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The projection algorithm for inverse quasi-variational inequalities with applications to traffic assignment and network equilibrium control problems

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ABSTRACT

We consider a first-order dynamical system for solving inverse quasi-variational inequalities in finite dimensional spaces. Through an explicit time discrete version of the proposed dynamical system, we investigate the linear convergence of a projection algorithm under suitable conditions of parameters. Moreover, we investigate the application of inverse quasi-variational inequalities in traffic assignment problem and network equilibrium control problem. The numerical experiments for these practical problems confirm the linear convergence of the theoretical part. In particular, the obtained results provide a positive answer to an open question posted by S. Dey and S. Reich in Optimization, DOI: 10.1080/02331934.2023.2173525 (2023).

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1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and a generated norm $\| \cdot \|$. Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping, we recall the variational inequality problems (VIPs) which consist of finding a point $x^* \in K$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall\, x \in K.$$

The VIPs have been appeared in many theoretical and applied fields, such as optimization problems, complementary problems, saddle-point (min–max) problems, Nash equilibrium problems and fixed point problems [1]. Hence, many methods, especially projection-type methods, have been studied for solving the VIPs (see, for instance [2–7]).

In the case where an explicit formula of F is not available, but defined as the inverse of a given mapping f, i.e. $F(x) = f^{-1}(x) = u$, the VIP becomes an inverse

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variational inequality problem (IVIP) which calls for finding $u^* \in \mathbb{R}^n$ such that

$$f(u^*) \in K \text{ and } \langle u^*, v - f(u^*)\rangle \geq 0, \quad \forall\, v \in K.$$

Although the IVIPs arise in various fields, such as traffic network problems [8] and economic equilibrium problems [9], there is a limited number of theoretical and numerical methods for solving them. Some researchers have paid attention and generalized IVIPs in various ways. This leads to one of the important generalizations of IVIPs called the inverse quasi-variational inequality problems [10,11].

Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed, convex point values and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a single-value mapping. Throughout this article, we consider the inverse quasi-variational inequality problem, denoted by IQVIP, which is to find $x^* \in \mathbb{R}^n$ such that

$$f(x^*) \in \Phi(x^*) \text{ and } \langle x^*, y - f(x^*)\rangle \geq 0 \quad \forall\, y \in \Phi(x^*). \tag{1}$$

In recent years, many researchers have widely studied dynamical systems for solving related optimization problems such as variational inequalities, fixed problems and monotone inclusions (see, e.g. [12–16]). Through the discrete forms of dynamical systems, numerous algorithms have been proposed for solving associated VIPs and monotone inclusions. Motivated by dynamical system approaches, in latest investigation [11], S. Dey and S. Reich have suggested an algorithm for solving IQVIP and simultaneously established linear convergence of the sequence generated by this algorithm under a restricted condition on Φ (which is called the *moving set* condition)

$$\Phi(x) = g(x) + \Omega, \tag{2}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is l-Lipschitz continuous and Ω is a nonempty closed convex subset of $\mathbb{R}^n$. In this paper, we revisit this proposed order dynamical system

$$\begin{cases} \dot{x}(t) + \alpha(t)\left(f(x) - P_{\Phi(x)}(f(x) - \lambda x)\right) = 0, \\ x(0) = x_0, \end{cases} \tag{3}$$

where x_0 is an arbitrary in $\mathbb{R}^n$ and study the projection algorithm derived from the discretization of (3)

$$x_{n+1} = x_n + \alpha_n \left[P_{\Phi(x_n)}(f(x_n) - \lambda x_n) - f(x_n)\right]. \tag{4}$$

The linear convergence of (4) was obtained in [11, Theorem 6.1] under the *moving set* condition (2) on Φ and the following question was posted:

Does Theorem 6.1 hold for more general set-valued mappings? This question remains open.

The first aim of this paper is to provide a positive answer for this question. We will prove the linear convergence of sequence generated by (4) to the unique solution for general set-valued mapping Φ.

The second aim of this paper is to investigate the applications of QIVIs in road pricing problems [8] and network equilibrium control problem [17]. We emphasize that in previous works, researchers simulated and tackled these problems in the form of IVI problems with the fixed constrain for link flows in traffic network (for road pricing problem) or amount of production and consumption (for network equilibrium control problem). However in the real life, these problems can become more complicated. For example, as mentioned in [10] for road pricing problem, we assume that the effects of traffic flow on environment are a function predetermined through statistics of data. First, the policy-makers attempt to manage the environment impact by limiting the traffic flow within a range of certain lower and upper bounds. To do this, some tolls are imposed on some lines in network. However, the taxes also effect traffic flows, hence cause the change on environment impact. Thus it is difficult to maintain the previous fixed bounds on environment impact function. Therefore, the policy-makers expect to adjust environment impact constraints depending on the imposed tolls on lines in network, which lead to a model of IQVI problem. This is a main motivation making us generalize the IVI problems to IQVI problems and study the algorithms for solving them. In our numerical experiment of road pricing problem, we take the flexible constraint of link flow function into account instead of environment impact function for simplicity. In addition, we discuss why the IQVI problem is more appropriate than IVI problem to express the network equilibrium control problem in the practical model.

In Section 2, we recall some basic definitions and results. In Section 3, we establish the linear convergence of the sequence generated by the projection algorithm. Finally, we discuss applications and provide numerical examples in Section 4.

2. Preliminaries

We will recall some definitions about Lipschitz continuity and monotonicity of f as follows (see, e.g. [18]):

- f is L-Lipschitz continuous on $\mathbb{R}^n$ if there exists $L > 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^n$.
- f is monotone on $\mathbb{R}^n$ if for any $x, y \in \mathbb{R}^n$ we have $\langle f(x) - f(y), x - y\rangle \geq 0$.
- f is γ-strongly monotone on $\mathbb{R}^n$ if for any $x, y \in \mathbb{R}^n$ we have $\langle f(x) - f(y), x - y\rangle \geq \gamma\|x - y\|^2$.Obviously, if f is strongly-monotone on $\mathbb{R}^n$, then f is monotone on $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ and K is a nonempty closed convex subset of $\mathbb{R}^n$. The metric projection of x on K, denoted by $P_K(x)$, is a unique element of K such that $\|x - P_K(x)\| \leq \|x - y\|$ for all $y \in K$. We recall two important properties of metric projection as follows [19].

Theorem 2.1: *For any $x, z \in \mathbb{R}^n$ we have*

(a) $\|P_K(x) - P_K(z)\| \leq \|x - z\|$ *(nonexpansivity of $P_K(.)$);*
(b) $\langle x - P_K(x), y - P_K(x)\rangle \leq 0, \ \forall\, y \in K.$

Remark 2.1: To the IQVIP (1), in the case where the set-valued mapping $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ has nonempty, closed and convex point values, it is not difficulty to check that x^* is a solution of IQVIP (1) if and only if it is a solution to the projection equation

$$f(x) = P_{\Phi(x)}(f(x) - \lambda x),$$

where $\lambda > 0$ is a fixed constant.

We recall conditions to establish the existence and uniqueness of solution to the inverse quasi-variational inequality problem (IQVIP) (1) (see [11], Theorem 3.2).

Theorem 2.2: *Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed and convex point values and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a L-Lipschitz continuous and γ-strongly monotone mapping. If there exists $\kappa > 0$ such that*

$$\|P_{\Phi(x)}(z) - P_{\Phi(y)}(z)\| \leq \kappa\|x - y\| \quad \forall\, x, y, z \in \mathbb{R}^n$$

and

$$\sqrt{L^2 - 2\gamma\lambda + \lambda^2} + \kappa < \lambda,$$

where $\lambda > 0$ is a constant, then the inverse quasi-variational inequality problem (1) has a unique solution.

The existence and uniqueness of the trajectory of dynamical system (3) are stated in the following result (see [11], Theorem 4.1).

