Humanities & Social Sciences Communications

ARTICLE

https://doi.org/10.1057/s41599-024-02991-w OPEN

Using social network analysis to investigate mathematical connections in U.S. and Chinese textbook problems

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Textbooks, as potentially implemented curricula, play an important role in supporting classroom teaching and learning. Mathematical connections, one of the essential and hot topics advocated in mathematics education, have been emphasized in national curriculum reforms in various countries. However, little is known about the connection networks represented in school textbooks; even less has been done to compare textbooks from different countries. In this study, we propose an innovative method for examining how connections are represented in two popular U.S. (the UCSMP series) and Chinese (the PEP-A series) high school textbook problems involving quadratic relations. By using social network analysis, we identified 1129 connections, characterized connection networks into dense, moderate, and sparse digraphs, identified influential, prominent, and dual concepts and representations, and evaluated the strength between typical and reverse connections. The results revealed that the Chinese series presented a denser network of balanced between-concept connections but limited within-concept connections. The U.S. series exhibited more within-concept connections but emphasized typical connections, thus validating the potential of this innovative method. From this study, we suggest that our novel method provides a theoretical contribution to textbook analysis and connection analysis, which has rich implications for practice, for example, examining the network of connections students construct as a way to assess and to promote their conceptual understanding, and our approach opens the possibility of adopting new and efficient analytical tools from social network analysis in mathematics education research.

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Introduction and rationale

Textbooks, as potentially implemented curricula (Valverde et al., 2002), still play a central role in classrooms and are currently of great research interest (Stein et al., 2007; Wang and Fan, 2021). Prior international comparisons of students' mathematical achievements (for example, the 2018 Program for International Student Assessment [PISA]) showed that Chinese students outperformed their U.S. counterparts (OECD, 2019). Even though students' mathematical performance has not been directly associated with textbooks, studies in which researchers explored possible contributing factors have indicated that the textbook was one potential impacting factor (Son and Diletti, 2017). Considering the consistently good mathematical performance of Chinese students in international comparisons, many researchers have conducted U.S. and Chinese textbook comparisons and have reported substantial differences in the integration of connections (e.g., Ding, 2016).

Fostering connections has been a major goal of mathematics education over the past 30 years in U.S. and Chinese curriculum reforms (Common Core State Standards Initiative [CCSSI], 2010; Ministry of Education [MOE], 2020; National Council of Teachers of Mathematics [NCTM], 1989, 2000), and such reforms have received increasing attention from researchers. As shown in Fig. 1, researchers have provided rich examples of within-concept and between-concept connections, and they have reported that translations between representations (for example, graph→equation, equation→graph) occurred in pairs (Janvier, 1987; Leinhardt et al., 1990), and connections between concepts (for example, multiplication↔division) occurred in pairs (Xin et al., 2011). In this study, we focused on (a) *within-concept connections* involving two representations of the same concept and (b) *between-concept connections* associating two distinct concepts. How students understand the concepts in and of themselves (within-concept connections) and how students connect concepts (between-concept connections) are considered to be vital for students' understanding of mathematics, as shown in various policy documents, research studies and theoretical reflections (e.g., Adu-Gyamfi et al., 2017; Selinski et al., 2014). The model has been used successfully in studying connections in linear algebra and could be used effectively in other mathematical content areas (Selinski et al., 2014).

NCTM (2000) emphasized that students must learn mathematics with active prior→new knowledge connections. Additionally, new→prior knowledge connections can significantly improve students' backward reasoning and meaningful learning (Hohensee, 2014). Accordingly, based on concepts and representations within the curriculum, we employed *typical connections* to depict a connection from prior to new knowledge, namely, from A to B, in which A and B are both concepts or representations of a concept and A appears earlier than B in a specific textbook series; a *reverse connection* refers to the new→prior knowledge connection (i.e., from B to A) within the context; and *bidirectional connections* (BC) portray a pair of typical and reverse connections (see Fig. 1). For example, in many textbooks, addition→subtraction is a typical between-concept connection, while subtraction→addition is a reverse between-concept connection. Generating a graph of a circle from its equation is a typical within-concept connection while generating an equation of a circle from its graph is a reverse one.

Researchers have shown a growing interest in bidirectional connections for their recognized benefits. These benefits include aiding conceptual understanding, attaining productive backward transfer, and supporting problem-solving (Hohensee, 2014; Nitsch et al., 2015; Piez and Voxman, 1997). Although the benefits of BC are widely endorsed, in prior studies it has been shown that students and even teachers have difficulties making BC, such as limited codomain→domain (Adu-Gyamfi and Bossé, 2014) and limited graph→equation of polynomial relations (Adu-Gyamfi et al., 2017). Researchers have indicated that limited learning opportunities for BC in textbooks may significantly contribute to learners' difficulties in making such connections. For example, routine textbook tasks require equation→graph. Relying solely on this approach produces limited opportunities for constructing graph→equation (Knuth, 2000). Additionally, reverse use of the distributive property rarely appeared in U.S. mathematics textbooks (Ding and Li, 2010); this finding was consistent with U.S. students' difficulties in such reverse use (Ding et al., 2021).

Given that problem-solving has been the central theme of mathematics education since the 1980s, researchers have shown renewed interest in problem analysis. From the perspective of connections, several attempts have been made to investigate BC represented in U.S. and Chinese textbook problems. However, most of the researchers examined only a narrow scope of connections or compared the number or percentage of BCs (e.g., Ding, 2016). This perspective failed to reveal the connection network, especially on how abundant concepts and representations are correlated to each other and how they play a role in such a giant network, thus diminishing the goal of viewing mathematics as an integrated whole. *Connectivism*, which sustains depicting connections by directed edges between two vertices, creates the possibility of using social network analysis (SNA) for connection analysis. As revealed in Siegal and Yovel's (2023) study, network analysis has revolutionized the investigation of complex systems, and it allows us to ask both quantitative and

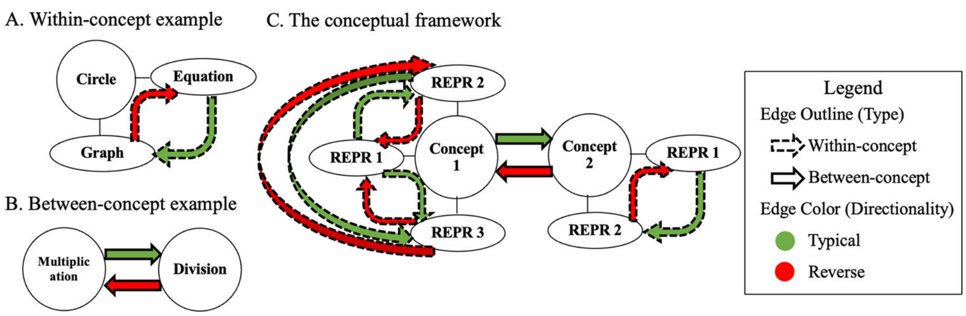


Fig. 1 A network of connections (adapted from Li and Fan (2023)). Several examples of within-concept connections, between-concept connections, and a combined conceptual framework are illustrated. **A** A within-concept connection of generating a graph of a circle from its equation is illustrated in the green edge with the dashed outline, and the reverse is shown by the red edge with the dashed outline. **B** A between-concept connection from multiplication to division is shown by the green edge with the solid outline, and the reverse is shown by the red edge with the solid outline. **C** The conceptual framework. REPR stands for representation.

qualitative questions about the system. However, there have been few attempts to investigate connections in textbook problems by using social network analysis.

Therefore, in this study, we proposed an innovative method by using SNA to characterize and visualize the network of typical and reverse between-concept and within-concept connections in U.S. and Chinese high school textbook problems dealing with quadratic relations (referred to as circles, ellipses, hyperbolas, and parabolas)—a challenging and important topic for students' smooth transition to college-level mathematics. The research question is as follows: What are the similarities and differences in the network of connections represented in popular U.S. and Chinese high school mathematics textbook problems that focus on the topic of quadratic relations?

By addressing this question, we hope that this study not only yields insights into promoting learning opportunities for balanced bidirectional connections in textbooks but also provides research evidence to consolidate the feasibility and potential contribution of using SNA as an analytical tool in mathematics education.

Literature review and analytical framework

Mathematics textbook analysis and connection analysis. Many researchers have compared U.S. and Chinese mathematics textbooks, and such research has indicated substantial differences in the integration of between-concept connections. For example, researchers compared U.S. and Chinese elementary school textbook problems on the typical and reverse use of the averaging algorithm (Cai et al., 2002), the distributive property (Ding and Li, 2010), and addition-subtraction and multiplication-division (Ding, 2016; Xin et al., 2011). Chinese textbooks in those studies were shown to embed these connections bidirectionally in deliberately constructed problems, whereas their U.S. counterparts lacked balanced bidirectional use. However, this is not the case for within-concept connections, as many U.S. and Chinese textbook problems lack bidirectional within-concept connections. For example, Wang et al. (2017) showed that worked-out examples from Grade 8 Chinese textbooks were more algebraic→graphical for linear functions than the reverse. Tran and Tarr (2018) reported that many association tasks involving bivariate data from U.S. textbooks restricted learning opportunities for bidirectional connections between two data representations.

Overall, most researchers have focused on content at the elementary and middle school levels. Very little attention has been given to high school textbooks (Wang and Lu, 2018), which may illuminate unexplored facets of cross-cultural differences. This can enrich our understanding and can facilitate a seamless transition for students progressing from high school to college-level mathematics. Furthermore, there are potentially different trends between high school and elementary/middle school mathematics textbooks (Hong and Choi, 2014). Thus, in this study, we addressed this gap by focusing on the presentation of connections in U.S. and Chinese high school textbooks to determine whether cross-cultural differences in connection integration still exist and to identify possible ways to strengthen students' learning opportunities for connections.

