

Proposition 2 [40]. *Let $\Delta(p, q, r)$ be a geodesic triangle in $\mathbb{P}$. Then*

$$d^2(p, q) + d^2(q, r) - 2\langle \exp_q^{-1} p, \exp_q^{-1} r\rangle \leqslant d^2(r, q) \tag{11}$$

and

$$d^2(p, q) \leqslant \langle \exp_p^{-1} r, \exp_p^{-1} q\rangle + \langle \exp_q^{-1} r, \exp_q^{-1} p\rangle. \tag{12}$$

Moreover, if θ is the angle at p, then we have

$$\langle \exp_p^{-1} q, \exp_p^{-1} r\rangle = d(q, p)d(p, r)\cos\theta. \tag{13}$$

Also,

$$\| \exp_p^{-1} q\|^2 = \langle \exp_p^{-1} q, \exp_p^{-1} q\rangle = d^2(p,q). \tag{14}$$

Remark 1 [25] If $x, y \in \mathbb{P}$ and $v \in T_y\mathbb{P}$, then

$$\langle v, -\exp_y^{-1} x\rangle = \langle v, \Gamma_{y,x} \exp_x^{-1} y\rangle = \langle \Gamma_{x,y} v, \exp_x^{-1} y\rangle. \tag{15}$$

Lemma 2 [21] *Let $\mathbb{P}$ be a Hadamard manifold and let $u, v, w \in \mathbb{P}$. Then,*

$$\| \exp_u^{-1} w - \Gamma_{u,v} \exp_v^{-1} w\| \leqslant d(u, v).$$

Lemma 3 [25] *Let $x_0 \in \mathbb{P}$ and $\{x_n\} \subset \mathbb{P}$ with $x_n \to x_0$. Then, the following assertions hold*:

(i) *For any $y \in \mathbb{P}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} x_n$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.*
(ii) *If $v_n \in T_{x_n}\mathbb{P}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathbb{P}$.*
(iii) *Given $u_n, v_n \in T_{x_n}\mathbb{P}$ and $u_0, v_0 \in T_{x_0}\mathbb{P}$, if $u_n \to u_0$, then $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$.*
(iv) *For any $u \in T_{x_0}\mathbb{P}$, the function $F : \mathbb{P} \to T\mathbb{P}$, defined by $F(x) = \Gamma_{x,x_0} u$ for each $x \in \mathbb{P}$ is continuous on $\mathbb{P}$.*

The next lemma presents the relationship between triangles in $\mathbb{R}^2$ and geodesic triangles in Riemannian manifolds (see [12]).

Lemma 4 [12]. *Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in $\mathbb{P}$. Then, there exists a triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ corresponding to $\Delta(x_1, x_2, x_3)$ such that $d(x_i, x_{i+1}) = \|\bar{x}_i - \bar{x}_{i+1}\|$ with the indices taken modulo 3. This triangle is unique up to isometries of $\mathbb{R}^2$.*

The triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Lemma 4 is said to be the comparison triangle for $\Delta(x_1, x_2, x_3) \subset \mathbb{P}$. The points $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are called comparison points to the points x_1, x_2 and x_3 in $\mathbb{P}$.

A function $h : \mathbb{P} \to \mathbb{R}$ is said to be geodesic if for any geodesic $\gamma \in \mathbb{P}$, the composition $h \circ \gamma : [u, v] \to \mathbb{R}$ is convex, that is,

$$h \circ \gamma(\lambda u + (1-\lambda)v) \leqslant \lambda h \circ \gamma(u) + (1-\lambda) h \circ \gamma(v), \quad u, v \in \mathbb{R}, \ \lambda \in [0, 1].$$

Lemma 5 [25] *Let $\Delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold $\mathbb{P}$ and $\Delta(p', q', r')$ be its comparison triangle.*

(i) *Let α, β, γ (resp. α', β', γ') be the angles of $\Delta(p, q, r)$ (resp. $\Delta(p', q', r')$) at the vertices p,q,r (resp. p', q', r'). Then, the following inequalities hold*:

$$\alpha' \geqslant \alpha, \ \beta' \geqslant \beta, \ \gamma' \geqslant \gamma.$$

(ii) *Let z be a point in the geodesic joining p to q and z' its comparison point in the interval $[p', q']$. Suppose that $d(z, p) = \|z' - p'\|$ and $d(z', q') = \|z' - q'\|$. Then, the following inequality holds*:

$$d(z, r) \leqslant \|z' - r'\|.$$

Lemma 6 [25] *Let $x_0 \in \mathbb{P}$ and $\{x_n\} \subset \mathbb{P}$ be such that $x_n \to x_0$. Then, for any $y \in \mathbb{P}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.*

The following propositions (see [17]) are very useful in our convergence analysis:

Proposition 3 *Let $\mathbb{P}$ be a Hadamard manifold and $d : \mathbb{P} \times \mathbb{P} :\to \mathbb{R}$ be the distance function. Then the function d is convex with respect to the product Riemannian metric. In other words, given any pair of geodesics $\gamma_1 : [0, 1] \to \mathbb{P}$ and $\gamma_2 : [0, 1] \to \mathbb{P}$, then for all $t \in [0, 1]$, we have*

$$d(\gamma_1(t), \gamma_2(t)) \leqslant (1 - t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in \mathbb{P}$, the function $d(\cdot, y) : \mathbb{P} \to \mathbb{R}$ is a convex function.