Theorem 2.3: *Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed and convex point values and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a L-Lipschitz continuous mapping. Assume that the parameter $\alpha(t) \in C([0, +\infty))$, the set of all continuous functions from $[0, +\infty)$ into itself, and there exists a number $\kappa > 0$ such that*

$$\|P_{\Phi(x)}(z) - P_{\Phi(y)}(z)\| \leq \kappa\|x - y\| \quad \forall\, x, y, z \in \mathbb{R}^n.$$

Then there exists a unique solution of the dynamical system (3).

3. Linear convergence analysis

A finite-difference scheme for dynamical system (3) with respect to time variable t, with stepsize $h_n > 0$ and initial point x_0, yields the following iterative scheme:

$$\frac{x_{n+1} - x_n}{h_n} = \alpha_n \{P_{\Phi(x)}(f(x_n) - \lambda x_n) - f(x_n)\},$$

which is equivalent to

$$x_{n+1} = x_n + \alpha_n h_n \{P_{\Phi(x)}(f(x_n) - \lambda x_n) - f(x_n)\}.$$

If $h_n = 1$, we can rewrite above scheme as

$$x_{n+1} = x_n + \alpha_n \left[P_{\Phi(x_n)}(f(x_n) - \lambda x_n) - f(x_n)\right]. \tag{5}$$

The linear convergence of the projection method (5) is established as follows.

Theorem 3.1: *Let $\Phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed and convex point values and $f : \mathbb{R}^n \to \mathbb{R}^n$ be L-Lipschitz continuous and γ-strongly monotone. Assume that*

$$\eta := \gamma - \kappa - \frac{1}{2} - \frac{1}{2}L^2 - \frac{1}{2}\lambda^2 + \lambda\gamma > 0, \tag{6}$$

where κ satisfies

$$\|P_{\Phi(x)}(z) - P_{\Phi(y)}(z)\| \leq \kappa\|x - y\| \quad \forall\, x, y, z \in \mathbb{R}^n \tag{7}$$

and

$$\sqrt{L^2 - 2\gamma\lambda + \lambda^2} + \kappa < \lambda. \tag{8}$$

Let $\{\alpha_n\}$ in scheme (5) such that

$$0 < M < \alpha_n < N, \quad \text{where} \tag{9}$$

$$0 < \frac{N^2}{M} < \frac{2\eta}{(2L + \kappa + \lambda)^2}. \tag{10}$$

Then the sequence $\{x_n\}$ generated by algorithm (5) converges linearly to the unique solution of the IQVIP (1).

Proof: First, under the conditions (7) and (8), it follows from Theorem 2.2 that the IQVIP (1) has a unique solution, denoted by x^*. On the one hand, using the nonexpansiveness of the projection and the Lipschitz continuity of f, for $x \in \mathbb{R}^n$

we have

$$\begin{aligned}
&\| \big(f(x) - P_{\Phi(x)}(f(x) - \lambda x)\big) - \big(f(x^*) - P_{\Phi(x^*)}(f(x^*) - \lambda x^*)\big) \| \\
&\quad \le \|f(x) - f(x^*)\| + \|P_{\Phi(x)}(f(x) - \lambda x) - P_{\Phi(x^*)}(f(x^*) - \lambda x^*)\| \\
&\quad \le L\|x - x^*\| + \|P_{\Phi(x)}(f(x) - \lambda x) - P_{\Phi(x)}(f(x^*) - \lambda x^*)\| \\
&\qquad + \|P_{\Phi(x)}(f(x^*) - \lambda x^*) - P_{\Phi(x^*)}(f(x^*) - \lambda x^*)\| \\
&\quad \le L\|x - x^*\| + \|(f(x) - \lambda x) - (f(x^*) - \lambda x^*)\| + \kappa\|x - x^*\| \\
&\quad \le (L + \kappa)\|x - x^*\| + \|f(x) - f(x^*)\| + \lambda\|x - x^*\| \\
&\quad \le (2L + \kappa + \lambda)\|x - x^*\|.
\end{aligned} \tag{11}$$

On the other hand, since f is L-Lipschitz and γ-strongly monotone, κ satisfies (7), we have

$$\begin{aligned}
&\langle f(x) - P_{\Phi(x)}(f(x) - \lambda x), x - x^*\rangle \\
&\quad = \langle \big(f(x) - P_{\Phi(x)}(f(x) - \lambda x)\big) - \big(f(x^*) - P_{\Phi(x^*)}(f(x^*) - \lambda x^*)\big), x - x^*\rangle \\
&\quad = \langle f(x) - f(x^*), x - x^*\rangle - \langle P_{\Phi(x)}(f(x) - \lambda x) \\
&\quad - P_{\Phi(x^*)}(f(x^*) - \lambda x^*), x - x^*\rangle \\
&\quad \ge \gamma\|x - x^*\|^2 - \langle P_{\Phi(x)}(f(x) - \lambda x) - P_{\Phi(x)}(f(x^*) - \lambda x^*), x - x^*\rangle \\
&\qquad - \langle P_{\Phi(x)}(f(x^*) - \lambda x^*) - P_{\Phi(x^*)}(f(x^*) - \lambda x^*), x - x^*\rangle \\
&\quad \ge \gamma\|x - x^*\|^2 - \langle P_{\Phi(x)}(f(x) - \lambda x) - P_{\Phi(x)}(f(x^*) - \lambda x^*), x - x^*\rangle \\
&\qquad - \|P_{\Phi(x)}(f(x^*) - \lambda x^*) - P_{\Phi(x^*)}(f(x^*) - \lambda x^*)\|\|x - x^*\| \\
&\quad \ge \gamma\|x - x^*\|^2 - \frac{1}{2}\|x - x^*\|^2 - \frac{1}{2}\|P_{\Phi(x)}(f(x) - \lambda x) \\
&\qquad - P_{\Phi(x)}(f(x^*) - \lambda x^*)\|^2 - \kappa\|x - x^*\|^2 \\
&\quad \ge \left(\gamma - \kappa - \frac{1}{2}\right)\|x - x^*\|^2 - \frac{1}{2}\|(f(x) - \lambda x) - (f(x^*) - \lambda x^*)\|^2 \\
&\quad = \left(\gamma - \kappa - \frac{1}{2}\right)\|x - x^*\|^2 - \frac{1}{2}\|f(x) - f(x^*)\|^2 - \frac{1}{2}\lambda^2\|x - x^*\|^2 \\
&\quad + \lambda\langle f(x) - f(x^*), x - x^*\rangle \\
&\quad \ge \left(\gamma - \kappa - \frac{1}{2} - \frac{1}{2}L^2 - \frac{1}{2}\lambda^2 + \lambda\gamma\right)\|x - x^*\|^2,
\end{aligned} \tag{12}$$

or equivalently

$$\langle P_{\Phi(x)}(f(x) - \lambda x) - f(x), x - x^*\rangle \le -\eta\|x - x^*\|^2. \tag{13}$$

Note that $f(x^*) - P_{\Phi(x^*)}(f(x^*) - \lambda x^*) = 0$, combining (11) and (12), we obtain

$$\begin{aligned}
&\|f(x) - P_{\Phi(x)}(f(x) - \lambda x)\|^2 \\
&\leq (2L + \kappa + \lambda)^2 \|x - x^*\|^2 \\
&\leq \frac{(2L + \kappa + \lambda)^2}{\eta} \langle f(x) - P_{\Phi(x)}(f(x) - \lambda x), x - x^* \rangle \\
&= -\frac{(2L + \kappa + \lambda)^2}{\eta} \langle P_{\Phi(x)}(f(x) - \lambda x) - f(x), x - x^* \rangle.
\end{aligned} \tag{14}$$

Let $\{x_n\}$ be the sequence generated by the scheme (5), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|x_n - x^* + \alpha_n \left[P_{\Phi(x_n)}(f(x_n) - \lambda x_n) - f(x_n)\right]\|^2 \\
&= \|x_n - x^*\|^2 + \alpha_n^2 \|P_{\Phi(x_n)}(f(x_n) - \lambda x_n) - f(x_n)\|^2 \\
&\quad + 2\alpha_n \langle x_n - x^*, P_{\Phi(x_n)}(f(x_n) - \lambda x_n) - f(x_n) \rangle.
\end{aligned} \tag{15}$$

