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Hyperbolicity and bounded-valued cohomology

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Abstract

We generalise a theorem of Gersten on surjectivity of the restriction map in ℓ^∞-cohomology of groups. This leads to applications on subgroups of hyperbolic groups, quasi-isometric distinction of finitely generated groups and ℓ^∞-cohomology calculations for some well-known classes of groups. Along the way, we obtain hyperbolicity criteria for groups of type $FP_2(\mathbb{Q})$ and for those satisfying a rational homological linear isoperimetric inequality, answering a question of Arora and Martínez-Pedroza.

1 Introduction

Bounded-valued cohomology, also known as ℓ^∞-cohomology, is a quasi-isometry invariant of finitely generated groups. It was introduced by Gersten as a tool to find lower bounds for the Dehn function of some finitely presented groups [23, 27]. Later, Gersten found a complete characterisation of hyperbolic groups in terms of ℓ^∞-cohomology [24].

Theorem 1.1 (Gersten) *Suppose that G is a finitely presented group. Then G is hyperbolic if and only if $H^2_{(\infty)}(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$.*

Recently, Milizia [36], extended this characterisation to relatively hyperbolic groups, and also gave a characterisation of amenable groups using the comparison map from ordinary group cohomology to ℓ^∞-cohomology. Another result of Gersten concerns the surjectivity of the restriction homomorphism in ℓ^∞-cohomology of groups [24, Theorem 6.3].

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Theorem 1.2 (Gersten) *Let G be the fundamental group of a finite aspherical 3-complex such that* cd $G = 2$. *Suppose H is a finitely presented subgroup of G. Then the restriction homomorphism $H^2_{(\infty)}(G; V) \to H^2_{(\infty)}(H; V)$ is surjective for all normed $\mathbb{Z}$-modules V.*

By combining Theorems 1.1 and 1.2, Gersten in [24] obtained a new proof of his remarkable theorem which states that a finitely presented subgroup of a hyperbolic group G with cd $G = 2$ is hyperbolic [26].

Our main result is the following generalisation of Theorem 1.2.

Theorem 1.3 *Let R be a subring of $\mathbb{Q}$ containing $\mathbb{Z}$. Suppose H is a group of type $FP_n(R)$ and is a subgroup of G with $\mathrm{cd}_R G = n < \infty$. Then the restriction homomorphism $H^n_{(\infty)}(G; V) \to H^n_{(\infty)}(H; V)$ is surjective for all normed R-modules V.*

Here, $\mathrm{cd}_R G$ stands for the *cohomological dimension* of a group G over a ring R. When $R = \mathbb{Z}$, we omit it from the notation and simply write cd G. This notion and properties F, $FP(R)$, $FP_n(R)$ and $FP_\infty(R)$ are reviewed is Sect. 2.
Theorem 1.3 generalises Theorem 1.2 in several ways: by allowing torsion in G, by removing the finiteness assumption on G, and by extending to higher cohomological dimensions. We can then apply Theorem 1.3 to subgroups of hyperbolic groups of type F, or more generally of type $FP(\mathbb{Q})$.

A major tool in the proof of Theorem 1.3 is our new characterisation of ℓ^∞-cohomology of a group G as the cohomology of orbitally-bounded cochains on any projective resolution of R over the group ring RG. The freedom to interpolate between different projective resolutions allows us to also generalise Gersten's Theorem 1.1, from finitely presented groups, to groups of type $FP_2(\mathbb{Q})$. Together, they lead to applications such as Corollary 1.5 below.

Corollary 1.4 *Let G be a hyperbolic group which is not virtually free. Suppose H is a subgroup of type F. Then $H^n_{(\infty)}(H; V) = 0$ for all normed $\mathbb{R}$-modules V, where $n = \mathrm{cd}_\mathbb{Q} G$.*

Corollary 1.5 *Let H be a subgroup of a hyperbolic group G such that $\mathrm{cd}_\mathbb{Q} G = 2$. Then H is hyperbolic if and only if H is of type $FP_2(\mathbb{Q})$.*

Weaker versions of Corollary 1.5 were previously established by Gersten when cd $G = 2$ and H is of type $FP_2(\mathbb{Z})$ [26], and more recently by Arora and Martínez-Pedroza [2] when $\mathrm{cd}_\mathbb{Q} G = 2$ and H is finitely presented. Gersten's result implied that homological coherence and coherence are equivalent for hyperbolic groups of cohomological dimension two. Corollary 1.5 strengthens this both in the direction of the hyperbolic group and the subgroup: for hyperbolic groups of rational cohomological dimension two, rational homological coherence is equivalent to coherence.

In [43], Rips constructed the first examples of finitely generated subgroups of hyperbolic (small cancellation) groups that are not finitely presented and hence not hyperbolic (see Theorem 5.3). Next corollary shows that these subgroups are not even of type $FP_2(\mathbb{Q})$.

Corollary 1.6 *Given any finitely presented group Q, let*

$$1 \to N \to G \xrightarrow{q} Q \to 1$$

be the Rips construction [43, Theorem 1], *for G being $C'(1/6)$, with N finitely generated. Suppose P is a finitely generated subgroup of Q which is not finitely presented. Then $q^{-1}(P)$ is finitely generated but not of type $FP_2(\mathbb{Q})$.*

An advantage of using rational cohomological dimension over the integral one is that it allows torsion. Corollary 1.5 can then be directly applied to one-relator groups.

Corollary 1.7 *A subgroup of a hyperbolic one-relator group is hyperbolic if and only if it is of type $FP_2(\mathbb{Q})$.*

Corollary 1.7 in particular applies to one-relator groups with torsion, since they are hyperbolic [4]. Note that Jaikin-Zapirain and Linton subsequently proved that one-relator groups are coherent [32], from which the above may be alternatively deduced in combination with earlier results (in fact, $FP_2(\mathbb{Q})$ may be relaxed to finitely generated).

Groups of rational cohomological dimension 2 play a prominent role in geometric group theory. Besides aforementioned examples, they also notably include random groups and fundamental groups of graphs of virtually free groups with virtually cyclic edge groups. Both of these classes of groups were shown to satisfy Gromov's Surface Subgroup Conjecture [13, 47].

Recall that a finitely presented group is necessarily of type $FP_2(\mathbb{Z})$, i.e. almost finitely presented, which in turn is stronger than being of type $FP_2(\mathbb{Q})$. In order to relax the finiteness assumption on the subgroup H in Corollary 1.5, we generalise in Sect. 4 Gersten's Theorem 1.1 and a result of Mineyev [38] from finitely presented groups to groups of type $FP_2(\mathbb{Q})$. This also leads us to answer positively a question of Arora and Martínez-Pedroza [2, Question 1.9] on hyperbolicity of groups of type $FP_2(\mathbb{Q})$ satisfying a rational homological linear isoperimetric inequality.

Theorem 1.8 *Suppose G is a group. Then G is hyperbolic if and only if it is of type $FP_2(\mathbb{Q})$ and its rational homological filling function of degree 2 is bounded above by a linear function.*

A similar result over other rings, but with a notion of an isoperimetric inequality that is not the usual one, appears in [33].

As Gersten states in [22], "Calculation of $H_{(\infty)}^n(G; \mathbb{R})$ poses an interesting but difficult problem". Indeed, there are only a few such calculations in the literature. Also, applications of these calculations have mostly been in degrees $n = 1, 2$. Next, we state some applications of Theorem 1.3 on (non-)vanishing of $H_{(\infty)}^n(G; \mathbb{R})$ in the top degree.

Corollary 1.9 *Let $A = \mathbb{Z}$ or $\mathbb{R}$. Suppose Γ is one of the following groups and $d = \mathrm{vcd}\,\Gamma \geqslant 1$:*

(i) *Γ is a finitely generated right-angled Artin group;*
(ii) *Γ is the Bestvina–Brady group of any finite flag complex L^n with $H^n(L; A) = 0$;*

(iii) Γ is the mapping class group of a closed orientable surface of genus 2;

(iv) $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ for $n \geqslant 2$;

(v) $\Gamma = \mathrm{Out}(\mathbb{F}_n)$ for $n \geqslant 2$.

Then the cokernel of the comparison map $\iota^d : H^d(\Gamma; A) \to H^d_{(\infty)}(\Gamma; A)$ contains a continuum of linearly independent elements.

In contrast, by a result of Wienhard [46], all uniform lattices in connected semisimple Lie groups with finite centre have vanishing ℓ^∞-cohomology with real coefficients in dimensions above the dimension of maximal flats of the associated symmetric space. In Section 5, we obtain an analogue of Wienhard's result for relatively hyperbolic groups (see Corollary 5.14).

Poincaré duality groups have vanishing ℓ^∞-cohomology with real coefficients in the top dimension precisely when they are not amenable (see Lemma 5.5). This can be seen from the fact that ℓ^∞-cohomology of a countable group is dual to the *uniformly finite homology* of Block and Weinberger and their fundamental characterisation of amenable groups via uniformly finite homology in degree zero [11]. The following are applications of this result and Theorem 1.3.

Corollary 1.10 *Let G be a finitely generated relatively hyperbolic group with $\mathrm{cd}_{\mathbb{Q}} G < \infty$. Let Γ be a finitely generated group with $2 \leqslant \mathrm{cd}_{\mathbb{Q}} \Gamma < \infty$. Suppose that Γ has an amenable Poincaré duality subgroup of dimension $\mathrm{cd}_{\mathbb{Q}} \Gamma$. If G is quasi-isometric to Γ, then there is a peripheral subgroup H of G such that $\mathrm{cd}_{\mathbb{Q}} H = \mathrm{cd}_{\mathbb{Q}} G$.*

Corollary 1.10 applies, for example, to groups Γ listed in Corollary 1.9; of note being the case when Γ is a Bestvina–Brady group, as the other cases were previously known.