Proposition 4 *Let $\mathbb{P}$ be a Hadamard manifold and $x \in \mathbb{P}$. The map $\Phi_x = d^2(x, y)$ satisfying the following*:

(1) *Φ_x is convex. Indeed, for any geodesic $\gamma : [0, 1] \to \mathbb{P}$, the following inequality holds for all $t \in [0, 1]$:*

$$d^2(x, \gamma(t)) \leqslant (1 - t)d^2(x, \gamma(0)) + td^2(x, \gamma(1)) - t(1 - t)d^2(\gamma(0), \gamma(1)).$$

(2) *Φ_x is smooth. Moreover, $\partial \Phi_x(y) = -2 \exp_y^{-1} x$.*

Lemma 7 [18] *Let $\{v_n\}$ and $\{\delta_n\}$ be nonnegative sequences which satisfy*

$$v_{n+1} = (1 + \delta_n)v_n + \delta_n v_{n-1}, \ n \geqslant 1.$$

Then,

$$v_{n+1} \leqslant M \cdot \prod_{j=1}^{n}(1 + 2\delta_j), \ \text{where } M = \max\{v_1, v_2\}.$$

Moreover, if $\sum_{n=1}^{\infty} \delta_n < +\infty$, then $\{v_n\}$ is bounded.

Lemma 8 [34] *Let $\{a_n\}$, $\{\varphi_n\}$ and $\{\beta_n\}$ be nonnegative sequences which satisfy*

$$a_{n+1} = (1 + \beta_n)a_n + \varphi_n, \ n \geqslant 1.$$

If $\sum_{n=1}^{\infty} \beta_n < +\infty$ and $\sum_{n=1}^{\infty} \varphi_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

3 Main Result

In this section, we present an iterative method for solving variational inclusion problem in the settings of Hadamard manifolds. We state the following assumptions:

Assumption 1

(L1) $\Phi \in \mathcal{H}(\mathcal{K})$ is monotone and L-Lipschitz continuous, and $\Psi \in \mathcal{X}(\mathcal{K})$ is maximal monotone.

(L2) The solution set $\Omega := (\Phi + \Psi)^{-1}(\mathbf{0})$ is nonempty.

(L3) $\{\lambda_k\}$ is a nonnegative real numbers sequence such that $\sum_{k=1}^{\infty} \lambda_k < \infty$.

Algorithm 1 *Self-adaptive method with two inertial steps for variational inclusion problem.* **Initialization:** *Choose* $\rho_0 > 0, \mu \in (0, 1)$, $\{\alpha_k\}, \{\theta_k\}$ *are real positive sequences. Let* $q_0, q_1 \in \mathbb{P}$ *be arbitrary.* **Iterative steps:** *Given the current iterate* q_k, *calculate* q_{k+1} *as follows*:

Step 1 *Compute*

$$\begin{cases} w_k = \exp_{q_k}(-\alpha_k \exp_{q_k}^{-1} q_{k-1}), \\ z_k = \exp_{w_k}(-\theta_k \exp_{w_k}^{-1} q_{k-1}), \end{cases} \tag{16}$$

and

$$\mathbf{0} \in \Gamma_{t_k,z_k}\Phi(z_k) + \Psi(t_k) - \frac{1}{\rho_k} \exp_{t_k}^{-1} z_k. \tag{17}$$

If $t_k = z_k$, *then stop and* $t_k \in \Omega$. *Else, proceed to step* 2.

Step 2 *Compute*

$$q_{k+1} = \exp_{t_k}\big(\rho_k(\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k))\big). \tag{18}$$

Update

$$\rho_{k+1} = \begin{cases} \min\left\{\dfrac{\mu d(z_k, t_k)}{\|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|}, \ \rho_k + \lambda_k\right\}, & \text{if } \Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k) \neq 0, \\ \rho_k + \lambda_k, & \text{otherwise}. \end{cases} \tag{19}$$

Stopping criterion *Set* $k := k + 1$ *and return to* **Iterative step 1**.

We start by establishing a technical lemma useful to our analysis.

Lemma 9 [2, 27] *Let* $\{q_k\}$ *be a sequence generated by Algorithm* 1 *and the sequence* $\{\rho_k\}$ *is generated by* (19). *Then we have that* $\lim_{k\to\infty} \rho_k = \rho$ *and* $\rho \in \left[\min\left\{\frac{\mu}{L}, \rho_0\right\}, \rho_0 + \lambda\right]$, *where* $\lambda = \sum_{k=0}^{\infty} \lambda_k$.

Remark 10 It is obvious that the stepsize in Algorithm 1 is allowed to increase from iteration to iteration and so (19) reduces the dependence on the initial stepsize ρ_0. Also, since $\{\lambda_k\}$ is summable, we obtain $\lim_{k\to\infty} \lambda_k = 0$. Thus the stepsize λ_k may be non-increasing when k is large. If $\lambda_k \equiv 0$, the step size in (1) reduces to the one in [20].

Theorem 1 *Suppose that Assumptions* (L1)-(L3) *holds and let* $\{q_k\}$ *be a sequence generated by Algorithm* 1. *If* $\sum_{k=1}^{\infty} \alpha_k < +\infty$ *and* $\sum_{k=1}^{\infty} \theta_k < +\infty$, *then*

(i) $d(q_{k+1}, p) \leqslant M \cdot \prod_{j=1}^{k} (1 + 2(\alpha_j + \theta_j(1 + \alpha_j)))$, *where* $M := \max\{d(q_1, p), d(q_2, p)\}$.

(ii) *The sequence* $\{q_k\}$ *converges to an element in* Ω.