Setting $y_n = P_{\Phi(x_n)}(f(x_n) - \lambda x_n)$ and combining with (14), it follows from (15) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \alpha_n^2 \frac{(2L + \kappa + \lambda)^2}{\eta} \langle x_n - x^*, y_n - f(x_n) \rangle \\
&\quad + 2\alpha_n \langle x_n - x^*, y_n - f(x_n) \rangle \\
&= \|x_n - x^*\|^2 + \left(2\alpha_n - \alpha_n^2 \frac{(2L + \kappa + \lambda)^2}{\eta}\right) \langle x_n - x^*, y_n - f(x_n) \rangle \\
&\leq \|x_n - x^*\|^2 - \eta \left(2\alpha_n - \alpha_n^2 \frac{(2L + \kappa + \lambda)^2}{\eta}\right) \|x_n - x^*\|^2 \\
&= \left(1 - \eta\alpha_n \left(2 - \alpha_n \frac{(2L + \kappa + \lambda)^2}{\eta}\right)\right) \|x_n - x^*\|^2,
\end{aligned} \tag{16}$$

where the last inequality is deduced from (9), (10) and (13). For every $\alpha_n > 0$, let $T(\alpha_n)$ defined by

$$T(\alpha_n) = \sqrt{1 - \eta\alpha_n \left(2 - \alpha_n \frac{(2L + \kappa + \lambda)^2}{\eta}\right)},$$

we can rewrite (16) as

$$\|x_{n+1} - x^*\| \leq T(\alpha_n) \|x_n - x^*\|.$$

Using the above inequality n times, we obtain

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{n} T(\alpha_k) \|x_0 - x^*\|. \tag{17}$$

Using the conditions (9) and (10), we get

$$\begin{aligned} T(\alpha_k)^2 &= 1 - 2\eta\alpha_k + (2L+\kappa+\lambda)^2\alpha_k^2 \\ &< 1 - 2\eta M + (2L+\kappa+\lambda)^2 N^2 = r < 1. \end{aligned} \tag{18}$$

Combining (17) with (18), we obtain

$$0 \leq \|x_{n+1} - x^*\| \leq r^{n/2}\|x_0 - x^*\|,$$

which implies that $\{x_n\}$ converges linearly to solution x^*. ■

Remark 3.1: In particular, we choose $\alpha_n = \alpha$ is a constant. From above proof, we obtain that the sequence $\{x_n\}$ converges linearly to the solution x^* with the rate $r = \sqrt{1 - \eta\alpha(2 - \alpha\frac{(2L+\kappa+\lambda)^2}{\eta}})$. Obviously, the rate r attains its minimum value when $q(\alpha) = 2\eta\alpha - (2L+\kappa+\lambda)^2\alpha^2$ takes its biggest value on the interval $(0, +\infty)$. It is not difficult to verify that $q(\alpha)$ attains its maximum value at $\alpha^* = \frac{\eta}{(2L+\kappa+\lambda)^2}$ and the best value of r is $r^* = \sqrt{1 - \frac{\eta^2}{(2L+\kappa+\lambda)^2}}$.

We demonstrate two small examples in $\mathbb{R}^2$ to show the linear convergence of the sequences generated by algorithm (5).

Example 3.1: In $\mathbb{R}^2$, let $\Phi_1(x_1, x_2) = R(x_1, x_2)$ where $R(x_1, x_2)$ is the closed rectangle restricted by four lines $x = |x_1|, x = -|x_1|, y = |x_2|, y = -|x_2|$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x) = Ax$ where $\left[A = \begin{smallmatrix} 3.2 & 2 \\ -0.6 & 1 \end{smallmatrix}\right]$. We consider the IQVIP (1).

First, we can easily prove that Φ satisfies (7) with $\kappa = 1$, which means

$$\|P_{\Phi_1(x)}(z) - P_{\Phi_1(y)}(z)\| \leq \|x - y\| \quad \forall\, x, y, z \in \mathbb{R}^2.$$

Besides, since the matrix $A = \left[\begin{smallmatrix} 3.2 & 2 \\ -0.6 & 1 \end{smallmatrix}\right]$ is positive definite with eigenvalues are 2.2 and 2, f is 2.2-Lipschitz continuous and 2-strongly monotone. Therefore we obtain $L = 2.2$ and $\gamma = 2$. We choose $\lambda = 2$ and verify the condition (6) by

$$\eta := \gamma - \kappa - \frac{1}{2} - \frac{1}{2}L^2 - \frac{1}{2}\lambda^2 + \lambda\gamma = 0.08 > 0.$$

It is easy to check that $(0, 0)$ is a solution of the IQVIP (5). Moreover, since

$$\lambda - \sqrt{L^2 - 2\gamma\lambda + \lambda^2} - \kappa \approx 0.083 > 0,$$

by Theorem 2.2, we obtain that the IQVIP (1) has a unique solution. Therefore, $(0, 0)$ is a unique solution of the IQVIP (1). We choose the parameter $\alpha_n = \alpha = 0.00146 < \frac{\eta}{(2L+\kappa+\lambda)^2}$ satisfying (9). Figure 1 illustrates the points generated by scheme (5) to the unique solution $x^* = (0, 0)$ with initial points $(5, 3), (-5, 3), (5, -3)$ and $(-5, -3)$.

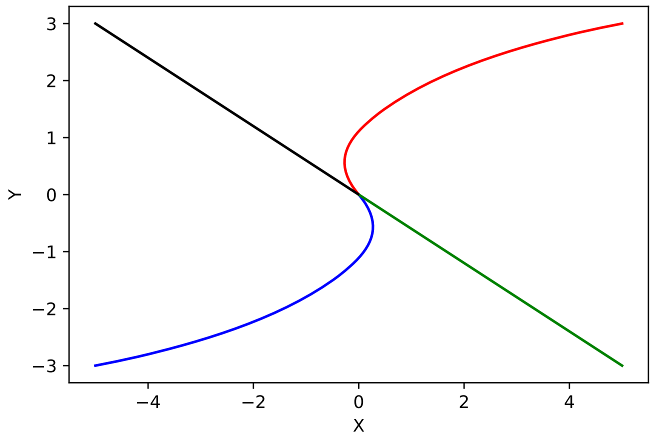

Figure 1. Performance of sequences generated by scheme (5) for different starting points.

Example 3.2: In $\mathbb{R}^2$, let $\Phi_2(x_1, x_2) = OR(x_1, x_2)$ where $OR(x_1, x_2)$ is the closed rectangle restricted by four lines $x = x_1, x = 0, y = x_2, y = 0$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x) = Ax$ where $A = \begin{bmatrix} 3 & 4 \\ -0.2 & 1.2 \end{bmatrix}$. We consider the IQVIP (1).

First, we can easily prove that Φ_2 satisfies (7) with $\kappa = 1$, which means

$$\|P_{\Phi_2(x)}(z) - P_{\Phi_2(y)}(z)\| \le \|x - y\| \quad \forall\, x, y, z \in \mathbb{R}^2.$$

Besides, since the matrix $\left[A = \begin{smallmatrix} 3 & 4 \\ -0.2 & 1.2 \end{smallmatrix}\right]$ is positive definite with eigenvalues are 2.2 and 2, f is 2.2-Lipschitz continuous and 2-strongly monotone. Therefore, we obtain $L = 2.2$ and $\gamma = 2$. We choose $\lambda = 2$ and verify the condition (6) by

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It is easy to check that $(0, 0)$ is a solution of the IQVIP (5). Moreover, since

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4. Applications and numerical experiments

4.1. *Traffic assignment problems*

In this section, we demonstrate a practical example as in [8,20] which is involved in traffic assignment, called road pricing problem. After that, we will