Corollary 1.11 *Let Γ be a finitely generated group with $2 \leqslant \mathrm{cd}_{\mathbb{Q}} \Gamma < \infty$. Suppose that Γ has an amenable Poincaré duality subgroup of dimension $\mathrm{cd}_{\mathbb{Q}} \Gamma$. Suppose Γ is quasi-isometric to the fundamental group G of a finite graph of hyperbolic groups. Then $\mathrm{cd}_{\mathbb{Q}} G = \mathrm{cd}_{\mathbb{Q}} \Gamma = 2$.*

Corollary 1.12 *Let Γ be a finitely generated group satisfying the hypothesis of Corollary 1.9. Suppose Γ is quasi-isometric to the fundamental group of a finite graph of hyperbolic groups. Then $\mathrm{cd}\, \Gamma \leqslant 2$, and Γ is either a right-angled Artin group or a Bestvina–Brady group.*

Besides our main result, Theorem 1.3 and its applications, in Sect. 5, we consider a long exact sequence in group cohomology arising from a group pair of a relatively hyperbolic group G and the collection of its peripheral subgroups. We use this sequence for calculations of ℓ^∞-cohomology of G. The following result is an application of such a calculation.

Theorem 1.13 *Let M be a complete, finite-volume Riemannian manifold of dimension n of negative sectional curvature bounded away from zero. Then $H^n_{(\infty)}(M; \mathbb{R}) = 0$. In addition, if M is non-compact, then $H^{n-1}_{(\infty)}(M; \mathbb{R})$ contains a continuum of linearly independent elements.*

Another motivation to study ℓ^∞-cohomology comes from its connection to Gromov's bounded cohomology via the comparison maps:

$$H_b^n(G; V) \xrightarrow{c^n} H^n(G; V) \xrightarrow{\iota^n} H_{(\infty)}^n(G; V). \tag{1}$$

The composition of the comparison maps is known to vanish [46] and Wienhard asks when the sequence is exact [46, Question 8]. Blank in [7, Question 6.3.10] also asks if the sequence (1) can be extended. Frigerio and Sisto in [21], show that the exactness of (1), when G is finitely presented, $V = \mathbb{R}$ and $n = 2$, is equivalent to a conjecture of Gromov on bounded primitives of differential forms on universal covers of closed Riemannian manifolds [28]. Using this reformulation, Gromov's conjecture was recently disproved by Ascari and Milizia [1].

In Sect. 3, we make the following observation.

Theorem 1.14 *Let G be a relatively hyperbolic group with amenable peripheral subgroups, then the sequence* (1) *is exact for all $n \geqslant 2$ and all dual normed $\mathbb{R}G$-modules V.*

We should note that when $n = 2$ and $V = \mathbb{R}$, Theorem 1.14 follows from Theorem 4 and Proposition 14 of [21].

In Sect. 5, we also define ℓ^∞-*cohomological dimension* $\mathrm{cd}_{(\infty)} G$ of a group G. It is a quasi-isometry invariant of finitely generated groups. As we have alluded to in Corollaries 1.10 and 1.12, $\mathrm{cd}_{(\infty)} G$ can be useful for distinguishing finitely generated groups up to quasi-isometry. By Theorem 1.1 and [37, Theorem 20], a finitely presented group G is hyperbolic if and only if $\mathrm{cd}_{(\infty)} G \leqslant 1$. It is natural to ask what other properties of a group G can be detected by $\mathrm{cd}_{(\infty)} G$. In Section 6, we end with some questions on $\mathrm{cd}_{(\infty)} G$ and its connections to acylindrical hyperbolicity, the dimension of maximal flats in cocompact CAT(0) complexes, and higher homological filling functions of G.

1.1 A few words on the proof of Theorem 1.3

In Gersten's proof of Theorem 1.2 [24, Theorem 6.3], he constructs a retraction from the relation module $Z_1(G)$ to the relation module $Z_1(H)$ which is a bounded linear map with respect to an ℓ^1-norm on $Z_1(G)$ and the standard norm on $Z_1(H)$. A key property used by Gersten is that $Z_1(G)$ can be constructed to be finitely generated and free. This follows from the assumptions that $\mathrm{cd}\, G = 2$ and that G admits a finite 3-dimensional $K(G, 1)$-complex. The ℓ^1-norm on $Z_1(G)$ is then equivalent to the standard norm, and the retraction becomes bounded with respect to the standard norms. As he states, "When translated into the language of ℓ^∞-cohomology, this means that the restriction homomorphism is surjective".

This method of using the standard norm and Gersten's original definition of ℓ^∞-cohomology does not work in our generality, because we have no finiteness assumptions on G. First, we make the key observation that the ℓ^∞-cohomology of a group can be defined using orbitally-bounded cochains on any projective resolution. For our purposes, this resolution is the truncation of a suitably constructed

free RG-resolution $F_* \twoheadrightarrow R$. In order to deal with torsion in G, we work over any ring $\mathbb{Z} \leqslant R \leqslant \mathbb{Q}$. The resolution $F_* \twoheadrightarrow R$ comes with an embedded RH-resolution $P_* \twoheadrightarrow R$ by finitely generated free RH-modules and such that $Z_{n-1}(F_*)$ is a free RG-module equipped with an ℓ^1-norm. We then construct a retraction $\rho : Z_{n-1}(F_*) \to Z_{n-1}(P_*)$ which is also a bounded linear map with respect to the corresponding norms. With this in hand, given an n-cocycle $\alpha \in Hom_R(Z_{n-1}(F_*), V)$ bounded on H-orbits, the retraction ρ defines an n-cocycle $\beta \in Hom_R(Z_{n-1}(F_*), V)$, which maps to α under the restriction map. The fact that ρ is a bounded linear map allows us to show that β is bounded on G-orbits. This shows that the restriction map in ℓ^∞-cohomology is surjective in degree n.

2 Preliminary results

First, we define the necessary finiteness notions and mention their relevant properties. For more details, we refer the reader to Chapter VIII of [10].

Let R denote a commutative ring with unit and G denote a discrete group. The *cohomological dimension of G over R* can be defined by

$$\text{cd}_R \, G = \sup\{n \in \mathbb{N} \mid H^n(G, M) \neq 0 \text{ for some } RG\text{-module } M\}.$$

When $R = \mathbb{Z}$, we omit it from the notation and denote the cohomological dimension by cd G. A group G is said to have finite *virtual cohomological dimension* equal to n, denoted vcd G, if G has a subgroup H of finite index with cd $H = n$. This is a well-defined notion by [10, VIII.2.4]. A group G is said to be of *type $FP_n(R)$* for some $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ if there is a projective RG-resolution

$$\cdots \to P_2 \to P_1 \to P_0 \twoheadrightarrow R$$

such that P_i is finitely generated and projective as an RG-module for all $i \leqslant n$. A projective RG-resolution $P_* \twoheadrightarrow R$ is said to be of *length n*, if $P_n \neq 0$ and $P_i = 0$ for all $i > n$. A group G is said to be of *type $FP(R)$*, if there is a projective RG-resolution of R by finitely generated projective RG-modules of finite length. It is immediate from the definitions that if G is of type $FP(R)$, then it is of type $FP_\infty(R)$.

Denote $\text{cd}_R \, G = n$. By [10, VIII.2.1], it follows that G is of type $FP(R)$ if and only if $n < \infty$ and G is of type $FP_n(R)$.

There are analogous topological finiteness properties which we explain next.

An Eilenberg-MacLane space or $K(G, 1)$-complex is an aspherical CW-complex with fundamental group equal to G. A group G is of *type F_n* if there is a $K(G, 1)$-complex X with finite n-skeleton and it is of *type F* if X is finite. If G is of type F_n or F, then it is of type $FP_n(\mathbb{Z})$ or FP, respectively. A group G is finitely generated if and only if it is of type F_1 or $FP_1(R)$ [6, 1.2.1]. A group G is finitely presented if and only if it is of type F_2.

Let H be a subgroup of G. Recall that given an RH-module M, the induced RG-module is defined as $\text{Ind}_H^G M := RG \otimes_{RH} M$. The left action of G on RG induces an

RG-module structure on $\mathrm{Ind}_H^G M$. Define an RH-module homomorphism

$$j_M : M \to \mathrm{Ind}_H^G M$$

by $j_M(m) = 1_G \otimes m$, $\forall m \in M$. The following result is a direct corollary of [10, III.5.1].

Lemma 2.1 *Suppose G is a group, H is a subgroup of G and M is an RH-module. Then j_M is a canonical embedding of M as a direct RH-module summand in $\mathrm{Ind}_H^G M$.*

Throughout the remainder of this section, R will denote a subring of $\mathbb{Q}$ containing $\mathbb{Z}$. Let F be a free RG-module with free RG-basis Λ. Then $\Omega = \{g\alpha \mid g \in G, \alpha \in \Lambda\}$ is a free R-basis of F. The ℓ^1-*norm on F with respect to Λ* is defined as

$$\left| \sum_{x \in \Omega} a_x x \right|_1 = \sum_{x \in \Omega} |a_x|.$$

Definition 2.2 Let $\rho : F \twoheadrightarrow M$ be a surjective homomorphism of RG-modules where F is a free RG-module given the ℓ^1-norm $|\cdot|_1$ with respect to a fixed basis. The *filling norm* on M (with respect to ρ) is defined as

$$\|m\| = \inf\big\{ |x|_1 \mid x \in F, \ \rho(x) = m \big\}.$$

When F is finitely generated over RG, we will say that this is the *standard norm* on M.

Remark 2.3 The filling norm on an RG-module is a pseudo-norm. It is a norm for example if $R = \mathbb{Z}$.