Proof Let $p \in \Omega$, then $-\Phi(q) \in \Psi(p)$. Using (16) of Algorithm 1, we get $\frac{1}{\rho_k} \exp_{t_k}^{-1} z_k - \Gamma_{t_k,z_k} \Phi(z_k) \in \Psi(t_k)$. By applying the monotonicity of Ψ, we deduce that

$$\Big\langle \frac{1}{\rho_k} \exp_{t_k}^{-1} z_k - \Gamma_{t_k,z_k} \Phi(z_k), \exp_{t_k}^{-1} p \Big\rangle \leqslant \langle -\Phi(p), -\exp_p^{-1} t_k \rangle$$

$$= \langle \Phi(p), \exp_p^{-1} t_k \rangle. \tag{20}$$

Since Φ is a monotone vector field, then

$$\langle \Phi(p), \exp_p^{-1} t_k \rangle \leqslant \langle -\Phi(t_k), \exp_{t_k}^{-1} p \rangle. \tag{21}$$

By combining (20) and (21), we have

$$\Big\langle \frac{1}{\rho_k} \exp_{t_k}^{-1} z_k - \Gamma_{t_k,z_k} \Phi(z_k), \exp_{t_k}^{-1} p \Big\rangle \leqslant \langle -\Phi(t_k), \exp_{t_k}^{-1} p \rangle,$$

thus

$$\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle \leqslant \rho_k \langle \Gamma_{t_k,z_k} \Phi(z_k) - \Phi(t_k), \exp_{t_k}^{-1} p \rangle. \tag{22}$$

Now, for $k \in \mathbb{N}$. Let $\Delta(z_k, t_k, p) \subseteq \mathbb{P}$ be a geodesic triangle with vertices z_k, t_k and p and let $\Delta(z'_k, t'_k, p') \subset \mathbb{R}^2$ be the corresponding comparison triangle, thus we have from Lemma 5 (ii) that $d(z_k, p) = \|z'_k - p'\|$, $d(t_k, p) = \|t'_k - p'\|$ and $d(t'_k, z'_k) = \|t'_k - z'_k\|$. Also, let $\Delta(q_{k+1}, t_k, p) \subseteq \mathbb{P}$ be a geodesic triangle with vertices q_{k+1}, t_k and p, then $\Delta(q'_{k+1}, t'_k, p') \subseteq \mathbb{R}^2$ is the corresponding comparison triangle. Hence, we have $d(q_{k+1}, p) = \|q'_{k+1} - p'\|$, $d(t_k, p) = \|t'_k - p'\|$ and $d(q_{k+1}, t_k) = \|q'_{k+1} - t'_k\|$.

Now,

$$\begin{aligned}
d^2(q_{k+1}, p) &\leqslant \|q'_{k+1} - p'\|^2 \\
&= \|q'_{k+1} - t'_k + t'_k - p'\|^2 \\
&= \|t'_k - p'\|^2 + \|q'_{k+1} - t'_k\|^2 + 2\langle q'_{k+1} - t'_k, t'_k - p'\rangle \\
&= \|(t'_k - z'_k) + (z'_k - p')\|^2 + \|q'_{k+1} - t'_k\|^2 + 2\langle q'_{k+1} - t'_k, t'_k - p'\rangle \\
&= \|t'_k - z'_k\|^2 + \|z'_k - p'\|^2 + \|q'_{k+1} - t'_k\|^2 + 2\langle t'_k - z'_k, z'_k - p'\rangle \\
&\quad + 2\langle t'_k - p', t'_k - p'\rangle - 2\|t'_k - p'\|^2 + 2\langle q'_{k+1} - t'_k, t'_k - p'\rangle \\
&\quad + 2\langle t'_k - z'_k, t'_k - z'_k\rangle \\
&\quad - 2\langle t'_k - z'_k, t'_k - z'_k\rangle \\
&= d^2(z_k, p) - d^2(t_k, z_k) + \|q'_{k+1} - t'_k\|^2 + 2\langle t'_k - z'_k, t'_k - p'\rangle \\
&\quad + 2\langle q'_{k+1} - t'_k, t'_k - p'\rangle \\
&\quad + 2\langle t'_k - p', t'_k - p'\rangle - 2d^2(t_k, p) \\
&= d^2(z_k, p) - d^2(t_k, z_k) + \|q'_{k+1} - t'_k\|^2 + 2\langle t'_k - z'_k, t'_k - p'\rangle \\
&\quad + 2\langle q'_{k+1} - p', t'_k - p'\rangle - 2d^2(t_k, p).
\end{aligned} \tag{23}$$

Let r and r' be the angles of the vertices t_k and t'_k, respectively. By Lemma 5 (i), we get $r' \geqslant r$. Therefore, we obtain from Lemma 4 and (13) that

$$\begin{aligned}
\langle t'_k - z'_k, t'_k - p'\rangle &= \|t'_k - z'_k\| \cdot \|t'_k - p'\| \cos r' \\
&= d(t_k, z_k) d(p, t_k) \cos r' \\
&\leqslant d(t_k, z_k) d(p, t_k) \cos r \\
&= \langle \exp^{-1}_{t_k} z_k, \exp^{-1}_{t_k} p\rangle.
\end{aligned} \tag{24}$$

Following the same argument as in (24), we have

$$\langle q'_{k+1} - p', t'_k - p'\rangle = \langle \exp^{-1}_{p} q_{k+1}, \exp^{-1}_{p} t_k\rangle. \tag{25}$$

Hence, we deduce from (18) that

$$\|q'_{k+1} - t'_k\|^2 \leqslant \rho_k^2 \|\Gamma_{t_k, z_k} \Phi(z_k) - \Phi(t_k)\|^2. \tag{26}$$

On substituting (24), (25) and (26) into (23), we obtain

$$\begin{aligned}
d^2(q_{k+1}, p) \leqslant d^2(z_k, p) - d^2(t_k, z_k) &+ \rho_k^2 \|\Gamma_{t_k, z_k} \Phi(z_k) - \Phi(t_k)\|^2 \\
&+ 2\langle \exp^{-1}_{t_k} z_k, \exp^{-1}_{t_k} p\rangle - 2d^2(t_k, p) \\
&+ 2\langle \exp^{-1}_{p} q_{k+1}, \exp^{-1}_{p} t_k\rangle.
\end{aligned} \tag{27}$$