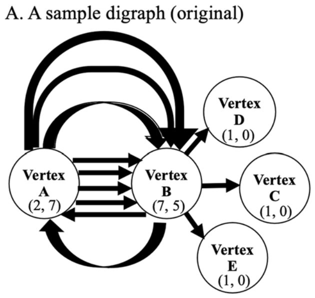
Problems have occupied a central place in mathematics textbooks. How examples and exercises are created is vital in textbook development; thus, in a plethora of studies, researchers delve into textbook problem analysis. Various schemes, such as mathematical features, contextual features, response types, representation forms, problem-solving heuristics, depth of knowledge, cognitive expectations, and cognitive demands (Fan et al., 2013; Son and Diletti, 2017), have been used in textbook problem analysis. Among these, cognitive demand is widely used to indicate the presence or absence of connections. For example, the popular Task Analysis Guide developed by Stein (2000) consists of two levels of cognitive demand: (a) high-level: doing mathematics and procedures with connections; and (b) low-level: memorization and procedures without connections. However, such an approach fails to reveal detailed information about concepts, representations, and connection networks. However, these are valuable data for designing textbook problems with learning opportunities for balanced connections.

Among the textbook problem studies in which researchers have explicitly examined specific connections, the total number, the number per page, the overall distribution by chapter or textbook, and the percentage of problems were employed as the prevailing analytic methods (Chang et al., 2016; Ding, 2016; Ding and Li, 2010; Wang et al., 2017). Regarding e-textbooks, Gueudet et al. (2018) reported a rating of 'yes', and examples such as 'both e-textbooks offer many connections between the different concepts (function, graph, image, etc.) and between the different representations: algebraic, graphic, and table of values' (p. 554), which addressed the connection network in an approximate manner. Additionally, many researchers have focused on either a specific between-concept (Xin et al., 2011) or a within-concept connection (Tran and Tarr, 2018; Wang et al., 2017), but few researchers have covered a large number of connections. Chang et al. (2016) examined 707 instances of 12 within-concept connections for functions in a U.S. calculus textbook. However, the exact functions (for example, exponential, logarithm, linear) involved were not differentiated. This was indeed critical since the graphical→symbolic of exponential functions may be more challenging than linear functions. Overall, these available methods fail to reveal the structure of connection networks in depth.

Considering other methods used in connection analysis, concept maps have been the dominant analytic method for examining connections students make between concepts, but concept maps may be inappropriate for probing connections in textbooks because of several limitations. First, a concept map usually covers fewer than 20 concepts, and thus, the method cannot be used for organizing a large number of concepts sufficiently, depicting the giant network in a concise form systematically, evaluating the directionality of connections effectively, and examining within-concept and between-concept connections simultaneously (Selinski et al., 2014). Second, the widely used scoring schemes (valid propositions, hierarchy levels, and cross-links) may not be valid for comparing complex maps across students (Williams, 1998).

According to a new learning theory, *connectivism* endorses connections as directed edges between two vertices and emphasizes connections in its learning principles that 'learning is a process of connecting specialized nodes', 'nurturing and maintaining connections is needed to facilitate continual learning' and 'ability to see connections between fields, ideas, and concepts is a core skill' (Siemens, 2005). Accordingly, connectivism supports depicting connections by directed edges from the source vertex to the target vertex; thus, the network of connections is a digraph (i.e., directed graph) with weak or strong directed edges. SNA, which maps and measures relationships between people, groups, and other entities (for example, mathematics concepts), has been used in investigating many networks within and between schools (Bokhove, 2018), and SNA also allows for examining connection networks in textbook problems. Hence, we opt to use SNA to examine connections represented in textbook problems.

Using SNA to analyze connections. We first briefly review some relevant definitions and metrics used in basic network analysis



Original	Schema-based
Total Edges: 12	Distinct Edges: 5
(Reciprocated: 9)	(Reciprocated: 2)
Centrality	Connectivity
In-degree	In-connection
Out-degree	Out-connection

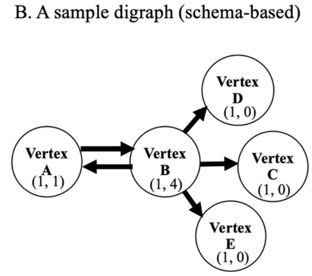


Fig. 2 A sample digraph. The original view and schema-based view of a sample digraph are shown, as well as their order, total edges, distinct edges, centrality, and connectivity. The order is five. There are seven multiple edges from vertex A to vertex B, two multiple edges from vertex B to vertex A, one edge from vertex B to vertex C, one edge from vertex B to vertex D, and one edge from vertex B to vertex E. **A** Original view: for each vertex, the ordered pair under the vertex name stands for (in-degree, outdegree). **B** Schema-based view: for each vertex in the schema-based digraph, the ordered pair stands for (in-connection, out-connection).

(Hanneman and Riddle, 2011; Smith et al., 2010; Strom et al., 2001).

(1) A *digraph* consists of a finite set of vertices and directed edges from the source vertex to the target vertex.
(2) The *order* is the number of vertices in a network;
(3) The *total number of edges* is the number of edges in the original view (multiple edges are all counted); the number of *distinct edges* is the number of edges in the schema-based view (multiple edges are counted only once);
(4) The *in-degree* is the number of edges leading to a vertex; the *out-degree* is the number of edges leading out of a vertex;
(5) The in-connection is the number of distinct edges leading to a vertex; the out-connection is the number of distinct edges leading out of a vertex;
(6) The reciprocated function filters out edges from vertex A to vertex B, which are joined by another edge from vertex B to vertex A.

Figure 2 shows a sample digraph in two views—original and schema-based—in which the original view reflects the quantity of connections, while the schema-based view demonstrates the diversity of connections. Although more edges start from vertex A than do from vertex B ($7 > 5$), there are 4 distinct target vertices for vertex B and only 1 for vertex A. Therefore, merely probing the original or the schema-based digraph may miss the edge with large weights (for example, 7 edges of $A \rightarrow B$) and the scope of connected vertices (i.e., the diversity of connections; for example, 4 target vertices for B). These facts strengthen the rationale for examining digraphs of the two views together.

In mathematics education, SNA is a relatively new analytical tool. A review of prior empirical research revealed that digraphs and adjacency matrices were used to examine connections mainly in classroom interactions or mathematical arguments during classroom discourses, lectures, and interviews (e.g., Bokhove, 2018). This approach has contributed valuable insights into (a) central concepts and representations (Jin and Wong, 2015; Weinberg et al., 2016); (b) shifts in connections over time (Wawro, 2014); (c) a schema-based view of the connection network (Strom et al., 2001); and (d) flexibility of within-concept and between-concept connections (Selinski et al., 2014). However, using SNA to examine connections is innovative in textbook problem analysis. As a part of our research project, Li and Fan (2023) concentrated on a subset of connections (bidirectional connections exclusively), by using both digraphs and adjacency matrices as analytical tools. To gain a more comprehensive view of the connection network, in the present study we employed digraphs to examine both typical and reverse within- and between-concept connections in depth. Below, we show how the use of digraphs in prior relevant research is similar to or different from our methods.

The primary aim of producing a clear visualization to reveal the structure of connection networks in textbook problems is like that in Strom et al. (2001). In their research, they present discourse analysis yielding digraphs (circular layout) of how various aspects of mathematics were connected in a chronological flow of argument. Similarly, in this study, we rendered connections identified in textbook problems as digraphs with (a) Vertices: concepts, concepts (representations); (b) Edges: connections; and (c) Arrows: directionality. In Strom et al.'s (2001) digraphs, a number on the edge represented the ordering of connections in the ebb and flow of argument, but multiple edges contributed to a 'messy' digraph and weakened the visibility of distinct edges and the number of total edges one digraph could sustain. Furthermore, many teachers customize the sequence of textbook problems to accommodate their teaching needs. Therefore, the ordering of connections is not that important in this study. To manage abundant connections, we utilized the width of the edges to reflect the number of multiple edges.

The second purpose of using SNA is to provide a nuanced characterization of connections. Prior connection analysis has been focused more on within-concept versus between-concept connections, and few researchers have analyzed between-concept and within-concept connections simultaneously (Selinski et al., 2014). Given this gap, it is important to examine the network of both within-concept and between-concept connections and to derive learning opportunities to see mathematics as a connected discipline. Inspired by the above, the proposed framework (see Table 1) attends to three levels: (a) Digraph: to examine the structure of connection networks and aggregations; (b) Vertex: to identify influential, dominant, or dual concepts and representations and their roles in the ego network; and (c) Edge: to explore the relative strength between typical and reverse directions; and exemplary examples.

At the digraph level, we adopted the density of a digraph (dense, moderate, and sparse) for a subtopic by its order, distinct edges, and total edges, as these three metrics reflect the quantity and the diversity of concepts and representations, as well as connections involved in a connection network, which reveals an overall evaluation of network robustness. Selinski et al. (2014) compared connections that students made within and between concepts in linear algebra, and they identified those concepts as dense (many between-concept and within-concept), sparse (mainly between-concept), and hub (mainly within-concept) adjacency matrices. In our pilot analysis of one subtopic, within-concept connections accounted for less than 10% of the total. Examining a digraph of rich between-concept and limited within-

Table 1 Analytical framework.