Lemma 2.4 ([39, Lemma 4.6]) *Let $f : M \to N$ be a homomorphism between finitely generated RG-modules with standard norms $\|\cdot\|_M$ and $\|\cdot\|_N$, respectively. Then, there is a constant $C > 0$ so that $\|f(m)\|_N \leqslant C\|m\|_M$ for all $m \in M$.*

Two filling norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on an RG-module M are called *equivalent* if there a constant $C > 0$ such that for all $m \in M$,

$$C^{-1}\|m\|_1 \leqslant \|m\|_2 \leqslant C\|m\|_1.$$

Lemma 2.4 implies the following result.

Lemma 2.5 *Any two standard norms on a finitely generated RG-module are equivalent.*

By Lemma 2.5, the standard norm on a finitely generated RG-module is well-defined up to linear equivalence.

We will need the following result of Gersten in [26].

Proposition 2.6 ([26, Proposition 4.4]) *Assume that there is a short exact sequence of RH-modules*

$$0 \to M \xrightarrow{\iota} N \xrightarrow{\pi} P \to 0,$$

where M is a finitely generated RH-module and equipped with the standard norm, N is RH-free equipped with the associated ℓ^1-norm, and P is projective. Then there is an RH-retraction $\rho : N \to M$ for ι and a constant $C > 0$ so that $\|\rho(x)\| \leqslant C|x|_1$ for all $x \in N$.

We are ready to prove the main result of this section.

Proposition 2.7 *Let G be a group with $\mathrm{cd}_R\, G = n \geqslant 1$. Suppose H is a subgroup of G of type $FP(R)$ and $P_* \twoheadrightarrow R$ is a resolution by finitely generated free RH-modules. Then, the following hold.*

(1) *There is a free RG-resolution $F_* \twoheadrightarrow R$ where $Z_{n-1}(F_*)$ is RG-free equipped with the associated ℓ^1-norm and an injective RH-morphism $\iota_* : P_* \to F_*$.*
(2) *There is a constant $C > 0$, and an RH-retraction $\rho : Z_{n-1}(F_*) \twoheadrightarrow Z_{n-1}(P_*)$ for $\iota_{n-1} : Z_{n-1}(P_*) \hookrightarrow Z_{n-1}(F_*)$ such that $\|\rho(y)\| \leqslant C|y|_1$ for all $y \in Z_{n-1}(F_*)$.*

Proof The induction of $P_* \twoheadrightarrow R$ gives the free RG-resolution $\mathrm{Ind}_H^G P_* \twoheadrightarrow \mathrm{Ind}_H^G R$. Let $F'_* \twoheadrightarrow R$ be a free RG-resolution. By the Fundamental Lemma of Homological Algebra [10, Lemma I.7.4], there is an RG-morphism $f_* : \mathrm{Ind}_H^G P_* \to F'_*$ extending the augmentation map $\mathrm{Ind}_H^G R \twoheadrightarrow R : g \otimes r \mapsto r$, $\forall g \in G$, $\forall r \in R$. Denote by $C(f)_*$ the algebraic mapping cylinder of f_* [45, 1.5.5]. Recall that

$$C(f)_i = \mathrm{Ind}_H^G P_i \oplus \mathrm{Ind}_H^G P_{i-1} \oplus F'_i$$

for all $i > 0$, $C(f)_0 = \mathrm{Ind}_H^G P_0 \oplus F'_0$. Since the natural inclusion $F'_* \hookrightarrow C(f)_*$ is a chain homotopy equivalence [45, 1.5.4], we have that $C(f)_* \twoheadrightarrow R$ is a free RG-resolution. Since $\mathrm{cd}_R\, G = n$, the submodule $Z_{n-1}(C(f)_*)$ of $(n-1)$-cycles in $C(f)_{n-1}$ is RG-projective. By applying the Eilenberg swindle [10, Lemma VIII.2.7], we can add an RG-free module N as a direct summand to both $C(f)_n$ and $C(f)_{n-1}$, so that the resulting complex F_* stays exact and $Z_{n-1}(F_*) \cong N$. The complex P_* embeds in F_* by the maps j_{P_i} that send P_i naturally into the $\mathrm{Ind}_H^G P_i$ summand in F_i for each i. Denote this embedding by $\iota_* : P_* \hookrightarrow F_*$. Observe that by construction and Lemma 2.1, ι_i is an embedding of P_i as a direct RH-module summand in F_i, for all $i \geqslant 0$. Hence ι_i splits for all $i \geqslant 0$. We form the quotient complex

$$Q_* := F_*/P_*,$$

and consider the short exact sequence of RH-modules:

$$0 \longrightarrow P_* \xrightarrow{\iota_*} F_* \longrightarrow\!\!\!\!\!\rightarrow Q_* \longrightarrow 0. \tag{2}$$

Since ι_i splits and F_i is RG-free and in particular RH-free, Q_i is RH-projective for all $i \geqslant 0$. The long exact sequence in homology associated to (2), shows that the complex Q_* is acyclic. Since

$$0 \longrightarrow Z_1(Q_*) \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0$$

is exact and Q_* is RH-projective, it follows that $Z_1(Q_*)$ is a projective RH-module. Proceeding by induction, and considering the short exact sequence

$$0 \longrightarrow Z_i(Q_*) \longrightarrow Q_i \longrightarrow Z_{i-1}(Q_*) \longrightarrow 0$$

for all $i \geqslant 2$, we obtain that $Z_i(Q_*)$ is a projective RH-module for all $i \geqslant 1$. Therefore, we obtain the short exact sequence:

$$0 \longrightarrow Z_{n-1}(P_*) \xrightarrow{\iota_{n-1}} Z_{n-1}(F_*) \longrightarrow Z_{n-1}(Q_*) \longrightarrow 0 \qquad (3)$$

of projective RH-modules. Since H is of type $FP(R)$, by [10, VIII.4.3], $Z_{n-1}(P_*)$ is a finitely generated RH-module. Since $Z_{n-1}(F_*)$ is a free RH-module, by Proposition 2.6, there is a constant $C > 0$, and an RH-retraction $\rho : Z_{n-1}(F_*) \twoheadrightarrow Z_{n-1}(P_*)$ of ι_{n-1} such that $\|\rho(y)\| \leqslant C|y|_1$ for all $y \in Z_{n-1}(P_*)$. $\qquad \square$

3 ℓ^∞-cohomology of groups

We start this section with preliminary results on ℓ^∞-cohomology of groups and derive our main result, Theorem 1.3.

Throughout, let R be a subring of $\mathbb{R}$ containing $\mathbb{Z}$. We equip R with the norm which is the restriction of the usual norm on $\mathbb{R}$. A *norm* on an R-module V is a function $\| \cdot \|_V : V \to \mathbb{R}$, so that for any $m, m' \in V$ and $r \in R$,

(1) $\|m\|_V \geqslant 0$ and $\|m\|_V = 0$ implies $m = 0$,
(2) $\|m + m'\|_V \leqslant \|m\|_V + \|m'\|_V$,
(3) $\|rm\|_V \leqslant |r| \|m\|_V$.

Let G be a group. A *normed RG-module* is a normed R-module with an isometric R-linear action of G. Let V be a normed RG-module. We denote by

$$\ell^\infty(G, V) = \{f : G \to V \mid f(G) \text{ is a bounded subset of } V\}$$

the RG-module of bounded functions defined on G with values in V. The ℓ^∞-*cohomology of a group G* can be defined as the ordinary (integral) cohomology of G with coefficients in $\ell^\infty(G, V)$,

$$H^*_{(\infty)}(G; V) := H^*(G; \ell^\infty(G, V)) = \text{Ext}_{\mathbb{Z}G}(\mathbb{Z}, \ell^\infty(G, V)).$$

This definition of ℓ^∞-cohomology in terms of ordinary group cohomology is due to Wienhard [46]. It reduces to Gersten's definition when G is assumed to be of type F_n, a necessary assumption in the original definition.

Remark 3.1 As noted in [36], two finitely generated quasi-isometric groups have isomorphic ℓ^∞-cohomologies. This can be seen by the fact that ℓ^∞-cohomology of a finitely generated group coincides with Elek's cohomology theory [18] of its Cayley graph. The latter is invariant under quasi-isometries of graphs [18, Theorem 2.1]. In fact, Elek defines ℓ^∞-cohomology only with trivial coefficients $V = \mathbb{R}$, but both [18, Definition 2.1] and [18, Theorem 2.1] generalise verbatim to any normed $\mathbb{R}$-module coefficients V. Alternatively, for general coefficients V, the quasi-isometry invariance of ℓ^∞-cohomology follows from Corollary 3.39 of [34]. When G is of type F_n, quasi-isometric invariance of $H^n_{(\infty)}(G; \mathbb{R})$ was already established in [22].

Definition 3.2 Suppose P_* be a projective RG-resolution of R. We say a cochain $\beta \in Hom_R(P_i, V)$ is *bounded on G-orbits* if for all $\sigma \in P_i$, the subset $\{\beta(g \cdot \sigma) \mid g \in G\} \subset V$ is bounded. We denote by $Hom^{bG}_R(P_i, V)$ the subspace containing all such co-chains.

The following observation is essential for our purposes.

Lemma 3.3 $Hom^{bG}_R(P_*, V)$ *is a co-chain complex whose cohomology is naturally isomorphic to* $H^*_{(\infty)}(G; V)$.

Proof The boundary operator on P_* gives a co-boundary operator on $Hom_R(P_*, V)$ which restricts to $Hom^{bG}_R(P_*, V)$.

We claim that $Hom^{bG}_R(P_i, V)$ is naturally isomorphic to $Hom_{RG}(P_i, \ell^\infty(G, V))$. For each $i \geqslant 0$, define

$$\Phi_i : Hom_{RG}(P_i, \ell^\infty(G, V)) \to Hom^{bG}_R(P_i, V), \ \ f \mapsto (x \mapsto f(x)(1_G)),$$
$$\Psi_i : Hom^{bG}_R(P_i, V) \to Hom_{RG}(P_i, \ell^\infty(G, V)), \ \ \theta \mapsto (x \mapsto (g \mapsto g\theta(g^{-1}x))).$$

One can check that Φ_i and Ψ_i are well-defined co-chain maps and are inverses of each other. This proves the claim. Thus, the cohomology of $Hom^{bG}_R(P_i, V)$ is naturally isomorphic to $H^*_R(G; \ell^\infty(G, V)) = \mathrm{Ext}_{RG}(R, \ell^\infty(G, V))$.