Using Remark 1, Lemma 2 and (27), we get

$$\begin{aligned}
d^2(q_{k+1}, p) &\leqslant d^2(z_k, p) - d^2(t_k, z_k) + \rho_k^2 \|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|^2 - 2d^2(t_k, p) \\
&\quad + 2\langle \exp_p^{-1} q_{k+1} - \Gamma_{p,t_k} \exp_{t_k}^{-1} q_{k+1} + \Gamma_{p,t_k} \exp_{t_k}^{-1} q_{k+1}, \exp_p^{-1} t_k \rangle \\
&\quad + 2\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle \\
&= d^2(z_k, p) - d^2(t_k, z_k) + \rho_k^2 \|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|^2 - 2d^2(t_k, p) \\
&\quad + 2\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle + 2\langle \exp_p^{-1} q_{k+1} - \Gamma_{p,t_k} \exp_{t_k}^{-1} q_{k+1}, \exp_p^{-1} t_k \rangle \\
&\quad + 2\langle \Gamma_{p,t_k} \exp_{t_k}^{-1} q_{k+1}, \exp_p^{-1} t_k \rangle \\
&\leqslant d^2(z_k, p) - d^2(t_k, z_k) + \rho_k^2 \|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|^2 - 2d^2(t_k, p) \\
&\quad + 2\| \exp_p^{-1} q_{k+1} - \Gamma_{p,t_k} \exp_{t_k}^{-1} q_{k+1}\| \| \exp_p^{-1} t_k\| + 2\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle \\
&\quad - 2\langle \exp_{t_k}^{-1} q_{k+1}, \exp_{t_k}^{-1} p \rangle, \qquad (28)
\end{aligned}$$

which also implies that

$$\begin{aligned}
d^2(q_{k+1}, p) &\leqslant d^2(z_k, p) - d^2(t_k, z_k) + \rho_k^2 \|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|^2 - 2d^2(t_k, p) \\
&\quad + 2d^2(p, t_k) + 2\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle - 2\langle \exp_{t_k}^{-1} q_{k+1}, \exp_{t_k}^{-1} p \rangle \\
&= d^2(z_k, p) - d^2(t_k, z_k) + \rho_k^2 \|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|^2 + 2\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle \\
&\quad - 2\langle \exp_{t_k}^{-1} q_{k+1}, \exp_{t_k}^{-1} p \rangle. \qquad (29)
\end{aligned}$$

It follows from the definition of q_{k+1} that $\exp_{t_k}^{-1} q_{k+1} = \rho_k(\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k))$. Using the last inequality, we obtain that

$$\begin{aligned}
d^2(q_{k+1}, p) &\leqslant d^2(z_k, p) - d^2(t_k, z_k) + \rho_k^2 \|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|^2 + 2\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle \\
&\quad - 2\rho_k \langle \Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k), \exp_{t_k}^{-1} p \rangle \\
&= d^2(z_k, p) - d^2(t_k, z_k) + \rho_k^2 \|\Gamma_{t_k,z_k}\Phi(z_k) - \Phi(t_k)\|^2 + 2\langle \exp_{t_k}^{-1} z_k, \exp_{t_k}^{-1} p \rangle \\
&\quad + 2\rho_k \langle \Phi(t_k) - \Gamma_{t_k,z_k}\Phi(z_k), \exp_{t_k}^{-1} p \rangle. \qquad (30)
\end{aligned}$$

By substituting (19) and (22) in (30), we get

$$\begin{aligned}
d^2(q_{k+1}, p) &\leqslant d^2(z_k, p) - d^2(t_k, z_k) + \mu^2 \frac{\rho_k^2}{\rho_{k+1}^2} d^2(t_k, z_k) \\
&\quad + 2\rho_k \langle \Phi(t_k) - \Gamma_{t_k,z_k}\Phi(z_k), \exp_{t_k}^{-1} p \rangle - 2\rho_k \langle \Phi(t_k) - \Gamma_{t_k,z_k}\Phi(z_k), \exp_{t_k}^{-1} p \rangle \\
&= d^2(z_k, p) - \big(1 - \mu^2 \frac{\rho_k^2}{\rho_{k+1}^2}\big) d^2(t_k, z_k). \qquad (31)
\end{aligned}$$

By utilizing the geodesic triangles $\triangle(w_k, q_k, p) \subset \mathbb{P}$ and $\triangle(q_k, q_{k-1}, p) \subset \mathbb{P}$ with their respective comparison triangles $\triangle(w'_k, q'_k, p') \subseteq \mathbb{R}^2$. Then, by Lemma 5 (ii), we have $d(w_k, q_k) = \|w'_k - q'_k\|$, $d(w_k, p) = \|w'_k - p'\|$ and $d(q_k, q_{k-1}) = \|q'_k - q'_{k-1}\|$. Similarly, using the geodesic triangles $\triangle(z_k, w_k, p) \subset \mathbb{P}$ and $\triangle(q_k, q_{k-1}, p) \subset \mathbb{P}$ with their respective comparison triangle $\triangle(z'_k, w'_k, p') \subseteq \mathbb{R}^2$. Then, by Lemma 5 (ii), we have $d(z_k, w_k) = \|z'_k - w'_k\|$, $d(z_k, q_k) = \|z'_k - q'_k\|$ and $d(z_k, p) = \|z'_k - p'\|$. From step 1 of Algorithm 1, we have that $w'_k = q'_k + \alpha_k(q'_k - q'_{k-1})$ and $z'_k = w'_k + \theta_k(w'_k - q'_{k-1})$, thus

$$\begin{aligned} d(w_k, p) &= \|w'_k - p'\| \\ &= \|q'_k + \alpha_k(q'_k - q'_{k-1}) - p'\| \\ &\leqslant \|q'_k - p'\| + \alpha_k\|q'_k - q'_{k-1}\| \\ &= d(q_k, p) + \alpha_k d(q_k, q_{k-1}). \end{aligned} \tag{32}$$