Level	Metrics	Application in textbook analysis
Digraph	Order	The number of concepts (representations) in a network.
	Total Edges	The number of connections in a network where multiple connections are all counted.
	Distinct Edges	The number of connections in a network where multiple connections are counted only once.
	Density	Dense, moderate, and sparse digraphs by order, distinct edges, and total edges.
	Aggregation	Key concepts and representations by aggregated arrows to a specific vertex. Central connections among broad categories by the aggregation of weighted edges.
Vertex	In-(Out-)degree	The number of connections leading to (out of) a concept (representation).
	Centrality	The sum of in-degree and out-degree divided by twice the total edges in the network.
	Ratio of In-/Out-degree	The ratio of in-degree to out-degree for a concept (representation).
	Ego Network Rating	An influential (higher out-degree), prominent (higher in-degree), or dual role in the ego network.
	In-(Out-)connection	The number of distinct connections leading to (out of) a concept (representation).
	Connectivity	The sum of in-connection and out-connection divided by twice the distinct edges in the network.
	Ratio of In-/Out-connection	The ratio of in-connection to out-connection for a concept (representation).
	Ego Network Rating (Schema-based)	An influential (higher out-connection), prominent (higher in-connection), or dual role in the schema-based ego network.
Bi Edge	Bi Edges (n, %)	The number of total bi connections; The percentage of bi edges to total edges.
	Bi Pairs (n, %)	The number of distinct bi connections; The percentage of bi pairs to distinct edges.
	Bi Gravity	The sum of typical and reverse connections divided by bi edges in the network.
	Ratio of Typical/Reverse	The ratio of typical connections to reverse connections for a specific bi pair.
	Directionality Rating	A typical, reverse, or balanced bi pair in the network.

Note. Bi stands for bidirectional; (n, %) stands for the number, and percentage. Bi pairs include self-loops. New metrics are in italics. "Network" means the corresponding network in the context.

concept connections may diminish the visibility of within-concept connections. Thus, we investigated digraphs for between-concept connections of each subtopic and one digraph for within-concept connections. For aggregations, in digraphs, key concepts, and representations dominating the network are self-evident by aggregating arrows to a specific vertex; key connections are apparent by the aggregated edges. This approach can provide clues about the trends of pivotal concepts and the representations and central connections in the connection network.

At the vertex level, in-degree, out-degree, centrality, in-connection, out-connection, and connectivity for each vertex were employed to identify key concepts and the representations in the global network, as well as the local importance of a vertex in its ego network. The centrality index—the sum of in-degree and out-degree divided by twice the total edges in the network—represents the specific vertex's advantage and disadvantage relative to vertices in their neighborhood (Hanneman and Riddle, 2011). The connectivity index—the sum of in-connection and out-connection divided by twice the number of distinct edges in the network—reveals the influence or prominence of a specific vertex in a schema-based view (Strom et al., 2001). In prior studies, researchers have shown the usefulness of centrality and connectivity. For example, Strom et al. (2001) used the in/out-degree and the connectivity index to identify central ideas in an argument. Similarly, Wawro (2014) demonstrated the centrality of span and linear independence in students' mathematical arguments by comparing in-/out-degree, in-/out-connection, and connectivity indices. However, simply adopting the above metrics fails to reflect the strength of directionality. Considering the sample digraph in Fig. 2, vertex A has the highest out-degree, and vertex B has the highest in-degree. Nevertheless, there is no quantitative measure of the dominant direction and its strength. Moreover, in the schema-based view, all vertices have one in-connection; however, the extent each vertex plays in two directions is unclear. Therefore, we calculated the ratio of in- to out-degree and the ratio of in- to out-connections to quantify the extent to which concepts and representations are more influential (higher out-, mainly exerting influence on other ideas), prominent (higher in-, mostly supported by other ideas), or are playing a dual role (showing influence and prestige) in the ego network of both original and schema-based views.

For the third level, in the present study, we adopted the number and percentage of bidirectional connections, both in original and schema-based views. Corresponding to the centrality and connectivity index, the bidirectional gravity—the sum of the typical and reverse connections of one pair divided by the number of total bidirectional pairs—was used to reveal a specific pair's strength and emphasis relative to the remaining bidirectional pairs. The ratio of typical to reverse connections was employed to quantify the extent of the emphasized direction of a specific bidirectional pair. All these metrics highlight weighted bidirectional pairs and the balance between typical and reverse directions; this approach can provide new perspectives on unbalanced learning opportunities for bidirectional connections in textbook problems.

Methods

Data collection. To demonstrate the potential utility of the SNA approach, quadratic relations were chosen as the sample topic for the following reasons. First, quadratics, a traditional core high school-level topic, is critical to students' success in high school and beyond. High school curriculum standards stress bidirectional within-concept connections of quadratics and between-concept connections in quadratic models or linear-quadratic models (CCSSI, 2010; MOE, 2020). Second, students demonstrate their limited understanding of bidirectional connections in quadratic relations, for example, within-concept connections of quadratic functions among the standard form, the vertex form, and the factored form (Parent, 2015) and connections between the vertex of quadratic relations and the derivative (Burns-Childers and Vidakovic, 2018). Finally, quadratic relations address the broader area of nonlinear polynomial equations with potentially plentiful connections. Table 2 lists the sample background.

For Chinese textbooks, we specifically chose the most widely used and circulated version, General High School Curriculum Standard Experimental Textbook Mathematics, A Version, named PEP-A. Among the most well-known standards-based

Table 2 Background of sample textbooks and chapters.

	Textbook	
	China	**U.S.**
Chapter	PEP-A Compulsory 2, Chapter 4 (Grade 10); PEP-A Elective 2-1, Chapter 2 (Grade 11)	UCSMP Advanced Algebra, Chapter 12 (Grade 9–12)
Sequence and Lessons	Circle, Ellipse, Hyperbola, Parabola; 6 lessons	Parabola, Circle, Ellipse, Hyperbola; 9 lessons
Publisher	People's Education Press, 2007	McGraw-Hill, 2010
Author	Shaoxue Liu, Peiling Qian, et al.	James Flanders, Zalman Usiskin, et al.

Note. The PEP-A Electives are prepared for high school students who take the China National College Entrance Examination; these books are required by most of the provinces and are treated essentially as compulsory.

high school textbooks in the U.S. (Stein et al., 2007), we selected the University of Chicago School Mathematics Project Grade 9–12, named UCSMP, with its explicit emphasis on representations and connections (Usiskin, 2018). Furthermore, we adopted the teachers' editions as supplementary materials. PEP-A and UCSMP teachers' editions include not only additional worked-out examples to accommodate students' needs but also detailed step-by-step solutions to exercises for which no solutions are provided in students' editions. These extra examples were valuable auxiliary data for identifying connections. Moreover, Ding and Li (2010) revealed the benefits of teachers' editions in resolving possible discrepancies in coding connections. Thus, worked-out examples, exercises, and solutions dealing with quadratic relations in students' and teachers' editions were all collected. We did not cover problems in projects, readings, explorations, chapter reviews, or self-tests due to their random sequence and frequency.

Data coding and analysis. First, we divided the collected data into separate instances. Textbook problems have one or two levels of sequence numbers. The first-level numbering was 1, 2, …; the second-level numbering was (1), (2), … in the PEP-A series and a, b, … in the UCSMP series. For problems with first-level numbering only, we divided the data into basic items by first-level numbering. For problems with second-level numbering, we divided the data into basic items by second-level numbering. Next, we assigned each item an item number, one by one. For example, the first two sample problems of the PEP-A series in Table 3 were divided and assigned: Item 1 for 1(1) and Item 2 for 1(2). In all, 537 problems were collected; there were more in the UCSMP series (61%).

Then, we collected the vocabulary checklist in the chapter review and glossary for each textbook to compile the Concepts Table (see supplementary material), and we constructed the Representations Table: written description (W), symbolic expressions (expanded to S1: standard form for a conic section; S2: $Ax^2+Bxy+Cx^2+Dx+Ey+F=0$; and S: other symbolic), tables (T), graphs (G), and numerals (N). Next, we determined the sequence of concepts and representations based on textbook content; we compiled all possible connections (an ordered pair of two concepts, or an ordered pair of two representations of one concept from the two tables) in a complete connection table; and, we coded solutions of each item step by step in terms of relevant connections in the connection table as well as their types: between-concept (BCC) or within-concept (WCC) and directionality: typical (T) or reverse (R). For example, in Table 3, Item 6 of the UCSMP series has one BCC, ellipse→foci, with the directionality T; Item 5 of the PEP-A series covers five BCCs, hyperbola→foci, angle→slope, slope→line, foci→line, line-parabola system→two intersections, with the directionality T, T, R, R, T.

To maintain neutral coding, we invited one U.S. and one Chinese mathematics teacher, who each had more than five years of teaching experience, to produce the final coding. We explained to them the coding rubrics, recorded their agreement or disagreement with our coding in their own language, discussed and resolved disputes and missed connections, and generated the final coding. Then, two graduate students majoring in mathematics education who are proficient in English and Chinese were invited to agree or to disagree with the final coding of one lesson randomly selected from the two series (74 items). As these two lessons were chosen randomly, fewer than 20% of the total lessons were chosen. We acknowledge this as a limitation. Overall, the percentage of agreement for each coder and the overall percentage of agreement for coder pairs across three dimensions (type, directionality, and connection (source and target vertex)) and textbooks all surpassed 80%, which satisfied the typical reliability requirement.