Let F_* be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. By [10, III.3.3], we get the natural isomorphism

$$Hom_{RG}(RG \otimes_{\mathbb{Z}G} F_i, \ell^\infty(G, V)) \cong Hom_{\mathbb{Z}G}(F_i, \ell^\infty(G, V)),$$

which shows that $H^*_R(G; \ell^\infty(G, V))$ is naturally isomorphic to $H^*(G; \ell^\infty(G, V))$. □

Remark 3.4 It follows from Lemma 3.3 that the G-action on the coefficient module V has no bearing on the ℓ^∞-cohomology of G, i.e. two different G-actions on V induce isomorphic cohomology groups, a fact that can also be seen directly from the definition. It therefore suffices to consider coefficients that are simply normed R-modules with trivial G-action.

We are now ready to prove our main result.

Theorem 3.5 *Let R be a subring of $\mathbb{Q}$ containing $\mathbb{Z}$. Suppose H is a group of type $FP_n(R)$ and is a subgroup of G with $\mathrm{cd}_R\, G = n < \infty$. Then the restriction homomorphism $H^n_{(\infty)}(G; V) \to H^n_{(\infty)}(H; V)$ is surjective for all normed R-modules V.*

Remark 3.6 Note that since H is a subgroup of G, we have $\mathrm{cd}_R\, H \leqslant \mathrm{cd}_R\, G = n$. By Sect. 2, it follows that H is of type $FP_n(R)$ if and only if it is of type $FP(R)$.

Proof of Theorem 3.5 Since the $n = 0$ case is trivially satisfied, we assume $n \geqslant 1$. Since H is of type of type $FP(R)$, it is of type $FP_\infty(R)$. So, there is a free RH-resolution $P_* \twoheadrightarrow R$ by finitely generated free RH-modules (possibly of infinite length) [10, VIII.4.5]. We use the notation of Proposition 2.7. Observe that the truncation of the resolutions P_* and F_* at n,

$$0 \to Z_{n-1}(P_*) \to P_{n-1} \to \cdots \to P_0 \twoheadrightarrow R$$

and

$$0 \to Z_{n-1}(F_*) \to F_{n-1} \to \cdots \to F_0 \twoheadrightarrow R$$

are RH- and RG-projective resolutions of R, respectively. Let $\alpha \in \mathrm{Hom}_R^{bH}(Z_{n-1}(P_*), V)$ be a cocycle which is bounded on H-orbits. Then $\beta := \alpha \circ \rho \in \mathrm{Hom}_R(Z_{n-1}(F_*), V)$. Since $\iota^{n-1}(\beta) = \alpha$, it is left to show that β is bounded on G-orbits.

Let $\sigma \in Z_{n-1}(F_*)$. By Proposition 2.7, there is a constant $C > 0$ such that $\forall g \in G$,

$$\|\rho(g \cdot \sigma)\| \leqslant C|g \cdot \sigma|_1 = C|\sigma|_1.$$

Now, let $\omega_1, \ldots, \omega_k$ be the standard generators of P_n as a free RH-module. Consider $\lambda \in P_n$ so that $\partial_n^P(\lambda) = \rho(g \cdot \sigma)$. Then $\lambda = q_1\omega_1 + \cdots + q_k\omega_k$ with $q_i \in RH$. We can write each $q_i = \sum_j r_{ij}h_{ij}$ where $h_{ij} \in H$ and $r_{ij} \in R$. Then

$$|\lambda|_1 = \sum_j |r_{1j}| + \cdots + \sum_j |r_{kj}|.$$

By the above inequality and the definition of the filling norm on $\rho(g \cdot \sigma)$, it follows that for any $\varepsilon > 0$, there is such λ so that

$$\sum_j |r_{1j}| + \cdots + \sum_j |r_{kj}| \leqslant C|\sigma|_1 + \varepsilon.$$

It follows that

$$
\begin{aligned}
\|\beta(g \cdot \sigma)\|_V &= \|\alpha(\rho(g \cdot \sigma))\|_V \\
&= \|\alpha(\partial_n^P(\lambda))\|_V \\
&= \|\alpha(q_1\partial_n^P(\omega_1)) + \cdots + \alpha(q_k\partial_n^P(\omega_k))\|_V
\end{aligned}
$$

$$\leqslant \|\alpha(q_1 \partial_n^P(\omega_1))\|_V + \cdots + \|\alpha(q_k \partial_n^P(\omega_k))\|_V$$

$$\leqslant \sum_j |r_{1j}| \|\alpha(h_{1j} \cdot \partial_n^P(\omega_1))\|_V + \cdots + \sum_j |r_{kj}| \|\alpha(h_{kj} \cdot \partial_n^P(\omega_k))\|_V$$

$$\leqslant \left(\sum_j |r_{1j}| + \cdots + \sum_j |r_{kj}| \right) \cdot \max\{\|\alpha(h_{ij} \cdot \partial_n^P(\omega_i))\|_V \mid i, j\}$$

$$\leqslant (C|\sigma|_1 + \varepsilon) \cdot \max\{\|\alpha(h_{ij} \cdot \partial_n^P(\omega_i))\|_V \mid i, j\},$$

Let $L = \sup\{\|\alpha(h \cdot \partial_n^P(\omega_i))\|_V \mid 1 \leqslant i \leqslant k, h \in H\}$. Since α is bounded on H-orbits, $L < \infty$. We have established that $\|\beta(g \cdot \sigma)\|_V \leqslant LC|\sigma|_1$ for all $g \in G$. This shows that β is bounded on G-orbits. $\qquad\square$

Remark 3.7 The first example of a non-hyperbolic subgroup H of type F of a hyperbolic group G was given in [30]. By Theorem 1.1, we know that $H_{(\infty)}^2(H; \ell^\infty(\mathbb{N}, \mathbb{R})) \neq 0$. Yet, since G is hyperbolic, $H_{(\infty)}^2(G; \ell^\infty(\mathbb{N}, \mathbb{R})) = 0$. This shows that Theorem 3.5 does not generalise to degrees below the top degree, which in this example is cd $G = 5$.

We conclude this section with an observation regarding relatively hyperbolic groups.

Theorem 3.8 *Let G be a relatively hyperbolic group with amenable peripheral subgroups, then the sequence*

$$H_b^n(G; V) \xrightarrow{c^n} H^n(G; V) \xrightarrow{\iota^n} H_{(\infty)}^n(G; V)$$

is exact for all $n \geqslant 2$ and all dual normed $\mathbb{R}G$-modules V.

Proof Denote by $\mathcal{H} = \{H_i\}_{i \in I}$ the collection of peripheral subgroups of G. There is a commutative diagram with exact middle row:

$$\begin{array}{ccccc}
H_b^n(G, \mathcal{H}; V) & \longrightarrow & H_b^n(G; V) & & \\
\downarrow{c'} & & \downarrow{c} & & \\
H^n(G, \mathcal{H}; V) & \longrightarrow & H^n(G; V) & \longrightarrow & \prod_{i \in I} H^n(H_i; V) \\
& & \downarrow{\iota} & & \uparrow{\iota''} \\
& & H_{(\infty)}^n(G; V) & \longrightarrow & \prod_{i \in I} H_{(\infty)}^n(H_i; V)
\end{array} \qquad (4)$$

We know that the homomorphism c' is surjective [20] and ι'' is injective [36, Theorem 1.1]. The result now follows by a diagram chase. $\qquad\square$

4 Hyperbolicity criteria

In [38], numerous equivalent conditions for a finitely presented group G to be hyperbolic are listed. In this section, we make the observation that (with a suitable restatement of the results) the finite presentability of G may be relaxed to $FP_2(\mathbb{Q})$.

A group G of type $FP_2(\mathbb{Q})$ is finitely generated, so we may fix a finite generating set and form the Cayley graph Γ. The property $FP_2(\mathbb{Q})$ now implies that $Z_1(\Gamma, \mathbb{Q})$ is finitely generated as a $\mathbb{Q}G$-module. We choose a finite generating set $\{\alpha_i\}_{i \in I}$ for $Z_1(\Gamma, \mathbb{Q})$ (as a $\mathbb{Q}G$-module) where each generator α_i lies in $C_1(\Gamma, \mathbb{Z})$ and hence is represented by a circuit w_i in Γ. Without loss of generality we may assume that there are no superfluous generators in this generating set. We attach 2-cells with attaching maps gw_i for each $g \in G$ and $i \in I$ and make the corresponding 2-complex X (where $X^{(1)} = \Gamma$). By construction, $H_1(X; \mathbb{Q}) = 0$ and G acts freely and cocompactly on X.

To distinguish filling norms with different coefficients, we adopt notation similar to [38]. For a 1-boundary with coefficients in $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, we define:

$$\|b\|_A := \inf \{|a|_1 : a \in C_2(X, A), \ \partial_2(a) = b\}.$$

We also have the norm $|\cdot|_1$ on 1-boundaries coming from the ℓ_1-norm on 1-chains.

The following result extends Theorem 7 of [38].