Similarly, it is easy to see that

$$\begin{aligned} d(w_k, q_{k-1}) &= \|w'_k - q'_{k-1}\| \\ &= \|q'_k + \alpha_k(q'_k - q'_{k-1}) - q'_{k-1}\| \\ &\leqslant \|q'_k - q'_{k-1}\| + \alpha_k\|q'_k - q'_{k-1}\| \\ &= d(q_k, q_{k-1}) + \alpha_k d(q_k, q_{k-1}) \\ &= (1 + \alpha_k) d(q_k, q_{k-1}). \end{aligned} \tag{33}$$

By definition of z_k, (32) and (33), we get

$$\begin{aligned} d(z_k, p) &= \|z'_k - p'\| \\ &= \|w'_k + \theta_k(w'_k - q'_{k-1}) - p'\| \\ &\leqslant \|w'_k - p'\| + \theta_k\|w'_k - q'_{k-1}\| \\ &= d(w_k, p) + \theta_k d(w_k, q_{k-1}) \\ &\leqslant d(q_k, p) + \alpha_k d(q_k, q_{k-1}) + \theta_k(1 + \alpha_k) d(q_k, q_{k-1}) \\ &= d(q_k, p) + (\alpha_k + \theta_k(1 + \alpha_k)) d(q_k, q_{k-1}). \end{aligned} \tag{34}$$

Since $\lim_{k\to\infty}\left(1 - \mu^2 \frac{\rho_k^2}{\rho_{k+1}^2}\right) = 1 - \mu^2 > 0$, this implies that there exists $N > 0$ such that $1 - \mu^2 \frac{\rho_k^2}{\rho_{k+1}^2} > 0, \forall\, k \geqslant \mathbb{N}$.
From (31) and (34), we deduce that

$$\begin{aligned} d(q_{k+1}, p) &\leqslant d(z_k, p) \\ &\leqslant d(q_k, p) + (\alpha_k + \theta_k(1 + \alpha_k)) d(q_k, q_{k-1}) \\ &\leqslant d(q_k, p) + (\alpha_k + \theta_k(1 + \alpha_k))\big(d(q_k, p) + d(q_{k-1}, p)\big) \\ &= (1 + \alpha_k + \theta_k(1 + \alpha_k)) d(q_k, p) + (\alpha_k + \theta_k(1 + \alpha_k)) d(q_{k-1}, p). \end{aligned} \tag{35}$$

By applying Lemma 7, we obtain that

$$d(q_{k+1}, p) \leqslant M \cdot \prod_{j=1}^{k}(1 + 2(\alpha_j + \theta_j(1 + \alpha_j))), \tag{36}$$

where $M = \max\{d(q_1, p), d(q_2, p)\}$. Hence, the proof completes.

To establish the second part of the proof, we need to show that $\{q_k\}$ converges to a point in Ω. Since $\sum_{k=1}^{\infty} \alpha_k < +\infty$ and $\sum_{k=1}^{\infty} \theta_k < +\infty$, by Lemma 7 and (36), the sequence $\{q_k\}$ is bounded. This also implies that $\sum_{k=1}^{\infty} \alpha_k d(q_k, q_{k-1}) < +\infty$ and $\sum_{k=1}^{\infty} \theta_k d(q_k, q_{k-1}) < +\infty$. Using Lemma 8 in (35), we can claim that $\lim_{k\to\infty} d(q_k, p)$ exists. We have from Lemma 5 (ii) and Proposition 4 that

$$\begin{aligned} d^2(w_k, p) &= \|w_k' - p'\|^2 \\ &= \|q_k' + \alpha_k(q_k' - q_{k-1}') - p'\|^2 \\ &= \|(1 + \alpha_k)(q_k' - p') - \alpha_k(q_{k-1}' - p')\|^2 \\ &= (1 + \alpha_k)d^2(q_k, p) - \alpha_k d^2(q_{k-1}, p) + \alpha_k(1 + \alpha_k)d^2(q_k, q_{k-1}). \end{aligned} \tag{37}$$

We also consider

$$\begin{aligned} d^2(w_k, q_{k-1}) &= \|w_k' - q_{k-1}'\|^2 \\ &= \|q_k' + \alpha_k(q_k' - q_{k-1}') - q_{k-1}'\|^2 \\ &= \|q_k' - q_{k-1}'\|^2 + 2\langle q_k' - q_{k-1}', \alpha_k(q_k' - q_{k-1}')\rangle \\ &\quad + \alpha_k^2\|q_k' - q_{k-1}'\|^2. \end{aligned} \tag{38}$$

But from (14), we have

$$\begin{aligned} \langle q_k' - q_{k-1}', q_k' - q_{k-1}'\rangle &\leqslant \langle \exp_{q_{k-1}}^{-1} q_k, \exp_{q_{k-1}}^{-1} q_k\rangle \\ &= \|\exp_{q_{k-1}}^{-1} q_k\|^2 \\ &= d^2(q_k, q_{k-1}). \end{aligned} \tag{39}$$

On substituting (39) into (38), we get

$$\begin{aligned} d^2(w_k, q_{k-1}) &\leqslant d^2(q_k, q_{k-1}) + 2\alpha_k d^2(q_k, q_{k-1}) + \alpha_k^2 d^2(q_k, q_{k-1}) \\ &= (1 + \alpha_k)^2 d^2(q_k, q_{k-1}). \end{aligned} \tag{40}$$