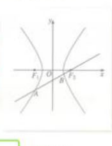
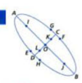
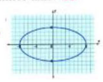
The final coding included 1129 connections (483 distinct), 91% of which were between-concept (89% distinct). NodeXL, a network analysis software package, was used to (a) generate digraphs for between-concept connections for four subtopics and digraphs for within-concept connections for both series and to (b) automatically produce the order, total edges, and distinct edges for each digraph. To make the trend of between-concept connections more visible, i.e., how each quadratic relation connects to prior knowledge (linear function-related concepts), its attributes, and other quadratic relations, we placed the concept vertices within a circle according to five categories: quadratic relations, linear function-related concepts (L), attributes of quadratic relations (A), two relations (SS), and other concepts (O). Moreover, to examine the density of connections in a particular subtopic, we established criteria to categorize dense, moderate, and sparse networks (see Table 4). The numeric cut-off for the dense digraph was calculated by the average number of vertices and distinct and total edges per digraph (for between-concept), which was consistent with the numeric cut-off (50%) of the strong connection by Jin and Wong (2015).

To explore emphasized influential, prominent, and dual concepts and representations and bidirectional connections, we calculated several graph metrics (see Table 1) for connections presented in each series, including in- and out-degree, centrality, ratio of in-/out-degree, in- and out-connection, connectivity, ratio of in-/out-connection, bi edges, bi pairs, bi gravity, ratio of typical/reverse, and we produced the ego network rating (original and schema-based) and directionality rating for each series.

Results

Digraph: dense, moderate, and sparse. To provide a clear visualization of 1,129 connections, 10 digraphs were produced and categorized as dense, moderate, or sparse according to the criteria listed in Table 4.

Table 3 Textbook problem samples and coding.

Textbook	Problem	#	Type	Connection		Directionality
				Source	Target	
PEP-A (C2, p. 123)	练 习 Exercises 1. Find the center and radius of the circle with equation: 1. 求下列各方程表示的圆的圆心坐标和半径长： (1) $x^2+y^2-6x=0$； (2) $x^2+y^2+2by=0$；	1	BCC	Circle	Center	T
			BCC	Circle	Radius	T
		2	BCC	Circle	Center	T
			BCC	Circle	Radius	T
PEP-A (E2-1, p. 42)	2. Write an equation of the standard form of the ellipse. 2. 写出适合下列条件的椭圆的标准方程： (1) $a=4$，$b=1$，焦点在 x 轴上； Foci on x-axis	3	BCC	a	Ellipse	R
			BCC	b	Ellipse	R
			BCC	Foci on x-axis	Ellipse	R
PEP-A (E2-1, p. 48)	7. There is a line passing through the left focus F_1 of ellipse $\frac{x^2}{2}+y^2=1$ with an inclination angle 60°, which intersects the hyperbola at A and B. Find | AB |. 7. 经过椭圆$\frac{x^2}{2}+y^2=1$的左焦点 F_1 作倾斜角为 60°的直线 l，直线 l 与椭圆相交于 A，B 两点，求 AB 的长.	4	BCC	Ellipse	Foci	T
			BCC	Angle	Slope	T
			BCC	Slope	Line	R
			BCC	Foci	Line	R
			BCC	Line; Ellipse	Two intersections	T
PEP-A (E2-1, p. 60)	Example 6. Refer to the graph 2.3-10, there is a line passing through the right focus F_2 of hyperbola $\frac{x^2}{3}-\frac{y^2}{6}=1$ with an inclination angle 30°, which intersects the hyperbola at A and B. Find | AB |. 例 6 如图 2.3-10，过双曲线$\frac{x^2}{3}-\frac{y^2}{6}=1$的右焦点 F_2，倾斜角为 30°的直线交双曲线于 A，B 两点，求$\|AB\|$. 解：由双曲线的方程得，两焦点分别为 $F_1(-3,0)$，$F_2(3,0)$. hyperbola→foci 因为直线 AB 的倾斜角是 30°，且直线经过右焦点 F_2，所以，直线 AB 的方程为 $y=\frac{\sqrt{3}}{3}(x-3)$. ① angle→slope slope→line foci→line  图 2.3-10 由 $\begin{cases} y=\frac{\sqrt{3}}{3}(x-3), \\ \frac{x^2}{3}-\frac{y^2}{6}=1, \end{cases}$ 消去 y，得 $5x^2+6x-27=0$. 解这个方程得 $x_1=-3$，$x_2=\frac{9}{5}$. 将 x_1，x_2 的值代入①，得 $y_1=-2\sqrt{3}$，$y_2=-\frac{2\sqrt{3}}{5}$. line-parabola system→two intersections 于是，A，B 两点的坐标分别为$(-3,-2\sqrt{3})$，$(\frac{9}{5},-\frac{2\sqrt{3}}{5})$. 所以，$\|AB\|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ $=\sqrt{(-3-\frac{9}{5})^2+(-2\sqrt{3}+\frac{2\sqrt{3}}{5})^2}$ $=\frac{16}{5}\sqrt{3}$. Applying the distance formula	5	BCC	Hyperbola	Foci	T
			BCC	Angle	Slope	T
			BCC	Slope	Line	R
			BCC	Foci	Line	R
			BCC	Line; Parabola	Two intersections	T
UCSMP (p. 808)	6. **Fill in the Blanks** The circle with equation $(x-h)^2+(y-k)^2=r^2$ has center _?_ and radius _?_.	1	BCC	Circle	Center	T
			BCC	Circle	Radius	T
	In 7 and 8, an equation for a circle is given. State the center and radius of the circle, sketch the circle, and solve the equation for *y*. 7. $x^2+y^2=121$	2	BCC	Circle	Center	T
			BCC	Circle	Radius	T
			WCC	Circle; S1	Circle; G	T
			WCC	Circle; S1	Circle S	T
UCSMP (p. 821)	**COVERING THE IDEAS** In 1 and 2, refer to the ellipse in the Activity. 1. What is the focal constant? 10	3	BCC	Ellipse	Focal constant	T
	2. What is the length of each a. semimajor axis? 5	4	BCC	Ellipse	a	T
	b. semiminor axis? 3	5	BCC	Ellipse	b	T
	 3. On the ellipse at the right, $OA=OB$, $OD=OC$ and $\overline{AB}\perp\overline{CD}$. Identify its a. foci. I, J	6	BCC	Ellipse	Foci	T
	b. major axis. $\overline{AB}$	7	BCC	Ellipse	Major axis	T
	c. minor axis. $\overline{CD}$	8	BCC	Ellipse	Minor axis	T
	APPLYING THE MATHEMATICS 11. Refer to the ellipse graphed at the right. a. Find an equation for the ellipse. $\frac{x^2}{64}+\frac{y^2}{16}=1$ 	9	WCC	Ellipse; G	Ellipse; S1	R

Note. *BCC* between-concept connections, *WCC* within-concept connections, *T* Typical, *R* Regular, *C2* Compulsory 2, *E2-1* Elective 2-1, # item number. The item number (the third column) is assigned based on the sample items included here, not the item number in the final codebook.

Table 4 Criteria for dense, moderate, and sparse digraphs.

	Digraph		
	Vertex	**Distinct edge**	**Total edge**
Dense	≥37	≥60	≥134
Moderate (Dense*75%)	≥28	≥45	≥100
Sparse (Dense*50%)	≥19	≥30	≥67

Note. The numeric cut-off for the density was calculated by the average number of vertices and the number of distinct and total edges per digraph (for between-concept). This was consistent with the numeric cut-off (50%) of strong connections determined by Jin and Wong (2015). If the digraph belongs to category x, it needs to satisfy at least two criteria.

Digraphs for BCC. Digraphs for the subtopic circle (see Fig. 3) and ellipse (see supplementary information) were all dense digraphs. The PEP-A series exhibited a denser network for circles with a similar number of vertices (48 and 49) but more distinct edges (96 > 78) and total edges (240 > 175); for ellipses, it also had more vertices (46 > 38), distinct edges (71 > 66), and total edges (135 > 128) than did the UCSMP series. This suggests that both series integrated diverse BCCs for circles and ellipses, especially for circles and the PEP-A series.

Comparatively, digraphs for the subtopic hyperbola (see supplementary information) and parabola (see Fig. 4) were moderate in the PEP-A series but sparse in the UCSMP series. The PEP-A series had