Theorem 4.1 *Suppose the group G is of type $FP_2(\mathbb{Q})$, and X is the complex above. Then the following are equivalent:*

(1) *G is hyperbolic.*
(2) *There exists $K \geqslant 0$ such that for any $b \in B_1(X, \mathbb{Z})$, we have the inequality:*

$$\|b\|_{\mathbb{Q}} \leqslant K|b|_1.$$

(3) *There exists $K \geqslant 0$ such that for any $b \in B_1(X, \mathbb{R})$, we have the inequality:*

$$\|b\|_{\mathbb{R}} \leqslant K|b|_1.$$

Note that since X is not necessarily simply-connected even when G is finitely presented, results such as Theorem 7 in [38] do not apply verbatim. Therefore we give a proof, although the arguments used are almost the same as in the proofs of the original results in the literature.

Proof Note that by construction of X, any circuit in Γ lies in $B_1(X, \mathbb{Q})$.
(1) $\implies$ (2): Without loss of generality, Γ is δ-hyperbolic for some positive integer δ. Since b is an integral 1-cycle, it can be represented by a circuit γ_b in $X^{(1)}$. We use induction on $|b|_1$. Since $X^{(1)}$ is locally finite, there are finitely many G-orbits of circuits of a given length, thus for the base case of $0 \leqslant |b|_1 \leqslant 16\delta$ we may take:

$$K := \max_{|b|_1 \leqslant 16\delta} \{\|b\|_{\mathbb{Q}}\}.$$

For the inductive step, suppose we have $|b|_1 > 16\delta$. If γ_b self-intersects, we may split it into two shorter circuits and the result follows by induction. Otherwise, we may invoke Lemma 2.6 in [5] (III.H) to represent $\gamma_b = \gamma_1 + \gamma_2$ as a sum in $C_1(X, \mathbb{Z})$ of
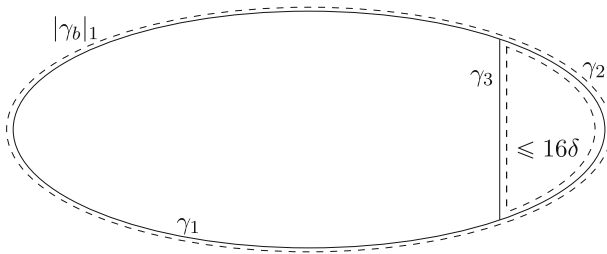
Fig. 1 Splitting up a filling of γ_b

a shorter circuit $\gamma_a = \gamma_1 + \gamma_3$ and a circuit $\gamma_c = \gamma_2 - \gamma_3$ of length at most 16δ (see Fig. 1, where dotted lines denote length). By induction,

$$\|\gamma_a\|_{\mathbb{Q}} \leqslant K|\gamma_a|_1.$$

Moreover, $\|\gamma_c\|_{\mathbb{Q}} \leqslant K$. Finally,

$$\|b\|_{\mathbb{Q}} \leqslant \|\gamma_a\|_{\mathbb{Q}} + \|\gamma_c\|_{\mathbb{Q}} \leqslant K(|b|_1 - 1) + K,$$

whence by induction the result holds.

(2) $\implies$ (3): Suppose that $b \in B_1(X, \mathbb{R})$. Applying Theorem 4 and the second part of Theorem 6 from [38], there exist $p_1, p_2, \ldots, p_n \in B_1(X, \mathbb{Z})$ and positive $\alpha_i \in \mathbb{R}$ such that

$$b = \sum_{i=1}^{n} \alpha_i p_i$$

and

$$|b|_1 = \sum_{i=1}^{n} \alpha_i |p_i|_1,$$

from which the required inequality follows (the same value for K works).

(3) $\implies$ (1): There are finitely many orbits of 2-cells in X, therefore there exists a constant M such that for all $a \in C_2(X, \mathbb{R})$,

$$|\partial_2(a)|_1 \leqslant M|a|_1$$

holds. Arguing by contradiction, suppose that G is not hyperbolic, but (3) holds. Theorem 1.4 of Papasoglu in [42] and its corollary, Lemma 5.2 of Gersten in [24], only rely on the group being finitely generated as they are results concerning the Cayley graph. We may therefore find arbitrarily long and thick quadrilaterals w in $X^{(1)}$, just like in the proof of Proposition 8 of Mineyev in [38]. Following the argument there,

which we can do since the first real homology of X vanishes, we can get:

$$\|w\|_{\mathbb{R}} \geqslant \frac{1}{800M^2} \left(|w|_1\right)^2.$$

Choosing $|w|_1$ large enough, this contradicts (3). □

Definition 4.2 (*Integral part*) We say that a $\mathbb{Z}G$-submodule I of a finitely generated $\mathbb{Q}G$-module M is an *integral part* of M if I is finitely generated as a $\mathbb{Z}G$-module as well as generating M as a $\mathbb{Q}G$-module.

Note that for a group G which is of type $FP_2(\mathbb{Q})$ but not of type $FP_2(\mathbb{Z})$, the relation module $Z_1(\Gamma, \mathbb{Z})$ is not finitely generated as a $\mathbb{Z}G$-module, thus is not an integral part of $Z_1(\Gamma, \mathbb{Q})$.

Lemma 4.3 *For $b \in B_1(X, \mathbb{Q})$ and any $r \in \mathbb{Q}$, we have $\|rb\|_{\mathbb{Q}} = |r|\|b\|_{\mathbb{Q}}$.*

Proof This is an extension of the second statement of Lemma 3.1 in [24], where X was assumed to be simply-connected. Recall that the generating set for 1-cycles of Γ does not have superfluous generators. Over the rationals, $|ra|_1 = |r||a|_1$ holds in $C_2(X, \mathbb{Q})$. □

Definition 4.4 (*Rational homological filling function,* [2]) Suppose that G is a group of type $FP_{n+1}(\mathbb{Q})$. Let

$$P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} P_0 \longrightarrow \mathbb{Q} \longrightarrow 0$$

be a partial projective resolution via finitely generated $\mathbb{Q}G$-modules. Note that by [10, VIII.4.3], ker ∂_n is finitely generated as a $\mathbb{Q}G$-module. Let K_n be an integral part of ker ∂_n. For $\gamma \in K_n$, define

$$\|\gamma\|_{\partial_{n+1}} := \inf \{\|\mu\| : \mu \in P_{n+1}, \partial_{n+1}(\mu) = \gamma\}.$$

Finally, define the nth-*filling function*

$$FV_{G,\mathbb{Q}}^{n+1}(k) := \sup \{\|\gamma\|_{\partial_{n+1}} : \gamma \in K_n, \|\gamma\| \leqslant k\},$$

where $\|\cdot\|$ denote the appropriate filling norms.

It is Theorem 3.3 in [2] that $FV_{G,\mathbb{Q}}^{n+1}$ is well-defined up to linear equivalence. In particular, being bounded above by a linear function is invariant under choice of a partial resolution P_* by finitely generated projective $\mathbb{Q}G$-modules, integral part or filling norms.

The following question was asked by Arora and Martínez-Pedroza.

Question 4.5 ([2, Question 1.9]) *Let G be a group of type $FP_2(\mathbb{Q})$ and suppose $FV_{G,\mathbb{Q}}^2$ is bounded above by a linear function. Is G a hyperbolic group?*

We give a positive answer to this question below.

Theorem 4.6 *Suppose that G is a group of type $FP_2(\mathbb{Q})$ and that $FV^2_{G,\mathbb{Q}}$ is bounded above by a linear function. Then G is hyperbolic.*

Proof By the construction of X above, we have that $H_1(X; \mathbb{Q}) = 0$. Suppose that G is not hyperbolic. Then by Theorem 4.1, for any $K \geqslant 0$ there exists some $b_K \in B_1(X, \mathbb{Z})$ such that

$$\|b_K\|_{\mathbb{Q}} > K|b_K|_1,$$

i.e.

$$\inf\{|a|_1 : a \in C_2(X, \mathbb{Q}), \partial_2(a) = b_K\} > K|b_K|_1.$$

By considering $K = 1, 2, 3, \ldots$, we form a sequence $(b_i)_{i \in \mathbb{N}}$. Since each b_i is in $B_1(X, \mathbb{Z})$, which is contained in $Z_1(X, \mathbb{Q})$, we can think of them as rational 1-cycles.

Let $z_1, z_2, \ldots, z_m$ be the generators of $Z_1(X, \mathbb{Q})$ from the construction of X; recall that these are integral 1-cycles. Consider the free $\mathbb{Q}G$-module N with generating set $\{e_1, e_2, \ldots, e_m\}$ and the epimorphism

$$\phi : N \to Z_1(X, \mathbb{Q}),$$

defined by $\phi(e_i) = z_i$ for $i = 1, 2, \ldots, m$. Let $f_i \in N$ be such that $\phi(f_i) = b_i$.

Consider the free $\mathbb{Z}G$-module M with generating set $\{e_1, e_2, \ldots, e_m\}$. We have a natural inclusion $M \subset N$. For each i, there exists some positive integer r_i such that $r_i f_i \in M$ and $\phi(r_i f_i) = r_i b_i$.

The image $I = \phi(M)$ inside $Z_1(X, \mathbb{Q})$ is finitely generated as a $\mathbb{Z}G$-module by $\phi(e_i) = z_i$, but it also generates all of $Z_1(X, \mathbb{Q})$ as a $\mathbb{Q}G$-module. Therefore I is an integral part for $Z_1(X, \mathbb{Q})$. Furthermore, it contains every $r_i b_i$.

We compute $FV^2_{G,\mathbb{Q}}(n)$ using the rational cellular structure on X and take I as the required integral part. By our assumptions, there exists a constant C such that

$$FV^2_{G,\mathbb{Q}}(n) \leqslant Cn \quad \forall n \in \mathbb{N}.$$

This means that for all positive integers n, we have

$$\sup\{\|b\|_{\mathbb{Q}} : b \in I, |b|_{C_1(X,\mathbb{Q})} \leqslant n\} \leqslant Cn.$$

This implies that for all $b \in I$ such that $|b|_{C_1(X,\mathbb{Q})} \leqslant m$, we have

$$\|b\|_{\mathbb{Q}} \leqslant Cm, \tag{5}$$

where $|\cdot|_{C_1(X,\mathbb{Q})}$ denotes the ℓ_1-norm on $C_1(X, \mathbb{Q})$. This is a filling norm because the module $C_1(X, \mathbb{Q})$ is free as a $\mathbb{Q}G$-module.