We deduce from Lemma 5, (37) and (40) that

$$
\begin{aligned}
d^2(z_k, p) &= \|z_k' - p'\|^2 \\
&= \|w_k' + \theta_k(w_k' - q_{k-1}') - p'\|^2 \\
&= \|(1+\theta_k)(w_k' - q_{k-1}') - \theta_k(q_{k-1}' - p')\|^2 \\
&= (1+\theta_k)d^2(w_k, q_{k-1}) - \theta_k d^2(q_{k-1}, p) + \theta_k(1+\theta_k)d^2(w_k, q_{k-1}) \\
&= (1+\theta_k)\big((1+\alpha_k)d^2(q_k, p) - \alpha_k d^2(q_{k-1}, p) + \alpha_k(1+\alpha_k)d^2(q_k, q_{k-1})\big) \\
&\quad - \theta_k d^2(q_{k-1}, p) + \theta_k(1+\theta_k)d^2(w_k, q_{k-1}) \\
&\leqslant (1+\theta_k)\big(d^2(q_k, p) + \alpha_k\big(d^2(q_k, p) - d^2(q_{k-1}, p)\big) + \alpha_k(1+\alpha_k)d^2(q_k, q_{k-1})\big) \\
&\quad - \theta_k d^2(q_{k-1}, p) + \theta_k(1+\theta_k)(1+\alpha_k)^2 d^2(q_k, q_{k-1}) \\
&= d^2(q_k, p) + \big(\theta_k + (1+\theta_k)\alpha_k\big)\big(d^2(q_k, p) - d^2(q_{k-1}, p)\big) \\
&\quad + \alpha_k(1+\alpha_k)(1+\theta_k)d^2(q_k, q_{k-1}) + \theta_k(1+\theta_k)(1+\alpha_k)^2 d^2(q_k, q_{k-1}). \qquad (41)
\end{aligned}
$$

On substituting (41) into (31), we obtain

$$
\begin{aligned}
d^2(q_{k+1}, p) \leqslant{}& d^2(q_k, p) + (\theta_k + (1+\theta_k)\alpha_k)\big(d^2(q_k, p) - d^2(q_{k-1}, p)\big) \\
&+ \alpha_k(1+\alpha_k)(1+\theta_k)d^2(q_k, q_{k-1}) + \theta_k(1+\theta_k)(1+\alpha_k)^2 d^2(q_k, q_{k-1}) \\
&- \Big(1 - \mu^2 \frac{\rho_k^2}{\rho_{k+1}^2}\Big)d^2(t_k, z_k). \qquad (42)
\end{aligned}
$$

The last inequality yields

$$
\begin{aligned}
\Big(1 - \mu^2 \frac{\rho_k^2}{\rho_{k+1}^2}\Big)d^2(t_k, z_k) &\leqslant d^2(q_k, p) + (\theta_k + (1+\theta_k)\alpha_k)\big(d^2(q_k, p) - d^2(q_{k-1}, p)\big) \\
&\quad + \alpha_k(1+\alpha_k)(1+\theta_k)d^2(q_k, q_{k-1}) + \theta_k(1+\theta_k)(1+\alpha_k)^2 d^2(q_k, q_{k-1}) \\
&\quad - d^2(q_{k+1}, p) \\
&= \big(d^2(q_k, p) - d^2(q_{k+1}, p)\big) + (\theta_k + (1+\theta_k)\alpha_k)\big(d^2(q_k, p) - d^2(q_{k-1}, p)\big) \\
&\quad + \alpha_k(1+\alpha_k)(1+\theta_k)d^2(q_k, q_{k-1}) + \theta_k(1+\theta_k)(1+\alpha_k)^2 d^2(q_k, q_{k-1}). \qquad (43)
\end{aligned}
$$

Since $\lim_{k\to\infty} d(q_k, p)$ exists, $\sum_{k=1}^{\infty} \alpha_k < +\infty$ and $\sum_{k=1}^{\infty} \theta_k < +\infty$. It follows from (43) that

$$
\lim_{k\to\infty} d(t_k, z_k) = 0. \qquad (44)
$$

Note that

$$\begin{aligned} d(w_k, q_k) &= \|w_k' - q_k'\| \\ &= \|q_k' + \alpha_k(q_k' - q_{k-1}') - q_k'\| \\ &= \alpha_k d(q_k, q_{k-1}) \to 0,\ k \to \infty. \end{aligned} \tag{45}$$

From (45), we get

$$\begin{aligned} d(z_k, q_k) &= \|z_k' - q_k'\| \\ &= \|w_k' + \theta_k(w_k' - q_{k-1}') - q_k'\| \\ &\leqslant \|w_k' - q_k'\| + \theta_k\|q_k' + \alpha_k(q_k' - q_{k-1}') - q_{k-1}'\| \\ &\leqslant d(w_k, q_k) + \theta_k d(q_k, q_{k-1}) + \theta_k\alpha_k d(q_k, q_{k-1}) \to 0,\ k \to \infty. \end{aligned} \tag{46}$$

From (44) and (46), we have

$$\lim_{k\to\infty} d(t_k, q_k) = 0. \tag{47}$$

Using (45) and (46), we deduce that

$$\lim_{k\to\infty} d(z_k, w_k) = 0. \tag{48}$$

Since $\{q_k\}$ is bounded, there exists a subsequence $\{q_{k_l}\}$ which converges to a cluster point $\overline{p}$. Also, from (47), there exists a subsequence $\{t_{k_l}\}$ of $\{t_k\}$ which converges weakly to $\overline{p} \in \mathbb{P}$. By (17), we deduce that

$$\Upsilon_{k_l} = -\Gamma_{t_{k_l}, z_{k_l}}\Phi(z_{k_l}) - \frac{1}{\rho_{k_l}}\exp^{-1}_{t_{k_l}} z_{k_l} \in \Psi(t_{k_l}). \tag{49}$$