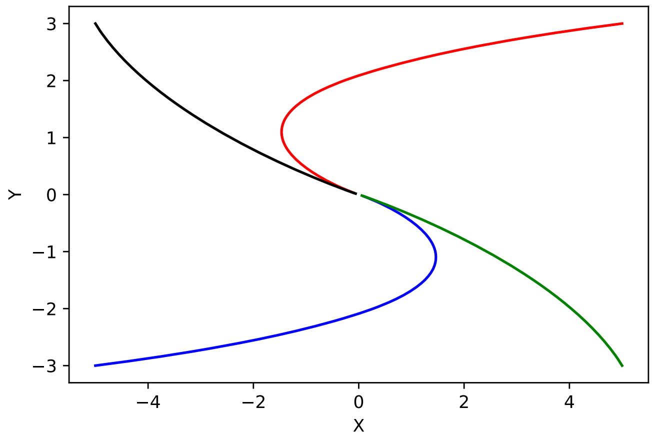

Figure 2. Performance of sequences generated by scheme (5) for different starting points.

use algorithm (5) to find the solution for this problem. Particularly, we take the continuous time road pricing into account, in which the government wants to manage some vehicle flows f_i on special links i in the network by charging the extra tolls x_i (with $i \in I$ is the subset of links in network). More details, initially the authority hope that the imposed taxes will control the link flows in predetermined range. However, as we explained in introduction, the government prefers to control the link flows within a range depending in imposed tolls. As mathematical interpreter, we can express that the link flow must satisfy: $f(x) \in \Phi(x)$, where $\Phi(x)$ is the multi-value function of charged tolls defined as $\Phi(x) = \{y : g_i(x) \leq y_i \leq h_i(x)\} = \{y : g(x) \leq y \leq h(x)\}$ (note that from now on, we use the notation '$\leq$' and '$\geq$' in $\mathbb{R}^n$ with point-wise order meaning). From that, we can understand the continuous pricing problem as: Find $x_i(t)$ such that $\lim_{t\to\infty} f(x(t)) \in \Phi(x(t))$. Using the discussions in [20, Section 5], we similarly obtain the Lagrangian function as

$$L(f, \nu, \theta) = \sum_{i\in I}[\nu_i(h_i - f_i)(x) + \theta_i(f_i - g_i)(x)],$$

where ν and θ are multiplier vectors. It follows from the Karush–Tuhn–Tucker (KKT) optimally condition that the equilibrium traffic flows f_i^* must satisfy

$$\begin{cases} \nu_i^* \geq 0; \quad h_i - f_i^* \geq 0; \quad \nu_i^*(h_i - f_i^*) = 0 \quad \forall\, i \in I, \\ \theta_i^* \geq 0; \quad f_i^* - g_i \geq 0; \quad \theta_i^*(f_i^* - g_i) = 0 \quad \forall\, i \in I. \end{cases} \tag{19}$$

We observe that $\nu_i > 0$ and $\theta_i > 0$ cannot occur simultaneously (at that time, $g_i = f_i = h_i$), so we can define the control variable (i.e. toll) as $x_i = \nu_i - \theta_i$ for each link $i \in I$. From (19), we have that, if $f_i^* = h_i$, then $f_i^* > g_i$, then $\theta_i^* = 0$. Since $x_i^* = \nu_i^* - \theta_i^*$, we have $x_i^* \geq 0$. On the other hand, if $f_i^* = g_i$, then $f_i^* < h_i$. Using (19) again, we can deduce $\nu_i^* = 0$. Since $x_i^* = \nu_i^* - \theta_i^*$, we get $x_i^* \leq 0$.

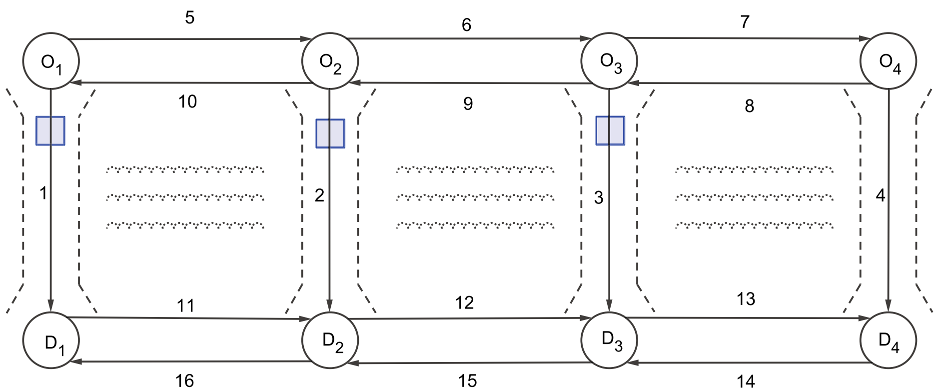

Figure 3. Road pricing problem with four bridge network.

Obviously, if $g_i < f_i^* < h_i$, then $\nu_i^* = \theta_i^* = 0$ or $x_i^* = 0$. In summary, in each situation, we always obtain $(y - f(x^*))^T x^* \leq 0$, $\forall y \in \Phi(x^*)$. Hence, this problem can be rewritten as: find the toll x^* such that

$$f(x^*) \in \Phi(x^*) \text{ and } (y - f(x^*))^T x^* \leq 0, \quad \forall\, y \in \Phi(x^*).$$

We will transfer above problem to basic form of IQVI problem (1). Set $F = -f$, we can rewrite above problem as

$$F(x^*) \in -\Phi(x^*) \text{ and } (y + F(x^*))^T x^* \leq 0, \quad \forall\, y \in \Phi(x^*),$$

or equivalently

$$F(x^*) \in -\Phi(x^*) \text{ and } (y - F(x^*))^T x^* \geq 0, \quad \forall\, y \in -\Phi(x^*). \tag{20}$$

We employ the detail traffic network in [8] shown in Figure 3. This network includes 8 nodes and 16 links connecting these nodes. As we can see, the links 1, 2, 3, 4 are four bridges, which connecting origin O_i with destination D_j. With the demands between OD pairs given in Table 1, we can see that the old bridge 2 is overload and the new bridge 3 is still have room for more vehicles in range of its capacity. Hence, the government's purpose is managing the flows in three links 1, 2, 3 (e.g.: reduces the flows in link 2 and increase the flows in link 3) by imposing tolls on them. In this case, we assume that the authorities attempt to maintain the link flow f satisfying $g(x) \leq f(x) \leq h(x)$, where $g(x) = x + G$ and $h(x) = x + H$, where $G = (40, 0, 100)^T$ and $H = (90, 50, 200)^T$. We verify easily that the $\Phi(x)$ satisfying the condition (7). As we can see in algorithm (5), we need the value of link flows f_i for calculating the next imposed tolls. In the real world, we can collect this data through observing the vehicle volumes passing the bridges after tolls are imposed. Assume that traffic flows satisfy user's equilibrium and the link performance function is strongly monotone. In this study, to simplify, we just compute the value of link flows by solving a fixed-demand user equilibrium traffic assignment. In this traffic assignment problem, the data

Table 1. Origin–destination demand table.

Demand	O_1	O_2	O_3	O_4
D_1	60	30	20	15
D_2	50	160	45	30
D_3	20	30	20	10
D_4	20	15	15	40

Table 2. Link free flow travel time and capacity.

Link i	1	2	3	4	5	6	7	8	9	10
t_i^0	60	40	60	20	20	20	20	20	20	20
c_i	150	100	300	200	300	300	300	300	300	300

about OD demands, the free flow travel times and the link capacities are given in Tables 1 and 2, respectively. Note that the free flow travel times and the link capacities in link 5-16 are the same for each term. To describe the impact of imposed tolls on link flows, we suppose the cost of a vehicle passing bridges i is the sum of travel time and imposed toll on this link. The relationship between link travel time t_i with the link flows f_i is the Bureau of Public Roads (BPR) function:

$$t_i(f_i) = t_i^0 \left[1 + 0.15 \left(\frac{f_i}{c_i} \right)^4 \right],$$

where f_i, t_i^0 and c_i denote link flow, free flow travel time and capacity on link i, respectively. Except for neighbourhood of zero, the BPR function satisfies the strongly monotone property. Hence, we will implement Algorithm (5) to solve the dynamic road pricing problem.