Now choose i such that $i > C$. For this i, let $n = |b_i|_1$. By construction, we have

$$\|b_i\|_{\mathbb{Q}} > i|b_i|_1,$$

which implies

$$|r_i|\|b_i\|_{\mathbb{Q}} > i|r_i||b_i|_1. \tag{6}$$

On the other hand, $|b_i|_{C_1(X,\mathbb{Q})} = n$. Consider $r_i b_i \in I$. We have

$$|r_i b_i|_{C_1(X,\mathbb{Q})} = |r_i||b_i|_{C_1(X,\mathbb{Q})} = |r_i|n,$$

so by setting $m = |r_i|n$ in (5), we get

$$\|r_i b_i\|_{\mathbb{Q}} \leqslant C|r_i|n.$$

Applying Lemma 4.3, we get

$$|r_i|\|b_i\|_{\mathbb{Q}} \leqslant C|r_i|n < i|r_i||b_i|_1,$$

which contradicts (6). Hence G is hyperbolic. $\qquad\square$

This proves Theorem 1.8, since the converse follows from hyperbolic groups being finitely presented and Theorem 1.8 in [2].

We end this section by giving a generalisation of a theorem of Gersten [24] from finitely presented groups to groups of type $FP_2(\mathbb{Q})$. It will be needed for applications to subgroups of hyperbolic groups in Sect. 5.

We use the shorthand $\ell_\infty := \ell^\infty(\mathbb{N}, \mathbb{R})$ to denote the injective Banach space of bounded real sequences, which can be thought of as a normed $\mathbb{R}G$-module for any group G with trivial G-action.

Theorem 4.7 *Suppose that G is a group of type $FP_2(\mathbb{Q})$ with $H^2_{(\infty)}(G; \ell_\infty) = 0$. Then G is hyperbolic.*

Proof Consider the two-dimensional X above. We can use its cellular complex, extended abstractly to an $\mathbb{R}G$-resolution $F_* \twoheadrightarrow \mathbb{R}$ (possibly infinite) where $F_i = C_i(X, \mathbb{R})$ for $i \leqslant 2$ and Lemma 3.3 to compute the ℓ_∞-cohomology in lower degrees of G. Thus, our assumption implies that the second cohomology of the co-chain complex

$$Hom_{\mathbb{R}}^{bG}(F_i, \ell_\infty)$$

vanishes.

For $i \in \{1, 2\}$, since X has finitely many G-orbits of both 1-cells and 2-cells, we may define the following norm on i-cochains bounded on G-orbits $Hom_{\mathbb{R}}^{bG}(C_i(X, \mathbb{R}), \mathbb{R})$:

$$\|f\|_\infty := \sup\{|f(x)| \ \ x \text{ is an } i\text{-cell of } X\}.$$

We claim that there exists $M > 0$ with the property that for all $z \in Hom_{\mathbb{R}}^{bG}(C_2(X, \mathbb{R}),$ $\mathbb{R})$ with $\delta(z) = 0$, there exists some $c \in Hom_{\mathbb{R}}^{bG}(C_1(X, \mathbb{R}), \mathbb{R})$ with $\delta(c) = z$ and $\|c\|_\infty \leqslant M\|z\|_\infty$.

If not, then there exist $z_i \in Hom_{\mathbb{R}}^{bG}(C_2(X, \mathbb{R}), \mathbb{R})$ for $i \in \mathbb{N}$ such that $\delta(z_i) = 0$ and $\|z_i\|_\infty = 1$ for all i, with the property that

$$\sup_{i \in \mathbb{N}} \inf \left\{ \|c_i\|_\infty \ \ c_i \in Hom_{\mathbb{R}}^{bG}(C_1(X, \mathbb{R}), \mathbb{R}), \ \delta(c_i) = z_i \right\} = \infty.$$

We can now combine the z_i to form a 2-cocycle $\bar{z} \in Hom_{\mathbb{R}}^{bG}(C_2(X, \mathbb{R}), \ell_\infty)$ via $\bar{z}(x)(i) = z_i(x)$ which is bounded on G-orbits, but which is not the image of any 1-cochain bounded on G-orbits. This contradicts the assumption that $H_{(\infty)}^2(G; \ell_\infty) = 0$, which proves the claim. Note that this is similar to the proof of Proposition 4.6 in [24], where this property was referred to as *strong vanishing*.

By the same argument as Proposition 4.7 in [24], it follows that the second cohomology of the co-chain complex

$$Hom_{\mathbb{R}}^{bG}(F_i, \ell^\infty(S, \mathbb{R}))$$

vanishes for all discrete sets S. We now follow the argument in the proof of Theorem 4.3 in [24]. By setting S to be the set of all bounded linear maps

$$g : B_1(X, \mathbb{R}) \to \mathbb{R}$$

of norm 1, we can deduce that the norms $\| \cdot \|_{\mathbb{R}}$ (denoted $N_{\mathbb{R}}$ in [24]) and $| \cdot |_1$ are equivalent on $B_1(X, \mathbb{R})$, which means there exists a constant K such that $\|\cdot\|_{\mathbb{R}} \leqslant K|\cdot|_1$.

Finally, by Theorem 4.1 we deduce that G is hyperbolic. $\qquad\square$

5 Applications

In this section, we give some applications of Sect. 3 and Theorem 3.5.

5.1 Subgroups of hyperbolic groups

Subgroups of hyperbolic groups inherit some of the properties of hyperbolic groups, but may not inherit hyperbolicity even when these subgroups have strong finiteness conditions [9, 30, 43]. We consider subgroups of type $FP(\mathbb{Q})$.

Corollary 5.1 *Let G be a hyperbolic group with $\mathrm{cd}_{\mathbb{Q}} G = n$, $n \geqslant 2$. Suppose H is a subgroup of type $FP(\mathbb{Q})$. Then $H_{(\infty)}^n(H; V) = 0$ for all normed $\mathbb{R}$-modules V.*

Proof Since G is hyperbolic, $H_{(\infty)}^n(G; V) = 0$ [37]. By Theorem 3.5, we have $H_{(\infty)}^n(H; V) = 0$. $\qquad\square$

When the hyperbolic group has rational cohomological dimension 2, we can say more.

Corollary 5.2 *Let G be a hyperbolic group such that $\mathrm{cd}_{\mathbb{Q}}\, G = 2$. Suppose H is a subgroup of G of type $FP_2(\mathbb{Q})$. Then H is hyperbolic.*

Proof By Theorems 3.5 and 1.1, $H^2_{(\infty)}(H; \ell_\infty) = 0$. By Theorem 4.7, H is hyperbolic. □

Rips constructed subgroups of hyperbolic small cancellation groups that are finitely generated but not finitely presented [43, Theorem 1]. Corollary 5.2 shows that those subgroups are not of type $FP_2(\mathbb{Q})$.

Theorem 5.3 (Rips) *Given any finitely presented group Q and any real number $\lambda > 0$, there is a $C'(\lambda)$ group G and a short exact sequence*

$$1 \to N \to G \xrightarrow{q} Q \to 1$$

such that N is finitely generated.

Proof of Corollary 1.6 Suppose Q is a finitely presented group that contains a finitely generated subgroup P which is not finitely presented. Consider the preimage $H = q^{-1}(P)$. Then H is finitely generated but it is not finitely presented, for otherwise P would be finitely presented. Set $\lambda = 1/6$. Then G is hyperbolic and moreover $\mathrm{cd}_{\mathbb{Q}}\, G \leqslant 2$ [40]. Since H is not finitely presented, Corollary 5.2 shows that H is not of type $FP_2(\mathbb{Q})$. □

5.2 Non-vanishing in top dimension

The following result was proven by Gersten in [25, Prop. 7.2] for groups of type F_n. The proof generalises verbatim to any group.

Lemma 5.4 (Gersten) *Let G be a group. Then $H^i_{(\infty)}(G; \mathbb{Z}) = H^i_{(\infty)}(G; \mathbb{R})$ for all $i \geqslant 2$.*

Lemma 5.5 *Let Γ be a Poincaré duality group of dimension $d \geqslant 1$. Then Γ is amenable if and only if $H^d_{(\infty)}(\Gamma, \mathbb{R}) \neq 0$ if and only if $H^d_{(\infty)}(\Gamma, \mathbb{R})$ has uncountably infinite dimension over $\mathbb{R}$.*

Proof Recall that ℓ^∞-cohomology is quasi-isometry invariant and a finite index subgroup of a Poincaré duality group is again a Poincaré duality group [31]. Passing to an index at most 2 subgroup of Γ, we can assume it is an orientable Poincaré duality group. By duality, we have

$$\begin{aligned}
H^d_{(\infty)}(\Gamma; \mathbb{R}) &= H^d(\Gamma; \ell^\infty(\Gamma, \mathbb{R})) \\
&= H_0(\Gamma; \ell^\infty(\Gamma, \mathbb{R})) \\
&= \ell^\infty(\Gamma, \mathbb{R})_\Gamma.
\end{aligned}$$

Moreover, $H_0(\Gamma; \ell^\infty(\Gamma, \mathbb{R}))$ is isomorphic to uniformly finite homology of G in degree 0 [8]. By the fundamental result of Block and Weinberger [11], amenable groups are

exactly those groups for which uniformly finite homology in degree 0 does not vanish. If Γ amenable, then combining Theorem 6.6.1 and Proposition 6.7.2 of [7], it follows that $\ell^\infty(\Gamma, \mathbb{R})_\Gamma$ has dimension at least $2^{2^{|\Gamma|}}$. □

The next proposition is our main tool for establishing non-vanishing of ℓ^∞-cohomology in the top degree.