Thus, by applying (44), we have

$$\lim_{l\to\infty}\frac{1}{\rho_{k_l}}\|\exp^{-1}_{t_{k_l}} z_{k_l}\| = \lim_{l\to\infty}\frac{1}{\rho_{k_l}} d(t_{k_l}, z_{k_l}) = 0,$$

hence,

$$\lim_{l\to\infty}\frac{1}{\rho_{k_l}}\exp^{-1}_{t_{k_l}} z_{k_l} = 0. \tag{50}$$

Since Φ is a Lipschitz continuous vector field and $z_{k_l} \to \overline{p}$ as $l \to \infty$. Combining (49) and (50), we obtain

$$\lim_{l\to\infty}\Upsilon_{k_l} = -\Gamma(\overline{p}). \tag{51}$$

Also, using the fact that Ψ is a maximal monotone vector field, so it is upper Kuratowski semicontinuous. Thus $-\Gamma(\overline{p}) \in \Psi(\overline{p})$, which implies that $\overline{p}$ solves Ω. Lastly by Lemma 1, we obtain that $\{q_k\}$ converges to a point in Ω.

4 Numerical Example

Example 1 Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{P} = (\mathbb{R}_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold with Riemannian metric defined by $\langle p, q\rangle = \frac{1}{x^2}pq \ \in \mathbb{R}_{++},\ p, q \in T_x\mathbb{P}$. The Riemannian distance $d : \mathbb{P}\times\mathbb{P} \to \mathbb{R}_+$ is given by $d(x, y) = |\ln \frac{y}{x}|$ for all $x, y \in \mathbb{P}$. Let $x \in \mathbb{P}$, then the exponential map $\exp_x : T_x\mathbb{P} \to \mathbb{P}$ is defined by $\exp_x sq = x\mathrm{e}^{\frac{qs}{x}}$ for all $q \in T_x\mathbb{P}$. The inverse of the exponential map, $\exp_x^{-1} : \mathbb{P} \to T_x\mathbb{P}$ is defined by $\exp_x^{-1} y = x \ln \frac{y}{x}$ for all $x, y \in \mathbb{P}$. The parallel transport is the identity on $T\mathbb{P}$. Let $\mathcal{K} = (0, 1]$, $\Psi : \mathcal{K} \to \mathbb{R}$ and $\Phi : \mathcal{K} \to T\mathbb{P}$ be defined by $\Psi(x) = x \ln x$ and $\Phi(x) = x(1 + \ln x)$, respectively. Then, Ψ is maximal monotone on $\mathcal{K}$ and Φ is a continuous and monotone vector field on $\mathcal{K}$. By simple calculation, we obtain that t_k in Algorithm 1 can be expressed as

$$t_k = \left(\frac{z_k}{\mathrm{e}^{\rho_k}}\right)^{\frac{1}{1+\rho_k}}, \ \ \rho_k > 0,$$

and $(\Phi + \Psi)^{-1}(0) = \frac{1}{\sqrt{\mathrm{e}}}$. We choose $\alpha_k = \frac{1}{k+1}$, $\theta_k = \frac{1}{2n+3}$, $\lambda_k = \frac{1}{k\sqrt{k}}$, $\mu = \frac{1}{2}$ and $\rho_0 = 0.3$. We terminate the execution of the process at $E_k = d(x_{k+1}, x_k) = 10^{-3}$ and make a comparison of Algorithm 1 with a step inertial and non-accelerated versions of the Algorithm. We test the convergence of the method with some initial values of x_0 and x_1. The result of this experiment is shown in Fig. 1.

Case I: $x_0 = 0.1$ and $x_1 = 0.18$.
Case II: $x_0 = 0.9$ and $x_1 = 0.5$.

Example 2 Let $\mathbb{R}^3_{++} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0, i = 1, 2, 3\}$, $\mathbb{P} = (\mathbb{R}^3_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold with the Riemannian metric is defined by

$$\langle p, q\rangle = pG(x)q^{\mathrm{T}}, \quad x \in \mathbb{R}^3_{++},\ p, q \in T_x\mathbb{R}^3_{++} = \mathbb{R}^3,$$

where $G(x)$ is a diagonal matrix defined $G(x) = \mathrm{diag}(x_1^{-2}, x_2^{-2}, x_3^{-2})$. The Riemannian $d : \mathbb{P} \times \mathbb{P} \to \mathbb{R}_+$ is defined by

$$d(x, y) = \sqrt{\left(\sum_{i=1}^{3} \ln^2 \frac{x_i}{y_i}\right)}, \ \forall\, x, y \in \mathbb{P}.$$

The sectional curvature of the Riemannian manifold $\mathbb{P}$ is 0. Thus $\mathbb{P} = (\mathbb{R}^3_{++}, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold. Let $x = (x_1, x_2, x_3) \in \mathbb{P}$. Then, the exponential map $\exp_x : T_x\mathbb{P} \to \mathbb{P}$ is defined by