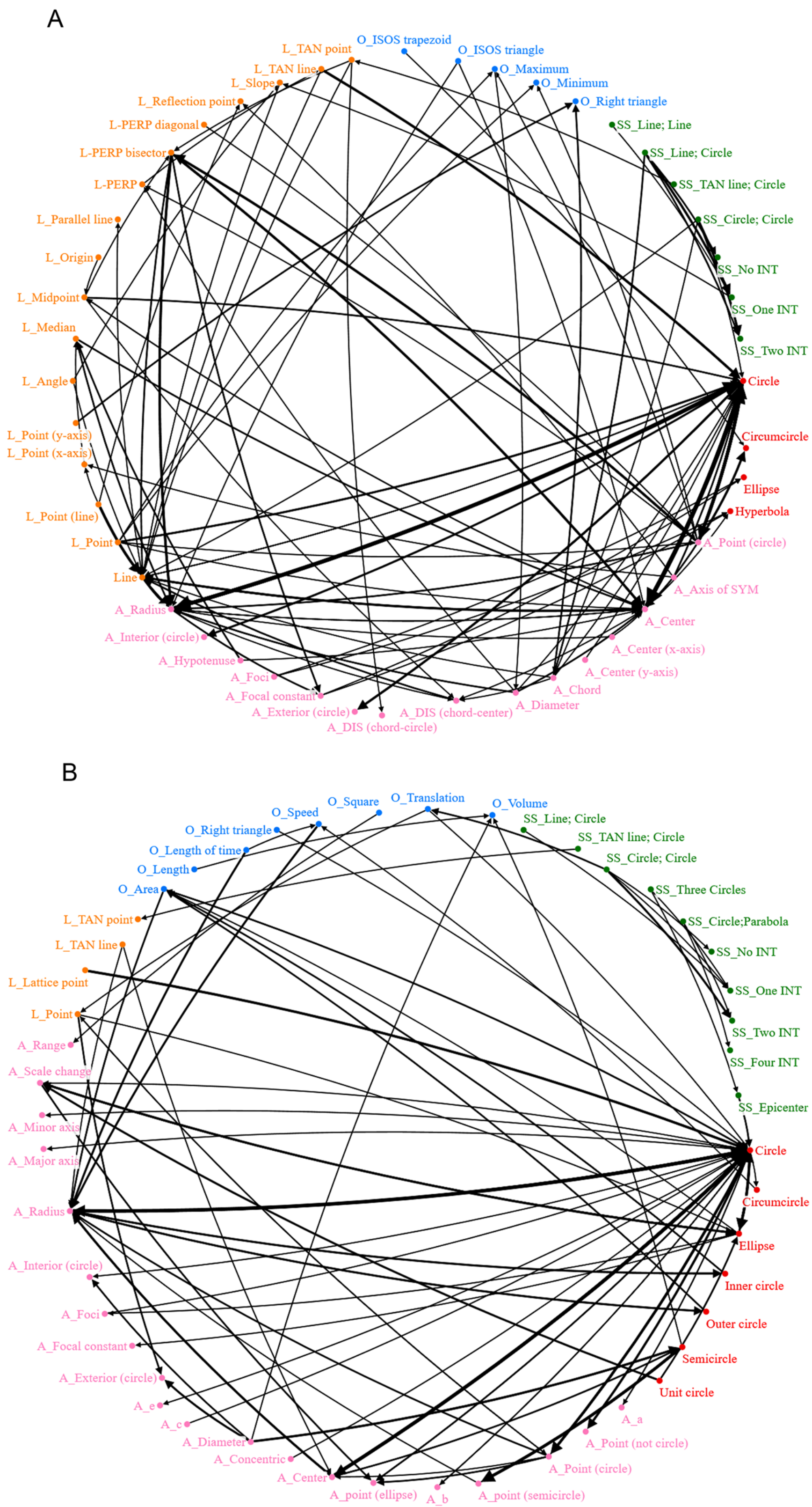


Fig. 3 Digraphs for BCC in the subtopic circle. A The PEP-A series. **B** The UCSMP series. Digraphs for BCC in the subtopic circle are dense in two series, in which vertices are placed within a circle according to five categories with different colors. Note. *TAN* tangent, *ISOS* isosceles, *PERP* perpendicular, *INT* intersections, *DIS* distance.

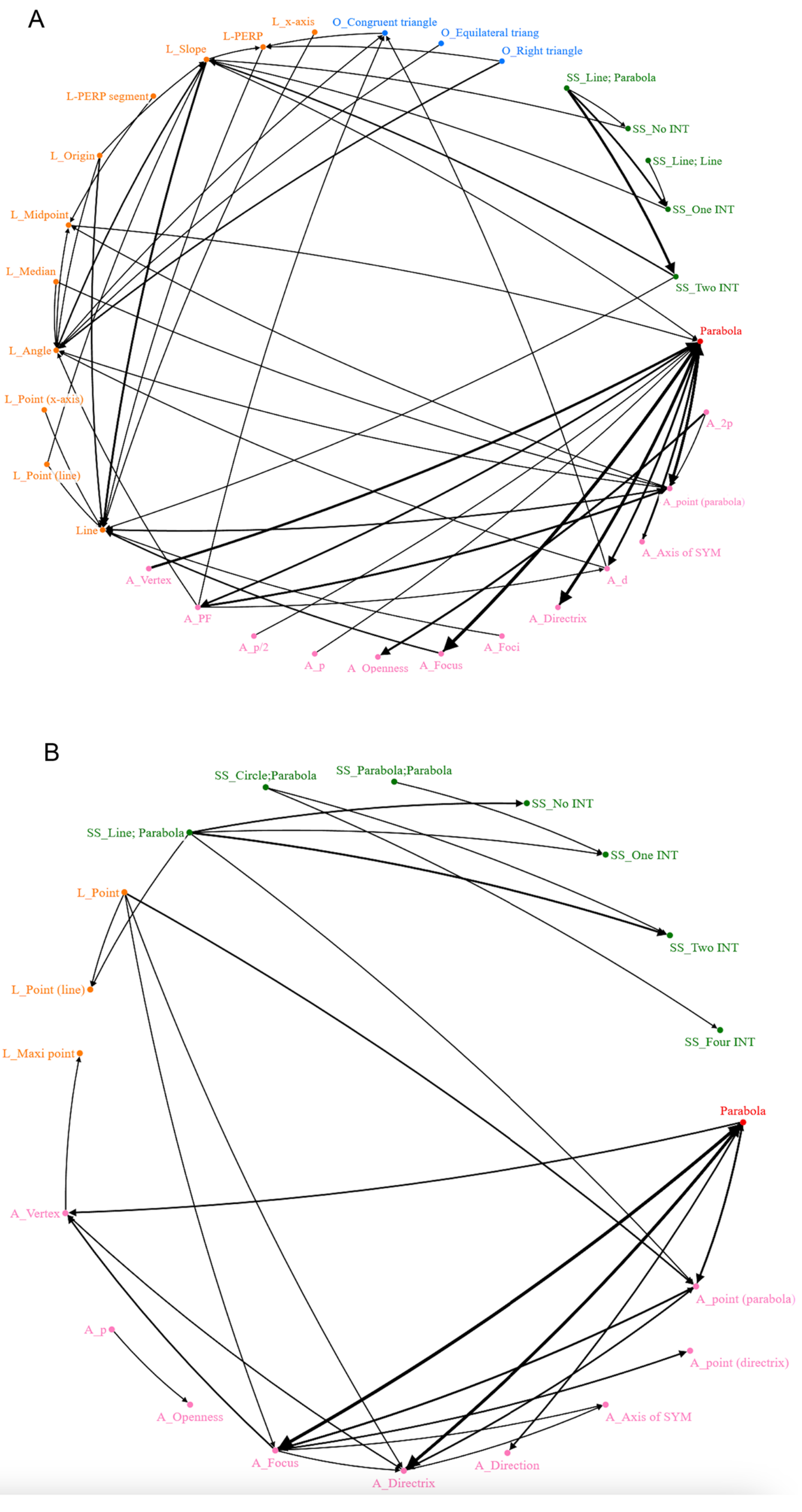


Fig. 4 Digraphs for BCC in the subtopic parabola. A The PEP-A series. **B** The UCSMP series. Digraphs for BCC in the subtopic parabola were moderate in the PEP-A series but sparse in the UCSMP series, in which vertices are placed within a circle according to five categories with different colors. Note. *INT* intersections, *PERP* perpendicular.

many more vertices ($42 > 24$; $32 > 20$), distinct edges ($51 > 30$; $56 > 33$), and total edges ($110 > 82$; $128 > 72$) than did the UCSMP series for the subtopic hyperbola and parabola, respectively. This indicates that the PEP-A series had more learning opportunities for BCCs in hyperbolas and parabolas than did the UCSMP series but was not as rich as the circles and ellipses for both series.

Considering aggregations, both series emphasized quadratic relations↔attributes, as indicated by aggregated edges between red and pink vertices. In the PEP-A series, extra attention is given to quadratic relations↔linear function-related concepts (evidenced by edges between red and orange vertices), whereas the UCSMP series emphasized connections in quadratic-quadratic systems (edges among red and green vertices).

For the subtopic circle, both series stressed circle, radius, and circle↔attributes. In the PEP-A series, extra attention is given to center and circle↔linear function-related concepts (for example, line and slope). In contrast, the UCSMP series stressed connections between special circles (semicircles, interior circles, exterior circles, inner circles, and outer circles) and their attributes and connections in quadratic-quadratic systems. This was directly related to two separate lessons in the UCSMP series, *Semicircles, Interiors, and Exteriors of Circles*, and the *Relationships between Ellipses and Circles*. For example, the problem 'Write an equation of the image of the circle $x^2 + y^2 = 1$ under the scale change $S_{5,1/2}$.' (p. 828) embedded connections from the circle and scale change to the ellipse were not addressed in the PEP-A series. This exemplary problem can be used to enhance circle↔ellipse in the PEP-A series. For the subtopic ellipse, the PEP-A series stressed ellipse↔attributes and ellipse↔linear function-related concepts, whereas the UCSMP series highlighted ellipse→attributes (especially the x-intercept, y-intercept, and foci).

For the subtopic hyperbola, both series emphasized hyperbola↔attributes. In the PEP-A series, extra attention is given to hyperbola↔linear function-related concepts. In terms of the ellipse-hyperbola system, the PEP-A series used eccentricity to enhance ellipse↔hyperbola, or the problem context in which an ellipse and a hyperbola share the same focus. In contrast, the UCSMP series simply required students to find all solutions to an ellipse-hyperbola system. Eccentricity can be used to enhance hyperbola↔ellipse in the UCSMP series. For the subtopic parabola, both series stressed parabola→focus and directrix. Only the PEP-A series highlighted connections between parabolas and linear function-related or triangle-related concepts (see examples later).