Proposition 5.6 *Let $A = \mathbb{Z}$ or $\mathbb{R}$. Let Γ be a group that has an amenable Poincaré duality subgroup N of dimension $d \geqslant 1$ and $d = \operatorname{vcd} \Gamma < \infty$ or $d = \operatorname{cd}_\mathbb{Q} \Gamma < \infty$. Then $H^d_{(\infty)}(\Gamma, A)$ contains a continuum of linearly independent elements.*

Proof By Lemma 5.4, it suffices to consider $A = \mathbb{R}$. Recall that ℓ^∞-cohomology is quasi-isometry invariant and a finite index subgroup of a Poincaré duality group is again a Poincaré duality group [31]. Passing to a finite index subgroup of Γ, we can either assume that $d = \operatorname{cd} \Gamma < \infty$ or $d = \operatorname{cd}_\mathbb{Q} \Gamma < \infty$. The result now follows from Theorem 3.5 and Lemma 5.5. □

It turns out that many well-known classes of groups contain free abelian or nilpotent subgroups of large Hirsch length.

Corollary 5.7 *Let $A = \mathbb{Z}$ or $\mathbb{R}$. Suppose Γ is one of the following groups and $d = \operatorname{vcd} \Gamma \geqslant 1$:*

(i) *Γ is a finitely generated right-angled Artin group;*
(ii) *Γ is the Bestvina–Brady group of any finite flag complex L^n with $H^n(L; A) = 0$;*
(iii) *Γ is the mapping class group $\operatorname{Mod}(S_2)$ of a closed orientable surface of genus 2;*
(iv) *$\Gamma = \operatorname{SL}(n, \mathbb{Z})$ for $n \geqslant 2$;*
(v) *$\Gamma = \operatorname{Out}(\mathbb{F}_n)$ for $n \geqslant 2$.*

Then the cokernel of the comparison map $\iota^d : H^d(\Gamma; A) \to H^d_{(\infty)}(\Gamma; A)$ contains a continuum of linearly independent elements.

Proof We first check that each case satisfies the hypothesis of Proposition 5.6 where N can be taken either abelian or nilpotent.

(i) The dimension of the flag complex L for Γ equals to $d - 1$ (see for example [35, Proposition 18]) and hence Γ contains a subgroup isomorphic to $\mathbb{Z}^d$.

(ii) Γ contains a free abelian subgroups of rank equal to the dimension of L which is n. By our hypothesis and [35, Theorem 22], $\operatorname{cd} \Gamma = n$.

(iii) In this case $d = 3$ and $\operatorname{Mod}(S_2)$ contains a group isomorphic to $\mathbb{Z}^3$ generated by Dehn twists [29].

(iv) It is known that $\operatorname{vcd} \Gamma = n(n-1)/2$ [10]. The unipotent subgroup of Γ of upper triangular matrices is a nilpotent subgroup of Hirsch length (and hence rational cohomological dimension) equal to $n(n-1)/2$.

(v) It is shown in [12] that $\operatorname{Out}(\mathbb{F}_n)$ contains a free abelian subgroup of rank d.

Proposition 5.6 implies that $H^d_{(\infty)}(\Gamma; A)$ contains a continuum of linearly independent elements. Our stronger assertion follows, since in each of the cases (i) and (iii)–(v), the groups are of type VF, in which case $H^d(\Gamma; A)$ is a finitely generated A-module. In case (ii), it follows by Corollary 12 of [35] that $H^d(\Gamma; A)$ is countably generated as an A-module. □

Lemma 5.8 *Let G be the fundamental group of a graph of groups with vertex groups $\{G_v\}_{v \in V}$ and edge groups $\{G_e\}_{e \in E}$. Then there is a long exact Mayer-Vietoris sequence:*

$$\cdots \to \prod_{e \in E} H^{i-1}_{(\infty)}(G_e; \ell^\infty(G_e \backslash G, W)) \to H^i_{(\infty)}(G; W) \to$$
$$\prod_{v \in V} H^i_{(\infty)}(G_v; \ell^\infty(G_v \backslash G, W)) \to \prod_{e \in E} H^i_{(\infty)}(G_e; \ell^\infty(G_e \backslash G, W)) \to \cdots$$

for all $i > 0$ and normed $\mathbb{R}G$-modules W.

Proof For any subgroup $H \leqslant G$, there is an $\mathbb{R}H$-module isomorphism

$$\ell^\infty(G, W) \cong \ell^\infty(H, \ell^\infty(H \backslash G, W)).$$

The result now follows from applying this isomorphism to the Mayer-Vietoris long exact sequence of G in group cohomology with $\ell^\infty(G, W)$ coefficients. $\qquad\square$

Corollary 5.9 *Let Γ be a finitely generated group with $2 \leqslant \mathrm{cd}_\mathbb{Q} \Gamma < \infty$. Suppose that Γ has an amenable Poincaré duality subgroup of dimension $\mathrm{cd}_\mathbb{Q} \Gamma$. Suppose Γ is quasi-isometric to the fundamental group G of a finite graph of hyperbolic groups. Then $\mathrm{cd}_\mathbb{Q} G = \mathrm{cd}_\mathbb{Q} \Gamma = 2$.*

Proof Since rational cohomological dimension is a quasi-isometry invariant amongst groups with finite rational cohomological dimension [44, Theorem 1.2], denote

$$n := \mathrm{cd}_\mathbb{Q} G = \mathrm{cd}_\mathbb{Q} \Gamma.$$

Suppose $n \geqslant 3$. Let $\{G_v\}_{v \in V}$ be the vertex groups and $\{G_e\}_{e \in E}$ be the edge groups of the finite graph of groups. We apply the Mayer-Vietoris sequence of Lemma 5.8:

$$\cdots \to \prod_{e \in E} H^{i-1}_{(\infty)}(G_e; \ell^\infty(G_e \backslash G, \mathbb{R})) \to H^i_{(\infty)}(G; \mathbb{R}) \to$$
$$\prod_{v \in V} H^i_{(\infty)}(G_v; \ell^\infty(G_v \backslash G, \mathbb{R})) \to \prod_{e \in E} H^i_{(\infty)}(G_e; \ell^\infty(G_e \backslash G, \mathbb{R})) \to \cdots$$

By [37, Theorem 0],

$$H^n_{(\infty)}(G_v; \ell^\infty(G_v \backslash G, \mathbb{R})) = H^{n-1}_{(\infty)}(G_e; \ell^\infty(G_e \backslash G, \mathbb{R})) = 0.$$

Then $H^n_{(\infty)}(G; \mathbb{R})$ vanishes, implying that $H^n_{(\infty)}(\Gamma; \mathbb{R})$ also vanishes, but by Proposition 5.6, $H^n_{(\infty)}(\Gamma; \mathbb{R}) \neq 0$. This is a contradiction. $\qquad\square$

We recall that by a result of Dunwoody, finitely generated groups of rational cohomological dimension at most one are virtually free [17]. Thus the assumption of Corollary 5.9 that $\mathrm{cd}_\mathbb{Q} \Gamma \geqslant 2$ is in fact equivalent to G not being virtually free.

Corollary 5.10 *Let Γ be a finitely generated group satisfying the hypothesis of Corollary* 1.9. *Suppose Γ is quasi-isometric to the fundamental group of a finite graph of hyperbolic groups. Then either Γ is a right angled Artin group or a Bestvina–Brady groups of cohomological dimension at most* 2.

Proof In the proof of Corollary 5.7, we have established that Γ has either an abelian or a nilpotent Poincaré duality subgroup of dimension equal to $\mathrm{cd}_{\mathbb{Q}}\,\Gamma$. By Corollary 5.9, $\mathrm{cd}_{\mathbb{Q}}\,\Gamma = 2$, which rules out ((iii))–((v)). $\square$

5.3 ℓ^∞-cohomological dimension

Suppose G is a group. Define the ℓ^∞-*cohomological dimension of G* by

$$\mathrm{cd}_{(\infty)}G := \sup\{n \mid H^n_{(\infty)}(G; V) \neq 0 \text{ for some normed } \mathbb{R}\text{-module } V\}.$$

Let V be a normed $\mathbb{R}$-module such that $H^n_{(\infty)}(G; V) \neq 0$. Viewing V as a normed $\mathbb{Q}$-module, by the last claim of the proof of Lemma 3.3, we obtain $H^n_{\mathbb{Q}}(G; \ell^\infty(G, V)) = H^n_{(\infty)}(G; V)$. This shows that $\mathrm{cd}_{\mathbb{Q}}\,G \geqslant n$ and hence $\mathrm{cd}_{\mathbb{Q}}\,G \geqslant \mathrm{cd}_{(\infty)}G$.

Note also that when G is finitely generated, then by Remark 3.1, $\mathrm{cd}_{(\infty)}G$ is a quasi-isometry invariant of G. We end this section with some applications to $\mathrm{cd}_{(\infty)}G$.

Lemma 5.11 *Let G be a group of type $FP_2(\mathbb{Q})$. Then G is an infinite hyperbolic group if and only if $\mathrm{cd}_{(\infty)}G = 1$.*

Proof If G satisfies $\mathrm{cd}_{(\infty)}G = 1$, then by Theorem 4.7, it is hyperbolic. Conversely, if G is hyperbolic, then $\mathrm{cd}_{(\infty)}G \leqslant 1$ [37]. Since G contains a quasi-convex infinite cyclic subgroup C, by Corollary 10.3 of [27], the restriction $C \hookrightarrow G$ induces a surjection

$$H^1_{(\infty)}(G; \mathbb{R}) \twoheadrightarrow H^1_{(\infty)}(C; \mathbb{R}) \neq 0.$$

This shows that $\mathrm{cd}_{(\infty)}G = 1$. $\square$

Here is another reformulation of Corollary 5.1.