In our numerical experiment, we use fixed stepsize λ and scaling factor α. Note that $f = -F, P_{-\Phi(x_n)}(F(x_n) - \lambda x_n) = -P_{\Phi(x_n)}(f(x_n) + \lambda x_n)$. Thus, for the IQVI problem in (20), the projection algorithm (5) becomes

$$\begin{aligned} x_{n+1} &= x_n + \alpha[P_{-\Phi(x_n)}(F(x_n) - \lambda x_n) - F(x_n)] \\ &= x_n + \alpha[f(x_n) - P_{\Phi(x_n)}(f(x_n) + \lambda x_n)]. \end{aligned}$$

Recall that in this case the solution x^* must satisfy the projection equation

$$F(x) = P_{-\Phi(x)}(F(x) - \lambda x),$$

which is equivalent to

$$f(x) = P_{\Phi(x)}(f(x) + \lambda x).$$

Therefore, we define the $r_n = \|\alpha(P_{\Phi(x_n)}(f(x_n) + \lambda x_n) - f(x_n))\|$ as the residual of the projection algorithm (5) and use it to demonstrate the convergence rate of this algorithm.

In first experiment, we apply the projection algorithm (5) with $\lambda = 0.5$ and $\alpha = 0.02$ and the result is depicted in Figure 4. In roundly initial 50 time step,

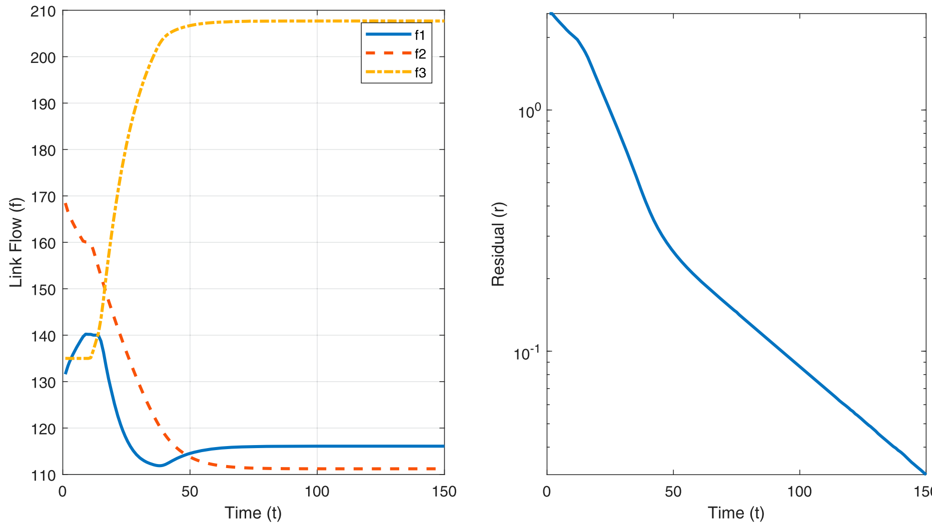

Figure 4. Flows on three bridges and convergence rate of the projection algorithm (5) with $\lambda = 0.5$ and $\alpha = 0.02$.

the traffic flows on three bridges change considerably. After that, the link flow on each bridge becomes more stable and after the step 90, the residual becomes lower than 0.1. The tolls in three bridges are 25.24, 60.36 and 6.84 correspondingly with the flows 116.10, 111.22 and 207.69. At the time step 150, we can observe that the residual is approximately 0.03 and the tolls x_i almost satisfy authority's condition with tiny differences. In comparison with toll on bridge 3, the toll on bridge 2 is significantly higher, therefore meets the requirement from policy-makers which is to restrict the vehicles flow on bridges 2 (from 170 to 111.22). Meanwhile, the toll in bridge 1 is reasonable to keep the link flow on bridge 1 in the range of its capacity. In the real life, the authority can adjust the condition Φ flexibly to match certain situation, such as reduce the link flow on bridge 2 to its maximum capacity, and simultaneously increase the link flows on bridges 1 and 3 to acceptable level.

In the second experiment, we run the projection algorithm (5) with larger step size $\lambda = 0.8$ and larger scalar factor $\alpha = 1/30$ and the result is shown in Figure 5. In comparison with the first, three of bridge flows converge considerably faster. Especially, they change dramatically with first 20 time steps. After about 35 time steps, the trajectories of three of link flows are stable and the residual becomes lower than 0.1 after 70 time steps. Besides, at the time step 150, the residual is relatively 0.005 which is significantly more precise than that in first experiment and the tolls meet the authority's requirement. Therefore, we can conclude that our second experiment is better than the first one.

If we keep the stepsize $\lambda = 0.5$ in first experiment and choose a larger scalar $\alpha = 1/6$, our bridge flows does not converge as shown in Figure 6. Note that from

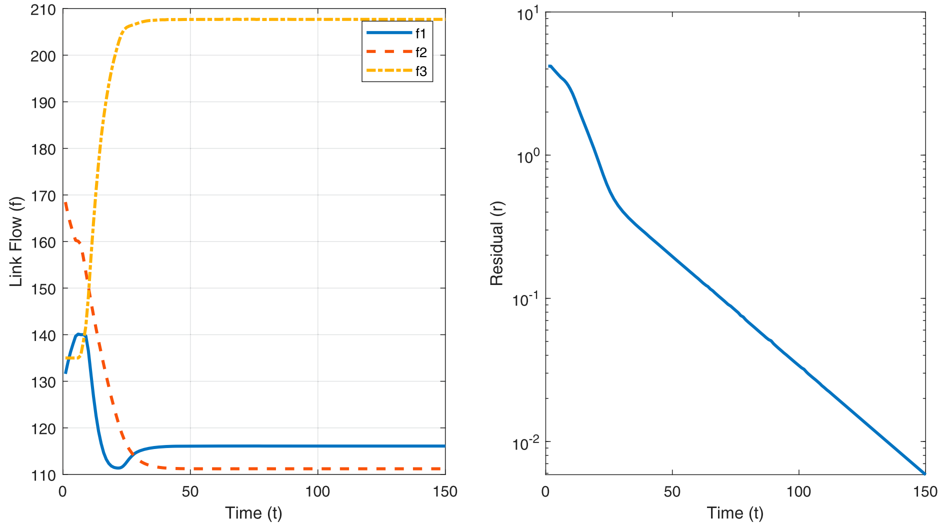

Figure 5. Flows on three bridges and convergence rate of the projection algorithm (5) with $\lambda = 0.8$ and $\alpha = 1/30$.

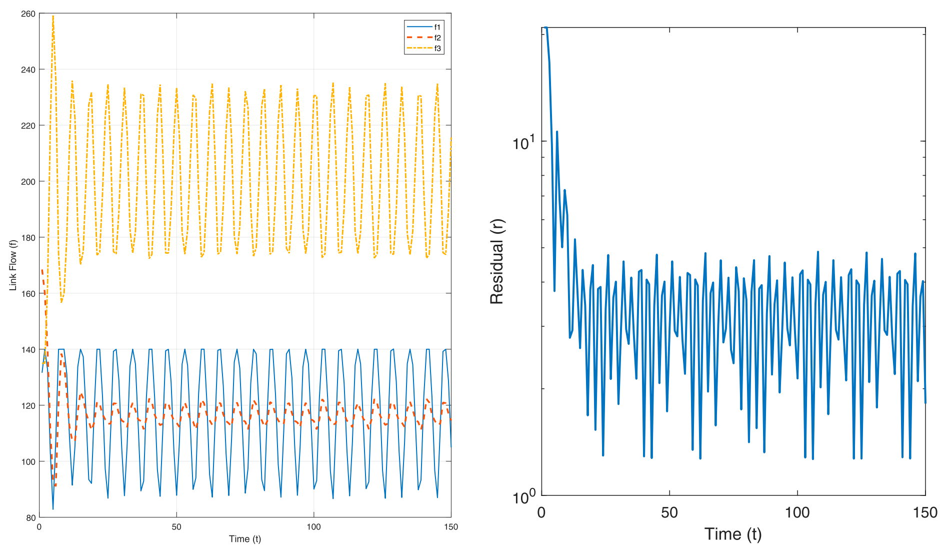

Figure 6. Flows on three bridges and convergence rate of the projection algorithm (5) with $\lambda = 0.5$ and $\alpha = 1/6$.

the conditions (9) and (10), we can deduce that

$$\alpha < \frac{\alpha^2}{M} < \frac{N^2}{M} < \frac{2\eta}{(2L + \kappa + \lambda)^2}. \tag{21}$$

Thus α in this experiment breaks above condition and causes the instability of trajectories of link flows.