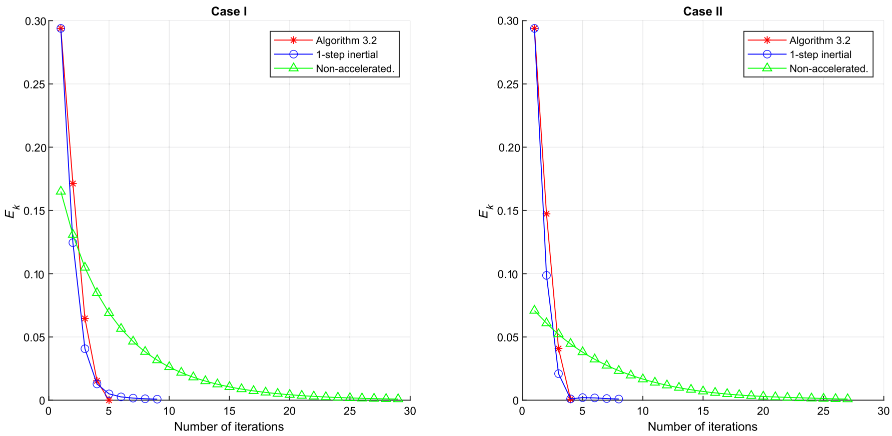

Fig. 1 Numerical report for Example 2

$$\exp_x(p) = \left(x_1 e^{\frac{p_1}{x_1}}, x_2 e^{\frac{p_2}{x_2}}, x_3 e^{\frac{p_3}{x_3}} \right)$$

for all $p = (p_1, p_p, p_3) \in T_x\mathbb{P}$. The inverse of the exponential map, $\exp_x^{-1} : \mathbb{P} \to T_x\mathbb{P}$ is defined by

$$\exp_x^{-1} y = \left(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2}, x_3 \ln \frac{y_3}{x_3} \right)$$

for all $x, y \in \mathbb{P}$. The parallel transport $\Gamma_{y,x} : T_x\mathbb{P} \to T_y\mathbb{P}$ is defined by

$$P_{y,x}(p) = \left(p_1 \frac{y_1}{x_1}, p_2 \frac{y_2}{x_2}, p_3 \frac{y_3}{x_3} \right)$$

for all $p = (p_1, p_2, p_3) \in T_x\mathbb{P}$. Let $\mathcal{K} = \{x = (x_1, x_2, x_3) \in \mathbb{P} : 0 < x_i \leqslant 1,\ \text{for } i = 1, 2, 3\}$ be the geodesic convex subset of $\mathbb{P}$. Let $\Phi : \mathbb{M} \to T\mathbb{P}$ be defined by

$$\Psi(x) = (-x_1, x_2 \ln x_2, 3x_3),\ \forall\, (x_1, x_2, x_3) \in \mathbb{P},$$

and $\Phi : \mathbb{M} \to T\mathbb{P}$ be defined by

$$\Phi(x_1, x_2, x_3) = (x_1 + x_1 \ln x_1, x_2, -3x_1 + 2x_3 \ln 2x_3),\ \forall\, (x_1, x_2, x_3) \in M.$$

Then, Ψ is maximal monotone vector field on $\mathcal{K}$ and Φ is continuous and monotone vector field on $\mathcal{K}$ (see [8, Example 1]). By simple calculation, we see that t_k in Algorithm 1 can be expressed as

$$t_k = \left(t_k^1 e^{\rho_k}, (t_k^2)^{\frac{1}{1+\rho_k}}, t_k^3 e^{-3\rho_k} \right).$$

Note that $(\Psi + \Phi)^{-1}(0) = \{(1, \frac{1}{e}, \frac{1}{2})\}$. Let $\alpha_k = \frac{1}{k+1}$, $\theta = \frac{1}{2k+3}$, $\lambda_k = \frac{1}{k\sqrt{k}}$, $\mu = \frac{1}{2}$ and $\rho_0 = 0.9$. We terminate the execution of the process at $E_k = d(x_{k+1}, x_k) = 10^{-4}$ and make a comparison of Algorithm 1 with one inertial and a non-accelerated versions of the Algorithm. The result of this experiment is shown in Fig. 2 for two initial values of x_0 and x_1.

Case 1: $x_0 = [1.5, 1.5, 1.5]'$ and $x_1 = [1.3, 1.2, 1.1]'$.
Case 2: $x_0 = [1.8, 1.8, 1.8]'$ and $x_1 = [1.5, 1.5, 1.6]'$.

5 Conclusion

In this manuscript, we proposed double inertial methods with a forward–backward method for solving variational inclusion problem in the settings of a Hadamard manifold. We establish a convergence result for solving variational inclusion problem and illustrate some numerical examples to show the performance of our method in comparison with some related ones in the literature. It can be seen from our figures that the two steps inertial extrapolation method illustrated in our manuscript converges faster

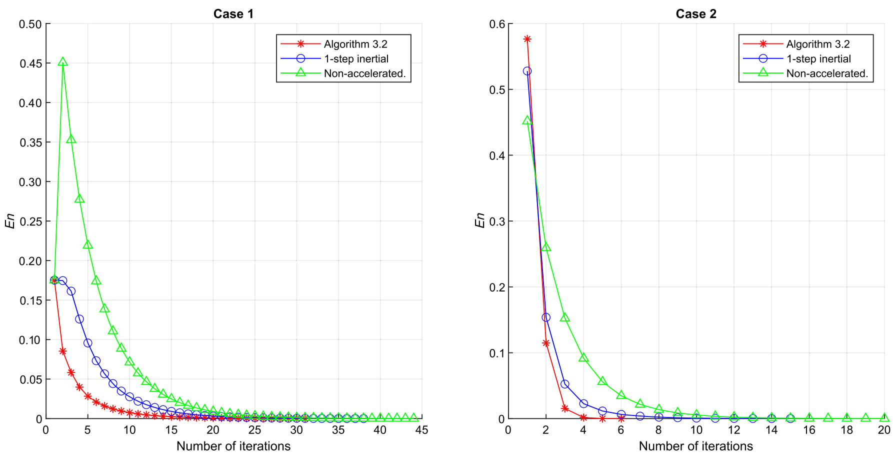

Fig. 2 Numerical report for Example 2

that the one step inertial method and the non-inertial iterative method. This result discussed in this manuscript is new in the settings of a Hadamard manifold.

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Competing interests The authors declare that there is no competing interest.

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