Corollary 5.12 *Let G be a hyperbolic group with $\mathrm{cd}_{\mathbb{Q}}\,G = n$, $n \geqslant 2$. Suppose H is a subgroup of type $FP(\mathbb{Q})$. Then $\mathrm{cd}_{(\infty)}H \leqslant n - 1$.*

Before stating the next result, we need the following lemma.

Lemma 5.13 *Let H be a subgroup of G and V be a normed $\mathbb{R}$-module. Then $\ell^\infty(H, V)$ is a direct summand of $\ell^\infty(G, V)$ as an $\mathbb{R}H$-module.*

Proof Let T be a right transversal of H in G. We define

$$\varphi : \ell^\infty(H, V) \to \ell^\infty(G, V), \ \ f \mapsto (ht \mapsto f(h)), \ \forall h \in H, \ \forall t \in T.$$

One can check that φ is an H-equivariant section of the restriction morphism from $\ell^\infty(G, V)$ to $\ell^\infty(H, V)$. $\square$

Corollary 5.14 *Let G be a finitely generated infinite group, and hyperbolic relative to $\mathcal{H} = \{H_i\}_{i \in I}$. Then*

$$\mathrm{cd}_{(\infty)} G = \max\{1, \mathrm{cd}_{(\infty)} H_i \mid i \in I\}.$$

Proof Denote $m := \max\{1, \mathrm{cd}_{(\infty)} H_i \mid i \in I\}$, which may be infinity. If G is hyperbolic, then so are all peripheral subgroups because they are undistorted. In this case, $\mathrm{cd}_{(\infty)} G = m = 1$.

Suppose $\mathrm{cd}_{(\infty)} G \geqslant 2$. Using the long exact sequence in group cohomology for the group pair $(G, \mathcal{H})$ with coefficients in $\ell^\infty(G, V)$, we have

$$\cdots \to H^j_{(\infty)}(G, \mathcal{H}; V) \to H^j_{(\infty)}(G; V) \to$$
$$\prod_{i \in I} H^j(H_i; \ell^\infty(G, V)) \to H^{j+1}_{(\infty)}(G, \mathcal{H}; V) \to \cdots$$

By Theorem 1.3 of [36], we know that $H^j_{(\infty)}(G, \mathcal{H}; V) = 0$ for all $j \geqslant 2$. So, the long exact sequence shows that

$$H^j_{(\infty)}(G; V) \cong \prod_{i \in I} H^j(H_i; \ell^\infty(G, V)), \ \forall j \geqslant 2.$$

By Lemma 5.13, $\ell^\infty(H_i, V)$ is a direct summand of $\ell^\infty(G, V)$; therefore, it follows that $H^j_{(\infty)}(H_i; V)$ is isomorphic to a direct summand of $H^j_{(\infty)}(G; V)$, for each $i \in I$. This gives us $\mathrm{cd}_{(\infty)} G \geqslant m$.

On the other hand, since

$$H^j_{(\infty)}(G; V) \cong \prod_{i \in I} H^j_{(\infty)}(H_i; \ell^\infty(H_i \backslash G, V)), \ \forall j \geqslant 2,$$

we obtain $\mathrm{cd}_{(\infty)} G = m$. $\square$

Corollary 5.15 *Suppose Γ is the fundamental group of a non-compact complete finite-volume Riemannian n-dimensional manifold M of negative sectional curvature bounded away from zero. Then $\mathrm{cd}_{(\infty)} \Gamma = n - 1$ and $H^{n-1}_{(\infty)}(\Gamma; \mathbb{R})$ contains a continuum of linearly independent elements.*

Proof We know that Γ is hyperbolic relative to the fundamental groups of the cusps, which are infra-nilmanifolds of dimension $(n - 1)$ [19]. Therefore, peripheral subgroups are torsion-free virtually nilpotent groups of Hirsch length $(n - 1)$. The first claim now follows from Corollary 5.14 and Lemma 5.5.

Let $\mathcal{H} = \{H_i\}_{i \in I}$ denote the collection of the fundamental groups of the cusps. For the second claim, observe that the long exact sequence in group cohomology for the group pair $(\Gamma, \mathcal{H})$ with coefficients in $\ell^\infty(\Gamma, V)$ gives a surjection from $H^{n-1}_{(\infty)}(\Gamma, \mathbb{R})$ to $\prod_{i \in I} H^{n-1}(H_i; \ell^\infty(\Gamma, \mathbb{R}))$. This in turn surjects onto $\prod_{i \in I} H^{n-1}_{(\infty)}(H_i; \mathbb{R})$ by Lemma 5.13. By Lemma 5.5, we know that each $H^{n-1}_{(\infty)}(H_i; \mathbb{R})$ has dimension at least $2^{2^{|H_i|}}$, hence the claim follows. $\square$

Proof of Theorem 1.13 There is an isomorphism $H^*_{(\infty)}(M; V) \cong H^*_{(\infty)}(\pi_1(M); V)$ [36, Remark 2.3]. If M is non-compact, then by Corollary 5.15, we obtain that $H^n_{(\infty)}(M; \mathbb{R}) = 0$ and $H^{n-1}_{(\infty)}(M; \mathbb{R})$ contains a continuum of linearly independent elements. If M is compact, then $\pi_1(M)$ is a Poincaré duality group. It is non-amenable, since $\pi_1(M)$ is a non-elementary hyperbolic group. By Lemma 5.5, it follows that $H^n_{(\infty)}(M; \mathbb{R}) = 0$. □

Corollary 5.16 *Let G be a finitely generated relatively hyperbolic group with* $\mathrm{cd}_{\mathbb{Q}} G < \infty$. *Let* Γ *be a finitely generated group with* $2 \leqslant \mathrm{cd}_{\mathbb{Q}} \Gamma < \infty$. *Suppose that* Γ *has an amenable Poincaré duality subgroup of dimension* $\mathrm{cd}_{\mathbb{Q}} \Gamma$. *If G is quasi-isometric to* Γ, *then there is a peripheral subgroup H of G such that* $\mathrm{cd}_{\mathbb{Q}} H = \mathrm{cd}_{\mathbb{Q}} G$.

Proof Suppose each peripheral subgroup H of G satisfies $\mathrm{cd}_{\mathbb{Q}} H < \mathrm{cd}_{\mathbb{Q}} G$. Since rational cohomological dimension is a quasi-isometry invariant amongst group with finite rational cohomological dimension [44, Theorem 1.2], $n := \mathrm{cd}_{\mathbb{Q}} G = \mathrm{cd}_{\mathbb{Q}} \Gamma$. By Corollary 5.14, $\mathrm{cd}_{(\infty)} G \leqslant n - 1$, implying $H^n_{(\infty)}(G; \mathbb{R}) = 0$. Since G and Γ are quasi-isometric, we have $H^n_{(\infty)}(\Gamma; \mathbb{R}) = 0$. On the other hand, by Proposition 5.6, $H^n_{(\infty)}(\Gamma; \mathbb{R}) \neq 0$, which is a contradiction. □

Corollary 5.16 applies to groups Γ satisfying the hypothesis of Corollary 5.7. We should mention that a finitely generated group which is quasi-isometric to a finitely generated relatively hyperbolic group is itself relatively hyperbolic [15]. It is known that, apart from perhaps some right-angled Artin groups and Bestvina–Brady groups, that the other classes of groups listed in Corollary 1.9 are not relatively hyperbolic. In the case of right-angled Artin groups, they are non-virtually cyclic relatively hyperbolic if and only if the associated flag complex is disconnected [3].

6 Some questions

We have only scratched the surface and many questions on the relationship between ℓ^∞-cohomology of a group and its geometric properties remain unanswered. We state some of these next.

Question 6.1 *Let H be a hyperbolically embedded subgroup of a finitely generated group G. Is* $\mathrm{cd}_{(\infty)} H \leqslant \mathrm{cd}_{(\infty)} G$?

For the definition of a hyperbolically embedded subgroup, we refer to [14]. There are many examples of such subgroups, such as parabolic subgroups of a relatively hyperbolic group; or the fact that a group is acylindrically hyperbolic if and only if it contains a non-trivial hyperbolically embedded virtually cyclic subgroup [41]. In the former case, the question has a positive answer by Corollary 5.14.

Question 6.2 *Let G be a group acting properly and cocompactly by simplicial isometries on a CAT(0)-simplicial complex X. Does* $\mathrm{cd}_{(\infty)} G$ *equal to the maximum dimension of flats in X?*

By [27, Theorem 7.1], it follows that $H^2_{(\infty)}(G; \ell^\infty) \neq 0$ if and only if X contains a flat plane.

Question 6.3 *Let G be the outer automorphism group of a finitely generated right-angled Artin group with* $\operatorname{vcd} G = d \geqslant 2$. *Is* $H_\infty^d(G, \mathbb{R}) \neq 0$?

By Corollary 5.7, we know that Question 6.3 has a positive answer in the extremal cases when the associated graph is either a set of points or it is complete. In [16], it is conjectured that G contains a poly-$\mathbb{Z}$ subgroup of Hirsch length equal to vcd G. If this conjecture holds, then Question 6.3 can be answered positively by Proposition 5.6.

Question 6.4 *Let G be a group of type $FP_n(\mathbb{Q})$ for $n \geqslant 2$. Is* $\operatorname{cd}_{(\infty)} G \leqslant n - 1$ *if and only if the filling function* $FV_{G,\mathbb{Q}}^n$ *is bounded above by a linear function?*

Question 6.4 has a positive answer for $n = 2$ by Theorems 4.7 and 4.6, and for all $n \geqslant 2$ when G is hyperbolic by [37]. We note that Question 3.9 of [2] asks if $FV_{G,\mathbb{Q}}^n(k) < \infty$ for each $k \in \mathbb{N}$ when G is of type $FP_n(\mathbb{Q})$.

Data Availability Not Applicable.

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