Digraphs for WCC. The UCSMP series showed a denser WCC (moderate) digraph, with many more vertices ($51 > 12$), distinct edges ($44 > 8$), and total edges ($79 > 21$), than did the PEP-A series (the sparsest one) (see Fig. 5). The PEP-A series had more learning opportunities for BCCs than for WCCs, which suggests that problems were more on relating concepts. Even though the UCSMP series integrated more and more diverse WCCs than did the PEP-A series, the density was still lower than that of the digraphs for BCC in the PEP-A series.

Regarding aggregation, symbolic representation was largely involved. The PEP-A series covered within-concept connections of points, circles, ellipses, and parabolas, mostly symbolic→graphical; none of them were bidirectional. In contrast, the UCSMP series stressed various within-concept connections in (a) circle, semicircle; (b) hyperbola, line-hyperbola systems; (c) ellipse, superellipse; and (d) parabola, line-parabola systems, as well as symbolic→graphical systems. Referring to Table 3, Item 9 of the UCSMP series presented graphical→symbolic of ellipse but still in limited settings. New technology, such as dynamic geometry software, is needed to enrich bidirectional WCC, especially in the PEP-A series.

Vertex: influential, prominent, and dual role. In Table 5, we report the concepts and representations (either the in- or out- is among the top 5 in the original BCC network or the top 2 in the schema-based BCC network and WCC network) in descending order, based on their centrality and connectivity.

Overall, the circle was the most central vertex exhibiting dual roles in the network of total BCCs with the highest centrality, of which 9.7% of BCCs in the PEP-A series and 9.9% of BCCs in the UCSMP series led to or out of circle. The top 5 in- and out-degree regions suggested that BCC leading to and out of circle, ellipse, hyperbola, and parabola (only out-) were emphasized in both, whereas center and line (only in-) were stressed in the PEP-A series, as well as the radius and two intersections (only in-) in the UCSMP series.

For the ego network, four quadratic relations played a dual role in the PEP-A series. However, this was not the case in the UCSMP series, as ellipses and parabolas were influential concepts, with a lower ratio of in- to out-degree (0.3, 0.5). This indicates that, on average, there were three times more connections linking the ellipse outwards to concepts other than connections leading to the ellipse. That is, the UCSMP series tended to accentuate the influence of ellipses on other concepts. According to Table 3, Items 3 to 8 in the UCSMP series, all simple single-step problems, covered six ellipse→attributes. The UCSMP series has four different sections in the exercises, in which the first section (covering the ideas, 40–50% of the total) usually contains more single-step problems with repetition. This suggests that the repetition of simple single-step problems may shift the balance of typical and reverse connections.

In addition, the line was prominent in the PEP-A series, and two intersections were prominent in the UCSMP series, with a higher ratio of in- to out-degree (2.6, 9.5). As shown in Fig. 6, the UCSMP series usually requires students to find solutions to a directly listed linear-quadratic or quadratic-quadratic system, while the PEP-A series contains many deliberately designed linear-quadratic problems. For example, in Table 3, Item 5 covered hyperbola→foci, angle→slope, slope→line, foci→line, line-parabola system→two intersections, which also reflected the influential role of lines.

Furthermore, the PEP-A series lacked learning opportunities for various connections in quadratic-quadratic systems, except for circle-circle and ellipse-hyperbola systems. More variations to linear function-related concepts for diverse connections in linear-quadratic systems can be embedded in the UCSMP series, whereas the PEP-A series can incorporate more learning opportunities for connections in quadratic-quadratic systems, such as the circle-ellipse system, as mentioned in the digraph analysis.

According to the schema-based view, the ellipse emerged as a pivotal vertex, exhibiting the greatest diversity of connected vertices with the highest connectivity in both series; the ellipse consistently played an influential role in the ego network of the UCSMP series. Approximately 7.4% of the distinct BCCs in the PEP-A series and 8.0% of the distinct BCCs in the UCSMP series led to or out of ellipse. This indicates that the ellipse was connected with the widest range of concepts but tended to be mainly connected outwards to other concepts in the UCSMP series. Furthermore, in the UCSMP series, the circle exhibited greater connectivity (7.8%), while in the PEP-A series, despite being involved in more than 100 connections, the circle exhibited limited diversity in connected vertices. This result suggests that the circle in the PEP-A series is somehow "separated" from other quadratic relations, as the PEP-A series covered the circle in one chapter and placed the ellipse, hyperbola, and parabola in another chapter. This finding implies that the separation of subtopics in two chapters may weaken the bidirectional connections between

Fig. 5 Digraphs for WCC. A The PEP-A series. **B** The UCSMP series. Digraphs for WCC were moderate in the UCSMP series and sparse in the PEP-A series (the sparsest one), in which vertices are placed within a circle for each subtopic with different colors. Note. *Q–Q* quadratic-quadratic, *INT* intersections, *QR* quadratic relations.

Table 5 Emphasized influential, prominent, and dual concepts and representations.

Network	PEP-A						UCSMP					
	Vertex	In	Out	%	I:O	Rating	Vertex	In	Out	%	I:O	Rating
Original	Circle	**64**	**53**	9.7	1.2	Dual	Circle	**46**	**38**	9.9	1.2	Dual
	Ellipse	**51**	**38**	7.4	1.3	Dual	Ellipse	**21**	**61**	9.7	0.3	Infl
	Hyperbola	**46**	**36**	6.8	1.3	Dual	Hyperbola	**25**	**32**	6.7	0.8	Dual
	Center	**39**	**28**	5.5	1.4	Dual	Radius	**22**	**19**	4.8	1.2	Dual
	Parabola	28	**37**	5.4	0.8	Dual	Parabola	13	**25**	4.5	0.5	Infl
	Line	**42**	16	4.8	2.6	Pro	Two INT	**19**	2	2.5	9.5	Pro
	Circle; G	***8***	*0*	*19.0*	*/*	*Pro*	*Circle; S1*	*2*	***16***	*11.4*	*0.1*	*Infl*
	Parabola; G	***6***	*0*	*14.3*	*/*	*Pro*	*Ellipse; G*	***8***	*4*	*7.6*	*2.0*	*Pro*
	Circle; S2	*0*	***6***	*14.3*	*0.0*	*Infl*	*Hyperbola; S1*	*0*	***9***	*5.7*	*0*	*Infl*
	Parabola; S1	*0*	***6***	*14.3*	*0.0*	*Infl*	*Circle; G*	***8***	*0*	*5.1*	*/*	*Pro*
							Hyperbola; G	***8***	*0*	*5.1*	*/*	*Pro*
Schema-based	Ellipse	**21**	**15**	7.4	1.4	Dual	Ellipse	**9**	**21**	8.0	0.4	Infl
	Line	**20**	8	5.7	2.5	Pro	Circle	**14**	**15**	7.8	0.9	Dual
	Hyperbola	19	**9**	5.7	2.1	Pro	*Circle; S1*	*1*	***4***	*3.2*	*0.3*	*Infl*

Note. % stands for centrality (original) and connectivity (schema-based); *I:O* In:Out, *Infl* Influential, *Pro* Prominent, *INT* intersections. Considering the connection network in each series, the top 5 concepts and representations (with concepts) are bolded in terms of in- and out-degree, respectively. For the schema-based network in each series, the top 2 concepts and representations (with concepts) are bolded based on in- and out-connection, respectively. Vertices with 0 to 3 for in-/out-degree/connection are not included here due to their minimal effect on the whole network .

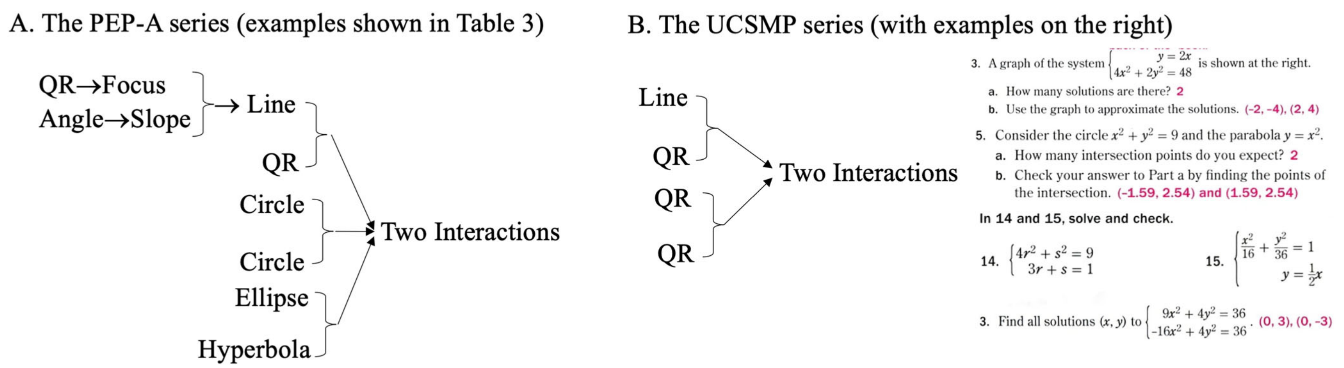


Fig. 6 Connections in problems of linear-quadratic and quadratic-quadratic systems. A The PEP-A series (examples shown in Table 3). **B** The UCSMP series (with examples). The UCSMP series usually requires students to find solutions to a directly listed linear–quadratic or quadratic–quadratic system, while the PEP-A series contains many deliberately designed linear–quadratic problems. Note. QR stands for quadratic relations. Sample problems in the UCSMP series come from p.847, p.848, p.854, and p.853.

these subtopics. Considering the notable differences in the approach to quadratic-quadratic systems and linear-quadratic systems, as shown in Fig. 6, circle↔ellipse and connections among special circles can be used to diversify connections involving circles in the PEP-A series.