4.2. Network equilibrium control problems

At first, we recall the spatial price problem which received many attentions from researchers, such as Nagurney [21,22], He et al. [17] and Thore [23]. In this problem, one seeks to compute the commodity supply prices, demand prices and costs from trade or transportation. The equilibrium condition states that the demand price is equal to the supply price plus cost from trade or transportation when we already know both supply and demand functions. When the demand price is smaller than supply price plus transportation cost, there will be no profit and merchants will not decide to trade. Thus this problem can be considered as the optimization problem of merchant's profit.

However, sometimes the authority wants to control the commodity flows from supply to demand markets for special reasons. For example, the government usually imposes tax or subsidy to adjust the amount of import or export of some special kinds of food for ensuring national food security. Another example is in the national energy problems, where the authority always attempts to encourage the production and consumption of clean energy and restrict the production and consumption of non-renewable resources. Hence, to reach such goal, one possible way for the policy-maker is to impose higher tax to reduce the production at supply markets and consumption at demand market or provide subsidy to get the opposite thing. Therefore, from the government's perspective, this problem becomes the network equilibrium problem in which the objective they attempt to control is the amount of trading through the policy implemented on supply and demand markets. Researchers are also interested in the models where the tax or subsidy works as policy intervention. For example, in the model of He et al. [17], they expect to control the amount of supply and demand resources in a given range, then formulate this control problem to an inverse variational inequality problem. In our study, we expect further to control the amount of demand and supply in a range depending in the imposed tax, which is more flexible in practical problem. After that, we rewrite this network control equilibrium problem as an inverse quasi-variational inequality problem and implement the projection algorithm (5) for solving it. This is the novelty value in our model.

Next, we will introduce a bipartite market equilibrium depicted in Figure 7 for one commodity. There are m supply markets and n demand markets denoted by S_i, $i = 1, 2, \dots, m$ and D_j, $j = 1, 2, \dots, n$, respectively. To simplify, for $i = 1, \dots, m$ and $j = 1, \dots, n$, we use following notations:

- y_{ij}: the amount of commodity transported from S_i to D_j
- s_i: amount of production at the supply market S_i
- d_j: amount of consumption at the demand market D_j

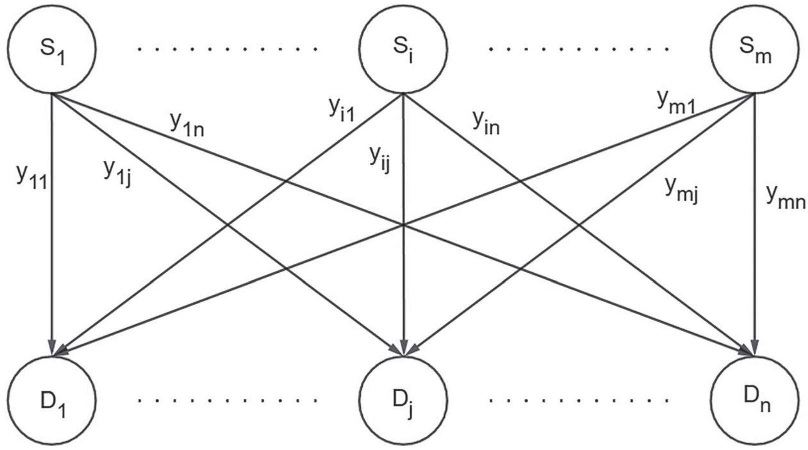

Figure 7. Bipartite market network equilibrium model.

- p_i, q_j : tax of the commodity on the supply market S_i and demand market D_j, respectively
- t_{ij}: the transportation cost from supply market S_i to demand market D_j
- k_i: purchase price of the commodity at supply market S_i
- l_j: sale price of the commodity at demand market D_j.

Obviously, the amount of commodity at supply market S_i is equal to sum of commodity shipments from S_i to all demand markets D_j. Hence $s_i = \sum_{j=1}^{n} y_{ij}$. Similarly, $d_j = \sum_{i=1}^{m} y_{ij}$. Besides, commodity's prices at the supply and demand markets are effected by the amount of production and consumption, respectively; and the costs of transportation t_{ij} depend on the amount of commodity shipment y_{ij}. Therefore, we assume $k_i = k_i(s_i)$, $l_j = l_j(d_j)$ and $t_{ij} = t_{ij}(y_{ij})$.

As above-mentioned, the market equilibrium condition means that between a pair of markets, if the purchase price at the supply market plus cost (consist of tax and transportation cost) is equal to the sale price at the demand market minus the commodity tax, then there is trade. Besides, if the purchase price at the supply market plus cost is higher than the sale price at the demand market minus the commodity tax, then there is no trade. Thus we express the equilibrium condition as follows:

$$k_i + p_i + t_{ij} \begin{cases} = l_j - q_j, & \text{if } y_{ij} \neq 0, \\ \geq l_j - q_j, & \text{if } y_{ij} = 0. \end{cases} \tag{22}$$

We can rewrite the condition (22) as

$$y_{ij} \geq 0, \quad y_{ij}^T(k_i + p_i + t_{ij} - (l_j - q_j)) \geq 0, \quad \forall\, i = \overline{1, m}, j = \overline{1, n}. \tag{23}$$

Let $E_n = (1, 1, \dots, 1)$ is a row matrix with n elements equal to 1 and I_n is the identity matrix in $\mathbb{R}^{n\times n}$. We denote

$$A = \begin{pmatrix} E_n & 0 & \dots & 0 \\ 0 & E_n & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & E_n \end{pmatrix} \in \mathbb{R}^{m\times mn} \quad \text{and} \quad B = (I_n, I_n, \dots, I_n) \in \mathbb{R}^{n\times mn}.$$

Since $s_i = \sum_{j=1}^{n} y_{ij}$ and $d_j = \sum_{i=1}^{m} y_{ij}$, we also demonstrate the supply and demand as

$$s = Ay \quad \text{and} \quad d = By.$$

Thus we rewrite the condition (23) as the form of a variational inequality problem

$$y \geq 0, \quad (y' - y)^T\{t(y) + A^T[k(s) + p] - B^T[l(d) - q]\} \geq 0, \; \forall\, y' \geq 0. \quad (24)$$

Now, the authority implements control policy to regulate the amount of production and consumption by imposing tax on supply and demand markets. Let $u = (u_1, u_2, \dots, u_m)^T$ and $v = (v_1, \dots, v_n)^T$ denote the changes of taxes on m supply and n demand markets. Combining with (24), we have a new VI problem as

$$y \geq 0, \quad (y' - y)^T\{t(y) + A^T[k(s) + p + u] - B^T[l(d) - (q + v)]\} \geq 0,$$
$$\forall\, y' \geq 0. \quad (25)$$

Let $x = (u, v)^T$ is the change of imposed tax, we will obtain the value of equilibrium shipment $y(x)$ by solving the VI problem (25). Moreover, the total supply and demand at the equilibrium state can be computed as

$$s(x) = Ay(x) \quad \text{and} \quad d(x) = By(x).$$

Let us convert the network equilibrium control problem under the government's perspective into the IQVI problem. Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be the function of amount of supply and demand at the markets, defined as $f(x) = (s(x), d(x))^T$. In our model, the policy-makers suppose to adjust the amount of supply and demand in flexible ranges depending on the change of imposed tax, given by $\Phi(x) = \{b \in \mathbb{R}^{m+n} : g(x) \leq b \leq h(x)\}$. As we explained above, once the new taxes are imposed, our market will determine new equilibrium state which is described in (25). We assume that total supply and demand $f(x)$ satisfies the upper bound depending on change of imposed tax $x \geq 0$ because the government wants to reduce the amount of commodity. Hence, the optimal control variable x^* and the amount of supply and demand $f(x^*)$ at the equilibrium state must satisfy

$$x \geq 0, \quad f(x) \leq h(x), \quad (f(x) - h(x))^T x = 0. \quad (26)$$

Denote $H_x = \{w \in \mathbb{R}^{m+n} : w \leq h(x)\}$, we can rewrite equivalently the problem (26) as

$$f(x^*) \in H_{x^*}, \quad (w - f(x^*))^T x^* \leq 0, \; \forall\, w \in H_{x^*}. \quad (27)$$