For circles and parabolas in the PEP-A series and circles, ellipses, and hyperbolas in the UCSMP series, the graphical representation was prominent. The symbolic representation was influential in the ego network, and these representations were stressed with higher centrality (more than 5%) in the whole network. This finding was consistent with the digraph analysis showing that more typical symbolic→graphical quadratic relations existed than do reverse relations. For the symbolic representation of a circle, the UCSMP series stressed the standard form for a circle: $(x–a)^2 + (y–b)^2 = r^2$, while the PEP-A series highlighted the standard form for a quadratic equation: $\mathrm{Ax}^2+\mathrm{Bxy}+\mathrm{Cx}^2 + \mathrm{Dx}+\mathrm{Ey}+\mathrm{F} = 0$; it is more challenging for students to make circle→radius and circle→center than in the other form. In the schema-based view, most representations of concepts shared values of 0 to 3, except for the out-connection of the standard symbolic form of circle (4) in the UCSMP series, which revealed extremely low diversity of WCCs.

Bidirectional edge: typical, reverse, and balanced. The majority of identified connections (55% for total, 74% for distinct) were still not bidirectional. A total of 503 bidirectional connections (127 distinct) were identified, most of which were BCC (95% for total, 88% for distinct). The UCSMP series showed a slightly greater bidirectional percentage for BCC (48% > 46% for total; 28% > 25% for distinct) and a much greater bidirectional percentage for WCC (30% > 0% for total; 34% > 0% for distinct) than did the PEP-A series. The PEP-A series included more distinct BCC pairs (30 > 26), whereas the UCSMP series contained more bidirectional WCCs (6 pairs and 3 self-loops; 0 for PEP-A). Table 6 lists the bidirectional pairs (either the typical or reverse is among the top 5 in the corresponding BCC network or the top 2 in the corresponding WCC network) in descending order, based on their bidirectional gravity.

Both series highlighted quadratic relations↔attributes (such as radius, focus/foci, directrix, and focal constant) with high bidirectional gravity; some of these connections showed large weights. The top two pairs were circle↔center, which accounted for 14.4% of the bidirectional BCC pairs in the PEP-A series (9.0% for UCSMP), and circle↔radius (11.2% for PEP-A and 9.5% for UCSMP).

Table 6 Emphasized bidirectional between-concept and within-concept connections.

Network	PEP-A Source	PEP-A Target	T	R	%	T:R	Di-	UCSMP Source	UCSMP Target	T	R	%	T:R	Di-
BCC	Circle	Center	**18**	**22**	14.4	0.8	B	Circle	Radius	6	**13**	9.5	0.5	R
	Circle	Radius	**16**	**15**	11.2	1.1	B	Circle	Center	5	**13**	9.0	0.4	R
	Circle	Point (circle)	**11**	**10**	7.6	1.1	B	Parabola	Focus	**10**	**6**	8.0	1.7	T
	Parabola	Focus	**13**	4	6.1	3.3	T	Hyperbola	Foci	7	**9**	8.0	0.8	B
	Parabola	Directrix	**10**	4	5.1	2.5	T	Hyperbola	Focal constant	5	**9**	7.0	0.6	R
	Hyperbola	Foci	8	**6**	5.1	1.3	B	Parabola	Directrix	**8**	**6**	7.0	1.3	B
	Parabola	Point (parabola)	6	**6**	4.3	1.0	B	Ellipse	x-intercept	**9**	1	5.0	9.0	T
								Ellipse	y-intercept	**8**	1	4.5	8.0	T
								Semicircle	Point (semicircle)	**8**	1	4.5	8.0	T
WCC								*Ellipse; S1*	*Ellipse; G*	**6**	**3**	*37.5*	*2.0*	*T*

Note. T Typical, *R* Reverse; % bidirectional gravity, *T:R* typical:reverse, *Di-* directionality rating. Considering the connection network in each series, the top 5 bidirectional between-concept connections are bolded in terms of typical and reverse connections, respectively; the top 2 bidirectional within-concept connections are bolded based on typical and reverse connections, respectively. Edges with weights of 0 to 3 are not included here due to their minimal effect on the whole network.

In terms of the ratio of typical to reverse connections, the PEP-A series had an embedded balanced circle↔center (0.8) and circle↔radius (1.1). On average, for one typical circle→radius, there was one learning opportunity with the reverse: radius→circle. By comparison, the UCSMP series deemphasized circle→radius (0.5) and circle→center (0.4). Referring to Items 1 and 2 in Table 3 and the vertex analysis, the UCSMP series usually adopts the standard form for a circle, $(x–a)^2 + (y–b)^2 = r^2$, in which the radius and center of the circle can be directly seen from the equation of the circle, while the PEP-A series uses step-by-step variations in the $Ax^2+Bxy+Cx^2+Dx+Ey+F=0$ form of a circle. The UCSMP series can diversify problems with 'Find the radius and center of a circle in the standard form for a quadratic relation.' Furthermore, the UCSMP series emphasized more ellipse→x-intercept, ellipse→y-intercept than the reverse, with an extremely high ratio of typical to reverse connections (9.0, 8.0). This ratio was determined by digraph analysis and vertex analysis. Both series highlighted ellipse→foci and parabola→focus rather than the reverse, but they integrated balanced hyperbola↔foci. Overall, the most emphasized BCC pairs in the PEP-A series exhibited a balanced rating, whereas the UCSMP series stressed attributes→circle, ellipse→attributes, and symbolic→graphical of ellipse rather than the reverse. This suggests that the PEP-A series embedded BCC pairs in a more balanced way than did the UCSMP series, and the UCSMP series embedded more WCC pairs than did the PEP-A series but in an unbalanced way.

Discussion

In this study, we proposed a novel method of using social network analysis as an analytical tool, and we applied our method to examine more than 1000 connections identified in more than 500 U.S. and Chinese high school textbook problems dealing with quadratic relations, with interesting results in terms of connections in textbook problems.

Specifically, the digraph analysis revealed that the PEP-A series presented denser digraphs in between-concept connections than did the UCSMP series. The PEP-A series showed slightly less attention to hyperbolas and parabolas (moderate) than to circles and ellipses (dense), whereas the UCSMP series placed much more attention on circles and ellipses (dense) than on parabolas and hyperbolas (sparse). This leads to some practical implications that the network density of a specific area, to some extent, is related to the nature of mathematics itself. Furthermore, different placement of subtopics might be an underlying reason for the relative robustness of connection networks. The PEP-A series first addressed circles in one chapter and then placed ellipses, hyperbolas, and parabolas in another chapter, whereas the UCSMP series placed all subtopics in one chapter in the order of parabolas, circles, ellipses, and hyperbolas. Both series placed linear functions and quadratic relations far apart. The density trend was consistent with the sequence of subtopics in the textbooks. In addition, the separation of subtopics in textbooks may weaken connections between them. For example, the UCSMP series embedded limited linear↔quadratic, and the PEP-A series lacked circle↔ellipse. Although some subtopics are placed far apart in textbooks, intentional bidirectional connections between concepts that are connected but placed far away (e.g., linear↔quadratic in the PEP-A series) are viable ways to enhance connectivity. Moreover, individual lessons on the relationships between two subtopics contribute to richer corresponding connections (e.g., circle↔ellipse in the UCSMP series), which can be used to boost connectivity.

For within-concept connections, the UCSMP series showed a denser network (moderate) than did the PEP-A series (the sparsest) but exhibited lower density than did digraphs for between-concept connections in the PEP-A series. Both series emphasized symbolic representation and lacked graphical→symbolic quadratic relations. This was consistent with previous studies on a Chinese middle school textbook conducted by Wang et al. (2017) and a U.S. university-level calculus textbook conducted by Chang et al. (2016). In both studies, the results indicated that most tasks had symbolic→graphical only. Over the past 30 years, researchers have consistently reported students' difficulties in graphical→symbolic (e.g., Adu-Gyamfi et al., 2017). The consistency between students' difficulties and the lack of corresponding learning opportunities in textbook problems indicates that a sparse network of connections presented in textbook problems may hinder learners' bidirectional connection-making moves, thus influencing their learning progress. Moreover, a possible explanation for the striking difference in within-concept connection integration might be the best-known SPUR (skills, properties, uses, and representations) approach in the UCSMP series, in which the *representation* objective explicitly stresses within-concept connections.

Vertex analysis revealed that four quadratic relations were emphasized and rated as dual concepts in the ego network of the PEP-A series, whereas ellipses and parabolas played an influential role in the ego network of the UCSMP series. By reviewing the problems in the UCSMP series, single-step problems of finding one attribute of an ellipse contributed to stronger ellipse→attributes than the reverse. This leads to some practical implications in that the UCSMP series can save space for repetitive simple single-step problems with ellipse↔attributes and present deliberately designed problems with flexible ellipse↔attributes. Furthermore, two intersections in the UCSMP series and one line in the PEP-A series showed greater prominence in the ego network. The diversity of connections involving circles was limited, mostly with large weights, especially in the PPE-A series. The PEP-A series had deliberately designed linear-quadratic system problems with rich connections among linear function-related concepts, which could be adopted to enhance linear↔quadratic relations in the UCMSP series. The UCSMP series embedded various quadratic-quadratic systems, which can be adopted to diversify connections involving circles in the PEP-A series. These results suggest that textbook comparisons yield valuable insights into supplementing exemplary problems for certain connections if they are not available in a specific textbook series.

Bidirectional edge analysis revealed that the PEP-A series exhibited balanced typical and reverse between-concept connections, whereas the UCSMP series included many unbalanced between-concept and within-concept pairs. For example, several typical connections, such as ellipse/parabola→attributes and symbolic→graphical of the ellipse, were stronger than the reverse. This finding was consistent with earlier studies showing that some U.S. elementary and middle school textbooks lacked reverse connections compared to those present in their Chinese counterparts (Ding and Li, 2010). Furthermore, some between-concept pairs exhibited extremely large weights, e.g., circle↔radius, circle↔center. They were balanced in the PEP-A series, while the UCSMP series stressed the reverse. Problems of finding the radius/center of a circle in the form $(x–a)^2 + (y–b)^2 = \mathrm{r}^2$ are not challenging, and it is easier to add variations than for the $\mathrm{Ax}^2+\mathrm{Bxy}+\mathrm{Cx}^2 + \mathrm{Dx}+\mathrm{Ey}+F = 0$ form of a circle. This approach leads to varied learning opportunities for circle↔radius/center, strengthening the argument for investigating concepts and representations at the same time.

Moreover, in this study, we suggest that the new method has rich implications for practice and opens the possibility of adopting new analytical tools from SNA in mathematics education research. Specifically:

(1) Our study broadens the scope of content and the scale of curriculum analysis.
(2) Our study optimizes visualization for the network of connections in terms of graph components and the number of connections it can handle.
(3) Our study provides a nuanced investigation of connections through a systematic three-level quantitative analysis supported by a qualitative analysis of exemplary problems.
(4) Our study shows the convenience and diminished complexity of using the given digital technologies, given the nature of written text data.

First, regarding curriculum analysis, this framework broadened the scope of content; scaled up the number of connections, concepts, and representations covered; and supplemented novel analytical tools. Previous textbook problem analysis has usually been focused on one or two within-concept connections (WCC) or between-concept connections (BCC) (Adu-Gyamfi et al., 2017). Moreover, researchers have examined more WCCs than BCCs and have rarely investigated them together (Selinski et al., 2014). The mere analysis of concepts or representations may result in the loss of other critical aspects and the overall network. Additionally, missing one connection/concept/representation can influence the rest of the network. We supplemented previous studies by examining 1,029 between-concept and 100 within-concept concepts together, and thus, we reached some generalized conjectures about the structure of connection networks.

Second, previous visualizations covered a limited number of concepts or were tangled with multiple edges (Jin and Wong, 2015; Strom et al., 2001). In our study, we collapsed multiple edges by using weighted edges, which can address a large number of concepts, representations, and connections in a digraph; this approach provided a clear and comparable visualization of what connections were exactly represented. Moreover, coloring vertices highlighted the structural features, making the concept component and connection trends much more visible.

Third, the systematic three-level quantitative analysis of the digraph, vertex, and bidirectional edge level accompanied by qualitative analyses of relevant exemplary problems is innovative in textbook research. In previous textbook analyses, researchers usually adopted the number or percentage of connections or concepts, or representations (Ding, 2016; Wang et al., 2017) as the dominant analytical tool, but this approach failed to reveal the type, directionality, and structure of connection networks. Furthermore, in previous textbook analyses, researchers often used (a) one specific connection (Ding and Li, 2010; Wang et al., 2017) or (b) the reverse direction to depict the direction in which students had difficulties (Adu-Gyamfi et al., 2017; Jin and Wong, 2015). In the present study, we made progress in providing a clear definition of typical and reverse connections in curriculum materials that can be used in future textbook analysis. In addition, previous U.S. and Chinese textbook comparisons often addressed problem types (Zhu and Fan, 2006) and elementary or middle school mathematics content and textbooks (Wang and Lu, 2018). In our study, we extended textbook research to a new perspective, connections represented in high school textbooks, which deserve more attention and further study.

Fourth, this study differed from prior empirical research by using graph theory based on the nature of the collected data (e.g., Strom et al., 2001). Textbook problems usually go through several rounds of revision in a certain format, whereas verbal data from discourses, lectures, and interviews reveal evoked connections at the time they are conducted—a more unstable and divergent nature compared to that of the written text. Correspondingly, data coding and analysis were somewhat complicated and labor intensive, while digitalization may diminish the complexity of using this method. Sun (2013) embedded NodeXL into Moodle and successfully collected and visualized the data (logs of interactions between students and teachers), thereby reducing the challenges of data collection and analysis. With precoded connections for each mathematics problem and preset algorithms, future researchers can design digital platforms to collect the data and to automatically conduct connection analysis. Digital affordance diminished the complexity of using this novel approach.

Conclusions and implications

In the present study, we examined more than 1,000 mathematical connections in U.S. (the UCSMP series) and Chinese (the PEP-A series) mathematics textbook problems that are related to quadratic relations. Typical and reverse connections in curriculum materials were clearly defined for the first time and can be used in future textbook research. The analysis presented here is the first time connections in textbooks have been examined by using graph theory as an analytical tool. Similarities and differences in typical and reverse connections between and within concepts are observed across the two series. This approach validates the potential of this method in cross-national textbook comparisons and elucidates (a) gaps between typical and reverse connections presented in textbooks and potential reasons for unbalanced connections; (b) good practices of integrating balanced between-concept or within-concept connections in textbooks; and (c) visualizations of connection networks depicting intricate details such as concepts, representations, and connections involved, which can be used to enrich learning opportunities for BC.

The results have important implications for education practice. For students, the framework can be adopted to examine connections they construct to assess and promote their conceptual understanding, either at the individual level or the whole class level. Vivid visualization of students' connection networks reflects and documents the progress of their conceptual understanding, i.e., learning trajectory, and thus assists students in acquiring the perception that mathematics is a discipline of connected ideas or an integrated whole and in producing valuable information (e.g., structural holes, flexibility in between-concept and within-concept connections) to improve the teaching and learning of mathematics. Furthermore, teachers can grasp students' learning difficulties in a particular concept, representation, and connection from the visualization and three-level analysis results of the network of students' constructed connections and then can adjust teaching strategies accordingly. Teachers are recommended to select problems with balanced bidirectional connections and to supplement problems from other textbooks if certain connections are not available; this is important since unbalanced opportunities may hinder students from grasping bidirectional connections.

Moreover, teachers and textbook authors are encouraged to pay attention to connections (a) between subtopics in one chapter, for example, circle↔ellipse by scale change, ellipse↔hyperbola by eccentricity; (b) between two subtopics that are connected by nature but split over two chapters or textbooks; and (c) leading out of the graphical and into the symbolic representation. It is also viable for teachers and textbook authors to consider increasing the diversity of representations to provide rich and balanced typical and reverse within-concept connections with the help of digital interactive software. For example, more learning opportunities for diverse within-concept connections have become available as e-textbooks and e-exercise platforms (e.g., Gueudet et al., 2018).

Regarding limitations, as our comparison covered two specific textbook series, the results should be taken with caution in generalizing to broader U.S. and Chinese textbook comparisons. However, both series were popular in their own countries in the past ten years and emphasized connections. In this sense, the data largely are representative. Additionally, even though textbook analysis contributes to our understanding of what students learn and how teachers teach in certain ways, we do indicate that there is some gap between what is intended in the textbooks and what happens in the classroom (also see Zhu and Fan, 2006), and these potential gaps exist between the textbooks and the curricula in each country, as well as their similarities and differences in connections.

For other future studies, the new method can be extended to study connections in (a) a unit plan; (b) other mathematical topics such as statistics or subjects such as physics; (c) textbooks or curriculum materials in other countries or different series in China and the U.S.; and (d) e-textbooks. Digital textbooks, which tend to be 'fertile soil' for cultivating richer connections than printed ones, have a high level of connectivity and coherence (O'Halloran et al., 2018); there is a call for new theoretical frameworks to analyze e-textbooks (Gueudet et al., 2018). Instead of merely counting connections, in future studies, researchers can adopt the present method to investigate overall connectivity in e-textbooks. The feasibility and the validity of the present method invite the possibility of adopting efficient analytical tools from SNA in future mathematics education research.

Data availability

The datasets generated during and/or analyzed during the current study are available in the Harvard Dataverse repository, https://doi.org/10.7910/DVN/9BFHP0.

Received: 25 July 2023; Accepted: 26 March 2024;
Published online: 08 April 2024

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Acknowledgements

This study was partly based on a portion of the first author's doctoral dissertation completed at Columbia University under the direction of Prof. Alexander Karp. Significant changes were made to this publication. This work was in part supported by a research grant from East China Normal University's 'Happy Flowers' Strategic Research Fund [Award No: 2019ECNU-XF2H004], a research grant from the Science and Technology Commission of Shanghai Municipality [Award No: 22DZ2229014], and another research grant from Ministry of Education of China [Award No: 22YJC880035].

Author contributions

SL and LF conceptualized the whole framework. SL carried out data collection, analysis, and visualization of this article. SL and LF participated in writing and revising the manuscript and approved the final manuscript.

Competing interests

The authors declare no competing interests.

Ethical approval

This article does not contain any studies with human participants performed by any of the authors.

Informed consent

This article does not contain any studies with human participants performed by any of the authors.

Additional information

Supplementary information The online version contains supplementary material available at https://doi.org/10.1057/s41599-024-02991-w.

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