Lemma 4.1: *The problems in (26) and (27) are equivalent.*

Proof: In fact, if x^* is a solution of (26), then $(f(x^*) - h(x^*))^T x^* = 0$ and $f(x^*) \leq h(x^*)$. Hence, $x^* \in H_{x^*}$. Furthermore, for any $w \in H_{x^*}$, we have

$$\begin{aligned}&(w - f(x^*))^T x^*\\&= (w - h(x^*))^T x^* + (h(x^*) - f(x^*))^T x^* = (w - h(x^*))^T x^* \leq 0,\end{aligned}$$

which shows that x^* is a solution of (27).

On another hand, if x^* is a solution of (27), then $f(x^*) \leq h(x^*)$. Besides, choose $w = h(x^*) \in H_{x^*}$, we have

$$(h(x^*) - f(x^*))^T x^* \leq 0.$$

If we choose $w = 2f(x^*) - h(x^*) \leq 2h(x^*) - h(x^*) = h(x^*)$, then $w \in H_{x^*}$. Thus we obtain

$$(f(x^*) - h(x^*))^T x^* \leq 0.$$

Hence, $(f(x^*) - h(x^*))^T x^* = 0$. Lastly, we assume that there exists $x_i^* < 0$ is the ith element of x^*. We choose w such that $w_j = f_j(x^*)$, $j \neq i$ and $w_i = f_i(x^*) - 1$. Then $w \in H_{x^*}$ and $(w - f(x^*))^T x^* > 0$ which contradicts (27). Hence, $x^* \geq 0$ is a solution of (26). ■

With the opposite purpose, if the policy-makers want to encourage the consumption and production then the imposed taxes need to be reduced. Therefore, the change of tax $x \leq 0$ and the amount of commodity $f(x)$ satisfy $f(x) \geq g(x)$. Similarly, we deduce that x^* is a solution of the following problem:

$$x \leq 0, \quad f(x) \geq g(x), \quad (f(x) - g(x))^T x = 0,$$

if and only if x^* is the solution of the problem

$$f(x^*) \in G_{x^*}, \quad (w - f(x^*))^T x^* \leq 0, \; \forall\, w \in G_{x^*},$$

where $G_x = \{w \in \mathbb{R}^{m+n}, g(x) \leq w\}$.Thus, with the condition $g(x) \leq f(x) \leq h(x)$, our network equilibrium control problem can be rewritten as: find the x^* such that

$$f(x^*) \in \Phi(x^*) \text{ and } (w - f(x^*))^T x^* \leq 0, \quad \forall\, w \in \Phi(x^*). \tag{28}$$

As in the traffic assignment problem, with $F = -f$, the problem (28) can be rewritten as an IQVI problem: Find the $x^* \in \mathbb{R}^{m+n}$ such that

$$F(x^*) \in -\Phi(x^*) \text{ and } (w - F(x^*))^T x^* \leq 0, \quad \forall\, w \in -\Phi(x^*). \tag{29}$$

As the matter of fact, due to the law of supply and demand, we can observe that the purchase price will increase with the higher demand while the sale price does the opposite, it will increase when the supply becomes smaller. Besides, if the shipment increases, one need to use greater number of means of transportation, then the transportation costs will not decrease. Thus we can assume that the functions k, l and t satisfy

$$(s - s')^T(k(s) - k(s')) \geq 0, \quad \forall\, s \in \mathbb{R}^m_+, \tag{30}$$

$$(d - d')^T(l(d) - l(d')) \leq 0, \quad \forall\, d \in \mathbb{R}^n_+ \tag{31}$$

and

$$(y - y')^T(t(y) - t(y')) \geq 0, \quad \forall\, y \in \mathbb{R}^{mn}_+. \tag{32}$$

Lemma 4.2: *Under the conditions (30), (31) and (32), the function F in (29) is monotone.*

Proof: In fact, let $x_1 = (u_1, v_1), x_2 = (u_2, v_2) \in \mathbb{R}^{m+n}$ and y_1, y_2 are, respectively, the solution of VI problem (25). Let $s_1 = Ay_1, d_1 = By_1, s_2 = Ay_2$ and $d_2 = By_2$. Since y_1 is the solution of (25), with $y' = y_2$, we have

$$(y_2 - y_1)^T t(y_1) + (s_2 - s_1)^T(k(s_1) + p + u_1) - (d_2 - d_1)^T(l(d_1) - q - v_1) \\ \geq 0. \tag{33}$$

Similarly, since y_2 is the solution of (25), with $y' = y_1$, we get

$$(y_1 - y_2)^T t(y_2) + (s_1 - s_2)^T(k(s_2) + p + u_2) - (d_1 - d_2)^T(l(d_2) - q - v_2) \\ \geq 0. \tag{34}$$

Adding (33) and (34) together and rearranging, we deduce

$$\begin{aligned} &(s_2 - s_1)^T(u_1 - u_2) + (d_2 - d_1)^T(v_1 - v_2) \\ &\geq (y_2 - y_1)^T(t(y_2) - t(y_1)) + (s_2 - s_1)^T(k(s_2) - k(s_1)) \\ &\quad - (d_2 - d_1)^T(l(d_2) - l(d_1)) \\ &\geq 0, \end{aligned} \tag{35}$$

where the second inequality is deduced from (30), (31) and (32). Note that the left-hand side of (35) is $(f(x_2) - f(x_1))^T(x_1 - x_2)$, therefore with $F = -f$, we get

$$(F(x_1) - F(x_2))^T(x_1 - x_2) \geq 0.$$

Hence, F is monotone. ■

Next, for the IQVI problem (29), we need more assumptions as follows.

(A1) The mapping F is strongly monotone and Lipschitz continuous.

(A2) The solution set of IQVI problem (29) is nonempty.
(A3) With given toll x on supply and demand markets, the amount of production and consumption $f(x)$ is computed through commodity shipment y, which is received exactly from solving the VI problem (25).

To implement the projection algorithm (5), we consider network equilibrium problem with $m = 10$, $n = 30$. To establish the functions t, k and l fulfilling the conditions (30), (31) and (32), respectively, we prioritize to use linear functions as follows:

$$t(x) = Tx + \mathcal{T}, \quad k(s) = Ks + \mathcal{K}, \quad l(d) = -Ld + \mathcal{L}, \tag{36}$$

where T and $\mathcal{T}$ are mn-diagonal matrices and elements t_{ii} and $\mathcal{T}_{ii}$ on diagonal are randomly given in $(0.1, 0.2)$ and $(10, 20)$, respectively; K and $\mathcal{K}$ are m-diagonal matrices and elements K_{ii} and $\mathcal{K}_{ii}$ on diagonal are randomly given in $(1, 2)$ and $(270, 370)$, respectively; L and $\mathcal{L}$ are n-diagonal matrices and elements L_{ii} and $\mathcal{L}_{ii}$ on diagonal are randomly given in $(1, 2)$ and $(620, 720)$, respectively. Besides, we set $p_i = 30$ and $q_j = 20$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. With the construction (36), note that $s = Ay$ and $d = By$, the VI problem (25) can be rewritten as: Find $y \geq 0$ such that

$$(y' - y)^T \left((T + A^TKA + B^TLB)y + A^T(\mathcal{K} + p + u) + B^T(q + v - \mathcal{L})\right) \geq 0,$$
$$\forall\, y' \geq 0. \tag{37}$$

Set $M = T + A^TKA + B^TLB$ and $N = A^T(\mathcal{K} + p + u) + B^T(q + v - \mathcal{L})$, the VI problem (37) becomes

$$y \geq 0, (y' - y)^T(My + N) \geq 0, \quad \forall\, y' \geq 0, \tag{38}$$

which is a linear VI problem. In summary, the projection algorithm (5) works as follows. For the given toll $x_n = (u_n, v_n)^T$, we employ the projection–contraction method in [24] to solve linear VI problem (38) (we take $\gamma = 1.8$ and error $\epsilon_1 = 0.1$), then gain the value $y(x_n)$. Compute $s(x_n) = Ay(x_n)$ and $d(x_n) = B(y_n)$, we obtain the value $f(x_n) = (s(x_n), d(x_n))^T$. The update of toll x_{n+1} with fixed scalar α and stepsize λ is

$$\begin{aligned} x_{n+1} &= x_n + \alpha \left[P_{-\Phi(x_n)}(F(x_n) - \lambda x_n) - F(x_n)\right] \\ &= x_n + \alpha \left[f(x_n) - P_{\Phi(x_n)}(f(x_n) + \lambda x_n)\right]. \end{aligned} \tag{39}$$

Let $\Phi(x) = \{w \in \mathbb{R}^{m+n} | g(x) \leq w \leq h(x)\}$, where $g_i(x) = 0$ and $h_i(x) = x_i + 160$, $i = 1, \ldots, m$ and $g_{m+j}(x) = 20$, $h_{m+j}(x) = x_{m+j} + 60$, $j = 1, \ldots, n$. It is not difficult to verify that Φ satisfies the condition (7) with $\kappa = 1$. Because the projection algorithm (5) follows the form of scheme (39), we continue using the term $r_n = \alpha\|f(x_n) - P_{\Phi(x_n)}(f(x_n) + \lambda x_n)\|$ to show the convergence rate of this algorithm.

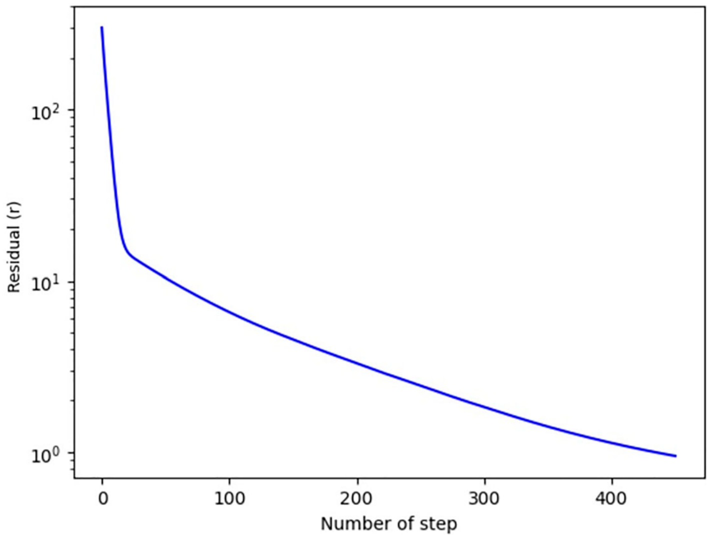

Figure 8. Residual of the projection algorithm (5) with $\alpha = 0.1$ and $\lambda = 0.005$.

We start with $x_0 = (0,0)^T$, which means that the toll has not been changed. Solving the linear VI problem (25), we obtain the supply and demand at the beginning with

$$\max_{1\le i\le 10} s_i(0) = 203.17, \qquad \max_{1\le j\le 30} d_j(0) = 102.04,$$

which is out of range of the constraint $\Phi(0)$. Thus the authority needs to adjust tax and makes the production and consumption meet the expected constraints.

First, we choose the scalar $\alpha = 0.1$ and the stepsize $\lambda = 0.005$, the performance of algorithm (5) within 450 step is depicted in Figure 8. We can see that the residual attains linear convergence and its value decreases rapidly after roughly first 20 steps. Then, it decreases gradually and becomes approximately 1 at the step 450.

Next, we choose both greater scalar $\alpha = 0.25$ and stepsize $\lambda = 0.01$. As demonstrated in Figure 9, the projection algorithm (5) still guarantees the linear convergence. Furthermore, in this case the residual converges significantly faster than that in the first experiment, especially within about first 100 steps. It takes the value about 0.01 at the step 100 and keep decreasing after that.

Finally, we keep the scalar $\alpha = 0.1$ as in the first implement and choose the greater stepsize $\lambda = 0.2$. The result is illustrated in Figure 10. In this situation, the residual does not converge and Algorithm (5) fails. As in the last experiment in Traffic assignment problem, the reason could be because α and λ break the condition in (21).

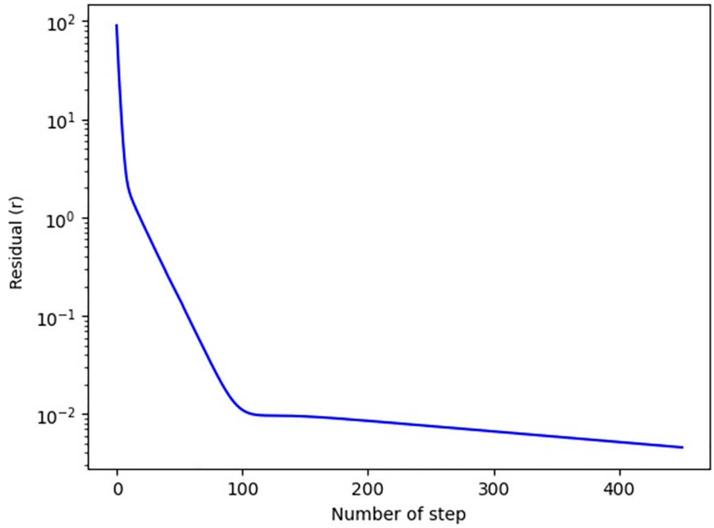

Figure 9. Residual of the projection algorithm (5) with $\alpha = 0.25$ and $\lambda = 0.01$.

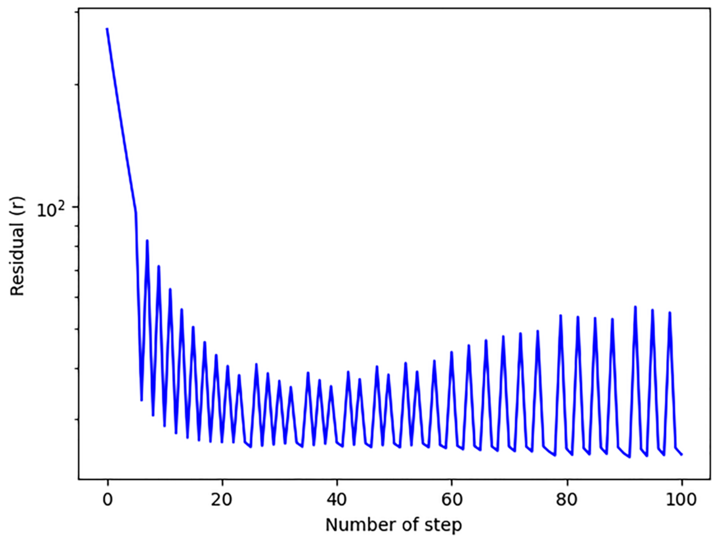

Figure 10. Residual of the projection algorithm (5) with $\alpha = 0.1$ and $\lambda = 0.2$.

5. Conclusion

We have revisited the projection algorithm for solving inverse quasi-variational inequalities with a general constraint function. We established the linear convergence of the iterations computed by the projection algorithm and demonstrated the convergent rate by numerical examples. We note that the theoretical results obtained in this paper still hold in infinite dimensional Hilbert spaces. In the application side, we discussed the inverse quasi-variational inequalities in traffic assignments and network equilibrium problems and provided some numerical experiments. Further applications of IQVI models in practical problems and extensive numerical tests are reserved for future research.

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